Equivariant cellular models in Lie theory

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Thèse de Doctorat

Mention : Mathématiques

présentée à l'***Ecole Doctorale en Sciences, Technologie, Santé (ED 585)***

de l'Université de Picardie Jules Verne

par

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pour obtenir le grade de Docteur de l'Université de Picardie Jules Verne

Modèles cellulaires équivariants en théorie de Lie

Soutenue le 10 décembre 2021, après avis des rapporteurs, devant le jury d'examen

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Remerciements

En premier lieu, je tiens à adresser mes plus vifs remerciements à mes directeurs Daniel et David, qui m'ont accompagné, suivi, épaulé, soutenu et conseillé tout au long de ce parcours fastidieux, parsemé d'embûches et de doutes, que sont les premiers pas dans la Recherche et la rédaction d'un mémoire de thèse. Vous n'avez pas compté votre temps ni vos efforts pour m'aider et je vous en suis infiniment reconnaissant. Je me souviens en particulier de ces sessions intenses de rédaction, relecture, re-rédaction et re-relectures qui m'ont fait prendre conscience des subtilités de l'écriture mathématique. Merci également à vous pour votre expertise et votre œil de lynx, ainsi que pour m'avoir encouragé à toujours aller un peu plus loin. David, merci pour ce partage d'enseignements, de m'avoir rappelé que des classes de cohomologie ça se représente et pour toute cette topologie en général ; Daniel, merci pour le joli argument de convexité de cube tronqué, pour m'avoir susurré qu'un groupe d'ordre 48 opérant sur une sphère c'est mieux qu'un groupe d'ordre 6 opérant sur une variété de drapeaux et pour toute cette géométrie ! Ce fut un plaisir et un honneur de réaliser ce travail sous votre aile.

Also a warm thanks goes out to Shizuo Kaji and Geordie Williamson for having accepted the tedious, meticulous and time-demanding work of refereeing my thesis; it is a great honor for me. Here and there, I have been a bit tough with unfriendly calculations and I hope you will forgive me for this. Shizuo, your paper on the three presentations of the equivariant cohomology of flag manifolds was very inspiring to work on and substancially made my interest for these objects grow. Again, thank you! Je remercie de même Antoine Touzé et Serge Bouc d'avoir bien voulu accepter la responsabilité de faire partie du jury. Je suis particulièrement redevable à Serge de m'avoir, au travers de son superbe cours sur les techniques fonctorielles en théorie des groupes finis, suggéré l'idée de l'élégante formulation de certains résultats sur les complexes cellulaires en termes de foncteurs inspirés des bïensembles.

Bien évidemment, je dois également un grand merci à tous les membres et ancien.ne.s du LAMFA pour m'avoir si chaleureusement accueilli et avec qui j'ai pu échanger durant ce parcours, aussi bien dans un contexte de Recherche que pour des questions d'enseignement. À ce propos, j'ai une énorme dette envers les enseignant.e.s qui m'ont accompagné et transmis tant de choses, de concepts et de points de vue durant toutes ces années à la fac. Merci à Ivan pour les représentations, merci à Olivier pour les distributions et Fourier, merci à Alexander pour les théorèmes de Sylow et l'homologie, merci à Thomas pour les fonctions holomorphes, merci à Yann pour l'arithmétique, merci à Véronique pour les équations aux dérivées partielles, merci à Alberto pour l'analyse fonctionnelle, merci à Stéphane pour la statistique et merci à Élise et Benoît pour les probabilités... Et puisqu'il s'agit ici d'enseignement, il n'est pas dithyrambique d'affirmer que mon goût de la Mathématique est né et a d'abord grandi par les soins de M. Aymeric Autin, envers lequel j'ai une gratitude infinie ; il restera mon premier Capitaine.

Un grand merci aux personnels de l'UPJV et du LAMFA et en particulier à Isabelle, Christelle, Mylène, Valérie et Caroline, sans qui je serais encore en train de me dépatouiller dans une administration qui m'échappe (voire me terrifie) totalement[1]. À cet égard, Ivan, Radu et David ont également toute ma gratitude, pour leur disponibilité et surtout pour leur patience.

Merci aux doctorant.e.s du laboratoire, avec qui j'ai passé de si bons moments, mathématiques ou non. Merci en particulier à Gauthier pour ses mots « désopilants », merci à Sylvain pour ces discussions mathématiques passionnantes et pour m'avoir remonté le moral lors de ma première conférence à Montpellier, merci à Guillaume pour ces échanges qui m'ont énormément apporté,

[1]Ivan en sait quelque-chose...

merci à Sebastian pour cette traversée nocturne Gare de Lyon-Bastille-Gare du Nord, merci à Arnaud pour ce JDR incroyable et merci à Afaf, Marouan, Henry, Clément, Alice, Arianne, Yohan, Ismaïl, Valérie, Clément, Jérémy... J'emploierai la formule consacrée : « Mes amis, ce fut un privilège de jouer avec vous ce soir. »[2].

La liste ne serait pas complète sans mes élèves, qui ont fait l'effort de m'écouter pendant ces séances parfois fastidieuses et qui, immanquablement, ont essuyé les plâtres de mes débuts dans ce monde merveilleux mais exigeant, qu'est celui de l'enseignement et de la transmission ; c'est notamment grâce à elleux que je suis motivé à progresser et m'améliorer dans ce domaine ! J'ai une pensée particulière pour Cécile, Élise, Michael, Ismaël et Arnaud qui se sont intéressé.e.s à ma thèse et qui ont participé à ces sympathiques (je l'espère!) sessions d'oraux d'agreg. Parlant pédagogie, un grand merci aussi à Ramla pour le serveur Discord spécial re-confinement.

Bien évidemment, un merci tout spécial à tous mes ami.e.s, de plus ou moins longue date, qui m'ont également soutenu, écouté et supporté durant ce parcours. Plus précisément, merci à Mathieu de m'avoir aidé à tenir le coup de la prépa et pour ne pas m'avoir laissé tomber d'une semelle depuis le CE2 ! Merci à Flo pour toutes ces discussions philosophiques, politiques et mathématiques qui ont fait évoluer nos visions du monde et nous ont permis de nous accorder sur le plus important : les définitions (et les « histoires qu'on veut bien se raconter », bien qu'on ne sache toujours pas ce qu'on va faire). Dans cette même veine, merci à Mathieu, Émile, Lucien, Mathilde, Romane, Ian, Maurane, ainsi que pour m'avoir toujours ouvert les bras avec tant de gentillesse et de bienveillance (et d'avoir toujours été partant.e.s pour une bonne bière et à propos, merci à Hafid, Nizar, Kerry, Ted, Paul, Steve, Olivier, Margarita, Sabri et tou.te.s celleux que j'oublie).

Il est bien entendu que je ne saurais trop remercier ma famille pour tout : son amour sans limite, son soutien permanent et sans faille durant ces trois dernières années bien sûr, mais aussi tout au long de mes études et en particulier pendant le marathon de l'agreg. C'est ainsi que ma plus sincère gratitude va à Marraine Régine et Tonton Jacques pour m'avoir reçu comme un prince pendant les oraux, ce qui fut d'un très grand secours. Je suis également infiniment redevable à mes grand-parents Danièle, Didier, Françoise ainsi qu'à Babeth pour m'avoir transmis le goût de la littérature et des belles lettres[3], l'amour de l'apprentissage et les joies de l'enseignement, qui ont fortement contribué à me forger. Un grand merci également à mon père pour m'avoir fait découvrir la Science et en particulier pour m'avoir appris les rudiments de l'électricité, des nombres complexes et de la loi normale. Cependant, pour ce qui est des récepteurs bêta-adrénergiques, ce n'est toujours pas clair pour moi[4]. Merci aussi à Laszlo pour ces moments de déconnades et pour ces échanges passionnants sur l'horlogerie.

Je ne sais comment te remercier, Maman, à ta juste valeur, car les mots seraient faibles ; toi qui nous as tout donné à Owen et à moi, qui t'es coupée en seize pour que nous ne manquions jamais de rien, toi qui nous as épaulés et accompagnés depuis le premier jour et qui as réussi à nous inculquer l'appétit d'apprendre, entre autres au travers de l'école et nous a patiemment écoutés et soutenus, même une fois nous être mis à faire des maths absconses[5]. Ô combien merci également pour toute cette poésie et ces échanges artistiques passionnants, ce n'est pas rien que de découvrir Baudelaire et Rimbaud de ta main ! Ni non plus d'avoir été initié à cet immense joyau qu'est la Musique Classique[6], c'est depuis un voyage de chaque instant. Nous te devons tant ! Y compris ce travail et jusqu'au pot que nous aurons tou.te.s le plaisir de partager.

[2] "Gentlemen, it has been a privilege playing with you tonight.", see [Titanic, J. Cameron, 1997]

[3] et à ce sujet, je ne puis oublier Madame Hélène Trépagne et sa regrettée sœur, Nathalie.

[4] pas plus que le culte d'Isis Marine dans le monde gréco-romain, d'ailleurs

[5] Dans ses Contemplations, Victor Hugo ne disait-il pas : « On me tordait depuis les ailes jusqu'au bec, sur l'affreux chevalet des X et des Y » ? Il affirmait aussi à propos des sciences que « C'est en les pénétrant d'explication tendre, en les faisant aimer, qu'on les fera comprendre. »

[6] J'en profite pour remercier Babeth pour les rudiments de l'écriture musicale.

De même, le verbe me manque pour te remercier, Owen. Tu as toujours été d'un inestimable secours à mon égard, fût-il ou non mathématique. Merci de m'avoir toujours permis d'essuyer les coups durs et d'avoir été mon épaule droite. Merci pour ces sessions endiablées sur League of Legends, Rocket League, Full Metal Furies, Minecraft, Cook Serve Delicious, Isaac et sans oublier le meilleur jeu que tu sais. Et puis ces soirées aux bars, passées à parler de tout et de rien, à refaire le monde : un plaisir ! Côté mathématique, tu auras toi aussi mangé de la cellule pendant trois ans. Tu as bien dû t'y faire à la topologie et tu t'en sors remarquablement bien. Pour les GRC, ne désespère pas : un jour tu trouveras une définition satisfaisante des groupes de réflexions sur tout corps en termes de schémas en groupes. J'ai hâte de poursuivre ces discussions amuse-hilarantes ! En attendant, je te souhaite une bonne exploration de B_{31} et de belles mathématiques en général[7].

Enfin, j'en arrive à remercier le plus chaleureusement et humblement du monde Delphine, l'Étoile, la Flamme qui illumine et partage ma vie depuis maintenant presque dix ans[8]. Merci tout d'abord de m'avoir suivi jusqu'ici, à Amiens. C'est un enchantement de chaque jour que de vivre à tes côtés. Tu as admirablement supporté ces nuits où, après avoir pesté sur mon code et sans avoir rien résolu du tout, je suis venu me coucher à 5h alors que tu te levais une heure après. Tu as aussi su accepter, et combattre quand nécessaire, mon humeur irritable et maussade lorsque j'étais bloqué sur un truc. Tu sais me faire rire et me donner de la force ; ce texte est empreint du courage que tu possèdes et que tu ne peux t'empêcher de communiquer autour de toi. Je t'en prie, conserve ce pouvoir ! Bien que, comme annoncé et à plus forte raison, les mots ne sauraient te rendre dignement hommage, je vais tout de même essayer pour eux.

Ces quelques lignes donc, pour modeste présent,
En gage de ma dette et de ma gratitude.
C'est ce soutien des cœurs, juste, jamais absent,
Qui élève l'Union en hautes altitudes.

Afin de, je l'espère, n'oublier personne, je conclurai par les artistes qui ont largement contribué à ma motivation. Une liste non exhaustive, dans un ordre quelconque : Jacqueline Maillan, François Rollin, Alexandre Astier, Michel Roux, Émile Zola, Benzaie, Antoine Daniel, MisterMV, Mister-JDay, The Beatles, The Smashing Pumpkins, Siouxsie, Bauhaus, Michel Petrucciani, Ryo Fukui, Emily Remler, Bill Evans, Beethoven, Mozart, Mendelssohn, Brahms, Schubert, Rachmaninov et, l'ami le plus proche de mon cœur, Chopin[9].

[7] « Je suis fier de combattre à tes côtés. »

[8] C'est l'occasion d'adresser mes vifs remerciements à ses parents et sœurs, qui m'ont toujours accepté et reçu comme l'un des leurs.

[9] sans oublier ses interprètes en général et les participant.e.s du concours éponyme, mais la liste est longue...

À Delphine, Owen, ma famille, Basile et Fryderyk.

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Introduction

Motivations

Dans de nombreux contextes mathématiques, un espace topologique ou géométrique est naturellement équipé d'une action d'un groupe. En particulier, les espaces d'orbites donnent des exemples d'espaces remarquables. On peut par exemple faire agir le groupe cyclique C_2 sur la n-sphère $\mathbb{S}^n$ par antipode et l'espace des orbites associé est l'espace projectif réel $\mathbb{RP}^n = \mathbb{S}^n/C_2$. Mais il y a également des actions de groupes infinis qui produisent des espaces agréables, tels que le n-tore $\mathbb{R}^n/\mathbb{Z}^n$. L'étude des structures et propriétés de tels espaces, et de l'action dont ils sont munis, forme le but de la *topologie algébrique équivariante*.

Pour fixer les idées, considérons un espace topologique X, muni d'une action d'un groupe W. On cherche classiquement à déterminer pour la paire (X, W) des *invariants algébriques* (fonctoriels si possible) décrivant l'action de W sur X. Par exemple, l'homologie $H_*(X, \mathbb{Z})$ possède une structure de $\mathbb{Z}[W]$-module. On peut donc toujours considérer la (co)homologie classique, mais comme une *représentation entière* de W plutôt que comme un simple groupe abélien ; gardant ainsi la trace de l'action de W. On peut également considérer l'*algèbre de cohomologie équivariante* $H^*_W(X, \mathbb{Z})$ de X (voir [Bor60] ou [Hsi75] entre autres). Quand l'action est libre, on obtient l'algèbre de cohomologie usuelle de l'espace des orbites. Notons que tout ceci a lieu pour tout anneau de coefficients, et pas seulement sur les entiers.

Par ailleurs, on peut naturellement considérer le problème inverse : étant donnée une représentation d'un groupe, peut-on trouver un espace muni d'une action du groupe et dont la (co)homologie donne la représentation souhaitée; lui donnant une interprétation géométrique. Par exemple, tout caractère complexe irréductible d'un groupe réductif fini apparait comme composant d'un *caractère de Deligne-Lusztig*, construit en prenant la cohomologie ℓ-adique d'une variété algébrique sur $\overline{\mathbb{F}_q}$ (voir [DL76]).

Ceci étant dit, dans le cas équivariant, la cohomologie donne parfois trop peu d'informations, comme nous le verrons plus bas. La *théorie des faisceaux équivariants* fournit pour cela un vocabulaire adapté. Dans le cas classique, le théorème [Bre97, Theorem III.1.1] assure que la cohomologie $H^*(X, \mathbb{Z})$ d'un espace raisonnable X (localement contractile et héréditairement paracompact, ce qui est le cas des espaces qui nous intéressent) se réalise comme *cohomologie des faisceaux* de X à coefficients dans le faisceau constant $\underline{\mathbb{Z}}$ (ici encore, ceci est vrai pour tout anneau de coefficients). C'est la cohomologie d'un complexe $R\Gamma(X, \underline{\mathbb{Z}})$, donnée en appliquant le *foncteur dérivé à droite* $R\Gamma(X, -)$ du foncteur des sections globales $\Gamma(X, -)$ au faisceau $\underline{\mathbb{Z}}$. Si on note $\mathbf{Ab}(X)$ la catégorie des faisceaux en groupes abéliens sur X, alors le foncteur exact à gauche $\Gamma(X, -) : \mathbf{Ab}(X) \longrightarrow \mathbf{Ab}$ induit un foncteur dérivé $R\Gamma(X, -) : \mathcal{D}^b(X) \longrightarrow \mathcal{D}^b(\mathbf{Ab})$, où l'on a noté $\mathcal{D}^b(X) := \mathcal{D}^b(\mathbf{Ab}(X))$ par souci de lisibilité. Dans le cas équivariant où un groupe W agit sur X, Bernstein et Lunts ([BL94]) ont défini la *catégorie dérivée équivariante* $\mathcal{D}_W(X)$ et si W est discret, alors le foncteur sections globales $\Gamma(X, -)$ induit un foncteur $\mathcal{D}_W(X) \longrightarrow \mathcal{D}^b(\mathbb{Z}[W]\text{-}\mathbf{Mod})$. De plus dans ce cas, la catégorie $\mathcal{D}_W(X)$ s'interprète comme la catégorie dérivée de la catégorie $\mathbf{Ab}_W(X)$ des faisceaux W-équivariants sur X et le foncteur $\mathcal{D}_W(X) \longrightarrow \mathcal{D}^b(\mathbb{Z}[W])$ coïncide alors avec le foncteur dérivé de $\Gamma(X, -) : \mathbf{Ab}_W(X) \longrightarrow \mathbb{Z}[W]\text{-}\mathbf{Mod}$.

Par ailleurs, dans le cas classique le complexe $R\Gamma(X, \underline{\mathbb{Z}})$ est représenté par le complexe des cochaînes singulières de X, formé de groupes abéliens et comme l'anneau $\mathbb{Z}$ est héréditaire, ce complexe est quasi-isomorphe à sa cohomologie. On n'obtient donc pas d'information plus précise avec $R\Gamma(X, \underline{\mathbb{Z}})$ qu'avec la cohomologie dans ce cas. Cependant, l'anneau $\mathbb{Z}[W]$ n'est pas héréditaire, donc il n'y a plus d'isomorphisme $R\Gamma(X, \underline{\mathbb{Z}}) \simeq H^*(X, \mathbb{Z})$ dans $\mathcal{D}^b(\mathbb{Z}[W])$. Ainsi, dans le cadre équivariant, le complexe $R\Gamma(X, \underline{\mathbb{Z}})$ donne en effet plus d'informations que la cohomologie, dans la catégorie dérivée $\mathcal{D}^b(\mathbb{Z}[W])$.

Ces foncteurs dérivés peuvent se révéler peu pratiques et difficiles à calculer explicitement. Aussi a-t-on besoin de méthodes effectives pour décrire $R\Gamma(X, \underline{\mathbb{Z}})$. Dans le cas classique, il est bien connu que déterminer une structure cellulaire sur X (en d'autres termes, décrire X comme un CW-complexe) induit un complexe de groupes abéliens libres représentant $R\Gamma(X, \underline{\mathbb{Z}})$ dans $\mathcal{D}^b(X)$. Le même raisonnement fonctionne dans le cas équivariant, à condition que le groupe agissant W soit discret et que certaines conditions de compatibilité entre la structure cellulaire et l'action soient satisfaites : W doit permuter les cellules entre elles et, si un élément de W stabilise une cellule, il doit la fixer ponctuellement. On obtient alors la notion de *W-CW-complexe* ; une telle structure sur l'espace X induit un complexe de cochaînes cellulaires de cohomologie, qui est bien un modèle pour $R\Gamma(X, \underline{\mathbb{Z}})$ dans $\mathcal{D}^b(\mathbb{Z}[W])$. Ainsi, la question de décrire $R\Gamma(X, \underline{\mathbb{Z}})$ dans la catégorie dérivée des $\mathbb{Z}[W]$-modules se ramène à déterminer une structure de W-CW-complexe sur le W-espace X, au moins quand le groupe W est supposé discret.

Un cas remarquable est celui où W est un groupe de Weyl agissant sur un espace X provenant de la théorie de Lie. Deux des principales classes d'espaces intervenant dans ce contexte sont les tores maximaux de groupes de Lie compacts et les variétés de drapeaux. Plus précisément, étant donné un groupe de Lie compact K et un tore maximal $T < K$ de K, le *groupe de Weyl* est le groupe fini $W := N_K(T)/T$, dont les éléments agissent naturellement sur T par conjugaison par des représentants dans $N_K(T)$ (cette action est bien définie car T est abélien). D'un autre côté, la *variété de drapeaux* est l'espace homogène K/T, muni de l'action à droite libre de W (par multiplication par un représentant dans $N_K(T)$).

Ces notations étant données, nous résumons donc le but de cette thèse sous forme de deux problèmes principaux. Le premier concerne les tores et leurs potentielles généralisations aux groupes de Coxeter finis :

Problème A. *Nous décomposons le problème de deux parties :*

1. *Exhiber une décomposition cellulaire W-équivariante du tore T et décrire le complexe d'homologie cellulaire équivariant associé.*
2. *Peut-on construire des espaces analogues aux tores maximaux des groupes de Lie compacts pour les groupes de Coxeter finis non-cristallographiques ?*

Le second problème est le problème central et concerne la variété de drapeaux K/T.

Problème B. *Exhiber une décomposition cellulaire W-equivariante de la variété de drapeaux K/T et décrire le complexe d'homologie cellulaire équivariant associé.*

Un exemple éclairant de variété de drapeaux est le type A_{n-1} (pour $n \geq 2$): soient $K = SU(n)$ le groupe spécial unitaire et T le groupe des matrices diagonales de K :

$$T := \left\{ \begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & * \end{pmatrix} \in SU(n) \right\} = S(U(1)^n).$$

Le normalisateur de T est donné par les matrices monomiales et le groupe de Weyl est le groupe symétrique $\mathfrak{S}_n$. Le groupe K agit naturellement sur l'ensemble des n-uplets de droites de $\mathbb{C}^n$ et il fixe (globalement) le sous-ensemble des droites orthogonales deux à deux. On s'aperçoit aisément que cette action induit une bijection

$$SU(n)/T \longrightarrow \{(L_1, \ldots, L_n)\ ;\ L_i \leq \mathbb{C}^n,\ \dim(L_i) = 1 \text{ et } L_1 \overset{\perp}{\oplus} L_2 \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} L_n = \mathbb{C}^n\}$$

et que $W = \mathfrak{S}_n$ agit par permutation des droites.

La raison pour laquelle on appelle K/T une *variété de drapeaux* est la suivante: à partir d'un n-uplet $(L_1, \ldots, L_n)$, on peut définir les sous-espaces emboîtés $V_i := L_1 \oplus \cdots \oplus L_i$ et définir le *drapeau* $(V_1, \ldots, V_n = \mathbb{C}^n)$. On obtient une application

$$SU(n)/T \longrightarrow \{(V_1, \ldots, V_n)\ ;\ V_i \leq V_{i+1} \leq \mathbb{C}^n \text{ et } \dim(V_i) = i\},$$

le second ensemble étant l'ensemble des drapeaux de $\mathbb{C}^n$. Le point important est que cette application est en fait une bijection. En effet, si $(V_1, \ldots, V_n)$ est un drapeau de $\mathbb{C}^n$, alors on peut considérer L_i l'orthogonal de V_{i-1} dans V_i (avec la convention $V_0 = 0$) et alors $L_1 \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} L_n = \mathbb{C}^n$. Ceci revient à choisir une base adaptée au drapeau et à lui appliquer le procédé de Gram-Schmidt et le n-uplet de droites obtenu est indépendant du choix d'une base adaptée. Par ailleurs, l'ensemble des drapeaux s'interprète naturellement comme un espace homogène : on peut pour s'en apercevoir considérer l'action transitive de $G := SL_n(\mathbb{C})$ sur les drapeaux (composante par composante) et le stabilisateur du drapeau standard est le sous-groupe

$$B := \left\{ \begin{pmatrix} * & * & \cdots & * \\ 0 & * & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & * \end{pmatrix} \in SL_n(\mathbb{C}) \right\}.$$

On a ainsi une application bijective

$$SU(n)/T \xrightarrow{\sim} SL_n(\mathbb{C})/B,$$

qui se trouve être un difféomorphisme. Voir la Section 7.4 pour plus de détails.

Revenant au cas général, le difféomorphisme précédent a un analogue pour tout groupe de Lie compact K : considérons la *complexification* G de K. Il s'agit d'un groupe algébrique complexe réductif contenant K comme sous-groupe compact maximal, dont l'algèbre de Lie est la complexification classique de l'algèbre de Lie de K, et on peut choisir un *sous-groupe de Borel* $B < G$ contenant T. Dans ce contexte, la *décomposition d'Iwasawa* induit un difféomorphisme $K/T \xrightarrow{\sim} G/B$ qui donne un moyen de faire agir W (non algébriquement) sur la variété projective $\mathcal{F} := G/B$. D'autre part, si $G_{\mathbb{R}}$ dénote une forme réelle *déployée* de G, alors celle-ci munit $\mathcal{F}$ d'une structure réelle et en posant $B_{\mathbb{R}} := B \cap G_{\mathbb{R}}$, on a une identification des points réels $\mathcal{F}(\mathbb{R}) \simeq G_{\mathbb{R}}/B_{\mathbb{R}}$. Par exemple, dans le cas $G = SL_n(\mathbb{C})$, le groupe $K = SU(n)$ est la forme réelle compacte de G et $G_{\mathbb{R}} := SL_n(\mathbb{R})$ est sa forme réelle déployée. Le groupe $B_{\mathbb{R}}$ est formé des matrices triangulaires supérieures dans $SL_n(\mathbb{R})$. En fait, nous avons deux involutions anti-holomorphes qui commutent $\theta_c(x) := x^* \overset{\text{def}}{=} {}^t\overline{x}$ et

$\theta_s(x) := \overline{x}$ définies sur $SL_n(\mathbb{C})$ et donnant le treillis suivant :

$$\begin{tikzcd}[column sep=small]
& SL_n(\mathbb{C}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
SU(n) = SL_n(\mathbb{C})^{\theta_c} \arrow[dr, dash] & SO(n,\mathbb{C}) = SL_n(\mathbb{C})^{\theta_c\theta_s} \arrow[d, dash] & SL_n(\mathbb{C})^{\theta_s} = SL_n(\mathbb{R}) \arrow[dl, dash] \\
& SO(n) = SL_n(\mathbb{C})^{\langle\theta_c,\theta_s\rangle} &
\end{tikzcd}$$

De plus, le procédé de Gram-Schmidt donne un difféomorphisme

$$SL_n(\mathbb{R})/B_{\mathbb{R}} \simeq SO(n)/S(O(1)^n),$$

où $S(O(1)^n)$ est le sous-groupe des matrices diagonales de $SO(n)$, qui est isomorphe à $(\mathbb{Z}/2)^{n-1}$.

La variété de drapeaux réelle $\mathcal{F}(\mathbb{R})$ est une première étape vers la résolution du problème général. Ensuite, on peut tenter de construire une décomposition cellulaire de $\mathcal{F}(\mathbb{C})$, équivariante pour l'action du groupe $W \rtimes \langle\theta_s\rangle$, à partir d'une décomposition W-équivariante de $\mathcal{F}(\mathbb{R})$.

Dans la suite, nous considérerons toujours la variété de drapeaux $\mathcal{F} = G/B$ comme étant munie de la structure réelle provenant de la structure déployée sur G et nous noterons $\mathcal{F}(\mathbb{R})$ les points réels de $\mathcal{F}$ pour cette structure.

Une motivation pour l'étude du complexe $R\Gamma(G/B, \underline{\mathbb{Z}})$ est la *théorie de Springer*, qui relie les représentations irréductibles de W à la géométrie du cône nilpotent $\mathcal{N} \subset \mathfrak{g} = \mathrm{Lie}(G)$. On note $\mathfrak{g}_{\mathrm{rs}}$ l'ouvert des éléments réguliers semi-simples de $\mathfrak{g}$. On définit $\widetilde{\mathfrak{g}} := \{(x, gB) \in \mathfrak{g} \times G/B \ ; \ x \in {}^g\mathfrak{b}\}$. La première projection $\pi : \widetilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ est appelée *résolution simultanée Grothendieck*. C'est un morphisme propre. On note $\widetilde{\mathcal{N}} := \pi^{-1}(\mathcal{N})$ et $\widetilde{\mathfrak{g}}_{\mathrm{rs}} := \pi^{-1}(\mathfrak{g}_{\mathrm{rs}})$. On obtient donc deux carrés cartésiens

$$\begin{tikzcd}
\widetilde{\mathcal{N}} \arrow[r, hook] \arrow[d, "\pi_{\mathcal{N}} := \pi_{|\widetilde{\mathcal{N}}}"'] \arrow[dr, phantom, "\lrcorner"] & \widetilde{\mathfrak{g}} \arrow[d, "\pi"] \arrow[dr, phantom, "\llcorner"] & \widetilde{\mathfrak{g}}_{\mathrm{rs}} \arrow[l, hook'] \arrow[d, "\pi_{|\widetilde{\mathfrak{g}}_{\mathrm{rs}}} =: \pi_{\mathrm{rs}}"] \\
\mathcal{N} \arrow[r, hook] & \mathfrak{g} & \mathfrak{g}_{\mathrm{rs}} \arrow[l, hook', "j_{\mathrm{rs}}"]
\end{tikzcd}$$

Le morphisme $\pi_{\mathcal{N}}$ est la *résolution de Springer* du cône nilpotent. Comme observé par Lusztig dans [Lus81], puisque le morphisme π est propre et petit (condition sur la dimension de ses fibres), le complexe décalé $\mathcal{G} := R\pi_*\underline{\mathbb{Q}}[\dim\mathfrak{g}]$ est égal au complexe d'intersection $\mathbf{IC}(\mathfrak{g}, \pi_{\mathrm{rs}*}\underline{\mathbb{Q}})$, alias le prolongement intermédiaire $j_{\mathrm{rs}!*}(\pi_{\mathrm{rs}*}\underline{\mathbb{Q}}[\dim\mathfrak{g}])$. Le système local $\pi_{\mathrm{rs}*}\underline{\mathbb{Q}}$ s'identifie à une représentation du groupe fondamental $\pi_1(\mathfrak{g}_{\mathrm{rs}})$, qui est le groupe de tresses de W. Cette représentation se factorise par le quotient W et c'est en fait la représentation régulière de W. Le foncteur $j_{\mathrm{rs}!*}$ étant pleinement fidèle, on en déduit un isomorphisme

$$\mathbb{Q}[W] \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}^b(\mathfrak{g})}(\mathcal{G}).$$

On en tire une action de W sur la cohomologie $H^*(\mathcal{B}_x, \underline{\mathbb{Q}})$ de la *fibre de Springer* $\mathcal{B}_x = \pi_{\mathcal{N}}^{-1}(x)$ en $x \in \mathcal{N}$. Lusztig conjecture que ceci mène à une nouvelle construction de la *correspondance de Springer*. Pour $x = 0$, on a $\mathcal{B}_0 = G/B$ et l'action sur $R\Gamma(G/B, \underline{\mathbb{Q}})$ coïncide avec celle induite par l'action de W sur G/B introduite plus haut.

Le *faisceau de Springer* est l'objet $\mathcal{S} := R\pi_{\mathcal{N}*}\underline{\mathbb{Q}}[\dim \mathcal{N}]$ de $\mathcal{D}^b(\mathcal{N})$. Puisque $\pi_{\mathcal{N}}$ est propre et semi-petit, le théorème de décomposition entraîne que $\mathcal{S}$ est un faisceau pervers semi-simple sur $\mathcal{N}$. Un point crucial prouvé dans [BM83] est que la restriction au cône nilpotent induit un isomorphisme

$$\mathbb{Q}[W] \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}^b(\mathfrak{g})}(\mathcal{G}) \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}^b(\mathcal{N})}(\mathcal{S})$$

Pour ce faire, Borho et MacPherson montrent que ces deux algèbres ont la même dimension, puis que le morphisme est injectif, en remarquant que l'action sur la cohomologie de la fibre en 0 est fidèle, puisque c'est le module régulier.

Dans [Jut09], Daniel Juteau a défini une version modulaire[1] de la correspondance de Springer, mais pour cela il a plutôt utilisé la transformation de Fourier-Deligne : en effet l'argument de Borho et MacPherson ne peut pas être appliqué tel quel, puisque la cohomologie de la fibre en 0 n'est plus fidèle, comme on peut le voir sur l'exemple de SL_2 : on a $H^*(SL_2(\mathbb{C})/B, \mathbb{Q}) = \mathbb{1} \oplus \varepsilon[-2]$, où ε est le caractère signe de $\mathfrak{S}_2$; et alors $H^*(SL_2(\mathbb{C})/B, \mathbb{F}_2) = \mathbb{1}_{\mathbb{F}_2} \oplus \mathbb{1}_{\mathbb{F}_2}[-2]$, qui n'est pas fidèle. Peut-être devrions-nous considérer le complexe $R\Gamma(SL_2(\mathbb{C})/B, \underline{\mathbb{F}_2})$, plutôt que sa cohomologie ?

On a un isomorphisme de variétés

$$\begin{array}{ccc} SL_2(\mathbb{C})/B & \xrightarrow{\sim} & \mathbb{CP}^1 \simeq \mathbb{S}^2 \\ \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) B & \longmapsto & [a:b] \end{array}$$

et en le pré-composant avec le difféomorphisme $SU(2) \xrightarrow{\sim} SL_2(\mathbb{C})/B$ où $T = S(U(1)^2)$, on obtient un autre difféomorphisme

$$\begin{array}{ccc} SU(2)/T & \xrightarrow{\sim} & \mathbb{CP}^1 \\ \left(\begin{smallmatrix} a & -\overline{b} \\ b & \overline{a} \end{smallmatrix}\right) T & \longmapsto & [a:b] \end{array}$$

et l'action de $\mathfrak{S}_2 = \{1, s\}$ sur $\mathbb{CP}^1$ obtenue en transportant l'action sur $SU(2)/T$ est donnée par $[a:b] \cdot s = [-\overline{b} : \overline{a}]$. Sur l'ouvert $\{ab \neq 0\}$, ceci donne $[1:z] \cdot s = [-\overline{z} : 1] = [1 : -1/\overline{z}]$ et l'action sur $\mathbb{S}^2$ obtenue en transportant ceci à nouveau et en utilisant la projection stéréographique $\mathbb{CP}^1 \simeq \mathbb{S}^2$ est l'*antipode*, c'est-à-dire que pour $x \in \mathbb{S}^2 \subset \mathbb{R}^3$ on a $x \cdot s = -x$. Il est maintenant facile de trouver une décomposition cellulaire $\mathfrak{S}_2$-équivariante de $\mathbb{S}^2$: prenons le point $e^0 := (0,0,1) \in \mathbb{S}^2$ (qui correspond à $\overline{1} \in SL_2(\mathbb{C})/B$). Il est envoyé sur $e^0 \cdot s := (0,0,-1)$ par s et ces deux points constituent une $\mathfrak{S}_2$-orbite: ils forment notre 0-squelette. Ensuite, nous définissons une 1-cellule en prenant l'arc géodésique $e^1 := \{(x,y,z) \in \mathbb{S}^2 \ ; \ z = 0, \ x > 0\}$ joignant e^0 et $e^0 \cdot s$. Il est envoyé sur son opposé $e^1 \cdot s$ par s. Ces deux 1-cellules forment notre 1-squelette. Ensuite, on définit e^2 comme étant l'hémisphère supérieure de $\mathbb{S}^2$, et son image par s est l'autre hémisphère $e_2 \cdot s$. Ces cellules forment notre 2-squelette et nous avons fini. La décomposition en résultant est illustrée dans la Figure A.

Le complexe d'homologie cellulaire associé (dont la cohomologie est $H^*(SL_2/B, \mathbb{Z})$) est donné par

$$\mathbb{Z}[\mathfrak{S}_2] \xrightarrow{1+s} \mathbb{Z}[\mathfrak{S}_2] \xrightarrow{1-s} \mathbb{Z}[\mathfrak{S}_2].$$

Maintenant, l'action de $\mathfrak{S}_2$ sur ce complexe est fidèle, même après réduction modulo 2. Ceci explique le slogan mentionné précédemment : "prendre la cohomologie fait perdre

[1] voir aussi [JMW12] et [AHJR14]

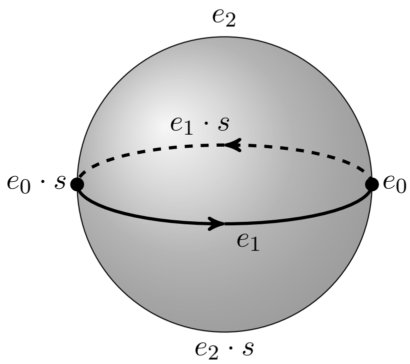


Figure A: Décomposition C_2-équivariante de $\mathbb{S}^2$.

trop d'information et l'on doit travailler au niveau dérivé". C'est pourquoi on doit calculer $R\Gamma(G/B, \underline{\mathbb{Z}})$; et décrire le complexe d'homologie des (co)chaînes associé à la structure cellulaire de G/B est un moyen naturel et efficace de répondre à cette question. Il est bon de mentionner à ce stade que l'existence abstraite d'une telle décomposition découle d'un résultat général dû à Matumoto ([Mat73]), puisque la *décomposition de Bruhat* (voir [Bum13]) donne à G/B la structure d'un CW-complexe ; mais cette dernière structure n'interagit malheureusement pas convenablement avec l'action de W.

Résultats principaux

Cette section présente quelques résultats principaux obtenus dans cette thèse. Afin de motiver l'étude des structures cellulaires équivariantes, on doit tout d'abord prouver que le complexe de cochaînes cellulaires permet effectivement de calculer $R\Gamma(X, \underline{\mathbb{Z}})$ dans la catégorie dérivée des $\mathbb{Z}[W]$-modules. C'est le but du résultat préliminaire suivant, qui affirme de plus que tous les complexes ainsi obtenus sont homotopiquement équivalents. Nous restreignons notre étude au cas d'un groupe discret, ce qui est suffisant pour la suite.

Théorème 0 (alias 2.1.11)**.** *Soient W un groupe discret et X un W-CW-complexe. Alors, le complexe de cochaînes cellulaires $C^*_{\mathrm{cell}}(X, W; \mathbb{Z})$ satisfait*

$$R\Gamma(X, \underline{\mathbb{Z}}) \simeq C^*_{\mathrm{sing}}(X, \mathbb{Z}) \simeq C^*_{\mathrm{cell}}(X, W; \mathbb{Z}) \text{ dans } \mathcal{D}^b(\mathbb{Z}[W]).$$

De plus, le complexe $C^*_{\mathrm{cell}}(X, W; \mathbb{Z})$ est indépendant de la structure de W-CW-complexe sur X, à homotopie équivariante près, i.e., deux telles structures quelconques donnent des complexes qui sont isomorphes dans la catégorie homotopique bornée $\mathcal{K}^b(\mathbb{Z}[W])$.

Comme mentionné plus haut, nous cherchons des structures cellulaires équivariantes sur les tores et les variétés de drapeaux. Nous résumons les principaux résultats de ce travail dans le tableau suivant :

Problème A : Tores maximaux	**Problème B :** Variétés de drapeaux
Théorème A1 : *Triangulation équivariante de $T < K$ et dg-anneau dans le cas où $\pi_1(K) = 1$.*	**Theorem B1 :** *Structure cellulaire équivariante sur $\mathcal{F}_3(\mathbb{R}) := SO(3)/S(O(1)^3)$ utilisant $\mathbb{P}(\overline{\mathcal{O}_{\min}})$ et le graphe de GKM.*
Théorème A2 : *Triangulation équivariante de $T < K$ dans le cas général.*	**Théorème B2 :** *Structure cellulaire équivariante sur $\mathcal{F}_3(\mathbb{R})$ à partir du groupe octaédral binaire $\mathcal{O} < \mathbb{S}^3$ d'ordre 48.*
Théorème A3 : *Construction d'un analogue W-triangulé des tores pour tout groupe de Coxeter fini irréductible.*	**Théorème B3 :** *Structure cellulaire équivariante sur $\mathcal{F}_3(\mathbb{R})$ à partir d'une métrique normale homogène et d'un domaine fondamental de Dirichlet-Voronoi.*
	Proposition B5 : *Détermination du rayon d'injectivité de $SO(n)/S(O(1)^n)$ et une estimation pour celui de $SU(n)/S(U(1)^n)$.*

Le point manquant B4 est une conjecture qui permettrait de généraliser l'approche de Dirichlet-Voronoi aux cas supérieurs ; la Proposition B5 est un premier résultat dans ce sens. Mentionnons de plus que nous fournissons deux paquets pour `GAP`. Le premier[2] permet de travailler avec des modules libres sur des algèbres de groupes en utilisant le méta-paquet `CAP`[3]. Dans le second[4] nous implémentons les complexes définis dans les Théorème A1, A2 et A3.

Tores maximaux des groupes de Lie compacts et extension aux groupes non-cristallographiques

D'abord, on étudie l'action des groupes de Weyl sur les tores (maximaux) des groupes de Lie compacts semi-simples. On utilise le vocabulaire des données radicielles, des groupes de Weyl affines et des alcôves pour formuler le premier résultat suivant, qui suppose le groupe de Lie simplement connexe. Les diagrammes de Dynkin affines sont donnés dans la Table 1.

Théorème A1 (alias 3.3.3)**.** *Soient K un groupe de Lie simple, compact et simplement connexe, $T < K$ un tore maximal et $W = N_K(T)/T$ le groupe de Weyl associé. Si W_a dénote le groupe de Weyl affine, alors l'alcôve fondamentale induit une triangulation W_a-equivariante de l'algèbre de Lie* $\mathrm{Lie}(T)$ *de T, dont le W_a-dg-anneau associé est décrit en termes de classes paraboliques. Ceci induit une triangulation W-équivariante de T et le W-dg-anneau associé est donné par*

$$C^*_{\mathrm{cell}}(T, W; \mathbb{Z}) = \mathrm{Def}^{W_\mathrm{a}}_W(C^*_{\mathrm{cell}}(\mathrm{Lie}(T), W_\mathrm{a}; \mathbb{Z})),$$

où $\mathrm{Def}^{W_\mathrm{a}}_W : \mathbb{Z}[W_\mathrm{a}]\textbf{-dgAlg} \to \mathbb{Z}[W]\textbf{-dgAlg}$ *est le foncteur de déflation.*

En particulier, on retrouve bien

$$H^\bullet(C^*_{\mathrm{cell}}(T, W; \mathbb{Z})) = H^\bullet(T, \mathbb{Z}) = \Lambda^\bullet(P).$$

[2]`https://github.com/arthur-garnier/FreeIntegralModules`
[3]`https://github.com/homalg-project/CAP_project`
[4]`https://github.com/arthur-garnier/Salvetti-and-tori-complexes`

Dans le cas général (où l'on ne suppose plus guère $\pi_1(K) = 1$), le réseau des cocaractères $Y(T)$ de T n'est plus égal au réseau des copoids $Q^\vee$ et la combinatoire précédente ne s'applique plus, puisque le groupe étendu $W_{Y(T)} := Y(T) \rtimes W$ n'est plus un groupe de Coxeter. Cependant, nous pouvons appliquer une subdivision barycentrique à l'alcôve fondamentale $\mathcal{A}$ (qui est un n-simplexe), ce qui induit dessus une triangulation $\Omega_{Y(T)}$-équivariante, où $\Omega_{Y(T)} := \{\widehat{w} \in W_{Y(T)} \ ; \ \widehat{w}(\mathcal{A}) = \mathcal{A}\} \simeq \pi_1(K)$. On a obtenu le résultat explicite suivant :

Théorème A2 (alias 4.2.3)**.** *La subdivision barycentrique de l'alcôve fondamentale du système de racine de (K,T) induit une triangulation $W_{Y(T)}$-équivariante de* $\mathrm{Lie}(T)$. *Nous décrivons la combinatoire du complexe des cochaînes cellulaires $C^*_{\mathrm{cell}}(\mathrm{Lie}(T), W_{Y(T)}; \mathbb{Z})$, ainsi que son cup-produit. Cette triangulation induit une triangulation W-équivariante de T et le W-dg-anneau associé est obtenu en appliquant le foncteur* $\mathrm{Def}_W^{W_{Y(T)}}$ *à* $C^*_{\mathrm{cell}}(\mathrm{Lie}(T), W_{Y(T)}; \mathbb{Z})$.

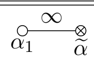
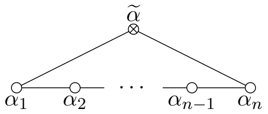
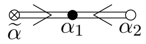
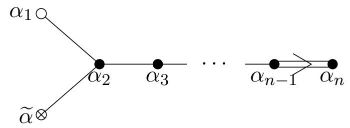
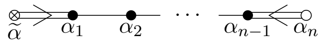
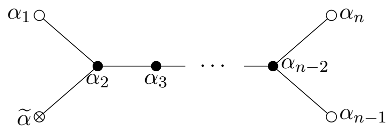
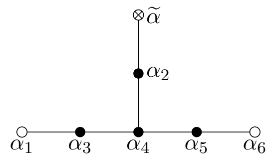
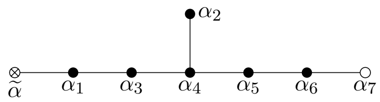
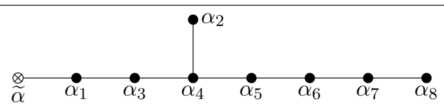
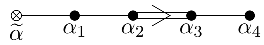
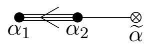
Type	Diagramme de Dynkin étendu
$\widetilde{A_1}$	
$\widetilde{A_n}$ $(n \geq 2)$	
$\widetilde{B_2} = \widetilde{C_2}$	
$\widetilde{B_n}$ $(n \geq 3)$	
$\widetilde{C_n}$ $(n \geq 3)$	
$\widetilde{D_n}$ $(n \geq 4)$	
$\widetilde{E_6}$	
$\widetilde{E_7}$	
$\widetilde{E_8}$	
$\widetilde{F_4}$	
$\widetilde{G_2}$	

Table 1: Diagrammes de Dynkin étendus des systèmes de racines irréductibles.
Les points blancs représentent les racines correspondantes aux poids minuscules et les croix représentent les plus basses racines $\widetilde{\alpha} := -\alpha_0$.

On remarque que la combinatoire du complexe dans le cas simplement connexe a un sens en réalité pour tout couple (W, r) avec W un groupe de Coxeter fini irréductible et $r \in W$ une réflexion de W. Le second point du Problème B est alors naturel : si W n'est pas cristallographique, est-il possible de choisir une telle réflexion $r \in W$ pour laquelle ce complexe est le complexe des chaînes simpliciales d'une certaine W-variété triangulée, de telle sorte que dans le cas cristallographique avec r la réflexion associée à la plus haute racine, on retrouve bien un tore maximal ? Le résultat suivant répond à la question de manière affirmative :

Théorème A3 (alias 5.3.3)**.** *Soit (W, S) un système de Coxeter fini irréductible de rang n. Étant donnée une réflexion $r \in W$, on peut considérer le système de Coxeter $(\widehat{W}, S \cup \{r\})$ dont le diagramme est celui de W muni du nœud additionnel correspondant à r, avec arêtes associées données par les ordres de sr pour $s \in S$. Alors, il existe une réflexion $r_W \in W$ telle que l'extension $\widehat{W}$ soit affine si W est un groupe de Weyl et hyperbolique compacte sinon. Si de plus $n > 2$, alors la réflexion r_W est unique pour cette propriété.*

Si $\widehat{W}$ est une telle extension, si on note $\widehat{\Sigma}$ le complexe de Coxeter de $\widehat{W}$ et $Q := \ker(\widehat{W} \twoheadrightarrow W)$, alors $\mathbf{T}(W) := \widehat{\Sigma}/Q$ est une W-variété riemannienne connexe, compacte, orientable, W-triangulée et de dimension n telle que

- *si W est un groupe de Weyl, alors $\mathbf{T}(W)$ est W-isométrique à un tore maximal du groupe de Lie compact simplement connexe dont le système de racine est celui de W,*
- *dans les autres cas, la variété $\mathbf{T}(W)$ est hyperbolique.*

Le cas particulier des groupes diédraux $I_2(m)$ présente des propriétés intéressantes, que l'on peut résumer dans l'énoncé suivant :

Corollaire (alias 5.5.1, 5.5.5 et 5.5.6)**.** *Pour $g \in \mathbb{N}^*$, les surfaces $\mathbf{T}(I_2(2g+1))$, $\mathbf{T}(I_2(4g))$ and $\mathbf{T}(I_2(4g+2))$ sont des surfaces de Riemann de genre g et définissables sur $\overline{\mathbb{Q}}$. En particulier, pour $g = 1$, ce sont des courbes elliptiques rationnelles. De plus, on a une isométrie $\mathbf{T}(I_2(2g+1)) \simeq \mathbf{T}(I_2(4g+2))$ et ces deux surfaces ne sont pas isométriques à $\mathbf{T}(I_2(4g))$.*

Notre approche permet de déterminer une présentation du groupe fondamental de $\mathbf{T}(W)$, en utilisant le théorème du domaine fondamental polyédral de Poincaré et de caractériser la représentation d'homologie de $\mathbf{T}(W)$, à l'aide de la formule de la trace de Hopf.

Proposition (alias 5.4.4, 6.2.2, 6.2.5, 6.2.6 and 6.2.7)**.** *Le groupe fondamental $\pi_1(\mathbf{T}(W)) \simeq Q$ admet une présentation explicite avec $[W : C_W(\widetilde{r})]$ générateurs, où $\widetilde{r}$ est la réflection additionnelle dans l'extension $\widehat{W}$. De cette présentation, on déduit que l'homologie $H_*(\mathbf{T}(W), \mathbb{Z})$ est sans torsion et donc les nombres de Betti sont palindromiques. De plus, nous obtenons une décomposition de la représentation d'homologie $H_*(\mathbf{T}(W), \Bbbk)$ en caractères irréductibles, où $\Bbbk$ est un corps de scindage pour W.*

Trois structures cellulaires équivariantes sur la variété de drapeaux de $SL_3(\mathbb{R})$

Après ceci, on étudie l'action du groupe de Weyl sur les variétés de drapeaux. Plus spécifiquement, on étudie la variété de drapeaux réelle $\mathcal{F}(\mathbb{R})$ de $SL_3(\mathbb{R})$, qui constitue déjà un exemple non-trivial à traiter. Utilisant le plongement $\mathcal{F}(\mathbb{R}) \hookrightarrow \mathbb{RP}^7$ induit par le plongement de $\mathcal{F} = \mathbb{P}(\overline{\mathcal{O}_{\min}})$ dans $\mathbb{P}(\mathfrak{sl}_3) \simeq \mathbb{CP}^7$, où $\mathcal{O}_{\min}$ est l'*orbite nilpotente minimale* de $SL_3(\mathbb{C})$, ainsi que le graphe de Goresky-Kottwitz-MacPherson[5] (GKM) de $W = \mathfrak{S}_3$, on obtient une première structure cellulaire sur $\mathcal{F}(\mathbb{R})$. Ceci est synthétisé dans le résultat suivant :

Théorème B1 (alias 8.3.8 et 9.2.2)**.** *La variété de drapeaux réelle $\mathcal{F}(\mathbb{R})$ de $SL_3(\mathbb{R})$ admet un structure cellulaire semi-algébrique régulière $\mathfrak{S}_3$-équivariante dont le complexe de chaînes cellulaires est donné par*

$$\mathbb{Z}[\mathfrak{S}_3]^4 \xrightarrow{\partial_3} \mathbb{Z}[\mathfrak{S}_3]^6 \xrightarrow{\partial_2} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{\partial_1} \mathbb{Z}[\mathfrak{S}_3]$$

où les bords ∂_i sont données par multiplication par les matrices suivantes

$$\partial_1 = \begin{pmatrix} 1 - s_\alpha & 1 - s_\beta & 1 - w_0 \end{pmatrix},$$

$$\partial_2 = \begin{pmatrix} -1 & 1 & 1 & s_\alpha & w_0 - s_\alpha s_\beta & s_\beta - s_\beta s_\alpha \\ s_\beta s_\alpha - s_\beta & s_\alpha - 1 & -w_0 & w_0 & s_\alpha s_\beta & s_\alpha s_\beta \\ s_\beta & s_\beta s_\alpha & s_\alpha - 1 & s_\alpha s_\beta - w_0 & -s_\beta & s_\beta s_\alpha \end{pmatrix},$$

$$\partial_3 = \begin{pmatrix} 0 & s_\alpha & 0 & 1 \\ -s_\beta s_\alpha & 0 & -w_0 & 0 \\ 0 & s_\beta s_\alpha & 1 & 0 \\ 1 & 0 & 0 & s_\beta s_\alpha \\ -s_\alpha s_\beta & s_\alpha s_\beta & 0 & 0 \\ 0 & 0 & s_\alpha s_\beta & -s_\alpha s_\beta \end{pmatrix},$$

où s_α et s_β sont les réflexions simples de $\mathfrak{S}_3$ et $w_0 := s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$ est son plus long élément.

Cette approche permet aussi de déterminer l'action de $\mathfrak{S}_3$ sur la (co)homologie $\mathcal{F}(\mathbb{R})$ et en particulier, on donne la structure de $\mathbb{F}_2$-algèbre $\mathfrak{S}_3$-équivariante sur $H^*(\mathcal{F}(\mathbb{R}), \mathbb{F}_2)$. Plus précisément, on a le résultat suivant :

Corollaire (alias 9.4.7)**.** *Soit $\mathbb{F}_2[x, y, z]_{\mathfrak{S}_3}$ l'algèbre des coïnvariants modulo 2 de $\mathfrak{S}_3$. Il existe un isomorphisme $\mathfrak{S}_3$-équivariant de $\mathbb{F}_2$-algèbres graduées*

$$\mathbb{F}_2[x, y, z]_{\mathfrak{S}_3} \xrightarrow{\sim} H^*(\mathcal{F}(\mathbb{R}), \mathbb{F}_2)$$

envoyant les indéterminées x, y et z sur des 1-cocycles algébriques irréductibles.

Ensuite, on jette un regard nouveau sur $\mathcal{F}(\mathbb{R})$. Plus spécifiquement, il se trouve que l'on a un difféomorphisme $\mathcal{F}(\mathbb{R}) = \mathbb{S}^3/\mathcal{Q}_8$, où $\mathcal{Q}_8$ est le groupe des quaternions d'ordre 8. Ceci fait de la variété $\mathcal{F}(\mathbb{R})$ une *spherical space form* et on se ramène donc à déterminer une décomposition cellulaire de la sphère $\mathbb{S}^3$, équivariante pour l'action du group octaédral binaire $\mathcal{O} = \mathcal{Q}_8 \rtimes \mathfrak{S}_3$, en utilisant la méthode de Chirivì-Spreafico. De plus, puisque le cas du groupe icosaédral binaire $\mathcal{I} \circlearrowright \mathbb{S}^3$ n'a pas été traité dans la littérature auparavant, nous l'étudions également. Dans le cas octaédral, quotienter par l'action du groupe des quaternions d'ordre 8 donne la conséquence suivante :

[5]Rappelons que le graphe de GKM d'un groupe de Weyl W a pour sommets les éléments de W et on met une arête entre w et w' s'il existe une réflexion $r \in W$ telle que $w' = wr$ et $\ell(w') > \ell(w)$.

Théorème B2 (alias 14.0.5)**.** *La variété de drapeaux réelle $\mathcal{F}(\mathbb{R})$ de $SL_3(\mathbb{R})$ admet une structure cellulaire $\mathfrak{S}_3$-équivariante, dont le complexe des chaînes cellulaires est le suivant*

$$\mathbb{Z}[\mathfrak{S}_3] \xrightarrow{\partial_3} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{\partial_2} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{\partial_1} \mathbb{Z}[\mathfrak{S}_3] \ ,$$

où

$$\partial_1 = \begin{pmatrix} 1 - s_\beta & 1 - w_0 & 1 - s_\alpha \end{pmatrix}, \quad \partial_2 = \begin{pmatrix} s_\alpha s_\beta & 1 & w_0 - 1 \\ s_\alpha - 1 & s_\alpha s_\beta & 1 \\ 1 & s_\beta - 1 & s_\alpha s_\beta \end{pmatrix}, \quad \partial_3 = \begin{pmatrix} 1 - s_\beta \\ 1 - w_0 \\ 1 - s_\alpha \end{pmatrix}.$$

Nous remarquons également que la structure cellulaire du précédent théorème possède quelques jolies propriétés par rapport à la métrique riemannienne sur $\mathcal{F}(\mathbb{R})$, induite par la métrique bi-invariante sur $SU(3)$.

Proposition (alias 16.2.2)**.** *On munit la variété algébrique complexe $\mathcal{F} = SU(3)/T$ de la métrique induite par la métrique einsteinienne bi-invariante sur $SU(3)$, puis la restreignons à $\mathcal{F}(\mathbb{R})$. Alors, les cellules de la décomposition cellulaire précédente de $\mathcal{F}(\mathbb{R})$ sont des unions de géodésiques minimales de $\mathcal{F}(\mathbb{R})$. En particulier, les 1-cellules sont les orbites de sous-groupes à un paramètre de $SO(3)$.*

Continuant l'étude de la géométrie riemannienne de $\mathcal{F}(\mathbb{R})$ et de manière à obtenir un énoncé plus intrinsèque, nous terminons par l'étude d'un *domaine de Dirichlet-Voronoi* pour $\mathfrak{S}_3$ agissant sur $\mathcal{F}(\mathbb{R})$:

$$\mathcal{DV} := \{x \in \mathcal{F}(\mathbb{R}) \ ; \ d(1,x) \leq d(w,x), \ \forall w \in W\},$$

où d est la distance géodésique sur $\mathcal{F}(\mathbb{R})$ associée à la métrique. Nous prouvons le résultat suivant :

Théorème B3 (alias 17.4.2)**.** *Le domaine de Dirichlet-Voronoi $\mathcal{DV}$ est un domaine fondamental pour $\mathfrak{S}_3$ agissant sur $\mathcal{F}(\mathbb{R})$ et admet une structure cellulaire induisant une décomposition cellulaire $\mathfrak{S}_3$-équivariante de $\mathcal{F}(\mathbb{R})$, dont le complexe d'homologie cellulaire associé est donné par*

$$\mathbb{Z}[\mathfrak{S}_3] \xrightarrow{\partial_3} \mathbb{Z}[\mathfrak{S}_3]^7 \xrightarrow{\partial_2} \mathbb{Z}[\mathfrak{S}_3]^{12} \xrightarrow{\partial_1} \mathbb{Z}[\mathfrak{S}_3]^6$$

avec bords

$$\partial_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & s_\beta & -s_\beta & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & s_\beta s_\alpha & 0 & 0 & -1 \\ -w_0 & 0 & 0 & 0 & 0 & 0 & 0 & w_0 & 0 & s_\beta & -w_0 & 0 \\ s_\beta s_\alpha & -s_\beta s_\alpha & 0 & 0 & s_\alpha & -s_\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & s_\beta s_\alpha & -s_\beta s_\alpha & 0 & 0 & 0 & 0 & 0 & -w_0 & 0 & w_0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & s_\beta & -s_\beta & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\partial_2 = \begin{pmatrix} 1 & 0 & w_0 & 0 & 0 & 0 & -w_0 \\ 1 & -s_\alpha s_\beta & 0 & 0 & -1 & 0 & 0 \\ 1 & s_\beta & 0 & -s_\beta & 0 & 0 & 0 \\ 1 & 0 & s_\alpha & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & -w_0 & 0 & 0 \\ 1 & s_\alpha & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -s_\beta & 0 \\ 1 & 0 & -s_\beta s_\alpha & -1 & 0 & 0 & 0 \\ 0 & -1 & -w_0 & 0 & -s_\beta & 0 & 0 \\ 0 & s_\beta & 1 & 0 & 0 & 0 & -s_\beta \\ 0 & w_0 & -w_0 & -1 & 0 & 0 & 0 \\ 0 & -s_\beta s_\alpha & 1 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad \partial_3 := \begin{pmatrix} 1-s_\alpha \\ 1-s_\beta \\ 1-w_0 \\ 1-s_\beta s_\alpha \\ 1-s_\alpha s_\beta \\ 1-s_\beta s_\alpha \\ 1-s_\alpha s_\beta \end{pmatrix}.$$

Chacune des trois décompositions de $SO(3)/S(O(1)^3)$ que nous avons trouvées possède ses avantages et inconvénients : la décomposition du Théorème B1 a un 1-squelette compatible avec le graphe de GKM de $\mathfrak{S}_3$ mais ne fonctionne que parce que la variété est de petite dimension et semble difficile à généraliser. Celle du Théorème B2 a peu de cellules et des degrés agréables comme nous le verrons ci-dessous mais utilise le difféomorphisme équivariant très particulier $\mathcal{F}(\mathbb{R}) \simeq \mathbb{S}^3/\mathcal{Q}_8$. Une telle identification entre une variété de drapeaux réelle et un espace d'orbites libre d'une sphère ne peut évidemment pas être attendue dans les cas supérieurs. La dernière du Théorème B3 a beaucoup de cellules, mais paraît généralisable aux autres variétés de drapeaux réelles, puisqu'elle ne repose que sur la géométrie intrinsèque de la variété considérée. Pour plus de détails à ce sujet, nous renvoyons le lecteur à la Conjecture B4 et à la Proposition B5 ci-dessous.

Quelques perspectives et conjectures

À l'issue de la rédaction de cette thèse, les thèmes restant à explorer sont nombreux.

Comme première perspective de recherche, on peut mentionner le *cas étale des tores.* Dans le deuxième chapitre on exhibe des triangulations des tores des groupes de Lie compacts, équivariantes par rapport à l'action du groupe de Weyl W. Une question naturellement reliée à ceci est le cas des groupes réductifs finis. Plus précisément, on prend un groupe réductif $\mathbf{G}$ sur un corps $\overline{\mathbb{F}_q}$ et défini sur $\mathbb{F}_q$, d'endomorphisme de Frobenius associé $F : \mathbf{G} \to \mathbf{G}$ et on se donne un tore F-stable $\mathbf{T} < \mathbf{G}$. Le *groupe réductif fini* associé est le groupe $\mathbf{G}^F$ des points F-fixes de $\mathbf{G}$ et le tore associé est $\mathbf{T}^F$. Dans ce cas, la notion de CW-complexe n'a plus de sens, mais l'on peut s'attendre à ce que le *complexe de Rickard* $R\Gamma_c(\mathbf{T}, \mathbb{Z}/n\mathbb{Z})$ (qui est un complexe de modules de permutations, analogue au complexe cellulaire provenant d'une CW-structure, voir [Ric94]) puisse être calculé via une combinatoire similaire à celle du cas des groupes de Lie. Dans le cas étale, on doit aussi prendre en compte l'action du Frobenius. Nous résumons ceci dans le problème suivant :

Problème A4. *Décrire la combinatoire du complexe de Rickard $R\Gamma_c(\mathbf{T}, \mathbb{Z}/n)$ en tant qu'objet de $\mathcal{D}^b(\mathbb{Z}/n[W \rtimes \langle F \rangle])$.*

Une deuxième piste de réflexion possible concerne notre extension de la construction des tores pour les groupes de Coxeter. Basiquement, le slogan est : “il existe des analogues des tores pour les groupes de Coxeter non-cristallographiques”. Ceci est à rapprocher de la notion de *spets.* Informellement, les spetses sont des “groupes algébriques fantômes”, d'abord associé aux groupes de Coxeter (voir les travaux pionniers de Lusztig dans [Lus93]) et plus tard aux groupes de réflexions complexes par Broué, Malle et Michel dans [BMM99]. On pourrait dire que $\mathbf{T}(W)$ est le “tore” du spets du groupe de Coxeter W. Ainsi, une question raisonnable est de demander si la construction de $\mathbf{T}(W)$ peut être étendue aux groupes de réflexions complexes (irréductibles), associant un “tore” à tout spets. Cependant, nous avons fait un très lourd usage de la *représentation de Tits* de (une extension de) W dans notre construction de $\mathbf{T}(W)$ et la méthode générale pour traiter du cas des groupes complexes n'est absolument pas claire et requiert un travail additionnel substantiel, si c'est toutefois possible. Nous formulons le problème suivant :

Problème A5. *Étant donné un groupe de réflexions complexe W, est-il possible de construire une W-variété (éventuellement compacte) généralisant la construction que nous avons donnée dans le cas Coxeter ?*

Ensuite, si T est un tore maximal dans un groupe de Lie compact K, puisque le groupe de Weyl W agit sur T, il agit aussi sur l'*espace classifiant* $B_T \simeq (\mathbb{CP}^\infty)^{\dim T}$ de T et on peut chercher une structure cellulaire W-équivariante sur B_T. En type A_1, on a $B_T = B_{\mathbb{S}^1} = \mathbb{CP}^\infty$ et l'élément non trivial s de $W = \mathfrak{S}_2$ agit comme la conjugaison complexe sur $\mathbb{CP}^\infty$. Premièrement, on partitionne $\mathbb{CP}^\infty$ en sous-espaces $A_d \simeq \mathbb{C}^d$ des éléments dont la dernière coordonnée non-nulle est la $d^{\text{ème}}$. Ensuite, on décompose A_d comme suit :

$$\mathbb{C}^d = \mathbb{R}^d \sqcup \bigsqcup_{k=1}^{d} \left(\mathbb{C}^{k-1} \times (\mathbb{C} \setminus \mathbb{R}) \times \mathbb{R}^{d-k}\right) = e_{d,0} \sqcup \bigsqcup_{k=1}^{d} (e^+_{d,k} \sqcup e^-_{d,k})$$

où $e_{d,0} = \mathbb{R}^d$ est la partie réelle et $e^\pm_{d,k} := \mathbb{C}^{k-1} \times H^\pm \times \mathbb{R}^{d-k}$ sont deux cellules de dimension $d+k$ échangées par s (on a noté $H^\pm$ les demi-plans supérieurs et inférieurs de $\mathbb{C}$). Il est pratique de numéroter les cellules par

$$\text{suites } a = (\underbrace{2, \ldots, 2}_{k\text{-fois}}, \underbrace{1, \ldots, 1}_{d-k\text{-fois}}, 0, \ldots), \text{ avec un signe } \varepsilon = \pm 1 \text{ si } k > 0;$$

on pose $e^\pm_a := e^\pm_{d,k}$, ou $e_a := e_{d,0} \subset \mathbb{RP}^\infty$ quand $k = 0$. Ainsi, e_a et $e^\pm_a$ sont des cellules de dimension $|a| = \sum_i a_i$; de plus, s fixe e_a ponctuellement et échange les $e^\pm_a$. Ensuite, on permet aux paramètres de prendre des valeurs complexes. De plus, on a une dualité de Koszul entre $H^*(T, \mathbb{Q}) = \Lambda^\bullet(\mathrm{Lie}(T)^*)$ et $H^*(B_T, \mathbb{Q}) = S^\bullet(\mathrm{Lie}(T))$ et il serait intéressant de voir si cette dualité apparaît en réalité à un niveau géométrique.

Concernant les variétés de drapeaux, une première chose potentielle à explorer est le plongement $G/B \hookrightarrow \mathbb{P}(V(\rho))$, où $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$ est la demi-somme des racines positives et $V(\rho)$ est le module irréductible de plus haut poids ρ (voir le théorème 8.1.3). Plus spécifiquement, puisque ρ est le plus petit poids dominant régulier de G, le plongement $G/B \hookrightarrow \mathbb{P}(V(\rho))$ est en quelque sorte minimal parmi les plongements de la variété de drapeaux dans un espace projectif. Il semble donc intéressant de l'étudier, par exemple pour relier l'action de W sur G/B et les propriétés de la représentation $V(\rho)$, mais cette approche semble difficile en général. En effet, on a $\dim V(\rho) = 2^{|\Phi^+|}$, donc le nombre de coordonnées explose avec le rang et de plus, l'expression de l'action du groupe de Weyl dans ces coordonnées est difficile à manipuler (voir la Proposition 8.2.5).

À propos de la combinatoire d'un potentiel complexe cellulaire $C^{\text{cell}}_*(K/T, W; \mathbb{Z})$, nous pouvons essayer de deviner une jolie formule plausible pour les rangs de ses composantes homogènes. Soit

$$P^{\mathbb{C}}_W(q) := \sum_i \#\{W\text{-orbites de } i\text{-cellules de } \mathcal{F}(\mathbb{C})\} q^i$$

et on considère de même $P^{\mathbb{R}}_W$ pour les points réels $\mathcal{F}(\mathbb{R})$. Le polynôme $P^{\mathbb{C}}_W$ doit vérifier $\deg(P^{\mathbb{C}}_W) = 2N$, où $N = |\Phi^+|$ est le nombre de réflexions de W, ainsi que $P^{\mathbb{C}}_W(-1) = 1$, puisque $\chi(\mathcal{F}) = |W|$. Une première supposition raisonnable pour $P^{\mathbb{C}}_W$ repose sur le graphe de GKM de W. À chaque racine positive $\alpha \in \Phi^+$ est associé le sous-groupe parabolique minimal $P_\alpha = \langle B, \dot{s}_\alpha \rangle$ et on a que $P_\alpha/B \simeq \mathbb{CP}^1$ est stable sous l'action du sous-groupe $\langle s_\alpha \rangle$; donc on a une "situation SL_2" pour chaque racine positive, ce que l'on peut représenter dans le diagramme suivant :

$$\begin{tikzcd} s_\alpha \arrow[d, bend right, dash] \arrow[d, bend left, dash] \\ 1 \end{tikzcd}$$

Faisant ceci pour toute réflexion de $W = \mathfrak{S}_3$ et prenant la clôture sous action de $\mathfrak{S}_3$, nous obtenons un diagramme tout-à-fait similaire au graphe de GKM :

(a) Le graphe de GKM (b) Plusieurs situations SL_2

Figure B: Le graphe de GKM de $\mathfrak{S}_3$ et le 1-squelette de $\mathcal{F}(\mathbb{R})$.

En extrapolant ceci en dimensions supérieures, nous pouvons espérer paramétrer les orbites de i-cellules de $\mathcal{F}(\mathbb{R})$ par les sous-ensembles des racines positives de cardinal i et les i-cellules seraient paramétrées par i paramètres réels (un pour chaque racine). Ceci donnerait $P_W^{\mathbb{R}}(q) = [2]_q^{|\Phi^+|}$, où $[k]_q = 1 + q + \cdots + q^{k-1}$. Pour trouver les cellules manquantes dans $\mathcal{F}(\mathbb{C})$, nous permettons à certains paramètres de prendre des valeurs complexes, de façon que chaque racine se voit attribuée une multiplicité 0, 1 (paramètre réel) ou 2 (paramètre complexe) et nous obtenons des multi-ensembles de racines positives avec multiplicités, menant à l'égalité $P_W^{\mathbb{C}}(q) = [3]_q^{|\Phi^+|}$. Cette formule a la saveur combinatoire du complexe de de Concini-Salvetti, qui est une résolution libre de $\mathbb{Z}$ sur $\mathbb{Z}[W]$, valable pour tout groupe de Coxeter fini W et construite en utilisant les chaînes croissantes de sous-ensembles des réflexions simples. Ici, on regarderait plutôt des chaînes de longueur au plus 2 de parties de Φ^+. Pour SL_3, nous obtiendrions

$$P_W^{\mathbb{R}}(q) = [2]_q^3 = q^3 + 3q^2 + 3q + 1 \quad \text{et} \quad P_W^{\mathbb{C}}(q) = [3]_q^3 = q^6 + 3q^5 + 6q^4 + 7q^3 + 6q^2 + 3q + 1.$$

Ceci donnerait une explication pour les rangs 1, 3, 3 et 1 du complexe du Théorème B2.

Une autre formule possible, n'utilisant que les racines simples, pour ce nombre d'orbites est donnée par $\prod_i [2d_i - 1]_q$, avec $(d_i)_i$ les degrés de W. Rappelons que les d_i sont les degrés des invariants fondamentaux de W et satisfont $\sum_i (d_i - 1) = N$ and $\prod_i d_i = |W|$ et donc on aurait bien $\deg\left(\prod_i [2d_i - 1]_q\right) = \sum_i (2d_i - 2) = 2N$ et $\prod_i [2d_i - 1]_{-1} = 1$. Sur $\mathbb{R}$, des considérations similaires donneraient $\prod_i [d_i]_q$. Pour SL_3, ceci donne

$$P_W^{\mathbb{R}}(q) = [2]_q [3]_q = q^3 + 2q^2 + 2q + 1 \quad \text{et} \quad P_W^{\mathbb{C}}(q) = [3]_q [5]_q = q^6 + 2q^5 + 3q^4 + 3q^3 + 3q^2 + 2q + 1.$$

La première formule $[3]_q^{|\Phi^+|}$ a un lien clair avec le graphe de GKM et semble plus simple à porter de $\mathbb{R}$ à $\mathbb{C}$, mais la seconde $\prod_i [2d_i - 1]_q$ présente moins de cellules. Malheureusement, comme le complexe du Théorème B2 est en réalité homotopiquement équivalent à un complexe avec degrés 1, 2, 2 et 1, ce cas ne nous permet pas de nous décider entre les deux possibilités. Cependant, nous n'avons pas (encore) de modèle géométrique pour ce complexe.

Cependant, la méthode la plus prometteuse pour construire une structure cellulaire équivariante sur les variétés de drapeaux réelles en général semble être l'approche riemannienne. En effet, si $\mathcal{F} := K/T$ désigne la variété de drapeaux munie d'une métrique *normale* *homogène* (i.e. une métrique provenant d'une métrique bi-invariante sur le groupe de Lie compact K) et W est le groupe de Weyl associé, alors on peut considérer le *domaine de Dirichlet-Voronoi* associé

$$\mathcal{DV} := \{x \in \mathcal{F} \ ; \ d(1,x) \leq d(w,x), \ \forall w \in W\},$$

où d est la distance géodésique sur $\mathcal{F}$. Remarquons que ceci est indépendant de la métrique normale homogène choisie puisqu'une telle métrique est unique à dilatation près. Alors, $\mathcal{DV}$ est un domaine fondamental pour W agissant sur $\mathcal{F}$ (voir Proposition 17.1.5) et l'on souhaite utiliser son intérieur et son bord pour construire une structure cellulaire sur $\mathcal{F}$. On découpe le bord en utilisant les *murs*, i.e. les intersections de $\mathcal{DV}$ avec les *hypersurfaces médiatrices* $H_w := \{x \ ; \ d(1,x) = d(w,x)\}$ et les intersections des murs sont supposées être les cellules de dimension inférieure de la décomposition recherchée. Par exemple, dans le cas des variétés hyperboliques et en particulier pour les groupes fuchsiens, le domaine de Dirichlet-Voronoi est un polyèdre (géodésique) de facettes $H_w \cap \mathcal{DV}$ et son treillis des faces fournit une décomposition cellulaire équivariante de la variété. Malheureusement, les variétés de drapeaux sont à courbure positive (non-minorée en générale) et on a alors besoin d'une condition, à savoir que le domaine $\mathcal{DV}$ doit être inclus dans une boule métrique fermée centrée en 1 et de rayon $\rho < \mathrm{inj}(\mathcal{F})$ strictement inférieur au *rayon d'injectivité* $\mathrm{inj}(\mathcal{F})$ de $\mathcal{F}$. Rappelons que ce rayon est défini comme étant le supremum des rayons de boules centrées en $0 \in T_1\mathcal{F}$ sur lesquelles l'exponentielle riemannienne est injective. Si cette condition est remplie, alors on sait au moins que le bord de $\mathcal{DV}$ est homéomorphe à la sphère $\mathbb{S}^{\dim_{\mathbb{R}} \mathcal{F}-1}$. Cependant, estimer le rayon d'injectivité d'une variété est un problème très ardu et montrer que le domaine $\mathcal{DV}$ est inclus dans une boule suffisamment petite est un problème difficile également ; et même dans le cas où la condition serait satisfaite, ceci ne garantit pas que les murs seront des cellules. Par exemple, si l'on fait agir le groupe cyclique C_2 sur $\mathbb{S}^2$ par l'antipode, le bord d'un domaine de Dirichlet-Voronoi centré en un pôle est un équateur $\mathbb{S}^1$. Néanmoins, si l'on se restreint à la sous-variété totalement géodésique $\mathcal{F}(\mathbb{R})$ de $\mathcal{F}$ et si l'on considère un domaine de Dirichlet-Voronoi dans $\mathcal{F}(\mathbb{R})$, alors ceci donne effectivement une décomposition cellulaire $\mathfrak{S}_2$-équivariante pour $\mathcal{F}(\mathbb{R}) \simeq \mathbb{S}^1$, puis pour $\mathcal{F}(\mathbb{C}) \simeq \mathbb{S}^2$. Nous conjecturons que ceci demeure en général pour les autres variétés de drapeaux.

Pour résumer, nous formulons les conjectures suivantes :

Conjecture B4 (alias 17.2.3, 17.2.1 et 17.2.2)**.** *Nous munissons $\mathcal{F} = K/T$ de la métrique induite par la forme de Killing.*

1. *Le rayon d'injectivité de $\mathcal{F}$ est la distance minimale entre deux éléments de W, réalisée par une réflexion de W.*
2. *Le domaine de Dirichlet-Voronoi $\mathcal{DV}$ associé à $\mathcal{F}$ et W est inclus dans la boule ouverte centrée en 1 et de rayon $\mathrm{inj}(\mathcal{F})$.*
3. *Si la dernière conjecture est vérifiée et si $I \subset W$, alors $\mathcal{F}(\mathbb{R}) \cap \bigcap_{w \in I} H_w$ est une union (possiblement vide) de cellules de dimension $(N - |I|)$.*

Se concentrant sur le type A où $K = SU(n)$, $\mathcal{F}_n := SU(n)/S(U(1)^n)$ et $W = \mathfrak{S}_n$, nous démontrons le résultat suivant, qui établit la première conjecture ci-dessus pour $\mathcal{F}_n(\mathbb{R})$:

Proposition B5 (alias 17.3.1 et 17.3.4)**.** *Les rayons d'injectivité de $\mathcal{F}_n$ et de $\mathcal{F}_n(\mathbb{R})$ vérifient*

$$\mathrm{inj}(\mathcal{F}_n, g_n) \geq \pi\sqrt{\frac{n}{2}} \quad \textit{et} \quad \mathrm{inj}(\mathcal{F}_n(\mathbb{R}), g_n) = \pi\sqrt{n}.$$

Guide du lecteur

Le but du présent travail est de construire des structures cellulaires sur les tores des groupes de Lie compacts et sur les variétés de drapeaux, qui soient équivariantes par rapport à l'action du groupe de Weyl.

En guise de mise en train, on rappelle la définition et quelques propriétés basiques des faisceaux équivariants. En se concentrant sur le cas où le groupe agissant est discret (ce qui est le cas des groupes que nous considérons dans la suite), on donne différentes définitions équivalentes des faisceaux équivariants et l'on définit la *catégorie dérivée équivariante* comme la catégorie dérivée des faisceaux équivariants sur notre espace. On définit ensuite la notion de *CW-complexe équivariant* telle qu'introduite par Matumoto dans [Mat71]. Là encore, profitant du caractère discret du groupe agissant, on montre que cette définition se reformule comme la donnée d'un CW-complexe, avec des conditions supplémentaires concernant l'action du groupe (voir la Definition 2.1.1). On montre dans le Corollaire 2.1.11 le fait essentiel que si W est un groupe discret et X est un W-CW-complexe, alors le complexe des chaînes cellulaires $C_*^{\text{cell}}(X, W; \mathbb{Z})$ est un complexe de $\mathbb{Z}[W]$-modules de permutation et que $R\Gamma(X, \underline{\mathbb{Z}}) = C^*_{\text{cell}}(X, W; \mathbb{Z})$ dans $\mathcal{D}^*(\mathbb{Z}[W])$. De plus, deux telles structures distinctes sur X donnent deux complexes isomorphes dans la catégorie homotopique $\mathcal{K}_b(\mathbb{Z}[W])$. On clôt ce chapitre préliminaire en étudiant le comportement du complexe cellulaire $C_*^{\text{cell}}(X, W; \mathbb{Z})$ par rapport aux sous-groupes et aux quotients de W.

Dans le chapitre suivant, on détermine des structures cellulaires sur les tores des groupes de Lie compacts via des données radicielles et des groupes de Weyl affines (étendus), qui ne sont plus finis, mais toujours discrets. Plus précisément, si K est un groupe de Lie compact simple, $T < K$ un tore maximal et Φ le système de racines associé, alors le *réseau des caractères* $X(T)$ et le *réseau des cocaractères* $Y(T)$ de T sont en dualité parfaite et le quadruplet $(X(T), \Phi, Y(T), \Phi^\vee)$ est une *donnée radicielle* (voir la Definition 3.1.1) qui détermine complètement le couple (K, T) à isomorphisme près. On a de plus un W-isomorphisme de tores $V^*/Y(T) \xrightarrow{\sim} T$, on peut ainsi oublier le groupe K et juste travailler avec une donnée radicielle irréductible donnée $(X, \Phi, Y, \Phi^\vee)$, d'espace ambiant $V := \mathbb{R} \otimes_{\mathbb{Z}} X$ et on doit déterminer une structure de W-CW-complexe sur V^*/Y. Dans ce but, on recherche une structure de W_Y-CW-complexe sur l'espace vectoriel V^*, où $W_Y := Y \rtimes W$ est le *groupe de Weyl affine étendu.* Dans le cas où $Y = Q^\vee$ est le réseau des coracines (qui correspond au cas où le groupe K est simplement connexe), le groupe $W_Y = W_{\text{a}}$ est le groupe de Weyl affine classique, qui est un groupe de Coxeter. On peut donc appliquer la combinatoire des *alcôves* et des *murs* pour obtenir une triangulation W_{a}-équivariante de V^* pour laquelle le complexe des chaînes cellulaires peut être calculé explicitement (voir le Théorème 3.2.2). De plus, on donne une formule explicite pour le cup-produit sur le complexe dual (voir le Théorème 3.3.2) et on obtient dans le Corollaire 3.3.3 le complexe pour la W-triangulation quotient de $V^*/Q^\vee$ en appliquant un foncteur de déflation à $C^*_{\text{cell}}(V^*, W_{\text{a}}; \mathbb{Z})$ depuis W_{a} vers $W = W_{\text{a}}/Q^\vee$, ce qui donne un W-dg-anneau, dont la cohomologie est $H^*(V^*/Q^\vee, \mathbb{Z}) = \Lambda^\bullet(P)$. Dans le cas général, le groupe W_Y n'est pas de Coxeter, le problème est alors que l'alcôve fondamentale possède un groupe de symétries $\Omega_Y \leq W_Y$, qui n'est pas un groupe de réflexions. Par ailleurs, dans cette situation, la *subdivision barycentrique* de l'alcôve fondamentale (qui est un simplexe) donne toujours une triangulation Ω_Y-équivariante de l'alcôve. Même si la W-triangulation de V^*/Y ainsi obtenue a un grand nombre de simplexes, cette construction a l'avantage de fonctionner pour toute donnée radicielle en plus d'être effective. Le résultat principal est résumé dans le Théorème

4.2.3. Enfin, le paquet `Salvetti-and-tori-complexes`[6] que nous avons développé pour `GAP` permet de calculer les complexes sus-mentionné pour toute donnée radicielle (irréductible).

Le but du troisième chapitre est d'étendre la combinatoire du chapitre précédent à n'importe quel groupe de Coxeter fini. Plus précisément, étant donné un groupe de Coxeter fini W, on construit une W-variété $\mathbf{T}(W)$ qui joue en quelque sorte le rôle d'un tore pour W. Plus spécifiquement, on choisit une réflexion convenable r_W de W et l'on considère le groupe de Coxeter $\widehat{W}$, dont le diagramme de Coxeter est celui de W, pourvu d'un nœud supplémentaire correspondant à la réflexion r_W, c'est-à-dire que l'on ajoute des arêtes selon les ordres des sr_W, pour s parcourant les réflexions simples de W. Nous choisissons la réflexion r_W de telle sorte que $\widehat{W}$ soit un groupe de Coxeter affine si W est un groupe de Weyl, auquel cas r_W est la réflexion associée à la *plus haute racine* du système de racines de W et $\widehat{W} = W_{\mathrm{a}}$ est alors le groupe de Weyl affine. Dans les cas restants (autrement dit les cas non-cristallographiques), on choisit r_W pour que le groupe obtenu $\widehat{W}$ soit un *groupe de Coxeter hyperbolique compact*[7]. Ceci est résumé dans la Proposition-Définition 5.1.3. On introduit ensuite le sous-groupe $Q := \ker(\widehat{W} \twoheadrightarrow W)$ et on définit la variété $\mathbf{T}(W)$ comme l'espace quotient du *complexe de Coxeter* $\Sigma(\widehat{W})$ de $\widehat{W}$ sous l'action de Q. Ceci est bien défini car Q agit librement et proprement discontinûment sur $\Sigma(\widehat{W})$ et ce, car il intersecte trivialement tous les sous-groupes paraboliques propres de $\widehat{W}$ (voir les Lemmes 5.2.1 et 5.3.2). On montre dans le Théorème 5.3.3 que $\mathbf{T}(W)$ est une W-variété fermée, connexe, orientable, compacte, W-triangulée de dimension $\mathrm{rk}(W)$. De plus, si W est un groupe de Weyl, alors ceci coïncide en effet avec un $\mathrm{rk}(W)$-tore, et il s'agit d'une *variété hyperbolique* dans le cas non-cristallographique. Après la rédaction de ce chapitre, j'ai pris connaissance des travaux de Zimmermann et Davis ([Zim93] et [Dav85]), qui ont défini les mêmes variétés pour les types H_3 et H_4, avec des approches totalement différentes. On poursuit notre étude plus avant en donnant une présentation générale pour le groupe fondamental $\pi_1(\mathbf{T}(W)) \simeq Q$ dans le Théorème 5.4.4 que l'on applique ensuite à H_3 et H_4, pour lesquels les calculs complets se trouvent dans l'Appendice B. On se penche ensuite sur le cas des groupes diédraux : dans le Corollaire 5.5.1 et les Propositions 5.5.5 et 5.5.6, on montre que $\mathbf{T}(I_2(m))$ est une *surface de Riemann arithmétique* (et même une *courbe elliptique* si $I_2(m)$ est un groupe de Weyl, ce qui donne un point de vue inhabituel sur ces tores) et on les classifie à isométrie près. Il est bon de noter que, dans le cas cristallographique (i.e quand $m \in \{3, 4, 6\}$), les surfaces $\mathbf{T}(I_2(m))$ correspondent aux points du domaine fondamental classique de Poincaré dont le stabilisateur sous l'action de $PSL_2(\mathbb{Z})$ sont non triviaux (deux d'entre eux sont dans la même orbite sous $PSL_2(\mathbb{Z})$). Enfin, on utilise la W-triangulation de $\mathbf{T}(W)$ mentionnée plus tôt pour obtenir le complexe de chaînes cellulaires $C_*^{\mathrm{cell}}(\mathbf{T}(W), W; \mathbb{Z})$ et calculer le cup-produit de son dual (voir Corollaire 6.1.2) et on termine en détaillant la représentation d'homologie de $\mathbf{T}(W)$. En utilisant la présentation de $Q = \pi_1(\mathbf{T}(W))$ obtenue précédemment, on montre que l'homologie est sans torsion et que les nombres de Betti sont donc *palindromiques*, tout comme pour un véritable tore. Finalement, on utilise la *formule de trace de Hopf* pour décomposer le caractère d'homologie en somme de caractères irréductibles de W (voir les Théorèmes 6.2.5, 6.2.6 et 6.2.7). Ceci vient enrichir les résultats de Zimmermann et Davis pour H_3 et H_4.

Dans la suite, on en vient aux variétés de drapeaux. Aussi, le quatrième chapitre est dédié à la construction d'une première structure cellulaire W-équivariante sur la variété de drapeaux $\mathcal{F}(\mathbb{R})$ de $SL_3(\mathbb{R})$. On débute par quelques rappels d'ordre général sur les groupes semi-simples et les variétés de drapeaux. En particulier, on définit et étudie l'algèbre de

[6]`https://github.com/arthur-garnier/Salvetti-and-tori-complexes`

[7]voir la Proposition-Definition 5.1.1

cohomologie rationnelle (T-équivariante) de K/T, via les trois descriptions de Schubert, Borel et Goresky-Kottwitz-MacPherson et on rappelle que la cohomologie $H^*(K/T, \mathbb{Q})$ est le $\mathbb{Q}[W]$-module régulier. Rappelons de plus qu'en général on a un difféomorphisme $\mathcal{F} = K/T \simeq G/B$, ce dernier étant une variété projective complexe lisse et la variété de drapeaux réelle $\mathcal{F}(\mathbb{R})$ s'interprète alors comme ses points réels (pour la structure réelle induite par la structure réelle déployée sur G). La première étape vers la suite est de réaliser la variété de drapeaux complexe G/B comme une variété projective, ce qui peut se faire en utilisant des *modules irréductibles de plus haut poids* et la *demi-somme des racines positives* (voir le Théorème 8.1.3 et le Corollaire 8.1.4). Ensuite, après avoir traité le cas trivial de SL_2, on se concentre sur le cas spécial de la variété de drapeaux $\mathcal{F}(\mathbb{R})$ de $SL_3(\mathbb{R})$. En utilisant le plongement précédent, on peut réaliser la variété complexe $\mathcal{F}$ comme une sous-variété de $\mathbb{CP}^7$ et on donne un ensemble complet d'équations la décrivant, comme une base de Gröbner de l'idéal associé dans l'anneau de polynômes ambiant, voir la Proposition 8.2.1. Dans la Proposition 8.2.5, on donne des équations pour l'action de $W = \mathfrak{S}_3$ sur $SL_3(\mathbb{C})/B$ dans les cartes locales. Pour construire une structure cellulaire équivariante sur $\mathcal{F}(\mathbb{R})$, on démarre avec les éléments de $W = \mathfrak{S}_3$ comme sommets. Ensuite, on utilise le graphe de GKM de $\mathfrak{S}_3$ (voir Figure B). Inspiré par le cas de SL_2 et par le graphe de GKM, on définit des 1-cellules pour $\mathcal{F}(\mathbb{R})$ que l'on peut représenter dans le graphe comme sur la Figure B. On remarque que le "1-squelette" ainsi obtenu est une union de sous-variétés fermées de la variété algébrique (réelle) $\mathcal{F}(\mathbb{R})$. En relâchant une partie des équations de définition et en imposant des conditions intermédiaires de positivité on obtient des 2-cellules et le "2-squelette" ainsi obtenu est une sous-variété algébrique réelle de $\mathcal{F}(\mathbb{R})$. Finalement, on prend les composantes connexes du complémentaire et nous prouvons que ce sont bien là des cellules, fournissant ainsi une structure cellulaire sur $\mathcal{F}(\mathbb{R})$ dont on montre qu'elle est effectivement $\mathfrak{S}_3$-équivariante (voir le Théorème 8.3.8). Ensuite, quelques calculs nous permettent de déterminer les $\mathfrak{S}_3$-orbites des cellules ainsi que les bords de ces cellules, donnant ainsi le complexe d'homologie des chaînes cellulaires et le Théorème principal 9.2.2. On utilise par la suite ce complexe pour déterminer la structure de $\mathfrak{S}_3$-module sur $H^*(\mathcal{F}(\mathbb{R}), \mathbb{Z})$. Une situation particulièrement agréable se produit en prenant la cohomologie à coefficients dans $\mathbb{F}_2$. On réalise également les classes de première cohomologie modulo 2 par neuf sous-variétés *transverses* deux à deux de $\mathcal{F}(\mathbb{R})$ (voir la Définition 9.4.1). Par transversalité et formule du produit de Poincaré, on détermine dans le Théorème 9.4.6 et le Corollaire 9.4.7 la structure de $\mathbb{F}_2$-algèbre $\mathfrak{S}_3$-équivariante sur $H^*(\mathcal{F}(\mathbb{R}), \mathbb{F}_2)$, qui se trouve être l'*algèbre des coinvariants modulo 2* de $\mathfrak{S}_3$. En tant $\mathbb{F}_2$-algèbre, ce résultat est connu depuis [Bor53a], mais ici on prend également en compte l'action de $\mathfrak{S}_3$. On termine en expliquant le lien entre nos cycles algébriques et les classes de Stiefel-Whitney des fibrés en droites universels sur $\mathcal{F}(\mathbb{R})$ (voir la Remarque 9.4.9).

Dans le cinquième chapitre, on exhibe une structure cellulaire $\mathfrak{S}_3$-équivariante sur $\mathcal{F}(\mathbb{R})$ via une méthode radicalement différente. C'est un travail en commun avec R. Chirivì et M. Spreafico [CGS20]. Plus précisément, on écrit $\mathcal{F}(\mathbb{R}) \simeq SO(3)/S(O(1)^3)$, où

$$S(O(1)^3) = \langle \mathrm{diag}(-1,-1,1), \mathrm{diag}(1,-1,-1) \rangle \simeq C_2 \times C_2$$

est le groupe de Klein. Mais comme le revêtement universel de $SO(3)$ est la sphère $\mathbb{S}^3$, on obtient une tour de revêtements, dans laquelle $\mathcal{Q}_8$ est le groupe des quaternions d'ordre 8 et $\mathcal{O} \simeq \mathcal{Q}_8 \rtimes \mathfrak{S}_3$ est le *groupe octaédral binaire* d'ordre 48, un groupe au dessus d'une flèche désignant un revêtement admettant ce groupe pour fibre,

$$\begin{tikzcd}
\mathbb{S}^3 \arrow[d, "C_2"] \arrow[dd, bend right, "\mathcal{Q}_8"'] \arrow[ddd, bend left, "\mathcal{O}"] \\
SO(3) \arrow[d, "C_2^2"] \\
\mathcal{F}(\mathbb{R}) = \mathbb{S}^3/\mathcal{Q}_8 \arrow[d, "\mathfrak{S}_3"] \\
\mathcal{F}(\mathbb{R})/\mathfrak{S}_3 = \mathbb{S}^3/\mathcal{O}.
\end{tikzcd}$$

Ainsi, au lieu de l'action $\mathfrak{S}_3 \circlearrowright \mathcal{F}(\mathbb{R})$, on peut étudier l'action du groupe $\mathcal{O}$ sur $\mathbb{S}^3$, qui est un espace plus simple. L'espace $\mathbb{S}^3/\mathcal{O}$ est une "spherical space form", c'est-à-dire une variété riemanienne compacte de courbure sectionnelle positive et constante. Cette classe d'espace est bien étudiée (voir par exemple [Mil57; Wol67; ST31]). En s'inspirant de la série d'articles [MMS13; FGMS13; FGMS16], Chirivì et Spreafico ont développé dans [CS17] une méthode générale pour construire des structures cellulaires sur les sphères, équivariantes pour l'action libre d'un groupe fini d'isométries. En particulier, ceci donne une structure cellulaires sur les "spherical space forms". Les cas restants pour $\mathbb{S}^3$ pour lesquels les déterminations explicites de décomposition équivariante n'avaient pas déjà été menées dans la littérature étaient le groupe octaédral binaire $\mathcal{O}$ et le groupe binaire icosaédral $\mathcal{I}$ d'ordre 120. Dans [CGS20], nous appliquons la méthode des *polytopes d'orbites* de Chirivì-Spreafico à ces deux cas afin d'obtenir des structures cellulaires équivariantes. L'idée principale est d'exhiber des domaines fondamentaux polytopaux dans l'enveloppe convexe du groupe (vu comme sous-ensemble de $\mathbb{R}^4$) pour les projeter ensuite sur la sphère $\mathbb{S}^3$. On commence par rappeler les résultats fondamentaux et le principe de la méthode et en particulier, on introduit le *joint courbe* et les *groupes polyédraux binaires*. On applique ensuite la méthode à $\mathcal{O}$ et $\mathcal{I}$, ainsi qu'au groupe binaire tétraédral $\mathcal{T}$ (pour lequel le résultat était déjà connu). On suit le même plan pour ces différents groupes : on trouve un domaine fondamental pour le groupe $\mathcal{G} \in \{\mathcal{T}, \mathcal{O}, \mathcal{I}\}$ dans $\mathbb{S}^3$ en y projetant un domaine fondamental polytopal et on construit une structure cellulaire en utilisant son treillis des faces. Ensuite, on calcule le complexe d'homologie cellulaire associé et on l'utilise pour trouver une résolution libre 4-périodique de $\mathbb{Z}$ sur l'algèbre de groupe entière du groupe (voir les Théorèmes 11.4.1, 12.3.1, 13.3.1 et les Corollaires 11.4.2, 12.3.3, 13.3.2). En particulier, on retrouve la cohomologie de ces groupes. Enfin, on utilise le *joint courbe* pour généraliser ceci au cas des sphères $\mathbb{S}^{4n-1}$ et interpréter la résolution libre comme une limite (quand $n \to \infty$) du complexe d'homologie cellulaire des revêtements universels des "spherical space forms" $\mathbb{S}^{4n-1}/\rho^{\oplus n}(\mathcal{G})$, où ρ est l'inclusion de $\mathcal{G}$ dans $SU(2)$ et $\rho^{\oplus n} : \mathcal{G} \hookrightarrow SU(2n)$ (voir les Théorèmes 11.4.7, 12.3.6 et 13.3.4). Il est encore bon de noter que pour $\mathcal{G} = \mathcal{I}$, la structure cellulaire $\mathcal{I}$-équivariante sur $\mathbb{S}^3$ induit une structure de CW-complexe sur la *sphère de Poincaré* $\mathbb{S}^3/\mathcal{I}$. On termine en appliquant le cas octaédral à la variété de drapeaux $\mathcal{F}(\mathbb{R})$ de $SL_3(\mathbb{R})$ pour en obtenir une structure cellulaire $\mathfrak{S}_3$-équivariante, et l'on calcule son complexe d'homologie cellulaire dans le Théorème 14.0.5. On remarque que cette décomposition a bien moins de cellules que celle obtenue au quatrième chapitre. De plus, ce complexe présente une jolie symétrie et est compatible avec la première formule attendue sur les rangs des composantes homogènes.

Le dernier chapitre présente certains espoirs de construction d'une structure cellulaire dans les cas supérieurs, au moins pour les variétés de drapeaux réelles. Plus précisément, partant du fait que le groupe compact K admet une métrique riemannienne bi-invariante, on obtient une métrique riemannienne sur sa variété de drapeaux, que l'on peut restreindre aux points réels. La variété de drapeaux $\mathcal{F}(\mathbb{R})$ de $SL_3(\mathbb{R})$ admet deux métriques naturelles : la métrique bi-invariante héritée de $SO(3)$ et celle induite en quotientant la métrique standard de $\mathbb{S}^3$ par le groupe des quaternions $\mathcal{Q}_8$. Après avoir rappelé certains résultats élémentaires de la géométrie riemannienne et des métriques bi-invariantes sur les groupes de Lie compacts, on montre dans la Proposition 15.4.2 que ces deux métriques sont proportionnelles. On décrit ensuite les géodésiques de $\mathcal{F}(\mathbb{R})$ comme des orbites de sous-groupes à un paramètre de $SO(3)$ (Proposition 16.1.1) ce qui nous permet d'interpréter les cellules de $\mathcal{F}(\mathbb{R})$ construites au chapitre 5 comme des unions de géodésiques (minimales) dans $\mathcal{F}(\mathbb{R})$ (voir le Corollaire 16.1.2 et le Théorème 16.2.2) et en particulier, les 1-cellules sont des (translatées de) géodésiques minimales entre 1 et les réflexions de $\mathfrak{S}_3$, vus comme points de $\mathcal{F}(\mathbb{R})$. Dans le cas plus général où $W \leq \mathrm{Isom}(M)$ est un groupe discret d'isométries d'une variété riemannienne connexe complète (M, g), on introduit le domaine de Dirichlet-Voronoi

$$\mathcal{DV} := \{x \in M \ ; \ \forall w \in W,\ d(x_0, x) \leq d(wx_0, x)\},$$

avec d la distance géodésique sur M et $x_0 \in M$ est un point régulier. On prouve en général (voir Proposition 17.1.5) que $\mathcal{DV}$ est un domaine fondamental connexe par arcs pour W agissant sur M. Ensuite, nous nous concentrons sur le cas où $M = \mathcal{F} = K/T$, avec K un groupe de Lie compact et W le groupe de Weyl, $\mathcal{F}$ étant munie d'une métrique normale homogène. Nous énonçons alors la Conjecture B4 sur le rayon d'injectivité de $\mathcal{F}(\mathbb{R})$. Un premier pas vers la construction effective d'une décomposition cellulaire est que, sous la condition d'injectivité, le domaine de Dirichlet-Voronoi ouvert est une cellule de dimension $2N$. Par la suite, nous nous restreignons au cas où $\mathcal{F}_n := SU(n)/T$ est de type A_{n-1} et nous donnons une estimation du rayon d'injectivité de $\mathcal{F}_n$ et $\mathcal{F}_n(\mathbb{R})$ dans le Proposition 17.3.1 et le Lemme 17.3.4. Nous voyons de plus que la métrique g_n sur $\mathcal{F}_n$ est (proportionnelle à) la restriction de la métrique produit de Fubini-Study sur $(\mathbb{CP}^{n-1})^n$, où le plongement $\mathcal{F}_n \longleftrightarrow (\mathbb{CP}^{n-1})^n$ est donné en envoyant une matrice unitaire sur le n-uplet des droites orthogonales dans $\mathbb{C}^n$ correspondant à ses colonnes, chacune d'elles étant vue comme un élément de l'espace projectif $\mathbb{CP}^{n-1}$. La distance induite d_{FS} sur $\mathcal{F}_n$ (qui est inférieure à d) se comporte bien par rapport à l'action du groupe de Weyl $\mathfrak{S}_n$. Plus précisément, nous avons une formule pour calculer des distances $d_{FS}(1, x)$ et $d_{FS}(1, xw)$ pour $w \in \mathfrak{S}_n$, en termes des coefficients de la matrice sous-jacente à $x \in \mathcal{F}$, voir le Lemme 17.3.8. Nous concluons notre étude par l'inspection du cas de $\mathcal{F}_3(\mathbb{R}) = SO(3)/\{\pm 1\}^2$. En particulier, nous donnons dans la Proposition 17.4.1 la distance maximale entre 1 et un élément $x \in \mathcal{DV}$. Nous prouvons en particulier qu'il y a exactement vingt-quatre points réalisant cette distance. Ce seront quelques-unes des 0-cellules d'une troisième décomposition cellulaire $\mathfrak{S}_3$-équivariante sur $\mathcal{F}_3(\mathbb{R})$. En effet, dans ce cas le domaine est combinatoirement équivalent à un *cube tronqué* et nous en déduisons facilement une décomposition cellulaire (polyédrale) de $\mathcal{F}_3(\mathbb{R})$. Cependant, par manque de temps nous avons uniquement vérifié ceci numériquement. Nous avons trouvé une orbite de 3-cellules, sept orbites de 2-cellules, douze orbites de 1-cellules et six orbites de 0-cellules. Nous terminons en décrivant le complexe de chaînes qui en résulte.

Introduction

Setting the stage

There are plenty of mathematical circumstances where a topological or geometric space naturally comes with an action of some group. In particular, the associated orbit spaces provide interesting examples of spaces. For instance, we can make the cyclic group C_2 act on the n-sphere $\mathbb{S}^n$ via the antipode and the resulting orbit space is the real projective space $\mathbb{RP}^n = \mathbb{S}^n/C_2$. But there are also occurrences of infinite groups acting and producing nice spaces, such as the n-torus $\mathbb{R}^n/\mathbb{Z}^n$. Studying the structure and properties of such spaces and the actions they carry is the goal of *equivariant algebraic topology*.

To fix ideas, consider a topological space X, acted on by a group W. As for classical topology, we aim to attach to the pair (X, W) *algebraic invariants* (functorial if possible) which describe the action of W on X. As an example, the homology $H_*(X, \mathbb{Z})$ is endowed with the structure of a $\mathbb{Z}[W]$-module. Hence, we still look at the classical (co)homology of X, but as an *integral representation* of W rather than only as an abelian group; therefore keeping track of the action of W. We could also consider the *equivariant cohomology algebra* $H^*_W(X, \mathbb{Z})$ of X (see [Bor60] or [Hsi75] among others). When the action is free, this is the usual cohomology algebra of the orbit space. Notice that all this holds for any ring of coefficients, not only the integers.

Besides, it is natural to consider the inverse problem: given a representation of a group, can we find a space on which the group acts and whose (co)homology yields the desired representation; giving it a geometric or topological interpretation. As an example, every complex irreducible character of a finite reductive group occurs as a component of a *Deligne-Lusztig character*, constructed taking the ℓ-adic cohomology of algebraic varieties over $\overline{\mathbb{F}_q}$ (see [DL76]).

This being said, cohomology sometimes doesn't bestow enough information, as we shall see later on. A suitable framework lies in the *theory of equivariant sheaves*. In the classical case, the theorem [Bre97, Theorem III.1.1] ensures that for a reasonable space X (locally contractible and hereditarily paracompact, which is the case for all spaces we are looking at) the cohomology $H^*(X, \mathbb{Z})$ of a reasonable space X can be obtained as the *sheaf cohomology* of X with coefficients in the constant sheaf $\underline{\mathbb{Z}}$ (here again, this holds for any coefficient ring). This is the cohomology of a complex $R\Gamma(X, \underline{\mathbb{Z}})$, given by applying the *right derived functor* $R\Gamma(X, -)$ of the global section functor $\Gamma(X, -)$ to the sheaf $\underline{\mathbb{Z}}$. If $\mathbf{Ab}(X)$ denotes the category of sheaves of abelian groups on X, then the left exact functor $\Gamma(X, -) : \mathbf{Ab}(X) \longrightarrow \mathbf{Ab}$ gives rise to a derived functor $R\Gamma(X, -) : \mathcal{D}^b(X) \longrightarrow \mathcal{D}^b(\mathbf{Ab})$, where we have denoted $\mathcal{D}^b(X) := \mathcal{D}^b(\mathbf{Ab}(X))$ for short. In the equivariant setting where a group W acts on X, Bernstein and Lunts ([BL94]) have defined the *equivariant derived category* $\mathcal{D}_W(X)$ and if W is discrete, the global section functor $\Gamma(X, -)$ yields a functor $\mathcal{D}_W(X) \longrightarrow \mathcal{D}^b(\mathbb{Z}[W]\text{-}\mathbf{Mod})$. Moreover in this case, the category $\mathcal{D}_W(X)$ can be interpreted as the derived category of the category $\mathbf{Ab}_W(X)$ of W-equivariant sheaves on X and the functor $\mathcal{D}_W(X) \longrightarrow \mathcal{D}^b(\mathbb{Z}[W])$ then coincides with the derived functor of $\Gamma(X, -) : \mathbf{Ab}_W(X) \longrightarrow \mathbb{Z}[W]\text{-}\mathbf{Mod}$.

Besides, in the classical case the complex $R\Gamma(X, \underline{\mathbb{Z}})$ is represented by the singular cochain complex of X, consisting of abelian groups and since the ring $\mathbb{Z}$ is hereditary, this complex is therefore quasi-isomorphic to its cohomology. Hence, we do not get more precise information by looking at $R\Gamma(X, \underline{\mathbb{Z}})$ rather than by looking at its cohomology. However, the ring $\mathbb{Z}[W]$

is not hereditary and there is no longer an isomorphism $R\Gamma(X, \underline{\mathbb{Z}}) \simeq H^*(X, \mathbb{Z})$ in $\mathcal{D}^b(\mathbb{Z}[W])$. Henceforth, in the equivariant setting, the complex $R\Gamma(X, \underline{\mathbb{Z}})$ indeed yields more information than the cohomology, in the derived category $\mathcal{D}^b(\mathbb{Z}[W])$.

These derived functors tend to be unwieldy and hard to calculate explicitly. Also, we need practical methods to help describe $R\Gamma(X, \underline{\mathbb{Z}})$. In the classical case, it is well-known that exhibiting a cellular structure on X (in other words, describing X as a CW-complex) yields a complex of free abelian groups representing $R\Gamma(X, \underline{\mathbb{Z}})$ in $\mathcal{D}^b(X)$. The same idea works in the equivariant case, provided that the acting group W is discrete and under some compatibility conditions between the cellular structure and the action; namely W should permute the cells and, if an element of W stabilizes a cell, then it should fix it pointwise. The resulting notion is that of a *W-CW-complex* and such a structure on the space X yields a cellular homology cochain complex, which indeed is a model for $R\Gamma(X, \underline{\mathbb{Z}})$ in $\mathcal{D}^b(\mathbb{Z}[W])$. Henceforth, the question of describing $R\Gamma(X, \underline{\mathbb{Z}})$ in the derived category of $\mathbb{Z}[W]$-modules reduces to determine a W-CW-complex structure on the W-space X, at least when the group W is assumed to be discrete.

An interesting case to emphasize is when W is a Weyl group acting on a space X coming from Lie theory. Two of the main classes of spaces arising in this context are maximal tori of compact Lie groups and flag manifolds. More precisely, given a compact Lie group K and a maximal torus $T < K$ of K, the *Weyl group* is the finite group $W := N_K(T)/T$ whose elements act naturally on T, by conjugation by representative elements in $N_K(T)$ (this is well-defined as T is abelian). On the other hand, the *flag manifold* is the homogeneous space K/T, endowed with a free right action of W (by multiplication by a representative element in $N_K(T)$).

This notation being settled, we summarize the aim of this thesis in two main problems. The first one concerns tori and their potential generalizations to finite Coxeter groups:

Problem A. *We split the problem into two parts:*

1. *Exhibit a W-equivariant cellular decomposition of the torus T and describe the associated equivariant cellular homology chain complex.*
2. *Is it possible to construct spaces that are analogous to tori of maximal compact Lie groups for non-crystallographic finite Coxeter groups?*

The second problem is the central one and is about the flag manifold K/T.

Problem B. *Exhibit a W-equivariant cellular decomposition of the flag manifold K/T and describe the associated equivariant cellular homology chain complex.*

An enlightening example of flag manifolds is in type A_{n-1} (for $n \geq 2$): let $K = SU(n)$ be the special unitary group and take T to be the group of diagonal matrices of K, that is

$$T := \left\{ \begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & * \end{pmatrix} \in SU(n) \right\} = S(U(1)^n).$$

The normalizer of T consists of monomial matrices and the Weyl group is the symmetric group $\mathfrak{S}_n$. Now, the group K acts naturally on the set of n-tuples of lines in $\mathbb{C}^n$ and it (globally) stabilizes the subset of pairwise orthogonal lines. It is easy to see that this action yields a bijection

$$SU(n)/T \longrightarrow \{(L_1,\dots,L_n)\ ;\ L_i \leq \mathbb{C}^n,\ \dim(L_i)=1 \text{ and } L_1 \overset{\perp}{\oplus} L_2 \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} L_n = \mathbb{C}^n\}$$

and that $W = \mathfrak{S}_n$ acts on it by simply permuting the lines.

The reason for K/T to be called a *flag manifold* lies in the following fact: from an n-tuple $(L_1,\dots,L_n)$, we can define the nested subspaces $V_i := L_1 \oplus \cdots \oplus L_i$ and form the *flag* $(V_1,\dots,V_n = \mathbb{C}^n)$ and we get a map

$$SU(n)/T \longrightarrow \{(V_1,\dots,V_n)\ ;\ V_i \leq V_{i+1} \leq \mathbb{C}^n \text{ and } \dim(V_i) = i\},$$

the later set being the set of flags of $\mathbb{C}^n$. The point is that this map is in fact a bijection. Indeed, if $(V_1,\dots,V_n)$ is a flag in $\mathbb{C}^n$, then we may consider the orthogonal L_i of V_{i-1} in V_i (with $V_0 = 0$, by convention) and then $L_1 \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} L_n = \mathbb{C}^n$. This can also be constructed by choosing a basis of $\mathbb{C}^n$ which is adapted to the flag and apply the Gram-Schmidt process to it. The resulting n-tuple of lines is independent of the chosen adapted basis. On the other hand, there also is a natural way of interpreting the set of flags as a homogeneous space. To do this we can consider the transitive action of $G := SL_n(\mathbb{C})$ on flags (on each component) and the stabilizer of the standard flag is the subgroup

$$B := \left\{ \begin{pmatrix} * & * & \cdots & * \\ 0 & * & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & * \end{pmatrix} \in SL_n(\mathbb{C}) \right\}.$$

Therefore, we have a bijective map

$$SU(n)/T \xrightarrow{\sim} SL_n(\mathbb{C})/B,$$

which turns out to be a diffeomorphism. See Section 7.4 for more details.

Back to the general case, the latter diffeomorphism has an analogue for every compact Lie group K: consider the *complexification* G of K. This is a reductive complex algebraic group containing K as a maximal compact subgroup, whose Lie algebra is the classical complexification of the Lie algebra of K and we may choose a *Borel subgroup* $B < G$ containing T. In this context, the *Iwasawa decomposition* yields a diffeomorphism $K/T \xrightarrow{\sim} G/B$ and this gives a way of making W actually act (non algebraically) on the projective variety $\mathcal{F} := G/B$. On another hand, if $G_{\mathbb{R}}$ denotes a *split* real form of G, then it endows $\mathcal{F}$ with a real structure and letting $B_{\mathbb{R}} := B \cap G_{\mathbb{R}}$, we get an identification of the real points $\mathcal{F}(\mathbb{R}) \simeq G_{\mathbb{R}}/B_{\mathbb{R}}$. For instance, in the case where $G = SL_n(\mathbb{C})$, the group $K = SU(n)$ is the compact real form of G and $G_{\mathbb{R}} := SL_n(\mathbb{R})$ is its split real form. The group $B_{\mathbb{R}}$ consists of the upper-triangular matrices in $SL_n(\mathbb{R})$. In fact, we have two commuting anti-holomorphic involutions $\theta_c(x) := x^* \overset{\text{df}}{=} {}^t\overline{x}$ and $\theta_s(x) := \overline{x}$ defined on $SL_n(\mathbb{C})$ and yielding the following lattice:

$$\begin{tikzcd}[column sep=small] & SL_n(\mathbb{C}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ SU(n) = SL_n(\mathbb{C})^{\theta_c} \arrow[dr, dash] & SO(n,\mathbb{C}) = SL_n(\mathbb{C})^{\theta_c\theta_s} \arrow[d, dash] & SL_n(\mathbb{C})^{\theta_s} = SL_n(\mathbb{R}) \arrow[dl, dash] \\ & SO(n) = SL_n(\mathbb{C})^{\langle\theta_c,\theta_s\rangle} & \end{tikzcd}$$

Furthermore, the Gram-Schmidt process induces a diffeomorphism

$$SL_n(\mathbb{R})/B_{\mathbb{R}} \simeq SO(n)/S(O(1)^n),$$

where $S(O(1)^n)$ is the subgroup of diagonal matrices in $SO(n)$, which is isomorphic to $(\mathbb{Z}/2)^{n-1}$.

The real flag manifold $\mathcal{F}(\mathbb{R})$ is a first step toward a solution of the general problem. Next, we can try to construct a cellular decomposition of $\mathcal{F}(\mathbb{C})$, equivariant with respect to the action of the group $W \rtimes \langle \theta_s \rangle$, from a W-equivariant decomposition of $\mathcal{F}(\mathbb{R})$.

In the sequel, we always endow the flag manifold $\mathcal{F} = G/B$ with the real structure induced by the split real structure on G and we shall denote by $\mathcal{F}(\mathbb{R})$ the real points of $\mathcal{F}$, with respect to this structure.

Notice moreover that the fact that G is the complexified group of K endows it with a real structure, therefore $\mathcal{F}$ is also endowed with a real structure and we may consider its real points $\mathcal{F}(\mathbb{R})$. We will adopt this notation in the sequel.

A motivation for studying the complex $R\Gamma(G/B, \underline{\mathbb{Z}})$ is *Springer theory*, which relates the irreducible representations of W with the geometry of the nilpotent cone $\mathcal{N} \subset \mathfrak{g} = \mathrm{Lie}(G)$. We denote by $\mathfrak{g}_{\mathrm{rs}}$ the open subset of regular semisimple elements of $\mathfrak{g}$. We define $\widetilde{\mathfrak{g}} := \{(x, gB) \in \mathfrak{g} \times G/B \ ; \ x \in {}^g\mathfrak{b}\}$. The first projection $\pi_{\mathfrak{g}} : \widetilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ is called the *Grothendieck simultaneous resolution*. It is a proper morphism. We denote $\widetilde{\mathcal{N}} := \pi^{-1}(\mathcal{N})$ and $\widetilde{\mathfrak{g}}_{\mathrm{rs}} := \pi^{-1}(\mathfrak{g}_{\mathrm{rs}})$. We then get two Cartesian squares

$$\begin{tikzcd}
\widetilde{\mathcal{N}} \arrow[r, hook] \arrow[d, "\pi_{\mathcal{N}} := \pi_{|\widetilde{\mathcal{N}}}"'] \arrow[dr, phantom, "\lrcorner"] & \widetilde{\mathfrak{g}} \arrow[d, "\pi"] \arrow[dr, phantom, "\llcorner"] & \widetilde{\mathfrak{g}}_{\mathrm{rs}} \arrow[l, hook'] \arrow[d, "\pi_{|\widetilde{\mathfrak{g}}_{\mathrm{rs}}} =: \pi_{\mathrm{rs}}"] \\
\mathcal{N} \arrow[r, hook] & \mathfrak{g} & \mathfrak{g}_{\mathrm{rs}} \arrow[l, hook', "j_{\mathrm{rs}}"]
\end{tikzcd}$$

The morphism $\pi_{\mathcal{N}}$ is the *Springer resolution* of the nilpotent cone. As observed by Lusztig in [Lus81], since the morphism π is proper and small (a condition on the dimension of its fibers), the shifted complex $\mathcal{G} := R\pi_* \underline{\mathbb{Q}}[\dim \mathfrak{g}]$ equals the intersection complex $\mathbf{IC}(\mathfrak{g}, \pi_{\mathrm{rs}*}\underline{\mathbb{Q}})$, alias the intermediate extension $j_{\mathrm{rs}!*}(\pi_{\mathrm{rs}*}\underline{\mathbb{Q}}[\dim \mathfrak{g}])$. The local system $\pi_{\mathrm{rs}*}\underline{\mathbb{Q}}$ identifies with a representation of the fundamental group $\pi_1(\mathfrak{g}_{\mathrm{rs}})$, which is the braid group of W. This representation factorizes through the quotient W and this turns out to be the regular representation of W. The functor $j_{\mathrm{rs}!*}$ being fully faithful, we deduce an isomorphism

$$\mathbb{Q}[W] \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}^b(\mathfrak{g})}(\mathcal{G}).$$

From this we get an action of W on the cohomology $H^*(\mathcal{B}_x, \underline{\mathbb{Q}})$ of the *Springer fiber* $\mathcal{B}_x = \pi^{-1}(x)$ at $x \in \mathcal{N}$. Lusztig conjectures that this leads to a new construction of the *Springer correspondence*. For $x = 0$, we have $\mathcal{B}_0 = G/B$ and the action on $R\Gamma(G/B, \underline{\mathbb{Q}})$ coincides with the one induced by the action of W on G/B mentioned above.

The *Springer sheaf* is the object $\mathcal{S} := R\pi_{\mathcal{N}*}\underline{\mathbb{Q}}[\dim \mathcal{N}]$ of $\mathcal{D}^b(\mathcal{N})$. As $\pi_{\mathcal{N}}$ is proper and semi-small, by the decomposition theorem, $\mathcal{S}$ is a semisimple perverse sheaf on $\mathcal{N}$. A crucial point proved in [BM83] is that the restriction to the nilpotent cone induces an isomorphism

$$\mathbb{Q}[W] \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}^b(\mathfrak{g})}(\mathcal{G}) \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}^b(\mathcal{N})}(\mathcal{S})$$

To do this, Borho and MacPherson show that these two algebras have the same dimension and then that the morphism is injective, by noticing that the action on the cohomology of the 0-stalk is faithful, since it is the regular module.

In [Jut09], Daniel Juteau establishes a modular version[8] of the Springer correspondence, but for this he rather uses the Fourier-Deligne transform: indeed Borho-MacPherson's argument cannot be applied directly, because the cohomology of the 0-stalk is no longer faithful, as we can see on the example of SL_2: we have $H^*(SL_2(\mathbb{C})/B, \mathbb{Q}) = \mathbb{1} \oplus \varepsilon[-2]$, where ε is the sign character of $\mathfrak{S}_2$; and thus $H^*(SL_2(\mathbb{C})/B, \mathbb{F}_2) = \mathbb{1}_{\mathbb{F}_2} \oplus \mathbb{1}_{\mathbb{F}_2}[-2]$, which is not faithful. Perhaps we should consider the complex $R\Gamma(SL_2(\mathbb{C})/B, \underline{\mathbb{F}_2})$ rather than its cohomology?

There is an isomorphism of varieties

$$\begin{array}{ccc} SL_2(\mathbb{C})/B & \xrightarrow{\sim} & \mathbb{CP}^1 \simeq \mathbb{S}^2 \\ \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) B & \longmapsto & [a:b] \end{array}$$

and pre-composing this with the diffeomorphism $SU(2)/T \xrightarrow{\sim} SL_2(\mathbb{C})/B$ where $T = S(U(1)^2)$, we obtain another diffeomorphism

$$\begin{array}{ccc} SU(2)/T & \xrightarrow{\sim} & \mathbb{CP}^1 \\ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} T & \longmapsto & [a:b] \end{array}$$

and the action of $\mathfrak{S}_2 = \{1, s\}$ on $\mathbb{CP}^1$ obtained by transporting the action on $SU(2)/T$ is given by $[a:b] \cdot s = [-\overline{b} : \overline{a}]$. On the open subset $\{ab \neq 0\}$, this reads $[1:z] \cdot s = [-\overline{z} : 1] = [1 : -1/\overline{z}]$ and the action on $\mathbb{S}^2$ obtained by transporting this again using the stereographic projection $\mathbb{CP}^1 \simeq \mathbb{S}^2$ is the *antipode*, that is, for $x \in \mathbb{S}^2 \subset \mathbb{R}^3$ we have $x \cdot s = -x$. It is now easy to find an $\mathfrak{S}_2$-equivariant cellular structure on $\mathbb{S}^2$: take the particular point $e^0 := (0,0,1) \in \mathbb{S}^2$ (which corresponds to $\overline{1} \in SL_2(\mathbb{C})/B$). It is sent to $e^0 \cdot s := (0,0,-1)$ by s and these two points form a $\mathfrak{S}_2$-orbit: they shall form our 0-skeleton. Next, we define a 1-cell by taking the geodesic arc $e^1 := \{(x,y,z) \in \mathbb{S}^2 \ ; \ z = 0, \ x > 0\}$ joining e^0 and $e^0 \cdot s$. It is sent by s to its opposite $e^1 \cdot s$. These two 1-cells form the 1-skeleton. Then we take e^2 to be the upper spherical cap of $\mathbb{S}^2$, and its image under s is the other cap $e_2 \cdot s$. These cells form our 2-skeleton and we are done. The resulting decomposition is depicted in Figure A.

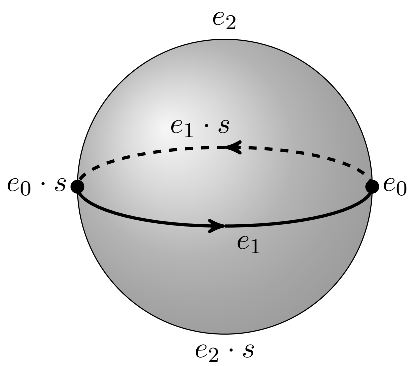


Figure A: A C_2-equivariant decomposition of $\mathbb{S}^2$.

The associated cellular complex (with cohomology $H^*(SL_2/B, \mathbb{Z})$) is given by

$$\mathbb{Z}[\mathfrak{S}_2] \xrightarrow{1+s} \mathbb{Z}[\mathfrak{S}_2] \xrightarrow{1-s} \mathbb{Z}[\mathfrak{S}_2].$$

Now, the action of $\mathfrak{S}_2$ on this complex is faithful, even after reduction modulo 2. This explains the slogan mentioned earlier: "taking cohomology looses too much information

[8] see also [JMW12] and [AHJR14]

and we have to work at a derived level". This is why we have to compute $R\Gamma(G/B, \underline{\mathbb{Z}})$; and describing the homology (co)chain complex associated to a cellular structure on G/B is a natural and efficient way to achieve this. It should be noted that such a structure abstractly exists by a general result of Matumoto ([Mat73]), since the *Bruhat decomposition* (see [Bum13]) gives G/B the structure of a CW-complex; but this structure of course doesn't behave well under the action of W.

Main results

This section presents some of the main results obtained in this thesis. In order to motivate the study of equivariant cellular structures, we first have to show that the associated cohomology cochain complex indeed computes $R\Gamma(X, \underline{\mathbb{Z}})$ in the derived category of $\mathbb{Z}[W]$-modules. This is done in the following preliminary result, which states further that all the complexes obtained this way are homotopy equivalent. We restrict our study to the action of a discrete group, which is enough for the sequel.

Theorem 0 (alias 2.1.11)**.** *Let W be a discrete group and X be a W-CW-complex. Then, the associated cellular cohomology cochain complex $C^*_{\mathrm{cell}}(X, W; \mathbb{Z})$ satisfies*

$$R\Gamma(X, \underline{\mathbb{Z}}) \simeq C^*_{\mathrm{sing}}(X, \mathbb{Z}) \simeq C^*_{\mathrm{cell}}(X, W; \mathbb{Z}) \text{ in } \mathcal{D}^b(\mathbb{Z}[W]).$$

Furthermore, the complex $C^*_{\mathrm{cell}}(X, W; \mathbb{Z})$ is independent of the chosen W-CW-structure on X, up to equivariant homotopy, i.e., any two such structures give complexes that are isomorphic in the bounded homotopy category $\mathcal{K}^b(\mathbb{Z}[W])$.

As mentioned before, we look for equivariant cellular structures on tori and flag manifolds. The following table summarizes the main results of this work:

Problem A: Maximal tori	**Problem B:** Flag manifolds
Theorem A1: *Equivariant triangulation of $T < K$ and dg-ring in the case where $\pi_1(K) = 1$.*	**Theorem B1:** *Equivariant cell structure on $\mathcal{F}_3(\mathbb{R}) := SO(3)/S(O(1)^3)$ using $\mathbb{P}(\overline{\mathcal{O}_{\min}})$ and the GKM graph.*
Theorem A2: *Equivariant triangulation of $T < K$ in the general case.*	**Theorem B2:** *Equivariant cell structure on $\mathcal{F}_3(\mathbb{R})$ from the binary octahedral group $\mathcal{O} < \mathbb{S}^3$ of order 48.*
Theorem A3: *Construction of a W-triangulated analogue of tori for all finite irreducible Coxeter groups.*	**Theorem B3:** *Equivariant cell structure on $\mathcal{F}_3(\mathbb{R})$ from a normal homogeneous metric and a Dirichlet-Voronoi fundamental domain.*
	Proposition B5: *Determination of the injectivity radius of $SO(n)/S(O(1)^n)$ and an estimate for the one of $SU(n)/S(U(1)^n)$.*

The missing point B4 is a conjecture, that allows to generalize the Dirichlet-Voronoi approach to higher cases; Proposition B5 is a first result in this direction. Moreover, we provide two packages for `GAP`. The first one[9] allows to work with free modules over a group algebra using the meta-package `CAP`[10]. In the second one[11] we implement the complexes defined in Theorems A1, A2 and A3.

Maximal tori of compact Lie groups and extension to non-crystallographic Coxeter groups

First, we study the case of Weyl groups acting on (maximal) tori of semisimple compact Lie groups. We use the vocabulary of root data, affine Weyl groups and alcoves to formulate the following first result, which assumes that the ambient Lie group is simply-connected. The affine Dynkin diagrams are displayed in Table 1.

Theorem A1 (alias 3.3.3)**.** *Let K be a simply-connected simple compact Lie group, $T < K$ be a maximal torus and $W = N_K(T)/T$ be the associated Weyl group. If W_{a} denotes the affine Weyl group, then the fundamental alcove induces a W_{a}-equivariant triangulation of the Lie algebra $\mathrm{Lie}(T)$ of T, whose W_{a}-dg-ring $C^*_{\mathrm{cell}}(\mathrm{Lie}(T), W_{\mathrm{a}}; \mathbb{Z})$ is described in terms of parabolic cosets. This induces a W-equivariant triangulation of T and the associated W-dg-ring is given by*

$$C^*_{\mathrm{cell}}(T, W; \mathbb{Z}) = \mathrm{Def}_W^{W_{\mathrm{a}}}(C^*_{\mathrm{cell}}(\mathrm{Lie}(T), W_{\mathrm{a}}; \mathbb{Z})),$$

where $\mathrm{Def}_W^{W_{\mathrm{a}}} : \mathbb{Z}[W_{\mathrm{a}}]$-**dgAlg** $\to \mathbb{Z}[W]$-**dgAlg** *is the deflation functor.*

In particular, we retrieve indeed

$$H^\bullet(C^*_{\mathrm{cell}}(T, W; \mathbb{Z})) = H^\bullet(T, \mathbb{Z}) = \Lambda^\bullet(P).$$

In the general case (where we no longer assume $\pi_1(K) = 1$), the cocharacter lattice $Y(T)$ of T does no longer equal the coroot lattice $Q^\vee$ and the previous combinatorics doesn't apply, because the extended group $W_{Y(T)} := Y(T) \rtimes W$ is no longer a Coxeter group. However, we may apply a barycentric subdivision to the fundamental alcove $\mathcal{A}$ (which is an n-simplex), which induces an $\Omega_{Y(T)}$-equivariant triangulation of it, where $\Omega_{Y(T)} := \{\widehat{w} \in W_{Y(T)} \ ; \ \widehat{w}(\mathcal{A}) = \mathcal{A}\} \simeq \pi_1(K)$. We have obtained the following explicit result:

Theorem A2 (alias 4.2.3)**.** *The barycentric subdivision of the fundamental alcove of the root system of (K, T) induces an $W_{Y(T)}$-equivariant triangulation of $\mathrm{Lie}(T)$. We describe the combinatorics of the resulting cohomology cochain complex $C^*_{\mathrm{cell}}(\mathrm{Lie}(T), W_{Y(T)}; \mathbb{Z})$, as well as its cup product. This triangulation induces a W-equivariant triangulation of T and the associated W-dg-ring is given by applying the functor $\mathrm{Def}_W^{W_{Y(T)}}$ to $C^*_{\mathrm{cell}}(\mathrm{Lie}(T), W_{Y(T)}; \mathbb{Z})$.*

We notice that the combinatorics of the complex in the simply-connected case makes sense for every pair (W, r) with W a finite irreducible Coxeter group and $r \in W$ is a reflection in W. The second part of Problem B is quite natural: if W is non-crystallographic, is it possible to choose such a reflection $r \in W$ for which this complex is the simplicial chain complex of some triangulated W-manifold, in such a way that in the crystallographic case, with r the reflection associated to the highest root, we find indeed a maximal torus? The following result affirmatively answers the question:

[9]`https://github.com/arthur-garnier/FreeIntegralModules`
[10]`https://github.com/homalg-project/CAP_project`
[11]`https://github.com/arthur-garnier/Salvetti-and-tori-complexes`

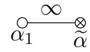
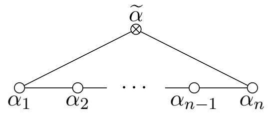
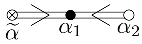
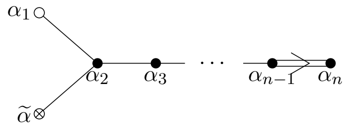
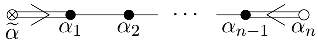
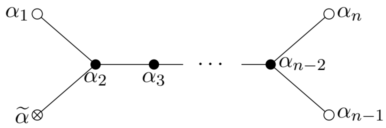
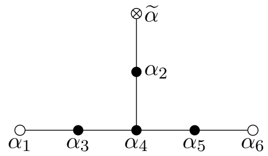
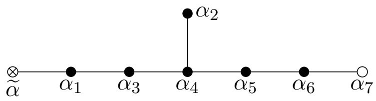
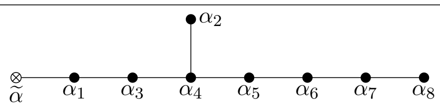
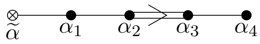
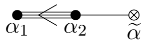
Type	Extended Dynkin diagram
$\widetilde{A_1}$	
$\widetilde{A_n}$ $(n \geq 2)$	
$\widetilde{B_2} = \widetilde{C_2}$	
$\widetilde{B_n}$ $(n \geq 3)$	
$\widetilde{C_n}$ $(n \geq 3)$	
$\widetilde{D_n}$ $(n \geq 4)$	
$\widetilde{E_6}$	
$\widetilde{E_7}$	
$\widetilde{E_8}$	
$\widetilde{F_4}$	
$\widetilde{G_2}$	

Table 1: Extended Dynkin diagrams of irreducible root systems.
The white dots stand for the roots corresponding to minuscule weights and the crossed dots represent the lowest root $\widetilde{\alpha} := -\alpha_0$.

Theorem A3 (alias 5.3.3). *Let (W, S) be a finite irreducible Coxeter system of rank n. Given a reflection $r \in W$, we consider the Coxeter system $(\widehat{W}, S \cup \{\widetilde{r}\})$ whose diagram is the one of W, with the additional node $\widetilde{r}$ corresponding to r and with associated edges given by the orders of sr for $s \in S$. Then there is a reflection $r_W \in W$ such that the extension $\widehat{W}$ is affine if W is a Weyl group and compact hyperbolic otherwise. If moreover $n > 2$, then the reflection r_W is unique with this property.*

If $\widehat{W}$ is such an extension, if we denote by $\widehat{\Sigma}$ the Coxeter complex of $\widehat{W}$ and $Q := \ker(\widehat{W} \twoheadrightarrow W)$, then $\mathbf{T}(W) := \widehat{\Sigma}/Q$ is a connected, orientable, compact, W-triangulated Riemannian W-manifold of dimension n such that,

- *if W is a Weyl group, then $\mathbf{T}(W)$ is W-isometric to a maximal torus of the simply-connected compact Lie group with root system that of W,*
- *otherwise, the manifold $\mathbf{T}(W)$ is hyperbolic.*

The particular case of the dihedral groups $I_2(m)$ has some interesting features, which we summarize in the following statement:

Corollary (alias 5.5.1, 5.5.5 and 5.5.6)**.** *For $g \in \mathbb{N}^*$, the surfaces $\mathbf{T}(I_2(2g+1))$, $\mathbf{T}(I_2(4g))$ and $\mathbf{T}(I_2(4g+2))$ are Riemann surfaces of genus g and definable over $\overline{\mathbb{Q}}$. In particular, for $g = 1$ these are rational elliptic curves. Moreover, we have an isometry $\mathbf{T}(I_2(2g+1)) \simeq \mathbf{T}(I_2(4g+2))$ and these two are not isometric to $\mathbf{T}(I_2(4g))$.*

Our approach also allows to determine a presentation for the fundamental group of $\mathbf{T}(W)$, using Poincaré's fundamental polyhedral domain theorem and to characterize the homology representation of $\mathbf{T}(W)$, using Hopf's trace formula.

Proposition (alias 5.4.4, 6.2.2, 6.2.5, 6.2.6 and 6.2.7)**.** *The fundamental group $\pi_1(\mathbf{T}(W)) \simeq Q$ admits an explicit presentation with $[W : C_W(\widetilde{r})]$ generators, where $\widetilde{r}$ is the additional reflection in the extension $\widehat{W}$. From this presentation, we derive that the homology $H_*(\mathbf{T}(W), \mathbb{Z})$ is torsion-free and so the Betti numbers are palindromic. Moreover we obtain a decomposition of the homology representation $H_*(\mathbf{T}(W), \Bbbk)$ into irreducible characters, where $\Bbbk$ is a splitting field for W.*

Three equivariant cell structures on the flag manifold of $SL_3(\mathbb{R})$

After this, we study the action of the Weyl group on flag manifolds. More specifically, we study the real flag manifold $\mathcal{F}(\mathbb{R})$ of $SL_3(\mathbb{R})$, which already is a non-trivial example to treat. Using the embedding $\mathcal{F}(\mathbb{R}) \hookrightarrow \mathbb{RP}^7$ induced by the embedding of $\mathcal{F} = \mathbb{P}(\overline{\mathcal{O}_{\min}})$ into $\mathbb{P}(\mathfrak{sl}_3) \simeq \mathbb{CP}^7$, where $\mathcal{O}_{\min}$ is the *minimal nilpotent orbit* of $SL_3(\mathbb{C})$, as well as the Goresky-Kottwitz-MacPherson (GKM) graph[12] of $W = \mathfrak{S}_3$, we obtain a first equivariant cellular structure on $\mathcal{F}(\mathbb{R})$. This is summarized in the following result:

Theorem B1 (alias 8.3.8 and 9.2.2)**.** *The real flag variety $\mathcal{F}(\mathbb{R})$ of $SL_3(\mathbb{R})$ admits a semi-algebraic regular $\mathfrak{S}_3$-equivariant cellular structure whose cellular homology chain complex is given by*

$$\mathbb{Z}[\mathfrak{S}_3]^4 \xrightarrow{\partial_3} \mathbb{Z}[\mathfrak{S}_3]^6 \xrightarrow{\partial_2} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{\partial_1} \mathbb{Z}[\mathfrak{S}_3]$$

where the boundaries ∂_i are given by left multiplication by the following matrices

$$\partial_1 = \begin{pmatrix} 1 - s_\alpha & 1 - s_\beta & 1 - w_0 \end{pmatrix},$$

$$\partial_2 = \begin{pmatrix} -1 & 1 & 1 & s_\alpha & w_0 - s_\alpha s_\beta & s_\beta - s_\beta s_\alpha \\ s_\beta s_\alpha - s_\beta & s_\alpha - 1 & -w_0 & w_0 & s_\alpha s_\beta & s_\alpha s_\beta \\ s_\beta & s_\beta s_\alpha & s_\alpha - 1 & s_\alpha s_\beta - w_0 & -s_\beta & s_\beta s_\alpha \end{pmatrix},$$

$$\partial_3 = \begin{pmatrix} 0 & s_\alpha & 0 & 1 \\ -s_\beta s_\alpha & 0 & -w_0 & 0 \\ 0 & s_\beta s_\alpha & 1 & 0 \\ 1 & 0 & 0 & s_\beta s_\alpha \\ -s_\alpha s_\beta & s_\alpha s_\beta & 0 & 0 \\ 0 & 0 & s_\alpha s_\beta & -s_\alpha s_\beta \end{pmatrix},$$

[12] Recall that the GKM graph of a Weyl group W has the elements of W as vertices and we put an edge between w and w' if there is a reflection $r \in W$ such that $w' = wr$ and $\ell(w') > \ell(w)$.

where s_α and s_β are the simple reflections of $\mathfrak{S}_3$ and $w_0 := s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$ is its longest element.

This approach also allows to determine the action of $\mathfrak{S}_3$ on the (co)homology of $\mathcal{F}(\mathbb{R})$ and in particular, we give the $\mathfrak{S}_3$-equivariant $\mathbb{F}_2$-algebra structure on $H^*(\mathcal{F}(\mathbb{R}), \mathbb{F}_2)$. More precisely, we have the following result:

Corollary (alias 9.4.7)**.** *Let $\mathbb{F}_2[x,y,z]_{\mathfrak{S}_3}$ be the mod 2 coinvariant algebra of $\mathfrak{S}_3$. There is an $\mathfrak{S}_3$-equivariant isomorphism of graded $\mathbb{F}_2$-algebras*

$$\mathbb{F}_2[x,y,z]_{\mathfrak{S}_3} \xrightarrow{\sim} H^*(\mathcal{F}(\mathbb{R}), \mathbb{F}_2)$$

sending the indeterminates x, y and z to irreducible algebraic 1-cocycles.

Next, we take a new look at $\mathcal{F}(\mathbb{R})$. Specifically, it turns out that there is a diffeomorphism $\mathcal{F}(\mathbb{R}) \simeq \mathbb{S}^3/\mathcal{Q}_8$, where $\mathcal{Q}_8$ is the quaternion group of order 8. This makes the manifold $\mathcal{F}(\mathbb{R})$ into a *spherical space form* and we thus have to determine a cellular decomposition of the sphere $\mathbb{S}^3$, equivariant with respect to the action of the binary octahedral group $\mathcal{O} = \mathcal{Q}_8 \rtimes \mathfrak{S}_3$, using the method of Chirivì-Spreafico. Furthermore, since the case of the binary icosahedral group $\mathcal{I} \circlearrowright \mathbb{S}^3$ has not been treated in the literature before, we study it as well. In the octahedral case, modding out by the quaternion group of order 8 yields the following consequence:

Theorem B2 (alias 14.0.5)**.** *The real flag manifold $\mathcal{F}(\mathbb{R})$ of $SL_3(\mathbb{R})$ admits an $\mathfrak{S}_3$-equivariant cellular structure, whose cellular homology chain complex is given by*

$$\mathbb{Z}[\mathfrak{S}_3] \xrightarrow{\partial_3} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{\partial_2} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{\partial_1} \mathbb{Z}[\mathfrak{S}_3] ,$$

where

$$\partial_1 = \begin{pmatrix} 1 - s_\beta & 1 - w_0 & 1 - s_\alpha \end{pmatrix}, \qquad \partial_2 = \begin{pmatrix} s_\alpha s_\beta & 1 & w_0 - 1 \\ s_\alpha - 1 & s_\alpha s_\beta & 1 \\ 1 & s_\beta - 1 & s_\alpha s_\beta \end{pmatrix}, \qquad \partial_3 = \begin{pmatrix} 1 - s_\beta \\ 1 - w_0 \\ 1 - s_\alpha \end{pmatrix}.$$

We also notice that the cellular structure from the previous theorem has some nice features, regarding the Riemannian metric on $\mathcal{F}(\mathbb{R})$ induced by the bi-invariant Riemannian metric on $SU(3)$.

Proposition (alias 16.2.2)**.** *We endow the complex manifold $\mathcal{F} = SU(3)/T$ with the metric induced by the bi-invariant Einstein metric on $SU(3)$ and we restrict it to $\mathcal{F}(\mathbb{R})$. Then the cells of the previous cellular structure on $\mathcal{F}(\mathbb{R})$ are unions of minimal geodesics of $\mathcal{F}(\mathbb{R})$. In particular, the 1-cells are orbits of one-parameter subgroups of $SO(3)$.*

Taking our study of the Riemannian geometry of the manifold $\mathcal{F}(\mathbb{R})$ further and in order to obtain a more intrinsic statement, we finish by studying a *Dirichlet-Voronoi domain* for $\mathfrak{S}_3$ acting on $\mathcal{F}(\mathbb{R})$:

$$\mathcal{DV} := \{x \in \mathcal{F}(\mathbb{R}) \ ; \ d(1,x) \leq d(w,x), \ \forall w \in W\},$$

where d is the geodesic distance on $\mathcal{F}(\mathbb{R})$ associated to the metric. We prove the following result:

Theorem B3 (alias 17.4.2)**.** *The Dirichlet-Voronoi domain $\mathcal{DV}$ is a fundamental domain for $\mathfrak{S}_3$ acting on $\mathcal{F}(\mathbb{R})$ and admits a cellular structure inducing an $\mathfrak{S}_3$-equivariant cellular decomposition on $\mathcal{F}(\mathbb{R})$, whose associated cellular homology chain complex is given by*

$$\mathbb{Z}[\mathfrak{S}_3] \xrightarrow{\partial_3} \mathbb{Z}[\mathfrak{S}_3]^7 \xrightarrow{\partial_2} \mathbb{Z}[\mathfrak{S}_3]^{12} \xrightarrow{\partial_1} \mathbb{Z}[\mathfrak{S}_3]^6$$

with boundaries

$$\partial_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & s_\beta & -s_\beta & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & s_\beta s_\alpha & 0 & 0 & -1 \\ -w_0 & 0 & 0 & 0 & 0 & 0 & 0 & w_0 & 0 & s_\beta & -w_0 & 0 \\ s_\beta s_\alpha & -s_\beta s_\alpha & 0 & 0 & s_\alpha & -s_\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & s_\beta s_\alpha & -s_\beta s_\alpha & 0 & 0 & 0 & 0 & 0 & -w_0 & 0 & w_0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & s_\beta & -s_\beta & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\partial_2 = \begin{pmatrix} 1 & 0 & w_0 & 0 & 0 & 0 & -w_0 \\ 1 & -s_\alpha s_\beta & 0 & 0 & -1 & 0 & 0 \\ 1 & s_\beta & 0 & -s_\beta & 0 & 0 & 0 \\ 1 & 0 & s_\alpha & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & -w_0 & 0 & 0 \\ 1 & s_\alpha & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -s_\beta & 0 \\ 1 & 0 & -s_\beta s_\alpha & -1 & 0 & 0 & 0 \\ 0 & -1 & -w_0 & 0 & -s_\beta & 0 & 0 \\ 0 & s_\beta & 1 & 0 & 0 & 0 & -s_\beta \\ 0 & w_0 & -w_0 & -1 & 0 & 0 & 0 \\ 0 & -s_\beta s_\alpha & 1 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad \partial_3 := \begin{pmatrix} 1-s_\alpha \\ 1-s_\beta \\ 1-w_0 \\ 1-s_\beta s_\alpha \\ 1-s_\alpha s_\beta \\ 1-s_\beta s_\alpha \\ 1-s_\alpha s_\beta \end{pmatrix}.$$

Each one of the three decompositions we found for $SO(3)/S(O(1)^3)$ has its own advantages and caveats: the first one from Theorem B1 has a 1-skeleton that fits in the GKM graph of $\mathfrak{S}_3$ but works because of the small dimension of the manifold and seems hard to generalize. The second one from Theorem B2 has few cells and nice degrees as we shall see below but uses the very special equivariant diffeomorphism $\mathcal{F}(\mathbb{R}) \simeq \mathbb{S}^3/\mathcal{Q}_8$. Such an identification between a real flag manifold and a free orbit space of a sphere can of course not be hoped in higher cases. The last one in Theorem B3 has too many cells, but is expected to be generalized to other real flag manifolds, as it only relies on the intrinsic geometry of the considered manifold. For more details about this, we refer the reader to Conjecture B4 and Proposition B5 below.

Some perspectives and conjectures

After writing this work, many themes are to be studied further.

As a first perspective of research, we can mention the *étale case for tori.* In the second chapter we exhibit a triangulations of tori of compact Lie groups, equivariant with respect to the action of the Weyl group W. A naturally related question is to see what could be said in the case of finite reductive groups. More precisely, we take a reductive group $\mathbf{G}$ over a field $\overline{\mathbb{F}_q}$ and defined over $\mathbb{F}_q$, with associated Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ and a given F-stable torus $\mathbf{T} < \mathbf{G}$. The associated *finite reductive group* is the group $\mathbf{G}^F$ of F-fixed points of $\mathbf{G}$, and the associated torus is $\mathbf{T}^F$. In this case, the notion of CW-structure does no longer make any sense, but we can expect that the *Rickard complex* $R\Gamma_c(\mathbf{T}, \mathbb{Z}/n\mathbb{Z})$ (which is a complex of permutation modules, analogous to the cellular complex coming from a CW-structure, see [Ric94]) can be computed using similar combinatorics as for the Lie group case. In the étale case, one should also take the action of the Frobenius into account. We summarize this in the following problem:

Problem A4. *Describe the combinatorics of the Rickard complex $R\Gamma_c(\mathbf{T}, \mathbb{Z}/n)$ as an object of $\mathcal{D}^b(\mathbb{Z}/n[W \rtimes \langle F\rangle])$.*

A second possible track concerns our extension of the construction of tori for Coxeter groups. Basically, the slogan is: "there are analogues of tori for non-crystallographic Coxeter groups, but they are no longer Lie groups". This has the flavour of *spetses*. Roughly speaking, spetses are "fake algebraic groups", first associated to Coxeter groups (see the pioneer work of Lusztig in [Lus93]) and later to complex reflection groups by Broué, Malle and Michel in [BMM99]. We could say that $\mathbf{T}(W)$ is the "torus" of the spets of the Coxeter group W. Therefore, it is a reasonable question to ask if the construction of $\mathbf{T}(W)$ could be extended to (irreducible) complex reflection groups, yielding a "torus" for any spets. However, we made heavy use of the *Tits representation* of (an extension of) W in our construction of $\mathbf{T}(W)$ and the general method to deal with complex reflection groups is not clear at all and requires substantial additional work, if even possible. We formulate the following problem:

Problem A5. *Given a complex reflection group W, is it possible to construct a (possibly compact) W-manifold generalizing the construction we gave in the Coxeter case?*

Next, if T is a maximal torus in a compact Lie group K, as the Weyl group W acts on T, it also acts on the *classifying space* $B_T \simeq (\mathbb{CP}^\infty)^{\dim T}$ of T and we can look for a W-equivariant cellular structure on B_T. In type A_1, we have $B_T = B_{\mathbb{S}^1} = \mathbb{CP}^\infty$ and the non-trivial element s of $W = \mathfrak{S}_2$ acts as complex conjugation on $\mathbb{CP}^\infty$. First, we partition $\mathbb{CP}^\infty$ by subspaces $A_d \simeq \mathbb{C}^d$ of elements whose last non-zero coordinate is the d^{th} one. Then we decompose A_d as follows:

$$\mathbb{C}^d = \mathbb{R}^d \sqcup \bigsqcup_{k=1}^{d} \left(\mathbb{C}^{k-1} \times (\mathbb{C} \setminus \mathbb{R}) \times \mathbb{R}^{d-k}\right) = e_{d,0} \sqcup \bigsqcup_{k=1}^{d} (e^+_{d,k} \sqcup e^-_{d,k})$$

where $e_{d,0} = \mathbb{R}^d$ is the real part and $e^\pm_{d,k} := \mathbb{C}^{k-1} \times H^\pm \times \mathbb{R}^{d-k}$ are two cells of dimension $d+k$ exchanged by s (we have denoted by $H^\pm$ the upper and lower open half planes in $\mathbb{C}$). It is convenient to label the cells by

$$\text{sequences } a = (\underbrace{2, \ldots, 2}_{k \text{ times}}, \underbrace{1, \ldots, 1}_{d-k \text{ times}}, 0, \ldots), \text{ with a sign } \varepsilon = \pm 1 \text{ if } k > 0;$$

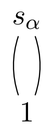
we set $e^\pm_a := e^\pm_{d,k}$, or $e_a := e_{d,0} \subset \mathbb{RP}^\infty$ when $k = 0$. Thus e_a and $e^\pm_a$ are cells of dimension $|a| = \sum_i a_i$; furthermore, s fixes e_a pointwise and exchanges the $e^\pm_a$. Then we allow some parameters to take complex values. Moreover, there is a Koszul duality between $H^*(T, \mathbb{Q}) = \Lambda^\bullet(\mathrm{Lie}(T)^*)$ and $H^*(B_T, \mathbb{Q}) = S^\bullet(\mathrm{Lie}(T))$ and it would be interesting to see if this duality actually occurs at a geometric level.

Concerning flag varieties, a first potential thing to explore is the embedding $G/B \hookrightarrow \mathbb{P}(V(\rho))$, where $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$ is half the sum of positive roots and $V(\rho)$ is the irreducible highest weight module of highest weight ρ (see theorem 8.1.3). More specifically, since ρ is the smallest regular dominant weight of G, the embedding $G/B \hookrightarrow \mathbb{P}(V(\rho))$ is somewhat minimal among the embeddings of the flag variety $\mathcal{F} = G/B$ into projective spaces. Therefore, it seems interesting to study it, in order for instance to relate the action of W on $\mathcal{F}$ and the representation $V(\rho)$, but this approach seems difficult to carry in general. Indeed, we have $\dim V(\rho) = 2^{|\Phi^+|}$, so the number of coordinates explodes with the rank and moreover, the expression of the action of W using these coordinates is hard to handle (see Proposition 8.2.5).

About the combinatorics of a potential cellular complex $C_*^{\mathrm{cell}}(K/T, W; \mathbb{Z})$, we can try to guess a nice and plausible formula for the ranks of its homogeneous components. Let

$$P_W^{\mathbb{C}}(q) := \sum_i \#\{W\text{-orbits of } i\text{-cells of } \mathcal{F}(\mathbb{C})\} q^i$$

and similarly consider $P_W^{\mathbb{R}}$ for the real points $\mathcal{F}(\mathbb{R})$. The polynomial $P_W^{\mathbb{C}}$ must verify $\deg(P_W^{\mathbb{C}}) = 2N$ where $N = |\Phi^+|$ is the number of reflections of W, as well as $P_W^{\mathbb{C}}(-1) = 1$, since $\chi(\mathcal{F}) = |W|$. A first reasonable guess for $P_W^{\mathbb{C}}$ starts with the GKM graph of W. To each positive root $\alpha \in \Phi^+$ is associated the *minimal parabolic subgroup* $P_\alpha = \langle B, \dot{s}_\alpha \rangle$ and we have that $P_\alpha / B \simeq \mathbb{CP}^1$ is stable under the action of the subgroup $\langle s_\alpha \rangle$; so there is an "SL_2 situation" for each positive root, which we may represent in the following diagram:



Doing this for any reflection of $W = \mathfrak{S}_3$ and taking the closure under the action of $\mathfrak{S}_3$, we obtain a diagram which is very similar to the GKM graph:

(a) The GKM graph

(b) Many SL_2 situations

Figure B: The GKM graph of $\mathfrak{S}_3$ and the 1-skeleton of $\mathcal{F}(\mathbb{R})$

Extrapolating this to higher dimensions, we can hope to parametrize the orbits of i-cells of $\mathcal{F}(\mathbb{R})$ by subsets of positive roots of cardinal i and the i-cells would be parametrized by i real parameters (one for each root). This would give $P_W^{\mathbb{R}}(q) = [2]_q^{|\Phi^+|}$, where $[k]_q = 1 + q + \cdots + q^{k-1}$. To find the missing cells in $\mathcal{F}(\mathbb{C})$, we allow some parameters to take complex values, so that each positive root would have a multiplicity 0, 1 (real parameter) or 2 (complex parameter) and we get multisets of positive roots with multiplicities, yielding $P_W^{\mathbb{C}}(q) = [3]_q^{|\Phi^+|}$. This formula has the combinatorial flavour of the de Concini-Salvetti complex, which is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[W]$, valid for any finite Coxeter group W and constructed using increasing chains of subsets of simple reflections. Here, we should rather look at chains of subsets of Φ^+ of length at most 2. For SL_3, we would get

$$P_W^{\mathbb{R}}(q) = [2]_q^3 = q^3 + 3q^2 + 3q + 1 \quad \text{and} \quad P_W^{\mathbb{C}}(q) = [3]_q^3 = q^6 + 3q^5 + 6q^4 + 7q^3 + 6q^2 + 3q + 1.$$

This would give an explanation for the ranks 1, 3, 3 and 1 of the complex in Theorem B2.

Another possible formula, involving only the simple roots, for this number of orbits is given by $\prod_i [2d_i - 1]_q$, with $(d_i)_i$ the degrees of W. Recall that the d_i's are the degrees of fundamental invariants of W and satisfy $\sum_i (d_i - 1) = N$ and $\prod_i d_i = |W|$ and we would have indeed $\deg\left(\prod_i [2d_i - 1]_q\right) = \sum_i (2d_i - 2) = 2N$ and $\prod_i [2d_i - 1]_{-1} = 1$. Over $\mathbb{R}$, similar considerations would give $\prod_i [d_i]_q$. For SL_3, we would get

$$P_W^{\mathbb{R}}(q) = [2]_q[3]_q = q^3+2q^2+2q+1 \text{ and } P_W^{\mathbb{C}}(q) = [3]_q[5]_q = q^6+2q^5+3q^4+3q^3+3q^2+2q+1$$

The first formula $[3]_q^{|\Phi^+|}$ has a clear link with the GKM graph and seems easier to pass from $\mathbb{R}$ to $\mathbb{C}$, but the second one $\prod_i [2d_i - 1]_q$ yields fewer cells. Unfortunately, since the complex in Theorem B2 is in fact homotopy equivalent to a complex with degrees 1, 2, 2 and 1, this case doesn't let us decide between the two possibilities. However, we have found no geometric model for this complex (so far).

But the most promising method to build an equivariant cellular structure on real flag manifolds in general seems to be the Riemannian approach. Indeed, if the flag manifold $\mathcal{F} = K/T$ is equipped with a *normal homogeneous* metric (i.e. a metric coming from a bi-invariant metric on the compact Lie group K) and W is the associated Weyl group, then we may consider the associated *Dirichlet-Voronoi domain*

$$\mathcal{DV} := \{x \in \mathcal{F} \ ; \ d(1,x) \leq d(w,x), \ \forall w \in W\},$$

where d is the geodesic distance on $\mathcal{F}$. Notice that this is independent of the chosen normal homogeneous metric, since such a metric is unique up to scaling. Then $\mathcal{DV}$ is a fundamental domain for W acting on $\mathcal{F}$ (see Proposition 17.1.5) and we want to use its interior and boundary to construct a cell structure on $\mathcal{F}$. We split the boundary using *walls*, i.e. intersection of $\mathcal{DV}$ with the *dissecting hypersurfaces* $H_w := \{x \ ; \ d(1,x) = d(w,x)\}$ and the intersections of walls are supposed to be the lower cells of the wanted decomposition. For example, in the case of hyperbolic manifolds and in particular, for Fuchsian groups, the Dirichlet-Voronoi domain is a (geodesic) polyhedron with facets $H_w \cap \mathcal{DV}$ and its face lattice yields an equivariant cellular structure on the manifold. However, flag manifolds are non-negatively curved and we need a technical condition, namely the domain $\mathcal{DV}$ should be included in a closed metric ball with center 1 and of radius $\rho < \mathrm{inj}(\mathcal{F})$ smaller than the *injectivity radius* $\mathrm{inj}(\mathcal{F})$ of $\mathcal{F}$. Recall that this radius is defined to be the supremum of the radii of balls centered at $0 \in T_1\mathcal{F}$ on which the Riemannian exponential map is injective. If this condition holds, then we know at least that the boundary of $\mathcal{DV}$ is homeomorphic to the sphere $\mathbb{S}^{\dim_{\mathbb{R}} \mathcal{F} - 1}$. Yet, estimating the injectivity radius of a manifold is a very hard problem and proving that the domain $\mathcal{DV}$ is included in a small enough ball around the central point is a difficult problem too; and even if the condition is satisfied, it doesn't guarantee the walls to be cells. As an example, letting the cyclic group C_2 act on $\mathbb{S}^2$ as the antipode, the boundary of a Dirichlet-Voronoi domain centered at a pole is an equatorial line $\mathbb{S}^1$. Nevertheless, if we restrict our attention to the totally geodesic submanifold $\mathcal{F}(\mathbb{R})$ of $\mathcal{F}$ and consider a Dirichlet-Voronoi domain on $\mathcal{F}(\mathbb{R})$, then it gives indeed an $\mathfrak{S}_2$-equivariant cellular decomposition of $\mathcal{F}(\mathbb{R}) \simeq \mathbb{S}^1$ and then of $\mathcal{F}(\mathbb{C}) \simeq \mathbb{S}^2$. We conjecture that this is still the case for other real flag manifolds.

Summarizing, we formulate the following conjectures:

Conjecture B4 (alias 17.2.3, 17.2.1 and 17.2.2)**.** *We endow $\mathcal{F} = K/T$ with the metric induced by the Killing form.*

1. *The injectivity radius of $\mathcal{F} = K/T$ is the minimal distance between two elements of W, realized by a simple reflection in W.*
2. *The Dirichlet-Voronoi domain $\mathcal{DV}$ associated to $\mathcal{F}$ and W is included in the open ball centered at* 1 *and of radius* $\operatorname{inj}(\mathcal{F})$*.*
3. *If the later holds and if $I \subset W$, then the wall $\mathcal{F}(\mathbb{R}) \cap \bigcap_{w\in I} H_w$ is a (possibly empty) union of $(N - |I|)$-cells.*

Focusing on the type A where $K = SU(n)$, $\mathcal{F}_n := SU(n)/S(U(1)^n)$ and $W = \mathfrak{S}_n$, we prove the following result, which establishes the first conjecture above for $\mathcal{F}_n(\mathbb{R})$:

Proposition B5 (alias 17.3.1 and 17.3.4)**.** *The injectivity radii of $\mathcal{F}_n$ and $\mathcal{F}_n(\mathbb{R})$ verify*

$$\operatorname{inj}(\mathcal{F}_n, g_n) \geq \pi\sqrt{\frac{n}{2}} \quad \textit{and} \quad \operatorname{inj}(\mathcal{F}_n(\mathbb{R}), g_n) = \pi\sqrt{n}.$$

A reader's guide

This work's aim is to build cellular structures on tori of compact Lie groups and flag manifolds, that are equivariant with respect to the Weyl group actions.

As a warm-up, we recall the definition and some basic facts about equivariant sheaves. Focusing on the case where the acting group is discrete (which is the case for the groups we shall consider in the sequel) we give equivalent definitions of an equivariant sheaf and define the *derived equivariant category* as the derived category of equivariant sheaves on the space. Next, we define the notion of *equivariant CW-complex* as introduced by Matumoto in [Mat71] and here again, taking advantage of the discreteness of the acting group, we rephrase this as just being a CW-complex, with an additional condition regarding the action of the group (see Definition 2.1.1). We prove in Corollary 2.1.11 the essential fact that if W is a discrete group and X is a W-CW-complex, then the cellular chain complex $C_*^{\mathrm{cell}}(X, W; \mathbb{Z})$ is a complex of permutation $\mathbb{Z}[W]$-modules and that $R\Gamma(X, \underline{\mathbb{Z}}) = C^*_{\mathrm{cell}}(X, W; \mathbb{Z})$ in $\mathcal{D}^b(\mathbb{Z}[W])$. Moreover, any two such W-CW-complex structures on X give two complexes that are isomorphic in the homotopy category $\mathcal{K}_b(\mathbb{Z}[W])$. We finish this preliminary chapter by studying the behaviour of the cellular complexes $C_*^{\mathrm{cell}}(X, W; \mathbb{Z})$ with respect to subgroups and quotients of W.

In the next chapter, we solve the question of finding equivariant cellular structures on tori of compact Lie groups by means of root data and (extended) affine Weyl groups, which are no longer finite, but still discrete. More precisely, if K is a simple compact Lie group, if $T < K$ is a maximal torus and if Φ denotes the associated root system, then the *character lattice* $X(T)$ and the *cocharacter lattice* $Y(T)$ of T are in perfect duality and the quadruple $(X(T), \Phi, Y(T), \Phi^\vee)$ is a *root datum* (see Definition 3.1.1) which completely determines the pair (K, T) up to isomorphism. Moreover, we have a W-isomorphism of tori $V^*/Y(T) \xrightarrow{\sim} T$; thus we may drop the group K and just work with a given irreducible root datum $(X, \Phi, Y, \Phi^\vee)$, with ambient space $V := \mathbb{R} \otimes_{\mathbb{Z}} X$. We have to find a W-CW-complex structure on V^*/Y. To do this, we look for a W_Y-CW-complex structure on the vector space V^*, where $W_Y := Y \rtimes W$ is the *extended affine Weyl group.* In the case when $Y = Q^\vee$ is the coroot lattice (which corresponds to the case where the group K is simply-connected), the group $W_Y = W_{\mathrm{a}}$ is the classical affine Weyl group and this is a Coxeter group. Hence we may apply the combinatorics of *alcoves* and *walls* to obtain a W_{a}-equivariant triangulation of V^* for which the cellular chain complexe can be explicitly computed (see Theorem 3.2.2). Moreover, we give an explicit formula for the cup product on the dual complex (see Theorem 3.3.2) and we obtain in Corollary 3.3.3 the complex for the quotient W-triangulation of $V^*/Q^\vee$ by applying a deflation functor to $C^*_{\mathrm{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z})$ from W_{a} to $W = W_{\mathrm{a}}/Q^\vee$ and this gives a W-dg-ring, whose cohomology is $H^*(V^*/Q^\vee, \mathbb{Z}) \simeq \Lambda^\bullet(P)$. In the general case, the group W_Y is no longer Coxeter and the issue is that the fundamental alcove has a group of symmetries $\Omega_Y \leq W_Y$, which is not a reflection group. However, it is a general fact that, in this situation, the *barycentric subdivision* of the fundamental alcove (which is a simplex) forms an Ω_Y-equivariant triangulation of the alcove. Though the resulting W-triangulation of V^*/Y has many simplices, this construction has the advantage to work for every root datum and is quite effective. The main result is summarized in Theorem 4.2.3. Finally, the package `Salvetti-and-tori-complexes`[13] we have developed for `GAP` allows to compute the previously mentioned complexes for any (irreducible) root datum.

As for the third chapter, we extend the combinatorics from the second chapter to any finite Coxeter group. More precisely, given an irreducible finite Coxeter group W, we construct a W-manifold $\mathbf{T}(W)$, which somehow plays the role of a torus for W. Specifically, we pick a suitable reflection $r_W \in W$ and consider the Coxeter group $\widehat{W}$, whose Coxeter diagram is the one of W, with one more node corresponding to the reflection r_W that is, we add edges according to the orders of sr_W, for s a simple reflection of W. We choose the reflection r_W in such a way that the group $\widehat{W}$ is an affine Coxeter group if W is a Weyl group, in which case r_W is the reflection associated to the *highest root* of the root system of W and $\widehat{W} = W_{\mathrm{a}}$ is the affine Weyl group. In other cases, (the non-crystallographic ones), we choose r_W in order for the group $\widehat{W}$ to be a *compact hyperbolic Coxeter group*[14]. This is summarized in Proposition-Definition 5.1.3. Next, we introduce the subgroup $Q := \ker(\widehat{W} \twoheadrightarrow W)$ and we define the manifold $\mathbf{T}(W)$ as the orbit space of the *Coxeter complex* $\Sigma(\widehat{W})$ of $\widehat{W}$ under the action of Q. This is well-defined since Q acts freely and properly discontinuously on $\Sigma(\widehat{W})$, because it trivially intersects each proper parabolic subgroup of $\widehat{W}$ (see Lemmas 5.2.1 and 5.3.2). We prove in Theorem 5.3.3 that $\mathbf{T}(W)$ is a W-triangulated, closed, connected, orientable, compact W-manifold of dimension $\mathrm{rk}(W)$. Moreover, if W is a Weyl group, then this indeed coincides with an $\mathrm{rk}(W)$-torus and $\mathbf{T}(W)$ is a *hyperbolic manifold* in the non-crystallographic cases. After writing this chapter, I learned about the work of Zimmermann and Davis ([Zim93] and [Dav85]), in which they define the same manifolds in types H_3 and H_4, with totally different approaches. We carry our study further by giving a general presentation of the fundamental group $\pi_1(\mathbf{T}(W)) \simeq Q$ in Theorem 5.4.4 and then we specialize it to H_3 and H_4, for which the full computations are available in Appendix B. Next we investigate the case of dihedral groups: in Corollary 5.5.1 and Propositions 5.5.5 and 5.5.6, we show that $\mathbf{T}(I_2(m))$ is an *arithmetic Riemann surface* (and even an *elliptic curve* if $I_2(m)$ is a Weyl group, which gives an unusual point of view on these tori) and we classify them up to isometry. It should be noted that, in the crystallographic case (i.e. when $m \in \{3, 4, 6\}$), the surfaces $\mathbf{T}(I_2(m))$ correspond to the points in the classical Poincaré fundamental domain which have non-trivial stabilizers in $PSL_2(\mathbb{Z})$ (two of which are in the same $PSL_2(\mathbb{Z})$-orbit). Finally, we use the W-triangulation of $\mathbf{T}(W)$ mentioned earlier to derive the homology chain complex $C_*^{\mathrm{cell}}(\mathbf{T}(W), W; \mathbb{Z})$ and compute the cup product on its dual (see Corollary 6.1.2) and we finish by detailing the homology representation of $\mathbf{T}(W)$. Using the previously found presentation of $Q = \pi_1(\mathbf{T}(W))$, we prove that the homology is torsion-free and the Betti numbers are therefore *palindromic*, just as for actual tori. Lastly, we use the *Hopf trace formula* to decompose the homology character as sum

[13] `https://github.com/arthur-garnier/Salvetti-and-tori-complexes`
[14] see Proposition-Definition 5.1.1

of irreducible characters of W (see Theorems 6.2.5, 6.2.6 and 6.2.7). These complete the results of Zimmermann and Davis for H_3 and H_4.

In the sequel, we come to flag manifolds. Also, the fourth chapter is dedicated to the construction of a first W-equivariant cellular structure on the flag manifold $\mathcal{F}(\mathbb{R})$ of $SL_3(\mathbb{R})$. We start with some general reminders on semisimple groups and flag manifolds. In particular, we define and study the (T-equivariant) rational cohomology algebra of K/T, through the three descriptions of Schubert, Borel and Goresky-Kottwitz-MacPherson and we recall that the cohomology $H^*(K/T, \mathbb{Q})$ is the regular $\mathbb{Q}[W]$-module. Recall moreover that, in general, we have a diffeomorphism $\mathcal{F} = K/T \simeq G/B$, the later being a smooth projective complex variety and the real flag manifold $\mathcal{F}(\mathbb{R})$ can be interpreted as the set of its real points (with respect to the real structure induced by the split real structure on G). The first step toward the sequel is to realize the complex flag variety G/B as a projective variety, and this can be done using *irreducible highest weight modules* and *half the sum of the positive roots* (see Theorem 8.1.3 and Corollary 8.1.4). Next, after having investigated the trivial case of SL_2, we focus on the special case of the flag manifold $\mathcal{F}(\mathbb{R})$ of $SL_3(\mathbb{R})$. Using the previous embedding, we can realize the complex variety $\mathcal{F}$ as a subvariety of $\mathbb{CP}^7$ and we give a complete set of equations defining it, as a Gröbner basis of the defining ideal in the ambient polynomial ring, see Proposition 8.2.1. In Proposition 8.2.5, we give equations for the action of $W = \mathfrak{S}_3$ on $SL_3(\mathbb{C})/B$ in local charts. To build an equivariant cellular structure on $\mathcal{F}(\mathbb{R})$, we begin with the elements of $W = \mathfrak{S}_3$ as vertices. Then, we use the GKM graph of $\mathfrak{S}_3$ (see Figure B). Inspired by the case of SL_2 and the GKM graph, we define 1-cells for $\mathcal{F}(\mathbb{R})$ that we can represent in the graph as in Figure B. We notice that the resulting "1-skeleton" is a union of closed subvarieties of the (real) algebraic variety $\mathcal{F}(\mathbb{R})$. Relaxing a part of the defining equations and imposing some intermediate positivity conditions we obtain 2-cells and the resulting "2-skeleton" is a real algebraic subvariety of $\mathcal{F}(\mathbb{R})$. Finally, we take the connected components of the complement and prove that these are cells, therefore providing a cellular structure on $\mathcal{F}(\mathbb{R})$ and we show that this structure is indeed $\mathfrak{S}_3$-equivariant (see Theorem 8.3.8). Next, some calculations allow us to compute the $\mathfrak{S}_3$-orbits of cells as well as the boundaries of the cells, yielding the associated cellular homology chain complex and the main Theorem 9.2.2. Next, we use this complex to determine the $\mathfrak{S}_3$-module structure on $H^*(\mathcal{F}(\mathbb{R}), \mathbb{Z})$. A particularly nice situation occurs when taking the cohomology with coefficients in $\mathbb{F}_2$. Also, we realize the mod 2 first cohomology classes by nine pairwise *transverse* 2-dimensional subvarieties of $\mathcal{F}(\mathbb{R})$ (see Definition 9.4.1). By transversality and Poincaré's product formula, in Theorem 9.4.6 and Corollary 9.4.7, we derive the $\mathfrak{S}_3$-equivariant $\mathbb{F}_2$-algebra structure on $H^*(\mathcal{F}(\mathbb{R}), \mathbb{F}_2)$, which turns out to be the *coinvariant mod 2 algebra* of $\mathfrak{S}_3$. As an $\mathbb{F}_2$-algebra, this result is known since [Bor53a], but here the action of $\mathfrak{S}_3$ is also taken into account. We finish by explaining the link between our algebraic cycles and the Stiefel-Whitney classes of the universal line bundles over $\mathcal{F}(\mathbb{R})$ (see Remark 9.4.9).

In the fifth chapter, we exhibit an $\mathfrak{S}_3$-equivariant cellular structure on $\mathcal{F}(\mathbb{R})$ using a deeply different method. This is a joint work with R. Chirivì and M. Spreafico [CGS20]. More precisely, we write $\mathcal{F}(\mathbb{R}) \simeq SO(3)/S(O(1)^3)$, where

$$S(O(1)^3) = \langle \mathrm{diag}(-1,-1,1), \mathrm{diag}(1,-1,-1) \rangle \simeq C_2 \times C_2$$

is the Klein four-group. But since the universal cover of $SO(3)$ is the sphere $\mathbb{S}^3$, we get a tower of covering spaces, where $\mathcal{Q}_8$ is the quaternion group of order 8 and $\mathcal{O} \simeq \mathcal{Q}_8 \rtimes \mathfrak{S}_3$ is the *binary octahedral group* of order 48 and a group over an arrow denotes the covering space with this group as fiber,

$$
\begin{tikzcd}
\mathbb{S}^3 \arrow[d, "C_2"] \arrow[dd, bend right, "\mathcal{Q}_8"'] \arrow[ddd, bend left, "\mathcal{O}"] \\
SO(3) \arrow[d, "C_2^2"] \\
\mathcal{F}(\mathbb{R}) = \mathbb{S}^3/\mathcal{Q}_8 \arrow[d, "\mathfrak{S}_3"] \\
\mathcal{F}(\mathbb{R})/\mathfrak{S}_3 = \mathbb{S}^3/\mathcal{O}.
\end{tikzcd}
$$

Therefore, instead of the action $\mathfrak{S}_3 \circlearrowright \mathcal{F}(\mathbb{R})$, we can study the action of the group $\mathcal{O}$ on $\mathbb{S}^3$, which is a simpler space. The space $\mathbb{S}^3/\mathcal{O}$ is a *spherical space form*, that is, a compact Riemannian manifold with constant positive sectional curvature. This class of spaces is well-known (see for instance [Mil57; Wol67; ST31]). Inspired by the series of papers [MMS13; FGMS13; FGMS16], Chirivì and Spreafico developed in [CS17] a general method for constructing cellular structures on spheres, that are equivariant for the free action of a finite isometry group. In particular, this provides a cellular structure on spherical space forms. The remaining cases for $\mathbb{S}^3$ for which the determination of an explicit decomposition had not yet been done in the literature were the binary octahedral group $\mathcal{O}$ and the binary icosahedral group $\mathcal{I}$ of order 120. In [CGS20], we apply the *orbit polytope* method of Chirivì-Spreafico to these two cases and exhibit equivariant cellular structures. The key idea is to exhibit a polytopal fundamental domain in the convex hull of the group (as a subset of $\mathbb{R}^4$) and then project it on the sphere $\mathbb{S}^3$. We start by recalling the main results and principles of the method and in particular, we introduce the *curved join* and the *binary polyhedral groups*. Next, we apply the method to $\mathcal{O}$, $\mathcal{I}$ and the binary tetrahedral group $\mathcal{T}$ for completeness (this case is not new). For each of those, we follow the same outline: we find a fundamental domain for the group $\mathcal{G} \in \{\mathcal{O}, \mathcal{I}, \mathcal{T}\}$ in $\mathbb{S}^3$ by projecting a polytopal fundamental domain and we derive a cell structure using its face lattice. Then, we calculate the associated cellular homology complex and use it to find a 4-periodic free resolution of $\mathbb{Z}$ over the integral group algebra of the group (see Theorems 11.4.1, 12.3.1, 13.3.1 and Corollaries 11.4.2, 12.3.3, 13.3.2). In particular, we recover the cohomology of these groups. Finally, we use the *curved join* to generalize it to the spheres $\mathbb{S}^{4n-1}$ and interpret the free resolution as a limit (when $n \to \infty$) of the cellular homology complexes of the universal covers of the spherical space forms $\mathbb{S}^{4n-1}/\rho^{\oplus n}(\mathcal{G})$, where ρ is the inclusion of $\mathcal{G}$ into $SU(2)$ and $\rho^{\oplus n} : \mathcal{G} \hookrightarrow SU(2n)$ (see Theorems 11.4.7, 12.3.6 and 13.3.4). It is worth noticing that when $\mathcal{G} = \mathcal{I}$, the $\mathcal{I}$-equivariant cellular structure on $\mathbb{S}^3$ induces a CW-complex structure on the *Poincaré sphere* $\mathbb{S}^3/\mathcal{I}$. We finish by applying the octahedral case to the flag manifold $\mathcal{F}(\mathbb{R})$ of $SL_3(\mathbb{R})$ and obtain an $\mathfrak{S}_3$-equivariant cellular structure on it, and we compute its cellular homology chain complex in Theorem 14.0.5. We notice that this decomposition has much fewer cells than the one given in the fourth chapter. Moreover, the resulting complex has a nice symmetry and is compatible with the first expected formula for the ranks of the components.

The final chapter presents some hope of construction of a cell structure in the higher cases, at least for real flag manifolds. More precisely, starting with the fact that the compact group K admits a bi-invariant Riemannian metric, we obtain a Riemannian metric on its flag manifold and we can restrict it to the real points. The flag manifold $\mathcal{F}(\mathbb{R})$ of $SL_3(\mathbb{R})$ carries two natural metrics: the bi-invariant one inherited from $SO(3)$ and the one induced by modding out the (standard) round metric on $\mathbb{S}^3$ by the quaternion group $\mathcal{Q}_8$. After recalling some elementary facts from Riemannian geometry and bi-invariant metrics on compact Lie groups, we prove in Proposition 15.4.2 that these two metrics are proportional. Next, we describe the geodesics of $\mathcal{F}(\mathbb{R})$ as orbits of one-parameter subgroups of $SO(3)$ (Proposition 16.1.1) and this allows us to interpret the cells of $\mathcal{F}(\mathbb{R})$ constructed in the chapter 5 as unions of (minimal) geodesics in $\mathcal{F}(\mathbb{R})$ (see Corollary 16.1.2 and Theorem 16.2.2) and in particular, the 1-cells are (translates of) minimal geodesics between 1 and the reflections of $\mathfrak{S}_3$, seen as points of $\mathcal{F}(\mathbb{R})$. In the general case where $W \leq \mathrm{Isom}(M)$ is a discrete isometry group of a connected complete Riemannian manifold (M, g), we introduce the Dirichlet-Voronoi domain

$$\mathcal{DV} := \{x \in M \ ; \ \forall w \in W, \ d(x_0, x) \leq d(wx_0, x)\},$$

where d is the geodesic distance on M and $x_0 \in M$ is a regular point. We prove in general (see Proposition 17.1.5) that $\mathcal{DV}$ is a path-connected fundamental domain for W acting on M. Next, we focus on the case where $M = \mathcal{F} = K/T$, where K is a compact Lie group and W is the Weyl group, $\mathcal{F}$ being equipped with a normal homogeneous metric. Then, we state the Conjecture B4 on the injectivity radius of $\mathcal{F}(\mathbb{R})$. A first step toward constructing a cell decomposition is that, under the injectivity condition, the open Dirichlet-Voronoi domain is a $2N$-cell. Next, we focus further on the case where $\mathcal{F}_n := SU(n)/T$ is of type A_{n-1} and we give estimates on the injectivity radius of $\mathcal{F}_n$ and $\mathcal{F}_n(\mathbb{R})$ in Proposition 17.3.1 and Lemma 17.3.4. We see moreover that the metric g_n on $\mathcal{F}_n$ is (proportional to) the restriction of the product Fubini-Study metric on $(\mathbb{CP}^{n-1})^n$, where the embedding $\mathcal{F}_n \hookrightarrow (\mathbb{CP}^{n-1})^n$ is given by sending a unitary matrix to the n-tuple of orthogonal lines in $\mathbb{C}^n$ corresponding to its columns, each one of which being seen as an element of $\mathbb{CP}^{n-1}$. The induced distance d_{FS} on $\mathcal{F}_n$ (which is lower or equal to d) is well-behaved with respect to the Weyl group $\mathfrak{S}_n$. More precisely, we have a formula for distances $d_{FS}(1, x)$ and $d_{FS}(1, xw)$ for $w \in \mathfrak{S}_n$, in terms of the entries of the underlying matrix of $x \in \mathcal{F}$, see Lemma 17.3.8. We conclude our study by investigating the case of $\mathcal{F}_3(\mathbb{R}) = SO(3)/\{\pm 1\}^2$. In particular, we give in Proposition 17.4.1 the maximal distance from 1 to an element $x \in \mathcal{DV}$. We prove in particular that there are exactly twenty-four points realizing this distance. These will be some of the 0-cells of a third $\mathfrak{S}_3$-equivariant cellular structure on $\mathcal{F}_3(\mathbb{R})$. Indeed, in this case the domain is combinatorially equivalent to a *truncated cube* and we easily derive a (polyhedral) cellular decomposition of $\mathcal{F}_3(\mathbb{R})$. However, by lack of time we only have verified this numerically. We found one orbit of 3-cells, seven orbits of 2-cells, twelve orbits of 1-cells and six orbits of 0-cells. We finish by describing the resulting chain complex.

Part I

Representing derived global sections using equivariant cellular structures for discrete group actions

In this preliminary part, we consider a topological group G acting on a space X, and we recall the definition of the category of *G-equivariant sheaves* on X. In case the group G is discrete, we also remind the definition of the *equivariant derived category* $\mathcal{D}_G(X)$ and the derived global sections functor can then be seen as a functor

$$R\Gamma(X,-) : \mathcal{D}_G(X) \to \mathcal{D}(\mathbb{Z}[G]).$$

Next, we introduce the notion of a *G-CW-complex.* Focusing again on the case where G is discrete and given a G-CW-complex X, we consider its *cellular homology chain complex* $C_*^{\mathrm{cell}}(X, G; \mathbb{Z})$, which is a complex of permutation $\mathbb{Z}[G]$-modules. Dually, we may consider the cochain complex $C^*_{\mathrm{cell}}(X, G; \mathbb{Z})$. The main result is the Corollary 2.1.11, which states that $C^*_{\mathrm{cell}}(X, G; \mathbb{Z})$ is independent of the chosen cell structure on X, up to isomorphism in the homotopy category $\mathcal{K}^b(\mathbb{Z}[G])$, and is isomorphic to $R\Gamma(X, \underline{\mathbb{Z}})$ in $\mathcal{D}^b(\mathbb{Z}[G])$. We finish by studying the behaviour of the cellular (co)chain complexes with respect to subgroups and quotients. For the quotients, we use the deflation functor.

1 Preliminaries on equivariant sheaves

In this first section, we review some basic facts about equivariant sheaves. There are at least three different definitions of an equivariant sheaf that we shall review, which are equivalent in case the acting group is *discrete*. A standard reference for this topic is [BL94].

We fix a topological group G, a G-space X, with anti-action map $a : G \times X \to X$ given by $(g, x) \mapsto g^{-1}x$. Define $p : G \times X \to X$ by $p(g, x) = x$ and $\iota : X \to G \times X$ by $\iota(x) = (1, x)$ and define the following maps

$$\begin{array}{ccc} G\times G\times X & \xrightarrow{\eta} & G\times X \\ (g,h,x) & \mapsto & (h,g^{-1}x) \end{array} \qquad \begin{array}{ccc} G\times G\times X & \xrightarrow{\mu} & G\times X \\ (g,h,x) & \mapsto & (gh,x) \end{array} \qquad \begin{array}{ccc} G\times G\times X & \xrightarrow{\pi} & G\times X \\ (g,h,x) & \mapsto & (g,x) \end{array}$$

In [BL94], these maps are respectively called d_0, d_1 and d_2 and are viewed as the first face maps of the simplicial set $X//G = EG \times_G X$. We can now define what is meant by an equivariant sheaf.

Definition 1.0.1 (*[BL94, §0.2]*)**.** *Let X be a G-space.*

1. *A G-*equivariant sheaf *on X is a pair $(\mathcal{F}, \theta)$ where $\mathcal{F} \in \mathbf{Ab}(X)$ is an abelian sheaf on X and θ is an isomorphism*
$$\theta : p^*\mathcal{F} \xrightarrow{\sim} a^*\mathcal{F}$$
such that
$$\eta^*\theta \circ \pi^*\theta = \mu^*\theta \quad \textit{and} \quad \iota^*\theta = id_{\mathcal{F}}.$$

2. *We say that a morphism $f : (\mathcal{F}, \theta^{\mathcal{F}}) \to (\mathcal{G}, \theta^{\mathcal{G}})$ between equivariant sheaves is a* morphism of equivariant sheaves *if the following square is commutative*

$$\begin{tikzcd} p^*\mathcal{F} \arrow[r, "\theta^{\mathcal{F}}"] \arrow[d, "p^*f"'] & a^*\mathcal{F} \arrow[d, "a^*f"] \\ p^*\mathcal{G} \arrow[r, "\theta^{\mathcal{G}}"'] & a^*\mathcal{G} \end{tikzcd}$$

3. *The subcategory of $\mathbf{Ab}(X)$ whose objects are equivariant sheaves and whose maps are morphisms of equivariant sheaves is denoted by $\mathbf{Ab}_G(X)$.*

Remark 1.0.2. *In [Let05, Proposition 4.2.7], it is proved that if G is connected, then the category $\mathbf{Ab}_G(X)$ is a full subcategory of $\mathbf{Ab}(X)$. In other words, for a sheaf $\mathcal{F} \in \mathbf{Ab}(X)$, if there exists some isomorphism $\theta : p^*\mathcal{F} \xrightarrow{\sim} a^*\mathcal{F}$ satisfying the conditions of the previous definition, then it must be unique. Hence in this case, the fact of being "equivariant" for a sheaf is more like a property of the sheaf, rather than an additional structure.*

This definition is as general as possible, though not very handy. However, one has the following result:

Proposition 1.0.3. *Let G be a topological group, X be a G-space and $\mathcal{F}$ be an abelian sheaf on X. Then, the following first condition implies the other two, which are equivalent to each other:*

(i) There exists an isomorphism $\theta : p^*\mathcal{F} \to a^*\mathcal{F}$ such that $(\mathcal{F}, \theta)$ is an equivariant sheaf on X,

(ii) There are isomorphisms $\alpha_g : \mathcal{F} \xrightarrow{\sim} g^*\mathcal{F}$ for all $g \in G$, verifying the cocycle condition

$$\forall g, h \in G,\ \alpha_{gh} = h^*(\alpha_g) \circ \alpha_h \quad \text{and} \quad \alpha_1 = id_{\mathcal{F}},$$

(iii) The group G acts on the espace étalé $\mathrm{Et}(\mathcal{F})$ *and the projection* $\mathrm{Et}(\mathcal{F}) \twoheadrightarrow X$ *is a G-bundle.*

In the second statement, a morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$ is G-equivariant if $\alpha_g^{\mathcal{G}} \circ f = g^*(f) \circ \alpha_g^{\mathcal{F}}$ for all $g \in G$ and in the third statement, f is a G-equivariant if $\mathrm{Et}(f) : \mathrm{Et}(\mathcal{F}) \to \mathrm{Et}(\mathcal{G})$ *is a homomorphism of G-bundles.*

Moreover, if the group G is discrete, then the three conditions above are equivalent.

Proof. First, we prove that $(ii) \Leftrightarrow (iii)$. So assume that we have isomorphisms $\alpha_g : \mathcal{F} \to g^*\mathcal{F}$ satisfying the cocycle condition. Recall that, set-theoretically we have $\mathrm{Et}(\mathcal{F}) = \coprod_{x \in X} \mathcal{F}_x$. Given $x \in X$ and $g \in G$, we can consider the homomorphism between stalks $(\alpha_g)_x : \mathcal{F}_x \to (g^*\mathcal{F})_x = \mathcal{F}_{gx}$ and these induce a continuous map $\widetilde{\alpha_g} : \mathrm{Et}(\mathcal{F}) \to \mathrm{Et}(\mathcal{F})$. The cocycle condition applied to the stalks at x gives the identity $(\alpha_{gh})_x = (\alpha_g)_{hx} \circ (\alpha_h)_x$, hence $\widetilde{\alpha_{gh}} = \widetilde{\alpha_g} \circ \widetilde{\alpha_h}$ and the fact that $\alpha_1 = id_{\mathcal{F}}$ gives of course $\widetilde{\alpha_1} = id_{\mathrm{Et}(\mathcal{F})}$. This makes $\mathrm{Et}(\mathcal{F})$ into a G-space. Note that, by construction, if $q : \mathrm{Et}(\mathcal{F}) \to X$ denotes the projection, then we have $q \circ \widetilde{\alpha_g} = a(g^{-1}, -) \circ q$ so that $\mathrm{Et}(\mathcal{F}) \to X$ is indeed a G-bundle. Moreover, if two sheaves $\mathcal{F}$ and $\mathcal{G}$ verify condition (ii) and if $f : \mathcal{F} \to \mathcal{G}$ verifies $\alpha_g^{\mathcal{G}} \circ f = g^*(f) \circ \alpha_g^{\mathcal{F}}$, then reading this equation on stalks shows that $\mathrm{Et}(f) : \mathrm{Et}(\mathcal{F}) \to \mathrm{Et}(\mathcal{G})$ is a G-bundle homomorphism.

Conversely, suppose that $\mathrm{Et}(\mathcal{F})$ is equipped with the structure of a G-bundle. Given $g \in G$, we have to define an isomorphism α_g. Note that for any sheaf $\mathcal{E} \in \mathbf{Ab}(X)$, as $g = a(g, -) : X \to X$ is a homeomorphism, the presheaf

$$U \mapsto \varinjlim_{V \supset g(U)} \mathcal{E}(V) = \mathcal{E}(gU)$$

is a sheaf, so the inverse image $g^*\mathcal{E}$ is given by $(g^*\mathcal{E})(U) = \mathcal{E}(gU)$. Take an open subset $U \subset X$. Since $\mathcal{F}(U)$ is the group of continuous local sections of $q : \mathrm{Et}(\mathcal{F}) \to X$,

$$\mathcal{F}(U) = \{\varphi : U \to \mathrm{Et}(\mathcal{F}) \ ; \ q \circ \varphi = id_U\},$$

we may define $\alpha_g(U)$ at a local section φ as

$$\alpha_g(U)(\varphi) := \begin{pmatrix} gU & \to & \mathrm{Et}(\mathcal{F}) \\ x & \mapsto & g\varphi(g^{-1}x) \end{pmatrix}.$$

It is straightforward, using the fact that g sends a stalk $\mathcal{F}_x$ to $\mathcal{F}_{gx}$, to check that $\alpha_g(U)(\varphi) \in \mathcal{F}(gU) = (g^*\mathcal{F})(U)$ and that this induces a homomorphism of sheaves $\alpha_g : \mathcal{F} \to g^*\mathcal{F}$ and it is clear that $\alpha_1 = id_{\mathcal{F}}$. We have to verify the cocycle condition. Interpreting the stalk at $x \in X$ as germs of sections around x, we find that $(\alpha_g)_x : \mathcal{F}_x \to \mathcal{F}_{gx}$ is given by the action of g on $\mathcal{F}_x$. Thus, we have $(h^*(\alpha_g) \circ \alpha_h)_x = (\alpha_g)_{hx} \circ (\alpha_h)_x = (\alpha_{gh})_x$ and thus $h^*(\alpha_g) \circ \alpha_h = \alpha_{gh}$, as required. Furthermore, if a morphism $f : \mathcal{F} \to \mathcal{G}$ between sheaves verifying (iii) induces a homomorphism of G-bundles $\mathrm{Et}(\mathcal{F}) \to \mathrm{Et}(\mathcal{G})$, then we have

$$\forall g \in G,\ \forall x \in X,\ (\alpha_g^{\mathcal{G}} \circ f)_x = (\alpha_g^{\mathcal{G}})_x \circ f_x = (g\cdot)_x \circ f_x = f_{gx} \circ (g\cdot)_x = (g^*(f) \circ \alpha_g^{\mathcal{F}})_x,$$

and thus $\alpha_g^{\mathcal{G}} \circ f = g^*(f) \circ \alpha_g^{\mathcal{F}}$.

Now, we prove $(i) \Rightarrow (ii)$. For $g \in G$, define the map

$$\begin{array}{cccc} \phi_g & : & X & \to & G \times X \\ & & x & \mapsto & (g, x) \end{array}$$

and define also $\alpha_g := \phi_{g^{-1}}^*\theta$. Since $\phi_{g^{-1}}^* p^* = (p\phi_{g^{-1}})^* = id_{\mathcal{F}}$ and $\phi_{g^{-1}}^* a^* = (a\phi_{g^{-1}})^* = g^*$, this gives a morphism $\mathcal{F} \to g^*\mathcal{F}$ and since $\phi_1 = \iota$, we have that $\alpha_1 = \iota^*\theta = id_{\mathcal{F}}$. Now, given $g, h \in G$ and $x \in X$, we compute

$$(\alpha_{gh} - h^*(\alpha_g) \circ \alpha_h)_x = (\phi_{(gh)^{-1}}^*\theta)_x - (\phi_{g^{-1}}^*\theta)_{hx} \circ (\phi_{h^{-1}}^*\theta)_x = \theta_{(h^{-1}g^{-1},x)} - \theta_{(g^{-1},hx)} \circ \theta_{(h^{-1},x)}$$

$$= (\mu^*\theta)_{(h^{-1},g^{-1},x)} - (\eta^*\theta)_{(h^{-1},g^{-1},x)} \circ (\pi^*\theta)_{(h^{-1},g^{-1},x)} = (\mu^*\theta - \eta^*\theta \circ \pi^*\theta)_{(h^{-1},g^{-1},x)},$$

so if $\mu^*\theta = \eta^*\theta \circ \pi^*\theta$, then $\alpha_\bullet$ satisfies the cocycle condition. Moreover, if $f : \mathcal{F} \to \mathcal{G}$ is a morphism of equivariant sheaves, then we have

$$\forall g \in G,\ \forall x \in X,\ (g^*(f) \circ \alpha_g^{\mathcal{F}} - \alpha_g^{\mathcal{G}} \circ f)_x = (a^*f \circ \theta^{\mathcal{F}} - \theta^{\mathcal{G}} \circ p^*f)_{(g^{-1},x)},$$

thus f verifies (ii) if and only if it is a morphism of equivariant sheaves.

Finally, suppose (ii) and that G is discrete. We have to define $\theta : p^*\mathcal{F} \to a^*\mathcal{F}$. Since $p^*\mathcal{F} \in \mathbf{Ab}(G \times X)$, it suffices to define $\theta(\Omega \times U)$ for $\Omega \subset G$ and $U \subset X$ two open subsets. As G is discrete, using coproducts we may assume that $\Omega = \{g\}$ is a singleton. Denote by $(p^*\mathcal{F})^-$ the presheaf $U \mapsto \varinjlim_{V \supset p(U)} \mathcal{F}(V)$, so that $p^*\mathcal{F}$ is the sheafification of $(p^*\mathcal{F})^-$. Consider

$$(p^*\mathcal{F})^-(\{g\} \times U) = \mathcal{F}(U) \xrightarrow{\alpha_{g^{-1}}(U)} \mathcal{F}(g^{-1}U) \xrightarrow{+} (a^*\mathcal{F})(\{g\} \times U).$$

This gives a map $(p^*\mathcal{F})^- \to a^*\mathcal{F}$ and the universal property of the sheafification yields a map $\theta : p^*\mathcal{F} \to a^*\mathcal{F}$. We have $\theta_{(g,x)} = (\alpha_{g^{-1}})_x$ so θ is an isomorphism and the following equation still holds

$$\forall g, h \in G,\ \forall x \in X,\ (\mu^*\theta - \eta^*\theta \circ \pi^*\theta)_{(h^{-1},g^{-1},x)} = (\alpha_{gh} - h^*(\alpha_g) \circ \alpha_h)_x,$$

showing that $\mu^*\theta = \eta^*\theta \circ \pi^*\theta$ and the fact that $\theta_{(1,x)} = (\alpha_1)_x = id$ ensures that $\iota^*\theta = id_{\mathcal{F}}$ and this finishes the proof. □

Remark 1.0.4. *In general, if $\mathcal{F} \in \mathbf{Ab}_G(X)$ is an equivariant sheaf on X, then $\Gamma(X, \mathcal{F})$ is naturally a $\mathbb{Z}[G]$-module through the following composition of isomorphisms*

$$\Gamma(X, \mathcal{F}) \xrightarrow{\Gamma(a^*)} \Gamma(G \times X, a^*\mathcal{F}) \xrightarrow{\Gamma(\theta^{-1})} \Gamma(G \times X, p^*\mathcal{F}) = \mathbb{Z}[G] \otimes_{\mathbb{Z}} \Gamma(X, \mathcal{F}).$$

More generally, if $U \subset X$ is open, then we have a map

$$\mathbb{Z}[G] \otimes \Gamma(U, \mathcal{F}) \to \Gamma(gU, \mathcal{F}),$$

this can be seen using the condition (ii) in the above Proposition. Hence, we obtain a functor

$$\Gamma(X, -) : \mathbf{Ab}_G(X) \to \mathbb{Z}[G]\text{-}\mathbf{Mod}.$$

Notice that this can be extended to any (commutative) coefficient ring k.

Proposition 1.0.5. *Let G be a discrete group and X be a paracompact locally contractible G-space. The singular cochains restricted to open subsets of X gives a complex of presheaves C^* on X. We denote by $\mathcal{C}^*$ its sheafification. Then, $\mathcal{C}^*$ is a flasque resolution of the constant sheaf $\underline{\mathbb{Z}}$ in $\mathbf{Ab}_G(X)$.*

Proof. For an open subset $U \subset X$, the presheaf $C^n(U)$ is defined by $C^n(U) := \mathrm{Hom}_{\mathbb{Z}}(C_n(U), \mathbb{Z})$ where $C_n(U) = \mathbb{Z}\langle \Delta^n \to U \rangle$ is the group of singular chains in U. The restriction maps being obvious, this defines a presheaf C^n on X and singular differentials yield that C^* is a complex of abelian presheaves on X. The sheafification of this complex is $\mathcal{C}^*$. The fact that this gives a resolution of the constant sheaf $\underline{\mathbb{Z}}$ is well-known, see for instance [Bre97, Theorem III.1.1]. We have to see that $\underline{\mathbb{Z}}$ and $\mathcal{C}^n$ are naturally equivariant sheaves and that the complex $\mathcal{C}^*$ is a complex in $\mathbf{Ab}_G(X)$. For every $x \in X$, we have $\underline{\mathbb{Z}}_x = \mathbb{Z}$, so $\mathrm{Et}(\underline{\mathbb{Z}}) = \coprod_x \mathbb{Z}$ and we may simply consider the trivial action of G on $\mathbb{Z}$ and extend it to an action on $\mathrm{Et}(\underline{\mathbb{Z}})$. This endows $\underline{\mathbb{Z}}$ with the structure of an equivariant sheaf on X. Now, G acts naturally on singular chains, sending a local singular simplex $\sigma : \Delta^n \to U$ to $g\sigma : \Delta^n \to gU$ simply defined by $(g\sigma)(x) := g \cdot \sigma(x)$. This induces a map $g\cdot : C^n(U) \to C^n(gU)$ which in turn, by sheafification, yields a map $\alpha_g : \mathcal{C}^n \to g^*\mathcal{C}^n$ and we immediately see that α satisfies the cocycle condition. As G is discrete, this gives an equivariant structure to $\mathcal{C}^n$. Now, by construction of $C^n(U) \to C^n(gU)$, we see that the singular differential $C^n(U) \xrightarrow{d} C^{n+1}(U)$ fits into a commutative square

$$\begin{tikzcd}
C^n(U) \arrow[r] \arrow[d, "d"'] & C^n(gU) \arrow[d, "d"] \\
C^{n+1}(U) \arrow[r] & C^{n+1}(gU)
\end{tikzcd}$$

and hence, the differential $d : \mathcal{C}^n \to \mathcal{C}^{n+1}$ is a morphism of equivariant sheaves. □

Definition 1.0.6. *If X is a G-space, with G a discrete group, then the* equivariant derived category $\mathcal{D}_G(X)$ *is the derived category of the abelian category of equivariant sheaves on X*

$$\mathcal{D}_G(X) := \mathcal{D}(\mathbf{Ab}_G(X)).$$

We can similarly define the bounded equivariant derived category $\mathcal{D}^b_G(X) := \mathcal{D}^b(\mathbf{Ab}_G(X))$.

Remark 1.0.7. *We still can define the derived equivariant category in the general case where G is not supposed to be discrete, but this is far more subtle. In this context, one needs to use n-acyclic resolutions of X and define $\mathcal{D}_G(X)$ as a limit (see [BL94, I §2]). It is worth noticing that, if the G-space X is free, then we have a natural derived equivalence*

$$\mathcal{D}_G(X) \simeq \mathcal{D}(X/G).$$

Corollary 1.0.8. *If G is a discrete group and if X is a G-space, then the constant sheaf $\underline{\mathbb{Z}}$ is naturally endowed with the structure of a G-equivariant sheaf on X and, denoting by $\mathcal{C}^*$ the sheafification of the local singular cochain complex, in the equivariant derived category $\mathcal{D}_G(X)$, one can compute the total derived global section functor on $\underline{\mathbb{Z}}$ as*

$$R\Gamma(X, \underline{\mathbb{Z}}) = \Gamma(X, \mathcal{C}^*) \in \mathcal{D}^b(\mathbb{Z}[G]).$$

Proof. This is clear using the Proposition 1.0.5 and the fact that flasque sheaves are acyclic for $\Gamma(X, -)$. $\square$

2 Equivariant cellular and simplicial structures

2.1 Definitions of equivariant structures and related (co)chain complexes

In this section we recall the definition of a G-CW-complex X, for a given group G. We shall be most interested in the case where G is discrete, or even finite since we are dealing with at most extended affine Weyl groups. For more on G-CW-complexes and their use in the homotopy theory of G-spaces, see [May93], [Sha10] or [Die87].

The notion of G-CW-complex was first introduced in [Bre67] for discrete groups and later generalized to arbitrary groups in [Mat71].

Definition 2.1.1. *Let G be a topological group and X be a G-space.*

1. *The G-space X is a* G-CW-complex *if X is the colimit $X = \varinjlim_{n\in\mathbb{N}} X_n$ of G-spaces X_n with inclusions $i_n : X_n \hookrightarrow X_{n+1}$ such that X_0 is a disjoint union of orbits G/H (for $H \leq G$ a closed subgroup) and X_n is obtained from X_{n-1} by attaching equivariant n-cells $G/H \times \mathbb{D}^n$ via G-maps $G/H \times \mathbb{S}^{n-1} \to X_{n-1}$ ($\mathbb{D}^n$ and $\mathbb{S}^{n-1}$ are considered as trivial G-spaces) as in the following pushout diagram*

$$\begin{tikzcd} \coprod_{i\in I} G/H_i \times \mathbb{S}^{n-1} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & X_{n-1} \arrow[d, hook, "i_{n-1}"] \\ \coprod_{i\in I} G/H_i \times \mathbb{D}^n \arrow[r] & X_n \end{tikzcd}$$

2. *If X is a CW-complex, we say that it is a G-*cellular CW-complex *if G acts on the set of n-cells of X for all n and if, as soon as $g \in G$ lets a cell globally invariant, then it restricts to the identity on this cell.*

Remark 2.1.2. *Any G-map $\phi : G/H \times \mathbb{S}^n \to X$ determines a map $\phi' : \mathbb{S}^n \to X^H$ by letting $\phi'(x) := \phi(H, x)$ and conversely, one has $\phi(gH, x) = g\phi'(x)$. This allows sometimes to reduce the equivariant theory to the non-equivariant case.*

Proposition 2.1.3 *([Die87, Propositions 1.15 and 1.16 and Exercise 1.17 (2)])*. *Let G be a discrete group and let X be a G-space.*

1. *If X is a G-cellular CW-complex, then it is also a G-CW-complex with the same skeleton.*
2. *Let $H \leq G$ be a subgroup of G. If X is a G-CW-complex, then considered as an H-space, is a H-CW-complex with the same skeleton.*
3. *In particular, the two notions of G-cellular CW-complex and G-CW-complex coincide.*
4. *If X is a G-CW-complex with n-skeleton X_n such that the orbit space X/G is Hausdorff, then X/G is a CW-complex with n-skeleton X_n/G.*

Of course, we have a simplicial version of this notion:

Definition 2.1.4. *Let G be a topological group. A G-*simplicial complex *is a simplicial complex (V, Σ) such that V and Σ are G-sets and such that for $S \in \Sigma$ and $g \in G_S$, we have $gv = v$ for all $v \in S$.*

Of course, for a discrete group G, the geometric realization $|V|$ of a G-simplicial complex is a *regular* G-CW-complex. Here, the term *regular* means that the closure of each cell in $|V|$ is homeomorphic to a closed ball.

Lemma 2.1.5 *([Wan80, Lemma 4.3])*. *For a topological group G, any G-CW-complex is G-homotopy equivalent to a colimit of finite dimensional G-simplicial complexes.*

Though hard to determine explicitly, equivariant cellular structures arise frequently.

Proposition 2.1.6 *([Mat73, Proposition 0.5])*. *If G is a compact Lie group, then any compact G-manifold has a finite G-equivariant CW-complex structure.*

For a given CW-complex X, we can consider its *cellular homology chain complex* $C_*^{\text{cell}}(X, \mathbb{Z})$, where each $C_n^{\text{cell}}(X, Z) = \bigoplus_{i \in I} \mathbb{Z} e_i$ with e_i the n-cells of X (see [Hat02]). If X is a G-CW-complex (with G discrete), then its cellular chain complex $C_*^{\text{cell}}(X, \mathbb{Z})$ becomes a chain complex of $\mathbb{Z}[G]$-modules, which we denote by $C_*^{\text{cell}}(X, G; \mathbb{Z})$ if the acting group G is ambiguous. Moreover, if $\mathcal{E}_n$ is the (possibly infinite) set of n-cells of X, with $n \in \mathbb{N}$, then G acts on $\mathcal{E}_n$ and the $\mathbb{Z}$-module $C_n^{\text{cell}}(X, \mathbb{Z})$ is free with basis $\mathcal{E}_n$, i.e.

$$C_n^{\text{cell}}(X, \mathbb{Z}) = \mathbb{Z}[\mathcal{E}_n].$$

This means by definition that $C_n^{\text{cell}}(X, G; \mathbb{Z})$ is a *permutation module*. Furthermore, decomposing $\mathcal{E}_n = \bigsqcup_i G/H_i$ into orbits, we have

$$C_n^{\text{cell}}(X, G; \mathbb{Z}) \simeq \bigoplus_i \mathbb{Z}[G/H_i],$$

where H_i runs through a representative set of stabilizers of n-cells of X. Since the action of G on X is cellular, this implies that each H_i is in fact the stabilizer of any point of the corresponding cell.

We may describe the dual complex $C^*_{\mathrm{cell}}(X, G; \mathbb{Z})$ in a similar way, but we have to take care of the dualisation when the number of cells is infinite. For an arbitrary set S, we denote by $\mathbb{Z}[[S]]$ the set of families $x = (x_s)_{s\in S}$ of integers, indexed by S. It will be convenient to prefer the formal notation $x = \sum_{s\in S} x_s s$. Notice that, for an arbitrary group G and $H \leq G$, we have a canonical isomorphism of right $\mathbb{Z}[G]$-modules

$$\begin{array}{rcl} \mathbb{Z}[G/H]^\vee \overset{\mathrm{df}}{=} \mathrm{Hom}(\mathbb{Z}[G/H], \mathbb{Z}) & \longrightarrow & \mathbb{Z}[[H\backslash G]] \\ (gH)^* & \longmapsto & Hg^{-1} \end{array}$$

and this yields an isomorphism $\mathbb{Z}[G/H]^\vee \to \mathbb{Z}[H\backslash G]$ in case H is of finite index. This allows to give a general description for the homogeneous components of the dual complex:

$$C^n_{\mathrm{cell}}(X, G; \mathbb{Z}) = \prod_i \mathbb{Z}[[H_i\backslash G]]$$

where the H_i's are as above. This is indeed a right $\mathbb{Z}[G]$-module, but it is a permutation module only when the number of cells is finite. If we only suppose G finite, then it is a product of permutation modules.

Summarizing, we have obtained the following standard result, which is a cellular version of [Ric94, Theorem 3.2]:

Proposition 2.1.7. *Let G be a discrete group and X be a G-CW-complex. Then the cellular homology chain complex $C_*^{\mathrm{cell}}(X, G; \mathbb{Z})$ is a chain complex of permutation $\mathbb{Z}[G]$-modules, each of which being of the form $\mathbb{Z}[G/H]$ where H runs through the stabilizers of points of X. In particular, if the G-space X is free, then $C_*^{\mathrm{cell}}(X, G; \mathbb{Z})$ is a chain complex of free $\mathbb{Z}[G]$-modules.*

We also have the following version of [Ric94, Corollary 3.3], which roughly says that, if we have some restrictions on the subgroups of G that can occur as point stabilizers of X, then we get some conditions on the modules that can appear in $C_*^{\mathrm{cell}}(X, G; \mathbb{Z})$.

Recall that for a subgroup $H \leq G$, a $\mathbb{Z}[G]$-module M is said to be *relatively H-projective* if every epimorphism of $\mathbb{Z}[G]$-modules $A \twoheadrightarrow M$ that splits in $\mathbb{Z}[H]$-**Mod**, splits in $\mathbb{Z}[G]$-**Mod** as well.

Corollary 2.1.8. *Let G be a discrete group and X be a G-CW-complex. Assume that the stabilizer of every point of X is conjugate to some fixed subgroup $H \leq G$. Then, every homogeneous component of the complex $C_*^{\mathrm{cell}}(X, G; \mathbb{Z})$ is a direct sum of relatively H-projective modules.*

Proof. For a G-set S, we denote by G_s the stabilizer of $s \in S$. Notice that, since G acts cellularly on X, if $e \in \mathcal{E}_n$ is an n-cell of X and if $x \in e$, then $G_e = G_x$. Thus, the module $C_n^{\mathrm{cell}}(X, G; \mathbb{Z})$ is a direct sum of modules of the form $\mathbb{Z}[G/G_e]$, where $e \in \mathcal{E}_n$. Thus, it suffices to prove that each $\mathbb{Z}[G/G_e]$ is relatively H-projective. By hypothesis, there exists $g \in G$ such that $gG_eg^{-1} \leq H$ and we have

$$\mathbb{Z}[G/G_e] \simeq \mathbb{Z}[G/(gG_eg^{-1})],$$

so we may assume that $K := G_e \leq H$.

Suppose that $\pi : A \to \mathbb{Z}[G/K]$ is a $\mathbb{Z}[G]$-linear map and let $s : \mathbb{Z}[G/K] \to A$ be a $\mathbb{Z}[H]$-section of π. Since $K \leq H$, the element $x := s(1K)$ is in A^K and we may define $\widetilde{s} : \mathbb{Z}[G/K] \to A$ by $\widetilde{s}(gK) := gx$. This is well-defined since $x \in A^K$ and $\widetilde{s} \in \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G/K], A)$ by construction. It is now clear that $\widetilde{s}$ is a $\mathbb{Z}[G]$-section of π. $\square$

We shall see the link between the complex $C^*_{\text{cell}}(X, G; \mathbb{Z})$ and the complex $R\Gamma(X, \underline{\mathbb{Z}}) \in \mathcal{D}^b(\mathbb{Z}[G])$ introduced in the previous section. Note that we have chosen to work with coefficients in $\mathbb{Z}$, but the same arguments apply with any ring.

Given a CW-complex X, it is well-known that the resulting chain complex $C_*^{\text{cell}}(X, \mathbb{Z})$ is homotopy equivalent to the singular chain complex $C_*(X, \mathbb{Z})$. This relies on the fact that they are quasi-isomorphic (see for instance [Hat02, Theorem 2.35]) and that quasi-isomorphic bounded below complexes of free modules are homotopy equivalent. Indeed, one may construct a homotopy inverse using bases of the free modules.

In the equivariant setting, if the action of G on X is not free, then the induced $\mathbb{Z}[G]$-modules $C_n(X, \mathbb{Z})$ are not free neither. However, the result is still true. First, let us recall the equivariant versions of Whitehead's theorem and the cellular approximation theorem.

Theorem 2.1.9 (*[Die87, II, §2, Theorem 2.1 and Proposition 2.7]*)**.** *Let G be a locally compact (Hausdorff) group.*

1. *If $f : X \to Y$ is a G-map between G-CW-complexes, then f is G-homotopy equivalent to a G-cellular map $f' : X \to Y$ (*cellular *meaning that $f'(X_n) \subset Y_n$ for all n).*
2. *Let $f : X \to Y$ be a G-map between G-CW-complexes. For a closed subgroup $H \leq G$, f induces a map between H-fixed points $f^H : X^H \to Y^H$. If f^H is a (classical) homotopy equivalence for every closed subgroup $H \leq G$, then f is a G-homotopy equivalence.*

Theorem 2.1.10. *Let G be a discrete group and X be a G-CW-complex. Then, the cellular homology chain complex $C_*^{cell}(X, G; \mathbb{Z})$ and the singular chain complex $C_*(X, \mathbb{Z})$ are complexes of $\mathbb{Z}[G]$-modules which are isomorphic in the homotopy category $\mathcal{K}_b(\mathbb{Z}[G])$.*

Proof. We first give some reminders on the classical (non-equivariant) case. Recall that the category s**Set** (resp. **Top**) of simplicial sets (resp. of topological spaces) is endowed with the structure of a model category with fibrations being Kan fibrations, cofibrations being monomorphisms and weak equivalences being simplicial maps inducing a topological weak equivalence between realizations (resp. with fibrations being Serre fibrations, cofibrations being retracts of relative CW-complexes and weak equivalences being maps inducing isomorphisms on all homotopy groups). In this context, we have a Quillen adjunction (see [GJ99, Proposition 2.2]) $|\cdot| \dashv \mathrm{Sing}$, with $|\cdot| : s\mathbf{Set} \to \mathbf{Top}$ being the geometric realization of a simplicial set and $\mathrm{Sing}(X)$ being the total singular set of a topological space X, that is, the simplicial set with $\mathrm{Hom}(\Delta^n, X)$ as n-simplices. In fact, this is a Quillen equivalence ([GJ99, Theorem 11.4]). This implies that, for a space X, the counit of the Quillen adjunction

$$|\mathrm{Sing}(X)| \xrightarrow{\varepsilon_X} X$$

is a weak equivalence. Denote $\Gamma X := |\mathrm{Sing}(X)|$. Since $\mathrm{Sing}(X)$ is a simplicial set, ΓX is a CW-complex, with n-cells given by non-degenerate singular n-simplices of X (see [May99, Chap. 16, §2]), hence there is a natural isomorphism of chain complexes of abelian groups

$$C_*^{\mathrm{cell}}(\Gamma X, \mathbb{Z}) \xrightarrow{\sim} C_*(X, \mathbb{Z}).$$

In fact, ΓX is a fibrant replacement of X in **Top**. On the other hand, if X is already a CW-complex, then by Whitehead's theorem ([Hat02, Theorem 4.5]), the map $\varepsilon : \Gamma X \to X$ is a (strong) homotopy equivalence. Furthermore, by the cellular approximation theorem ([Hat02, Theorem 4.8]), there exists a cellular map $\varepsilon' : \Gamma X \to X$ which is a homotopy equivalence. Hence this map ε' induces a homotopy equivalence between complexes

$$C^{\mathrm{cell}}(\varepsilon') : C_*^{\mathrm{cell}}(\Gamma X, \mathbb{Z}) \to C_*^{\mathrm{cell}}(X, \mathbb{Z})$$

and thus, we obtain a homotopy equivalence of chain complexes

$$C_*(X, \mathbb{Z}) \to C_*^{\mathrm{cell}}(X, \mathbb{Z}).$$

We can do the same for a G-CW-complex X. Let $H \leq G$ be a subgroup of G. Then we have $\mathrm{Sing}(X)^H = \mathrm{Sing}(X^H)$. Indeed, since G acts on a singular simplex $\sigma : \Delta^n \to X$ by $(g\sigma)(x) := g\sigma(x)$, it is clear that $\sigma : \Delta^n \to X$ is H-invariant if and only if it factors through $\widetilde{\sigma} : \Delta^n \to X^H$. Hence, we also have $(\Gamma X)^H = |\mathrm{Sing}(X)|^H = |\mathrm{Sing}(X)^H| = |\mathrm{Sing}(X^H)| = \Gamma(X^H)$. By naturality of ε_X, we have a commutative square

$$\begin{tikzcd} {|\mathrm{Sing}(X^H)|} \arrow[r, hook] \arrow[d, "\varepsilon_{X^H}"'] & {|\mathrm{Sing}(X)|} \arrow[d, "\varepsilon_X"] \\ X^H \arrow[r, hook] & X \end{tikzcd}$$

that is,

$$\begin{tikzcd} (\Gamma X)^H \arrow[r, hook] \arrow[d, "\varepsilon_{X^H}"'] & \Gamma X \arrow[d, "\varepsilon_X"] \\ X^H \arrow[r, hook] & X \end{tikzcd}$$

Now, since $(\Gamma X)^H = \Gamma(X^H) \xrightarrow{\varepsilon_{X^H}} X^H$ is a weak homotopy equivalence between CW-complexes, it is a homotopy equivalence. Since this is true for every subgroup $H \leq G$, the equivariant Whitehead theorem 2.1.9, (2) ensures that $\varepsilon_X : \Gamma X \to X$ is a G-homotopy equivalence. If we denote by $\eta : X \to \Gamma X$ its G-homotopy inverse, then by (1) of the Theorem 2.1.9, we may find cellular maps $\varepsilon' : \Gamma X \to X$ and $\eta' : X \to \Gamma X$ that are homotopy equivalent to ε and η, respectively. Then, $\varepsilon' : \Gamma X \to X$ is a G-cellular G-homotopy equivalence, with G-homotopy inverse η'. Hence, we obtain a $\mathbb{Z}[G]$-homotopy equivalence of complexes

$$C^{\mathrm{cell}}(\varepsilon') : C_*^{\mathrm{cell}}(\Gamma X, G; \mathbb{Z}) \to C_*^{\mathrm{cell}}(X, G; \mathbb{Z}).$$

Now, since $C_*(X, \mathbb{Z})$ is naturally isomorphic to $C_*^{\mathrm{cell}}(\Gamma X, G; \mathbb{Z})$, this isomorphism is thus an isomorphism of complexes of $\mathbb{Z}[G]$-modules. Thus, we have obtained a $\mathbb{Z}[G]$-homotopy equivalence

$$C_*(X, \mathbb{Z}) \to C_*^{\mathrm{cell}}(\Gamma X, G; \mathbb{Z}) \to C_*^{\mathrm{cell}}(X, G; \mathbb{Z}),$$

as claimed. □

Combining this result and the Corollary 1.0.8 yields the following result:

Corollary 2.1.11. *Let G be a discrete group and X be a G-CW-complex. Then, the associated cellular chain complex $C_*^{\mathrm{cell}}(X, G; \mathbb{Z})$ is isomorphic to the singular complex $C_*(X, \mathbb{Z})$ in the homotopy category $\mathcal{K}_b(\mathbb{Z}[G])$. By duality, the same holds for the cellular cochain complex. Moreover, one has*

$$R\Gamma(X, \underline{\mathbb{Z}}) = C^*(X, \mathbb{Z}) = C^*_{\mathrm{cell}}(X, G; \mathbb{Z}) \quad \text{in} \quad \mathcal{D}^b(\mathbb{Z}[G]).$$

In particular, any G-CW-structure on X gives a cellular cochain complex which is well-defined up to isomorphism in $\mathcal{K}^b(\mathbb{Z}[G])$ (i.e. is independent of the chosen structure, up to equivariant homotopy) and computes the derived functor $R\Gamma(X, \underline{\mathbb{Z}})$ in $\mathcal{D}^b(\mathbb{Z}[G])$.

Proof. The only thing left to be proven is the fact that the sheafification morphism

$$C^*(X, \mathbb{Z}) \xrightarrow{+} \Gamma(X, \mathcal{C}^*)$$

is an equivariant quasi-isomorphism. The fact that it is G-equivariant is quite clear by definition of the action of G on singular chains and on $\Gamma(X, \mathcal{C}^*)$. To prove that it is a quasi-isomorphism, we use an argument due to Ramanan (see [Ram05, Chapter 4, Proposition 4.12]). First, note that we have an isomorphism of abelian groups

$$\Gamma(X, \mathcal{C}^n) \simeq C^n(X)/C^n(X)_0,$$

where

$$C^n(X)_0 := \{\varphi \in C^n(X) \ ; \ \exists \mathcal{V} = (V_i)_{i \in I} \text{ open cover such that } \varphi_{|C_n(V_i)} = 0, \ \forall i \in I\}.$$

Furthermore, for an open cover $\mathcal{U} = (U_i)_{i \in I}$ of X, if $C^n_{\mathcal{U}}(X)$ denote the group of $\mathcal{U}$-small n-cochains of X (that is, the dual of the group of $\mathcal{U}$-small n-chains), we have an obvious (surjective) morphism $\pi_{\mathcal{U}} : C^*(X) \to C^*_{\mathcal{U}}(X)$, which is a homotopy equivalence (see [Vic94, Appendix I, Theorem 1.14]). Hence we have a short exact sequence of complexes

$$0 \longrightarrow \ker \pi_{\mathcal{U}} \longrightarrow C^*(X) \xrightarrow{\pi_{\mathcal{U}}} C^*_{\mathcal{U}}(X) \longrightarrow 0$$

The fact that $\pi_{\mathcal{U}}$ is a homotopy equivalence shows that $\ker \pi_{\mathcal{U}}$ is exact. Hence, taking the colimit over all open covers of X, ordered by refinement, yields and exact sequence of complexes

$$0 \longrightarrow \varinjlim \ker \pi_{\mathcal{U}} \longrightarrow C^*(X) \longrightarrow C^*(X)/C^*(X)_0 \longrightarrow 0$$

and since $H^*(\varinjlim \ker \pi_{\mathcal{U}}) = \varinjlim H^*(\ker \pi_{\mathcal{U}}) = 0$, we obtain that the projection $C^*(X) \twoheadrightarrow C^*(X)/C^*(X)_0$ is a quasi-isomorphism. $\square$

2.2 Compatibility with subgroups and quotients

As in [Ric94, Theorem 4.1], we have a compatibility result for $C^*_{\mathrm{cell}}(X, G; \mathbb{Z})$ with respect to quotient and fixed points. First, recall the notion of the (co)invariant module of a $\mathbb{Z}[G]$-module.

Definition 2.2.1. *Let G be a finite group and M a $\mathbb{Z}[G]$-module.*

- *The* module of invariants *of M is the abelian group*

$$M^G := \{m \in M \ ; \ gm = m, \ \forall g \in G\} \simeq \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

- *The* module of coinvariants *is the abelian group*

$$M_G := M/\langle gm - m,\ m \in M,\ g \in G\rangle \simeq \mathbb{Z} \otimes_{\mathbb{Z}[G]} M.$$

Remark 2.2.2. *Observe that for any $\mathbb{Z}[G]$-module M, the dual group $M^\vee$ is naturally a $\mathbb{Z}[G]$-module and we have an natural isomorphism of abelian groups $(M^\vee)^G \simeq (M_G)^\vee$.*

Lemma 2.2.3. *Let G be a finite group, S be a G-set and denote by $p : S \to S/G$ the natural projection. Then the free $\mathbb{Z}$-module $\mathbb{Z}[S]$ generated by S is a permutation $\mathbb{Z}[G]$-module and the map*

$$\begin{array}{ccc} \mathbb{Z}[S]_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} \mathbb{Z}[S] & \xrightarrow{p_*} & \mathbb{Z}[S/G] \\ 1 \otimes s & \longmapsto & p(s) \end{array}$$

is an isomorphism.

Proof. Observe first that the map $p : S \to S/G$ induces a map $\mathbb{Z}[S] \to \mathbb{Z}[S/G]$ which factors through $p_* : \mathbb{Z}[S]_G \to \mathbb{Z}[S/G]$. On the other hand, if $t \in S/G$, there is some $s \in S$ such that $t = p(s)$ and we may define $q_*(t) := 1 \otimes s$. This is well-defined because if $p(s) = p(s')$, then $s' = gs$ for some $g \in G$ and $1 \otimes s' = 1 \otimes gs = (g \cdot 1) \otimes s = 1 \otimes s$. Then we have $p_* q_* = id_{\mathbb{Z}[S/G]}$ and $q_* p_* = id_{\mathbb{Z}[S]_G}$, proving that p_* is an isomorphism. ☐

Proposition 2.2.4. *Let G be a finite group, $H \le G$ be a subgroup and X be a G-CW-complex. If the orbit space X/H is Hausdorff, then it is naturally endowed with the structure of a $N_G(H)$-CW-complex and the projection map $\pi : X \to X/H$ induces isomorphisms*

$$\pi_* : C_*^{\mathrm{cell}}(X, G; \mathbb{Z})_H \xrightarrow{\sim} C_*^{\mathrm{cell}}(X/H, N_G(H); \mathbb{Z}) \quad \textit{in} \quad \mathrm{Ch}_{\ge 0}(\mathbb{Z}[N_G(H)])$$

and

$$\pi^* : C^*_{\mathrm{cell}}(X/H, N_G(H); \mathbb{Z}) \xrightarrow{\sim} C^*_{\mathrm{cell}}(X, G; \mathbb{Z})^H \quad \textit{in} \quad \mathrm{Coch}^{\ge 0}(\mathbb{Z}[N_G(H)^{op}]).$$

Proof. Since X is a G-CW-complex, by Proposition 2.1.3 (4), the orbit space X/H is a CW-complex with n-skeleton X_n/H. Moreover, we see that $N_G(H)$ (and $N_G(H)/H$) acts on the n-cells of X/H, so this space is indeed a $N_G(H)$-CW-complex and $\pi : X \to X/H$ is a $N_G(H)$-cellular map. Let $\mathcal{E}_n$ be the set of n-cells of X. Since the action of G on X is cellular, if $e \in \mathcal{E}_n$ is a cell of X, then $\pi(e) \simeq e$ and so $\pi(e)$ is a cell in X/H. Moreover, every n-cell of X/H may be obtained in this way and thus the set of n-cells of X/H is in natural bijection with $\mathcal{E}_n/H$ and the projection map $\mathcal{E}_n \to \mathcal{E}_n/H$ is given by π. By the Lemma 2.2.3, we obtain an isomorphism

$$\pi_* : C_n^{\mathrm{cell}}(X, G; \mathbb{Z})_H = \mathbb{Z}[\mathcal{E}_n]_H \xrightarrow{\sim} \mathbb{Z}[\mathcal{E}_n/H] = C_n^{\mathrm{cell}}(X/H, N_G(H); \mathbb{Z}).$$

Moreover, π^* is a morphism of $\mathbb{Z}[N_G(H)]$-modules because $\pi : X \to X/H$ is a morphism of $N_G(H)$-spaces and since $\pi_* : C_*^{\mathrm{cell}}(X, G; \mathbb{Z}) \to C_*^{\mathrm{cell}}(X/H, N_G(H); \mathbb{Z})$ is a morphism of complexes, it is indeed an isomorphism in $\mathrm{Ch}_{\ge 0}(\mathbb{Z}[N_G(H)])$. The second statement is dual. ☐

Remark 2.2.5. *This result still holds for every G-space X, replacing C_*^{cell} (resp. C^*_{cell}) by the singular chain (resp. cochain) complex of X. For this, see [Wei94, Lemma 6.10.2] or [Bro82, II, §2, Proposition 2.4].*

Recall that if G acts on a set X and if $N \trianglelefteq G$, then we may consider the *deflation* of X:

$$\mathrm{Def}_{G/N}^G(X) := X/N,$$

with the induced action of G/N.

If $\pi : G \twoheadrightarrow G/N$ is the projection map and if $X = G/H$ is a transitive G-set, then we have a canonical isomorphism of G/N-sets

$$\mathrm{Def}^G_{G/N}(G/H) \simeq \pi(G)/\pi(H).$$

This gives a functor $\mathrm{Def}^G_{G/N} : G\text{-}\mathbf{Set} \to G/N\text{-}\mathbf{Set}$ and linearizing it gives a commutative square of functors

$$\begin{tikzcd} G\text{-}\mathbf{Set} \arrow[r, "\mathrm{Def}^G_{G/N}"] \arrow[d] & G/N\text{-}\mathbf{Set} \arrow[d] \\ \mathbb{Z}[G]\text{-}\mathbf{Mod} \arrow[r, "\mathrm{Def}^G_{G/N}"'] & \mathbb{Z}[G/N]\text{-}\mathbf{Mod} \end{tikzcd}$$

yielding the usual linear deflation

$$\begin{array}{rcccc} \mathrm{Def}^G_{G/N} & : & \mathbb{Z}[G]\text{-}\mathbf{Mod} & \longrightarrow & \mathbb{Z}[G/N]\text{-}\mathbf{Mod} \\ & & U & \longmapsto & U_N := U/\langle nu - u \rangle \end{array}$$

and we may extend this functor to (co)chain complex categories.

Lemma 2.2.6. *Let G be a discrete group, written as a semi-direct product $G = N \rtimes H$ and X be an G-CW-complex. Denote by $p : X \twoheadrightarrow X/N$ and by $\pi : G \twoheadrightarrow H$ the natural projections. If the quotient space X/N is Hausdorff, then it is an H-CW-complex such that, for all $k \in \mathbb{N}$,*

$$\mathcal{E}_k(X/N) = \{p(e),\ e \in \mathcal{E}_k(X)\}$$

and the map π induces a natural isomorphism

$$C^{\mathrm{cell}}_*(X/N, H; \mathbb{Z}) \simeq \mathrm{Def}^G_H\left(C^{\mathrm{cell}}_*(X, G; \mathbb{Z})\right).$$

Proof. The map

$$\begin{array}{ccccc} \mathcal{E}_k(X)/N & \to & \mathcal{E}_k(X/N) & \twoheadrightarrow & \mathcal{E}_k(X/N)/H \\ N \cdot e & \mapsto & p(e) & & \end{array}$$

together with the decomposition $G = N \rtimes H$ gives a natural bijection

$$\mathcal{E}_k(X)/G \xrightarrow{\approx} \mathcal{E}_k(X/N)/H$$

and we have $H_{p(e)} \simeq \pi(G_e)$, yielding the desired isomorphism. □

Remark 2.2.7. *Similar arguments show that we may replace deflations in the previous statement by inflation, restriction or induction. More precisely, letting G be finite again, if $N \trianglelefteq G$ and if X is a G/N-CW-complex, then it is also a G-CW-complex by letting N act trivially on X and we have*

$$C^{\mathrm{cell}}_*(X, G; \mathbb{Z}) = \mathrm{Inf}^G_{G/N}\left(C^{\mathrm{cell}}_*(X, G/N; \mathbb{Z})\right),$$

where $\mathrm{Inf}^G_{G/N} : \mathbb{Z}[G/N]\text{-}\mathbf{Mod} \to \mathbb{Z}[G]\text{-}\mathbf{Mod}$ is the linear inflation functor. Similarly, if $K \leq G$ is any subgroup and if X is a G-CW-complex, then it is trivially a K-CW-complex for the restricted action and we have

$$C^{\mathrm{cell}}_*(X, K; \mathbb{Z}) = \mathrm{Res}^G_K\left(C^{\mathrm{cell}}_*(X, G; \mathbb{Z})\right).$$

Now, if X is a K-space, we may define $\mathrm{Ind}_K^G(X) := G \times_K X = (G \times X)/K$*, see [Bou10, §2.3]. If X is a K-CW-complex with cells e, then this is a G-CW-complex with cells $\{g\} \times_K e$ for $g \in G$ and we get*

$$C_*^{\mathrm{cell}}(\mathrm{Ind}_K^G(X), G; \mathbb{Z}) = \mathrm{Ind}_K^G \left(C_*^{\mathrm{cell}}(X, K; \mathbb{Z}) \right).$$

Remark 2.2.8. *Some precision is to be given to the link between the action of G and the ring structure on $H^*(X, R)$ (with R a commutative ring). For $g \in G$ and $x, y \in H^*(X, R)$, the naturality of the cup product ensures that $(x \cup y)g = (xg) \cup (yg)$, so $H^*(X, R)$ is not an $R[G]$-algebra.*

In the sequel, an R-algebra A which is also a (left) $R[G]$-module with the additional property that $g(a \cdot b) = (ga) \cdot (gb)$ for $g \in G$ and $a, b \in A$ will be referred to as a G-equivariant R-algebra.

In the same fashion, the R-dg-algebra $(C^*_{\mathrm{cell}}(X, G; R), \cup)$ is not an $R[G]$-dg-algebra. We shall call such a dg-algebra a G-equivariant R-dg-algebra *and, when $R = \mathbb{Z}$, we simply name it a G-dg-ring.*

Part II

Weyl-equivariant triangulations of tori of compact Lie groups and related W-dg-rings

The material of this part and the next one is taken from [Gar21]. Our first purpose is to define a W-equivariant simplicial structure on a maximal torus T in a semisimple compact Lie group K, which we may in fact assume to be simple and $W = N_K(T)/T$ is the Weyl group. As the situation is totally encoded in the *root datum* of the pair (K, T), we will adopt this vocabulary and work in the ambient space of the given root datum. In the simply-connected case, a simplicial structure is constructed using the W_{a}-triangulation of the (dual of the) ambient space induced by the *fundamental alcove*, where W_{a} is the *affine Weyl group*. We describe the associated (co)chain complexes and give a formula for the cup product in Corollary 3.3.3.

Next, we study the general case and we prove that the *barycentric subdivision* of the alcove gives a triangulation of the ambient space, which is equivariant for the *extended affine Weyl group*. We also give a description of the resulting complexes and study an example. As the computations may be heavy, we provide a `GAP4` package[15] that can be used to define the complexes.

3 The simply-connected case

3.1 Prerequisites: root data, affine Weyl group and alcoves

We start by briefly recalling some basic facts about *root data* and why this is the suitable framework for our study. Standard references for what follows are [MT11] and [KK05].

Definition 3.1.1 ([MT11, Definition 9.10]). *A* root datum *is a quadruple* $(X, \Phi, Y, \Phi^\vee)$ *where*

(RD1) the objects X and Y are free abelian groups of finite rank, together with a perfect pairing $\langle \cdot, \cdot \rangle : Y \times X \longrightarrow \mathbb{Z}$,

(RD2) the subsets $\Phi \subset X$ and $\Phi^\vee \subset Y$ are (abstract) reduced root systems in $\mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbb{Z}\Phi^\vee \otimes_{\mathbb{Z}} \mathbb{R}$, respectively,

(RD3) there is a bijection $\Phi \longrightarrow \Phi^\vee$ (denoted by $\alpha \longmapsto \alpha^\vee$) such that $\langle \alpha^\vee, \alpha \rangle = 2$ for every $\alpha \in \Phi$,

(RD4) the reflections s_α of the root system Φ and $s_{\alpha^\vee}$ of $\Phi^\vee$ are respectively given by

$$\forall x \in X,\ s_\alpha(x) := x - \left\langle \alpha^\vee, x \right\rangle \alpha$$

and

$$\forall y \in Y,\ s_{\alpha^\vee}(y) := y - \langle y, \alpha \rangle \alpha^\vee.$$

[15] `https://github.com/arthur-garnier/Salvetti-and-tori-complexes`

The Weyl group W of the root system Φ (which is isomorphic to the Weyl group of $\Phi^\vee$ via the map $s_\alpha \longmapsto s_{\alpha^\vee}$) is called the Weyl group of the root datum. *Moreover, we say that the root datum $(X, \Phi, Y, \Phi^\vee)$ is* irreducible *if the root system Φ is.*

For a root datum $(X, \Phi, Y, \Phi^\vee)$, we denote by $V := \mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$ the ambient space and $V^* := \mathbb{Z}\Phi^\vee \otimes_{\mathbb{Z}} \mathbb{R}$. As usual, we denote by $\Phi^+ \subset \Phi$ a set of positive roots and by $\Pi \subset \Phi^+$ the corresponding set of simple roots. Define the fundamental weights $\varpi_\alpha \in V$, indexed by $\alpha \in \Pi$, by $\langle \beta^\vee, \varpi_\alpha \rangle = \delta_{\alpha,\beta}$. Dually, the fundamental coweights are elements $\varpi_\alpha^\vee \in V^*$ such that $\langle \varpi_\alpha^\vee, \beta \rangle = \delta_{\alpha,\beta}$. We also consider respectively

$$Q := \mathbb{Z}\Phi = \bigoplus_{\alpha \in \Pi} \mathbb{Z}\alpha \subset V \quad \text{and} \quad Q^\vee := \mathbb{Z}\Phi^\vee = \bigoplus_{\alpha \in \Pi} \mathbb{Z}\alpha^\vee \subset V^*$$

the *root lattice* and the *coroot lattice* of Φ. Further, we have the respective *weight lattice* and *coweight lattice*:

$$P := (Q^\vee)^\wedge = \{x \in V \ ; \ \forall \alpha \in \Phi, \ \langle \alpha^\vee, x \rangle \in \mathbb{Z}\} = \bigoplus_{\alpha \in \Pi} \mathbb{Z}\varpi_\alpha \subset V \quad \text{and} \quad P^\vee := \bigoplus_{\alpha \in \Pi} \mathbb{Z}\varpi_\alpha^\vee \subset V^*.$$

Thus, the abelian group X is a W-lattice between Q and P and if we enumerate the simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, $(n = \dim(V))$ and if $C := (\langle \alpha_i^\vee, \alpha_j \rangle)_{1 \leq i,j \leq n}$ is the Cartan matrix of Φ, then we have

$$\det(C) = [P : Q] = [P^\vee : Q^\vee].$$

Then, the Chevalley classification theorem (see [MT11, §9.2]) says that, given a connected reductive (complex) group G and T a maximal torus of G, if Φ denotes the root system of (G, T), if $X(T) := \mathrm{Hom}(T, \mathbb{G}_m(\mathbb{C}))$ and $Y(T) := \mathrm{Hom}(\mathbb{G}_m(\mathbb{C}), T)$ are the respective character and cocharacter lattice of T, then $(X(T), \Phi, Y(T), \Phi^\vee)$ is a root datum that characterizes G (and T) up to isomorphism.

This is also true for Lie groups. Let K be a simple compact Lie group, T a maximal torus of K, we denote by $\mathfrak{k}$ and $\mathfrak{t}$ their respective Lie algebras. The complexification of $\mathfrak{k}$ is the reductive complex algebra $\mathfrak{g} := \mathfrak{k} + i\mathfrak{k}$ and $\mathfrak{h} := \mathfrak{t} + i\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\Phi \subset \mathfrak{h}^*$ be the root system of $(\mathfrak{g}, \mathfrak{h})$, with simple system Π. Since $\mathfrak{t} = \mathrm{span}_{\mathbb{R}}(i\alpha^\vee)_{\alpha \in \Pi}$, we have $\Phi \subset i\mathfrak{t}^* =: V$ (see [KK05, §3.2] or [Želobenko73, §103]) and we may take

$$X(T) = \{d\lambda : \mathfrak{t} \to i\mathbb{R} \ ; \ \lambda \in \mathrm{Hom}(T, \mathbb{S}^1)\} \subset i\mathfrak{t}^* = V$$

and $Y(T) := X(T)^\wedge \subset V^*$, so that $(X(T), \Phi, Y(T), \Phi^\vee)$ is a root datum. Since T is abelian, the Weyl group $W = W(\Phi) \simeq N_K(T)/T$ acts on T by conjugation by a representative in the normalizer.

By [KK05, Lemma 1], the normalized exponential map defines a W-isomorphism of Lie groups

$$V^*/Y(T) \xrightarrow{\sim} T. \tag{†}$$

Moreover, we have the following isomorphisms

$$P/X(T) \simeq \pi_1(K) \quad \text{and} \quad X(T)/Q \simeq Z(K).$$

This shows that we may reformulate the initial problem as follows: given an irreducible root datum $(X, \Phi, Y, \Phi^\vee)$ with Weyl group W and ambient space $V := \mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$, find a W-equivariant triangulation of the torus V^*/Y. As mentioned above, this will depend on the *fundamental group* P/X of the root datum.

Notation. *Throughout this chapter we fix, once and for all, an irreducible root datum* $(X, \Phi, Y, \Phi^\vee)$*, with ambient space* $V = \mathbb{Z}\Phi \otimes \mathbb{R}$*, simple roots* $\Pi \subset \Phi^+$*, Weyl group* $W = \langle s_\alpha,\ \alpha \in \Pi\rangle$*, fundamental (co)weights* $(\varpi_\alpha)_{\alpha\in\Pi}$ *and* $(\varpi_\alpha^\vee)_{\alpha\in\Pi}$*, (co)root lattices* Q *and* $Q^\vee$ *and (co)weight lattices* P *and* $P^\vee$*.*

Recall that Φ has a unique *highest root*, i.e. a positive root $\alpha_0 = \sum_{\alpha\in\Pi} n_\alpha \alpha \in \Phi^+$ such that $\alpha \leq \alpha_0$ for all $\alpha \in \Phi$ (see [Hum72, §10.4, Lemma A] or [Kan01, §11.2]). We consider the affine transformation

$$s_0 := \mathrm{t}_{\alpha_0^\vee} s_{\alpha_0} : \lambda \longmapsto s_{\alpha_0}(\lambda) + \alpha_0^\vee = \lambda - (\langle \lambda, \alpha_0\rangle - 1)\alpha_0^\vee.$$

Then, the group $W_{\mathrm{a}} := \langle s_0, s_1, \ldots, s_n\rangle \leq \mathrm{Aff}(V^*)$ is a Coxeter group, called the *affine Weyl group*. It splits as

$$W_{\mathrm{a}} = Q^\vee \rtimes W.$$

Moreover, for $\alpha \in \Phi$ and $k \in \mathbb{Z}$, we consider the affine hyperplanes $H_{\alpha,k} := \{\lambda \in V^*\ ;\ \langle \lambda, \alpha\rangle = k\}$ and we call *alcove* any connected component of $V^* \setminus \bigcup_{\alpha,k} H_{\alpha,k}$. The *fundamental alcove* is

$$\mathcal{A}_0 := \{\lambda \in V^*\ ;\ \forall \alpha \in \Phi^+,\ 0 < \langle \lambda, \alpha\rangle < 1\} = \{\lambda \in V^*\ ;\ \forall \alpha \in \Pi,\ \langle \lambda, \alpha\rangle > 0,\ \langle \lambda, \alpha_0\rangle < 1\}.$$

Then, by [Bou02, V, §2.2, Corollaire], its closure is a standard simplex

$$\overline{\mathcal{A}_0} = \mathrm{conv}\left(\{0\} \cup \left\{\frac{\varpi_\alpha^\vee}{n_\alpha}\right\}_{\alpha\in\Pi}\right) \simeq \Delta^n$$

and by [Hum92, §4.5 and 4.8], $\overline{\mathcal{A}_0}$ is a fundamental domain for W_{a} in V^* and moreover, W_{a} acts simply transitively on the set of open alcoves.

3.2 The W_{a}-triangulation of V^* associated to the fundamental alcove

The problem of finding a W-equivariant triangulation of $T = V^*/Y$ lifts to finding a $(Y \rtimes W)$-equivariant triangulation of V^*. In the 1-connected case, we have $Y = Q^\vee$ and $Q^\vee \rtimes W = W_{\mathrm{a}}$. As the alcove $\overline{\mathcal{A}_0}$ is a fundamental domain for W_{a} acting on V^*, it suffices to have a triangulation of $\overline{\mathcal{A}_0}$, which is compatible with the action of W_{a} in the sense that if a face is fixed globally by some $w \in W_{\mathrm{a}}$, then w induces the identity on this face. The fundamental result is the Theorem (or more precisely, its proof) from [Hum92, §4.8], which ensures that the natural polytopal structure on the r-simplex $\overline{\mathcal{A}_0}$ is W_{a}-equivariant.

Now we introduce some notation about faces of polytopes.

Given a polytope $\mathcal{P} \subset \mathbb{R}^n$ and an integer $k \geq -1$, we denote by $F_k(\mathcal{P})$ the set of k-dimensional faces of $\mathcal{P}$. In particular, we have $F_{-1}(\mathcal{P}) = \{\emptyset\}$, $F_{\dim(\mathcal{P})}(\mathcal{P}) = \{\mathcal{P}\}$, $F_{\dim(\mathcal{P})-1}(\mathcal{P})$ is the set of facets of $\mathcal{P}$ and $F_0(\mathcal{P}) = \mathrm{vert}(\mathcal{P})$ is its set of vertices. Moreover, we let $F(\mathcal{P}) := \bigcup_k F_k(\mathcal{P})$ be the *face lattice* of $\mathcal{P}$. It is indeed a lattice for the inclusion relation.

Notice that the above vocabulary also applies to a polytopal complexes, that is, a finite family of polytopes glued together along common faces.

Resuming to root data, for each $i \in S := \{1, \ldots, n\}$ we consider the hyperplane

$$H_i := H_{\alpha_i,0} = \{\lambda \in V^*\ ;\ \langle \lambda, \alpha_i\rangle = 0\}$$

and

$$H_0 := H_{\alpha_0,1} = \{\lambda \in V^* \ ; \ \langle \lambda, \alpha_0 \rangle = 1\}$$

with $\alpha_0 = \sum_i n_i \alpha_i$ the highest root. We also take the following notation for the vertices of $\overline{\mathcal{A}_0}$, where $i \in S$,

$$v_i := \frac{\varpi_i^\vee}{n_i} \quad \text{and} \quad v_0 := 0 \quad \text{so that} \quad \mathrm{vert}(\overline{\mathcal{A}_0}) = \{v_0, v_1, \ldots, v_n\}.$$

The hyperplanes H_i for $i \in S_0 := S \cup \{0\}$ give a complete set of bounding hyperplanes for the n-simplex $\overline{\mathcal{A}_0}$. Furthermore, by definition, for every face $f \in F_k(\overline{\mathcal{A}_0})$ there exists a subset $I \subseteq S_0$ of cardinality $|I| = \mathrm{codim}_{\overline{\mathcal{A}_0}}(f) = n - k$ such that

$$f = f_I := \overline{\mathcal{A}_0} \cap \bigcap_{i \in I} H_i$$

and we readily have

$$\mathrm{vert}(f_I) = \{v_i \ ; \ i \in S_0 \setminus I\}.$$

Recall also that for $i \in S_0$, denoting by s_i the reflection with respect to the hyperplane H_i, we have $s_i = s_{\alpha_i^\vee} \in W$ for $i \geq 1$ and $s_0 = \mathrm{t}_{\alpha_0^\vee} s_{\alpha_0^\vee}$ and the group W_{a} is Coxeter, with generating system $\{s_0, s_1, \ldots, s_n\}$. For $I \subseteq S_0$, we may consider the *(standard) parabolic subgroup* $(W_{\mathrm{a}})_I$ of W_{a} generated by the subset $\{s_i, \ i \in I\}$. If $0 \notin I$, then $(W_{\mathrm{a}})_I$ is in fact a parabolic subgroup of W.

Lemma 3.2.1. *Let $0 \leq k \leq n$ and $I \subseteq S_0$ with $|I| = n - k$. Then the stabilizer of the face $f_I \in F_k(\overline{\mathcal{A}_0})$ is the parabolic subgroup of W_{a} associated to I. In other words,*

$$(W_{\mathrm{a}})_{f_I} = (W_{\mathrm{a}})_I.$$

Proof. As $\mathrm{vert}(f_I) = \{v_i, \ i \notin I\}$ is $(W_{\mathrm{a}})_{f_I}$-stable, the Theorem from [Hum92, §4.8] ensures that

$$(W_{\mathrm{a}})_{f_I} = \bigcap_{i \in S_0 \setminus I} (W_{\mathrm{a}})_{v_i}.$$

Moreover, each group $(W_{\mathrm{a}})_{v_j}$ is generated by the reflections it contains, so that $v_j \in H_i$. A reflection s_i fixes 0 if and only if it is linear, so $(W_{\mathrm{a}})_{v_0} = \langle s_i \ ; \ i \neq 0 \rangle = (W_{\mathrm{a}})_S = W$. Let now $j \in S$. Since

$$\{v_j\} = \left\{ \frac{\varpi_j^\vee}{n_j} \right\} = \bigcap_{j \neq i \in S_0} H_i,$$

we have that $s_i(v_j) = v_j$ if and only if $i \in S_0 \setminus \{j\}$ and hence, for every $j \in S_0$, we have

$$(W_{\mathrm{a}})_{v_j} = (W_{\mathrm{a}})_{S_0 \setminus \{j\}}$$

and thus

$$(W_{\mathrm{a}})_{f_I} = \bigcap_{i \in S_0 \setminus I} (W_{\mathrm{a}})_{v_i} = \bigcap_{i \in S_0 \setminus I} (W_{\mathrm{a}})_{S_0 \setminus \{i\}} = (W_{\mathrm{a}})_{\bigcap_{i \notin I} S_0 \setminus \{i\}} = (W_{\mathrm{a}})_I.$$

$\square$

Hence, we have a triangulation

$$V^* = \coprod_{\substack{f \in F(\overline{\mathcal{A}_0}) \\ \widetilde{w} \in W_{\mathrm{a}}/(W_{\mathrm{a}})_f}} \widetilde{w} \cdot f$$

which is W_{a}-equivariant and following the notation from the beginning of this section, we have $\mathcal{E}_k(V^*)/W_{\mathrm{a}} = F_k(\overline{\mathcal{A}_0})$ for all k. Therefore, we get isomorphisms of $\mathbb{Z}[W_{\mathrm{a}}]$-modules

$$C_k^{\mathrm{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z}) \simeq \bigoplus_{f \in F_k(\overline{\mathcal{A}_0})} \mathbb{Z}[W_{\mathrm{a}}/(W_{\mathrm{a}})_f] = \bigoplus_{\substack{I \subset S_0 \\ |I| = n-k}} \mathbb{Z}[W_{\mathrm{a}}/(W_{\mathrm{a}})_I].$$

We have to fix an orientation of the cells in V^* and determine their boundary. But each one of them is a simplex, so its orientation is determined by an orientation on its vertices. We choose to orient them as the index set $(S_0, \leq)$. For $I \subseteq S_0$ with corresponding k-face $f_I = \mathrm{conv}(\{v_i \ ; \ i \in S_0 \setminus I\})$, we write

$$f_I = [v_{j_1}, \ldots, v_{j_{k+1}}] \ \text{ with } \ \{j_1 < j_2 < \ldots < j_{k+1}\} = S_0 \setminus I$$

to make its orientation explicit. The oriented boundary of f_I is then simply given by the formula

$$\partial_k(f_I) = \sum_{u=1}^{k+1} (-1)^u \underbrace{[v_{j_1}, \ldots, \widehat{v_{j_u}}, \ldots, v_{j_{k+1}}]}_{=\mathrm{conv}(\{v_j \ ; \ j_u \neq j \in S_0 \setminus I\})} = \sum_{u=1}^{k+1} (-1)^u f_{I \cup \{j_u\}}$$

We have thus obtained the following result:

Theorem 3.2.2. *The face lattice of the n-simplex $\overline{\mathcal{A}_0}$ induces a W_{a}-equivariant triangulation of V^*, whose cellular complex $C_*^{\mathrm{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z})$ is given (in homogeneous degrees k and $k-1$) by*

$$\cdots \longrightarrow \bigoplus_{\substack{I \subset S_0 \\ |I| = n-k}} \mathbb{Z}\left[W_{\mathrm{a}}^I\right] \xrightarrow{\partial_k} \bigoplus_{\substack{I \subset S_0 \\ |I| = n-k+1}} \mathbb{Z}\left[W_{\mathrm{a}}^I\right] \longrightarrow \cdots$$

where $W_{\mathrm{a}}^I \approx W_{\mathrm{a}}/(W_{\mathrm{a}})_I$ is the W_{a}-set of minimal length left coset representatives, modulo the parabolic subgroup $(W_{\mathrm{a}})_I$ and boundaries are defined as follows: for $k \in \mathbb{N}$ and $I \subset S_0$, letting $\{j_1 < \cdots < j_{k+1}\} := S_0 \setminus I$,

$$(\partial_k)_{|\mathbb{Z}[W_{\mathrm{a}}^I]} = \sum_{u=1}^{k+1} (-1)^u p^I_{I \cup \{j_u\}},$$

where, for $I \subset J$, p^I_J denotes the projection

$$p^I_J : W_{\mathrm{a}}^I = W_{\mathrm{a}}/(W_{\mathrm{a}})_I \longrightarrow\!\!\!\!\!\rightarrow W_{\mathrm{a}}/(W_{\mathrm{a}})_J = W_{\mathrm{a}}^J.$$

Example 3.2.3. *We look at the case of the group $SU(3)$ in type A_2. We denote by $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$ a root system of type A_2, with simple system $\Pi = \{\alpha, \beta\}$. The Figure 1 depicts the (dual) root system of type A_2 and its fundamental alcove. The chain complex $C_*^{\mathrm{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z})$ is readily given by*

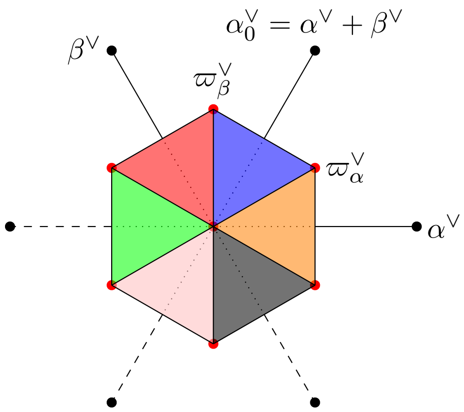
$$\mathbb{Z}[W_{\mathrm{a}}] \xrightarrow{\partial_2} \mathbb{Z}[W_{\mathrm{a}}/\langle s_\beta\rangle] \oplus \mathbb{Z}[W_{\mathrm{a}}/\langle s_0\rangle] \oplus \mathbb{Z}[W_{\mathrm{a}}/\langle s_\alpha\rangle] \xrightarrow{\partial_1} \mathbb{Z}[W_{\mathrm{a}}/\langle s_\alpha, s_\beta\rangle] \oplus \mathbb{Z}[W_{\mathrm{a}}/\langle s_\beta, s_0\rangle] \oplus \mathbb{Z}[W_{\mathrm{a}}/\langle s_\alpha, s_0\rangle] \ ,$$

where the boundaries are

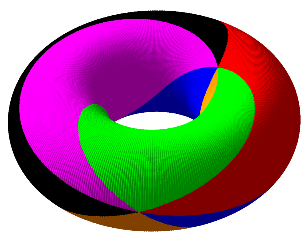
$$\partial_2 = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}, \ \partial_1 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Applying the deflation functor $\mathrm{Def}_W^{W_\mathrm{a}}$*, we obtain the complex* $C_*^{\mathrm{cell}}(T, W; \mathbb{Z})$ *where* $T = S(U(1)^3) \leq SU(3)$ *as*

$$\mathbb{Z}[W] \xrightarrow{\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}} \mathbb{Z}[W/\langle s_\beta\rangle] \oplus \mathbb{Z}[W/\langle s_\alpha s_\beta s_\alpha\rangle] \oplus \mathbb{Z}[W/\langle s_\alpha\rangle] \xrightarrow{\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}} \mathbb{Z}^3 .$$



(a) The fundamental alcove $\mathcal{A}_0$ (in blue) in type A_2, and its $\mathfrak{S}_3$-translates.



(b) The resulting $\mathfrak{S}_3$-equivariant triangulation of $S(U(1)^3) \simeq (\mathbb{S}^1)^2$.

Figure 1: Triangulation of the torus $S(U(1)^3)$ of $SU(3)$ from the fundamental alcove.

3.3 The W-dg-ring structure

We now make the cup product on $C_*^{\mathrm{cell}}(V^*, W_\mathrm{a}; \mathbb{Z})$ more explicit. For a cell $e \in C_*^{\mathrm{cell}}(V^*, W_\mathrm{a}; \mathbb{Z})$, we denote by $e^* \in C^*_{\mathrm{cell}}(V^*, W_\mathrm{a}; \mathbb{Z}) = C_*^{\mathrm{cell}}(V^*, W_\mathrm{a}; \mathbb{Z})^\vee$ its dual. Recall that the cup product of two (dual) simplices is given by the formula

$$[u_0, \ldots, u_k]^* \cup [v_0, \ldots, v_l]^* = \delta_{u_k, v_0}[u_0, \ldots, u_k, v_1, \ldots, v_l]^*.$$

We may express this product on $C_*^{\mathrm{cell}}(V^*, W_\mathrm{a}; \mathbb{Z})$ in terms of parabolic double cosets. We write

$$C^k_{\mathrm{cell}}(V^*, W_\mathrm{a}; \mathbb{Z}) = \prod_{\substack{I \subset S_0 \\ |I| = n-k}} \mathbb{Z}\left[W_\mathrm{a}/(W_\mathrm{a})_I\right]^\vee \simeq \bigoplus_{\substack{I \subset S_0 \\ |I| = n-k}} \mathbb{Z}\left[\left[{}^I W_\mathrm{a}\right]\right],$$

where

$${}^I W_\mathrm{a} \overset{\mathrm{df}}{=} \{w \in W_\mathrm{a} \ ; \ \ell(s_i w) > \ell(w), \ \forall i \in I\} \approx (W_\mathrm{a})_I \backslash W_\mathrm{a}$$

is the set of minimal length right coset representatives. Recall the following general result about double cosets:

Lemma 3.3.1 ([BKPST16, §3, Proposition 2 and Corollary 3]). *Let (W,S) be a Coxeter system and $I,J \subset S$. Denote as usual*

$$W^I := \{w \in W \ ; \ \ell(ws) > \ell(w), \ \forall s \in I\} \approx W/W_I,$$
$${}^IW := \{w \in W \ ; \ \ell(sw) > \ell(w), \ \forall s \in I\} \approx W_I\backslash W$$

and

$$J \subset I \ \Rightarrow \ W_I^J := \{w \in W_I \ ; \ \ell(ws) > \ell(w), \ \forall s \in J\} \approx W_I/W_J.$$

1. *Each double coset in $W_I\backslash W/W_J$ has a unique element of minimal length.*
2. *An element $w \in W$ is of minimal length in its double coset if and only if $w \in {}^IW \cap W^J$. In particular, we have a bijection*
$$W_I\backslash W/W_J \approx {}^IW \cap W^J.$$
3. *As a consequence, if $w \in {}^IW \cap W^J$ and $x \in W_I$, then $xw \in W^J$ if and only if $x \in W_I^{I\cap^w J}$. Hence, we have the following property:*
$$\forall x \in W_IwW_J, \ \exists!(u,v) \in W_I^{I\cap^w J} \times W_J \ ; \ \begin{cases} x = uwv, \\ \ell(x) = \ell(u) + \ell(w) + \ell(v). \end{cases}$$

We can now formulate the main result:

Theorem 3.3.2. *The $\mathbb{Z}[W_{\mathrm{a}}]$-cochain complex $C^*_{\text{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z})$ associated to the W_{a}-triangulation of V^* is a W_{a}-dg-ring with homogenous components*

$$\forall 0 \leq k \leq n, \ C^k_{\text{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z}) = \bigoplus_{\substack{I\subset S_0 \\ |I|=n-k}} \mathbb{Z}\left[\left[(W_{\mathrm{a}})_I\backslash W_{\mathrm{a}}\right]\right] \simeq \bigoplus_{\substack{I\subset S_0 \\ |I|=n-k}} \mathbb{Z}\left[\left[{}^IW_{\mathrm{a}}\right]\right]$$

and differentials defined, for any $I \subset S_0$ and $w \in W_{\mathrm{a}}$, by

$$d^k({}^Iw) = \sum_{\substack{0\leq u\leq k+1 \\ j_{u-1}<j<j_u}} (-1)^u \epsilon_I^{I\backslash\{j\}} w$$

where $\{j_0 < \cdots < j_k\} := S_0 \setminus I$ and, by convention, $j_{-1} = -1$, $j_{k+1} = n+1$ and for $J \subset I$,

$${}_I^JW_{\mathrm{a}} := \{w \in (W_{\mathrm{a}})_I \ ; \ \ell(s_jw) > \ell(w), \ \forall j \in J\} \ \text{ and } \ \epsilon_I^J := \sum_{x\in {}_I^JW_{\mathrm{a}}} x \in \mathbb{Z}\left[{}^JW_{\mathrm{a}}\right].$$

Moreover, the cup product

$$C^p_{\text{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z}) \otimes C^q_{\text{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z}) \xrightarrow{\cup} C^{p+q}_{\text{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z})$$

is induced by the unique map

$$\mathbb{Z}\left[\left[{}^IW_{\mathrm{a}}\right]\right] \otimes \mathbb{Z}\left[\left[{}^JW_{\mathrm{a}}\right]\right] \longrightarrow \mathbb{Z}\left[\left[{}^{I\cap J}W_{\mathrm{a}}\right]\right].$$

satisfying the formula

$${}^Ix \cup {}^Jy = \delta_{\max(I^\complement),\min(J^\complement)} \times \begin{cases} {}^{I\cap J}((xy^{-1})_Jy) & \text{if } xy^{-1} \in (W_{\mathrm{a}})_I(W_{\mathrm{a}})_J, \\ 0 & \text{otherwise,} \end{cases}$$

where $\complement$ denotes the complementary of a subset in S_0 and, given $w \in W_{\mathrm{a}}$, we denote by ${}^{I\cap J}w \in {}^{I\cap J}W_{\mathrm{a}}$ its minimal length right coset representative and if $(W_{\mathrm{a}})_Iw(W_{\mathrm{a}})_J = (W_{\mathrm{a}})_I(W_{\mathrm{a}})_J$, we let w_J be the unique element $v \in (W_{\mathrm{a}})_J$ such that $w = uv$, with $u \in (W_{\mathrm{a}})_I^{I\cap J}$ and $\ell(w) = \ell(u) + \ell(v)$.

Proof. Take a k-simplex $\sigma = [j_0, \dots, j_k] \subset \overline{\mathcal{A}_0}$ with $j_u \in S_0$ and set $j_{-1} := -1$ and $j_{k+1} := n+1$. By definition of the cochain differential d^k, we have

$$d^k(\sigma^*)_{|\overline{\mathcal{A}_0}} = \sum_{u=0}^{k+1} \sum_{j_{u-1}<j<j_u} (-1)^u [j_0, \dots, j_{u-1}, j, j_u, \dots, j_k]^*.$$

Letting $I := S_0 \setminus \{j_0, \dots, j_k\}$, we have $(W_\mathrm{a})_{\sigma^*} = (W_\mathrm{a})_I$ and the above formula reads

$$d^k((W_\mathrm{a})_I \cdot 1)_{|\overline{\mathcal{A}_0}} = \sum_{u=0}^{k+1} \sum_{j_{u-1}<j<j_u} (-1)^u ((W_\mathrm{a})_{I\setminus\{j\}} \cdot 1).$$

Therefore, as $\overline{\mathcal{A}_0}$ is a fundamental domain for W_a in V^*, this yields

$$d^k((W_\mathrm{a})_I \cdot 1) = \sum_{\substack{0 \le u \le n+1-|I| \\ j_{u-1}<j<j_u}} \left(\sum_{w \in {}_I^{I\setminus\{j\}}W_\mathrm{a}} (-1)^u ((W_\mathrm{a})_{I\setminus\{j\}} \cdot w) \right),$$

which leads to the stated formula.

To compute the cup product, using the bijection ${}^I W_\mathrm{a} \approx (W_\mathrm{a})_I \backslash W_\mathrm{a}$, the stated formula is

$$(W_\mathrm{a})_I x \cup (W_\mathrm{a})_J y = \delta_{\max I^\complement, \min J^\complement} \delta_{(W_\mathrm{a})_I x y^{-1} (W_\mathrm{a})_J, (W_\mathrm{a})_I (W_\mathrm{a})_J} (W_\mathrm{a})_{I\cap J}((xy^{-1})_J y).$$

Let $x, y \in W_\mathrm{a}$. As W_a acts simplicially on $\overline{\mathcal{A}_0}$, we have

$$(W_\mathrm{a})_I x \cup (W_\mathrm{a})_J y = ((W_\mathrm{a})_I x y^{-1} \cup (W_\mathrm{a})_J) y,$$

hence we may assume that $y = 1$ and we just have to compute $(W_\mathrm{a})_I w \cup (W_\mathrm{a})_J$.

First, we compute $\sigma^* \cup \tau^*$ for $\sigma, \tau \subset \overline{\mathcal{A}_0}$. As $\overline{\mathcal{A}_0} \simeq \Delta^r$ is a simplex, we may write $\sigma = [i_0, \dots, i_a]$ with $a = \dim \sigma$ and $I^\complement := \{i_0, \dots, i_a\} \subset \mathrm{vert}(\overline{\mathcal{A}_0}) \simeq S_0$. Write similarly $\tau = [j_0, \dots, j_b]$. We have $(W_\mathrm{a})_\sigma = (W_\mathrm{a})_{S_0 \setminus \{i_0, \dots, i_a\}} = (W_\mathrm{a})_I$, $(W_\mathrm{a})_\tau = (W_\mathrm{a})_J$ and

$$\sigma^* \cup \tau^* = \delta_{i_a, j_0} [i_0, \dots, i_a, j_1, \dots, j_b]^*$$

and the stabilizer in W_a of this last dual cell is $(W_\mathrm{a})_{S_0 \setminus \{i_0, \dots, i_a, j_1, \dots, j_b\}} = (W_\mathrm{a})_{I \cap J}$. Moreover, if $\sigma^* \cup \tau^* \neq 0$ then we must have $i_a = j_0$, that is, $\max(I^\complement) = \min(J^\complement)$. We make this assumption for the rest of this proof and we have indeed

$$\sigma^* \cup \tau^* = (W_\mathrm{a})_I \cup (W_\mathrm{a})_J = (W_\mathrm{a})_{I \cap J}.$$

<u>Claim:</u> For $\tau \subset \overline{\mathcal{A}_0}$ a simplex and $P \in F_k(V^*)$ a k-cell of V^*, if $\tau \subset P$ then $P \in (W_\mathrm{a})_\tau \cdot F_k(\overline{\mathcal{A}_0})$.

Indeed, we may assume that $\dim P = n = \dim \mathcal{A}_0$ so that there is some $z \in W_\mathrm{a}$ such that $P = z(\overline{\mathcal{A}_0})$ and so $\tau \subset \overline{\mathcal{A}_0} \cap z(\overline{\mathcal{A}_0})$, thus $z \in (W_\mathrm{a})_\tau$ (see [Hum92, §4.8]).

We are left to compute $\sigma^* w \cup \tau^*$ for $w \in W_\mathrm{a}$. If $\sigma^* w \cup \tau^* \neq 0$, then $w^{-1}\sigma$ and τ are included is some common simplex $P \in F_k(V^*)$ and by the claim we may choose $w_\tau \in (W_\mathrm{a})_\tau = (W_\mathrm{a})_J$ such that $w^{-1}\sigma \subset w_\tau(\overline{\mathcal{A}_0})$. But then $\sigma \subset \overline{\mathcal{A}_0} \cap w w_\tau(\overline{\mathcal{A}_0})$ and so $w_\tau \sigma = w^{-1}\sigma$. This yields

$$\sigma^* w \cup \tau^* = \sigma^* w_\tau^{-1} \cup \tau^* = \sigma^* w_\tau^{-1} \cup \tau^* w_\tau^{-1} = (\sigma^* \cup \tau^*) w_\tau^{-1} = (W_\mathrm{a})_{I\cap J} \cdot w_\tau^{-1}.$$

Furthermore, if $\sigma^* w \cup \tau^* \neq 0$ then we must have $ww_\tau \in (W_{\mathrm{a}})_I$, so $w \in (W_{\mathrm{a}})_I (W_{\mathrm{a}})_J$. In this case, the parabolic double coset decomposition from Lemma 3.3.1 applied to the trivial double coset $(W_{\mathrm{a}})_I w (W_{\mathrm{a}})_J$ allows one to write uniquely w as $w = uw_J$ with $u \in (W_{\mathrm{a}})_I^{I\cap J}$ and $w_J \in (W_{\mathrm{a}})_J$ such that $\ell(w) = \ell(u) + \ell(w_J)$. We obtain $w_J w_\tau \in (W_{\mathrm{a}})_J$ as well as $w_J w_\tau = u^{-1} w w_\tau \in (W_{\mathrm{a}})_I$. Hence $w_J w_\tau \in (W_{\mathrm{a}})_I \cap (W_{\mathrm{a}})_J = (W_{\mathrm{a}})_{I\cap J}$ and

$$\sigma^* w \cup \tau^* = (W_{\mathrm{a}})_{I\cap J} \cdot w_\tau^{-1} = (W_{\mathrm{a}})_{I\cap J} \cdot w_J.$$

The only thing remaining to be proved is that the formula

$${}^I x \cup {}^J y = \delta_{\max(I^\complement),\min(J^\complement)} \times \begin{cases} {}^{I\cap J}((xy^{-1})_J y) & \text{if } xy^{-1} \in (W_{\mathrm{a}})_I (W_{\mathrm{a}})_J, \\ 0 & \text{otherwise,} \end{cases}$$

indeed induces a well-defined map $\mathbb{Z}\left[\left[{}^I W_{\mathrm{a}}\right]\right] \otimes \mathbb{Z}\left[\left[{}^J W_{\mathrm{a}}\right]\right] \to \mathbb{Z}\left[\left[{}^{I\cap J} W_{\mathrm{a}}\right]\right]$. To see this, we show that for a given $z \in {}^{I\cap J} W_{\mathrm{a}}$, there are only finitely many pairs $({}^I x, {}^J y)$ for which $z = {}^I x \cup {}^J y$. Indeed, given $x, y \in W_{\mathrm{a}}$, if $x', y' \in W_{\mathrm{a}}$ are such that ${}^I x \cup {}^J y = {}^I x' \cup {}^J y'$, then $(xy^{-1})_J y$ and $(x'y'^{-1})_J y'$ are in the same class modulo $(W_{\mathrm{a}})_{I\cap J}$, hence in the same class modulo $(W_{\mathrm{a}})_J$ and therefore ${}^J y = {}^J y'$. Since $(W_{\mathrm{a}})_J$ is finite, there are only finitely many possibilities for y' and the same goes for $x' \in (W_{\mathrm{a}})_I (W_{\mathrm{a}})_J y'$. Therefore, if $a = \sum_{x \in {}^I W_{\mathrm{a}}} a_x x$ and $b = \sum_{y \in {}^J W_{\mathrm{a}}} b_y y$ with $a_x, b_y \in \mathbb{Z}$ (we use the formal series notation for simplicity), we can define

$$a \cup b := \sum_{z \in {}^{I\cap J} W_{\mathrm{a}}} \left(\sum_{\substack{(x,y) \in {}^I W_{\mathrm{a}} \times {}^J W_{\mathrm{a}} \\ x \cup y = z}} a_x b_y \right) z.$$

It is obvious that this is the only way of defining a bilinear map $\mathbb{Z}\left[\left[{}^I W_{\mathrm{a}}\right]\right] \times \mathbb{Z}\left[\left[{}^J W_{\mathrm{a}}\right]\right] \to \mathbb{Z}\left[\left[{}^{I\cap J} W_{\mathrm{a}}\right]\right]$ satisfying the stated formula. $\square$

Corollary 3.3.3. *The $\mathbb{Z}[W]$-cochain complex $C^*_{\mathrm{cell}}(T, W; \mathbb{Z})$ associated to the W-triangulation of $T = V^*/Q^\vee$ induced by the W_{a}-triangulation of V^* is given by*

$$C^*_{\mathrm{cell}}(T, W; \mathbb{Z}) = \mathrm{Def}_W^{W_{\mathrm{a}}} \left(C^*_{\mathrm{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z}) \right).$$

In other words, if $\pi : W_{\mathrm{a}} \twoheadrightarrow W$ is the projection, then

$$C^k_{\mathrm{cell}}(T, W; \mathbb{Z}) = \bigoplus_{\substack{I \subset S_0 \\ |I| = n-k}} \mathbb{Z}\left[\pi\left({}^I W_{\mathrm{a}}\right)\right] \simeq \bigoplus_{\substack{I \subset S_0 \\ |I| = n-k}} \mathbb{Z}\left[\pi((W_{\mathrm{a}})_I) \backslash W\right],$$

with differentials given, for any $I \subset S_0$ and $w \in W_{\mathrm{a}}$, by

$$d^k(\pi({}^I w)) = \sum_{\substack{0 \le u \le k+1 \\ j_{u-1} < j < j_u}} (-1)^u \pi\left(\epsilon_I^{I \setminus \{j\}} w\right), \; \epsilon_I^J = \sum_{x \in {}_I^J W_{\mathrm{a}}} x$$

where $\{j_0 < \cdots j_k\} := S_0 \setminus I$. Its product is given by the formula

$$\pi({}^I x) \cup \pi({}^J y) = \delta_{\max(I^\complement),\min(J^\complement)} \times \begin{cases} \pi\left({}^{I\cap J}((xy^{-1})_J y)\right) & \textit{if } xy^{-1} \in (W_{\mathrm{a}})_I (W_{\mathrm{a}})_J \\ 0 & \textit{otherwise.} \end{cases}$$

In particular, we have

$$H^\bullet(C^*_{\mathrm{cell}}(T, W; \mathbb{Z})) = H^\bullet(T, \mathbb{Z}) = \Lambda^\bullet(P).$$

4 The general case

4.1 The fundamental group as symmetries of an alcove

The *extended affine Weyl group* $\widehat{W}_{\mathrm{a}} := P^\vee \rtimes W$ acts on alcoves (transitively since $W_{\mathrm{a}} \trianglelefteq \widehat{W}_{\mathrm{a}}$ does) but not simply-transitively. We introduce the stabilizer

$$\Omega := \{\widehat{w} \in \widehat{W}_{\mathrm{a}} \ ; \ \widehat{w}(\mathcal{A}_0) = \mathcal{A}_0\}$$

and we see that we have a decomposition $\widehat{W}_{\mathrm{a}} \simeq W_{\mathrm{a}} \rtimes \Omega$ and in particular,

$$\Omega \simeq \widehat{W}_{\mathrm{a}}/W_{\mathrm{a}} \simeq P^\vee/Q^\vee \simeq P/Q.$$

Thus, Ω is a finite abelian group. The following table details the fundamental groups of the irreducible root systems:

Type	$\Omega \simeq P/Q$
A_n $(n \geq 1)$	$\mathbb{Z}/(n+1)\mathbb{Z}$
B_n $(n \geq 2)$	$\mathbb{Z}/2\mathbb{Z}$
C_n $(n \geq 3)$	$\mathbb{Z}/2\mathbb{Z}$
D_{2n} $(n \geq 2)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
D_{2n+1} $(n \geq 2)$	$\mathbb{Z}/4\mathbb{Z}$
E_6	$\mathbb{Z}/3\mathbb{Z}$
E_7	$\mathbb{Z}/2\mathbb{Z}$
E_8	1
F_4	1
G_2	1

Table 2: Fundamental groups of irreducible root systems

The description of Ω given in [Bou02, VI, §2.3] is useful. Given the highest root $\alpha_0 = \sum_{i=1}^n n_i\alpha_i$ of Φ, recall that a weight ϖ_i is called *minuscule* if $n_i = 1$ and that minuscule weights form a set of representatives of the non-trivial classes in P/Q (see [Bou02, Chapter VI, Exercise 24]). Dually, we have the same notion and result for *minuscule coweights*. Let

$$M := \{i \in S \ ; \ n_i = 1\}.$$

Proposition-Definition 4.1.1. ([Bou02, VI, §2.3, Proposition 6])
Let $\alpha_0 = \sum_{i \in S} n_i\alpha_i$ be the highest root of Φ and $w_0 \in W$ be the longest element. For $i \in S$, denote by $W_i \leq W$ the Weyl group of the subsystem of Φ generated by $\{\alpha_j \ ; \ j \neq i\} \subset \Pi$. For $i \in M$, let $w_0^i \in W_i$ be the longest element of W_i and $w_i := w_0^i w_0$.
Then the element $\mathrm{t}_{\varpi_i^\vee} w_i \in \widehat{W}_{\mathrm{a}}$ is in Ω and the map

$$\begin{array}{ccc} M & \longrightarrow & \Omega \setminus \{1\} \\ i & \longmapsto & \omega_i := \mathrm{t}_{\varpi_i^\vee} w_i \end{array}$$

is a bijection.

We now have to see what happens if the W-lattice Y is such that $Q^\vee \subsetneq Y \subsetneq P^\vee$. To simplify notation of this section, we identify a lattice $\Lambda \subset V^*$ with its translation group

$\mathrm{t}(\Lambda) \subset \mathrm{Aff}(V^*)$ and for such a lattice Λ, we define the *intermediate affine Weyl group* $W_\Lambda := \Lambda \rtimes W$. There is a correspondence between W-lattices $Q^\vee \subseteq \Lambda \subseteq P^\vee$ and the subgroups of Ω. In order to state this correspondence properly, we temporarily drop the letter Y and we work in the root system Φ only. Though straightforward, the following result is key:

Proposition 4.1.2. *Recall that* $\widehat{W_{\mathrm{a}}} \simeq W_{\mathrm{a}} \rtimes \Omega$ *and denote by*

$$\pi : \widehat{W_{\mathrm{a}}} \twoheadrightarrow \Omega$$

the natural projection. We have a bijective correspondence

$$\begin{array}{ccc} \{\Lambda \ ; \ Q^\vee \subseteq \Lambda \subseteq P^\vee \textit{ is a } W\textit{-lattice}\} & \xleftrightarrow{1-1} & \{H \leq \Omega\} \\ \Lambda & \longmapsto & \Omega_\Lambda := \pi(W_\Lambda) \\ \pi^{-1}(H) \cap P^\vee =: \Lambda(H) & \longleftarrow & H \end{array}$$

Moreover, for a W*-lattice* $Q^\vee \subset \Lambda \subset P^\vee$*, we have*

$$[\Omega : \Omega_\Lambda] = [P^\vee : \Lambda], \ \textit{equivalently}, \ |\Omega_\Lambda| = [\Lambda : Q^\vee].$$

Finally, we have a decomposition

$$W_\Lambda \simeq W_{\mathrm{a}} \rtimes \Omega_\Lambda.$$

4.2 A $\widehat{W_{\mathrm{a}}}$-triangulation of V^* from the barycentric subdivision of an alcove

In order to obtain a W_Y-triangulation of the torus V^*/Y, we just have to exhibit an Ω_Y-triangulation of the alcove $\overline{\mathcal{A}_0}$. As the group Ω_Y acts by affine automorphisms of $\overline{\mathcal{A}_0}$, the construction follows from the next easy result about simplicial subdivisions.

Recall that, given a polytope $\mathcal{P}$, its *barycentric subdivision* is the simplicial complex $\mathrm{Sd}(\mathcal{P})$ whose k-simplices are increasing chains of non-empty faces of $\mathcal{P}$ of length $k+1$. A k-simplex $(f_0, f_1, \ldots, f_k)$ of $\mathrm{Sd}(\mathcal{P})$ may be geometrically realized as $\mathrm{conv}(\mathrm{bar}(f_0), \ldots, \mathrm{bar}(f_k))$, where $\mathrm{bar}(f_i)$ stands for the barycenter of the face f.

Lemma 4.2.1. *If* $\mathcal{P}$ *is a polytope, then* $\mathrm{Sd}(\mathcal{P})$ *is an* $\mathrm{Aut}(\mathcal{P})$*-triangulation of* $\mathcal{P}$.

Proof. It is well-known that $\mathrm{Sd}(\mathcal{P})$ triangulates $\mathcal{P}$ and it is clear that $\Gamma := \mathrm{Aut}(\mathcal{P})$ permutes the simplices of $\mathrm{Sd}(\mathcal{P})$. We have to prove that, for a simplex $\sigma = (f_0, \ldots, f_k)$ of $\mathrm{Sd}(\mathcal{P})$ and $\gamma \in \Gamma$, if $\gamma\sigma = \sigma$, then $\gamma x = x$ for each $x \in |\sigma|$.

Take $0 \leq i \leq k$. The point $\mathrm{bar}(f_i)$ is taken by γ to some $\mathrm{bar}(f_j)$ and since the barycenter of a polytope lies in its relative interior, we have $\gamma(\mathring{f_i}) \cap \mathring{f_j} \neq \emptyset$ (where $\mathring{\cdot}$ is the relative interior) and as γ acts as an automorphism of $\mathcal{P}$, this forces $\gamma(f_i) = f_j$ and $\dim(f_i) = \dim(\gamma(f_i)) = \dim(f_j)$. But the sequence $(\dim f_0, \ldots, \dim f_k)$ is increasing, so $f_i = f_j$ and $\mathrm{bar}(f_i) = \mathrm{bar}(f_j) = \gamma(\mathrm{bar}(f_i))$. The conclusion now follows from the equality $|\sigma| = \mathrm{conv}(\mathrm{bar}(f_0), \ldots, \mathrm{bar}(f_k))$. ☐

From this we deduce that $W_{\mathrm{a}} \cdot \mathrm{Sd}(\overline{\mathcal{A}_0})$ is a W_Y-triangulation of V^* for all $Q^\vee \subset Y \subset P^\vee$ at once. We can describe the associated $\mathbb{Z}[W_Y]$-complex $C^*_{\mathrm{cell}}(V^*, W_Y; \mathbb{Z})$ using the face lattice of $\mathrm{Sd}(\overline{\mathcal{A}_0})$, but the description is rather tedious and not as nice as for the simply-connected case, as the combinatorics of parabolic subgroups doesn't make sense anymore.

There is a bijection $\mathrm{vert}(\overline{\mathcal{A}_0}) \approx S_0 = \{0, \dots, n\}$ and $\overline{\mathcal{A}_0} \simeq \Delta^n$, so that the face lattice of $\overline{\mathcal{A}_0}$ is $F(\overline{\mathcal{A}_0}) \simeq (\mathscr{P}(S_0), \subset)$. This gives a description of the face lattice of $\mathrm{Sd}(\overline{\mathcal{A}_0})$: for $0 \leq d \leq n$, we have

$$F_d(\mathrm{Sd}(\overline{\mathcal{A}_0})) = \{Z_\bullet = (Z_0, Z_1, \dots, Z_d) \ ; \ \forall i, \ \emptyset \neq Z_i \subset S_0, \ Z_i \subsetneq Z_{i+1}\}$$

and $Z_\bullet \subset Z'_\bullet$ if $Z_\bullet$ is a subsequence of $Z'_\bullet$.

Lemma 4.2.2. *The group Ω_Y acts on $\overline{\mathcal{A}_0}$ and this induces an action on S_0. The resulting action on $F(\mathrm{Sd}(\overline{\mathcal{A}_0}))$ corresponds to the action of Ω_Y on $|\mathrm{Sd}(\overline{\mathcal{A}_0})| = \overline{\mathcal{A}_0}$. Moreover, for $Z_\bullet \in F_d(\mathrm{Sd}(\overline{\mathcal{A}_0}))$, the stabilizer of $Z_\bullet$ in W_Y decomposes as*

$$(W_Y)_{Z_\bullet} = (W_{\mathrm{a}})_{Z_\bullet} \rtimes (\Omega_Y)_{Z_\bullet} = (W_{\mathrm{a}})_{S_0 \setminus Z_d} \rtimes (\Omega_Y)_{Z_\bullet} \quad \textit{and} \quad (\Omega_Y)_{Z_\bullet} = \bigcap_{i=0}^{d} \Omega_{Z_i}.$$

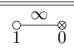
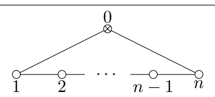
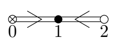
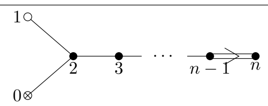
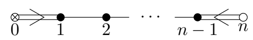
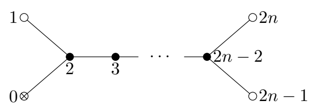
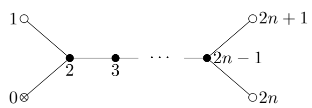
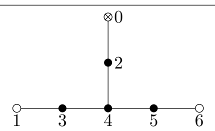
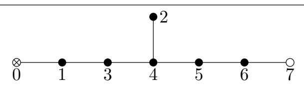
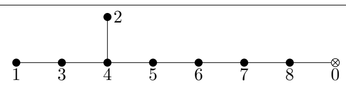
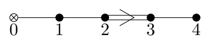
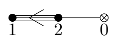
Type	Extended Dynkin diagram	Fundamental group $\Omega \leq \mathrm{Aut}(\mathcal{D}ynkin_0)$
$\widetilde{A_1}$		$\omega_1 = (0,1)$
$\widetilde{A_n}$ $(n \geq 2)$		$\omega_1 = (0,1,2,\cdots,n)$ $\omega_i = (\omega_1)^i, \ 0 \leq i \leq n$
$\widetilde{B_2} = \widetilde{C_2}$		$\omega_1 = (0,2)$
$\widetilde{B_n}$ $(n \geq 3)$		$\omega_1 = (0,1)$
$\widetilde{C_n}$ $(n \geq 3)$		$\omega_n = (0,n) \prod_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} (i, n-i)$
$\widetilde{D_{2n}}$ $(n \geq 2)$		$\omega_1 = (0,1)(2n-1,2n)$ $\omega_{2n-1} = (0,2n-1)(1,2n) \prod_{i=2}^{n-1}(i,2n-i)$ $\omega_{2n} = (0,2n)(1,2n-1) \prod_{i=2}^{n-1}(i,2n-i) = \omega_1\omega_{2n-1}$
$\widetilde{D_{2n+1}}$ $(n \geq 2)$		$\omega_1 = (0,1)(2n,2n+1)$ $\omega_{2n} = (0,2n,1,2n+1) \prod_{i=2}^{n}(i,2n+1-i)$ $\omega_{2n+1} = (0,2n+1,1,2n) \prod_{i=2}^{n}(i,2n+1-i)$
$\widetilde{E_6}$		$\omega_1 = (0,1,6)(2,3,5)$ $\omega_6 = (1,0,6)(3,2,5) = \omega_1^{-1}$
$\widetilde{E_7}$		$\omega_7 = (0,7)(1,6)(3,5)$
$\widetilde{E_8}$		$\varnothing$
$\widetilde{F_4}$		$\varnothing$
$\widetilde{G_2}$		$\varnothing$

Table 3: Extended Dynkin diagrams and fundamental groups elements, represented as permutations of the nodes.

Proof. The first statement is obvious. Write $Z_\bullet = (Z_0 \subsetneq \cdots \subsetneq Z_d)$ and let $\widehat{w} := w\omega_j \in (W_Y)_{Z_\bullet}$ with $w \in W_{\mathrm{a}}$ and $\omega_j \in \Omega_Y$. Then, for every $x \in |Z_\bullet|$, we have $\widehat{w}(x) = w(\omega_j(x)) = x$ and $\omega_j(x) \in \overline{\mathcal{A}_0}$ so $x = \omega_j(x)$ and $\omega_j \in (\Omega_Y)_{Z_\bullet}$. On another hand we get $w(x) = x$ so $w \in (W_{\mathrm{a}})_{Z_\bullet}$.

Now, an element $w \in W_{\mathrm{a}}$ fixes $Z_\bullet$ if and only if it fixes the maximal face of $Z_\bullet$, i.e. Z_d. This is indeed the parabolic subgroup $(W_{\mathrm{a}})_{S_0 \setminus Z_d}$. The last equality holds in general and is straightforward. □

To avoid too many choices, we fix a total ordering $\prec$ on $F(\mathrm{Sd}(\overline{\mathcal{A}_0}))$. For instance, the lexicographical order $<_{lex}$ induced by the order on $\mathscr{P}(S_0) = 2^{S_0}$ inherited from the natural order on S_0.

As the barycentric subdivision of $\overline{\mathcal{A}_0}$ is simplicial, the boundaries of the complex and the cup product are easily determined and lead to the following result:

Theorem 4.2.3. *For $0 \le d \le n$, decompose the Ω_Y-set $F_d(\mathrm{Sd}(\overline{\mathcal{A}_0}))$ into orbits*

$$F_d(\mathrm{Sd}(\overline{\mathcal{A}_0}))/\Omega_Y \approx \{Z_{d,1} \prec \cdots \prec Z_{d,k_d}\}, \quad \textit{where} \quad Z_{d,i} = \min_{\prec}(\Omega_Y \cdot Z_{d,i}).$$

Denote further, for $0 \le p \le d$ and $1 \le i \le k_d$,

$$Z_{d,i}^{(p)} := ((Z_{d,i})_0, \ldots, \widehat{(Z_{d,i})_p}, \ldots, (Z_{d,i})_d).$$

Then the complex $C_*^{\mathrm{cell}}(V^*, W_Y; \mathbb{Z})$ is given by

$$C_d^{\mathrm{cell}}(V^*, W_Y; \mathbb{Z}) = \bigoplus_{i=1}^{k_d} \mathbb{Z}\left[W_Y/(W_Y)_{Z_{d,i}}\right],$$

with

$$(W_Y)_{Z_{d,i}} = (W_{\mathrm{a}})_{(Z_{d,i})_d^{\complement}} \rtimes \bigcap_{j=0}^{d} (\Omega_Y)_{(Z_{d,i})_j}.$$

The boundaries are given by

$$\partial_d(Z_{d,i}) = \sum_{p=0}^{d} (-1)^p \omega_{p,i}(Z_{d-1,u_i}), \ \textit{where} \ u_i \in S_0 \ ; \ Z_{d-1,u_i} = \min_{\prec}(\Omega_Y \cdot Z_{d,i}^{(p)}) \ \textit{and} \ \omega_{p,i}(Z_{d-1,u_i}) = Z_{d,i}^{(p)}.$$

Moreover, the dual complex $C^*_{\mathrm{cell}}(V^*, \widehat{W}_{\mathrm{a}}; \mathbb{Z})$ is a W_Y-dg-ring with product

$$Z_{d,i}^* \cup Z_{e,j}^* = \delta_{(Z_{d,i})_d, (Z_{e,j})_0} \omega(Z_{d+e,k})^*,$$

where

$$Z_{d+e,k} = \min_{\prec}(\Omega_Y \cdot ((Z_{d,i})_0, \ldots, (Z_{d,i})_d, (Z_{e,j})_0, \ldots, (Z_{e,j})_e)) \ \textit{and} \ \omega(Z_{d+e,k}) = ((Z_{d,i})_0, \ldots, (Z_{e,j})_e).$$

Finally, the complex for the torus V^*/Y is given by

$$C_*^{\mathrm{cell}}(V^*/Y, W; \mathbb{Z}) = \mathrm{Def}_W^{W_Y}(C_*^{\mathrm{cell}}(V^*, W_Y; \mathbb{Z})).$$

Example 4.2.4. *Continuing the Example 3.2.3, we treat the extended type A_2, which is fairly computable by hand. We have $S_0 = \{0,1,2\} = J$ and*

$$\Omega = \Omega_{P^\vee} = \{1, \underbrace{t_{\varpi_\alpha^\vee} s_\alpha s_\beta}_{\omega_\alpha}, \underbrace{t_{\varpi_\beta^\vee} s_\beta s_\alpha}_{\omega_\beta}\} \simeq \mathbb{Z}/3\mathbb{Z}.$$

In this case, $W_{P^\vee} = \widehat{W}_{\mathrm{a}}$ is the classical extended affine Weyl group. Geometrically, the element ω_α acts as the rotation with angle $2\pi/3$ around the barycenter of $\overline{\mathcal{A}_0} = \mathrm{conv}(0, \varpi_\alpha^\vee, \varpi_\beta^\vee) =: [0,1,2] \simeq \Delta^2$. The situation can be visualized in Figure 2.

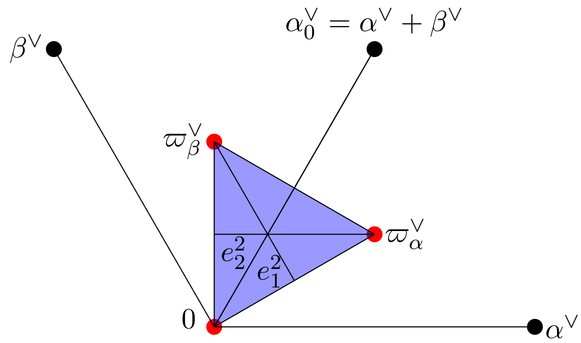


Figure 2: Barycentric subdivision $|\mathrm{Sd}(\overline{\mathcal{A}_0})|$ of the fundamental alcove $\overline{\mathcal{A}_0}$.

There are three $\widehat{W}_{\mathrm{a}}$-orbits of points in $|\mathrm{Sd}(\overline{\mathcal{A}_0})|$ and we represent them by the points

$$e_1^0 := (\{0\}) = 0,\ e_2^0 := (\{0,1\}) = \frac{\varpi_\alpha^\vee}{2},\ e_3^0 := (\{0,1,2\}) = \frac{\varpi_\alpha^\vee + \varpi_\beta^\vee}{3}.$$

Remember that we order $\mathscr{P}(S_0)$ lexicographically and these are lex-minimal in their orbits. There are also four orbits of 1-cells represented by

$$e_1^1 := (\{0\},\{0,1\}),\ e_2^1 := (\{0\},\{0,2\}),\ e_3^1 := (\{0\},\{0,1,2\}),\ e_4^1 := (\{0,1\},\{0,1,2\}).$$

Finally, there are two orbits of 2-cells represented by

$$e_1^2 := (\{0\},\{0,1\},\{0,1,2\}),\ e_2^2 := (\{0\},\{0,2\},\{0,1,2\}).$$

Now, we have

$$\forall e \in \{e_1^0, e_2^0, e_i^1, e_j^2\},\ \Omega_e = 1 \quad and \quad \Omega_{e_3^0} = \Omega$$

and we obtain the non-trivial stabilizers in $\widehat{W}_{\mathrm{a}}$:

$$(\widehat{W}_{\mathrm{a}})_{e_3^0} = \Omega,\ (\widehat{W}_{\mathrm{a}})_{e_1^0} = W,\ (\widehat{W}_{\mathrm{a}})_{e_2^0} = (\widehat{W}_{\mathrm{a}})_{e_1^1} = \langle s_\beta \rangle, (\widehat{W}_{\mathrm{a}})_{e_2^1} = \langle s_\alpha \rangle.$$

The boundaries are readily computed, with for instance

$$\partial_2(e_2^2) = -e_2^1 + e_3^1 - (\{0,2\},\{0,1,2\}) = -e_2^1 + e_3^1 - \omega_\beta e_4^1.$$

Therefore, the complex $C_*^{\mathrm{cell}}(V^*, \widehat{W}_{\mathrm{a}}; \mathbb{Z})$ is given by

$$\mathbb{Z}[\widehat{W}_{\mathrm{a}}]^2 \xrightarrow{\partial_2} \mathbb{Z}[\widehat{W}_{\mathrm{a}}/\langle s_\beta \rangle] \oplus \mathbb{Z}[\widehat{W}_{\mathrm{a}}/\langle s_\alpha \rangle] \oplus \mathbb{Z}[\widehat{W}_{\mathrm{a}}]^2 \xrightarrow{\partial_1} \mathbb{Z}[\widehat{W}_{\mathrm{a}}/W] \oplus \mathbb{Z}[\widehat{W}_{\mathrm{a}}/\langle s_\beta \rangle] \oplus \mathbb{Z}[\widehat{W}_{\mathrm{a}}/\Omega],$$

with

$$\partial_2 = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -\omega_\beta \end{pmatrix}, \quad \partial_1 = \begin{pmatrix} -1 & 1 & 0 \\ -1 & \omega_\beta & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Moreover, the root datum $(P, \Phi, P^\vee, \Phi^\vee)$ may be realized by the Lie group $PSU(3) = SU(3)/\mu_3$ with torus $T = T_0/\mu_3 \simeq V^*/P^\vee$, where $T_0 = S(U(1)^3)$ is the standard torus consisting of diagonal matrices of $SU(3)$. The complex

$$C_*^{\mathrm{cell}}(T, W; \mathbb{Z}) = \mathrm{Def}_W^{\widehat{W}_{\mathrm{a}}}(C_*^{\mathrm{cell}}(V^*, \widehat{W}_{\mathrm{a}}; \mathbb{Z}))$$

then becomes

$$\mathbb{Z}[W]^2 \xrightarrow{\overline{\partial_2}} \mathbb{Z}[W/\langle s_\beta\rangle] \oplus \mathbb{Z}[W/\langle s_\alpha\rangle] \oplus \mathbb{Z}[W]^2 \xrightarrow{\overline{\partial_1}} \mathbb{Z} \oplus \mathbb{Z}[W/\langle s_\beta\rangle] \oplus \mathbb{Z}[W/\langle s_\alpha s_\beta\rangle],$$

with

$$\overline{\partial_2} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -s_\beta s_\alpha \end{pmatrix}, \quad \overline{\partial_1} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & s_\beta s_\alpha & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

The complexes $C_*^{\mathrm{cell}}(T_0, W; \mathbb{Z})$ and $C_*^{\mathrm{cell}}(T, W; \mathbb{Z})$ may be obtained using the commands `ComplexForFiniteCoxeterGroup("A",2)` *and* `CellularComplexT("A",2,[0,1,2])` *provided by the package* `Salvetti-and-tori-complexes`[16].

Remark 4.2.5. *The complex $C_*^{\mathrm{cell}}(V^*, \widehat{W}_{\mathrm{a}}; \mathbb{Z})$ in the previous example can be reduced. Indeed, we can take $e^2 := e_1^2 \cup_{e_3^1} e_2^2$ as 2-cell. This deletes the 1-cell e_3^1 and the complex reduces to*

$$\mathbb{Z}[\widehat{W}_{\mathrm{a}}] \xrightarrow{{}^t\begin{pmatrix} 1 \\ -1 \\ 1-\omega_\beta \end{pmatrix}} \mathbb{Z}[\widehat{W}_{\mathrm{a}}/\langle s_\beta\rangle] \oplus \mathbb{Z}[\widehat{W}_{\mathrm{a}}/\langle s_\alpha\rangle] \oplus \mathbb{Z}[\widehat{W}_{\mathrm{a}}] \xrightarrow{\begin{pmatrix} -1 & 1 & 0 \\ -1 & \omega_\beta & 0 \\ 0 & -1 & 1 \end{pmatrix}} \mathbb{Z}[\widehat{W}_{\mathrm{a}}/W] \oplus \mathbb{Z}[\widehat{W}_{\mathrm{a}}/\langle s_\beta\rangle] \oplus \mathbb{Z}[\widehat{W}_{\mathrm{a}}/\Omega].$$

We recognize the closure $\overline{e^2} = \mathrm{conv}(e_1^0, e_2^0, \omega_\beta e_2^0, e_3^0)$ as the fundamental polytope $F_{P^\vee}$ for $\widehat{W}_{\mathrm{a}}$ acting on V^*.

More generally, we have proved that a fundamental polytope for the action of the extended affine Weyl group $\widehat{W}_{\mathrm{a}}$ is given by

$$\begin{aligned} F_{P^\vee} :&= \{\lambda \in \overline{\mathcal{A}_0} \; ; \; \langle \lambda, \alpha + \alpha_0\rangle \leq 1, \; \forall \alpha \in \Pi \; ; \; n_\alpha = 1\} \\ &= \{\lambda \in V^* \; ; \; \langle \lambda, \alpha_0\rangle \leq 1 \; \forall \alpha \in \Pi, \; \langle \lambda, \alpha\rangle \geq 0 \text{ and } n_\alpha = 1 \implies \langle \lambda, \alpha + \alpha_0\rangle \leq 1\}. \end{aligned}$$

where we have written the highest root α_0 as $\alpha_0 = \sum_{\alpha\in\Pi} n_\alpha \alpha$. Moreover, if we let $\Pi_0 := \{\alpha \in \Pi \; ; \; n_\alpha = 1\}$, then the vertices of $F_{P^\vee}$ are the isobarycenters of points in $\{0\} \cup \{\varpi_\alpha^\vee\}_{\alpha\in\Pi_0}$ with a non-zero coefficient with respect to the origin, together with the vertices of $\overline{\mathcal{A}_0}$ corresponding to the other simple roots. In other words, if we denote

$$\begin{aligned} \mathcal{B}_m :&= \left\{ \frac{1}{|A|+1} \sum_{\alpha\in A} \varpi_\alpha^\vee \; ; \; A \subset \Pi_0 \right\} \\ &= \{0\} \cup \left\{ \frac{\varpi_{\alpha_1}^\vee + \cdots + \varpi_{\alpha_k}^\vee}{k+1} \; ; \; 1 \leq k \leq |\Pi_0| \text{ and } \alpha_i \in \Pi_0, \; \forall 1 \leq i \leq k \right\}, \end{aligned}$$

[16] https://github.com/arthur-garnier/Salvetti-and-tori-complexes

then we have

$$\operatorname{vert}(F_{P^\vee}) = \mathcal{B}_m \cup \left\{ \frac{\varpi_\alpha^\vee}{n_\alpha} \right\}_{\alpha \in \Pi \setminus \Pi_0}.$$

After proving this, we have realized that the description of $F_{P^\vee}$ as an intersection of hyperplanes was first discovered by Komrakov and Premet [KP84]. It would be nice to obtain a cell decomposition from this polytope. However, this approach fails in general. Take the example of type C_3, whose positive coroots are depicted in Figure 3; denote by $\Pi = \{\alpha, \beta, \gamma\}$ a simple system and by $\varpi_\alpha^\vee = \alpha^\vee + \beta^\vee + \gamma^\vee$, $\varpi_\beta^\vee = \alpha^\vee + 2\beta^\vee + 2\gamma^\vee$, $\varpi_\gamma^\vee = \frac{1}{2}\alpha^\vee + \beta^\vee + \frac{3}{2}\gamma^\vee$ the corresponding coweights. The highest root is $\alpha_0 = 2\alpha + 2\beta + \gamma$ and from the Table 3, the fundamental group is

$$\Omega = \{1, \underbrace{\mathrm{t}_{\varpi_\gamma^\vee}(s_\gamma s_\alpha)^{s_\beta s_\gamma}}_{\omega_\gamma}\} \simeq \mathbb{Z}/2\mathbb{Z}.$$

Moreover, the fundamental alcove is $\overline{\mathcal{A}_0} = \operatorname{conv}(0, \varpi_\alpha^\vee/2, \varpi_\beta^\vee/2, \varpi_\gamma^\vee)$ and from this we see that the non trivial element $\omega_\gamma \in \Omega$ acts on the vertices of $\overline{\mathcal{A}_0}$ by exchanging $\varpi_\alpha^\vee/2$ and $\varpi_\beta^\vee/2$, as well as 0 and $\varpi_\gamma^\vee$. Therefore, the affine facet of $F_{P^\vee} = \operatorname{conv}(0, \varpi_\alpha^\vee/2, \varpi_\beta^\vee/2, \varpi_\gamma^\vee/2)$ given by

$$F_{P^\vee} \cap \{\lambda \in V^* \; ; \; 1 = \langle \lambda, \alpha_0 + \gamma \rangle = 2\,\langle \lambda, \alpha + \beta + \gamma \rangle\} = \frac{1}{2}\operatorname{conv}(\varpi_\alpha^\vee, \varpi_\beta^\vee, \varpi_\gamma^\vee)$$

is taken to itself by ω_γ but is not fixed pointwise since two of its three vertices are exchanged. Therefore, the triangulation of V^* induced by translating the simplex $F_{P^\vee}$ is not *$\widehat{W}_{\mathrm{a}}$-equivariant.*

From Table 3, we see that the same issue occurs for $C_{n\geq 3}$, $D_{n\geq 4}$, E_6 and E_7. We have tried to identify the faces that are non-pointwise fixed by some element of their stabilizers. However, after many computations, there are examples where no facet present a problem, but some higher codimensional faces do. To conclude, even though the barycentric subdivision yields many simplices, at least it works for any lattice and is rather simple to implement.

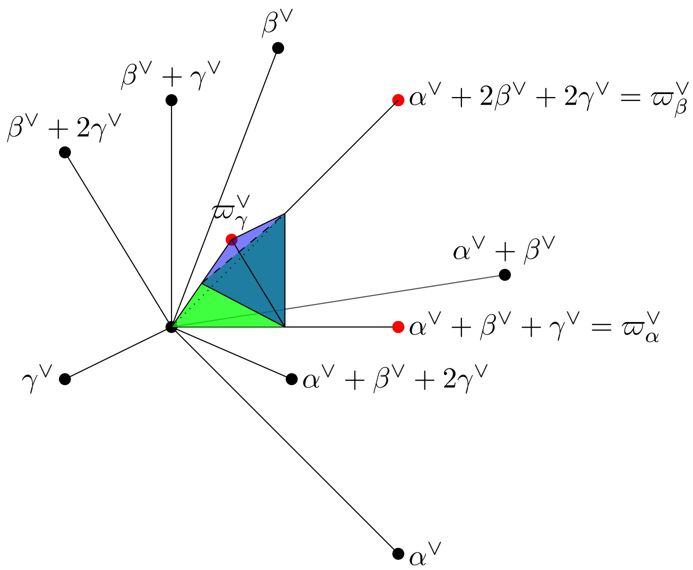


Figure 3: The Komrakov-Premet polytope $F_{P^\vee}$ (in green) inside the fundamental alcove $\overline{\mathcal{A}_0}$ (in blue) in type C_3.

Part III

Hyperbolic tori for non-crystallographic Coxeter groups

The goal of this part is to construct a smooth manifold affording a dg-algebra with a similar combinatorics as the one in Theorem 3.3.2 and playing the role of a torus for non-crystallographic Coxeter groups. Observe that if W is the Weyl group of a pair (K, T) as usual, then T is W-diffeomorphic to the orbit space of the *Coxeter complex* of the affine Weyl group W_{a}, under the action of the complement of W in W_{a}. Hence, for a non-crystallographic group W, we have to define an extension of W which plays the role of the affine Weyl group. We shall see that in this context, the suitable extensions to consider are *compact hyperbolic* Coxeter groups and we construct these extensions from a suitable reflection in W.

Then, we prove that the action of the complement of W in its extension on its Coxeter complex is properly discontinuous (this is where the fact that the extension is compact hyperbolic is crucial) and we define the manifold $\mathbf{T}(W)$ to be the resulting orbit space. This is a hyperbolic W-triangulated compact manifold. Next, we study some properties of this manifold. In particular, we prove that the manifolds $\mathbf{T}(I_2(m))$ are arithmetic Riemann surfaces. We finish by computing the homology representation of $\mathbf{T}(W)$.

5 Construction of the hyperbolic extensions and the hyperbolic torus

The non-crystallographic finite irreducible Coxeter groups are listed in the following table:

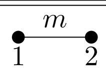
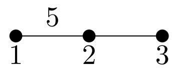
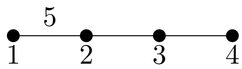
Type	Coxeter diagram
$I_2(m)$ $(5 \leq m \neq 6)$	
H_3	
H_4	

Table 4: Coxeter diagrams of finite non-crystallographic Coxeter systems.

Although we shall focus on the non-crystallographic case, what follows applies to all finite irreducible Coxeter groups. In particular, in the $I_2(m)$ case, we only assume that $m \geq 3$.

A key ingredient in the construction of the W-equivariant triangulation of the torus of a Weyl group (in fact of the simply-connected compact Lie group of type W) is the reflection s_0 associated to the highest root α_0 in the root system Φ of W. Thus, we first have to find a suitable reflection in the non-crystallographic cases. Moreover, this should come with an infinite extension of the finite group, which should again be a Coxeter group. By "suitable", we mean that each proper parabolic subgroup of the resulting infinite Coxeter group should be finite. The good class of Coxeter groups we shall consider for this matter is no longer the affine groups, but the compact hyperbolic groups.

5.1 Compact hyperbolic extensions of $I_2(m)$, H_3 and H_4

Let us first recall some basic terminology concerning Coxeter groups. For more detailed discussions, the reader is referred to [Bou02] and [Hum92].

Let (W, S) be an irreducible Coxeter system of rank n. We write

$$W = \langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{i,j}} = 1 \rangle ,$$

with $M = (m_{i,j})_{1 \leq i,j \leq n}$ the Coxeter matrix of (W, S). Recall ([Bou02, V, §4] or [Hum92, Chap. 5]) that on the formal vector space $V := \mathrm{span}_{\mathbb{R}}(\alpha_i,\ 1 \leq i \leq n)$ we may define a symmetric bilinear form by

$$B(\alpha_i, \alpha_j) := -\cos\left(\frac{\pi}{m_{i,j}}\right)$$

as well as the linear mappings

$$\forall 1 \leq i \leq n,\ \sigma_i := (v \mapsto v - 2B(\alpha_i, v)\alpha_i).$$

Then the assignment $s_i \mapsto \sigma_i$ extends uniquely to a faithful irreducible representation

$$\sigma : W \longrightarrow GL(V),$$

known as the *geometric representation* of W.

Moreover, W is finite (resp. affine) if and only if the form B is positive definite (resp. positive semidefinite) (see [Bou02, V, §4.8 and 4.9]).

Proposition-Definition 5.1.1 ([Hum92, §6.8])**.** *The followings are equivalent:*

(i) The form B has signature $(n-1, 1)$ and $B(\lambda, \lambda) < 0$ for λ in the open Weyl chamber,

(ii) The form B is non-degenerate but not positive and the graph obtained by removing any vertex from the graph of W is of non-negative type (i.e. its group is finite or affine).

If these conditions occur, then W is said to be hyperbolic*. If the second condition is enhanced by requiring that any such sub-graph is of positive definite type (i.e. its group is finite), then W is said to be* compact hyperbolic*.*

Remark 5.1.2. *As mentioned in [Hum92], the terminology comes from the fact that the homogeneous space $O(V, B)/W$, equipped with the induced measure coming from the Haar measure on $O(V, B)$, is of finite volume if and only if W is finite or hyperbolic and, in the hyperbolic case, a component of $\{\lambda \in V\ ;\ B(\lambda, \lambda) = -1\}$ gives a model for the hyperbolic $(n-1)$-space $\mathbb{H}^n$. Moreover, the space $O(V, B)/W$ is compact if and only if W is compact hyperbolic. Moreover, W is compact hyperbolic if and only if B is non-degenerate non-positive and every proper parabolic subgroup of W is finite.*

We are ready to define the compact hyperbolic extensions of non-crystallographic groups. From now on, we let (W, S) be a finite irreducible non-crystallographic Coxeter system of rank n with Coxeter matrix M. As mentioned at the beginning of this part, we have to find a reflection $r_W \in W$ yielding a compact hyperbolic extension of W. This is done in the following result, where the notation are as in the Table 5.

Proposition-Definition 5.1.3. *Let W be non-crystallographic and choose $r_W \in W$ to be the following reflection in W:*

$$r_W := \begin{cases} (s_1 s_2)^{\lfloor \frac{m-1}{2} \rfloor} s_1 & \text{if} \quad W = I_2(m),\ m \geq 3, \\ s_3^{(s_2 s_1)^2} & \text{if} \quad W = H_3, \\ s_4^{\left(s_1^{s_2 s_3}(s_1 s_2)^2 s_3 s_4\right)^2} & \text{if} \quad W = H_4. \end{cases}$$

Define

$$\widehat{W} := \left\langle \widehat{s}_0, \widehat{s}_1, \ldots, \widehat{s}_r \;\middle|\; \forall i, j \geq 1,\ (\widehat{s}_i \widehat{s}_j)^{m_{i,j}} = (\widehat{s}_0 \widehat{s}_i)^{o(r_W s_i)} = \widehat{s}_0^2 = 1 \right\rangle,$$

where $o(x)$ is the order of the element x and $\widehat{S} := \{\widehat{s}_0, \ldots, \widehat{s}_n\}$. Then the pair $(\widehat{W}, \widehat{S})$ is a compact hyperbolic Coxeter system, whose Coxeter graph is as in the following table:

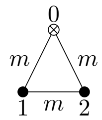
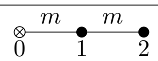
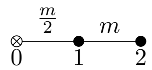
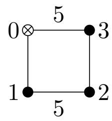
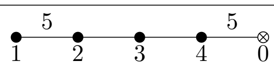
Extension	Coxeter graph
$\widehat{I_2(m)}$ $(m \equiv 1[2])$	
$\widehat{I_2(m)}$ $(m \equiv 0[4])$	
$\widehat{I_2(m)}$ $(m \equiv 2[4])$	
$\widehat{H_3}$	
$\widehat{H_4}$	

Table 5: Compact hyperbolic extensions of $I_2(m)$, H_3 and H_4.

Moreover, in type H, the reflection r_W is the only one for which the resulting group $\widehat{W}$ is compact hyperbolic.

Proof. The expression we give for r_W indicates that r_W indeed is a reflection of W. As r_W has order 2, the matrix $\widehat{M} := (\widehat{m}_{i,j})_{0 \leq i,j \leq n}$ defined by

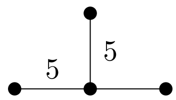
$$\forall i, j \geq 1,\ \widehat{m}_{i,j} = m_{i,j},\ \widehat{m}_{0,i} = \widehat{m}_{i,0} := o(r_W s_i),\ \widehat{m}_{0,0} := 1$$

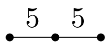
is indeed a Coxeter matrix and $\widehat{W}$ is the associated Coxeter group. Moreover, we may compute the integers $o(r_W s_i)$ directly and find the above Coxeter graphs and these are indeed graphs of compact hyperbolic groups, as all those graphs are well-known, see [Che69, Appendice].

The second statement comes from a tedious, but elementary verification on the 15 (resp. 60) reflections of H_3 (resp. H_4): only the reflection r_W from the statement gives a graph which appears in the table of [Che69]. $\square$

Remark 5.1.4. *A (non-crystallographic) root system Φ may be associated to W. More precisely, Φ is the orbit under W of the vectors α_i spanning V. Then Φ forms a (non-Euclidean) root system in V, which is non-crystallographic in the sense that the condition $\langle \alpha^\vee, \beta\rangle \in \mathbb{Z}$ does no longer hold. We still may choose a highest root in Φ. If $W \neq H_3$, then the reflection associated to this highest root is indeed r_W.*

The extension of H_3 with r_W the highest reflection has been considered in [PT02]. It has the following Coxeter graph



However, the sub-graph  *is of negative type, hence this extension is neither affine or hyperbolic and the sequel does not apply.*

Using the very definitions of W and $\widehat{W}$ as finitely presented groups, we obtain the following result:

Corollary 5.1.5. *For any W, the assignment*

$$\begin{cases} \widehat{s}_0 & \longmapsto & r_W \\ \widehat{s}_i & \longmapsto & s_i \end{cases}$$

extends (uniquely) to a surjective reflection-preserving group homomorphism

$$\widehat{W} \overset{\pi}{\twoheadrightarrow} W.$$

Moreover, if $r_W = s_{i_1} \cdots s_{i_k}$ is a reduced expression of r_W, then the element $\widehat{r_W} = \widehat{s}_{i_1} \cdots \widehat{s}_{i_k} \in \widehat{W}$ is well defined and we have

$$\ker \pi = \left\langle (\widehat{s}_0 \widehat{r_W})^{\widehat{W}} \right\rangle,$$

that is, $\ker(\pi)$ *is the normal closure of $\widehat{s}_0 \widehat{r_W}$ in $\widehat{W}$.*

Proof. In every reduced expression as in the statement, we have $i_j \geq 1$ so that the element $\widehat{r_W} = \widehat{s}_{i_1} \cdots \widehat{s}_{i_k}$ is in the parabolic subgroup $\widehat{W}_{\{1,\ldots,n\}} \simeq W$ and thus $\widehat{r_W}$ doesn't depend on the chosen reduced expression for r_W.

Because π sends a simple reflection of $\widehat{W}$ to a reflection of W, it is clear that is sends any reflection to a reflection.

We have $\pi(\widehat{s}_0 \widehat{r_W}) = r_W^2 = 1$ so that the subgroup $N := \left\langle (\widehat{s}_0 \widehat{r_W})^{\widehat{W}} \right\rangle$ is certainly contained in $\ker(\pi)$. Furthermore, we easily find a presentation of $\widehat{W}/N$ by adding the relation $\widehat{s}_0 = \widehat{s}_{i_1} \cdots \widehat{s}_{i_k}$ for $r_W = s_{i_1} \cdots s_{i_k}$ as above to the already known relations for $\widehat{W}$. The composite

$$\langle \overline{s}_0, \overline{s}_1, \ldots, \overline{s}_n \mid \forall i,j \geq 1,\ (\overline{s}_i \overline{s}_j)^{m_{i,j}} = 1,\ \overline{s}_0 = \overline{s}_{i_1} \cdots \overline{s}_{i_k} \rangle \simeq \widehat{W}/N \twoheadrightarrow \widehat{W}/\ker \pi = W$$

maps $\overline{s}_i$ to s_i and is an isomorphism. In particular, this yields an isomorphism of $\widehat{W}$-sets

$$\widehat{W}/N \simeq \widehat{W}/\ker\pi,$$

forcing $\ker(\pi)$ and N to be conjugate in $\widehat{W}$, hence equal. $\square$

Definition 5.1.6. *We denote the kernel of the projection from the previous Corollary by*

$$Q := \ker\pi = \left\langle (\widehat{s}_0\widehat{r_W})^{\widehat{W}} \right\rangle.$$

Corollary 5.1.7. *With the notation of the above theorem, we have*

$$\widehat{W} = Q \rtimes W.$$

Proof. The map $s_i \mapsto \widehat{s}_i$ $(i \geq 1)$ extends to a splitting $W \to \widehat{W}$ of π. $\square$

Remark 5.1.8. *Let Φ denote the (non-crystallographic) root system of W and $\widetilde{\alpha} \in \Phi^+$ be the (positive) root associated to the reflection r_W, i.e. such that $r_W = s_{\widetilde{\alpha}^\vee}$. If $W \neq H_3$, then $\widetilde{\alpha} = \alpha_0$ is the highest root of Φ. Denote by $\mathrm{t}_{\widetilde{\alpha}^\vee}$ the translation by $\widetilde{\alpha}^\vee$ and by $\sigma^* : W \to GL(V^*)$ the dual of the geometric representation of W. We can define a homomorphism*

$$\widehat{W} \xrightarrow{\mathbf{a}} \mathrm{Aff}(V^*)$$

by sending $\widehat{s}_i$ to $\sigma^*(s_i)$ for $i \geq 1$ and $\mathbf{a}(\widehat{s}_0) := \mathrm{t}_{\widetilde{\alpha}^\vee}\sigma^*(r_W)$. If W is a Weyl group, then $\mathbf{a}$ is injective and identifies $\widehat{W}$ with $W_{\mathrm{a}} \leq \mathrm{Aff}(V^*)$. Moreover, in this case we have

$$Q \simeq \mathbf{a}(Q) = \mathbf{a}\left(\left\langle (\widehat{s}_0\widehat{s_{\alpha_0^\vee}})^{\widehat{W}} \right\rangle\right) = \left\langle (\mathbf{a}(\widehat{s}_0)\mathbf{a}(\widehat{s_{\alpha_0^\vee}}))^{\mathbf{a}(\widehat{W})} \right\rangle$$

$$= \left\langle (t_{\alpha_0^\vee})^{W_{\mathrm{a}}} \right\rangle = \langle t_{\alpha^\vee},\ \alpha \in \Phi \rangle \simeq \mathbb{Z}\Phi^\vee \overset{df}{=} Q^\vee \simeq \mathbb{Z}^n.$$

This is the coroot lattice of Φ and in particular, the group Q is abelian.

However, a relatively recent result ([Qi07, Corollary 1.6]) states that an irreducible, infinite Coxeter group is affine if and only if it contains an abelian subgroup of finite index and, as $[\widehat{W} : Q] = |W| < \infty$, the group Q cannot be abelian in the hyperbolic case.

Moreover, in the non-crystallographic case, the image of $\mathbf{a}$ is no longer discrete because $\mathbb{Z}\Phi^\vee \subset V^*$ is dense in V^* and also, the morphism $\mathbf{a}$ has no reason to be injective, because we cannot relate the length function on $\widehat{W}$ with separating reflection hyperplanes in V any longer.

The morality is that we should take the geometry of $\widehat{W}$ into account, which is not affine but hyperbolic in the non-crystallographic case.

5.2 A key property of the subgroup Q

The following result will be crucial in the sequel.

Lemma 5.2.1. *The normal subgroup Q trivially intersects every proper parabolic subgroup of $\widehat{W}$, i.e.*

$$\forall I \subsetneq \widehat{S},\ Q \cap \widehat{W}_I = 1.$$

Proof. This is clear in the crystallographic case because $Q \simeq \mathbb{Z}\Phi^\vee \simeq \mathbb{Z}^n$.

Recall the morphism $\pi : \widehat{W} \twoheadrightarrow W$. The statement may be rephrased as follows:

$$\forall s \in \widehat{S},\ \ker\left(\widehat{W}_{\widehat{S}\setminus\{s\}} \xrightarrow{\pi} W\right) = 1.$$

For $s = \widehat{s}_0$, this is obvious since $\widehat{W}_{\widehat{S}\setminus\{\widehat{s}_0\}} \xrightarrow{\pi} W$ is an isomorphism.

Let $s \in \widehat{S}\setminus\{\widehat{s}_0\}$. Since $\widehat{W}$ is compact hyperbolic, the parabolic subgroup $\widehat{W}_{\widehat{S}\setminus\{s\}}$ is finite. Hence, to prove that the morphism

$$\widehat{W}_{\widehat{S}\setminus\{s\}} \overset{\pi}{\twoheadrightarrow} \pi\left(\widehat{W}_{\widehat{S}\setminus\{s\}}\right)$$

is injective, it suffices to prove that

$$\left|\pi\left(\widehat{W}_{\widehat{S}\setminus\{s\}}\right)\right| = \left|\widehat{W}_{\widehat{S}\setminus\{s\}}\right|. \tag{$\star_s$}$$

The right-hand side is easily computed using the Coxeter diagram of $\widehat{W}$ (see Table 5). To compute the left-hand side, we proceed by a case-by-case analysis. For H_4, we will need the following trick:

Denote by

$$R := \bigcup_{w\in W} wSw^{-1} = \bigcup_{w\in W} S^w$$

the set of reflections of W and

$$\forall w \in W,\ N(w) := \{r \in R\ ;\ \ell(rw) < \ell(w)\}.$$

If $H \leq W$ is a reflection subgroup of W (i.e. if $H = \langle H \cap R\rangle$), then the set

$$D(H) := \{r \in R\ ;\ N(r) \cap H = \{r\}\}$$

is a set of Coxeter generators of H (see [Dye90, Theorem 3.3]). In our situation, we find the Coxeter generators $D(\pi(\widehat{W}_{\widehat{S}\setminus\{s\}}))$ and determine the resulting Coxeter diagram, giving the order of $\pi(\widehat{W}_{\widehat{S}\setminus\{s\}})$.

- $W = I_2(m)$ with $m = 2k+1$. We have defined $r_W = (s_1s_2)^k s_1$ and we readily compute $s_2 = {s_1}^{r_W}$ and $s_1 = {s_2}^{r_W}$ so that $\pi(\widehat{W}_{\widehat{s}_0,\widehat{s}_1}) = \pi(\widehat{W}_{\widehat{s}_0,\widehat{s}_2}) = W$. On the other hand, we get from the diagram $|\widehat{W}_{\widehat{s}_0,\widehat{s}_1}| = |\widehat{W}_{\widehat{s}_0,\widehat{s}_1}| = 2m = |W|$. This proves $(\star_s)$ for $s = \widehat{s}_1, \widehat{s}_2$.
- $W = I_2(m)$ with $m = 4k$. In this case we have $r_W = (s_1s_2)^{2k-1}s_1$ and since $s_2 = (s_1r_W)^{2k-1}s_1$, we also have $\langle r_W, s_1\rangle = W$ and $(\star_s)$ is thus true for $s = \widehat{s}_2$ as $\widehat{W}_{\widehat{s}_0,\widehat{s}_1} \simeq W$. Because $s_2r_W = (s_2s_1)^{2k} = (s_1s_2)^{2k} = r_Ws_2$, we have $\langle s_2, r_W\rangle = A_1 \times A_1$ and $\widehat{W}_{\widehat{s}_0,\widehat{s}_2} \simeq A_1 \times A_1$ so $(\star_s)$ also holds for $s = \widehat{s}_2$.
- $W = I_2(m)$ with $m = 4k+2$. Here, $r_W = (s_1s_2)^{2k}s_1$ and we compute $r_Ws_1r_W = (s_1s_2)^{4k}s_1 = s_2s_1s_2 = s_1^{s_2}$. In the same way, we get $(s_1(s_1^{s_2}))^k s_1 = (s_1s_2)^{2k}s_1 = r_W$. This implies $\langle s_1, r_W\rangle = \langle s_1, s_1^{s_2}\rangle \simeq I_2(2k+1) \simeq \widehat{W}_{\widehat{s}_0,\widehat{s}_1}$. In fact, we have $D(\langle s_1, r_W\rangle) = \{s_1, s_1^{s_2}\}$. Now, as above we have $s_2r_W = r_Ws_2$ and $\widehat{W}_{\widehat{s}_0,\widehat{s}_2} \simeq A_1 \times A_1 \simeq \langle s_2, r_W\rangle$.

- $W = H_3$. Special relations among reflections occur in this case. Namely

$$r_W = s_3^{(s_2s_1)^2},\ s_3 = r_W^{(s_1s_2)^2},\ s_2 = s_3(r_Ws_3s_1)^2r_Ws_3,\ s_1 = (r_Ws_3s_2)^2r_Ws_3r_W.$$

  Hence, for $s \in \widehat{S}$, we have $\pi\left(\widehat{W}_{\widehat{S}\setminus\{s\}}\right) = W \simeq \widehat{W}_{\widehat{S}\setminus\{s\}}$, this last isomorphism being given by the diagram of $\widehat{H_3}$. Therefore, all the relations $(\star_s)$ hold in this case.

- $W = H_4$. The additional reflection is

$$r_W = s_4^{(s_3s_2s_1s_2s_3(s_1s_2)^2s_3s_4)^2}.$$

  We notice the following relation

$$s_1 = s_2s_3(s_4r_W)^2(s_3s_4r_Ws_2(s_3s_4r_W)^2s_2)^3s_3s_4r_Ws_4s_3s_2.$$

  This proves that $s_1 \in \langle r_W, s_2, s_3, s_4\rangle$ so $\pi(\widehat{W}_{\widehat{s}_0,\widehat{s}_2,\widehat{s}_3,\widehat{s}_4}) = W \simeq \widehat{W}_{\widehat{s}_0,\widehat{s}_2,\widehat{s}_3,\widehat{s}_4}$. We treat the remaining cases by determining the Dyer generators of the reflection subgroups. Calculations can be done on the sixty reflections of H_4 (though easier using [GAP4]). We obtain

$$\pi\left(\widehat{W}_{\widehat{s}_0,\widehat{s}_1,\widehat{s}_3,\widehat{s}_4}\right) = \langle r_W, s_1, s_3, s_4\rangle = \left\langle s_1^{s_2s_3(s_1s_2)^2}, s_1, s_3, s_4\right\rangle \simeq A_1 \times H_3 \simeq \widehat{W}_{\widehat{s}_0,\widehat{s}_1,\widehat{s}_3,\widehat{s}_4}$$

  and

$$\pi\left(\widehat{W}_{\widehat{s}_0,\widehat{s}_1,\widehat{s}_2,\widehat{s}_4}\right) = \langle r_W, s_1, s_2, s_4\rangle = \left\langle s_3^{s_4s_2s_1s_2s_3(s_1s_2)^2s_3}, s_1, s_2, s_4\right\rangle \simeq I_2(5)^2 \simeq \widehat{W}_{\widehat{s}_0,\widehat{s}_1,\widehat{s}_2,\widehat{s}_4}$$

  and finally,

$$\pi\left(\widehat{W}_{\widehat{s}_0,\widehat{s}_1,\widehat{s}_2,\widehat{s}_3}\right) = \langle r_W, s_1, s_2, s_3\rangle \simeq H_3 \times A_1 \simeq \widehat{W}_{\widehat{s}_0,\widehat{s}_1,\widehat{s}_2,\widehat{s}_3}.$$

  This establishes the relations $(\star_s)$ for $W = H_4$, finishing the proof.

□

Corollary 5.2.2. *The group Q is torsion-free.*

Proof. Let $q \in Q$ be of finite order. By a theorem of Tits (see [Qi07, Theorem 3.10]), there are $w \in \widehat{W}$ and $J \subset \widehat{S}$ such that $q \in w\widehat{W}_Jw^{-1}$ and $\widehat{W}_J$ is finite. This last condition implies $J \neq \widehat{S}$ and since Q is normal in $\widehat{W}$, from Lemma 5.2.1 we get $q^w \in Q \cap \widehat{W}_J = 1$, so $q = 1$. □

5.3 The hyperbolic torus $\mathbf{T}(W)$ of W and its first properties

Before defining the manifold $\mathbf{T}(W)$, we have to study the action of the subgroup $Q \trianglelefteq \widehat{W}$ on the Tits cone of $\widehat{W}$. Recall some notation: define $\widehat{V} := \mathrm{span}_{\mathbb{R}}(\alpha_s,\ s \in \widehat{S})$ and the bilinear form $\widehat{B}$ given by

$$\widehat{B}(\alpha_s, \alpha_t) = -\cos\left(\frac{\pi}{\widehat{m}_{s,t}}\right),$$

with $(\widehat{m}_{s,t})_{s,t\in\widehat{S}}$ the Coxeter matrix of $(\widehat{W}, \widehat{S})$. As $\widehat{W}$ is hyperbolic, the form $\widehat{B}$ has signature $(n-1, 1)$. We also have the geometric representation $\widehat{\sigma} : \widehat{W} \hookrightarrow O(\widehat{V}, \widehat{B})$ and consider its contragredient representation

$$\widehat{\sigma}^* : \widehat{W} \hookrightarrow GL(\widehat{V}^*) \qquad (\circledast)$$

and define $(\alpha_s^\vee)_{s\in\widehat{S}}$ to be the dual basis of $\widehat{V}^*$ associated to $(\alpha_s)_{s\in\widehat{S}}$. We have $\widehat{\sigma}^*(w) = {}^t\widehat{\sigma}(w^{-1})$, that is

$$\forall s,t \in \widehat{S},\ \widehat{\sigma}^*(s)(\alpha_t^\vee) = \alpha_t^\vee - 2\delta_{s,t}\widehat{B}(-,\alpha_s).$$

The duality pairing of $\widehat{V}$ is denoted $\langle\cdot,\cdot\rangle$ as usual. For $s\in\widehat{S}$, let moreover

$$H_s := \{\lambda\in\widehat{V}^*\ ;\ \langle\lambda,\alpha_s\rangle = 0\} \text{ and } A_s := \{\lambda\in\widehat{V}^*\ ;\ \langle\lambda,\alpha_s\rangle > 0\}$$

and consider the respective *fundamental chamber* and *Tits cone*

$$C := \{\lambda\in\widehat{V}^*\ ;\ \langle\lambda,\alpha_s\rangle > 0,\ \forall s\in\widehat{S}\} = \bigcap_{s\in\widehat{S}} A_s \text{ and } X := \bigcup_{w\in\widehat{W}} w(\overline{C}).$$

This is indeed a convex cone and $\overline{C}$ is a fundamental domain for the action of $\widehat{W}$ on X. Finally, for $I\subseteq\widehat{S}$ we let

$$C_I := \left(\bigcap_{s\in I} H_s\right) \cap \left(\bigcap_{s\notin I} A_s\right) \subset \overline{C},$$

in particular $C_\emptyset = C$, $C_{\widehat{S}} = \{0\}$ and we have $\overline{C} = \bigsqcup_{I\subseteq\widehat{S}} C_I$.

In this context, we have the *Coxeter complex*

$$\widehat{\Sigma} := \Sigma(\widehat{W},\widehat{S}) = (X\setminus\{0\})/\mathbb{R}_+^*.$$

This is a $\widehat{W}$-pseudomanifold and we have a decomposition

$$\widehat{\Sigma} = \bigcup_{\substack{w\in\widehat{W}\\ I\subsetneq\widehat{S}}} \mathbb{R}_+^* w(\overline{C_I})$$

which is in fact a $\widehat{W}$-triangulation since $\mathbb{R}_+^* w(\overline{C_I})$ may be identified with the standard $(n-|I|)$-simplex: $\mathbb{R}_+^* w(\overline{C_I}) \simeq \Delta^{n-|I|}$. Moreover, since $\widehat{W}$ is infinite, $\widehat{\Sigma}$ is contractible and by [Bro89, III, §2, Corollary 3], as every proper parabolic subgroup of $\widehat{W}$ is finite, the pseudomanifold $\widehat{\Sigma}$ is in fact a smooth n-manifold.

Remark 5.3.1. *The construction of the Coxeter complex makes sense for any Coxeter group. If the group is finite, then its Coxeter complex is homeomorphic to a sphere.*

We can give a natural simplicial structure to the Coxeter complex (see [BR04, Corollary 2.6]). Consider the set of parabolic cosets of $\widehat{W}$

$$P(\widehat{W},\widehat{S}) := \{w\widehat{W}_I\ ;\ w\in\widehat{W},\ I\subsetneq\widehat{S}\}.$$

We partially order this set as follows:

$$w\widehat{W}_I \preceq w'\widehat{W}_J \overset{\text{df}}{\iff} w\widehat{W}_I \supseteq w'\widehat{W}_J.$$

Notice that $w\widehat{W}_I \preceq w'\widehat{W}_J$ implies $w\widehat{W}_I = w'\widehat{W}_I$ and $J\subset I$. We define the simplicial complex $\Delta(\widehat{W},\widehat{S})$ as the nerve of this poset:

$$\Delta(\widehat{W},\widehat{S}) := \mathcal{N}(P(\widehat{W},\widehat{S}),\preceq).$$

If we denote by $\mathcal{P}(\widehat{\Sigma})$ the poset of faces of $\widehat{\Sigma}$ with respect to the triangulation described above. Then we have an isomorphism of posets

$$\begin{array}{ccc} (P(\widehat{W}, \widehat{S}), \preceq) & \xrightarrow{\sim} & (\mathcal{P}(\widehat{\Sigma}), \subseteq) \\ w\widehat{W}_I & \longmapsto & \mathbb{R}_+^* w(\overline{C_I}) \end{array}$$

and this yields a $\widehat{W}$-equivariant homeomorphism

$$|\Delta(\widehat{W}, \widehat{S})| \xrightarrow{\sim} \widehat{\Sigma}.$$

Now, recall that an action of a group G on a space Z is said to be *properly discontinuous* (or a *covering space action*, see [Hat02, §1.3]) if every point $z \in Z$ has an open neighbourhood $z \in U \subset Z$ such that if $g \in G$ is such that $gU \cap U \neq \emptyset$, then $g = 1$. In other words, such that

$$O_G(U) := \{g \in G \ ; \ g(U) \cap U \neq \emptyset\} = \{1\}.$$

Lemma 5.3.2. *Recall from (⊛) the representation $\widehat{\sigma}^*$. The action of the discrete subgroup $\widehat{\sigma}^*(Q) \leq GL(\widehat{V}^*)$ on the Coxeter complex $\widehat{\Sigma}$ is free and properly discontinuous.*

Proof. Of course, we identify the group $\widehat{W}$ with $\widehat{\sigma}^*(\widehat{W})$. Let $\overline{x} \in \widehat{\Sigma}$ (with $x \in X \setminus \{0\}$). First, we prove that $q(\overline{x}) \neq \overline{x}$ for $q \in Q \setminus \{1\}$. To say that $q(\overline{x}) = \overline{x}$ amounts to say that $q(x) = ax$ for some $a \in \mathbb{R}_+^*$ and we may assume that $x \in \overline{C} \setminus \{0\}$ since $Q \trianglelefteq \widehat{W}$. There is some $I \subsetneq \widehat{S}$ such that $x \in C_I$. Because C_I is a cone, we have $ax \in C_I \cap q(C_I) \neq \emptyset$ and by [Bou02, V, §4, Proposition 5], we obtain $q(C_I) = C_I$ so $q \in \widehat{W}_I \cap Q = 1$ by Lemma 5.2.1.

To prove that the action is properly discontinuous at $\overline{x}$, we have to find an open neighbourhood U of $\overline{x}$ in $\widehat{\Sigma}$ such that, for $1 \neq q \in Q$, we have $U \cap q(U) = \emptyset$, i.e. $O_Q(U) = \{1\}$. By definition of the topology on the Coxeter complex, it suffices to prove the statement for $X \setminus \{0\}$.

First, we show that the action of $\widehat{W}$ is *wandering* at x, that is, we can find an open neighbourhood A of x such that $O_{\widehat{W}}(A)$ is finite. We may assume that $x \in \overline{C} \setminus \{0\}$, say $x \in C_I$ with $I \subsetneq \widehat{S}$. Define A to be the interior in $X \setminus \{0\}$ of the subset $\bigcup_{v \in \widehat{W}_I} v(\overline{C})$. We prove that there are only finitely many $w \in \widehat{W}$ such that $A \cap w(A) \neq \emptyset$. Suppose that $w \in O_{\widehat{W}}(A)$ and choose $a \in A$ with $w(a) \in A$. Notice that we have

$$A \subseteq \bigcup_{u \in \widehat{W}_I} u(C) \cup \bigcup_{\substack{v \in \widehat{W}_I \\ s \in I}} v(H_s \cap \partial \overline{C}).$$

Thus, we distinguish four cases:

- As $\widehat{W}$ acts on $X \setminus \bigcup_{s \in \widehat{S}} H_s$, we cannot have $a \in \bigcup_{v,s} v(H_s \cap \partial \overline{C})$ and $w(a) \in \bigcup_u u(C)$.
- Similarly, we cannot have $a \in \bigcup_u u(C)$ and $w(a) \in \bigcup_{v,s} v(H_s \cap \partial \overline{C})$.
- Suppose that $a \in \bigcup_v v(C)$ and $w(a) \in \bigcup_v v(C)$, say $a \in u(C)$ and $w(a) \in v(C)$. This implies $u^{-1}(a) \in C$ and $v^{-1}w(a) = v^{-1}wu(u^{-1}(a)) \in C$, thus $uv^{-1}w(C) \cap C \neq \emptyset$ and so $w = vu^{-1} \in \widehat{W}_I$ by Tits' lemma.

- Suppose now that we have $a \in \bigcup_{v,s} v(H_s \cap \partial\overline{C})$ and $w(a) \in \bigcup_{v,s} v(H_s \cap \partial\overline{C})$, say $a \in u(H_s \cap \partial\overline{C})$ and $w(a) \in v(H_t \cap \partial\overline{C})$. This implies $u^{-1}(a) \in \overline{C}$ and $v^{-1}wu(u^{-1}(a)) = v^{-1}w(a) \in \overline{C}$ and by [Bou02, V, §4, Proposition 6] we get $v^{-1}w(a) = u^{-1}(a)$ and thus $uv^{-1}w \in (\widehat{W})_a = u\widehat{W}_J u^{-1}$ for some $J \subsetneq \widehat{S}$ (in fact, J is defined by the condition $\widehat{W}_J = (\widehat{W})_{u^{-1}(a)}$). Therefore, we have $w \in v\widehat{W}_J u^{-1}$.

In any case, we have

$$O_{\widehat{W}}(A) \stackrel{\text{df}}{=} \{w \in \widehat{W} \ ; \ w(A) \cap A \neq \emptyset\} \subset \bigcup_{\substack{u,v \in \widehat{W}_I \\ J \subsetneq \widehat{S}}} u\widehat{W}_J v.$$

However, as $\widehat{W}$ is compact, any proper parabolic subgroup is finite and so this last subset is finite and $O_{\widehat{W}}(A)$ is then finite as well.

The rest of the proof is very standard. For each $w \in O_{\widehat{W}}(A) \setminus \widehat{W}_I$ we have $w(x) \neq x$ and we may choose an open subset A_w such that $x \in A_w \subset A$ and $w(A_w) \cap A_w = \emptyset$ and define

$$B := \bigcap_{w \in O_{\widehat{W}}(A) \setminus \widehat{W}_I} A_w \subset A.$$

Because $O_{\widehat{W}}(A)$ is finite, B is open and let $w' \in O_{\widehat{W}}(B) \subset O_{\widehat{W}}(A)$. We must have $w' \in \widehat{W}_I$ because otherwise, $\emptyset \neq B \cap w'(B) \subset A_{w'} \cap w'(A_{w'}) = \emptyset$ and thus $O_{\widehat{W}}(B) \subset \widehat{W}_I$.

Consider the open subset

$$U := \bigcap_{w \in \widehat{W}_I} w(B) \subset B.$$

We have $O_{\widehat{W}}(U) \subset O_{\widehat{W}}(B) \subset \widehat{W}_I$ and U is $\widehat{W}_I$-stable (i.e. U is a $\widehat{W}$-*slice* at x). In particular, if $q \in Q \setminus \{1\}$, then $q \notin \widehat{W}_I$ by Lemma 5.2.1 and thus $q \notin O_{\widehat{W}}(U)$. □

We arrive then to the main result of this section. Remark that the Tits form $\widehat{B}$ induces a Riemannian metric on the Coxeter complex $\widehat{\Sigma}$.

Theorem 5.3.3. *Let (W, S) be a finite irreducible Coxeter group of rank n and $(\widehat{W}, \widehat{S})$ be either the affine Weyl group associated to W if W is crystallographic, or the extension constructed above otherwise, with $Q := \ker(\widehat{W} \twoheadrightarrow W)$. If $\widehat{\sigma}^*$ denotes the contragredient geometric representation (as in (⊛)), then the orbit space*

$$\mathbf{T}(W) := \widehat{\Sigma}/\widehat{\sigma}^*(Q)$$

is a closed, connected, orientable, compact smooth W-manifold of dimension n.

If W is a Weyl group, then we have a diffeomorphism $\widehat{\Sigma} \simeq \mathbb{R}^n$ and the manifold $\mathbf{T}(W)$ is W-diffeomorphic to a maximal torus of the simply-connected compact Lie group with root system that of W. Otherwise, the Riemannian manifold $\widehat{\Sigma}$ is isometric to the hyperbolic n-space $\mathbb{H}^n$ and $\mathbf{T}(W) \simeq \mathbb{H}^n/Q$ is a hyperbolic W-manifold.

Furthermore, the canonical projection yields a covering space

$$Q \hookrightarrow \widehat{\Sigma} \twoheadrightarrow \mathbf{T}(W)$$

and the quotient simplicial complex $\Delta(\widehat{W}, \widehat{S})/Q$ is a regular W-triangulation of $\mathbf{T}(W)$.

Proof. Since $\widehat{\Sigma}$ is a closed smooth manifold and the action $\widehat{\sigma}^*(Q) \circlearrowright \widehat{\Sigma}$ is properly discontinous by Lemma 5.3.2, the quotient manifold theorem ensures that $\mathbf{T}(W)$ is indeed a closed smooth manifold and by [Hat02, Proposition 1.40], the projection $\widehat{\Sigma} \twoheadrightarrow \mathbf{T}(W)$ is a covering map. Moreover, $\mathbf{T}(W)$ is connected since the Coxeter complex is and, as $(\overline{C} \setminus \{0\})/\mathbb{R}_+^* \simeq \overline{C} \cap \mathbb{S}^n$ is a $\widehat{W}$-fundamental domain on the Coxeter complex, its projection onto $\mathbf{T}(W)$ is a W-fundamental domain, hence $\mathbf{T}(W)$ is compact (W is finite). Since Q is normally generated by $\widehat{s}_0\widehat{r_W}$ and because $\ell(\widehat{r_W})$ is odd, we have $\varepsilon(\widehat{s}_0\widehat{r_W}) = 1$ and so $Q \leq \ker(\varepsilon)$. This proves that the action of Q on $\widehat{\Sigma}$ preserves the orientation, ensuring the orientability of $\mathbf{T}(W)$. The comparison with a torus of a Lie group follows directly from the Remark 5.1.8.

In the non-crystallographic case, let $v^* \in V^*$ be a normalized eigenvector for the negative eigenvalue of $\widehat{B}$. Then the subset $\mathcal{H} := \{\lambda \in \widehat{V}^* \ ; \ \widehat{B}(\lambda, \lambda) = -1, \ \widehat{B}(v^*, \lambda) < 0\}$, together with the metric induced by the restriction of $\widehat{B}$ is a Riemannian manifold isometric to the hyperbolic space $\mathbb{H}^n$. We have $\mathbf{T}(W) = \widehat{\Sigma}/Q \simeq \mathcal{H}/Q \simeq \mathbb{H}^n/Q$ and since Q preserves the form $\widehat{B}$, the manifold $\mathbf{T}(W)$ naturally inherits a hyperbolic Riemannian metric. □

Remark 5.3.4. *After we did this work, we realized that the manifolds $\mathbf{T}(H_3)$ and $\mathbf{T}(H_4)$ have already been discovered in [Zim93] and [Dav85]. Zimmermann and Davis construct them by taking the orbit under the action of Q (which is defined slightly differently) of hyperbolic polyhedra. However, our approach has the advantages of being more systematic and to work with any finite Coxeter group. The Zimmermann manifold $\mathbf{T}(H_3)$ has the particularity of being* maximally symmetric *among hyperbolic 3-manifolds with Heegaard genus 11, in the sense of [Zim92]. On the other hand, the Davis manifold $\mathbf{T}(H_4)$ has a* spin structure *(equivalently, its second Stiefel-Whitney class w_2 vanishes) and seems to be the only closed 4-manifold for which the intersection form has been explicitly determined, see [RT01] and [Mar15].*

Recall that, as $\widehat{W}$ is infinite, the Coxeter complex is contractible.

Corollary 5.3.5. *The covering space*

$$Q \hookrightarrow \widehat{\Sigma} \twoheadrightarrow \mathbf{T}(W)$$

is a universal principal Q-bundle. In particular, $\mathbf{T}(W)$ is a classifying space for Q and an Eilenberg-MacLane space

$$\mathbf{T}(W) \simeq B_Q \simeq K(Q, 1).$$

5.4 Presentation on the fundamental group of $\mathbf{T}(W)$

In this section, we derive a presentation of the group $\pi_1(\mathbf{T}(W)) \simeq Q \trianglelefteq \widehat{W}$ from Poincaré's fundamental polyhedron theorem (see [Rat06, Theorem 11.2.2]). Recall from [Rat06] that if P is a convex fundamental polyhedron for a discrete group Γ of isometries of $\mathbb{H}^n$ such that the associated tessellation $\{\gamma P, \ \gamma \in \Gamma\}$ is *exact* (i.e. each facet of P may be written as $P \cap \gamma P$ for some $\gamma \in \Gamma$), then by [Rat06, Theorem 6.7.5], for each facet $S \in F_{n-1}(P)$ of P, there is a unique $\gamma_S \in \Gamma$ such that $S = P \cap \gamma_S P$ and moreover, $\gamma_S^{-1}(S) =: S'$ is also a facet of P, with the *side relation* $\gamma_{S'} = \gamma_S^{-1}$. On the other hand, if T is a facet of S, then we may define a sequence $\{S_i\}_{i=1}^{\infty}$ of facets of P inductively as follows:

1. Let $S_1 := S$.

2. Let S_2 be the facet of P adjacent to S_1' such that $\gamma_{S_1}(S_1' \cap S_2) = T$.
3. For $i > 1$, let S_{i+1} be the facet of P adjacent to S_i' such that $\gamma_{S_i}(S_i' \cap S_{i+1}) = S_{i-1}' \cap S_i$.

Then, by [Rat06, Theorem 6.8.7], the sequence $\{S_i\}_{i=1}^{\infty}$ is periodic, with period k say, and we have the *cycle relation* $\gamma_{S_1} \cdots \gamma_{S_k} = 1$. In this case, we call $\{S_i\}_{i=1}^{k}$ a *cycle of sides* of P. The theorem of Poincaré states that the elements $\{g_S\}_S$, together with the side relations and the cycle relations form a presentation of Γ. More precisely, if we let $\Psi := \{g_S,\ S \in F_{n-1}(P)\}$ be an abstract set indexed by $F_{n-1}(P)$ and

$$R_{\text{side}} := \{g_S g_{S'},\ S, S' \in F_{n-1}(P) \text{ such that } S = \gamma_S(S')\}$$

as well as

$$R_{\text{cycle}} := \{g_{S_1} \cdots g_{S_k}\ ;\ \{S_i\}_{i=1}^{k} \text{ is a cycle of sides determined by } S \in F_{n-1}(P) \text{ and } T \in F_{n-2}(S)\}.$$

Then, we have an isomorphism

$$\begin{array}{ccc} \langle \Psi \mid R_{\text{side}} \cup R_{\text{cycle}} \rangle & \xrightarrow{\sim} & \Gamma \\ g_S & \longmapsto & \gamma_S \end{array}$$

In our case, the tessellation $\Delta(\widehat{W}, \widehat{S})$ of $\widehat{\Sigma} \simeq \mathbb{H}^n$ is indeed exact and yields a fundamental polyhedron for Q acting on $\widehat{\Sigma}$. Choose $v^* \in V^*$ a normalized eigenvector of the Tits form $\widehat{B}$ for its unique negative eigenvalue and consider the subset

$$\mathcal{H} := \{\lambda \in V^*\ ;\ \widehat{B}(\lambda, \lambda) = -1,\ \widehat{B}(v^*, \lambda) < 0\} \subset V^*.$$

As already noted in the proof of Theorem 5.3.3, the form $\widehat{B}$ induces a Riemannian metric on $\mathcal{H}$ and we have an isometry $\mathcal{H} \simeq \mathbb{H}^n$, where $\mathbb{H}^n$ is the standard hyperbolic n-space. By Remark 5.1.2, the fundamental chamber C is included in the subset $\{\lambda\ ;\ \widehat{B}(\lambda, \lambda) < 0\}$, hence we can project the punctured Tits cone $X \setminus \{0\}$ on the sheet $\mathcal{H}$ of the hyperbola $\{\lambda\ ;\ \widehat{B}(\lambda, \lambda) = -1\}$ and we get $\widehat{\Sigma} \simeq X \cap \mathcal{H}$. Consider the n-simplex

$$\Delta_0 := (\overline{C} \setminus \{0\})/\mathbb{R}_+^* \simeq \overline{C} \cap \mathcal{H} \subset \widehat{\Sigma}.$$

Recall that we have denoted $H_s := \{\lambda\ ;\ \langle \lambda, \alpha_s \rangle = 0\}$ for $s \in \widehat{S}$. As the subset $L_0 := \overline{C} \cap \bigcap_{s \neq \widehat{s}_0} H_s$ is a line, its intersection with $\mathcal{H}$ is a vertex of the tessellation $\Delta(\widehat{W}, \widehat{S})$ and we may consider its star

$$\Delta := \mathrm{St}\,(L_0 \cap \mathcal{H}) \overset{\text{df}}{=} \bigcup_{\substack{\sigma \in F_n(\Delta(\widehat{W}, \widehat{S})) \\ L_0 \cap \mathcal{H} \subset \sigma}} \sigma = \bigcup_{w \in W} w(\Delta_0).$$

We will describe the generators and relations for $\pi_1(\mathbf{T}(W))$ in terms of side-pairing and cycle relations, as in [Rat06, §6.8]. It is easy to see that the facets of Δ are the W-translates of the facet

$$\sigma_0 := H_{\widehat{s}_0} \cap \Delta \in F_{n-1}(\Delta),$$

in other words, $\partial\Delta = \bigcup_w w(\sigma_0)$. By [Rat06, Theorem 6.8.3], the group $Q = \pi_1(\mathbf{T}(W))$ is generated by the set

$$\Psi := \{q \in Q\ ;\ \Delta \cap q\Delta \in F_{n-1}(\Delta)\}.$$

Lemma 5.4.1. *The set Ψ of generators of Q is given by the W-conjugates of the normal generator of Q. In other words, if $r_W \in W$ is the chosen reflection and if $q_0 := \widehat{s}_0 r_W \in \widehat{W}$ then we have*

$$\Psi = \{{}^w q_0,\ w \in W\} = \{w q_0 w^{-1},\ \overline{w} \in W/C_W(q_0)\}.$$

Proof. Let $1 \neq q \in Q$ be such that $\Delta \cap q\Delta$ is a facet of Δ, say $w(\sigma_0)$ for some $w \in W$. We have

$$w(\sigma_0) = \Delta \cap q\Delta = \bigcup_{u,v\in W} u(\Delta_0) \cap qv(\Delta_0) = \bigcup_{u,v\in W} u(\Delta_0 \cap u^{-1}qv(\Delta_0)).$$

Since any term of the last union is (empty or) a closed simplex, this means that one of them has to be the whole of $w(\sigma_0)$, so we can find $u, v \in W$ such that

$$u^{-1}w(\sigma_0) = \Delta_0 \cap u^{-1}qv(\Delta_0).$$

In particular, we have $u^{-1}w(\sigma_0) \subset \Delta_0$ and since any $\widehat{W}$-orbit meets Δ_0 in only one point, this implies that $u^{-1}w(\sigma_0) = \sigma_0$ and so $u^{-1}w \in \widehat{W}_{\sigma_0} = \langle \widehat{s}_0 \rangle$ but as $u^{-1}w \in W$, this is possible only when $u = w$. Hence we get

$$\sigma_0 = \Delta_0 \cap u^{-1}qv(\Delta_0).$$

This implies in turn that $u^{-1}qv \in \langle \widehat{s}_0 \rangle$ and since $q \neq 1$, we must have $u^{-1}qv = \widehat{s}_0$, i.e. $q = u\widehat{s}_0 v^{-1}$. Finally, because $q \in Q$, applying the projection $\pi : \widehat{W} \twoheadrightarrow W$ to this equality yields $1 = ur_W v^{-1}$, so $v = ur_W$ and $q = u\widehat{s}_0 v^{-1} = uq_0 u^{-1}$. □

We can formulate the side-pairing and cycle relations using the combinatorics of W. To do this, we need a technical lemma on the centralizer of q_0.

Lemma 5.4.2. *The centralizer of $q_0 = \widehat{s}_0 r_W$ in W is given by*

$$C_W(q_0) = C_W(\widehat{s}_0) = \langle s \in S\ ;\ s\widehat{s}_0 = \widehat{s}_0 s \rangle .$$

In particular, this is (standard) parabolic.

Proof. First, we borrow an argument due to Sebastian Schoennenbeck[17] to prove the second equality above. Let $w = s_{i_1} \cdots s_{i_r}$ be a reduced expression of an element $w \in C_W(\widehat{s}_0)$. To show that w is in the parabolic subgroup of the statement, since the elements of $C_W(\widehat{s}_0)$ of length 1 are the simple reflections of W commuting with $\widehat{s}_0$, by induction it is enough to show that $\widehat{s}_0 s_{i_r} = s_{i_r}\widehat{s}_0$. We have $\ell(w\widehat{s}_0) = \ell(w) + 1$ and $\ell(w\widehat{s}_0 w^{-1}) = \ell(\widehat{s}_0) = 1$, so $\ell(w\widehat{s}_0 s_{i_r}) = \ell(w\widehat{s}_0 w^{-1} w s_{i_r}) \leq 1 + \ell(ws_{i_r}) = \ell(w)$ and thus $\ell(w\widehat{s}_0 s_{i_r}) = \ell(w)$. Thus, by the exchange condition, there is a reduced expression $w\widehat{s}_0 = s_{j_1} \cdots s_{j_r} s_{i_r}$ for $w\widehat{s}_0$ and since $s_{i_1} \cdots s_{i_r}\widehat{s}_0$ is already a reduced expression, by Matsumoto's lemma, there is a finite series of braid-moves from the second to the first. The expression $s_{i_1} \cdots s_{i_r}\widehat{s}_0$ satisfies the property

> The expression contains only one occurrence of $\widehat{s}_0$ and there is no simple reflection appearing on the right of $\widehat{s}_0$ that does not commute with it. $(*)$

Consider a braid relation $sts\cdots = tst\cdots$ connecting the two expressions of $w\widehat{s}_0$, with m factors on each side and suppose that we apply it to a reduced expression of $w\widehat{s}_0$ verifying $(*)$. If $s, t \neq \widehat{s}_0$, then the resulting expression still satisfies $(*)$. Now, if $s = \widehat{s}_0$ say, then t has to commute with $\widehat{s}_0$. Indeed, if not, then the left-hand side of the braid relation contains

[17] https://mathoverflow.net/questions/200433/centralizers-of-reflections-in-special-subgroups-of-coxeter-groups

at least two occurrences of $\widehat{s}_0$ (one on each side of t) and, in the right-hand side there is at least one occurrence of t on the right of $\widehat{s}_0$, but none of these occur in the considered reduced expression. Therefore, the reduced expression resulting from the application of the braid move still verifies $(*)$. In particular, the expression $s_{j_1}\cdots s_{j_r}s_{i_r}$ satisfies $(*)$ and thus, every simple reflection appearing on the right of $\widehat{s}_0$ must commute with it. In particular, this is the case of s_{i_r}, as required.

We now prove that $C_W(q_0) = C_W(\widehat{s}_0)$. Let $w = s_{i_1}\cdots s_{i_r}$ be a reduced expression of an element $w \in C_W(q_0)$. Since $wq_0 = q_0w$, we get $\widehat{s}_0 w\widehat{s}_0 = r_W w r_W \in W$. Let $\widehat{s}_0 w\widehat{s}_0 = s_{j_1}\cdots s_{j_k}$ be a reduced expression in W. Since $\ell(w\widehat{s}_0) = \ell(w)+1 = \ell(\widehat{s}_0 w)$, we have $\ell(\widehat{s}_0 w\widehat{s}_0) \in \{\ell(w), \ell(w)+2\}$. But taking length in the equality $\widehat{s}_0 s_{i_1}\cdots s_{i_r} = s_{j_1}\cdots s_{j_k}\widehat{s}_0$ gives $k=r$, that is $\ell(\widehat{s}_0 w\widehat{s}_0) = \ell(w)$. In particular, $\ell(\widehat{s}_0 w\widehat{s}_0) < \ell(w\widehat{s}_0)$ and by the exchange condition, there is a reduced expression $\widehat{s}_0 w\widehat{s}_0 = s_{i_1}\cdots\widehat{s_{i_l}}\cdots s_{i_r}\widehat{s}_0$ (the reflection s_{i_l} is omitted) and since this last expression is in W, we must have $s_{i_l} = \widehat{s}_0$, thus $\widehat{s}_0 w\widehat{s}_0 = s_{i_1}\cdots s_{i_r} = w$ and $w \in C_W(\widehat{s}_0)$. The reverse inclusion can be directly checked case by case using the parabolic description of $C_W(\widehat{s}_0)$. □

Remark 5.4.3. *From the diagrams of the hyperbolic extensions we get therefore*

$$C_{I_2(2g+1)}(q_0) = 1,\ C_{I_2(4g+2)}(q_0) = \langle s_2\rangle,\ C_{I_2(4g)}(q_0) = \langle s_2\rangle,$$

$$C_{H_3}(q_0) = \langle s_2\rangle,\ C_{H_4}(q_0) = \langle s_1, s_2, s_3\rangle \simeq H_3.$$

Theorem 5.4.4. *Let W be non-crystallographic and $U := \{w \in W\ ;\ \ell(ws) > \ell(w),\ \forall s \in S\ ;\ s\widehat{s}_0 = \widehat{s}_0 s\} \approx W/C_W(q_0)$ be the set of minimal length coset representatives modulo the parabolic subgroup $C_W(q_0)$ of W. The transitive action of W on $W/C_W(q_0)$ induces an action of W on U. Then the fundamental group $\pi_1(\mathbf{T}(W)) \simeq Q$ admits the following presentation*

$$\pi_1(\mathbf{T}(W)) = \langle q_u,\ u \in U \mid R_{side} \cup R_{cycle}\rangle,$$

where

$$R_{side} = \{q_u q_v,\ u, v \in U\ ;\ u^{-1}vr_W \in C_W(\widehat{s}_0)\}$$

and

$$R_{cycle} = \Big\{ q_{w(u_1)}q_{w(u_2)}\cdots q_{w(u_r)},\ w \in W,\ u_0, u_1, \ldots, u_r, u_{r+1} \in U \text{ such that } u_0 = u_{r+1} = 1$$
$$\text{and, for } i>0,\ \left\langle \widehat{s}_0, \widehat{s}_0^{\,u_{i+1}^{-1}u_i r_W}\right\rangle \text{ and } \left\langle \widehat{s}_0, \widehat{s}_0^{\,r_W u_{i-1}^{-1}u_i}\right\rangle \text{ are conjugate under } C_W(\widehat{s}_0)\Big\}.$$

Proof. Drop the presentation notation and, for $u \in U$, denote $q_u := {}^u q_0 = uq_0u^{-1}$, $\sigma_u := u(\sigma_0) = \Delta \cap q_u(\Delta)$ and $\sigma'_u := q_u^{-1}(\sigma_u) = uq_0^{-1}(\sigma_0)$. To say that for some $u, v \in U$ we have $q_uq_v = 1$ amounts to say that ${}^u q_0 = {}^v q_0^{-1} = {}^{vr_W} q_0$, i.e. $u^{-1}vr_W \in C_W(q_0)$.

For the cycle relations, recall that each facet of σ_0 is of the form σ_u for some $u \in U$. Choose $\sigma \in F_{n-1}(\Delta) \subset W\cdot\sigma_0$ and $\tau \in F_{n-2}(\sigma) \subset F_{n-2}(\Delta)$ and let $\{\sigma_{u_j}\}_{j\in\mathbb{N}^*}$ denote the associated cycle of sides, with period ℓ, say. We have the relation $q_{u_1}\cdots q_{u_\ell} = 1$. Up to conjugation by an element of W, we may assume that $\sigma = \sigma_0$ and so $q_{u_1} = q_0$. Let $i > 1$ be such that we have some relation

$$q_{u_i}(\sigma'_{u_i} \cap \sigma_{u_{i+1}}) = \sigma'_{u_{i-1}} \cap \sigma_{u_i} \neq \emptyset.$$

We write

$$\begin{aligned} q_{u_i}(\sigma'_{u_i} \cap \sigma_{u_{i+1}}) = \sigma_{u_i} \cap \sigma'_{u_{i-1}} &\iff \sigma_{u_i} \cap q_{u_i}(\sigma_{u_{i+1}}) = \sigma_{u_i} \cap q_{u_{i-1}}^{-1}(\sigma_{u_{i-1}}) \\ &\iff u_i(\sigma_0) \cap q_{u_i} u_{i+1}(\sigma_0) = u_i(\sigma_0) \cap q_{u_{i-1}}^{-1} u_{i-1}(\sigma_0) \\ &\iff u_i(\sigma_0 \cap u_i^{-1} q_{u_i} u_{i+1}(\sigma_0)) = u_i(\sigma_0 \cap u_i^{-1} u_{i-1} q_0^{-1}(\sigma_0)) \\ &\iff \sigma_0 \cap q_0 u_i^{-1} u_{i+1}(\sigma_0) = \sigma_0 \cap u_i^{-1} u_{i-1} r_W(\sigma_0), \end{aligned}$$

and the two sides of the last equality are simplices of the tessellation $\Delta(\widehat{W}, \widehat{S})$, whose face lattice is the lattice of standard parabolic subgroups of $\widehat{W}$. Hence these two coincides if and only if their stabilizers in $\widehat{W}$ are equal. Though this condition depends on the choice of the elements of U, it is straightforward to check that different choices give conjugate stabilizers in $C_W(q_0)$. □

Corollary 5.4.5. *The group $\pi_1(\mathbf{T}(H_3))$ (resp. $\mathbf{T}(H_4)$) admits a presentation with 11 (resp. 24) generators , all of whose relations are products of commutators. In particular, we have*

$$H_1(\mathbf{T}(H_3), \mathbb{Z}) = \pi_1(\mathbf{T}(H_3))^{\mathrm{ab}} \simeq \mathbb{Z}^{11} \quad \textit{and} \quad H_1(\mathbf{T}(H_4), \mathbb{Z}) = \pi_1(\mathbf{T}(H_4))^{\mathrm{ab}} \simeq \mathbb{Z}^{24}.$$

Proof. We apply the above result. For H_3, beside the side-pairing relations (which we can immediately simplify by removing half of the $[H_3 : C_{H_3}(q_0)] = [H_3 : \langle s_2 \rangle] = 60$ generators), we find only one primitive cycle relation (*primitive* meaning starting by q_0) of length 3 and one of length 5. Taking the H_3-conjugates of these gives 120 relations of length 3 and 120 relations of length 5. But the inverse of each of these relations appears so we can simplify them. We can also remove any cyclic permutation of these relations, which finally yields a presentation for $\pi_1(\mathbf{T}(H_3))$ with 30 generators, 20 relations of length 3 and 12 relations of length 5.

We do the same for H_4, where there is only one primitive cycle relation of length 5, which gives a presentation for $\pi_1(\mathbf{T}(H_4))$ with $\frac{1}{2}[H_4 : C_{H_3}(q_0)] = 60$ generators and 144 relations of length 5.

Using the relations, we can check that some of the generators are superfluous and that the simplified presentation has the stated number of generators (all among the original generators) and that the relations become trivial, once abelianized. All the formulas are given in Appendix B. □

Remark 5.4.6. *The intermediate presentations of $\pi_1(\mathbf{T}(H_3))$ and $\pi_1(\mathbf{T}(H_4))$ (with 30 generators and 32 relations for H_3 and 60 generators and 144 relations for H_4) are precisely (up to relabelling) the presentations given in [Zim93] and [RT01].*

5.5 The manifolds $\mathbf{T}(I_2(m))$ as Riemann surfaces

A little bit more can be said about the case of the surfaces $\mathbf{T}(I_2(m))$. Recall that by a theorem of Gauss (see [Jos02, Theorem 3.11.1]), any Riemannian metric on an oriented 2-manifold M induces a complex structure on M (making M a Riemann surface), called the *conformal structure* induced by the metric.

Corollary 5.5.1. *For any $g \in \mathbb{N}^*$ the surfaces $\mathbf{T}(I_2(2g+1))$, $\mathbf{T}(I_2(4g))$ or $\mathbf{T}(I_2(4g+2))$ are closed compact Riemann surfaces of genus g. In particular, we have homeomorphisms*

$$\mathbf{T}(I_2(2g+1)) \simeq \mathbf{T}(I_2(4g)) \simeq \mathbf{T}(I_2(4g+2)).$$

Proof. Since the surfaces are orientable, the Riemannian metric induced by the one on the Coxeter complex induces a conformal structure on them. To obtain the genus, we only have to compute the Euler characteristic.

Let m be either $2g+1$, $4g$ or $4g+2$ and

$$W := I_2(m) = \left\langle s, t \mid s^2 = t^2 = (st)^m = 1 \right\rangle .$$

We will detail the combinatorics of the W-triangulation $\Delta(\widehat{W}, \widehat{S})/Q$ in the next section, however we only have to compute the Euler characteristic χ and very little information is needed. The rational chain complex associated to the simplicial complex $\Delta(\widehat{W}, \widehat{S})$ has the following shape:

$$\mathbb{Q}[\widehat{W}] \longrightarrow \mathbb{Q}[\widehat{W}/\langle s\rangle] \oplus \mathbb{Q}[\widehat{W}/\langle t\rangle] \oplus \mathbb{Q}[\widehat{W}/\langle \widehat{s}_0\rangle] \longrightarrow \mathbb{Q}[\widehat{W}/\langle s,t\rangle] \oplus \mathbb{Q}[\widehat{W}/\langle s,\widehat{s}_0\rangle] \oplus \mathbb{Q}[\widehat{W}/\langle t,\widehat{s}_0\rangle]$$

Now, by Lemma 2.2.6, the complex for the surface $\mathbf{T}(I_2(m))$ is the image of the previous one by the deflation functor $\mathrm{Def}_W^{\widehat{W}}$. Thus, it is of the form

$$\mathbb{Q}[W] \longrightarrow \mathbb{Q}[W/\langle s\rangle] \oplus \mathbb{Q}[W/\langle t\rangle] \oplus \mathbb{Q}[W/\langle r\rangle] \longrightarrow \mathbb{Q} \oplus \mathbb{Q}[W/\langle s,r\rangle] \oplus \mathbb{Q}[W/\langle t,r\rangle] ,$$

where $r = r_W := (st)^{\lfloor (m-1)/2 \rfloor} s \in W$. Therefore the Euler characteristic is given by

$$\chi(\mathbf{T}_g) = 1 + [W : \langle s,r\rangle] + [W : \langle t,r\rangle] - [W : \langle s\rangle] - [W : \langle t\rangle] - [W : \langle r\rangle] + |W|$$

$$= 1 + [W : \langle s,r\rangle] + [W : \langle t,r\rangle] - 3[W : \langle s\rangle] + 2m = 1 - m + [W : \langle s,r\rangle] + [W : \langle t,r\rangle].$$

Now, we distinguish the three possible cases for m to determine the last two indices.

If $m = 2g+1$ is odd, then we have $r = (st)^g s$, so $s^r = t$ and $t^r = s$ so $\langle s,r\rangle = \langle t,r\rangle = W$.

If $m = 4g$ is divisible by 4, then $r = (st)^{2g-1}s$. From the Coxeter diagram of $\widehat{W} = \widehat{I_2(4g)}$, we see that the map interchanging $\widehat{s}_0$ and t and leaving s invariant extends to a non-trivial outer automorphism of $\widehat{I_2(4g)}$ and descends to an outer automorphism of $I_2(4g)$. Taking the image of the relation $r = (st)^{2g-1}s$ under this automorphism yields $t = (sr)^{2g-1}s$ and thus $\langle s,r\rangle = W$. Now, since the element $rt = (st)^{2g}$ has order 2, we have $\langle t,r\rangle = \{1,t,r,tr\} \simeq C_2 \times C_2$.

Now, if $m = 4g+2$, then $r = (st)^{2g}s$ and $\langle s,r\rangle = \left\langle s,(st)^{2g}\right\rangle = \left\langle s,(st)^2\right\rangle = \langle s, tst\rangle \simeq I_2(2g+1)$ and because $(rt)^2 = (st)^{2g}st(st)^{2g}st = (st)^{4g+2} = 1$, we also have $\langle t,r\rangle \simeq C_2 \times C_2$.

Gathering everything we get

$$[I_2(m) : \langle s,r\rangle] = \begin{cases} 2 & \text{if } m = 4g+2, \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad [I_2(m) : \langle t,r\rangle] = \begin{cases} 1 & \text{if } m \text{ is odd}, \\ m/2 & \text{otherwise} \end{cases}$$

thus, the Euler characteristic is given by

$$\chi(\mathbf{T}(I_2(m))) = \begin{cases} 3-m & \text{if } m = 2g+1, \\ 3-m/2 & \text{if } m = 4g+2, \\ 2-m/2 & \text{if } m = 4g, \end{cases}$$

in other words,

$$\chi(\mathbf{T}(I_2(m))) = 2 - 2g$$

and the genus of $\mathbf{T}(I_2(m))$ is indeed g for $m \in \{2g+1, 4g+2, 4g\}$. □

As the fundamental group of a Riemann surface of genus $g \geq 1$ is well-known (see [Hat02, §1.2]), we obtain a presentation for the group Q in the dihedral case.

Corollary 5.5.2. *Let $g \in \mathbb{N}^*$ and m be either $2g+1$, $4g$ or $4g+2$. Let also Q be the subgroup of $\widehat{I_2(m)}$ constructed in the previous section (see Definition 5.1.6). Then we have*

$$Q \simeq \pi_1(\mathbf{T}(I_2(m))) \simeq \langle x_1, \ldots, x_g, y_1, \ldots, y_g \mid [x_1, y_1] \cdots [x_g, y_g] = 1 \rangle$$

and in particular, $Q^{\mathrm{ab}} \simeq \mathbb{Z}^{2g}$.

In the cases where $g = 1$ that is, if $I_2(m)$ is one of the Weyl groups $I_2(3) = A_2$, $I_2(4) = B_2$ or $I_2(6) = G_2$, then $\mathbf{T}(I_2(m))$ is naturally an *elliptic curve.* More precisely, recalling the notation of the previous section, we have a preferred point

$$v_0 := \overline{C} \cap \mathcal{H} \cap \bigcap_{s \neq \widehat{s}_0} H_s \in \widehat{\Sigma},$$

and the pair $(\mathbf{T}(I_2(m)), [v_0])$ is a Riemann surface of genus 1 with a marked point, hence an elliptic curve. Notice that, under the diffeomorphism $\mathbf{T}(I_2(m)) \simeq \mathbb{R}^2/\mathbb{Z}^2$ induced by quotienting the 3-space $\widehat{V}$ by the radical of the Tits form $\widehat{B}_m$ of $\widehat{I_2(m)} = I_2(m)_{\mathrm{a}}$, the point $[v_0]$ corresponds to the origin.

We can easily identify the elliptic curves $\mathbf{T}(I_2(m))$ (for $m = 3, 4, 6$) in the moduli space $\mathcal{M}_{1,1} \simeq \mathbb{H}^2/PSL_2(\mathbb{Z})$ of complex elliptic curves, where $\mathbb{H}^2 = \{z \in \mathbb{C} \ ; \ \Im(z) > 0\}$ is the Poincaré half plane (see [Hai14, §2]). Recall that to $\tau \in \mathbb{H}^2$ we can associated a $j(\tau) \in \mathbb{C}$ and we have isomorphisms

$$\begin{array}{ccccc} \mathbb{C} & \xleftarrow{\sim} & \mathbb{H}^2/PSL_2(\mathbb{Z}) & \xrightarrow{\sim} & \mathcal{M}_{1,1} \\ j(\tau) & \longleftarrow\!\shortmid & \tau & \longmapsto & \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \end{array}$$

Recall also from [Ser70, Chapitre VII, §1.2] that $D := \{z \in \mathbb{H}^2 \ ; \ |\Re(z)| \leq 1/2, \ |z| \geq 1\}$ is a fundamental domain for $PSL_2(\mathbb{Z})$ acting on $\mathbb{H}^2$. We just have to determine a corresponding element $\tau \in D$ for each case. We have the following proposition:

Proposition 5.5.3. *Let $m \in \{3, 4, 6\}$ and let $\{\alpha^\vee, \beta^\vee\}$ be the simple coroots of the root system of type $I_2(m)$ and $V^* := \mathbb{R}\langle \alpha^\vee, \beta^\vee \rangle$. We denote by $\phi : V^* \to \mathbb{C}$ the unique isometry sending $\alpha^\vee$ to 1 and $\beta^\vee$ to an element of the upper-half plane $\mathbb{H}^2$. Then we have*

$$\phi(\beta^\vee) = \begin{cases} \exp\left(\frac{2i\pi}{3}\right) & \text{if } \ m = 3, \\ \sqrt{2}\exp\left(\frac{3i\pi}{4}\right) & \text{if } \ m = 4, \\ \sqrt{3}\exp\left(\frac{5i\pi}{6}\right) & \text{if } \ m = 6. \end{cases}$$

In particular, the corresponding lattice is $\mathbb{Z} \oplus \tau\mathbb{Z}$ where $\tau \in D$ equals $e^{\frac{2i\pi}{3}}$ for A_2 and G_2 (so that $j(\tau) = 0$) and equals i for B_2 (so that $j(\tau) = 1728$). Hence, the curves $\mathbf{T}(A_2)$, $\mathbf{T}(B_2)$ and $\mathbf{T}(G_2)$ are defined over $\mathbb{Q}$ and correspond to the orbifold points *of D, that is, the points in D having a non-trivial stabilizer in $PSL_2(\mathbb{R})$.*

Proof. We normalize the roots in such a way that the short simple roots have norm 2. For $I_2(3) = A_2$, we have $|\beta| = |\alpha| = 2$ and $\langle \alpha^\vee, \beta \rangle = \langle \beta^\vee, \alpha \rangle = -1$. Therefore, since ϕ is an isometry we should have

$$-\frac{1}{2} = \frac{1}{2}\left\langle \alpha^\vee, \beta \right\rangle = \left\langle \alpha^\vee, \beta^\vee \right\rangle = \left\langle \phi(\alpha^\vee), \phi(\beta^\vee) \right\rangle = \left\langle 1, \phi(\beta^\vee) \right\rangle \stackrel{\mathrm{df}}{=} \Re(\overline{\phi(\beta^\vee)}) = \Re(\phi(\beta^\vee)).$$

On the other hand, we have $1 = \langle \beta^\vee, \beta^\vee \rangle = |\phi(\beta^\vee)|^2$ and this implies $\phi(\beta^\vee) \in \{-1/2 \pm i\sqrt{3}/2\}$ and if we impose that $\phi(\beta^\vee) \in \mathbf{H}^2$, then it should have a positive imaginary part and the only possibility is $\phi(\beta^\vee) = -1/2 + i\sqrt{3}/2 = \exp\left(\frac{2i\pi}{3}\right)$. In the case of $I_2(4) = B_2$, we have $|\alpha| = \sqrt{2}|\beta| = 2$ and $\langle \beta^\vee, \alpha \rangle = 2\langle \alpha^\vee, \beta \rangle = -2$, so $\langle \alpha^\vee, \beta^\vee \rangle = \langle \alpha^\vee, \beta \rangle = -1 = \Re(\phi(\beta^\vee))$ and since $|\phi(\beta^\vee)|^2 = |\beta^\vee|^2 = 2$, we obtain $\phi(\beta^\vee) = \sqrt{2}e^{\frac{3i\pi}{4}}$. Finally, for $I_2(6) = G_2$, we have $|\alpha| = \sqrt{3}|\beta| = 2$ and $\langle \beta^\vee, \alpha \rangle = 3\langle \alpha^\vee, \beta \rangle = -3$, so $\langle \alpha^\vee, \beta^\vee \rangle = \frac{1}{2}\langle \beta^\vee, \alpha \rangle = -\frac{3}{2} = \Re(\phi(\beta^\vee))$ and $|\phi(\beta^\vee)|^2 = |\beta^\vee|^2 = 3$ and thus $\phi(\beta^\vee) = \sqrt{3}e^{\frac{5i\pi}{6}}$. □

Remark 5.5.4. *In Weierstrass forms, an equation for $\mathbf{T}(A_2)$ and $\mathbf{T}(G_2)$ is $y^2 = x^3 - 1$ and for $\mathbf{T}(B_2)$, we can take $y^2 = x^3 - x$. This is an unusual point of view on 2-dimensional tori. Indeed, they are first defined as Lie groups, hence as differentiable manifolds diffeomorphic to $(\mathbb{S}^1)^2$ and it turns out that they carry a natural rational elliptic curve structure. Moreover, they can be distinguished among complex elliptic curves by the fact that they correspond to the orbifold points of the Dirichlet domain.*

We now focus on the hyperbolic case where $g > 1$. We first notice the following coincidence between the Riemann surface $\mathbf{T}(I_2(m))$.

Proposition 5.5.5. *If $g > 1$, then we have an isometry (in particular, an isomorphism of Riemann surfaces)*

$$\mathbf{T}(I_2(4g+2)) \simeq \mathbf{T}(I_2(2g+1))$$

and these two are not isometric to the surface $\mathbf{T}(I_2(4g))$.

Proof. Using [Rat06, Theorem 8.1.5], it suffices to show that the groups Q_{2g+1} and Q_{4g+2} are conjugate in the positive Lorentz group $PO(1,2) \simeq \mathrm{Isom}(\mathbb{H}^2) \simeq PSL_2(\mathbb{R})$ and are not conjugate to Q_{4g}.

Let $m := 2g+1$. We first prove that Q_m and Q_{2m} are conjugate in $PO(1,2)$. Denote $I_2(2m) = \langle s, t \mid s^2 = t^2 = (st)^{2m} = 1 \rangle$ and $\widehat{I_2(2m)} = \langle s, t, \widehat{s}_0 \rangle$ its hyperbolic extension. Let $s' := s$, $t' := tst = s^t$ and $\widehat{s}'_0 := \widehat{s}_0$. Then $\langle s', t', \widehat{s}'_0 \rangle = \widehat{I_2(m)}$ and $\langle s', t' \rangle = I_2(m)$. Recall moreover that we have the reflection $r_{2m} = (st)^{2g}s = ((st)^2)^g s = (s't')^g s' = r_m$. Let α, β and γ denote the simple roots of $\widehat{I_2(2m)}$ and $V_{2m} := \mathrm{span}_{\mathbb{R}}(\alpha, \beta, \gamma)$. We have the representation

$$\widehat{I_2(2m)} \overset{\sigma_{2m}}{\hookrightarrow} O(V_{2m}, B_{2m}),$$

where

$$B_{2m} = \begin{pmatrix} 1 & -\cos(\pi/2m) & -\cos(\pi/m) \\ -\cos(\pi/2m) & 1 & 0 \\ -\cos(\pi/m) & 0 & 1 \end{pmatrix}.$$

In the same way, denote $V_m := \mathrm{span}_{\mathbb{R}}(\alpha', \beta', \gamma')$ and $\sigma_m : \widehat{I_2(m)} \hookrightarrow O(V_m, B_m)$, where

$$B_m = \begin{pmatrix} 1 & -\cos(\pi/m) & -\cos(\pi/m) \\ -\cos(\pi/m) & 1 & -\cos(\pi/m) \\ -\cos(\pi/m) & \cos(\pi/m) & 1 \end{pmatrix}.$$

Consider the linear map $P : V_{2m} \to V_m$ with matrix

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2\cos(\pi/2m) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have $B_m = {}^tPB_{2m}P$, so P induces an isomorphism $O(V_{2m}, B_{2m}) \to O(V_m, B_m)$ fitting in a commutative diagram

$$\begin{tikzcd}
\widehat{I_2(m)} \arrow[r, hook, "\sigma_m"] \arrow[d, hook] & O(V_m, B_m) \arrow[r, "\sim"] & PO(1,2) \\
\widehat{I_2(2m)} \arrow[r, hook, "\sigma_{2m}"'] & O(V_{2m}, B_{2m}) \arrow[r, "\sim"] \arrow[u, "\sim"] & PO(1,2) \arrow[u, "\sim"]
\end{tikzcd}$$

and thus the group $\sigma_m(\widehat{I_2(m)})$ is conjugate in $PO(1,2)$ to a subgroup of $\sigma_{2m}(\widehat{I_2(2m)})$. Therefore, identifying $\widehat{I_2(m)}$ with its image in $\widehat{I_2(2m)}$, it suffices to prove that $Q_{2m} = Q_m$. Recall that $q_{2m} \overset{\mathrm{df}}{=} \widehat{s}_0 r_{2m} = \widehat{s}_0 r_m = q_m$, so $q_m^{\widehat{I_2(m)}} \subset q_{2m}^{\widehat{I_2(2m)}}$ and thus $Q_m \leq Q_{2m}$. Since we have

$$\begin{aligned}
2m[\widehat{I_2(2m)} : \widehat{I_2(m)}] &= [\widehat{I_2(2m)} : \widehat{I_2(m)}][\widehat{I_2(m)} : Q_m] = [\widehat{I_2(2m)} : Q_m] \\
&= [\widehat{I_2(2m)} : Q_{2m}][Q_{2m} : Q_m] = 4m[Q_{2m} : Q_m],
\end{aligned}$$

we are left to show that $[\widehat{I_2(2m)} : \widehat{I_2(m)}] = 2$. Let $w \in \widehat{I_2(2m)}$. By induction on $\ell(w)$ and because t and $\widehat{s}_0$ commute, we immediately see that $w \in \widehat{I_2(m)}$ if and only if the number of occurrences of t in any reduced expression of w is even. Hence we have $[\widehat{I_2(2m)} : \widehat{I_2(m)}] \leq 2$ and since st and $\widehat{s}_0 t$ have even order, the map $\widehat{I_2(2m)} \to \mathbb{Z}/2\mathbb{Z}$ sending s and $\widehat{s}_0$ to 0 and t to 1 is a homomorphism whose kernel contains $\widehat{I_2(m)}$, hence the result.

We now prove that Q_{2g+1} and Q_{4g} are not conjugate in $PO(1,2)$. It is enough to prove that the elements $\sigma_{2g+1}(q_{2g+1}) \in PO(1,2)$ and $\sigma_{4g}(q_{4g})$ have different traces. Write $\widehat{I_2(2g+1)} = \langle s, t, \widehat{s}_0 \rangle$ and $\widehat{I_2(4g)} = \langle s', t', \widehat{s}'_0 \rangle$. We have $q_{2g+1} = \widehat{s}_0 (st)^g s$ and $q_{4g} = \widehat{s}'_0 (st)^{2g} s'$ and we can write explicitly the matrices of the simple reflections in the geometric representation. We diagonalize $st = PdP^{-1}$ and compute $\mathrm{tr}(q_{2g+1}) = \mathrm{tr}(d^g P^{-1} s \widehat{s}_0)$. After calculations, we find

$$\mathrm{tr}(q_{2g+1}) = 8\left(1 + \cos\left(\frac{\pi}{2g+1}\right)\right)\cot^2\left(\frac{\pi}{2g+1}\right) - 1.$$

Doing the same for g_{4g}, we find

$$\mathrm{tr}(q_{4g}) = 4\cot^2\left(\frac{\pi}{4g}\right) - 1.$$

And indeed, we get $\mathrm{tr}(q_{2g+1}) \neq \mathrm{tr}(q_{4g})$ for $g > 1$. $\square$

Recall that a *Belyi function* on a Riemann surface X is a holomorphic map $\beta : X \to \widehat{\mathbb{C}}$ which is ramified only over three points of $\widehat{\mathbb{C}}$. Since $\widehat{I_2(m)}$ is a compact triangle group and $Q_m \trianglelefteq \widehat{I_2(m)}$ is torsion-free and of finite index. Thus, by [JW16, Theorem 3.10], the projection

$$\beta : \mathbf{T}(I_2(m)) = \mathbb{H}^2/Q_m \twoheadrightarrow \mathbb{H}^2/\widehat{I_2(m)} \simeq \widehat{\mathbb{C}}$$

is a Belyi function on $\mathbf{T}(I_2(m))$ of degree $[\widehat{I_2(m)} : Q_m] = 2m$. Using [JW16, Theorem 1.3], this implies the following result:

Proposition 5.5.6. *For any $m \geq 3$, the Riemann surface $\mathbf{T}(I_2(m))$ may be defined over a number field (or equivalently, may be defined over $\overline{\mathbb{Q}}$). Moreover, if $m = 5$ or $m \geq 7$, then the 1-skeleton of the tessellation $\Delta(\widehat{I_2(m)})/Q_m$ defines a dessin d'enfant on $\mathbf{T}(I_2(m))$.*

Remark 5.5.7. *It is a reasonable to expect that $\mathbf{T}(I_2(m))$ is definable over $\mathbb{Q}(\cos(2\pi/m))$. This is coherent with the isomorphism $\mathbf{T}(I_2(2g+1)) \simeq \mathbf{T}(I_2(4g+2))$ and with the vertices of the tessellation of $\mathbb{H}^2$, whose coordinates may be chosen in this field. However, we haven't found a proof of this yet.*

This would give a geometric interpretation of the splitting field of a dihedral group.

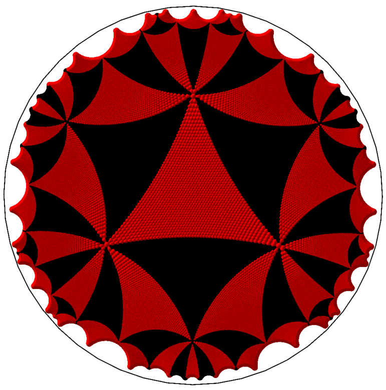
Example 5.5.8. *The triangulation $\Delta(\widehat{I_2(5)})$ is the classical tessellation $\{3,10\}$ of the Poincaré disk. More precisely, the Tits form $\widehat{B}$ is given by*

$$\widehat{B} = \begin{pmatrix} 1 & -c & -c \\ -c & 1 & -c \\ -c & -c & 1 \end{pmatrix} \quad \textit{with} \quad c = \cos(\pi/5)$$

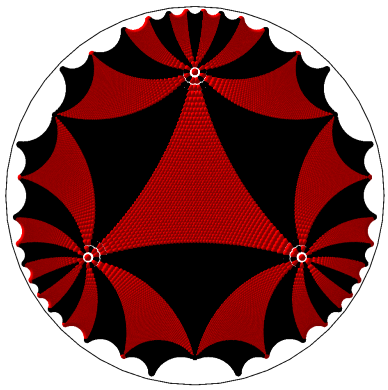
and, if $v^* \in V^*$ is a normalized eigenvector for the unique negative eigenvalue of $\widehat{B}$, then we have an identification with the hyperbolic plane

$$\mathcal{H} := \{\lambda \in V^* \ ; \ \widehat{B}(\lambda,\lambda) = -1, \ \widehat{B}(v^*,\lambda) < 0\} \simeq \mathbb{H}^2$$

and the stereographic projection on the hyperplane $\widehat{B}(v^*,-) = 0$ with pole λ_0 gives the Poincaré disk model for $\mathbb{H}^2$. Under this representation we represent the tessellation $\Delta(\widehat{I_2(5)}) = \{3,10\}$ of $\mathcal{H}$ as in Figure 4a, where the black triangles are the images of the fundamental triangle $\overline{C}/\mathbb{R}_+^* \simeq \overline{C} \cap \mathcal{H}$ under elements of odd length.



(a) The $\{3,10\}$-tessellation of $\Sigma(\widehat{I_2(5)}) \simeq \mathbb{H}^2$.



(b) The $\{3,14\}$-tessellation of $\Sigma(\widehat{I_2(7)}) \simeq \mathbb{H}^2$.

Figure 4: Two regular tessellations of the Poincaré disk.

In this tessellation, we can identify the triangles that are in the Q-orbit of $\overline{C} \cap \mathcal{H}$. These are displayed in green in Figure 5. Collapsing these triangles in one gives the surface $\mathbf{T}(I_2(5))$. We remark that we can extract a fundamental domain for Q on $\mathbf{T}(I_2(5))$ as the projection of the domain displayed in Figure 6a. Rearranging the figure we obtain the triangulation displayed in the Figure 6b, where the points with the same name (resp. the edges with the same color) are identified. We notice that the resulting space is indeed a closed surface of genus 2. The case of $I_2(m)$ for m odd is pretty similar and we obtain the $\{3,2m\}$-tessellation of the Poincaré disk. For instance, the Figure 4b shows the case of $I_2(7)$.

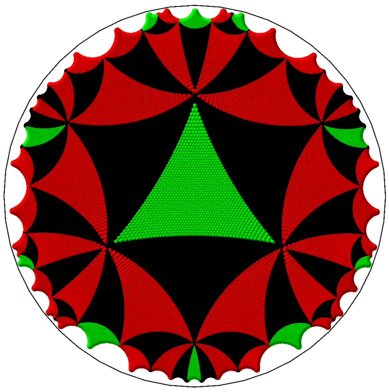


Figure 5: The green triangles form the Q-orbit of the fundamental triangle $\overline{C} \cap \mathcal{H}$ inside the Poincaré disk.

(a) Fundamental domain for Q in the Poincaré disk.

(b) Fundamental domain for $I_2(5)$ in $\mathbf{T}(I_2(5)) = \mathbf{T}_2$.

Figure 6: Fundamental domain for Q and its image in $\mathbf{T}(I_2(5))$.

6 Equivariant chain complex of $\mathbf{T}(W)$ and computation of homology

6.1 The W-dg-ring of $\mathbf{T}(W)$

The combinatorics of the complex $C_*^{\text{cell}}(\mathbf{T}(W), W; \mathbb{Z})$ is fairly similar to the one of the complex $C_*^{\text{cell}}(T, W; \mathbb{Z})$ we constructed in the first part and the proofs given above can be applied *verbatim* to this new situation. We obtain the following results:

Theorem 6.1.1. *The quotient simplicial complex $\Delta(\widehat{W}, \widehat{S})/Q$ is a regular W-equivariant triangulation of the manifold $\mathbf{T}(W)$. Recalling the projection $\pi : \widehat{W} \twoheadrightarrow W$, the resulting homology $\mathbb{Z}[W]$-chain complex $C_*^{\text{cell}}(\mathbf{T}(W), W; \mathbb{Z})$ is given (in homogeneous degrees k and $k-1$) by*

$$\cdots \longrightarrow \bigoplus_{\substack{I \subset \widehat{S} \\ |I| = n-k}} \mathbb{Z}[\pi(\widehat{W}^I)] \xrightarrow{\partial_k} \bigoplus_{\substack{I \subset \widehat{S} \\ |I| = n-k+1}} \mathbb{Z}[\pi(\widehat{W}^I)] \longrightarrow \cdots$$

with boundaries defined as follows: for $k \in \mathbb{N}$ and $I \subset \widehat{S}$, letting $\{j_1 w \cdots < j_{k+1}\} := \widehat{S} \setminus I$

we have

$$(\partial_k)_{|\mathbb{Z}[\pi(\widehat{W}^I)]} = \sum_{u=1}^{k+1} (-1)^u p^I_{I\cup\{j_u\}},$$

where, for $J \subset I$, p^J_I denotes the projection

$$p^J_I : \pi(\widehat{W}^J) = \pi(\widehat{W}/\widehat{W}_J) \twoheadrightarrow \pi(\widehat{W}/\widehat{W}_I) = \pi(\widehat{W}^I).$$

Corollary 6.1.2. *The W-dg-ring $C^*_{\text{cell}}(\mathbf{T}(W), W; \mathbb{Z})$ associated to the W-triangulation $\Delta(\widehat{W}, \widehat{S})/Q$ of $\mathbf{T}(W)$ has homogeneous components*

$$C^k_{\text{cell}}(\mathbf{T}(W), W; \mathbb{Z}) = \bigoplus_{\substack{I\subset\widehat{S}\\ |I|=n-k}} \mathbb{Z}[\pi({}^I\widehat{W})] \simeq \bigoplus_{\substack{I\subset\widehat{S}\\ |I|=n-k}} \mathbb{Z}[\pi(\widehat{W}_I)\backslash\widehat{W}],$$

differentials given, for any $I \subset \widehat{S}$ and $w \in \widehat{W}$, by

$$d^k(\pi({}^I w)) = \sum_{\substack{0\le u\le k+1\\ j_{u-1}<j<j_u}} (-1)^u \pi\left(\epsilon_I^{I\backslash\{j\}} w\right), \ \epsilon^J_I = \sum_{x\in {}^J_I\widehat{W}} x,$$

where $\{j_0 < \cdots < j_k\} := \widehat{S} \setminus I$. Its product

$$C^p_{\text{cell}}(\mathbf{T}(W), W; \mathbb{Z}) \otimes_{\mathbb{Z}} C^q_{\text{cell}}(\mathbf{T}(W), W; \mathbb{Z}) \xrightarrow{\cup} C^{p+q}_{\text{cell}}(\mathbf{T}(W), W; \mathbb{Z})$$

is induced by the deflation from $\widehat{W}$ to W of the unique map

$$\mathbb{Z}[[{}^I\widehat{W}]] \otimes_{\mathbb{Z}} \mathbb{Z}[[{}^J\widehat{W}]] \longrightarrow \mathbb{Z}[[{}^{I\cap J}\widehat{W}]]$$

satisfying the formula

$$\forall x, y \in \widehat{W}, \ {}^I x \cup {}^J y = \delta_{\max(I^{\complement}),\min(J^{\complement})} \times \begin{cases} {}^{I\cap J}((xy^{-1})_J y) & \text{if } xy^{-1} \in \widehat{W}_I\widehat{W}_J \\ 0 & \text{otherwise.} \end{cases}$$

Remark 6.1.3. *We make several observations on the previous results.*

- *As explained in [BR04, §2.3], a quotient simplicial complex of the form $\Delta(\widehat{W}, \widehat{S})/H$ (with $H \le \widehat{W}$) has a an interpretation in terms of double cosets. In our case, we have an isomorphism of posets*

$$\begin{array}{ccc} \left(\mathcal{P}(\Delta(\widehat{W}, \widehat{S})/Q), \subseteq\right) & \xrightarrow{\sim} & \left(\{(I, Qw\widehat{W}_I)\}_{I\subsetneq\widehat{S},\ w\in\widehat{W}}, \preceq\right) \\ \pi(\widehat{W}^I) \ni \pi(w^I) & \longmapsto & (I, Qw\widehat{W}_I) \end{array}$$

  where the order $\preceq$ on the second factor is defined by

$$(I, Qw\widehat{W}_I) \preceq (J, Qw'\widehat{W}_J) \overset{df}{\iff} \begin{cases} I \supseteq J \\ Qw\widehat{W}_I \supseteq Qw'\widehat{W}_J \end{cases}$$

  and we may rephrase the above results using this poset.

- *As the subgroup Q is torsion-free, it contains no reflection and we have $Q \le \ker(\varepsilon)$, so by the general result [Rei92, Proposition 2.4.2], the quotient $\mathbf{T}(W)$ is an orientable pseudomanifold. The Theorem 5.3.3 can be seen as a refinement of this result in our particular setting.*

- *The triangulation* $\Delta(\widehat{H_3})/Q$ *(resp.* $\Delta(\widehat{H_4})/Q$*) has f-vector* $(4, 124, 240, 120)$ *(resp.* $(266, 7920, 29280, 36000, 14400)$*). In particular, the Euler characteristics are given by*

$$\chi(\mathbf{T}(H_3)) = 0 \quad \textit{and} \quad \chi(\mathbf{T}(H_4)) = 26.$$

We notice that the f-vector of our triangulation of $\mathbf{T}(H_4)$ *is far bigger than the one found in [RT01, §3], which is* $(1, 60, 144, 60, 1)$ *but of course, this last one doesn't correspond to an equivariant triangulation. Notice finally that the general Gauss-Bonnet formula gives* $\mathrm{Vol}(\mathbf{T}(H_4)) = \frac{4\pi^2}{3}\chi(\mathbf{T}(H_4)) = 104\pi^2/3 \approx 342.15$.

6.2 The homology W-representation of $\mathbf{T}(W)$

We can now determine the action of W on $H_*(\mathbf{T}(W), \mathbb{Z})$. In fact, as for the classical tori, we will show that there is no torsion in $H_*(\mathbf{T}(W), \mathbb{Z})$ but we shall decompose the representations $H_*(\mathbf{T}(W), \mathbb{Q}(W))$ over a splitting field $\mathbb{Q}(W)$ of W, which is bigger than $\mathbb{Q}$ in the non-crystallographic cases.

Recall from [GP00, Theorem 5.3.8] that a splitting field for W is given by

$$\mathbb{Q}(W) = \mathbb{Q}(\cos(2\pi/m_{s,t}),\ s,t \in S) = \mathbb{Q}(\chi_\sigma(w),\ w \in W) \subset \mathbb{R},$$

where $\chi_\sigma = \mathrm{tr}(\sigma)$ is the character of the geometric representation $\sigma : W \to GL(V)$ of W. If W is a Weyl group, then $\mathbb{Q}(W) = \mathbb{Q}$ and we have

$$\mathbb{Q}(I_2(m)) = \mathbb{Q}(\cos(2\pi/m)) \quad \text{and} \quad \mathbb{Q}(H_3) = \mathbb{Q}(H_4) = \mathbb{Q}(\sqrt{5}).$$

We suppose from now on that W is one of the groups H_3, H_4 or $I_2(m)$, with $m \geq 3$ and we keep the notation of the previous section. The first groups to be determined are the top and bottom homology of $\mathbf{T}(W)$. Recall that we have $n = \mathrm{rk}(W) = \dim \mathbf{T}(W)$.

Proposition 6.2.1. *Let* $\mathbb{1}$ *and* ε *be the trivial and signature modules over* $\mathbb{Z}[W]$*, respectively. We have isomorphisms of* $\mathbb{Z}[W]$*-modules*

$$\begin{cases} H_0(\mathbf{T}(W), \mathbb{Z}) \simeq \mathbb{1}, \\ H_n(\mathbf{T}(W), \mathbb{Z}) \simeq \varepsilon. \end{cases}$$

Proof. Since $\widehat{\Sigma}$ is path-connected, its quotient $\mathbf{T}(W)$ is path-connected too and is orientable by Theorem 5.3.3. Thus, we have an isomorphism of abelian groups

$$H_0(\mathbf{T}(W), \mathbb{Z}) \simeq \mathbb{Z} \simeq H_n(\mathbf{T}(W), \mathbb{Z}).$$

It is clear that $H_0(\mathbf{T}(W), \mathbb{Z})$ is the trivial module and, as $\mathbb{Z}[W]$-modules we have $H_n(\mathbf{T}(W), \mathbb{Z}) = \ker(\partial_n)$ with

$$\begin{array}{rccc} \partial_n & : & \mathbb{Z}[W] & \longrightarrow & \bigoplus_{i=0}^n \mathbb{Z}[W/\langle s_i \rangle] \\ & & w & \longmapsto & \sum_{i=0}^n (-1)^i w \langle s_i \rangle \end{array}$$

where $s_i = \pi(\widehat{s}_i)$ is a simple reflection of W for $i \geq 1$ and $s_0 := r_W = \pi(\widehat{s}_0)$. Define $e := \sum_w \varepsilon(w) w \in \mathbb{Z}[W]$ with $\varepsilon(w) = (-1)^{\ell(w)}$ and notice that $we = \varepsilon(w)e$ for $w \in W$ and $\partial_n(e) = 0$. Let $x = \sum_w x_w w \in \mathbb{Z}[W]$ such that $\partial_n(x) = 0$. Then, for all $0 \leq i \leq n$, we have

$\sum_w x_w w \langle s_i \rangle = 0$. Fixing $1 \le i \le n$, we can choose a set $\{w_1, \dots, w_k\}$ of representatives of the left coset $W/\langle s_i \rangle$ (the minimal length representatives for instance). We have

$$0 = \sum_{w \in W} x_w w \langle s_i \rangle = \sum_{j=1}^{k} x_{w_j} w_j \langle s_i \rangle + \sum_{j=1}^{k} x_{w_j s_i} w_j s_i \langle s_i \rangle = \sum_j (x_{w_j} + x_{w_j s_i}) w_j \langle s_i \rangle ,$$

hence $x_{w_j} + x_{w_j s_i} = 0$ for all $1 \le j \le k$. This implies $x_w + x_{ws_i} = 0$ for all $w \in W$ and doing this for every $i \ge 1$ gives $x_w + x_{ws} = 0$ for all $w \in W$ and $s \in S$, in other words, $x_w = \varepsilon(w) x_1$ for $w \in W$ and $x = x_1 e \in \mathbb{Z}e$. $\square$

Proposition 6.2.2. *The homology $H_*(\mathbf{T}(W), \mathbb{Z})$ is torsion-free and the Poincaré duality on $\mathbf{T}(W)$ induces isomorphisms of $\mathbb{Z}[W]$-modules*

$$H_{n-i}(\mathbf{T}(W), \mathbb{Z})^\vee \simeq H_i(\mathbf{T}(W), \mathbb{Z})^\vee \otimes_{\mathbb{Z}} \varepsilon.$$

Proof. It suffices to prove that $H_*(\mathbf{T}(W), \mathbb{Z})$ is torsion-free, the Poincaré pairing $H^i(\mathbf{T}(W), \mathbb{Z}) \otimes H_{n-i}(\mathbf{T}(W), \mathbb{Z}) \to H_n(\mathbf{T}(W), \mathbb{Z}) = \varepsilon$ and the universal coefficient theorem implying the second one.

For simplicity, if A is an abelian group, we denote by $\mathrm{Tors}(A)$ its torsion subgroup.

We also denote $H_i := H_i(\mathbf{T}(W), \mathbb{Z})$ and $H^i := H^i(\mathbf{T}(W), \mathbb{Z})$. Again using Poincaré duality and the universal coefficients theorem, we get

$$\mathrm{Tors}(H^{n-i}) = \mathrm{Tors}(H_i) = \mathrm{Tors}(H^{i+1}).$$

Since $n \le 4$, it remains to show that $\mathrm{Tors}(H_1) = 0$. This is always true for an orientable surface and still holds for H_3 and H_4, by Corollary 5.4.5. $\square$

The above Lemma, combined with the Hopf trace formula (see below) provides enough information to determine the homology representation of $\mathbf{T}(W)$.

Lemma 6.2.3 (Hopf trace formula, [Spa81, Chap. 4, §7, Theorem 6] or [Lin19, Lemma 2.4])**.** *Let $\Bbbk$ be a field, $C_\bullet$ be a bounded chain complex of finite dimensional $\Bbbk$-vector spaces and $f \in \mathrm{End}_\Bbbk(C_\bullet)$ be an endomorphism of $C_\bullet$. If $H_*(f) \in \mathrm{End}_\Bbbk(H_*(C))$ denotes the induced endomorphism in homology, then we have the formula*

$$\sum_i (-1)^i \mathrm{tr}_\Bbbk(f_i) = \sum_i (-1)^i \mathrm{tr}_\Bbbk(H_i(f)).$$

This can be readily applied to our situation to obtain the following formula. As a reminder, if G is a (discrete) group, $H \le G$ is a subgroup and if M is an H-module, we denote by $M{\uparrow}_H^G$ the *induced module* of M; it is a G-module. Similarly, the *restricted module* of a G-module N is denoted $N{\downarrow}_H^G$. Observe that we have a canonical isomorphism of $\mathbb{Q}[G]$-modules $\mathbb{Q}[G/H] \simeq \mathbb{1}{\uparrow}_H^G$. Recall also that if $N \trianglelefteq G$ and if M is a G-module, then its *deflation* $\mathrm{Def}^G_{G/N}(M)$ is a G/N-module.

In our context, we have isomorphisms of $\mathbb{Q}[\widehat{W}]$-modules

$$C_k^{\mathrm{cell}}(\widehat{\Sigma}, \widehat{W}; \mathbb{Q}) = \bigoplus_{I \subset \widehat{S}\ ;\ |I| = n-k} \mathbb{1}{\uparrow}_{\widehat{W}_I}^{\widehat{W}}.$$

Thus
$$C_k^{\mathrm{cell}}(\mathbf{T}(W), W; \mathbb{Q}) = \mathrm{Def}_W^{\widehat{W}}(C_k^{\mathrm{cell}}(\widehat{\Sigma}, \widehat{W}; \mathbb{Q})) = \bigoplus_{I \subset \widehat{S}\ ;\ |I| = n-k} \mathbb{1}\uparrow_{\pi(\widehat{W}_I)}^{W},$$

and we obtain the following result:

Lemma 6.2.4. *We have the following equality of virtual rational characters of W*

$$\sum_{I \subsetneq \widehat{S}} (-1)^{|I|} \mathrm{Def}_W^{\widehat{W}} \left(\mathbb{1}\uparrow_{\widehat{W}_I}^{\widehat{W}} \right) = \sum_{I \subsetneq \widehat{S}} (-1)^{|I|} \mathbb{1}\uparrow_{\pi(\widehat{W}_I)}^{W} = (-1)^n \sum_{i=0}^{n} (-1)^i H_i(\mathbf{T}(W), \mathbb{Q}).$$

For notation simplicity, we shall use the conventions of [GP00] to denote the irreducible characters of W. We start with $I_2(m)$.

Theorem 6.2.5. *Let $m \geq 3$. Following [GP00, §5.3.4], for $1 \leq j \leq \lfloor (m-1)/2 \rfloor$, we consider the following representation of $I_2(m) = \langle s, t \mid s^2 = t^2 = (st)^m = 1 \rangle$*

$$\widetilde{\rho_j} : I_2(m) \to GL_2(\mathbb{R}) \text{ defined by } \widetilde{\rho_j}(s) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \widetilde{\rho_j}(st) := \begin{pmatrix} \cos(j\theta_m) & -\sin(j\theta_m) \\ \sin(j\theta_m) & \cos(j\theta_m) \end{pmatrix},$$

where $\theta_m := 2\pi/m$ and we let ρ_j be a realization of $\widetilde{\rho_j}$ on the splitting field $\mathbb{Q}(\theta_m)$ of $I_2(m)$.

Then, the first homology representation of $\mathbf{T}(I_2(m))$ is given by

$$H_1(\mathbf{T}(I_2(m)), \mathbb{Q}(\theta_m)) = \begin{cases} \displaystyle\bigoplus_{1 \leq j \leq (m-1)/2} \rho_j & \text{if } m \text{ is odd,} \\ \displaystyle\bigoplus_{\substack{1 \leq j \leq m/2-1 \\ j \text{ odd}}} \rho_j & \text{if } m \text{ is even.} \end{cases}$$

Recall also that $H_0(\mathbf{T}(I_2(m)), \mathbb{Q}) = \mathbb{1}$ and $H_2(\mathbf{T}(I_2(m)), \mathbb{Q}) = \varepsilon$.

Proof. We already have obtained the last statement above in Proposition 6.2.1. For the first homology, we let $\chi_j := \mathrm{tr}(\rho_j)$ be the character of ρ_j and, denoting by $\mathrm{Reg}_{\mathbb{Q}(\theta_m)} = \mathbb{Q}(\theta_m)[I_2(m)]$ the regular module, lemma 6.2.4 yields the following equality of virtual characters of $I_2(m)$

$$H_1(\mathbf{T}(I_2(m)), \mathbb{Q}(\theta_m)) = \mathbb{1} + \varepsilon - \mathrm{Reg}_{\mathbb{Q}(\theta_m)} - \sum_{\emptyset \neq I \subsetneq \{\widehat{s}_0, s, t\}} (-1)^{|I|} \mathbb{1}\uparrow_{\pi(\widehat{I_2(m)}_I)}^{I_2(m)}.$$

We deal with each case separately. Recall the computations of the images in $I_2(m)$ of the parabolic subgroups of $\widehat{I_2(m)}$ from the proof of the Corollary 5.5.1.

- $m = 2k+1$ is odd. We have $r := r_W = (st)^k s$. Hence $s^r = t$ and $t^r = s$ so $I_2(m) = \langle s, r \rangle = \langle t, r \rangle$. Furthermore, in this case we have (cf [GP00, §5.3.4]) $\mathrm{Reg}_{\mathbb{Q}(\theta_m)} = \mathbb{1} + \varepsilon + \sum_j 2\chi_j$ and the above formula reduces to

$$H_1(\mathbf{T}(I_2(m)), \mathbb{Q}(\theta_m)) = 3 \cdot \mathbb{1}\uparrow_{\langle s \rangle}^{I_2(m)} - 3 \cdot \mathbb{1} - \sum_j 2\chi_j.$$

Now, by [GP00, §6.3.5] we have $\mathbb{1}\uparrow_{\langle s\rangle}^{I_2(m)} = \mathbb{1} + \sum_j \chi_j$ and thus

$$H_1(\mathbf{T}(I_2(m)), \mathbb{Q}(\theta_m)) = \sum_j \chi_j.$$

- $m = 4k$. In this case we have $r = (st)^{2k-1}s$ and let $a := st$. The conjugacy classes of $I_2(m)$ are given as follows

Representative	1	a	a^2	$\cdots$	a^{2k}	a^{2k+1}	s	t
Cardinality	1	2	2	$\cdots$	2	1	$2k+1$	$2k+1$

First, we determine the characters $\mathbb{1}\uparrow_{\langle x,r\rangle}^{I_2(m)}$ for $x = s, t$. In the proof of 5.5.1 we have seen that $t = (sr)^{2k-1}s$, so $\langle s, r\rangle = I_2(m)$. Next, as detailed in [GP00, §5.3.4], the character χ_j is given by

$$\chi_j(a^i) = 2\cos\left(\frac{2\pi ij}{m}\right) \quad \text{and} \quad \chi_j(sa^i) = 0.$$

We have $\langle t, r\rangle = \{1, t, r, a^{2k}\} \simeq C_2 \times C_2$ and by Frobenius reciprocity

$$\forall j,\ \left(\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)}, \chi_j\right)_W = \left(\mathbb{1}, \chi_j\downarrow_{\langle t,r\rangle}^{I_2(m)}\right)_{\langle t,r\rangle} = \frac{\chi_j(1) + \chi_j(t) + \chi_j(r) + \chi_j(a^{2k})}{4}$$

$$= \frac{\chi_j(1) + 2\chi_j(t) + \chi_j(a^{2k})}{4} = \frac{1 + \cos(\pi j)}{2} = \begin{cases} 1 & \text{if } j \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

The 1-dimensional irreducible representations of $I_2(m)$ other that $\mathbb{1}$ and ε are given by $\varepsilon_s(s) = \varepsilon_t(t) = 1$ and $\varepsilon_s(t) = \varepsilon_t(s) = -1$. Therefore, $\mathrm{Reg}_{\mathbb{Q}(\theta_m)} = \mathbb{1}+\varepsilon+\varepsilon_s+\varepsilon_t+\sum_j 2\chi_j$. We directly compute using Frobenius reciprocity

$$\left(\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)}, \varepsilon_s\right)_{I_2(m)} = \left(\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)}, \varepsilon\right)_{I_2(m)} = 0$$

and

$$\left(\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)}, \varepsilon_t\right)_{I_2(m)} = \left(\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)}, \mathbb{1}\right)_{I_2(m)} = 1$$

and hence

$$\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)} = \mathbb{1} + \varepsilon_t + \sum_{j \text{ even}} \chi_j.$$

On the other hand, by [GP00, §6.3.5], we have $\mathbb{1}\uparrow_{\langle s\rangle}^{I_2(m)} = \mathbb{1}+\varepsilon_s+\sum_j \chi_j$ and $\mathbb{1}\uparrow_{\langle t\rangle}^{I_2(m)} = \mathbb{1}+\varepsilon_t+\sum_j \chi_j$. Putting everything together and remembering that t and r are conjugate yields

$$H_1(\mathbf{T}(I_2(m)), \mathbb{Q}(\theta_m)) = \mathbb{1} + \varepsilon - \mathrm{Reg}_{\mathbb{Q}(\theta_m)} - \sum_{\emptyset \neq I \subsetneq \{\widehat{s}_0, s, t\}} (-1)^{|I|}\mathbb{1}\uparrow_{\pi(\widehat{I_2(m)}_I)}^{I_2(m)}$$

$$= \mathbb{1} + \varepsilon - \mathrm{Reg}_{\mathbb{Q}(\theta_m)} + \mathbb{1}\uparrow_{\langle s\rangle}^{I_2(m)} + 2\cdot\mathbb{1}\uparrow_{\langle t\rangle}^{I_2(m)} - \mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)} - 2\cdot\mathbb{1} = \sum_{j \text{ odd}} \chi_j.$$

- $m = 4k+2$. We proceed in the same way, noticing that $r = (st)^{2k}s = a^{2k}s$. The characters $\mathbb{1}\uparrow_{\langle s\rangle}^{I_2(m)}$ and $\mathbb{1}\uparrow_{\langle t\rangle}^{I_2(m)}$ are determined as above. We compute

$$\left(\mathbb{1}\uparrow_{\langle s,r\rangle}^{I_2(m)}, \varepsilon_s\right)_{I_2(m)} = \left(\mathbb{1}, \varepsilon_s\downarrow_{\langle s,r\rangle}^{I_2(m)}\right)_{\langle s,r\rangle} = 1 = \left(\mathbb{1}\uparrow_{\langle s,r\rangle}^{I_2(m)}, \mathbb{1}\right)_{I_2(m)}$$

but since $\deg(\mathbb{1}\uparrow_{\langle s,r\rangle}^{I_2(m)}) = [I_2(m) : \langle s,r\rangle] = [I_2(m) : \langle s, a^{2k}\rangle] = 2$ this implies $\mathbb{1}\uparrow_{\langle s,r\rangle}^{I_2(m)} = \mathbb{1} + \varepsilon_s$. Now, we have $\langle t,r\rangle = \{1,t,r,a^{2k+1}\} \simeq C_2 \times C_2$ and using again the Frobenius reciprocity we obtain

$$\left(\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)}, \chi_j\right)_{I_2(m)} = \left(\mathbb{1}, \chi_j\downarrow_{\langle t,r\rangle}^{I_2(m)}\right)_{\langle t,r\rangle} = \frac{\chi_j(1)+\chi_j(t)+\chi_j(r)+\chi_j(a^{2k+1})}{4}$$

$$= \frac{\chi_j(1)+\chi_j(a^{2k+1})}{4} = \frac{1+\cos(\pi j)}{2}.$$

Since $\mathbb{1}\downarrow_{\langle t,r\rangle}^{I_2(m)} \neq \mathbb{1} \neq \mathbb{1}\downarrow_{\langle t,r\rangle}^{I_2(m)}$ we also get

$$\left(\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)}, \varepsilon_s\right)_{I_2(m)} = \left(\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)}, \varepsilon_t\right)_{I_2(m)} = \left(\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)}, \varepsilon\right)_{I_2(m)} = 0$$

and

$$\left(\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)}, \mathbb{1}\right)_{I_2(m)} = 1.$$

Finally,

$$\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)} = \mathbb{1} + \sum_{j \text{ even}} \chi_j.$$

As above, we conclude that

$$H_1(\mathbf{T}(I_2(m)), \mathbb{Q}(\theta_m)) = \varepsilon - \mathrm{Reg}_{\mathbb{Q}(\theta_m)} + 2\cdot\mathbb{1}\uparrow_{\langle s\rangle}^{I_2(m)} + \mathbb{1}\uparrow_{\langle t\rangle}^{I_2(m)} - \mathbb{1}\uparrow_{\langle s,r\rangle}^{I_2(m)} - \mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)} = \sum_{j \text{ odd}} \chi_j,$$

as claimed.

□

Theorem 6.2.6. *With the notation of [GP00, Appendix C, Table C.1], we have*

$$\forall 0 \leq i \leq 3,\ H_i(\mathbf{T}(H_3), \mathbb{Q}(\sqrt{5})) = \begin{cases} \mathbb{1} & \text{if } i = 0, \\ 3'_s \oplus \overline{3'_s} \oplus 5_r & \text{if } i = 1, \\ 3_s \oplus \overline{3_s} \oplus 5'_r & \text{if } i = 2, \\ \varepsilon & \text{if } i = 3. \end{cases}$$

Proof. Consider the virtual character $\chi_H := \sum_{I \subsetneq \widehat{S}} (-1)^{|I|+1} \mathbb{1}\uparrow_{\pi(\widehat{H_3}_I)}^{H_3}$. For $\chi \in \mathrm{Irr}(H_3)$, we may compute

$$(\chi, \chi)_{H_3} = \sum_{I \subsetneq \widehat{S}} (-1)^{|I|+1} \left(\mathbb{1}\uparrow_{\pi(\widehat{H_3}_I)}^{H_3}, \chi_H\right)_{H_3}$$

$$= \sum_{I \subsetneq \widehat{S}} (-1)^{|I|+1} \left(\mathbb{1}, \chi\downarrow_{\pi(\widehat{H_3}_I)}^{H_3}\right)_{\pi(\widehat{H_3}_I)} = \sum_{\substack{I \subsetneq \widehat{S} \\ w \in \pi(\widehat{H_3}_I)}} (-1)^{|I|+1} \chi(w).$$

We obtain

$$\chi_H = \varepsilon - \mathbb{1} - 3_s - \overline{3_s} + 3'_s + \overline{3'_s} + 5_r - 5'_r$$

and therefore, using lemma 6.2.4

$$H_2(\mathbf{T}(H_3)) - H_1(\mathbf{T}(H_3)) = -3_s - \overline{3_s} + 3'_s + \overline{3'_s} + 5_r - 5'_r.$$

But from Lemma 5.4.5, we have $\dim(H_1(\mathbf{T}(H_3))) = \dim(H_2(\mathbf{T}(H_3))) = 11 = \dim(3_s + \overline{3_s} + 5'_r)$, so

$H_1(\mathbf{T}(H_3), \mathbb{Q}(\sqrt{5})) = 3'_s + \overline{3'_s} + 5_r$ and $H_2(\mathbf{T}(H_3), \mathbb{Q}(\sqrt{5})) = 3_s + \overline{3_s} + 5'_r = H_1(\mathbf{T}(H_3), \mathbb{Q}(\sqrt{5})) \otimes \varepsilon$.

□

Finally we treat the case of H_4.

Theorem 6.2.7. *With the notation of [GP00, Appendix C, Table C.2], we have*

$$\forall 0 \leq i \leq 4,\ H_i(\mathbf{T}(H_4), \mathbb{Q}(\sqrt{5})) = \begin{cases} \mathbb{1} & \text{if } i = 0, \\ 4_t \oplus \overline{4_t} \oplus 16'_r & \text{if } i = 1, \\ 6_s \oplus \overline{6_s} \oplus 30_s \oplus \overline{30_s} & \text{if } i = 2, \\ 4'_t \oplus \overline{4'_t} \oplus 16_r & \text{if } i = 3, \\ \varepsilon & \text{if } i = 4. \end{cases}$$

Proof. As for the previous proof, we let $\chi_H := \sum_{I \subsetneq \widehat{S}} (-1)^{|I|} \mathbb{1}\uparrow^{H_4}_{\pi(\widehat{H_4}_I)}$ and

$$\forall \chi \in \mathrm{Irr}(H_4),\ (\chi, \chi_H)_{H_4} = \sum_{I \subsetneq \widehat{S},\ w \in \pi(\widehat{H_4}_I)} (-1)^{|I|} \chi(w).$$

This leads to

$$\chi_H = \mathbb{1} + \varepsilon - 4_t - \overline{4_t} - 4'_t - \overline{4'_t} + 6_s + \overline{6_s} - 16_r - 16'_r + 30_s + \overline{30_s}.$$

Since $\dim(H_1(\mathbf{T}(H_4))) = \dim(H_3(\mathbf{T}(H_4))) = 24$ we obtain

$$H_2(\mathbf{T}(H_4), \mathbb{Q}(\sqrt{5})) = 30_s + \overline{30_s} + 6_s + \overline{6_s}$$

and

$$H_1(\mathbf{T}(H_4)) + H_3(\mathbf{T}(H_4)) = 4_t + 4'_t + \overline{4_t} + \overline{4'_t} + 16_r + 16'_r.$$

But since the representations $H_1(\mathbf{T}(H_4))$ and $H_3(\mathbf{T}(H_4))$ must be realizable over $\mathbb{Q}$ and because of the Poincaré duality pairing between the two, we are left with the following four possibilities:

$H_1(\mathbf{T}(H_4))$	$4_t + \overline{4_t} + 16_r$	$4'_t + \overline{4'_t} + 16_r$	$4_t + \overline{4_t} + 16'_r$	$4'_t + \overline{4'_t} + 16'_r$
$H_3(\mathbf{T}(H_4))$	$4'_t + \overline{4'_t} + 16'_r$	$4_t + \overline{4_t} + 16'_r$	$4'_t + \overline{4'_t} + 16_r$	$4_t + \overline{4_t} + 16_r$

However, the $\mathbb{Q}[H_4]$-module $H_1(\mathbf{T}(H_4), \mathbb{Q})$ is a sub-quotient of the module

$$C_1^{\mathrm{cell}}(\mathbf{T}(H_4), H_4; \mathbb{Q}) = \sum_{\substack{I \subsetneq \widehat{S} \\ |I|=3}} \mathbb{1}\uparrow^{H_4}_{\pi(\widehat{H_4}_I)}$$

and we compute

$$\left(C_1^{\mathrm{cell}}(\mathbf{T}(H_4)), 16_r\right)_{H_4} = 0 \implies (H_1(\mathbf{T}(H_4)), 16_r)_{H_4} = 0.$$

Hence, only $16'_r$ can be a direct factor of $H_1(\mathbf{T}(H_4), \mathbb{Q}(\sqrt{5}))$. In the same fashion we compute

$$\left(C_1^{\mathrm{cell}}(\mathbf{T}(H_4)), 4'_t\right)_{H_4} = 0 \implies \left(H_1(\mathbf{T}(H_4)), 4'_t\right)_{H_4} = 0$$

and thus only the third column of the table above is possible. □

Remark 6.2.8. *In [RT01, §3] and [Mar15, §2.2], the homology of $\mathbf{T}(H_4)$ is also described, but only as a $\mathbb{Z}$-module.*

Finally, we exhibit another algebraic meaning of the Euler characteristic of $\mathbf{T}(W)$. The Poincaré series of $\widehat{I_2(m)}$, $\widehat{H_3}$ and $\widehat{H_4}$ can be found in [CLS10, §3.1, Table 7.4 and Table 7.5]. Using these expressions, we immediately obtain the following corollary:

Corollary 6.2.9. *Let W be a finite irreducible Coxeter group. If $W(q)$ (resp. $\widehat{W}(q)$) denotes the Poincaré series of W (resp. of its extension $\widehat{W}$), then the Euler characteristic of $\mathbf{T}(W)$ is given by*

$$\chi(\mathbf{T}(W)) = \left.\frac{W(q)}{\widehat{W}(q)}\right|_{q=1}$$

Moreover, the geometric representation σ of W is always a direct summand of $H_1(\mathbf{T}(W), \mathbb{Q}(W))$ for every W and the two are equal if and only if W is crystallographic. In particular

$$\bigoplus_{\alpha \in \mathrm{Gal}(\mathbb{Q}(W)/\mathbb{Q})} \sigma^{\alpha} \text{ is a direct summand of } H_1(\mathbf{T}(W), \mathbb{Q}).$$

Remark 6.2.10. *With [CLS10] it can be seen that the quotient $W(q)/\widehat{W}(q)$ is a polynomial in q, but we cannot hope for a generalization of the Bott factorization theorem [Hil82, Theorem 6.3] as in the affine case, i.e. a formula of the form*

$$\frac{W(q)}{W_{\mathrm{a}}(q)} = \prod_{i=1}^{n} 1 - q^{d_i - 1},$$

with $\{d_i\}$ the degrees of W. Indeed, the polynomial $H_4(q)/\widehat{H_4}(q)$ is irreducible of degree 60.

Part IV

An $\mathfrak{S}_3$-equivariant cellular structure on the flag manifold of $SL_3(\mathbb{R})$ from the GKM graph of $\mathfrak{S}_3$

In this part we start our study of flag manifolds. We first recall some classical facts about flag manifolds and in particular, we describe their (T-equivariant) cohomology, with the three descriptions of Borel, Schubert and Goresky-Kottwitz-MacPherson.

Afterwards, we explicitly describe the flag variety $\mathcal{F} := SL_3(\mathbb{C})/B$ of type A_2 as a closed subvariety of $\mathbb{P}^7(\mathbb{C})$ and we identify the action of the Weyl group $W = \mathfrak{S}_3$ under this embedding. Inspired by the case of SL_2 and the GKM graph of $\mathfrak{S}_3$, we define 1-cells on the real points $\mathcal{F}(\mathbb{R})$. Inductively, this leads to an $\mathfrak{S}_3$-equivariant (semialgebraic) cellular decomposition of $\mathcal{F}(\mathbb{R})$. We obtain the associated chain complex in Theorem 9.2.2 and then we describe the resulting homology representation. In particular, we prove that the mod 2 cohomology of $\mathcal{F}(\mathbb{R})$ is the mod 2 coinvariant algebra of $\mathfrak{S}_3$, equipped with its $\mathfrak{S}_3$-action. This is achieved by studying suitable transverse subvarieties of $\mathcal{F}(\mathbb{R})$.

7 Reminders on flag manifolds and their T-equivariant cohomology

7.1 Background and notation

We first recall some notation. We let as usual G be a simply-connected semisimple complex algebraic group and we choose a Borel subgroup B of G. We also consider K a maximal compact subgroup of G, with maximal torus T contained in B. The Lie algebra $\mathfrak{h}$ of $T^{\mathbb{C}}$ is a Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of G. We denote by Φ the root system of $(G, T^{\mathbb{C}})$ and by Φ^+ the positive roots associated to the pair (G, B). Finally, we let Π be the set of simple roots of Φ and $W := N_K(T)/T \simeq N_G(T^{\mathbb{C}})/T^{\mathbb{C}} \simeq W(\Phi)$ be the Weyl group. We have the root spaces decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha.$$

Recall that, given $\alpha \in \Phi^+$, we have an $\mathfrak{sl}_2$-triple $(x_\alpha, y_\alpha, h_\alpha)$ in $\mathfrak{g}$ and the vectors $(x_\alpha, y_\alpha, h_\alpha)_{\alpha\in\Phi^+}$ form the natural basis of $\mathfrak{g}$. For $\alpha \in \Pi$, the $\mathfrak{sl}_2$-triplet $(x_\alpha, y_\alpha, h_\alpha)$ induces an embedding of Lie algebras $\mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{g}$, which lifts via the exponential (since $SL_2(\mathbb{C})$ is simply connected) to an embedding of algebraic groups $j_\alpha : SL_2(\mathbb{C}) \hookrightarrow G$. Define

$$\dot{s_\alpha} := j_\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in G.$$

Then, $\dot{s_\alpha}$ represents the simple reflection s_α modulo $T^{\mathbb{C}}$. Moreover, if $w = s_{\alpha_1} \cdots s_{\alpha_m} \in W$ is a *reduced decomposition* for w, then the element $\dot{w} := \dot{s_{\alpha_1}} \cdots \dot{s_{\alpha_m}} \in G$ is independent of the chosen decomposition and represents w modulo $T^{\mathbb{C}}$. Then, W acts on K/T by

$$(kT) \cdot w := (k\dot{w})T.$$

On the other hand, the Iwasawa decomposition [Bum13, Theorem 26.4] reads $G = BK$. Hence, we obtain a diffeomorphism

$$G/B \simeq BK/B \simeq K/(B \cap K) \simeq K/T$$

and transporting the action of W on K/T using this diffeomorphism gives a (non-algebraic[18]) action of W on G/B. From now on, we let $\mathcal{F} := K/T \simeq G/B$ be the flag manifold of K.

The torus $T \simeq (\mathbb{S}^1)^n$ acts on $\mathcal{F}$, and we obtain then the corresponding rational T-equivariant cohomology algebra $H_T^*(\mathcal{F}, \mathbb{Q})$. The Bruhat decomposition ([Bum13, Theorem 27.2])

$$G = \bigcup_{w \in W} BwB$$

induces a decomposition

$$G/B = \bigcup_{w \in W} BwB/B.$$

Furthermore, each component BwB/B is an affine space, more precisely, one has $BwB/B \simeq \mathbb{C}^{\ell(w)}$ ([Bum13, Theorem 27.3]). Hence, BwB/B is a real cell of dimension $2\ell(w)$, called *Schubert cell* associated to w and the above decomposition makes $\mathcal{F}$ a cellular complex. If $w_0 \in W$ is the longest element, let $B_- := B^{w_0} = w_0 B w_0$ and consider the closure (with respect to the Zariski topology) of $B_- wB/B \simeq \mathbb{C}^{\ell(w_0)-\ell(w)}$, which we denote by $\Omega_w := \overline{B_- wB/B}$; we call it the *(dual) Schubert variety* associated to w. This is a closed (singular!) subvariety of the smooth projective variety $\mathcal{F} \simeq G/B$, of real codimension $2\ell(w)$. Its fundamental cohomology class, denoted by $X_w := [\Omega_w] \in H^{2\ell(w)}(\mathcal{F}, \mathbb{Q})$ is called a *Schubert class* (we look at $B_- wB/B$ instead of BwB/B in order to obtain a class in $H^{2\ell(w)}(G/B)$ rather than in $H^{2\ell(w_0)-2\ell(w)}(G/B)$). We have also an equivariant cohomology class $\mathcal{X}_w := [\Omega_w]_T \in H_T^{2\ell(w)}(\mathcal{F}, \mathbb{Q})$, called an *equivariant Schubert class*. The *basis theorem* ([Kaj15, §1]) tells us that

$$H^*(\mathcal{F}, \mathbb{Q}) = \bigoplus_{w \in W} \mathbb{Q} \langle X_w \rangle$$

and that

$$H_T^*(\mathcal{F}, \mathbb{Q}) = \bigoplus_{w \in W} H_T^*(\mathrm{pt}, \mathbb{Q}) \langle \mathcal{X}_w \rangle .$$

In other words, the Schubert classes (resp. the equivariant Schubert classes) form a $\mathbb{Q}$-basis (resp. a $H_T^*(\mathrm{pt})$-basis) of the cohomology (resp. the T-equivariant cohomology) of $\mathcal{F}$.

A fundamental question (sometimes referred as the Littlewood-Richardson problem) is to find the *structure constants* $c_{u,v}^w \in H_T^*(\mathrm{pt})$ verifying

$$\forall u, v \in W,\ \mathcal{X}_u \cdot \mathcal{X}_v = \sum_{w \in W} c_{u,v}^w \mathcal{X}_w.$$

To find such constants means to describe the multiplicative structure of the equivariant cohomology algebra of the flag manifold. We may of course ask the same question in the non-equivariant case.

[18] In the case of $SL_2(\mathbb{C})/B \simeq \mathbb{CP}^1$, the action of $s \in \mathfrak{S}_2 \setminus \{1\}$ is given by $z \cdot s = -1/\overline{z}$.

7.2 Identifying the action of W on $H_T^*(\mathrm{pt}, \mathbb{Q})$ and on the (T-equivariant) cohomology of $\mathcal{F}$

We have already studied the action of W on T by conjugation. This induces an action of W on B_T, so W acts on the cohomology algebras $H^*(B_T, \mathbb{Q})$ and $H^*(\mathcal{F}, \mathbb{Q})$ as well as on $H_T^*(\mathcal{F}, \mathbb{Q})$. Our aim is to describe these actions as actions on polynomial rings.

A choice of an isomorphism $T \xrightarrow{\sim} (\mathbb{S}^1)^n$ induces an isomorphism $T^{\mathbb{C}} \xrightarrow{\sim} (\mathbb{C}^\times)^n$ and a Lie algebra isomorphism $\mathfrak{h} \xrightarrow{\sim} \mathbb{C}^n$. We therefore obtain coordinates in $\mathfrak{h}$; hence a choice of an isomorphism $T \simeq (\mathbb{S}^1)^n$ determines the choice of graded algebra isomorphism $S(\mathfrak{h}_{\mathbb{Q}}^*) \simeq \mathbb{Q}[t_1, \dots, t_n]$, the t_i's having degree 2. Hence, with this choice, the action of $W = W(\Phi)$ on $\mathfrak{h}^*$ induces an action of W on $\mathbb{Q}[t_1, \dots, t_n]$. On the other hand, a choice of $T \xrightarrow{\sim} (\mathbb{S}^1)^n$ induces a choice of $B_T \xrightarrow{\sim} (\mathbb{CP}^\infty)^n$ and denoting by θ_i the Poincaré dual of the fundamental homology class of the i^{th} copy of $\mathbb{CP}^\infty$ in $(\mathbb{CP}^\infty)^n$, we obtain an isomorphism of graded algebras $\mathbb{Q}[\theta_1, \dots, \theta_n] \xrightarrow{\sim} H^*(B_T, \mathbb{Q})$, the θ_i's having degree 2 and here again, the action of W on $H^*(B_T, \mathbb{Q})$ yields an action of W on $\mathbb{Q}[\theta_1, \dots, \theta_n]$. Finally, a choice of an isomorphism $T \xrightarrow{\sim} (\mathbb{S}^1)^n$ determines the choice of an isomorphism $S(\mathfrak{h}_{\mathbb{Q}}^*) \simeq H^*(B_T, \mathbb{Q})$. One proves that this isomorphism is W-equivariant. More precisely, the isomorphism $H^*(\mathcal{F}, \mathbb{Q}) \simeq S(\mathfrak{h}_{\mathbb{Q}}^*) \,/\, S(\mathfrak{h}_{\mathbb{Q}}^*)_+^W$ is W-equivariant[19] and this gives the result.

From these considerations we may conclude that, once a choice of an isomorphism $T \simeq (\mathbb{S}^1)^n$ is made, we get a system of coordinates $(t_1, \dots, t_n)$ on $\mathfrak{h}$ and that the isomorphism

$$H^*(B_T, \mathbb{Q}) \xrightarrow{\sim} S(\mathfrak{h}_{\mathbb{Q}}^*) \xrightarrow{\sim} \mathbb{Q}[t_1, \dots, t_n]$$

is W-equivariant.

On the other hand, we can identify the representation of W on the (ungraded) ring $H^*(\mathcal{F}, \mathbb{Q})$.

Proposition 7.2.1. *The ungraded $\mathbb{Q}[W]$-module $H^*(\mathcal{F}, \mathbb{Q})$ (resp. the ungraded $H_T^*(\mathrm{pt}, \mathbb{Q})$-module $H_T^*(\mathcal{F}, \mathbb{Q})$) is the regular module.*

Proof. We treat the non-equivariant case using an argument due to Hsiang and can be found in [Hsi75, III, §1, Lemma 1.1]. The $|W|$ cells appearing in the Burhat decomposition of $\mathcal{F}$ are all even dimensional, so $\mathcal{F}$ has cohomology even in even degrees and its Euler characteristic is $\chi(\mathcal{F}) = |W|$. As $\mathcal{F}$ is a CW-complex and W is finite, [Mat73, Proposition 0.5] ensures the existence of a W-equivariant cellular structure on $\mathcal{F}$ and let $C^* := C^*_{\text{cell}}(\mathcal{F}, W; \mathbb{Q})$ be the associated cellular homology chain complex. In the Grothendieck group $K_0(\mathbb{Q}[W])$ we have $\sum_{i \geq 0} (-1)^i [C^i] = \sum_{i \geq 0} (-1)^i [H^i(\mathcal{F}, \mathbb{Q})] = \sum_{i \geq 0} [H^{2i}(\mathcal{F}, \mathbb{Q})] = |H^*(\mathcal{F}, \mathbb{Q})|$, but since W acts freely on $\mathcal{F}$, the modules C^i are free $\mathbb{Q}[W]$-modules and there exists $k \in \mathbb{Z}$ such that $[H^*(\mathcal{F}, \mathbb{Q})] = k[\mathbb{Q}[W]]$ but the equality $\chi(\mathcal{F}) = |W|$ forces $k = 1$ and thus $H^*(\mathcal{F}, \mathbb{Q})$ is indeed the regular $\mathbb{Q}[W]$-module.

On the other hand, recall that $\mathcal{F}$ is *T-equivariantly formal* (i.e. the Serre spectral sequence associated to the Bore fiber bundle $\mathcal{F} \hookrightarrow \mathcal{F} \times_T E_T \twoheadrightarrow B_T$ collapses, see [GKM97, §1.2]). By the Leray-Hirsch theorem [Hat02, Theorem 4D.1], this implies that there is an isomorphism of $H_T^*(\mathrm{pt})$-modules

$$H_T^*(\mathcal{F}, \mathbb{Q}) \simeq H_T^*(\mathrm{pt}) \otimes_{\mathbb{Q}} H^*(\mathcal{F}, \mathbb{Q})$$

[19] See [BGG73, Proposition 1.3], proven in [Bor53b] directly, and in [Ree95] by using De Rham cohomology of W-invariant differentiable forms on the manifold $\mathcal{F}$.

and moreover this isomorphism is W-equivariant, if we let W act trivially on $H_T^*(\mathrm{pt})$. We obtain $H_T^*(\mathcal{F},\mathbb{Q}) \simeq H_T^*(\mathrm{pt}) \otimes \mathbb{Q}[W] \simeq H_T^*(\mathrm{pt})[W]$, as required. □

Remark 7.2.2. *A more constructive proof can be found in [Tym07, Lemma 4.4 and Theorem 4.5], using (equivariant) Schubert classes. In fact, the W-translates of the sum $\sum_{w\in W} \mathcal{X}_w$ of all the equivariant Schubert classes is a $H_T^*(\mathrm{pt})$-basis of $H_T^*(\mathcal{F},\mathbb{Q})$. The same for the classical cohomology: the W-translates of $\sum_w X_w$ form a $\mathbb{Q}$ basis of $H^*(\mathcal{F},\mathbb{Q})$.*

7.3 The three descriptions of $H_T^*(G/B,\mathbb{Q})$ after S. Kaji

In order to solve the structure constants problem and hence describe the multiplicative structure of the equivariant cohomology algebra, we have three descriptions of this algebra. Of course we already have one description (the one of *Schubert*, also known as the *Chevalley description*), which is to write a cohomology class as a $H_T^*(\mathrm{pt})$-linear combination of Schubert classes. On the other hand, one can notice that (once an isomorphism $T \simeq (\mathbb{S}^1)^n$ is chosen)

$$R := H_T^*(\mathrm{pt},\mathbb{Q}) \overset{\mathrm{df}}{=} H^*(B_T,\mathbb{Q}) = H^*((\mathbb{CP}^\infty)^n,\mathbb{Q}) = \mathbb{Q}[\theta_1,\dots,\theta_n] \overset{\mathrm{def}}{=} \mathbb{Q}[\theta],$$

with $\deg(\theta_i) = 2$. In practice, we have a better understanding of the action of W on the root system Φ and we shall rather consider the W-equivariant isomorphism (see the previous section)

$$R := H_T^*(\mathrm{pt},\mathbb{Q}) \overset{\mathrm{df}}{=} H^*(B_T,\mathbb{Q}) \simeq S(\mathfrak{h}_{\mathbb{Q}}^*) \simeq \mathbb{Q}[t_1,\dots,t_n] =: \mathbb{Q}[t],$$

with $\deg(t_i) = 2$. Then, we get a polynomial algebra. We denote by R^W the W-invariant polynomyals. The *Borel description* ([Kaj15, Theorem 2.3]) consists in writing

$$H_T^*(\mathcal{F},\mathbb{Q}) \simeq R \otimes_{R^W} R \simeq \mathbb{Q}[t,x] \big/ (f(t)-f(x),\ f \in \mathbb{Q}[z]^W).$$

Hence, we may see a cohomology class as the class of a polynomial $\mathbb{Q}[t,x]$ modulo the ideal $I_W := (f(t)-f(x),\ f \in R^W)$. This allows us to easily calculate the product in the cohomology algebra. Moreover, the right action of W on $\mathcal{F}$ commutes with the one of T, hence W acts on $H_T^*(\mathcal{F},\mathbb{Q})$ and the isomorphism between this algebra and $R\otimes_{R^W} R$ shows that W acts on $R \otimes_{R^W} R$ on the second factor and we deduce that the action of W on $\mathbb{Q}[t,x]\,/\,I_W$ induced by the action on $\mathcal{F}$ is given by

$$\forall w \in W,\ \forall f(t,x) \in \mathbb{Q}[t,x] \big/ I_W,\ (w\cdot f)(t,x) = f(t, w^{-1}(x)).$$

We also have a combinatorial description of this algebra, called the *Goresky-Kottwitz-MacPherson description*, which makes fine use of the combinatorial properties of the Weyl group. Let us recall that if $u,v \in W$ and $\alpha \in \Phi^+$, we write $u <_\alpha v$ for $u = s_\alpha v$ and $\ell(u) < \ell(v)$. This is a binary relation and the reflexive transitive closure of the relation $\bigcup_\alpha <_\alpha$ is the Bruhat order. The GKM algebra is then ([Kaj15, Theorem 2.5])

$$H_T^*(\mathcal{F},\mathbb{Q}) \simeq H^*(\mathcal{G},\mathbb{Q}) := \left\{ h = (h_v)_v \in \bigoplus_{v\in W} \mathbb{Q}[t]\ ;\ \forall \alpha \in \Phi^+,\ u <_\alpha v \ \Rightarrow\ h_u - h_v \in \langle \alpha(t) \rangle \right\},$$

the notation $H^*(\mathcal{G},\mathbb{Q})$ making reference to the *GKM graph* of W. Recall that the GKM graph $\mathcal{G} := (V,E)$ has elements of W as vertices, and we set and edge between u and v if there is some $\beta \in \Phi^+$ such that $v <_\beta u$ or $u <_\beta v$. As a result, an equivariant cohomology class is represented by a family of polynomials in $\mathbb{Q}[t]$, indexed by the Weyl group. Here, the product is defined component-wise and there is no quotient modulo an ideal to care about, which makes the calculations easier. Furthermore, the action of the Weyl group on $H^*(\mathcal{G}, \mathbb{Q})$ is nicely given by (see [Kaj15, Proposition 3.3])

$$\forall v, w \in W,\ \forall h \in H^*(\mathcal{G}, \mathbb{Q}),\ (w \cdot h)_v = h_{vw}.$$

Finally, we can determine (see [Kaj15, Proposition 3.3]) the action of W on Schubert classes: if $s_\alpha \in W$ is a simple reflection associated to the simple root $\alpha \in \Pi$, then we have

$$s_\alpha \cdot \mathcal{X}_w = \begin{cases} \mathcal{X}_w - w(\alpha)(t)\mathcal{X}_{ws_\alpha} - \sum_{\beta \in \Phi^+ \ ;\ \ell(ws_\alpha s_\beta) = \ell(w)} 2\frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \mathcal{X}_{ws_\alpha s_\beta} & \text{if } \ \ell(ws_\alpha) = \ell(w) - 1 \\ \mathcal{X}_w & \text{if } \ \ell(ws_\alpha) = \ell(w) + 1 \end{cases}$$

Moreover, we have nice formulae describing Schubert classes in the Borel and GKM descriptions (we speak about *Schubert polynomials*, see [Kaj15, §3]). Finally, we have conversion algorithms allowing us to go from a description to another one and this can be used to solve the structure constants problem by writing the product of Schubert classes as polynomials (GKM or Borel), then compute the usual product of these polynomials, and eventually write the result back as $\mathbb{Q}[t]$-linear combination of Schubert classes. We have therefore entirely described the T-equivariant cohomology algebra of $\mathcal{F}$ and we have three nice ways to consider it.

7.4 The case of $SL_{n+1}(\mathbb{C})$

In this section, we fix $n \geq 1$ and we focus on the simple compact Lie group $K := SU(n+1)$ of type A_n, that is

$$K = SU(n+1) := \{A \in GL_{n+1}(\mathbb{C})\ ;\ AA^* = I_{n+1} \text{ and } \det(A) = 1\}.$$

Recall that the subgroup $T := S(U(1)^{n+1})$ of diagonal matrices in $SU(n+1)$ is a maximal torus. By [Bum13, §24], the Lie group $K = SU(n+1)$ admits $G := SL_{n+1}(\mathbb{C})$ as a complexification and the complex torus $T^{\mathbb{C}} \simeq (\mathbb{C}^\times)^n$ is formed with diagonal matrices in $SL_{n+1}(\mathbb{C})$. Furthermore, we may choose the Borel subgroup of G to be the group of upper-triangular matrices. The Lie algebra of G is $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ of traceless matrices of $\mathfrak{gl}_{n+1}(\mathbb{C})$:

$$\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C}) = \{x \in \mathfrak{gl}_{n+1}(\mathbb{C})\ ;\ \mathrm{tr}(x) = 0\}$$

and the Cartan subalgebra $\mathfrak{h} = \mathrm{Lie}(T^{\mathbb{C}})$ is given by

$$\mathfrak{h} = \left\{ \begin{pmatrix} x_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & x_{n+1} \end{pmatrix},\ x_i \in \mathbb{C}\ ;\ \sum_{i=1}^{n+1} x_i = 0 \right\} \simeq \mathbb{C}^n.$$

Since the Lie bracket on $\mathfrak{g}$ is the commutator, if we denote by $E_{i,j}$ the matrix of $\mathfrak{gl}_{n+1}$ whose (k,j) entry is 1 if $(i,j) = (k,l)$ and 0 else (i.e. $E_{i,j} = (\delta_{(k,l),(i,j)})_{1 \leq k,l \leq n})$. The roots of $\mathfrak{h}$ are the $\alpha_{i,j} \in \mathfrak{h}^*$ defined by $\alpha_{i,j}(x) = x_i - x_j$ for $i \neq j$; we have $\mathfrak{g}_\alpha = \mathbb{C}E_{i,j}$ and the root space decomposition is given by

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbb{C}E_{i,j}.$$

Since B is the group of triangular matrices of $SL_{n+1}(\mathbb{C})$, the positive roots are those $\alpha_{i,j}$ with $i<j$ and the Borel subalgebra of $\mathfrak{g}$ is $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{i<j} \mathbb{C}E_{i,j}$ and therefore, the simple roots are the $\alpha_i := \alpha_{i,i+1} : x \mapsto x_i - x_{i+1}$. Finally, the Weyl group is $W = \mathfrak{S}_n$ and acts on $\mathfrak{h}^*$ by permuting the variables and the simple reflections are $s_i := s_{\alpha_{i,i+1}} = (i, i+1)$ for $1 \leq i \leq n$.

Remark 7.4.1. *Since the normalizer in $GL_{n+1}(\mathbb{C})$ of the group T_{GL} of invertible diagonal matrices consists of monomial matrices, the normalizer $N_{SL_{n+1}(\mathbb{C})}(T)$ consists of normalized monomial matrices (that is, matrices of the form $\mu^{-1}x$ where $x \in GL_{n+1}(\mathbb{C})$ is monomial with determinant* 1 *and μ is a n-root of* $\det(x)$ *in* $\mathbb{C}$*) and we recover that the Weyl group is $W = N_{SL_{n+1}(\mathbb{C})}(T)/T = N_{GL_{n+1}(\mathbb{C})}(T_{GL})/T_{GL} \simeq \mathfrak{S}_n$.*

Since we have the description $\mathfrak{h} = \{(x_1, \dots, x_{n+1}) \in \mathbb{C}^{n+1} \ ; \ x_1 + \cdots + x_{n+1} = 0\}$, we may write

$$R := H^*(B_T, \mathbb{Q}) \simeq \frac{\mathbb{Q}[t_1, \dots, t_{n+1}]}{(t_1 + \cdots + t_{n+1})} (\simeq \mathbb{Q}[\alpha_1, \dots, \alpha_n])$$

Of course, the corresponding action of W on R consists of permuting the variables. Hence, if $e_i := \sum_{1 \leq k_1 < \cdots < k_i \leq n+1} x_{k_1} \cdots x_{k_i}$ denotes the i^{th} elementary symmetric polynomial in the variables x_i, then one has the well-known isomorphism

$$R^W \simeq \frac{\mathbb{Q}[t_1, \dots, t_{n+1}]^{\mathfrak{S}_n}}{(t_1 + \cdots t_{n+1})} \simeq \mathbb{Q}[e_2, \dots, e_{n+1}]$$

and hence

$$H_T^*(SU(n+1)/T, \mathbb{Q}) \simeq R \otimes_{R^W} R \simeq \frac{\mathbb{Q}[t_1, \dots, t_{n+1}, x_1, \dots, x_{n+1}]}{(e_1(t), e_1(x), e_i(t) - e_i(x), \ 2 \leq i \leq n+1)}$$

Recall from the Introduction the geometric descriptions of $SU(n+1)/T$ and $SL_{n+1}(\mathbb{C})/B$. As a reminder, define

$$\mathcal{F}\ell := \{F_\bullet = (F_1, \dots, F_{n+1}) \ ; \ F_i \leq \mathbb{C}^{n+1}, \ F_i \leq F_{i+1}, \ \dim(F_i) = i+1\}$$

the set of *flags* in $\mathbb{C}^{n+1}$. It is a Zariski closed subset of the variety $\prod_{1 \leq k \leq n+1} \mathrm{Gr}(k, n+1)$, where $\mathrm{Gr}(k, n+1)$ is the Grassmannian variety of subspaces of $\mathbb{C}^{n+1}$ of dimension k. Hence, $\mathcal{F}\ell$ is a (smooth) projective variety. We let $SL_{n+1}(\mathbb{C})$ act naturally on $\mathcal{F}\ell$ and this action is transitive. On the other hand, the subgroup B is precisely the stabilizer of the flag $(F_\bullet^0)$ associated to the canonical basis $(u_i)_i$ of $\mathbb{C}^{n+1}$ (i.e. $F_k^0 = \langle u_1, \dots, u_k \rangle$) and hence we obtain an isomorphism of complex varieties

$$\mathcal{F}\ell \simeq SL_{n+1}(\mathbb{C})/B = \mathcal{F}.$$

In particular, $\mathcal{F}\ell$ is a smooth irreducible variety of dimension $n(n+1)/2$. Next, define

$$\mathcal{D} := \{(L_1, \dots, L_{n+1}) \ ; \ L_i \leq \mathbb{C}^{n+1}, \ \dim(L_i) = 1, \ \mathbb{C}^{n+1} = L_1 \oplus \cdots \oplus L_{n+1}\}$$

and

$$\mathcal{D}_\perp := \{(L_1, \dots, L_{n+1}) \in \mathcal{D} \ ; \ \forall i \neq j, \ L_i \perp L_j\},$$

that is, $\mathcal{D}$ (resp. $\mathcal{D}_\perp$) is the space of ordered decompositions of $\mathbb{C}^{n+1}$ as a direct sum of lines (resp. pairwise orthogonal lines). These are equipped with the induced usual Hausdorff topology of $(\mathbb{CP}^1)^{n+1}$. We let $SU(n+1)$ (resp. $SL_{n+1}(\mathbb{C})$) act naturally on $\mathcal{D}_\perp$ (resp. $\mathcal{D}$) on each component of an $(n+1)$-tuple of (orthogonal) lines. If we let

$D_0 := (\langle u_1 \rangle, \dots, \langle u_{n+1} \rangle)$, then $D_0 \in \mathcal{D}_\perp$ and its stabilizer for the action of $SU(n+1)$ on $\mathcal{D}_\perp$ (resp. for the action of $SL_{n+1}(\mathbb{C})$ on $\mathcal{D}$) is the subgroup T (resp. the subgroup $T^{\mathbb{C}}$). We thus obtain homeomorphisms

$$\mathcal{D}_\perp \simeq SU(n+1)/T \ \text{ and } \ \mathcal{D} \simeq SL_{n+1}(\mathbb{C})/T^{\mathbb{C}}.$$

Recall also that the Iwasawa diffeomorphism $SU(n+1)/T \simeq SL_{n+1}(\mathbb{C})/B$ can be defined in this case using the Gram-Schmidt algorithm. Moreover, we have a homotopy equivalence

$$\mathcal{D} \sim \mathcal{F}\ell.$$

Indeed, we have a fiber bundle $U \hookrightarrow SL_{n+1}(\mathbb{C})/T^{\mathbb{C}} \overset{\pi}{\twoheadrightarrow} SL_{n+1}(\mathbb{C})/B$, where $U \simeq B/T^{\mathbb{C}}$ is the subgroup of B consisting of matrices with ones on the diagonal. Since $U \simeq \mathbb{C}^{n(n+1)/2}$ is contractible, the map π is a weak homotopy equivalence and in fact a strong homotopy equivalence by Whitehead's theorem, since $SL_{n+1}(\mathbb{C})/T^{\mathbb{C}}$ and $SL_{n+1}(\mathbb{C})/B$ have the homotopy type of CW-complexes.

As a consequence, we get

$$H^*(SU(n+1)/T, \mathbb{Z}) \simeq H^*(SL_{n+1}(\mathbb{C})/B, \mathbb{Z}) \simeq H^*(SL_{n+1}(\mathbb{C})/T^{\mathbb{C}}, \mathbb{Z}).$$

Remark 7.4.2. *In the case $n = 2$, we have $K = SU(2)$ and*

$$T = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix},\ |\lambda| = 1 \right\} \simeq \mathbb{S}^1 \ \ \textit{and} \ \ B = \left\{ \begin{pmatrix} z & a \\ 0 & z^{-1} \end{pmatrix},\ a \in \mathbb{C},\ z \neq 0 \right\}$$

as well as

$$\mathfrak{h} \simeq \{(x,y) \in \mathbb{C}^2 \ ;\ x + y = 0,\ W \simeq \mathfrak{S}_2 = \{1, s = s_\alpha\},$$

where $\alpha(x_1, x_2) = x_1 - x_2$ is the only simple root of Φ and $s = (12)$. Furthermore, we have

$$H^*(B_T, \mathbb{Q}) \simeq \frac{\mathbb{Q}[t_1, t_2]}{(t_1 + t_2)} \simeq \mathbb{Q}[\alpha],\ H^*_T(SU(2)/T, \mathbb{Q}) \simeq \frac{\mathbb{Q}[t_1, t_2, x_1, x_2]}{(t_1 + t_2, x_1 + x_2, t_1 t_2 - x_1 x_2)}.$$

Here, we can take $\mathcal{X}_e(t,x) = 1$ and $\mathcal{X}_s(t,x) = t_1 - x_1$ in the Borel description. Setting $P_e := \mathcal{X}_e + \mathcal{X}_s = 1 + t_1 - x_1$ and $P_s := s \cdot P_e = 1 + t_1 - x_2$, we recover the fact that $H^*_T(SU(2)/T, \mathbb{Q}) = H^*(B_T, \mathbb{Q}) \langle P_e, P_s \rangle$ is the regular representation of $\mathfrak{S}_2$ on $\frac{\mathbb{Q}[t_1, t_2]}{(t_1+t_2)}$.

8 Construction of the cell structure on the flag manifold of $SL_3(\mathbb{R})$ via a closed embedding in a projective space

In this part, we exhibit an $\mathfrak{S}_3$-equivariant cell structure on the real points of the flag manifold $SL_3(\mathbb{C})/B$. This structure is in fact regular and semi-algebraic. By *regular*, we mean that the closure of each i-cell is homeomorphic to a closed i-ball and, by a *semi-algebraic subset* of a real irreducible algebraic variety X we mean a subset S of X such that there exists an affine open cover $(U_j)_j$ of X such that for each j, the trace $S \cap U_j \subset \mathbb{A}^{\dim X}(\mathbb{R})$ is a semialgebraic subset of $\mathbb{R}^{\dim X}$. By the Tarski-Seidenberg principle ([Cos02], Theorem 2.3 and Corollary 2.4), this is independent of the chosen atlas on X.

8.1 Embedding the flag variety into the projective closure of a highest weight module

Using highest weight representations, one can see the flag variety G/B as a projective variety. In fact, it will be a closed subvariety of $\mathbb{P}^{2^\ell - 1}(\mathbb{C})$, where $\ell := |\Phi_+|$.

First recall a fundamental result of algebraic geometry that we shall use a couple of times and which is a direct corollary of Zariski's main theorem ([TY05, Theorem 17.4.3]):

Theorem 8.1.1 *([TY05, Corollary 17.4.8]). If $\phi : X \to Y$ is a bijective morphism between irreducible complex algebraic varieties and if Y is normal, then ϕ is an isomorphism.*

Recall further that the set $X(T^{\mathbb{C}}) := \{\lambda \in \mathfrak{h}^* \ ; \ \forall \alpha \in \Pi, \ \lambda(\alpha^\vee) \in \mathbb{Z}\}$ of *integral weights* is in natural bijection with $\mathrm{Hom}(T^{\mathbb{C}}, \mathbb{G}_m) = \mathrm{Hom}((\mathbb{G}_m)^n, \mathbb{G}_m) \simeq \mathbb{Z}^n$ and that the *dominant* integral weights $X(T^{\mathbb{C}})_+ := \{\lambda \in \mathfrak{h}^* \ ; \ \forall \alpha \in \Pi, \ \lambda(\alpha^\vee) \in \mathbb{N}\}$ corresponds to $\mathbb{N}^n \subset \mathbb{Z}^n \simeq \mathrm{Hom}(T^{\mathbb{C}}, \mathbb{G}_m)$. Finally, define $X(T^{\mathbb{C}})_{++} := \{\lambda \in \mathfrak{h}^* \ ; \ \forall \alpha \in \Phi^+, \ \lambda(\alpha^\vee) \in \mathbb{N}^*\}$ the set of *regular dominant* weights.

For $\lambda \in X(T^{\mathbb{C}})_+$ we have an irreducible highest weight representation $V(\lambda)$ of G with highest weight λ.

Remark 8.1.2. *The representation $V(\lambda)$ may be constructed using the Borel-Weil theorem, i.e. by taking the space of global sections of an algebraic line bundle on G/B, see [Jan87, Part II, §5] for more details.*

Fix λ a dominant weight and $v_\lambda \in V(\lambda)$ a primitive vector of highest weight λ. Since $B = N \rtimes T^{\mathbb{C}}$ (with $N := [B, B]$ the connected subgroup of G with Lie algebra $\mathfrak{n} = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$, which is also the unipotent radical $R_u(B)$ of B) acts on v_λ as $T^{\mathbb{C}}$ does, that is by scalar multiplication, one has a well-defined map

$$\begin{array}{rccc} \iota_\lambda & : & G/B & \longrightarrow & \mathbb{P}(V(\lambda)) \\ & & gB & \longmapsto & [g \cdot v_\lambda] \end{array}$$

We have the following result:

Theorem 8.1.3 *([Kum11, §4]). Let $\lambda \in X(T^{\mathbb{C}})_+$ be a dominant weight and let $v_\lambda \in V(\lambda)$ be a primitive vector of highest weight λ. We make G act on the projective completion $\mathbb{P}(V(\lambda))$ and we consider the stabilizer $P := \mathrm{Stab}_G([v_\lambda])$ of the line that v_λ spans. Then P is a* parabolic subgroup *of G (i.e. contains the Borel subgroup B) and we have a closed embedding of algebraic varieties*

$$\begin{array}{rccc} \iota_\lambda & : & G/P & \longrightarrow & \mathbb{P}(V(\lambda)) \\ & & gP & \longmapsto & [g \cdot v_\lambda] \end{array}$$

Moreover, if $\lambda \in X(T^{\mathbb{C}})_{++}$ is regular dominant, then $P = B$ and we obtain a closed embedding

$$\iota_\lambda : \mathcal{F} \hookrightarrow \mathbb{P}(V(\lambda)).$$

Proof. First notice that P is indeed parabolic, since $B = [B, B] \rtimes T^{\mathbb{C}}$ acts on v_λ as $T^{\mathbb{C}}$ does, that is, by scalar multiplication and thus $B \leq P$. Because the composite map $G \to G/P \xrightarrow{\iota_\lambda}$

$\mathbb{P}V(\lambda)$ is a morphism of algebraic varieties, by the universal property of the geometric quotient G/P, the map ι_λ is a morphism too, which is injective by construction. To prove that $Z := \iota_\lambda(G/P)$ is a closed subvariety of $\mathbb{P}V(\lambda)$, first note that $\overline{Z}$ is a G-stable subspace of $\mathbb{P}V(\lambda)$ and it follows that $\overline{Z} \setminus Z$ is G-stable. If $\overline{Z} \setminus Z \neq \emptyset$, by Borel's fixed point theorem, $\overline{Z} \setminus Z$ must contain a P-fixed point, which contradicts the uniqueness of a primitive vector of highest weight λ. Hence $\overline{Z} \setminus Z = \emptyset$, so Z is a closed subset of $\mathbb{P}V(\lambda)$. Finally, $\iota_\lambda : G/P \to Z$ is an isomorphism. Indeed, since every variety contains smooth points and G preserves smoothness, the fact that X is a G-orbit implies immediately that Z is smooth, hence normal and we conclude using Theorem 8.1.1.

For the second statement, take $\lambda \in X(T^{\mathbb{C}})_{++}$. For $\alpha \in \Pi$ a simple root, we have $\lambda(\alpha^\vee) \neq 0$ and so $s_\alpha(\lambda) = \lambda - \lambda(\alpha^\vee)\alpha \neq \lambda$ so s_α does not stabilize λ and thus we have $\dot{s_\alpha} \notin P$. But if we write the parabolic subgroup P as $P = \langle B, (\dot{s_\alpha})_{\alpha\in\pi}\rangle$ for some $\pi \subset \Pi$, then we must have $\pi = \emptyset$ and so $P = B$. $\square$

Next, consider half the sum of positive roots $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha \in \mathfrak{h}^*$. If $\delta \in \Pi$, then the simple reflection s_δ permutes $\Phi^+ \setminus \{\delta\}$, hence $s_\delta(\rho) = \rho - \delta$ implying $\rho(\delta^\vee) = 1$ and then $\rho \in X(T^{\mathbb{C}})_{++}$. From this and the Weyl dimension formula, we deduce the following very useful tool:

Corollary 8.1.4. *If $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$, then ρ is a regular weight and hence the morphism $\varphi := \iota_\rho$ is a closed embedding*

$$\varphi : G/B \hookrightarrow \mathbb{P}(V(\rho)) \simeq \mathbb{P}^k(\mathbb{C}),$$

where $k = 2^{|\Phi^+|} - 1$.

Remark 8.1.5. *In the case of $SL_2(\mathbb{C})$, we retrieve the algebraic isomorphism*

$$\begin{array}{ccc} SL_2(\mathbb{C})/B & \xrightarrow{\sim} & \mathbb{P}^1(\mathbb{C}) \\ \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) B & \longmapsto & [a:b] \end{array}$$

Indeed, we have that $T^{\mathbb{C}} = \left\{ \left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right) \ ; \ x \in \mathbb{C}^\times \right\}$ is one-dimensional and $X(T^{\mathbb{C}})_{++} = \mathbb{N}^*$ and we have $\rho = \alpha/2$, where $\Phi = \{\pm\alpha\}$. It is well-known ([FH91, Lecture 11]) that irreducible SL_2-modules are parametrized by $\mathbb{N}$ and, if $\lambda \in X(T^{\mathbb{C}})_+$ we can take $V(\lambda) := \mathbb{C}[X,Y]_n$ the $(n+1)$-dimensional space of degree n homogeneous polynomials ; SL_2 acting by

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot P(X,Y) := P(aX + bY, cX + dY).$$

In the case $\lambda = \rho = \frac{\alpha}{2}$, we have the weight spaces decomposition $V(\rho) = \mathbb{C}[X,Y]_1 = \mathbb{C}X \oplus \mathbb{C}Y = V_1 \oplus V_{-1}$ and $v_\alpha := X$ is a primitive highest weight vector of weight α. Hence, the morphism φ is given by

$$\begin{array}{ccccccc} \varphi & : & SL_2/B & \hookrightarrow & \mathbb{P}(V(\alpha)) & \simeq & \mathbb{P}^1(\mathbb{C}) \\ & & \begin{pmatrix} a & c \\ b & d \end{pmatrix} B & \mapsto & [aX + bY] & \mapsto & [a:b] \end{array}$$

as expected. Moreover, recall the six cells $e^{0,1,2}$ and their images under s defined in the Introduction forming an $\mathfrak{S}_2$-equivariant cellular structure on $\mathbb{S}^2$ (see Figure A). Under the stereographic projection $\mathbb{S}^2 \simeq \mathbb{CP}^1$, these cells are described by

$$\begin{cases} e^0 = \{[1:0]\}, & e^0 \cdot s = \{[0:1]\}, \\ e^1 = \{[a:b] \in \mathbb{CP}^1 \ ; \ a\bar{b} \in \mathbb{R}_-^*\}, & e^1 \cdot s = \{[a:b] \in \mathbb{CP}^1 \ ; \ a\bar{b} \in \mathbb{R}_+^*\}, \\ e^2 = \{[a:b] \in \mathbb{CP}^1 \ ; \ \Im(a\bar{b}) > 0\}, & e^2 \cdot s = \{[a:b] \in \mathbb{CP}^1 \ ; \ \Im(a\bar{b}) < 0\}. \end{cases}$$

This will be helpful for guessing the cells for $SL_3/B(\mathbb{R})$. We also observe that the obtained CW-structure on $\mathbb{S}^2$ is semialgebraic and regular. Finally, the associated homology chain complex

$$\mathcal{K} := \left(\mathbb{Z}[\mathfrak{S}_2] \xrightarrow{1+s} \mathbb{Z}[\mathfrak{S}_2] \xrightarrow{1-s} \mathbb{Z}[\mathfrak{S}_2] \right).$$

verifies

$$\mathrm{End}_{\mathcal{D}^b(\mathbb{Z}[\mathfrak{S}_2])}(\mathcal{K}) \simeq \mathbb{Z}[\mathfrak{S}_2].$$

8.2 The closed subvariety $\mathcal{F} := SL_3/B$ of $\mathbb{P}^7(\mathbb{C})$

Before looking for a $\mathfrak{S}_3$-cellular structure on real points of SL_3/B, we describe the complex variety SL_3/B explicitly as a closed subvariety of $\mathbb{P}^7(\mathbb{C})$ defined by homogeneous equations using Theorem 8.1.3.

Here again, we naturally choose B to be given by upper-triangular matrices and in this case, we have $\Pi = \{\alpha, \beta\}$ and

$$W = \mathfrak{S}_3 = \left\langle s_\alpha, s_\beta \mid s_\alpha^2 = s_\beta^2 = 1,\ s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta \right\rangle$$

Moreover, one has $\rho = \alpha + \beta$ and this is the highest root, that is, the highest weight in the adjoint representation of $SL_3(\mathbb{C})$. From this we conclude that $V(\rho) = V(\alpha+\beta)$ is precisely the adjoint representation of SL_3. The vector $v_\rho := x_{\alpha+\beta} = \left(\begin{smallmatrix} 0&0&1\\0&0&0\\0&0&0 \end{smallmatrix}\right)$ is primitive of highest weight $\alpha+\beta$. Here again, we use the notation $(x_\delta, y_\delta, h_\delta)_{\delta\in\Phi^+}$ for the standard basis of $\mathfrak{sl}_3$. Furthermore, one has

$$\forall g = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & j \end{pmatrix} \in SL_3(\mathbb{C}),\ g \cdot v_\rho = g x_{\alpha+\beta} g^{-1} = \begin{pmatrix} a(bf-ce) & a(cd-af) & a(ae-bd) \\ b(bf-ce) & b(cd-af) & b(ae-bd) \\ c(bf-ce) & c(cd-af) & c(ae-bd) \end{pmatrix}.$$

Hence, the embedding given by Corollary 8.1.4 reads

$$\begin{array}{cccccc} \varphi : & SL_3/B & \hookrightarrow & \mathbb{P}(\mathfrak{sl}_3) & \xrightarrow{\sim} & \mathbb{P}^7(\mathbb{C}) \\ & \left(\begin{smallmatrix} a&d&g\\b&e&h\\c&f&j \end{smallmatrix}\right) B & \longmapsto & \left[\begin{smallmatrix} a(bf-ce) & a(cd-af) & a(ae-bd) \\ b(bf-ce) & b(cd-af) & b(ae-bd) \\ c(bf-ce) & c(cd-af) & c(ae-bd) \end{smallmatrix}\right] & \longmapsto & \left[\begin{smallmatrix} s&x&z\\u&t&y\\w&v&r \end{smallmatrix}\right] \end{array}$$

We use the matrix notation

$$[x:y:z:u:v:w:s:t] =: \begin{bmatrix} s & x & z \\ u & t & y \\ w & v & r \end{bmatrix}$$

(with $r = -s-t$) for homogeneous coordinates in $\mathbb{P}^7$. Note that using the Iwasawa decomposition, we can write φ in a simpler way, but which is no longer complex algebraic

$$\begin{array}{cccc} \varphi : & SU_3/T & \hookrightarrow & \mathbb{P}^7(\mathbb{C}) \\ & \left(\begin{smallmatrix} a&d&g\\b&e&h\\c&f&j \end{smallmatrix}\right) T & \longmapsto & \left[\begin{smallmatrix} a\bar{g} & a\bar{h} & a\bar{j} \\ b\bar{g} & b\bar{h} & b\bar{j} \\ c\bar{g} & c\bar{h} & c\bar{j} \end{smallmatrix}\right] \end{array}$$

Now, we have to describe $\operatorname{im}\varphi$ as a projective variety. In the $SL_3(\mathbb{C})$-module $V(\alpha+\beta)$, the orbit of the highest weight vector $v_{\alpha+\beta}$ is just the *minimal nilpotent orbit*, which consists of square zero matrices of rank 1. But, a 3×3-matrix is of rank at most 1 and square zero if and only if it is traceless and all of its 2×2-minors vanish. The minors give nine equations and the trace condition gives $s+t+r=0$. This is formalized in the following proposition:

Proposition 8.2.1. *Let I be the ideal of $R := \mathbb{C}[x, y, z, u, v, w, s, t]$ generated by the following homogeneous polynomials*

$$\begin{cases} zw + s^2 + st, \\ yv + t^2 + st, \\ yw + us + ut, \\ vz + xs + xt, \end{cases} \quad \textit{and} \quad \begin{cases} xu - st, \\ uz - ys, \\ uv - wt, \\ xw - vs, \\ xy - zt. \end{cases} \tag{$\clubsuit$}$$

Then, the projective variety $\mathrm{Proj}_{\mathbb{C}}(R/I)$ is a smooth irreducible subvariety of $\mathbb{P}^7(\mathbb{C})$. Furthermore, the natural inclusion $\varphi(SL_3/B) \hookrightarrow \mathbb{P}^7(\mathbb{C})$ factors through an isomorphism of algebraic varieties

$$\sigma : \varphi(SL_3/B) \xrightarrow{\sim} \mathrm{Proj}_{\mathbb{C}}(R/I).$$

Proof. The ideal I is obviously homogeneous so the variety $X := \mathrm{Proj}_{\mathbb{C}}(R/I)$ is well-defined and is a subvariety of $\mathbb{P}^7(\mathbb{C})$. We need to show that I is prime in R, for which we use a general technique inspired by Alex Becker[20]. First note that the class of s modulo I is not a zero divisor in R/I. Hence, localizing with respect to s doesn't change the integrality of R/I, and we can use this fact to eliminate variables and find at last the quotient of a polynomial ring (which is a UFD) by an ideal generated by a single element, which turns out to be irreducible, so the ideal is prime and hence I is prime too. Let's detail this : first, compute the localized ideal I_s and enumerate the polynomials ($\clubsuit$) from (1) to (9). Now, define three new variables :

$$\begin{cases} y' := y - uz/s, \\ v' := v - xw/s, \\ t' := t - xu/s \end{cases}$$

Then, polynomials (6), (8) and (5) simply become respectively y', v' and t', which are the variables to be eliminated. Next, the polynomial (7) reads $uv' - wt'$. In the same way, polynomials (1), (2), (3), (4) and (9) become respectively

$$\begin{cases} zw + xu + s^2, \\ \left(y' + \frac{uz}{s}\right)\left(v' + \frac{xw}{s}\right) + \left(t' + \frac{xu}{s}\right)^2 + st' + xu, \\ wy' + \frac{uwz}{s} + us + ut' + \frac{xu^2}{s}, \\ zv' + \frac{xwz}{s} + xs + xt' + \frac{x^2u}{s}, \\ xy' - zt' \end{cases}$$

Now, by removing the redundant polynomials and since localization is exact, one has

$$(R/I)_s \simeq R_s/I_s \simeq \mathbb{C}[s^{\pm 1}][x, z, u, w] \big/ (zw + xu + s^2)$$

and by dividing the variables by s yields an isomorphism

$$(R/I)_s \simeq \mathbb{C}[s^{\pm 1}][x, z, u, w] \big/ (zw + xu + 1).$$

We may apply the same process again to this ring: localizing with respect to x for instance yields

$$(R/I)_{s,x} \simeq \mathbb{C}[s^{\pm 1}, x^{\pm 1}, z, w]$$

[20]See `https://math.stackexchange.com/q/95525`

and this last ring is obviously integral. Hence, R/I is integral too and this proves that the variety X is irreducible. Now, since I is prime, it is radical and the Jacobian criterion easily implies that X is smooth.

Then, define a morphism

$$\begin{array}{cccc} f & : & SL_3(\mathbb{C}) & \longrightarrow & X \\ & & \left(\begin{smallmatrix} a & d & g \\ b & e & h \\ c & f & j \end{smallmatrix}\right) & \longmapsto & \left[\begin{smallmatrix} s & x & z \\ u & t & y \\ w & v & r \end{smallmatrix}\right] \end{array}$$

where

$$\begin{bmatrix} s & x & z \\ u & t & y \\ w & v & r \end{bmatrix} = \begin{bmatrix} a(bf-ce) & a(cd-af) & a(ae-bd) \\ b(bf-ce) & b(cd-af) & b(ae-bd) \\ c(bf-ce) & c(cd-af) & c(ae-bd) \end{bmatrix} \in \mathrm{Proj}_{\mathbb{C}}(R).$$

(note that $r = -s-t$). Direct calculations show that $f(g)$ is in X and depends only on gB, so, by universal property of geometric quotient, f factors through $\overline{f} : SL_3/B \to X$. Then, σ is defined by the commutative diagram

$$\begin{tikzcd} \varphi(SL_3/B) \arrow[rr, "\sigma"] & & X \\ & SL_3/B \arrow[ul, "\varphi"', "\sim"] \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

It remains to show that σ is an isomorphism. Given the expression of σ in local coordinates, it is straightforward to see that it is injective. Thus, we have to prove that it is surjective. For this purpose, we shall look for a matrix $\left(\begin{smallmatrix} a & d & g \\ b & e & h \\ c & f & j \end{smallmatrix}\right) \in SU_3(\mathbb{C})$ such that

$$\begin{bmatrix} a\overline{g} & a\overline{h} & a\overline{j} \\ b\overline{g} & b\overline{h} & b\overline{j} \\ c\overline{g} & c\overline{h} & c\overline{j} \end{bmatrix} = \begin{bmatrix} s & x & z \\ u & t & y \\ w & v & r \end{bmatrix}.$$

Take $p := \left[\begin{smallmatrix} s & x & z \\ u & t & y \\ w & v & r \end{smallmatrix}\right] \in X$ and suppose $s \neq 0$. Normalize coordinates by imposing

$$|x|^2 + |y|^2 + \cdots + |t|^2 + |s+t|^2 = 1$$

Define $\rho_0 := \sqrt{|s|^2 + |x|^2 + |t|^2}$. Then, the equations allow to define

$$\begin{cases} a := \rho_0, \\ b := \frac{u\rho_0}{s}, \\ c := \frac{w\rho_0}{s} \end{cases} \quad \text{and} \quad \begin{cases} g := \frac{\overline{s}}{\rho_0}, \\ h := \frac{\overline{x}}{\rho_0}, \\ j := \frac{\overline{z}}{\rho_0} \end{cases}$$

Then, the inner product formula (in $\mathbb{R}^3$) suggests to define

$$\begin{cases} d := \frac{\overline{u}z}{\overline{s}} - \frac{\overline{w}x}{\overline{s}}, \\ e := \frac{\overline{w}s}{\overline{s}} - z, \\ f := x - \frac{\overline{u}s}{\overline{s}} \end{cases}$$

By using *every* equation defining I, we obtain that

$$\begin{pmatrix} \rho_0 & \overline{u}z/\overline{s} - \overline{w}x/\overline{s} & \overline{s}/\rho_0 \\ u\rho_0/s & \overline{w}s/\overline{s} - z & \overline{x}/\rho_0 \\ w\rho_0/s & x - \overline{u}s/\overline{s} & \overline{z}/\rho_0 \end{pmatrix} \in SU_3(\mathbb{C})$$

and is sent to p by σ. The same trick works if $x \neq 0$ or $z \neq 0$. Now, if $u \neq 0$ or $t \neq 0$ or $y \neq 0$, we can analogously find a preimage in $SU_3(\mathbb{C})$, as well as if $w \neq 0$ or $v \neq 0$. This proves that σ is bijective.

Finally, since $\sigma : \varphi(SL_3/B) \to X$ is a bijective morphism between irreducible varieties and X is smooth (hence normal), theorem 8.1.1 implies that σ is an isomorphism, which finishes the proof. □

Corollary 8.2.2. *We have an isomorphism of complex algebraic varieties*

$$\psi := \sigma \circ \varphi : SL_3/B \xrightarrow{\sim} \mathrm{Proj}(R/I).$$

Remark 8.2.3. *The generating set of I given by Proposition 8.2.1 is in fact a Gröbner basis of I, as it may be checked using Magma [MS19].*

From now on, we shall denote

$$\mathcal{F} := \mathrm{Proj}_{\mathbb{C}}(R/I)$$

with R and I as in Proposition 8.2.1. We finally have a direct consequence of Proposition 8.2.1:

Corollary 8.2.4. *The isomorphism ψ of Corollary 8.2.2 induces a homeomorphism between $\mathbb{R}$-points*

$$\psi(\mathbb{R}) : SL_3/B(\mathbb{R}) \xrightarrow{\sim} \mathcal{F}(\mathbb{R}).$$

It remains to identify the action of $\mathfrak{S}_3$ on $\mathcal{F}$ (at least on affine charts) in order to only work with coordinates. As we will focus on $\mathcal{F}(\mathbb{R})$ and for notational simplicity, we identify the action of $\mathfrak{S}_3$ only on $\mathcal{F}(\mathbb{R})$. The method to obtain equations for the action on $\mathcal{F}$ is exactly the same.

Proposition 8.2.5. *For each variable $q \in \{x, y, z, u, v, w, s, t\}$, let $U_q := \{q \neq 0\} \subset \mathbb{P}^7(\mathbb{C})$ be the standard affine open subset associated to q. Let also $p := \left[\begin{smallmatrix} s & x & z \\ u & t & y \\ w & v & -s-t \end{smallmatrix}\right] \in \mathcal{F}(\mathbb{R})$. We have the following formulae*

$$p \cdot s_\alpha = \begin{bmatrix} s & x & z \\ u & t & y \\ w & v & r \end{bmatrix} \cdot s_\alpha = \begin{cases} \begin{bmatrix} s(y-v) & x(y-v) & z(y-v) \\ s(w-z) & x(w-z) & z(w-z) \\ s(x-u) & x(x-u) & z(x-u) \end{bmatrix} & \text{if } \; p \in U_s \cup U_x \cup U_z \\[2em] \begin{bmatrix} u(y-v) & t(y-v) & y(y-v) \\ u(w-z) & t(w-z) & y(w-z) \\ u(x-u) & t(x-u) & y(x-u) \end{bmatrix} & \text{if } \; p \in U_u \cup U_t \cup U_y \\[2em] \begin{bmatrix} w(y-v) & v(y-v) & r(y-v) \\ w(w-z) & v(w-z) & r(w-z) \\ w(x-u) & v(x-u) & r(x-u) \end{bmatrix} & \text{if } \quad p \in U_w \cup U_v \end{cases}$$

and

$$p \cdot s_\beta = \begin{bmatrix} s & x & z \\ u & t & y \\ w & v & r \end{bmatrix} \cdot s_\beta = \begin{cases} \begin{bmatrix} s(v-y) & s(z-w) & s(u-x) \\ u(v-y) & u(z-w) & u(u-x) \\ w(v-y) & w(z-w) & w(u-x) \end{bmatrix} & \textit{if} \quad p \in U_s \cup U_u \cup U_w \\[2ex] \begin{bmatrix} x(v-y) & x(z-w) & x(u-x) \\ t(v-y) & t(z-w) & t(u-x) \\ v(v-y) & v(z-w) & v(u-x) \end{bmatrix} & \textit{if} \quad p \in U_x \cup U_t \cup U_v \\[2ex] \begin{bmatrix} z(v-y) & z(z-w) & z(u-x) \\ y(v-y) & y(z-w) & y(u-x) \\ r(v-y) & r(z-w) & r(u-x) \end{bmatrix} & \textit{if} \quad p \in U_z \cup U_y \end{cases}$$

Proof. Suppose for instance that $s \neq 0$. Then we can define $\rho_0 := \sqrt{|s|^2 + |x|^2 + |z|^2}$ and we have seen that the matrix

$$\begin{pmatrix} \rho_0 & \overline{u}z/\overline{s} - \overline{w}x/\overline{s} & \overline{s}/\rho_0 \\ u\rho_0/s & \overline{w}s/\overline{s} - z & \overline{x}/\rho_0 \\ w\rho_0/s & x - \overline{u}s/\overline{s} & \overline{z}/\rho_0 \end{pmatrix} \in SU_3(\mathbb{C})$$

is sent to p by $\sigma \circ \varphi$. Since the coordinates of p are real, this matrix simplifies to

$$\begin{pmatrix} \rho_0 & y - v & \frac{s}{\rho_0} \\ \frac{u\rho_0}{s} & w - z & \frac{x}{\rho_0} \\ \frac{w\rho_0}{s} & x - u & \frac{z}{\rho_0} \end{pmatrix} \in SO(3).$$

Then, we make s_α act on this matrix with a standard representative in SL_3:

$$\begin{pmatrix} \rho_0 & y - v & \frac{s}{\rho_0} \\ \frac{u\rho_0}{s} & w - z & \frac{x}{\rho_0} \\ \frac{w\rho_0}{s} & x - u & \frac{z}{\rho_0} \end{pmatrix} \cdot s_\alpha = \begin{pmatrix} \rho_0 & y - v & \frac{s}{\rho_0} \\ \frac{u\rho_0}{s} & w - z & \frac{x}{\rho_0} \\ \frac{w\rho_0}{s} & x - u & \frac{z}{\rho_0} \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} y - v & -\rho_0 & \frac{s}{\rho_0} \\ w - z & -\frac{u\rho_0}{s} & \frac{x}{\rho_0} \\ x - u & -\frac{w\rho_0}{s} & \frac{z}{\rho_0} \end{pmatrix}$$

which is easily seen to be sent by ψ to

$$\begin{bmatrix} \frac{s(y-v)}{\rho_0} & \frac{x(y-v)}{\rho_0} & \frac{z(y-v)}{\rho_0} \\ \frac{s(w-z)}{\rho_0} & \frac{x(w-z)}{\rho_0} & \frac{z(x-u)}{\rho_0} \\ \frac{s(x-u)}{\rho_0} & \frac{x(x-u)}{\rho_0} & \frac{z(x-u)}{\rho_0} \end{bmatrix} = \begin{bmatrix} s(y-v) & x(y-v) & z(y-v) \\ s(w-z) & x(w-z) & z(w-z) \\ s(x-u) & x(x-u) & z(x-u) \end{bmatrix}$$

and this proves the first formula. The other cases are perfectly similar. □

Remark 8.2.6. *Denote by τ the transposition involution*

$$\begin{array}{cccc} \tau : & \mathcal{F} & \longrightarrow & \mathcal{F} \\ & \left[\begin{smallmatrix} s & x & z \\ u & t & y \\ w & v & r \end{smallmatrix}\right] & \longmapsto & \left[\begin{smallmatrix} s & u & w \\ x & t & v \\ z & y & r \end{smallmatrix}\right] \end{array}$$

Then we have the following remarkable relation: if $p \in \mathcal{F}(\mathbb{R})$, then we have $\tau(p \cdot s_\alpha) = \tau(p) \cdot s_\beta$ and therefore

$$\tau s_\alpha \tau = s_\beta \quad \textit{on} \quad \mathcal{F}(\mathbb{R}).$$

We can further compute that $\tau = s_\alpha s_\beta s_\alpha =: w_0$ is the longest element in $\mathfrak{S}_3$ and, in particular,

$$\mathfrak{S}_3 = \langle s_\alpha, s_\beta \rangle = \langle s_\alpha, \tau \rangle .$$

This will be very useful to simplify the proof that our decomposition of $\mathcal{F}(\mathbb{R})$ is indeed $\mathfrak{S}_3$-cellular.

8.3 The cells of $\mathcal{F}(\mathbb{R})$ as connected components of non-vanishing loci of coordinates

The goal of this section and the following two is to find cells in $\mathcal{F}(\mathbb{R})$ that form a $\mathfrak{S}_3$-cellular decomposition of $\mathcal{F}(\mathbb{R})$.

A natural idea for the 0-skeleton of $\mathcal{F}(\mathbb{R})$ is to take $\mathfrak{S}_3$-translates of the the trivial class $1B \in SL_3/B$. This gives

$$\begin{cases} 1B = \left[\begin{smallmatrix} 0&0&1\\0&0&0\\0&0&0 \end{smallmatrix}\right], & s_\alpha B = \left[\begin{smallmatrix} 0&0&0\\0&0&1\\0&0&0 \end{smallmatrix}\right], \quad s_\beta B = \left[\begin{smallmatrix} 0&1&0\\0&0&0\\0&0&0 \end{smallmatrix}\right] \\ s_\alpha s_\beta B = \left[\begin{smallmatrix} 0&0&0\\1&0&0\\0&0&0 \end{smallmatrix}\right], & s_\beta s_\alpha B = \left[\begin{smallmatrix} 0&0&0\\0&0&0\\0&1&0 \end{smallmatrix}\right], \quad w_0 B = \left[\begin{smallmatrix} 0&0&0\\0&0&0\\1&0&0 \end{smallmatrix}\right]. \end{cases}$$

We can regroup these cells in the following way

$$\{1B, s_\alpha s_\beta B, s_\beta s_\alpha B\} = \left\{ p = \left[\begin{smallmatrix} s&x&z\\u&t&y\\w&v&r \end{smallmatrix}\right] \in \mathcal{F}(\mathbb{R}) \ ; \ x = y = w = 0 \right\}$$

and

$$\{s_\alpha B, s_\beta B, w_0 B\} = \{p \in \mathcal{F}(\mathbb{R}) \ ; \ u = v = z = 0\}.$$

This gives an idea of what to do for higher cells: impose the vanishing of some of the coordinates in $\mathcal{F}(\mathbb{R})$ and decompose into connected components (in the Euclidean topology). If we impose to three of the coordinates to be zero for the 0-cells, we can imagine to take the vanishing sets of two coordinates to get 1-cells. But one has to be careful at this point: the pairs of coordinates that are supposed to be zero have to be chosen in a suitable way.

The idea for 1-cells is confirmed by the following remark: in the case of A_1, the 1-cells were given as connected components of

$$\left\{ \left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right) T \in SU_2(\mathbb{C})/T \ ; \ a\bar{b} \in \mathbb{R}^* \right\},$$

see Remark 8.1.5. Here, we can find copies of such cells in $\mathcal{F}(\mathbb{R})$, for instance

$$\left\{ \left(\begin{smallmatrix} 1&0&0\\0&a&-b\\0&b&a \end{smallmatrix}\right) \in SU_3/T(\mathbb{R}) \ ; \ ab \neq 0 \right\}.$$

This last subset is homeomorphic to the first one, so it is indeed a disjoint union of two 1-cells. It can be written as constraints in $\mathcal{F}(\mathbb{R})$:

$$\left\{ p = \left[\begin{smallmatrix} s&x&z\\u&t&y\\w&v&r \end{smallmatrix}\right] \in \mathcal{F}(\mathbb{R}) \ ; \ s = t = 0, \ xz \neq 0 \right\}.$$

We'll have to introduce some notation here.

Notation 8.3.1. *Given variables $\nu_1, \nu_2, \eta_1, \eta_2 \in \{x, y, z, u, v, w, s, t\}$, denote*

$$e^{\nu_1,\nu_2}_{\eta_1,\eta_2,\pm} := \{p \in \mathcal{F}(\mathbb{R}) \ ; \ \nu_1 = \nu_2 = 0, \ \pm\eta_1\eta_2 > 0\}.$$

For instance, one writes

$$e^{s,t}_{x,z,+} = \{p \in \mathcal{F}(\mathbb{R}) \ ; \ s = t = 0, \ xz > 0\}.$$

By direct calculations, we get

$$e^{s,t}_{u,w,+} = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ a & -b & 0 \\ b & a & 0 \end{pmatrix} T \ ; \ ab > 0 \right\}, \ e^{s,t}_{x,v,+} = \left\{ \begin{pmatrix} a & -b & 0 \\ 0 & 0 & 1 \\ b & a & 0 \end{pmatrix} T \ ; \ ab > 0 \right\}, \ e^{s,t}_{y,z,+} = \left\{ \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} T \ ; \ ab > 0 \right\},$$
$$e^{s,t}_{x,z,+} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{pmatrix} T \ ; \ ab < 0 \right\}, \ e^{s,t}_{y,u,+} = \left\{ \begin{pmatrix} 0 & a & -b \\ 1 & 0 & 0 \\ 0 & b & a \end{pmatrix} T \ ; \ ab < 0 \right\}, \ e^{s,t}_{v,w,+} = \left\{ \begin{pmatrix} 0 & a & -b \\ 0 & b & a \\ 1 & 0 & 0 \end{pmatrix} T \ ; \ ab < 0 \right\},$$
$$e^{x,u}_{t,v,+} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ a & 0 & -b \\ b & 0 & a \end{pmatrix} T \ ; \ ab > 0 \right\}, \ e^{x,u}_{s,w,+} = \left\{ \begin{pmatrix} a & 0 & -b \\ 0 & 1 & 0 \\ b & 0 & a \end{pmatrix} T \ ; \ ab > 0 \right\}, \ e^{z,w}_{x,t,+} = \left\{ \begin{pmatrix} a & 0 & -b \\ b & 0 & a \\ 0 & 1 & 0 \end{pmatrix} T \ ; \ ab > 0 \right\}.$$

We can interchange every positivity symbol above to get nine other 1-cells and we obtain at last eighteen 1-cells:

$$e^{s,t}_{u,w,\pm}, \ e^{s,t}_{x,v,\pm}, \ e^{s,t}_{y,z,\pm}, \ e^{s,t}_{x,z,\pm}, \ e^{s,t}_{y,u,\pm}, \ e^{s,t}_{v,w,\pm}, \ e^{x,u}_{t,v,\pm}, \ e^{x,u}_{s,w,\pm}, \ e^{z,w}_{x,t,\pm}.$$

Denote by $\mathcal{F}(\mathbb{R})_1$ the union of the above 0 and 1-cells (it will of course be the 1-skeleton of $\mathcal{F}(\mathbb{R})$, but we don't know it's a cellular complex yet. Using SU_3-matrices, we see immediately that these cells are indeed freely permuted by $\mathfrak{S}_3$. It is straightforward to see that, in the Euclidean topology,

$$\overline{e^{s,t}_{u,w,\pm}} = e^{s,t}_{u,w,\pm} \cup \{s_\alpha s_\beta B\} \cup \{w_0 B\}.$$

Thus, $e^{s,t}_{u,w,+}$ and $e^{s,t}_{u,w,-}$ are two 1-cells connecting $\{s_\alpha s_\beta B\}$ and $\{w_0 B\}$. Doing the same calculations for the other 1-cells leads to represent them as in Figure 7, in which we recognize the (doubled) Goresky, Kottwitz, MacPherson (GKM) graph of $\mathfrak{S}_3$ here [Kaj15, §2.3].

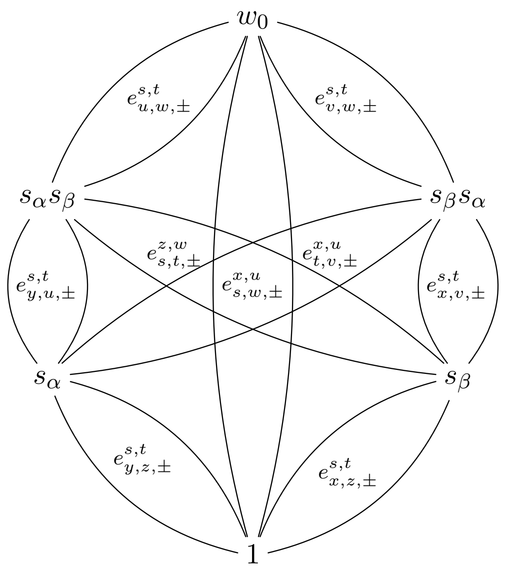


Figure 7: The 1-skeleton of $\mathcal{F}(\mathbb{R})$ in the GKM graph of $\mathfrak{S}_3$

Summarizing what we observed so far, we obtained the following lemma:

Lemma 8.3.2. *The closed subset*

$$\mathcal{F}(\mathbb{R})_1 = \bigcup_{\nu_1,\nu_2 \in \{x,y,z,u,v,w,s,t\}} \{\nu_1 = \nu_2 = 0\}$$

of $\mathcal{F}(\mathbb{R})$ is a $\mathfrak{S}_3$-CW-complex with six 0-cells given by

$$\mathcal{F}(\mathbb{R})_0 := \bigcup_{\nu \in \{x,y,z,u,v,w\}} \{s = t = 0, \ \forall \eta \in \{x,y,z,u,v,w\} \setminus \{\nu\}, \ \eta = 0\}$$

and eighteen 1*-cells given by*

$$e^{s,t}_{u,w,\pm},\ e^{s,t}_{x,v,\pm},\ e^{s,t}_{y,z,\pm},\ e^{s,t}_{x,z,\pm},\ e^{s,t}_{y,u,\pm},\ e^{s,t}_{v,w,\pm},\ e^{x,u}_{t,v,\pm},\ e^{x,u}_{s,w,\pm},\ e^{z,w}_{x,t,\pm}.$$

Moreover, the cellular structure on $\mathcal{F}(\mathbb{R})_1$ is regular and semialgebraic.

□

Continuing further down our line, we look for the 2-cells as Euclidean connected components of some subvarieties defined as vanishing sets of one coordinate in $\mathcal{F}(\mathbb{R})$. For 2-cells, we shall impose to one of the diagonal entries of $p = \left[\begin{smallmatrix} s & x & z \\ u & t & y \\ w & v & -s-t \end{smallmatrix}\right]$ to be zero. Note that, for example, the closed subvariety $\{s = 0\}$ of $\mathcal{F}$ is 2-dimensional and has two irreducible components given by $\{s = x = z = 0\}$ and $\{s = u = w = 0\}$. Concerning the subset $\{s = u = w = 0\}$, in order not to fall into a smaller cell, one has to impose $xy \neq 0$. Here again, we introduce some more notation:

Notation 8.3.3. *For variables $\nu, \nu', \eta_1, \eta_2 \in \{x, y, z, u, v, w, s, t\}$ and signs $\epsilon_1, \epsilon_2 \in \{\pm 1\}$, denote*

$$\begin{cases} e^{s}_{\eta_1^{\epsilon_1},\eta_2^{\epsilon_2}} := \{p \in \mathcal{F}(\mathbb{R})\ ;\ s = 0,\ \epsilon_i t \eta_i > 0,\ i = 1,2\}, \\ e^{s+t}_{\eta_1^{\epsilon_1},\eta_2^{\epsilon_2}} := \{p \in \mathcal{F}(\mathbb{R})\ ;\ s + t = 0,\ \epsilon_i t \eta_i > 0,\ i = 1,2\}, \\ e^{x}_{\eta_1^{\epsilon_1},\eta_2^{\epsilon_2}} := \{p \in \mathcal{F}(\mathbb{R})\ ;\ x = 0,\ \epsilon_i s \eta_i > 0,\ i = 1,2\}, \\ e^{u}_{\eta_1^{\epsilon_1},\eta_2^{\epsilon_2}} := \{p \in \mathcal{F}(\mathbb{R})\ ;\ u = 0,\ \epsilon_i s \eta_i > 0,\ i = 1,2\} \end{cases}$$

and

$$\begin{cases} e^{y=v}_{\eta_1^{\epsilon_1},\eta_2^{\epsilon_2}} := \{p \in \mathcal{F}(\mathbb{R})\ ;\ y = v,\ \epsilon_i t \eta_i > 0,\ i = 1,2\}, \\ e^{w=z}_{\eta_1^{\epsilon_1},\eta_2^{\epsilon_2}} := \{p \in \mathcal{F}(\mathbb{R})\ ;\ w = z,\ \epsilon_i s \eta_i > 0,\ i = 1,2\}, \\ e^{x=u}_{\eta_1^{\epsilon_1},\eta_2^{\epsilon_2}} := \{p \in \mathcal{F}(\mathbb{R})\ ;\ x = u,\ \epsilon_i t \eta_i > 0,\ i = 1,2\}. \end{cases}$$

For instance, we have

$$e^{s}_{x^+,y^-} = \{p \in \mathcal{F}(\mathbb{R})\ ;\ s = 0,\ xt > 0,\ yt < 0\} \text{ and } e^{y=v}_{x^+,z^+} = \{p \in \mathcal{F}(\mathbb{R})\ ;\ y = v,\ xt > 0,\ zt > 0\}.$$

Notice that

$$\{p \in \mathcal{F}\ ;\ s = x = z = 0\} = \left\{ \begin{pmatrix} 0 & d & g \\ b & e & h \\ c & f & j \end{pmatrix} T \in SU_3/T \right\}$$

and

$$\{p \in \mathcal{F}\ ;\ y = v\} = \left\{ \begin{pmatrix} a & 0 & g \\ b & e & h \\ c & f & j \end{pmatrix} T \in SU_3/T \right\}$$

and we have similar relations for $\{s = u = w = 0\}$, $\{x = t = v = 0\}$, $\{u = t = y = 0\}$, $\{z = y = s + t = 0\}$, $\{w = v = s + t = 0\}$, $\{w = z\}$ and $\{x = u\}$. For each one of them, one of the entries in the matrix class in SU_3/T has to be zero.

We consider the following *disjoint* subsets of $\mathcal{F}(\mathbb{R})$:

$$e^{s}_{x^\pm,y^\pm},\ e^{s}_{u^\pm,v^\pm},\ e^{x}_{y^\pm,u^\pm},\ e^{u}_{x^\pm,v^\pm},\ e^{s+t}_{x^\pm,y^\pm},\ e^{s+t}_{u^\pm,v^\pm},\ e^{y=v}_{x^\pm,z^\pm},\ e^{w=z}_{y^\pm,u^\pm},\ e^{x=u}_{v^\pm,w^\pm}.$$

These are the candidates for the 2-cells. We first have to show that they are indeed cells and then that they are freely permuted by $\mathfrak{S}_3$.

For example, take the subset $S := \{s = u = w = 0,\ xy \neq 0\} \subset \mathcal{F}(\mathbb{R})$ and let $p := \begin{bmatrix} 0 & x & z \\ 0 & t & y \\ 0 & v & -s-t \end{bmatrix} \in S$. We must have $t \neq 0$, because else, equations (♣) imply $x = 0$ or $y = 0$. Using the standard isomorphism

$$\begin{array}{ccc} \mathbb{P}^7(\mathbb{C}) \supset \{t \neq 0\} & \longrightarrow & \mathbb{A}^7(\mathbb{C}) \\ [x : y : z : u : v : w : s : t] & \longmapsto & \left(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}, \frac{u}{t}, \frac{v}{t}, \frac{w}{t}, \frac{s}{t}, 1\right) \end{array}$$

we find a homeomorphism (in real topology)

$$S \simeq \{(x, y, z, v) \in \mathbb{R}^4 \ ;\ xy = z,\ vz + x = 0,\ yv + 1 = 0,\ xy \neq 0\}.$$

Using the map $(x, y) \mapsto \left(x, y, xy, -\frac{1}{y}\right)$, this set is further homeomorphic to the set

$$\{(x, y) \in \mathbb{R}^2 \ ;\ xy \neq 0\}.$$

Thus, $S \simeq \{(x, y) \ ;\ xy \neq 0\}$ has four connected components in real topology, given by the $e^s_{x^\pm,y^\pm}$, and these are homeomorphic to $\mathbb{R}^2$, hence are 2-cells.

Remark 8.3.4. *Consider the set $S' := \{y = v, xz \neq 0\}$. For one of its points $p = \begin{bmatrix} s & x & z \\ u & t & y \\ w & y & -s-t \end{bmatrix}$, if $s = 0$, then equation (1) from (♣) implies $w = 0$ and (5) implies $u = 0$ and by (2), we get $y^2 + t^2 = 0$ and $y = t = 0$ because we are working with* real *points. So, only x and z are non-zero and we are in one of the 1-cells $e^{s,t}_{x,z,\pm}$. This explains why we impose $s \neq 0$ in the cells $e^{y=v}_{x^\pm,z^\pm}$, $e^{w=z}_{y^\pm,u^\pm}$ and $e^{x=u}_{v^\pm,w^\pm}$.*

As another example, take the set $S' := \{s \neq 0,\ y = v,\ xz \neq 0\}$. By using the homeomorphism

$$\begin{array}{ccc} \mathbb{P}^7(\mathbb{C}) \supset \{s \neq 0\} & \longrightarrow & \mathbb{A}^7(\mathbb{C}) \\ [x : y : z : u : v : w : s : t] & \longmapsto & \left(\frac{x}{s}, \frac{y}{s}, \frac{z}{s}, \frac{u}{s}, \frac{v}{s}, \frac{w}{s}, 1, \frac{t}{s}\right) \end{array}$$

we obtain a homeomorphism, as above,

$$\begin{aligned} S' &\simeq \left\{(x, y, z, u, w, t) \in \mathbb{R}^6 \ ;\ zw + t + 1 = 0,\ y^2 + t^2 + t = 0,\ t = xu,\ y = uz,\ xw = y,\ xz \neq 0\right\} \\ &\simeq \{(x, z, u, w) \in \mathbb{R}^4 \ ;\ zw + xu + 1 = 0,\ xw = uz,\ xz \neq 0\} \\ &\simeq \{(x, z, u) \in \mathbb{R}^4 \ ;\ uz^2 + x^2u + x = 0,\ xz \neq 0\} \\ &\simeq \{(x, z) \in \mathbb{R}^2 \ ;\ xz \neq 0\}. \end{aligned}$$

Thus, $S' \simeq \{(x, z) \ ;\ xz \neq 0\}$ has four connected components that are 2-cells. They are given by the $e^{y=v}_{x^\pm,z^\pm}$. The other examples can be treated in the same way. We get that the considered subsets are indeed 2-cells.

Now, since it is clear that $\mathfrak{S}_3$ acts on the 2-cells, we have to check that the induced action on cells is free. Take for instance a point p of the cell $e^s_{x^+,y^+}$. Note that $p \cdot \tau \in e^s_{u^+,v^+}$. Then the coordinates of p verify $s = u = w = 0$ and $x, y > 0$ and using Proposition 8.2.5, we see that the coordinates of $p \cdot s_\alpha$ are such that $s' = u' = w' = 0$ and $x't' = x^2(y - v)(w - z)$ and so, if $p \cdot s_\alpha \in e^s_{x^+,y^+}$, then $x't' > 0$, that is $z(v - y) > 0$. But, using equations (♣), we have $xy = zt$ and this implies $-xt = zv > zy > 0$, which contradicts the condition $xt > 0$. Next, if $p \cdot s_\alpha \tau \in e^s_{x^+,y^+}$, then $p \cdot s_\alpha = p \cdot s_\alpha \tau^2 \in e^s_{u^+,v^+}$, but since $p \cdot s_\alpha$ verifies $s' = u' = w' = 0$, this is excluded. Again by Proposition 8.2.5, we get that if $p \cdot s_\beta \in e^s_{x^+,y^+}$, then $v' = y'$ or $x' = t' = v' = 0$ and this is again impossible by construction. Last, if $p \cdot \tau s_\alpha \in e^s_{x^+,y^+}$, then $p \cdot \tau s_\alpha \tau = p \cdot s_\beta \in e^s_{u^\pm,v^\pm}$ and this last cell has no intersection with $e^s_{x^+,y^+}$ and we obtain again a contradiction.

In another hand, if $p \in e^{y=v}_{x^+,z^+}$, then $p \cdot s_\alpha \in e^s_{u^\pm,v^\pm}$ and $p \cdot s_\beta \in e^s_{x^\pm,z^\pm}$. Next, if $p \cdot s_\alpha \tau \in e^{y=v}_{x^+,z^+}$, then $e^s_{u^\pm,v^\pm} \ni p \cdot s_\alpha = p \cdot s_\alpha \tau^2 \in e^{y=v}_{*,*}$ which is excluded. In the same way, if $p \cdot \tau s_\alpha \in e^{y=v}_{x^+,z^+}$, then $e^s_{x^\pm,z^\pm} \ni p \cdot s_\beta = p \cdot \tau s_\alpha \tau \in e^{y=v}_{*,*}$, a contradiction.

The other examples are tedious to compute, but present no difficulty. By Remark 8.2.6, this allows to conclude that the induced action is indeed free. Thus, we have obtained the following result:

Lemma 8.3.5. *The closed subset*

$$\mathcal{F}(\mathbb{R})_2 := \{p \in \mathcal{F}(\mathbb{R}) \ ; \ st(s+t)(y-v)(w-z)(x-u) = 0\}$$

of $\mathcal{F}(\mathbb{R})$ is a $\mathfrak{S}_3$-CW-complex with the 0-cells and 1-cells given by the Lemma 8.3.2 and thirty-six 2-cells given by

$$e^s_{x^\pm,y^\pm},\ e^s_{u^\pm,v^\pm},\ e^x_{y^\pm,u^\pm},\ e^u_{x^\pm,v^\pm},\ e^{s+t}_{x^\pm,y^\pm},\ e^{s+t}_{u^\pm,v^\pm},\ e^{y=v}_{x^\pm,z^\pm},\ e^{w=z}_{y^\pm,u^\pm},\ e^{x=u}_{v^\pm,w^\pm}.$$

Moreover, this cellular structure is semialgebraic.

□

We are left to find the 3-cells in the open subset

$$\mathcal{F}(\mathbb{R}) \setminus \mathcal{F}(\mathbb{R})_2 = \{p \in \mathcal{F}(\mathbb{R}) \ ; \ st(s+t)(y-v)(w-z)(x-u) \neq 0\}.$$

The first observation to be made is that $\mathcal{F}(\mathbb{R}) \setminus \mathcal{F}(\mathbb{R})_2 \subset \{t \neq 0\}$, so we use the standard affine chart isomorphism to get

$$\mathcal{F}(\mathbb{R}) \setminus \mathcal{F}(\mathbb{R})_2 = \{[x:y:z:u:v:w:s:1] \in \mathcal{F}(\mathbb{R}) \ ; \ s(s+1)(y-v)(w-z)(x-u) \neq 0\}$$

$$\simeq \left\{ (x,y,z,u,v,w,s) \in \mathbb{A}^7(\mathbb{R}) \ ; \ \begin{cases} zw + s(s+1) = 0, \\ yv + s + 1 = 0, \\ yw + u(s+1) = 0, \\ vz + x(s+1) = 0, \end{cases} \text{and} \ \begin{cases} xu = s, \\ uz = ys, \\ uv = w, \\ xw = vs, \\ xy = z. \end{cases} \text{and} \ \begin{cases} s(s+1) \neq 0, \\ y \neq v, \\ w \neq z, \\ x \neq u \end{cases} \right\}$$

$$\simeq \left\{ (x,y,u,v) \in \mathbb{A}^4(\mathbb{R}) \ ; \ yv + ux + 1 = 0, \ ux(ux+1) \neq 0, \ y \neq v, \ x \neq u, \ uv \neq xy \right\}.$$

Then, using the Euclidean homeomorphism $(x,y,v) \mapsto \left(x, y, -\frac{yv+1}{x}, v\right)$, we obtain

$$\mathcal{F}(\mathbb{R}) \backslash \mathcal{F}(\mathbb{R})_2 \simeq \{(x,y,v) \in \mathbb{R}^3 \ ; \ x \neq 0, \ y \neq v, \ yv(yv+1) \neq 0, x^2 \neq -(yv+1), \ -v(yv+1) \neq x^2 y\},$$

thus

$$\mathcal{F}(\mathbb{R}) \setminus \mathcal{F}(\mathbb{R})_2 \simeq \{(x,y,v) \in \mathbb{R}^3 \ ; \ y \neq v, \ yv \neq -1, 0, \ x^2 y + v(yv+1) \neq 0, \ x^2 + yv + 1 \neq 0\}.$$

This open subset of $\mathbb{R}^3$ (or more exactly, its boundary) can be visualized using [Map19] (see Figure 8). Moreover, using its coordinate description in $\mathbb{R}^7$, it is easily seen to have twenty-four connected components. These will be the desired cells.

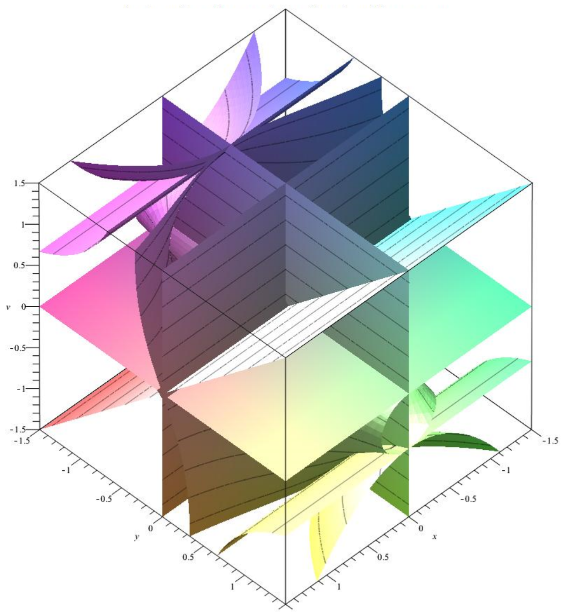


Figure 8: The open subset $\mathcal{F}(\mathbb{R}) \setminus \mathcal{F}(\mathbb{R})_2$

Notation 8.3.6. *Consider an index $i \in \{-1, 0, 1\}$, signs $\epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}$ and the following three real intervals:*

$$I_{-1} :=]-\infty, -1[, \ I_0 :=]-1, 0[, \ I_1 :=]0, +\infty[.$$

Then, denote

$$e^i_{\epsilon_1,\epsilon_2,\epsilon_3} := \left\{ [x:y:z:u:v:w:s:1] \in \mathcal{F}(\mathbb{R}) \setminus \mathcal{F}(\mathbb{R})_2 \ ; \ s \in I_i, \ \left\{ \begin{array}{l} \epsilon_1(y-v) > 0, \\ \epsilon_2(w-z) > 0, \\ \epsilon_3(x-u) > 0 \end{array} \right. \right\}$$

$$= \left\{ [x:y:z:u:v:w:s:t] \in \mathcal{F}(\mathbb{R}) \ ; \ \frac{s}{t} \in I_i, \ \left\{ \begin{array}{l} \epsilon_1 t(y-v) > 0, \\ \epsilon_2 t(w-z) > 0, \\ \epsilon_3 t(x-u) > 0 \end{array} \right. \right\}.$$

For instance, we have

$$e^0_{+,+,-} = \{[x:y:z:u:v:w:s:1] \in \mathcal{F}(\mathbb{R}) \ ; \ -1 < s < 0, \ y > v, \ w > z, \ x < u\}.$$

Take a subset $e^i_{\epsilon_1,\epsilon_2,\epsilon_3}$ as above. One has a homeomorphism

$$e^i_{\epsilon_1,\epsilon_2,\epsilon_3} \simeq \left\{ (x,y,v) \in \mathbb{R}^3 \ ; \ x \neq 0, \ yv \in I_i, \ \left\{ \begin{array}{l} \epsilon_1(y-v) > 0, \\ \epsilon_2 \mathrm{sgn}(x)(x^2 y + v(yv+1)) < 0, \\ \epsilon_3 \mathrm{sgn}(x)(x^2 + yv + 1) > 0 \end{array} \right. \right\}.$$

Hence, these open subsets are 3-cells in $\mathcal{F}(\mathbb{R})$. We still have to show that they are freely permuted by $\mathfrak{S}_3$. First, since the action on $\mathfrak{S}_3$ on $\mathcal{F}(\mathbb{R})$ is continuous (even smooth), the induced action on the open subset $\mathcal{F}(\mathbb{R}) \setminus \mathcal{F}(\mathbb{R})_2$ has to permute its connected components, which are the above 3-cells. It remains to show that this action is free. First, write down the action in normalized form

$$\begin{bmatrix} s & x & z \\ u & 1 & y \\ w & v & -s-1 \end{bmatrix} \cdot s_\alpha = \begin{bmatrix} \frac{s(y-v)}{x(w-z)} & \frac{y-v}{w-z} & \frac{z(y-v)}{x(w-z)} \\ \frac{s}{x} & 1 & \frac{z}{x} \\ \frac{s(x-u)}{x(w-z)} & \frac{x-u}{w-z} & \frac{z(x-u)}{x(w-z)} \end{bmatrix}$$

and

$$\begin{bmatrix} s & x & z \\ u & 1 & y \\ w & v & -s-1 \end{bmatrix} \cdot s_\beta = \begin{bmatrix} \frac{s(v-y)}{u(z-w)} & \frac{s}{u} & \frac{s(u-x)}{u(z-w)} \\ \frac{v-y}{z-w} & 1 & \frac{u-x}{z-w} \\ \frac{w(v-y)}{u(z-w)} & \frac{w}{u} & \frac{w(u-x)}{u(z-w)} \end{bmatrix}$$

and observe the following fact. If $p := [x : y : z : u : v : w : s : 1] \in \mathcal{F}(\mathbb{R}) \setminus \mathcal{F}(\mathbb{R})_2$, then $p \cdot s_\alpha := [x' : y' : z' : u' : v' : w' : s' : 1]$ verifies

$$w' - z' = \frac{s}{x}\left(\frac{x-u}{w-z}\right) - \frac{z}{x}\left(\frac{y-v}{w-z}\right) = \frac{xs - us - yz + zv}{x(w-z)} = -\frac{us + yz + x}{x(w-z)}$$

$$= -\frac{xu^2 + xy^2 + x}{x(w-z)} = -\frac{y^2 + u^2 + 1}{w-z}.$$

Hence, we get $\mathrm{sgn}(w' - z') = -\mathrm{sgn}(w - z)$. In the same way, if $p \cdot s_\alpha := [x' : y' : z' : u' : v' : w' : s' : 1]$, then

$$z' - w' = \frac{s}{u}\left(\frac{u-x}{z-w}\right) - \frac{w}{u}\left(\frac{v-y}{z-w}\right) = \frac{su - sx - vw + yw}{u(z-w)} = -\frac{sx + vw + u}{u(z-w)}$$

$$= -\frac{x^2u + uv^2 + u}{u(z-w)} = -\frac{x^2 + v^2 + 1}{z-w}$$

and $\mathrm{sgn}(z'-w') = -\mathrm{sgn}(z-w)$. This implies that, if $p := [x : y : z : u : v : w : s : 1] \in e^0_{+,+,+}$, then $p \cdot s_\alpha \notin e^0_{+,+,+}$ and $p \cdot s_\beta \notin e^0_{+,+,+}$. Also note that $p \cdot \tau = p \cdot w_0 \in e^0_{-,-,-}$. Next, using the formula $s_\beta s_\alpha = s_\alpha \tau$, to say that $p \cdot s_\beta s_\alpha \in e^0_{+,+,+}$ is equivalent to say that $p \cdot s_\alpha \in e^0_{-,-,-}$. If we still denote $p \cdot s_\alpha = [x' : y' : z' : u' : v' : w' : s' : 1]$ and if $p \cdot s_\alpha \in e^0_{-,-,-}$, then $y' < v'$ and $x' < u'$. This second equation reads $\frac{y-v}{w-z} < \frac{s}{x}$. Then, since $s < 0$, we must have $x < 0$, because else $0 < \frac{y-v}{w-z} < \frac{s}{x} < 0$, which is absurd. But in this case, the equation $s' < 0$ reads $\frac{s}{x}\frac{y-v}{w-z} < 0$ but this contradicts the fact that $\frac{y-v}{w-z} > 0$ and $\frac{s}{x} > 0$. In any case, we have $p \cdot s_\beta s_\alpha \notin e^0_{+,+,+}$. Finally, if $p \cdot s_\alpha s_\beta \in e^0_{+,+,+}$, then $p \cdot s_\beta \tau \in e^0_{+,+,+}$ and then $p \cdot s_\beta := [x' : y' : z' : u' : v' : w' : s' : 1] \in e^0_{-,-,-}$. We compute $x' - u' = \frac{s}{u} - \frac{v-y}{z-w} = x - \frac{y-v}{w-z}$ and, if $x' > u'$, then $x > \frac{y-v}{w-z} > 0$. But, since $s' = \frac{s}{u}\frac{v-y}{z-w} = x\frac{y-v}{w-z}$ and $s' < 0$, this implies $x < 0$ and this is absurd. Hence $p \cdot s_\alpha s_\beta \notin e^0_{+,+,+}$. We conclude that $p \cdot w \notin e^0_{+,+,+}$ for all $w \in \mathfrak{S}_3$ and so $e^0_{+,+,+} \cdot \mathfrak{S}_3 \cap e^0_{+,+,+} = \emptyset$. The other examples, though tedious, are treated in the same way.

We leave the boundary condition to the reader. Therefore, we have obtained the following result:

Lemma 8.3.7. *The open subset*

$$\mathcal{F}(\mathbb{R}) \setminus \mathcal{F}(\mathbb{R})_2 = \{p \in \mathcal{F}(\mathbb{R}) \ ; \ st(s+t)(y-v)(w-z)(x-u) \neq 0\}$$

of $\mathcal{F}(\mathbb{R})$ admits a partition into twenty-four semi-algebraic 3-cells given by

$$e^{-1,0,1}_{\pm,\pm,\pm}$$

that are freely permuted by $\mathfrak{S}_3$.

The previous Lemmas 8.3.2, 8.3.5 and 8.3.7 lead us to the following main result:

Theorem 8.3.8. *The real points $\mathcal{F}(\mathbb{R})$ of the projective variety $\mathcal{F}$ (isomorphic to SL_3/B) admits a semialgebraic $\mathfrak{S}_3$-cellular structure, which admits six 0-cells*

$$\begin{cases} \{s_\alpha B\} = \{[0:1:0:0:0:0:0:0]\},\ \{s_\beta B\} = \{[1:0:0:0:0:0:0:0]\}, \\ \{s_\alpha s_\beta B\} = \{[0:0:0:1:0:0:0:0]\},\ \{s_\beta s_\alpha B\} = \{[0:0:0:0:1:0:0:0]\}, \\ \{\{1B\} = \{[0:0:1:0:0:0:0:0]\},\ \{w_0 B\} = \{[0:0:0:0:0:1:0:0]\}, \end{cases}$$

eighteen 1-cells

$$e^{s,t}_{u,w,\pm},\ e^{s,t}_{x,v,\pm},\ e^{s,t}_{y,z,\pm},\ e^{s,t}_{x,z,\pm},\ e^{s,t}_{y,u,\pm},\ e^{s,t}_{v,w,\pm},\ e^{x,u}_{t,v,\pm},\ e^{x,u}_{s,w,\pm},\ e^{z,w}_{x,t,\pm},$$

thirty-six 2-cells

$$e^{s}_{x^\pm,y^\pm},\ e^{s}_{u^\pm,v^\pm},\ e^{x}_{y^\pm,u^\pm},\ e^{u}_{x^\pm,v^\pm},\ e^{s+t}_{x^\pm,y^\pm},\ e^{s+t}_{u^\pm,v^\pm},\ e^{y=v}_{x^\pm,z^\pm},\ e^{w=z}_{y^\pm,u^\pm},\ e^{x=u}_{v^\pm,w^\pm}$$

and twenty-four 3-cells

$$e^{-1,0,1}_{\pm,\pm,\pm}.$$

In particular, the associated cellular homology chain complex is a complex of free right $\mathbb{Z}[\mathfrak{S}_3]$-modules of the following shape:

$$\mathbb{Z}[\mathfrak{S}_3]^4 \longrightarrow \mathbb{Z}[\mathfrak{S}_3]^6 \longrightarrow \mathbb{Z}[\mathfrak{S}_3]^3 \longrightarrow \mathbb{Z}[\mathfrak{S}_3]\ .$$

Corollary 8.3.9. *The cellular homology complex of $\mathcal{F}(\mathbb{R})$ induced by the cellular structure given in Theorem 8.3.8 is a perfect complex of right $\mathbb{Z}[\mathfrak{S}_3]$-modules of the form*

$$\mathbb{Z}[\mathfrak{S}_3]^4 \longrightarrow \mathbb{Z}[\mathfrak{S}_3]^6 \longrightarrow \mathbb{Z}[\mathfrak{S}_3]^3 \longrightarrow \mathbb{Z}[\mathfrak{S}_3]\ .$$

9 The cellular homology complex of $\mathcal{F}(\mathbb{R})$ and its (co)homology

9.1 Cells representing orbits, boundaries and regularity

In order to determine explicitly the complex of Theorem 8.3.8, we have to compute the boundaries of the cells and for this, we first need to find cells that represent orbits of cells. After some tedious calculations using abundantly equations ($\clubsuit$) and the fact that $\tau = s_\alpha s_\beta s_\alpha$, or using the command “SamplePoints” form the package “RegularChains” in [Map19], one can obtain the following relations, which may be verified by hand calculations afterwards:

<u>Dimension 1</u>:

$$\begin{cases} e^{s,t}_{y,z,+}\cdot s_\alpha = e^{s,t}_{y,z,-} \\ e^{s,t}_{y,z,+}\cdot s_\beta = e^{z,w}_{x,t,+} \\ e^{s,t}_{y,z,+}\cdot s_\alpha s_\beta = e^{z,w}_{x,t,-} \\ e^{s,t}_{y,z,+}\cdot s_\beta s_\alpha = e^{s,t}_{v,w,-} \\ e^{s,t}_{y,z,+}\cdot w_0 = e^{s,t}_{v,w,+} \end{cases} \quad \begin{cases} e^{s,t}_{x,z,-}\cdot s_\alpha = e^{x,u}_{t,v,+} \\ e^{s,t}_{x,z,-}\cdot s_\beta = e^{s,t}_{x,z,+} \\ e^{s,t}_{x,z,-}\cdot s_\alpha s_\beta = e^{s,t}_{u,w,+} \\ e^{s,t}_{x,z,-}\cdot s_\beta s_\alpha = e^{x,u}_{t,v,-} \\ e^{s,t}_{x,z,-}\cdot w_0 = e^{s,t}_{u,w,-} \end{cases} \quad \begin{cases} e^{x,u}_{s,w,-}\cdot s_\alpha = e^{s,t}_{y,u,+} \\ e^{x,u}_{s,w,-}\cdot s_\beta = e^{s,t}_{x,v,-} \\ e^{x,u}_{s,w,-}\cdot s_\alpha s_\beta = e^{s,t}_{y,u,-} \\ e^{x,u}_{s,w,-}\cdot s_\beta s_\alpha = e^{s,t}_{x,v,+} \\ e^{x,u}_{s,w,-}\cdot w_0 = e^{x,u}_{s,w,+} \end{cases}$$

Dimension 2:

$$\begin{cases} e^s_{x^+,y^+}\cdot s_\alpha = e^s_{x^-,y^+} \\ e^s_{x^+,y^+}\cdot s_\beta = e^{y=v}_{x^+z^-} \\ e^s_{x^+,y^+}\cdot s_\alpha s_\beta = e^{y=v}_{x^-,z^+} \\ e^s_{x^+,y^+}\cdot s_\beta s_\alpha = e^s_{u^-,v^+} \\ e^s_{x^+,y^+}\cdot w_0 = e^s_{u^+,v^+} \end{cases} \quad \begin{cases} e^s_{x^+,y^-}\cdot s_\alpha = e^s_{x^-,y^-} \\ e^s_{x^+,y^-}\cdot s_\beta = e^{y=v}_{x^+,z^+} \\ e^s_{x^+,y^-}\cdot s_\alpha s_\beta = e^{y=v}_{x^-,z^-} \\ e^s_{x^+,y^-}\cdot s_\beta s_\alpha = e^s_{u^-,v^-} \\ e^s_{x^+,y^-}\cdot w_0 = e^s_{u^+,v^-} \end{cases} \quad \begin{cases} e^x_{y^+,u^+}\cdot s_\alpha = e^x_{y^-,u^-} \\ e^x_{y^+,u^+}\cdot s_\beta = e^{w=z}_{y^-,u^+} \\ e^x_{y^+,u^+}\cdot s_\alpha s_\beta = e^{w=z}_{y^+,u^-} \\ e^x_{y^+,u^+}\cdot s_\beta s_\alpha = e^u_{x^-,v^-} \\ e^x_{y^+,u^+}\cdot w_0 = e^u_{x^+,v^+} \end{cases}$$

$$\begin{cases} e^x_{y^+,u^-}\cdot s_\alpha = e^x_{y^-,u^+} \\ e^x_{y^+,u^-}\cdot s_\beta = e^{w=z}_{y^-,u^-} \\ e^x_{y^+,u^-}\cdot s_\alpha s_\beta = e^{w=z}_{y^+,u^+} \\ e^x_{y^+,u^-}\cdot s_\beta s_\alpha = e^u_{x^+,v^-} \\ e^x_{y^+,u^-}\cdot w_0 = e^u_{x^-,v^+} \end{cases} \quad \begin{cases} e^{s+t}_{u^+,v^+}\cdot s_\alpha = e^{s+t}_{u^+,v^-} \\ e^{s+t}_{u^+,v^+}\cdot s_\beta = e^{x=u}_{v^+,w^-} \\ e^{s+t}_{u^+,v^+}\cdot s_\alpha s_\beta = e^{x=u}_{v^-,w^+} \\ e^{s+t}_{u^+,v^+}\cdot s_\beta s_\alpha = e^{s+t}_{x^+,y^-} \\ e^{s+t}_{u^+,v^+}\cdot w_0 = e^{s+t}_{x^+,y^+} \end{cases} \quad \begin{cases} e^{s+t}_{u^-,v^+}\cdot s_\alpha = e^{s+t}_{u^-,v^-} \\ e^{s+t}_{u^-,v^+}\cdot s_\beta = e^{x=u}_{v^+,w^+} \\ e^{s+t}_{u^-,v^+}\cdot s_\alpha s_\beta = e^{x=u}_{v^-,w^-} \\ e^{s+t}_{u^-,v^+}\cdot s_\beta s_\alpha = e^{s+t}_{x^-,y^-} \\ e^{s+t}_{u^-,v^+}\cdot w_0 = e^{s+t}_{x^-,y^+} \end{cases}$$

Dimension 3:

$$\begin{cases} e^1_{+,+,+}\cdot s_\alpha = e^{-1}_{+,-,+} \\ e^1_{+,+,+}\cdot s_\beta = e^0_{+,-,-} \\ e^1_{+,+,+}\cdot s_\alpha s_\beta = e^0_{-,+,+} \\ e^1_{+,+,+}\cdot s_\beta s_\alpha = e^{-1}_{-,+,-} \\ e^1_{+,+,+}\cdot w_0 = e^1_{-,-,-} \end{cases} \quad \begin{cases} e^1_{+,+,-}\cdot s_\alpha = e^0_{+,-,+} \\ e^1_{+,+,-}\cdot s_\beta = e^{-1}_{+,-,-} \\ e^1_{+,+,-}\cdot s_\alpha s_\beta = e^{-1}_{-,+,+} \\ e^1_{+,+,-}\cdot s_\beta s_\alpha = e^0_{-,+,-} \\ e^1_{+,+,-}\cdot w_0 = e^1_{-,-,+} \end{cases} \quad \begin{cases} e^1_{+,-,-}\cdot s_\alpha = e^{-1}_{+,+,-} \\ e^1_{+,-,-}\cdot s_\beta = e^0_{+,+,+} \\ e^1_{+,-,-}\cdot s_\alpha s_\beta = e^0_{-,-,-} \\ e^1_{+,-,-}\cdot s_\beta s_\alpha = e^{-1}_{-,-,+} \\ e^1_{+,-,-}\cdot w_0 = e^1_{-,+,+} \end{cases}$$

$$\begin{cases} e^1_{+,-,+}\cdot s_\alpha = e^0_{+,+,-} \\ e^1_{+,-,+}\cdot s_\beta = e^{-1}_{+,+,+} \\ e^1_{+,-,+}\cdot s_\alpha s_\beta = e^{-1}_{-,-,-} \\ e^1_{+,-,+}\cdot s_\beta s_\alpha = e^0_{-,-,+} \\ e^1_{+,-,+}\cdot w_0 = e^1_{-,+,-} \end{cases}$$

From this, we deduce the following consequence:

Proposition 9.1.1. *In $\mathcal{F}(\mathbb{R})$, the cells*

$$\begin{cases} \{B\} = \{[0:0:1:0:0:0:0:0]\}, \\ e^{s,t}_{y,z,+},\ e^{s,t}_{x,z,-},\ e^{x,u}_{s,w,-}, \\ e^s_{x^+,y^+},\ e^s_{x^+,y^-},\ e^x_{y^+,u^+},\ e^x_{y^+,u^-},\ e^{s+t}_{u^+,v^+},\ e^{s+t}_{u^-,v^+}, \\ e^1_{+,+,+},\ e^1_{+,+,-},\ e^1_{+,-,-},\ e^1_{+,-,+} \end{cases}$$

respectively represent the set of i-cells $(0 \le i \le 3)$ under the right action of $\mathfrak{S}_3$.

We now have to compute the boundary of these cells.

Proposition 9.1.2. *For each representing cell e in Proposition 9.1.1, the one codimensional cells appearing in the boundary $\partial e = \overline{e} \setminus e$ are the following*

$$\begin{cases} \partial e^{s,t}_{y,z,+} = \{B\} \cup \{Bs_\alpha\}, \\ \partial e^{s,t}_{x,z,-} = \{B\} \cup \{Bs_\beta\}, \\ \partial e^{x,u}_{s,w,-} = \{B\} \cup \{Bw_0\} \end{cases}$$

$$\begin{cases} \partial e^{s}_{x^+,y^+} = e^{x,u}_{t,v,-} \cup e^{s,t}_{y,z,+} \cup e^{s,t}_{x,v,-} \cup e^{s,t}_{x,z,+}, \\ \partial e^{s}_{x^+,y^-} = e^{x,u}_{t,v,+} \cup e^{s,t}_{y,z,+} \cup e^{s,t}_{x,v,+} \cup e^{s,t}_{x,z,-}, \\ \partial e^{x}_{y^+,u^+} = e^{s,t}_{y,u,+} \cup e^{x,u}_{s,w,-} \cup e^{s,t}_{u,w,-} \cup e^{s,t}_{y,z,+}, \\ \partial e^{x}_{y^+,u^-} = e^{s,t}_{y,u,-} \cup e^{x,u}_{s,w,+} \cup e^{s,t}_{u,w,-} \cup e^{s,t}_{y,z,-}, \\ \partial e^{s+t}_{u^+,v^+} = e^{z,w}_{x,t,-} \cup e^{s,t}_{v,w,+} \cup e^{s,t}_{x,v,-} \cup e^{s,t}_{u,w,+}, \\ \partial e^{s+t}_{u^-,v^+} = e^{z,w}_{x,t,+} \cup e^{s,t}_{v,w,-} \cup e^{s,t}_{x,v,+} \cup e^{s,t}_{u,w,+} \end{cases} \qquad \begin{cases} \partial e^{1}_{+,+,+} = e^{s}_{u^-,v^-} \cup e^{x=u}_{v^-,w^+} \cup e^{x}_{y^+,u^-}, \\ \partial e^{1}_{+,+,-} = e^{s}_{x^-,y^+} \cup e^{x=u}_{v^-,w^+} \cup e^{u}_{x^-,v^-}, \\ \partial e^{1}_{+,-,-} = e^{s}_{u^+,v^-} \cup e^{x=u}_{v^-,w^-} \cup e^{x}_{y^+,u^+}, \\ \partial e^{1}_{+,-,+} = e^{s}_{x^+,y^+} \cup e^{x=u}_{v^-,w^-} \cup e^{u}_{x^+,v^-} \end{cases}$$

The proof of this proposition relies entirely on the following lemma:

Lemma 9.1.3. *The closures of the representing cells satisfy the following relations:*

$$\begin{cases} \overline{e^{s,t}_{y,z,+}} = e^{s,t}_{y,z,+} \cup \{B\} \cup \{Bs_\alpha\}, \\ \overline{e^{s,t}_{x,z,-}} = e^{s,t}_{x,z,-} \cup \{B\} \cup \{Bs_\beta\}, \\ \overline{e^{x,u}_{s,w,-}} = e^{x,u}_{s,w,-} \cup \{B\} \cup \{Bw_0\} \end{cases} \qquad \begin{cases} \overline{e^{s}_{x^+,y^+}} = e^{s}_{x^+,y^+} \cup \overline{e^{x,u}_{t,v,-}} \cup \overline{e^{s,t}_{y,z,+}} \cup \overline{e^{s,t}_{x,v,-}} \cup \overline{e^{s,t}_{x,z,+}}, \\ \overline{e^{s}_{x^+,y^-}} = e^{s}_{x^+,y^-} \cup \overline{e^{x,u}_{t,v,+}} \cup \overline{e^{s,t}_{y,z,+}} \cup \overline{e^{s,t}_{x,v,+}} \cup \overline{e^{s,t}_{x,z,-}}, \\ \overline{e^{x}_{y^+,u^+}} = e^{x}_{y^+,u^+} \cup \overline{e^{s,t}_{y,u,+}} \cup \overline{e^{x,u}_{s,w,-}} \cup \overline{e^{s,t}_{u,w,-}} \cup \overline{e^{s,t}_{y,z,+}}, \\ \overline{e^{x}_{y^+,u^-}} = e^{x}_{y^+,u^-} \cup \overline{e^{s,t}_{y,u,-}} \cup \overline{e^{x,u}_{s,w,+}} \cup \overline{e^{s,t}_{u,w,-}} \cup \overline{e^{s,t}_{y,z,-}}, \\ \overline{e^{s+t}_{u^+,v^+}} = e^{s+t}_{u^+,v^+} \cup \overline{e^{z,w}_{x,t,-}} \cup \overline{e^{s,t}_{v,w,+}} \cup \overline{e^{s,t}_{x,v,-}} \cup \overline{e^{s,t}_{u,w,+}}, \\ \overline{e^{s+t}_{u^-,v^+}} = e^{s+t}_{u^-,v^+} \cup \overline{e^{z,w}_{x,t,+}} \cup \overline{e^{s,t}_{v,w,-}} \cup \overline{e^{s,t}_{x,v,+}} \cup \overline{e^{s,t}_{u,w,+}} \end{cases}$$

$$\begin{cases} \overline{e^{1}_{+,+,+}} = e^{1}_{+,+,+} \cup \overline{e^{s}_{u^-,v^-}} \cup \overline{e^{x=u}_{v^-,w^+}} \cup \overline{e^{x}_{y^+,u^-}}, \\ \overline{e^{1}_{+,+,-}} = e^{1}_{+,+,-} \cup \overline{e^{s}_{x^-,y^+}} \cup \overline{e^{x=u}_{v^-,w^+}} \cup \overline{e^{u}_{x^-,v^-}}, \\ \overline{e^{1}_{+,-,-}} = e^{1}_{+,-,-} \cup \overline{e^{s}_{u^+,v^-}} \cup \overline{e^{x=u}_{v^-,w^-}} \cup \overline{e^{x}_{y^+,u^+}}, \\ \overline{e^{1}_{+,-,+}} = e^{1}_{+,-,+} \cup \overline{e^{s}_{x^+,y^+}} \cup \overline{e^{x=u}_{v^-,w^-}} \cup \overline{e^{u}_{x^+,v^-}} \end{cases}$$

Proof. We shall use the Fubini-Study metric on $\mathbb{P}^7(\mathbb{R})$. Recall that, for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, for $\langle \cdot, \cdot \rangle$ a scalar (Hermitian) product on $\mathbb{K}^{n+1}$ and for a line $\ell \in \mathbb{P}^n(\mathbb{K})$, one has $T_\ell \mathbb{P}^n(\mathbb{K}) = \mathrm{Hom}_{\mathbb{K}}(\ell, \ell^\perp)$ and this space is equipped with the dot product $(f, g) := \mathrm{tr}(f \circ g^*)$ and this gives $\mathbb{P}^n(\mathbb{K})$ a Riemannian (Kähler) metric, called the *Fubini-Study metric*. The distance associated to this metric is given by (see [AG07])

$$\forall p, q \in \mathbb{P}^n(\mathbb{K}),\ d_{FS}(p,q) = \arccos \sqrt{\frac{|\langle p,q\rangle|^2}{\langle p,p\rangle\langle q,q\rangle}}.$$

In our case, $\mathbb{K} = \mathbb{R}$ and

$$\forall p, q \in \mathbb{P}^7(\mathbb{R}),\ d_{FS}(p,q) = \arccos \frac{|\langle p,q\rangle|}{\|p\|\|q\|}.$$

The method for finding which cells appear in the closure of a given cell e will be the following: first, we determine which 0-cells do not appear in $\overline{e}$ using the distance d_{FS}. Then, we know that every k-cell in $\overline{e}$ must not contain these 0-cells in its closure. This will leave only a few possible cells and we can check each one of them using local charts. To simplify, we shall do this explicitly only for one cell per dimension, since the other can be obtained in the same way.

We start with 1-cells. Take for instance $e^{s,t}_{y,z,+}$. Recall that

$$\begin{aligned} e^{s,t}_{y,z,+} &= \{[x:y:z:u:v:w:s:t] \in \mathcal{F}(\mathbb{R})\ ;\ s = t = 0,\ yz > 0\} \\ &= \{[0:y:z:0:0:0:0:0] \in \mathbb{P}^7(\mathbb{R})\ ;\ yz > 0\}. \end{aligned}$$

Next, for $p := [x:y:z:u:v:w:s:t]$ and $q := [x':y':z':u':v':w':s':t']$ in $\mathcal{F}(\mathbb{R})$, the induced distance between p and q is given by

$$d_{FS}(p,q) = \arccos \sqrt{\frac{\langle p,q\rangle^2}{\|p\|^2\|q\|^2}}.$$

In order to simplify this, we assume for the rest of this proof that *the homogeneous coordinates of every point in $\mathcal{F}(\mathbb{R})$ are normalized*, that is, for every $p = [x:y:z:u:v:w:s:t] \in \mathcal{F}(\mathbb{R})$, we impose that $x^2+y^2+z^2+u^2+v^2+w^2+s^2+t^2 = 1$. This gives

$$d_{FS}(p,q) = \arccos|\langle p,q\rangle| = \arccos|xx'+yy'+zz'+uu'+vv'+ww'+ss'+tt'|$$

and thus, for $\varepsilon > 0$ small enough,

$$d_{FS}(p,q) < \varepsilon \iff \langle p,q\rangle^2 > \cos(\varepsilon)^2 \implies \langle p,q\rangle^2 > 1-\varepsilon^2.$$

Then, we get the following criterion :

$$\forall A \subset \mathcal{F}(\mathbb{R}),\ \forall p \in \mathcal{F}(\mathbb{R}),\ p \in \overline{A} \Leftrightarrow \forall \varepsilon > 0,\ \exists a \in A\ ;\ \langle p,a\rangle^2 > 1-\varepsilon. \qquad (\partial)$$

Take a 0-cell e^0. It is given by the non-vanishing of just one coordinate ν in $\{x,y,z,u,v,w\}$. Then, applying the above criterion (∂) to $A = e^{s,t}_{y,z,+}$ yields

$$e^0 \subset \overline{e^{s,t}_{y,z,+}} \iff \forall \varepsilon > 0,\ \exists [x:y:z:u:v:w:s:t] \in e^{s,t}_{y,z,+}\ ;\ \nu^2 > 1-\varepsilon.$$

For instance,

$$Bs_\beta \in \overline{e^{s,t}_{y,z,+}} \iff \forall \varepsilon > 0,\ \exists [x:y:z:u:v:w:s:t] \in e^{s,t}_{y,z,+}\ ;\ x^2 > 1-\varepsilon.$$

Using the description $e^{s,t}_{y,z,+} = \{[0:y:z:0:0:0:0:0] \in \mathbb{P}^7(\mathbb{R})\ ;\ yz > 0\}$, we obtain immediately that

$$B \cdot w \in \overline{e^{s,t}_{y,z,+}} \iff w \in \{1, s_\alpha\}.$$

and hence
$$\overline{e^{s,t}_{y,z,+}} = e^{s,t}_{y,z,+} \cup \{B\} \cup \{Bs_\alpha\}.$$

Now, take the 2-cell $e^s_{x^+,y^+}$ given by

$$\begin{aligned} e^s_{x^+,y^+} &= \{[x:y:z:u:v:w:s:t] \in \mathcal{F}(\mathbb{R}) \ ; \ s=0,\ xt>0,\ yt>0\} \\ &= \{[x:y:z:0:v:0:0:t] \in \mathcal{F}(\mathbb{R}) \ ; \ xt>0,\ yt>0\}. \end{aligned}$$

Now, using the criterion (∂), we obtain that

$$B \cdot w \in \overline{e^s_{x^+,y^+}} \iff w \in \{1, s_\alpha, s_\beta, s_\beta s_\alpha\}.$$

So, to find the 1-cells in $\overline{e^s_{x^+,y^+}}$, we look at 1-cells only having B, Bs_α, Bs_β or $Bs_\beta s_\alpha$ in their closure. Having computed the closure of every 1-cell using the action of $\mathfrak{S}_3$ gives that the only 1-cells that can appear in $\overline{e^s_{x^+,y^+}}$ are $e^{s,t}_{x,v,\pm}$, $e^{x,u}_{t,v,\pm}$, $e^{s,t}_{x,z,\pm}$ and $e^{s,t}_{y,z,\pm}$. Since the five cells $e^{s,t}_{x,v,\pm}$, $e^{x,u}_{t,v,\pm}$ and $e^s_{x^+,y^+}$ are fully contained in the local chart $\{v \neq 0\}$ we can compute the Euclidean boundary of the image of $e^s_{x^+,y^+}$ under this chart. Recall that we have a homeomorphism

$$\begin{array}{cccccc} \phi_v & : & \{v \neq 0\} & \longrightarrow & \mathbb{R}^7 \\ & & [x:y:z:u:v:w:s:t] & \longmapsto & \left(\frac{x}{v}, \frac{y}{v}, \frac{z}{v}, \frac{u}{v}, \frac{w}{v}, \frac{s}{v}, \frac{t}{v}\right) \end{array}$$

so,

$$\begin{aligned} e^s_{x^+,y^+} \simeq \phi_v(e^s_{x^+,y^+}) &= \{(x,y,z,0,0,0,t) \in \mathbb{R}^7 \ ; \ y+t^2=0,\ z+xt=0,\ xy=zt,\ xt>0,\ yt>0\} \\ &= \{(x,y,z,0,0,0,t) \in \mathbb{R}^7 \ ; \ y=-t^2,\ z=-xt,\ xt>0,\ yt>0\}. \end{aligned}$$

Define the homeomorphism

$$\begin{array}{cccccc} \psi_v & : & \{(x,y,z,0,0,0,t) \in \mathbb{R}^7 \ ; \ y=-t^2,\ z=-xt\} & \to & \mathbb{R}^2 \\ & & (x,y,z,0,0,0,t) & \mapsto & (x,t) \end{array}$$

Then,

$$\psi_v(\phi_v(e^s_{x^+,y^+})) = \{(x,t) \in \mathbb{R}^2 \ ; \ xt>0,\ -t^3>0\} = \{(x,t) \in \mathbb{R}^2 \ ; \ x<0,\ t<0\}.$$

Thus, the closure of $e^s_{x^+,y^+}$ in $\{v \neq 0\}$ is given by

$$\begin{aligned} \psi_v\big(\phi_v(\overline{e^s_{x^+,y^+}}^{v\neq 0})\big) &= \overline{\psi_v(\phi_v(e^s_{x^+,y^+}))}^{\mathbb{R}^2} \simeq \{(x,t) \in \mathbb{R}^2 \ ; \ x \le 0,\ t \le 0\} \\ &= \psi_v(\phi_v(e^s_{x^+,y^+})) \cup \{(0,t) \ ; \ t<0\} \cup \{(x,0) \ ; \ x<0\} \cup \{(0,0)\}. \end{aligned}$$

Whence,

$$\overline{e^s_{x^+,y^+}}^{v\neq 0} = e^s_{x^+,y^+} \cup \phi_v^{-1} \circ \psi_v^{-1}(\{(0,t) \ ; \ t<0\}) \cup \phi_v^{-1} \circ \psi_v^{-1}(\{(x,0) \ ; \ x<0\}) \cup \phi_v^{-1} \circ \psi_v^{-1}(0,0)$$

$$= e^s_{x^+,y^+} \cup \phi_v^{-1}(\{(0,-t^2,0,0,0,0,t) \ ; \ t<0\}) \cup \phi_v^{-1}(\{(x,0,0,0,0,0,0) \ ; \ x<0\}) \cup \phi_v^{-1}(0,0,0,0,0,0,0)$$

$$\begin{aligned} &= e^s_{x^+,y^+} \cup \left\{ \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & t & -t^2 \\ 0 & 1 & -t \end{smallmatrix}\right] \in \mathcal{F}(\mathbb{R}) \ ; \ t<0 \right\} \cup \left\{ \left[\begin{smallmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right] \in \mathcal{F}(\mathbb{R}) \ ; \ x<0 \right\} \cup \{Bs_\beta s_\alpha\} \\ &= e^s_{x^+,y^+} \cup \left\{ \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & t & y \\ 0 & v & -t \end{smallmatrix}\right] \in \mathcal{F}(\mathbb{R}) \ ; \ tv<0 \right\} \cup \left\{ \left[\begin{smallmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & v & 0 \end{smallmatrix}\right] \in \mathcal{F}(\mathbb{R}) \ ; \ xv<0 \right\} \cup \{Bs_\beta s_\alpha\} \\ &= e^s_{x^+,y^+} \cup e^{x,u}_{t,v,-} \cup e^{s,t}_{x,v,-} \cup \{Bs_\beta s_\alpha\} \end{aligned}$$

$$= e^s_{x^+,y^+} \cup \overline{e^{x,u}_{t,v,-}}^{v\neq 0} \cup \overline{e^{s,t}_{x,v,-}}^{v\neq 0}.$$

For the four remaining cells, we use the homeomorphisms

$$\begin{array}{cccc} \phi_z & : & \{z \neq 0\} & \longrightarrow & \mathbb{R}^7 \\ & & [x:y:z:u:v:w:s:t] & \longmapsto & \left(\frac{x}{z}, \frac{y}{z}, \frac{u}{z}, \frac{v}{z}, \frac{w}{z}, \frac{s}{z}, \frac{t}{z}\right) \end{array}$$

and

$$\begin{array}{cccc} \psi_z & : & \{(x,y,0,v,0,0,t)\ ;\ t^2+vy=0,\ tx+v=0,\ xy-t=0\} & \longrightarrow & \mathbb{R}^2 \\ & & (x,y,0,v,0,0,t) & \longmapsto & (x,y) \end{array}$$

in order to have

$$\psi_z(\phi_z(e^s_{x^+,y^+})) = \{(x,y)\in\mathbb{R}^2\ ;\ x^2y>0,\ xy^2>0\} = \{(x,y)\in\mathbb{R}^2\ ;\ x>0,\ y>0\}.$$

We compute

$$\overline{e^s_{x^+,y^+}}^{z\neq 0} = e^s_{x^+,y^+} \cup \phi_z^{-1}\circ\psi_z^{-1}(\{(0,y)\ ;\ y>0\}) \cup \phi_z^{-1}\circ\psi_z^{-1}(\{(x,0)\ ;\ x>0\}) \cup \phi_z^{-1}\circ\psi_z^{-1}(0,0)$$

$$= e^s_{x^+,y^+} \cup \left\{\left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{smallmatrix}\right] \in \mathcal{F}(\mathbb{R})\ ;\ y>0\right\} \cup \left\{\left[\begin{smallmatrix} 0 & x & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] \in \mathcal{F}(\mathbb{R})\ ;\ x>0\right\} \cup \{B\}$$

$$= e^s_{x^+,y^+} \cup e^{s,t}_{y,z,+} \cup e^{s,t}_{x,z,+} \cup \{B\}$$

$$= e^s_{x^+,y^+} \cup \overline{e^{s,t}_{y,z,+}}^{z\neq 0} \cup \overline{e^{s,t}_{x,z,+}}^{z\neq 0}.$$

Hence, we obtain

$$\overline{e^s_{x^+,y^+}} = \overline{e^s_{x^+,y^+}}^{v\neq 0} \cup \overline{e^s_{x^+,y^+}}^{z\neq 0} = e^s_{x^+,y^+} \cup \overline{e^{x,u}_{t,v,-}} \cup \overline{e^{s,t}_{x,v,-}} \cup \overline{e^{s,t}_{y,z,+}} \cup \overline{e^{s,t}_{x,z,+}}.$$

Finally, take the 3-cell $e^1_{+,+,+}$. First, we claim that $\{Bs_\beta\} \notin \overline{e^1_{+,+,+}}$. Indeed, using criterion (∂), one has

$$\{Bs_\beta\} \in \overline{e^1_{+,+,+}} \Rightarrow \forall\varepsilon>0,\ \exists[x:y:z:u:v:w:s:t] \in e^1_{+,+,+}\ ;\ x^2 > 1-\varepsilon.$$

Since, for every $[x:y:z:u:v:w:s:t] \in e^1_{+,+,+}$ we have $s\neq 0$ and $t\neq 0$, we must have $x\neq 0$ and we may rewrite the last condition, without normalizing, as

$$\forall\varepsilon>0,\ \exists p\in e^1_{+,+,+}\ ;\ x^2 > (1-\varepsilon)\|p\|^2$$

so

$$\forall 1>\varepsilon>0,\ \exists p\in e^1_{+,+,+}\ ;\ \|p\|^2 < \frac{x^2}{1-\varepsilon}.$$

Now, if $\{Bs_\beta\} \in \overline{e^1_{+,+,+}}$, then it can be seen in the chart $\{x\neq 0\}$. Then, we have a homeomorphism

$$e^1_{+,+,+} \simeq \left\{ (y,z,u,v,w,s,t)\in\mathbb{R}^7\ ;\ \begin{array}{l} zw+s(s+t) = yv+t(s+t) = yw+u(s+t) = vz+s+t = 0, \\ u=st,\ w=sv,\ y=tz, \\ st>0,\ t(y-v)>0,\ t(1-u)>0,\ t(w-z)>0 \end{array} \right\}$$

$$\simeq \{(z,v,s)\in\mathbb{R}^3\ ;\ s(s+vz)<0,\ s(v+z(s+vz))<0,\ s(1+s(s+vz))>0,\ s(sv-z)>0\} =: C$$

and, under this homeomorphism, the norm reads

$$x^2+y^2+z^2+u^2+v^2+w^2+s^2+t^2 = 1+(s+vz)^2(z^2+s^2+1)+s^2(v^2+1)+z^2.$$

Whence, we can rewrite the criterion as

$$\{Bs_\beta\} \in \overline{e^1_{+,+,+}} \Rightarrow \forall 1 > \varepsilon > 0,\ \exists (z,v,s) \in C\ ;\ (s+vz)^2(z^2+s^2+1)+s^2(v^2+1)+z^2 < \frac{\varepsilon}{1-\varepsilon}.$$

Now, take $p := (z,v,s) \in C$. We have $s(s+vz) < 0$, hence $svz < 0$. If $s > 0$, then $sv > z$ and this implies $v > 0$ and $z < 0$. But we have $s(v+z(s+vz)) < 0$, so $sv < -sz(s+vz)$ and so $0 < v < -z(s+vz) < 0$, a contradiction. Thus, we have $s < 0$, $z > 0$ and $v > 0$. Since $s(1+s(s+vz)) > 0$, one gets $s > -s^2(s+vz)$, so $1 < -s(s+vz)$ and hence, $s(s+vz) < -1$. This last inequation implies $s^2(s+vz)^2 > 1$, whence

$$\|p\|^2 = (s+vz)^2(z^2+s^2+1)+s^2(v^2+1)+z^2 > s^2(s+vz)^2 > 1.$$

Thus, we can't have $\|p\|^2 < \frac{\varepsilon}{1-\varepsilon}$ for every $\varepsilon > 0$. This proves that $\{Bs_\beta\} \notin \overline{e^1_{+,+,+}}$. In the same way, using the chart $\{u \neq 0\}$, one obtains

$$\begin{cases} \{Bs_\beta\} \notin \overline{e^1_{+,+,+}} \cup \overline{e^1_{+,-,-}} \\ \\ \{Bs_\alpha s_\beta\} \notin \overline{e^1_{+,+,-}} \cup \overline{e^1_{+,-,+}} \end{cases}$$

Henceforth, the only possible 2-cells appearing in $\overline{e^1_{+,+,+}}$ are the ones not containing $\{Bs_\beta\}$ in their closure. Thus, there are only twelve possibilities :

$$\overline{e^1_{+,+,+}} \subset e^1_{+,+,+} \cup \overline{e^s_{u^\pm,v^\pm}} \cup \overline{e^x_{y^\pm,u^\pm}} \cup \overline{e^{x=u}_{v^\pm,w^\pm}}.$$

Since each of these cells is included in the chart $\{u \neq 0\}$, we may compute the Euclidean boundary of the image of $e^1_{+,+,+}$ under this chart. Define

$$Y := \mathcal{F}(\mathbb{R}) \setminus \mathcal{F}(\mathbb{R})_2 = \{[x:y:z:u:v:w:s:t] \in \mathcal{F}(\mathbb{R})\ ;\ st(s+t)(y-v)(w-z)(x-u) \neq 0\}$$

as well as

$$\begin{array}{rccc} j_u & : & Y & \longrightarrow & \mathbb{R}^3 \\ & & [x:y:z:1:v:w:s:t] & \longmapsto & (y,w,t) \end{array}$$

We see that

$$j_u(Y) = \{(y,w,t) \in \mathbb{R}^3\ ;\ t(t+yw) \neq 0,\ yw \neq 0,\ y \neq tw,\ w+y(yw+t) \neq 0,\ t(yw+t) \neq -1\}$$

and the inverse of j_u reads

$$\begin{array}{ccc} j_u(Y) & \longrightarrow & Y \\ (y,w,t) & \longmapsto & [-t(yw+t):y:-y(yw+t):1:tw:w:-(yw+t):t] \end{array}$$

Then, we obtain that

$$j_u(e^1_{+,+,+}) = \{(y,w,t)\ ;\ t(yw+t) < 0,\ t(y-tw) > 0,\ t(w+y(yw+t)) > 0,\ t(t(yw+t)+1) < 0\}.$$

Now, if $(y,w,t) \in j_u(e^1_{+,+,+})$ and $t > 0$, since $tyw < 0$ this implies that $yw < 0$. Next, the inequation $t(y-tw) > 0$ implies $y > tw$ and so $y > 0$ and $w < 0$. But, on the other hand, $0 > tw > -ty(yw+t) > 0$, a contradiction. Thus, we must have $t < 0$, $y < 0$ and $w < 0$ and so,

$$\begin{aligned} j_u(e^1_{+,+,+}) &= \{(y,w,t) \in \mathbb{R}^3\ ;\ y,w,t < 0,\ yw+t > 0,\ t(yw+t)+1 > 0\} \\ &= \{(y,w,t) \in \mathbb{R}^3\ ;\ y,w < 0,\ -yw < t < 0,\ t(yw+t) > -1\}. \end{aligned}$$

Hence,

$$\overline{j_u(e^1_{+,+,+})} = \{(y,w,t) \in \mathbb{R}^3 \ ; \ y,w<0, \ -yw \leq t \leq 0, \ t(yw+t) \geq -1\}$$

$= j_u(e^1_{+,+,+}) \cup \{(y,w,-yw) \ ; \ y,w<0\} \cup \{(y,w,0) \ ; \ y,w<0\} \cup \{(y,w,t) \ ; \ y,w,t<0, \ t(yw+t)=-1\}$.

Therefore, using j_u^{-1}, we finally obtain

$$\overline{e^1_{+,+,+}} = e^1_{+,+,+} \cup \overline{e^s_{u^-,v^-}} \cup \overline{e^x_{y^+,u^-}} \cup \overline{e^{x=u}_{v^-,w^+}},$$

which finishes the proof. □

As an immediate important consequence of these computations, one has the following result:

Corollary 9.1.4. *The cellular structure on $\mathcal{F}(\mathbb{R})$ given by Theorem 8.3.8 is regular.*

We introduce some notation for representative cells in each orbit.

Notation 9.1.5. *Following Proposition 9.1.1, we denote*

$$\begin{cases} e^0 := \{B\} = \{[0:0:1:0:0:0:0:0]\}, \\ e^1_1 := e^{s,t}_{y,z,+}, \ e^1_2 := e^{s,t}_{x,z,-}, \ e^1_3 := e^{x,u}_{s,w,-}, \\ e^2_1 := e^s_{x^+,y^+}, \ e^2_2 := e^s_{x^+,y^-}, \ e^2_3 := e^x_{y^+,u^+}, \ e^2_4 := e^x_{y^+,u^-}, \ e^2_5 := e^{s+t}_{u^+,v^+}, \ e^2_6 := e^{s+t}_{u^-,v^+}, \\ e^3_1 := e^1_{+,+,+}, \ e^3_2 := e^1_{+,+,-}, \ e^3_3 := e^1_{+,-,-}, \ e^3_4 := e^1_{+,-,+} \end{cases}$$

Using Proposition 9.1.2 and the action of $\mathfrak{S}_3$ on cells detailed above, we obtain the following consequence:

Corollary 9.1.6. *The boundaries of the representing cells are given by*

$$\begin{cases} \partial e^1_1 = e^0 \cup e^0 s_\alpha, \\ \partial e^1_2 = e^0 \cup e^0 s_\beta, \\ \partial e^1_3 = e^0 \cup e^0 w_0, \end{cases} \quad \begin{cases} \partial e^2_1 = e^1_2 s_\beta s_\alpha \cup e^1_1 \cup e^1_3 s_\beta \cup e^1_2 s_\beta, \\ \partial e^2_2 = e^1_2 s_\alpha \cup e^1_1 \cup e^1_3 s_\beta s_\alpha \cup e^1_2, \\ \partial e^2_3 = e^1_3 s_\alpha \cup e^1_3 \cup e^1_2 w_0 \cup e^1_1, \\ \partial e^2_4 = e^1_3 s_\alpha s_\beta \cup e^1_3 w_0 \cup e^1_2 w_0 \cup e^1_1 s_\alpha, \\ \partial e^2_5 = e^1_1 s_\alpha s_\beta \cup e^1_1 w_0 \cup e^1_3 s_\beta \cup e^1_2 s_\alpha s_\beta, \\ \partial e^2_6 = e^1_1 s_\beta \cup e^1_1 s_\beta s_\alpha \cup e^1_3 s_\beta s_\alpha \cup e^1_2 s_\alpha s_\beta, \end{cases} \quad \begin{cases} \partial e^3_1 = e^2_2 s_\beta s_\alpha \cup e^2_5 s_\alpha s_\beta \cup e^2_4, \\ \partial e^3_2 = e^2_1 s_\alpha \cup e^2_5 s_\alpha s_\beta \cup e^2_3 s_\beta s_\alpha, \\ \partial e^3_3 = e^2_2 w_0 \cup e^2_6 s_\alpha s_\beta \cup e^2_3, \\ \partial e^3_4 = e^2_1 \cup e^2_6 s_\alpha s_\beta \cup e^2_4 s_\beta s_\alpha. \end{cases}$$

9.2 The cellular homology complex of $\mathcal{F}(\mathbb{R})$

We are now ready to compute the differentials in the cellular homology complex from Theorem 8.3.8

$$\mathbb{Z}[\mathfrak{S}_3]^4 \xrightarrow{d_3} \mathbb{Z}[\mathfrak{S}_3]^6 \xrightarrow{d_2} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{d_1} \mathbb{Z}[\mathfrak{S}_3] \ .$$

Since the cellular structure is regular, the coefficients of the matrix representing $d_i : \mathbb{Z}^{6k} \to \mathbb{Z}^{6\ell}$ are in $\{-1,0,1\}$ for all $i=1,2,3$. Thus, we just have to compute the signs using orientations of cells. We will impose to the action of $\mathfrak{S}_3$ to respect the given orientations. This implies that we just have to give an orientation to the representing cells e_j^i. We will further assume that the orientations of two adjacent cells are the same, as in a simplicial complex. Hence, we just have to give an orientation to the cells e_j^1 for $j = 1, 2, 3$ and to e_1^2 and e_1^3.

We choose arbitrarily to orient each of the 1-cells e_j^1 from the reflection to the identity of $\mathfrak{S}_3$. This gives the oriented doubled GKM graph as in Figure 9.

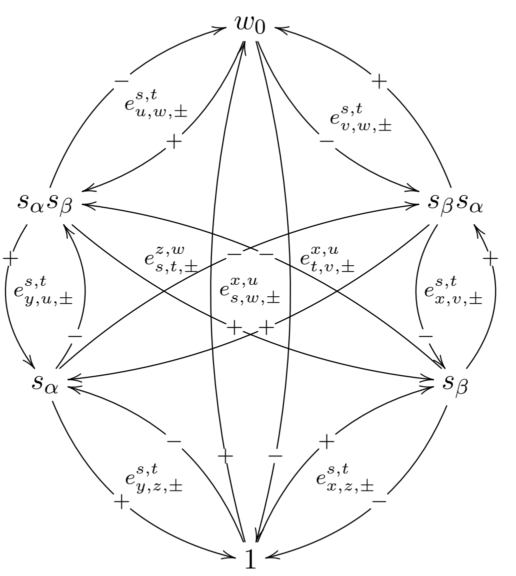


Figure 9: The oriented 1-skeleton of $\mathcal{F}(\mathbb{R})$

We then get for instance $d(e_1^1) = d(e^{s,t}_{y,z,+}) = 1 - s_\alpha$.

We now have to give an orientation to the 2-cells. Take for example $e^s_{x^+,y^+}$. Recall from the proof of Lemma 9.1.3 the diffeomorphisms

$$\begin{array}{cccc} \phi_v & : & \{v \neq 0\} & \longrightarrow & \mathbb{R}^7 \\ & & [x:y:z:t:u:v:w:s:t] & \longmapsto & \left(\frac{x}{v}, \frac{y}{v}, \frac{z}{v}, \frac{u}{v}, \frac{w}{v}, \frac{s}{v}, \frac{t}{v}\right) \end{array}$$

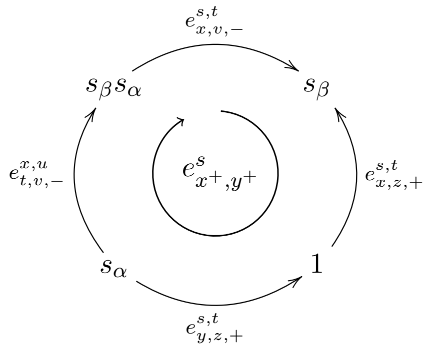
and

$$\begin{array}{ccccc} \psi_v & : & \{(x,y,z,0,0,0,t) \in \mathbb{R}^7 \ ; \ y = -t^2, \ z = -xt\} & \longrightarrow & \mathbb{R}^2 \\ & & (x,y,z,0,0,0,t) & \longmapsto & (x,t) \end{array}$$

Then, we have showed that $\psi_v(\phi_v(e^s_{x^+,y^+})) = \{x < 0, \ t < 0\}$. Post-composing $\psi_v \circ \phi_v$ with the orientation preserving diffeomorphism

$$\begin{array}{ccccc} \kappa & : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & & (x,t) & \longmapsto & \left(\frac{2}{\pi}\arctan(x), \frac{2}{\pi}\arctan(t)\right) \end{array}$$

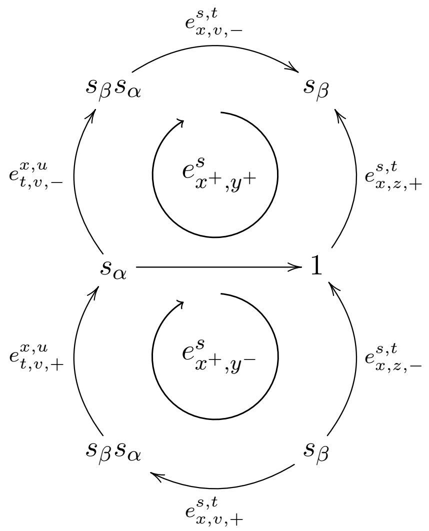
we obtain a diffeomorphism $\kappa \circ \psi_v \circ \phi_v : e^s_{x^+,y^+} \simeq [-1,0]^2$. We *arbitraty* choose to take the reversed orientation on $\mathbb{R}^2$ and we impose to the diffeomorphism $\kappa \circ \psi_v \circ \phi_v$ to preserve the orientation. Identifying the boundary of $[-1,0]^2$ with the one of $e^s_{x^+,y^+}$ gives the following picture:



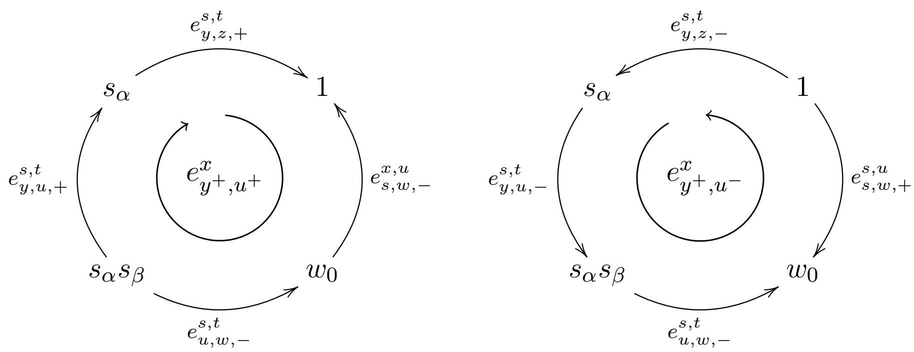
and using the chosen orientation of the 1-cells, this yields

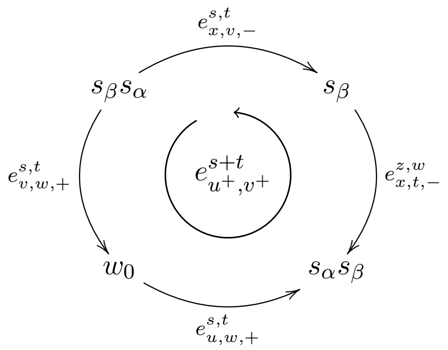
$$d(e^s_{x^+,y^+}) = e^{x,u}_{s,w,-} + e^{s,t}_{x,v,-} - e^{s,t}_{x,z,+} - e^{s,t}_{y,z,+}.$$

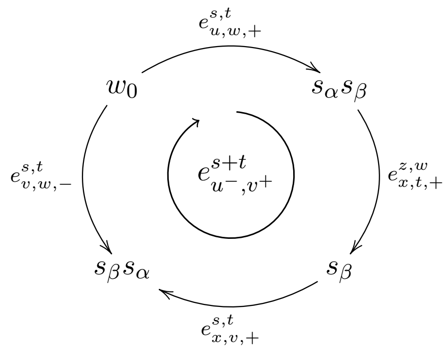
Now, $e^s_{x^+,y^-}$ shares a 1-cell in its boundary with $e^s_{x^+,y^+}$, we then orient it in a simplicial way:



Using the same method, we may orient the cells e^2_j for $j = 3, 5, 6$ and finally, this method applies to $e^2_4 \cdot s_\alpha$. Since we impose to the action of $\mathfrak{S}_3$ to preserve orientation, this gives the orientation of e^2_4. We obtain the following pictures for e^2_j, $j = 3, 4, 5, 6$:







We finally have to orient the 3-cells. Take for example $e^1_{+,+,+}$. Recall that in the proof of Lemma 9.1.3, we have defined

$$Y := \mathcal{F}(\mathbb{R})\setminus\mathcal{F}(\mathbb{R})_2 = \{[x:y:z:u:v:w:s:t] \in \mathcal{F}(\mathbb{R})\ ;\ st(s+t)(y-v)(w-z)(x-u) \neq 0\}$$

and

$$\begin{array}{cccc} j_u & : & Y & \longrightarrow & \mathbb{R}^3 \\ & & [x:y:z:1:v:w:s:t] & \longmapsto & (y,w,t) \end{array}$$

We had

$$j_u(Y) = \{(y,w,t) \in \mathbb{R}^3\ ;\ yw \neq 0,\ t(yw+t) \neq 0,-1,\ y \neq tw,\ x+y(yw+t) \neq 0\}$$

and $j_u : Y \to j_u(Y)$ is a diffeomorphism, which inverse is given by

$$\begin{array}{ccc} j_u(Y) & \longrightarrow & Y \\ (y,w,t) & \longmapsto & [-t(yw+t):y:-y(yw+t):1:tw:w:-(yw+t):t] \end{array}$$

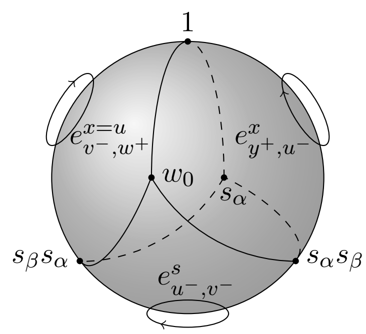
Furthermore, we have shown that

$$j_u(e^1_{+,+,+}) = \{(y,w,t) \in \mathbb{R}^3\ ;\ y,w<0,\ -yw<t<0,\ t(yw+t)+1>0\}.$$

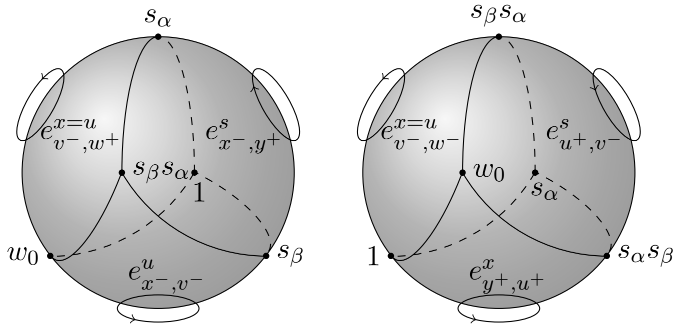
We *arbitrary* choose to take the reversed orientation on $\mathbb{R}^3$, we take the induced orientation on $j_u(e^1_{+,+,+})$ and we impose to j_u to preserve orientation. We obtain

$$d(e^1_{+,+,+}) = e^x_{y^+,u^-} - e^s_{u^-,v^-} - e^{x=u}_{v^-,w^+}$$

and the following picture for $e^3_1 = e^1_{+,+,+}$:



Then, we orient the cells e^3_j, $j = 2,3,4$ using the same method as for 2-cells: two cobording cells are oriented simplicially. This gives the following pictures for the respectives cells $e^1_{+,+,-}$, $e^1_{+,-,-}$ and $e^1_{+,-,+}$:



From this discussion, we can compute the boundary operators. For convenience, the orientations of the cells given above will be called the *reversed charts orientation*.

Lemma 9.2.1. *For $i = 0, 1, 2, 3$ denote by $\mathcal{E}_i$ the set of i-cells in $\mathcal{F}(\mathbb{R})$. Recall Notation 9.1.5. Then we have isomorphisms of* right *$\mathbb{Z}[\mathfrak{S}_3]$-modules*

$$\begin{cases} \mathbb{Z}\langle\mathcal{E}_0\rangle = \langle e^0\rangle\,\mathbb{Z}[\mathfrak{S}_3] \simeq \mathbb{Z}[\mathfrak{S}_3], \\ \mathbb{Z}\langle\mathcal{E}_1\rangle = \left\langle e^1_j,\ 1\le j\le 3\right\rangle \mathbb{Z}[\mathfrak{S}_3] \simeq \mathbb{Z}[\mathfrak{S}_3]^3, \\ \mathbb{Z}\langle\mathcal{E}_2\rangle = \left\langle e^2_j,\ 1\le j\le 6\right\rangle \mathbb{Z}[\mathfrak{S}_3] \simeq \mathbb{Z}[\mathfrak{S}_3]^6, \\ \mathbb{Z}\langle\mathcal{E}_3\rangle = \left\langle e^3_j,\ 1\le j\le 4\right\rangle \mathbb{Z}[\mathfrak{S}_3] \simeq \mathbb{Z}[\mathfrak{S}_3]^4. \end{cases}$$

Furthermore, the boundary operators $d_i : \mathbb{Z}\langle\mathcal{E}_i\rangle \to \mathbb{Z}\langle\mathcal{E}_{i-1}\rangle$ associated to the reversed charts orientation are homomorphisms of right $\mathbb{Z}[\mathfrak{S}_3]$-modules and satisfy the following relations

$$\begin{cases} d_1(e^1_1) = d_1(e^{s,t}_{y,z,+}) = e^0(1-s_\alpha), \\ d_1(e^1_2) = d_1(e^{s,t}_{x,z,-}) = e^0(1-s_\beta), \\ d_1(e^1_3) = d_1(e^{x,u}_{s,w,-}) = e^0(1-w_0). \end{cases}$$

$$\begin{cases} d_2(e^2_1) = d_2(e^{s}_{x^+,y^+}) = e^{x,u}_{t,v,-} + e^{s,t}_{x,v,-} - e^{s,t}_{x,z,+} - e^{s,t}_{y,z,+} = e^1_2(s_\beta s_\alpha - s_\beta) + e^1_3 s_\beta - e^1_1, \\ d_2(e^2_2) = d_2(e^{s}_{x^+,y^-}) = e^{s,t}_{y,z,+} - e^{s,t}_{x,z,-} + e^{s,t}_{x,v,+} + e^{x,u}_{t,v,+} = e^1_1 + e^1_2(s_\alpha - 1) + e^1_3 s_\beta s_\alpha, \\ d_2(e^2_3) = d_2(e^{x}_{y^+,u^+}) = e^{s,t}_{y,u,+} - e^{s,t}_{u,w,-} - e^{x,u}_{s,w,-} + e^{s,t}_{y,z,+} = e^1_3(s_\alpha - 1) - e^1_2 w_0 + e^1_1, \\ d_2(e^2_4) = d_2(e^{x}_{y^+,u^-}) = e^{s,t}_{y,u,-} + e^{s,t}_{y,z,-} - e^{x,u}_{s,w,+} + e^{s,t}_{u,w,-} = e^1_3(s_\alpha s_\beta - w_0) + e^1_1 s_\alpha + e^1_2 w_0, \\ d_2(e^2_5) = d_2(e^{s+t}_{u^+,v^+}) = e^{s,t}_{u,w,+} - e^{z,w}_{x,t,-} - e^{s,t}_{x,v,-} + e^{s,t}_{v,w,+} = e^1_2 s_\alpha s_\beta + e^1_1(w_0 - s_\alpha s_\beta) - e^1_3 s_\beta, \\ d_2(e^2_6) = d_2(e^{s+t}_{u^-,v^+}) = -e^{s,t}_{v,w,-} + e^{s,t}_{x,v,+} + e^{z,w}_{x,t,+} + e^{s,t}_{u,w,+} = e^1_1(s_\beta - s_\beta s_\alpha) + e^1_3 s_\beta s_\alpha + e^1_2 s_\alpha s_\beta. \end{cases}$$

$$\begin{cases} d_3(e_1^3) = d_3(e^1_{+,+,+}) = e^x_{y^+,u^-} - e^s_{u^-,v^-} - e^{x=u}_{v^-,w^+} = e^2_4 - e^2_2 s_\beta s_\alpha - e^2_5 s_\alpha s_\beta, \\ d_3(e_2^3) = d_3(e^1_{+,+,-}) = e^u_{x^-,v^-} + e^s_{x^-,y^+} + e^{x=u}_{v^-,w^+} = e^2_3 s_\beta s_\alpha + e^2_1 s_\alpha + e^2_5 s_\alpha s_\beta, \\ d_3(e_3^3) = d_3(e^1_{+,-,-}) = e^x_{y^+,u^+} - e^s_{u^+,v^-} + e^{x=u}_{v^-,w^-} = e^2_3 - e^2_2 w_0 + e^2_6 s_\alpha s_\beta, \\ d_3(e_4^3) = d_3(e^1_{+,-,+}) = e^u_{x^+,v^-} + e^s_{x^+,y^+} - e^{x=u}_{v^-,w^-} = e^2_4 s_\beta s_\alpha + e^1_1 - e^2_6 s_\alpha s_\beta. \end{cases}$$

Summarizing, we obtain the following result:

Theorem 9.2.2. *The cellular homology complex of $\mathcal{F}(\mathbb{R})$ associated to the cellular decomposition of Theorem 8.3.8 is a complex of free right $\mathbb{Z}[\mathfrak{S}_3]$-modules isomorphic to*

$$\mathcal{K}_{\mathfrak{S}_3}(\mathbb{R}) := \left(\mathbb{Z}[\mathfrak{S}_3]^4 \xrightarrow{d_3} \mathbb{Z}[\mathfrak{S}_3]^6 \xrightarrow{d_2} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{d_1} \mathbb{Z}[\mathfrak{S}_3] \right)$$

where the d_i's are given, in the canonical bases, by left multiplication by the following matrices

$$d_1 = \begin{pmatrix} 1 - s_\alpha & 1 - s_\beta & 1 - w_0 \end{pmatrix},$$

$$d_2 = \begin{pmatrix} -1 & 1 & 1 & s_\alpha & w_0 - s_\alpha s_\beta & s_\beta - s_\beta s_\alpha \\ s_\beta s_\alpha - s_\beta & s_\alpha - 1 & -w_0 & w_0 & s_\alpha s_\beta & s_\alpha s_\beta \\ s_\beta & s_\beta s_\alpha & s_\alpha - 1 & s_\alpha s_\beta - w_0 & -s_\beta & s_\beta s_\alpha \end{pmatrix},$$

$$d_3 = \begin{pmatrix} 0 & s_\alpha & 0 & 1 \\ -s_\beta s_\alpha & 0 & -w_0 & 0 \\ 0 & s_\beta s_\alpha & 1 & 0 \\ 1 & 0 & 0 & s_\beta s_\alpha \\ -s_\alpha s_\beta & s_\alpha s_\beta & 0 & 0 \\ 0 & 0 & s_\alpha s_\beta & -s_\alpha s_\beta \end{pmatrix}.$$

Corollary 9.2.3. *The integral homology of $\mathcal{F}(\mathbb{R})$ is given by*

$$\forall i \geq 0,\ H_i(\mathcal{F}(\mathbb{R}), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \textit{if } \ i = 0, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \textit{if } \ i = 1, \\ 0 & \textit{if } \ i = 2, \\ \mathbb{Z} & \textit{if } \ i = 3, \\ 0 & \textit{if } \ i \geq 4. \end{cases}$$

Proof. Just remark that, by the cellular homology theorem, we have

$$H_*(\mathcal{K}_{\mathfrak{S}_3}(\mathbb{R})) = H_*(\mathcal{F}(\mathbb{R}), \mathbb{Z})$$

and that the homology $H_*(\mathcal{K}_{\mathfrak{S}_3}(\mathbb{R}))$ may be computed using [GAP4] and the Kronecker product of matrices. More precisely, for $w \in \mathfrak{S}_3$, let $\mu^\ell_w : \mathbb{Z}[\mathfrak{S}_3] \to \mathbb{Z}[\mathfrak{S}_3]$ denote the left multiplication by w. Identifying the basis $\mathcal{B} := \{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, w_0\}$ with the canonical basis $\mathcal{B}_c$ of $\mathbb{Z}^6$, denote by $\mathrm{Mat}^\ell(w)$ the matrix of μ_w^ℓ with respect to this basis. It is an element of $\mathcal{M}_6(\mathbb{Z})$. For instance, one has

$$\mathrm{Mat}^\ell(s_\alpha) = \begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{pmatrix} \quad \text{and} \quad \mathrm{Mat}^\ell(s_\beta) = \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix}.$$

Thus, if a homomorphism of right $\mathbb{Z}[\mathfrak{S}_3]$-modules $\mathbb{Z}[\mathfrak{S}_3]^n \to \mathbb{Z}[\mathfrak{S}_3]^m$ is represented, in the canonical basis $\mathcal{B}$ by the matrix $d := (d_{i,j})_{i,j} \in \mathcal{M}_{n,m}(\mathbb{Z}[\mathfrak{S}_3])$ and if we decompose $d_{i,j} := \sum_{w\in\mathfrak{S}_3} d_{i,j}^w w \in \mathbb{Z}[\mathfrak{S}_3]$ in the canonical basis, denote $d_w := \left(d_{i,j}^w\right)_{i,j} \in \mathcal{M}_{n,m}(\mathbb{Z})$. Then, the corresponding homomorphism of abelian groups $\mathbb{Z}^{6n} \to \mathbb{Z}^{6m}$ is represented by the matrix

$$\widetilde{d} := \sum_{w\in\mathfrak{S}_3} d_w \otimes_K \mathrm{Mat}^\ell(w) \in \mathcal{M}_{6n,6m}(\mathbb{Z}),$$

where $\otimes_K$ is the Kronecker tensor product of matrices.

This may be implemented in [GAP4] to find the matrices $\widetilde{d}_i$, $i = 1, 2, 3$ and finally compute $H_*(\mathcal{K}_{\mathfrak{S}_3}(\mathbb{R}))$. □

A remark can be made concerning this last result. The same proof as for the Theorem 8.2.1 (some care is required here : the Zariski theorem is no longer true in the non-algebraically closed field $\mathbb{R}$) shows that we have a homeomorphism

$$\begin{array}{ccc} SL_3(\mathbb{R})/B & \xrightarrow{\sim} & \mathcal{F}(\mathbb{R}) \\ \begin{pmatrix} a&d&g\\ b&e&h\\ c&f&j \end{pmatrix} & \longmapsto & \begin{bmatrix} a(bf-ce) & a(cd-af) & a(ae-bd)\\ b(bf-ce) & b(cd-af) & b(ae-bd)\\ c(bf-ce) & c(cd-af) & c(ae-bd) \end{bmatrix} \end{array}$$

(here, and only here, the letter B denotes the Borel subgroup of real upper-triangular matrices of $SL_3(\mathbb{R})$). In fact, it is a homeomorphism since it is a bijective continuous map with compact domain. We also have the real Bruhat decomposition of $\mathcal{F}(\mathbb{R})$:

$$\mathcal{F}(\mathbb{R}) \simeq SL_3(\mathbb{R})/B = \bigcup_{w\in\mathfrak{S}_3} B\dot{w}B/B =: \bigcup_{w\in\mathfrak{S}_3} Y_w,$$

and this is a cellular decomposition of $\mathcal{F}(\mathbb{R})$, which is of course not $\mathfrak{S}_3$-equivariant. We have that $\dim(Y_w) = \ell(w)$ for each $w \in \mathfrak{S}_3$, where $\ell : \mathfrak{S}_3 \to \mathbb{Z}$ is the length function. Using the above homeomorphism, these cells may be described as

$$\left\{ \begin{array}{lll} Y_1 = \left\{ \begin{bmatrix} 0&0&1\\ 0&0&0\\ 0&0&0 \end{bmatrix} \right\}, & Y_{s_\alpha} = \left\{ \begin{bmatrix} 0&0&z\\ 0&0&1\\ 0&0&0 \end{bmatrix},\ z\in\mathbb{R} \right\}, & Y_{s_\beta} = \left\{ \begin{bmatrix} 0&1&z\\ 0&0&0\\ 0&0&0 \end{bmatrix},\ z\in\mathbb{R} \right\} \\ Y_{s_\alpha s_\beta} = \left\{ \begin{bmatrix} s&x&z\\ 1&-s&y\\ 0&0&0 \end{bmatrix} \in \mathcal{F}(\mathbb{R}) \right\}, & Y_{s_\beta s_\alpha} = \left\{ \begin{bmatrix} 0&x&z\\ 0&t&y\\ 0&1&-t \end{bmatrix} \in \mathcal{F}(\mathbb{R}) \right\}, & Y_{w_0} = \left\{ \begin{bmatrix} s&x&z\\ u&t&y\\ 1&v&r \end{bmatrix} \in \mathcal{F}(\mathbb{R}) \right\}. \end{array} \right.$$

Lonardo Rabelo and Luiz San Martin computed the boundary operators of this cellular structure (see [RM18], §2.2):

$$d_1(Y_{s_\alpha}) = d_1(Y_{s_\beta}) = 0,\ d_2(Y_{s_\alpha s_\beta}) = -2Y_{s_\beta},\ d_2(Y_{s_\beta s_\alpha}) = -2Y_{s_\alpha},\ d_3(Y_{w_0}) = 0.$$

This gives that the associated cellular homology complex is isomorphic (as a complex of abelian groups) to the complex

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z}^2 \xrightarrow{-\left(\begin{smallmatrix} 0&2\\ 2&0 \end{smallmatrix}\right)} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z}$$

and hence, we recover the homology of $\mathcal{F}(\mathbb{R})$ as in the Corollary 9.2.3.

9.3 The cohomology of $\mathcal{F}(\mathbb{R})$ as an $\mathfrak{S}_3$-module

We know that, since $\mathfrak{S}_3$ acts on the right of $\mathcal{F}(\mathbb{R})$ and since cohomology is a contravariant functor, $\mathfrak{S}_3$ acts on the left on $H^*(\mathcal{F}(\mathbb{R}), \mathbb{Z})$. The goal of this section is to identify this action in terms of $\mathbb{Z}[\mathfrak{S}_3]$ modules.

First of all, define the integral representation

$$\mathbf{2} : \mathfrak{S}_3 \longrightarrow GL_2(\mathbb{Z})$$

by

$$\mathbf{2}(s_\alpha) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{2}(s_\beta) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

Then, $\mathbf{2}$ is an integral form of the 2-dimensional irreducible complex representation of $\mathfrak{S}_3$. Its reduction modulo 2 is the irreducible $\mathbb{F}_2[\mathfrak{S}_3]$-module $\overline{\mathbf{2}}$ of dimension 2. Moreover, we let $\mathbf{2} \otimes \mathbb{F}_2$ be the representation $\mathbb{Z}[\mathfrak{S}_3] \to \mathrm{End}_{\mathbb{Z}}(\mathbb{F}_2^2)$.

We are now able to determine the $\mathbb{Z}[\mathfrak{S}_3]$-module structure on the integral cohomology of $\mathcal{F}(\mathbb{R})$. For convenience, we consider $\mathbb{Z}[\mathfrak{S}_3]$ as a graded algebra concentrated in degree zero.

Theorem 9.3.1. *The cohomology $H^*(\mathcal{F}(\mathbb{R}), \mathbb{Z})$ of $\mathcal{F}(\mathbb{R})$ is a graded commutative left $\mathbb{Z}[\mathfrak{S}_3]$-module such that*

$$H^i(\mathcal{F}(\mathbb{R}), \mathbb{Z}) = \begin{cases} \mathbb{1} & \text{if } i = 0, 3, \\ \mathbf{2} \otimes \mathbb{F}_2 & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the action of $\mathfrak{S}_3$ on $\mathcal{F}(\mathbb{R})$ preserves the orientation.

In particular, reducing modulo 2 gives

$$H^i(\mathcal{F}(\mathbb{R}), \mathbb{F}_2) = \begin{cases} \mathbb{1} & \text{if } i = 0, 3, \\ \overline{\mathbf{2}} & \text{if } i = 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Denote

$$\begin{cases} \sigma_+ := \sum_{w \in \mathfrak{S}_3} w, \\ \sigma_- := \sum_{w \in \mathfrak{S}_3} \varepsilon(w) w, \end{cases}$$

where $\varepsilon(w) = (-1)^{\ell(w)}$ is the sign character of $\mathfrak{S}_3$ and recall the cellular homology complex

$$\mathbb{Z}[\mathfrak{S}_3]^4 \xrightarrow{d_3} \mathbb{Z}[\mathfrak{S}_3]^6 \xrightarrow{d_2} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{d_1} \mathbb{Z}[\mathfrak{S}_3] .$$

We can directly compute that

$$H_0(\mathcal{F}(\mathbb{R}), \mathbb{Z}) = \operatorname{coker} d_1 = \mathbb{Z} \langle 1 \rangle \simeq \mathbb{Z}$$

and

$$H_3(\mathcal{F}(\mathbb{R}), \mathbb{Z}) = \ker d_3 = \mathbb{Z} \left\langle \sigma_+ {}^t\!\begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix} \right\rangle \simeq \mathbb{Z}.$$

We determine an orientation of $\mathcal{F}(\mathbb{R})$ by choosing as fundamental class

$$[\mathcal{F}(\mathbb{R})] := \sigma_+ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

Then, if $w \in \mathfrak{S}_3$, then $\sigma_+ \cdot w = \sigma_+$ and so $[\mathcal{F}(\mathbb{R})] \cdot w = [\mathcal{F}(\mathbb{R})]$. Thus, the right action of $\mathfrak{S}_3$ on $\mathcal{F}(\mathbb{R})$ preserves the orientation. Denoting by

$$\mathcal{D}^i := ([\mathcal{F}(\mathbb{R})] \cap -) : H^i(\mathcal{F}(\mathbb{R}), \mathbb{Z}) \xrightarrow{\sim} H_{3-i}(\mathcal{F}(\mathbb{R}), \mathbb{Z})$$

the associated Poincaré duality, the naturality theorem [Mun84, Theorem 67.2], resulting from the naturality of the cap product, yields

$$w_* \mathcal{D}^i w^* = \mathcal{D}^i.$$

For a right $\mathfrak{S}_3$-set X, we naturally write X^{op} for the left $\mathfrak{S}_3$-set X endowed with the action $w \cdot x := xw^{-1}$. Then, the last equation becomes a reformulation of the property

$$\mathcal{D}^i \in \mathrm{Hom}_{\mathbb{Z}[\mathfrak{S}_3]} \left(H^i(\mathcal{F}(\mathbb{R}), \mathbb{Z}), H_{3-i}(\mathcal{F}(\mathbb{R}), \mathbb{Z})^{op} \right)$$

and the left modules $H^i(\mathcal{F}(\mathbb{R}), \mathbb{Z})$ and $H_{3-i}(\mathcal{F}(\mathbb{R}), \mathbb{Z})^{op}$ are thus isomorphic.

Since the right $\mathfrak{S}_3$-modules $H_0(\mathcal{F}(\mathbb{R}), \mathbb{Z}) = \mathbb{Z}\langle \sigma_+ \rangle$ and $H_3(\mathcal{F}(\mathbb{R}), \mathbb{Z}) = \mathbb{Z}\langle [\mathcal{F}(\mathbb{R})] \rangle$ are trivial, it remains to show that $H_1(\mathcal{F}(\mathbb{R}), \mathbb{Z})^{op} \simeq \mathbf{2} \otimes \mathbb{F}_2$. Denote respectively by x and y the classes of $\left(\begin{smallmatrix} 0 \\ 0 \\ 1+w_0 \end{smallmatrix} \right) \in \ker d_1$ and $\left(\begin{smallmatrix} 0 \\ 0 \\ s_\alpha + s_\alpha s_\beta \end{smallmatrix} \right) \in \ker d_1$ in $H_1(\mathcal{F}(\mathbb{R}), \mathbb{Z})$. Then we have $H_1(\mathcal{F}(\mathbb{R}), \mathbb{Z}) = \mathbb{Z}\langle x, y \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^2$ and since

$$x + y + \left(\begin{smallmatrix} 0 \\ 0 \\ s_\beta + s_\beta s_\alpha \end{smallmatrix} \right) = \left(\begin{smallmatrix} 0 \\ 0 \\ \sigma_+ \end{smallmatrix} \right) = d_2 \left(\left(\frac{\sigma_+ + \sigma_-}{2} \right) {}^t\!\left(\begin{smallmatrix} 1 & 1 & 0 & 0 & 0 & 0 \end{smallmatrix} \right) \right)$$

we get

$$x \cdot s_\beta = \left(\begin{smallmatrix} 0 \\ 0 \\ s_\beta + s_\beta s_\alpha \end{smallmatrix} \right) = -x - y.$$

Next, it is easy to compute that $x \cdot s_\alpha = y$, $y \cdot s_\alpha = x$ and $y \cdot s_\beta = y$. These equations mean that, with respect to the canonical generating set $\{x, y\}$ of the torsion $\mathbb{Z}$-module $H_1(\mathcal{F}(\mathbb{R}), \mathbb{Z})^{op}$, the matrices of the action of s_α and s_β are given by

$$\mathrm{Mat}_{\{x,y\}}(s_\alpha) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{Mat}_{\{x,y\}}(s_\beta) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and these are indeed the matrices defining $\mathbf{2} \otimes \mathbb{F}_2$. □

We also get the following direct consequence:

Corollary 9.3.2. *The rational cohomology ring $H^*(\mathcal{F}(\mathbb{R}), \mathbb{Q})$ is a graded commutative $\mathbb{Q}$-algebra and we have an isomorphism of $\mathbb{Q}[\mathfrak{S}_3]$-modules*

$$H^*(\mathcal{F}(\mathbb{R}), \mathbb{Q}) \simeq \mathbb{Q} \oplus \mathbb{Q},$$

with one summand in each degree $i = 0, 3$.

Finally, using the notations from the proof of Theorem 9.3.1, we let

$$x := \left(\begin{smallmatrix} 0 \\ 0 \\ 1+w_0 \end{smallmatrix} \right) = [e^{x,u}_{s,w,-} \cup e^{x,u}_{s,w,+}] \in H_1(\mathcal{F}(\mathbb{R}), \mathbb{Z}),\ y := x \cdot s_\alpha = [e^{s,t}_{y,u,-} \cup e^{s,t}_{y,u,+}] \in H_1(\mathcal{F}(\mathbb{R}), \mathbb{Z}),$$

$$z := \sigma_+ = [\cup_w e^0 w] = [\mathcal{F}(\mathbb{R})_0] \in H_0(\mathcal{F}(\mathbb{R}), \mathbb{Z})$$

and defining

$$u := (\mathcal{D}^2)^{-1}(x),\ v := (\mathcal{D}^2)^{-1}(y),\ c := (\mathcal{D}^3)^{-1}(z),$$

we easily obtain the following description for the ring structure of the integral cohomology of $\mathcal{F}(\mathbb{R})$:

$$H^*(\mathcal{F}(\mathbb{R}), \mathbb{Z}) \simeq \frac{\mathbb{Z}[u, v, c]}{(2u, 2v, u^2, v^2, uv, uc, vc, c^2)}, \quad \text{where } |u| = |v| = 2 \text{ and } |c| = 3.$$

9.4 The mod 2 cohomology of $\mathcal{F}(\mathbb{R})$ as the coinvariant algebra of $\mathfrak{S}_3$

The description of the integral cohomology algebra is not really satisfying. Also, we shall give the ring structure on the mod 2 cohomology $H^*(\mathcal{F}(\mathbb{R}), \mathbb{F}_2)$, which turns out to be more interesting. Furthermore, viewing $\mathcal{F}(\mathbb{R})$ as a real algebraic variety, we shall express the ring generators as fundamental classes of closed subvarieties of $\mathcal{F}(\mathbb{R})$. More precisely, recalling the ring $R = \mathbb{C}[x, y, z, u, v, w, s, t]$ and the ideal I from Proposition 8.2.1, if one considers the complex conjugation involution $\sigma : R \to R$, $P \mapsto \overline{P}$, then one has $R^\sigma = \mathbb{R}[x, y, z, u, v, w, s, t]$ along with a homeomorphism

$$\mathcal{F}(\mathbb{R}) \xrightarrow{\sim} \mathrm{Proj}_{\mathbb{R}}(R^\sigma/I^\sigma).$$

Moreover, we will realize every homology class as the fundamental class of a closed subvariety of $\mathcal{F}(\mathbb{R})$.

Definition 9.4.1. *Denoting $p := [x : y : z : u : v : w : s : t]$ the homogeneous coordinates on $\mathbb{P}^7(\mathbb{R})$, for coordinates (or linear combinations thereof) $\lambda, \mu, \nu \in \{x, y, z, u, v, w, s, t\}$, consider the following closed algebraic subvarieties of $\mathcal{F}(\mathbb{R}) = \mathrm{Proj}_{\mathbb{R}}(R^\sigma/I^\sigma)$:*

$$\Omega(\lambda, \mu, \nu) := \{p \in \mathcal{F}(\mathbb{R}) \ ; \ \lambda = \mu = \nu = 0\}, \quad \Omega(\lambda - \mu) := \{p \in \mathcal{F}(\mathbb{R}) \ ; \ \lambda = \mu\},$$

and define

$$\begin{cases} \Omega_x := \Omega(x, t, v), & \Omega_y := \Omega(s + t, y, z), & \Omega_z := \Omega(s, x, z), \\ \Omega_u := \Omega(u, t, y), & \Omega_v := \Omega(s + t, v, w), & \Omega_w := \Omega(s, u, w), \\ \Omega_{y=v} := \Omega(y - v), & \Omega_{w=z} := \Omega(w - z), & \Omega_{u=x} := \Omega(u - x). \end{cases}$$

Recall that two submanifolds V and W of a given manifold M are said to be *transverse*, written $V \pitchfork W$, if one has

$$\forall x \in V \cap W, \ T_x M = T_x V + T_x W.$$

Remark 9.4.2. *Notice that, on a smooth algebraic variety M, the tangent space associated to the differentiable structure on M coincides with the Zariski tangent space of M. Thus, one can say that two smooth subvarieties of M are transverse if they are as smooth submanifolds of M and we can check the transversality condition on Zariski tangent spaces. More generally, if V and W are no longer required to be smooth (but M still is) then the transversality condition may be defined as*

$$\forall x \in V \cap W, \ \mathrm{codim}_{\mathcal{O}_{M,x}}(\mathcal{O}_{V,x} \cap \mathcal{O}_{W,x}) = \mathrm{codim}_M V + \mathrm{codim}_M W.$$

Using this notion, one can relate the cup product on $H^*(M)$ and intersection of transverse submanifolds. More precisely, the cup product is Poincaré dual to intersection, provided that the submanifolds are transverse.

Theorem 9.4.3 (*[Hut11, Theorem 1.1]*)**.** *If A is a ring and V, W are two transverse smooth A-oriented submanifolds of a smooth compact closed A-oriented m-manifold M of respective dimensions v and w, then $V \cap W$ is a smooth A-oriented submanifold of M of dimension $v + w - m$. Furthermore, denoting $\mathcal{D}^i : H^i(M, A) \to H_{m-i}(M, A)$ the Poincaré duality isomorphisms and $\mathcal{D}_i := (\mathcal{D}^{m-i})^{-1}$, one has*

$$\mathcal{D}_v([V]) \cup \mathcal{D}_w([W]) = \mathcal{D}_{v+w-m}([V \cap W]).$$

In a concise way, denoting $[Z]^*$ the Poincaré dual of the fundamental class of a submanifold Z, we obtain

$$V \pitchfork W \implies [V]^* \cup [W]^* = [V \cap W]^*.$$

In particular, this holds with $A = \mathbb{F}_2$ without any assumption on orientation.

First, we establish the following lemma:

Lemma 9.4.4. *The nine subvarieties defined in 9.4.1 are 2-dimensional, smooth, irreducible and pairwise transverse.*

Proof. As usual, we will only work out a few examples since the other ones may be treated in the same way. To prove irreducibility, we can use the same trick as in the proof of Proposition 8.2.1. For example, the subvariety Ω_x has coordinate ring

$$\mathbb{R}[\Omega_x] := \mathbb{R}[y,z,u,w,s] \big/ (uz - ys, yw + us, zw + s^2)$$

and localizing it with respect to s and defining the new variables $y' := y - uz/s$ and $u' := u + yw/s$, as well as $z' := z/s$ and $w' := w/s$ yields an isomorphism

$$\mathbb{R}[\Omega_x]_s \simeq \mathbb{R}[s^{\pm 1}][z',w'] \big/ (z'w' + 1) \simeq \left(\mathbb{R}[z',w',s] \big/ (z'w'+1) \right)_s$$

and this last ring is integral, hence the coordinate ring of Ω_x is integral too. As another example, consider $\Omega_{y=v}$. Localize its coordinate ring $\mathbb{R}[\Omega_{y=v}]$ with respect to s and changing variables to $y' := y - uz/s$, $v' := v - xw/s$ and $t' := t - xu/s$, we obtain

$$\mathbb{R}[\Omega_{y=v}]_s \simeq \mathbb{R}[s^{\pm 1}][x,z,u,w] \big/ (zw + xu + 1, xw - uz).$$

Localizing this again with respect to x and eliminating the variable w yields

$$\mathbb{R}[\Omega_{y=v}]_{s,x} \simeq \mathbb{R}[s^{\pm 1}, x^{\pm 1}][u,z] \big/ (ux^2 + uz^2 + x) \simeq \left(\mathbb{R}[x,u,z,s] \big/ (ux^2 + uz^2 + x) \right)_{s,x}$$

and the polynomial $ux^2 + uz^2 + x$ is easily seen to be irreducible in the polynomial ring $\mathbb{R}[x,u,z]$, hence the last ring above is integral and thus $\mathbb{R}[\Omega_{y=v}]$ is integral too. From the irreducibility, we can prove the smoothness using the Jacobian criterion by straightforward calculations.

Regarding the transversality conditions, take for instance Ω_y and Ω_w and start by noticing that, for two hyperplanes H, H' of $\mathbb{R}^n$, we have the obvious consequence of Grassmann formula

$$H + H' = \mathbb{R}^n \iff H \neq H'. \qquad (\sharp)$$

Recall the standard open affine subsets $U_\lambda := \{\lambda \neq 0\}$ for $\lambda \in \{x,y,z,u,v,w,s,t\}$. Then, one has $\Omega_y \cap \Omega_w \subset U_x \cup U_v$ and there are isomorphisms of varieties

$$\begin{array}{ccc} U_x & \xrightarrow{\varphi_x} & \mathbb{R}^3 \\ [1:y:z:u:v:w:s:t] & \longmapsto & (z,v,s) \\ [1:-z(s+vz):z:-s(s+vz):v:sv:s:-(s+vz)] & \longleftarrow & (z,v,s) \end{array}$$

and

$$\begin{array}{ccc} U_v & \xrightarrow{\varphi_v} & \mathbb{R}^3 \\ [x:y:z:u:1:w:s:t] & \longmapsto & (x,w,t) \\ [x:-t(t+wx):-x(t+wx):tw:1:w:wx:t] & \longleftarrow & (x,w,t) \end{array}$$

Now, let $p := [x : y : z : u : v : w : s : t] \in \Omega_y \cap \Omega_v$ and denote (a, b, c) the coordinates on $\mathbb{R}^3$. Suppose first that $p \in U_x$ and $x = 1$. We have that $\varphi_x(\Omega_y \cap U_x) = \{a = 0\} \subset \mathbb{R}^3$ and $\varphi_x(\Omega_w \cap U_x) = \{c = 0\} \subset \mathbb{R}^3$ are two distinct hyperplanes of $\mathbb{R}^3$ and by (♯) this implies that

$$d_p\varphi_x(T_p(\Omega_y \cap U_x)) + d_p\varphi_x(T_p(\Omega_w \cap U_x)) = T_{\varphi_x(p)}(\{a = 0\}) + T_{\varphi_x(p)}(\{c = 0\}) = \mathbb{R}^3 = T_p(\mathcal{F}(\mathbb{R}) \cap U_x).$$

Now, if $p \in U_v$ and $v = 1$ then one has $\varphi_x(\Omega_y \cap U_v) = \{ab+c = 0\}$ and $\varphi_x(\Omega_w \cap U_v) = \{b = 0\}$ and again, using Zariski tangent spaces,

$$d_p\varphi_v(T_p(\Omega_y \cap U_v)) + d_p\varphi_v(T_p(\Omega_w \cap U_v)) = \{bx+c = 0\} + T_{\varphi_x(p)}(\{b = 0\}) = \mathbb{R}^3 = T_p(\mathcal{F}(\mathbb{R}) \cap U_v).$$

This implies that $T_p\Omega_y + T_p\Omega_w = T_p(\mathcal{F}(\mathbb{R}))$ and $\Omega_y \pitchfork \Omega_w$. The cases concerning only the first six varieties Ω_λ for $\lambda \in \{x, y, z, u, v, w\}$ are treated in the same way.

Next, look at the case $\Omega_{y=v} \pitchfork \Omega_z$. We have $\varphi_v(\Omega_{y=v} \cap U_v) = \{1 + c(ab + c) = 0\}$ and $\varphi_v(\Omega_z \cap U_v) = \{a = 0\}$. Thus, if $p \in \Omega_{y=v} \cap \Omega_z \cap U_v$,

$$d_p\varphi_v(T_p(\omega_{y=v} \cap U_v)) = T_{\varphi_v(p)}(\varphi_v(\Omega_{y=v} \cap U_v)) = \{twa + txb + (2t + wx)c = 0\} \subset \mathbb{R}^3,$$

and

$$d_p\varphi_v(T_p(\Omega_z \cap U_v)) = T_{\varphi_x(p)}(\varphi_v(\Omega_z \cap U_v)) = \{a = 0\} \subset \mathbb{R}^3$$

and these hyperplanes cannot be equal because this would imply $tw \neq 0$ and $x = 2t+wx = 0$, which is impossible. Hence they sum to $\mathbb{R}^3$ by (♯), i.e.

$$T_p(\Omega_{y=v} \cap U_v) + T_p(\Omega_z \cap U_v) = T_p(\mathcal{F}(\mathbb{R}) \cap U_v)$$

and we can do the same for every chart U_λ intersecting $\Omega_{y=v} \cap \Omega_z$.

Finally, we look at the case $\Omega_{y=v} \pitchfork \Omega_{w=z}$ and take the open affine subset U_x. We have $\varphi_x(\Omega_{y=v} \cap U_x) = \{b + a(ab + c) = 0\}$ and $\varphi_x(\Omega_{w=z} \cap U_x) = \{ab = c\}$, so for every $p \in \Omega_{y=v} \cap \Omega_{w=z} \cap U_x$, one has

$$d_p\varphi_x(T_p(\Omega_{y=v} \cap U_x)) = T_{\varphi_x(p)}(\varphi_x(\Omega_{y=v} \cap U_x)) = \{(s + 2vz)a + (1 + z^2)b + zc = 0\} \subset \mathbb{R}^3,$$

and

$$d_p\varphi_x(T_p(\Omega_{w=z} \cap U_x)) = T_{\varphi_x(p)}(\varphi_x(\Omega_{w=z} \cap U_x)) = \{-a + sb + vc = 0\} \subset \mathbb{R}^3.$$

These two hyperplanes cannot be equal, because else there would exist $e \in \mathbb{R}$ such that $(s + 2vz, 1 + z^2, z) = e(-1, s, v)$ for some $p = [1 : y : z : u : y : z : s : t] \in \Omega_{y=v} \cap \Omega_{w=z} \cap U_x$. But, this implies $e \neq 0$ and $0 < -e^2 = e(s+2vz) = es+2evz = 1+3z^2 > 0$, a contradiction. Hence, they sum to $\mathbb{R}^3$, i.e.

$$T_p(\Omega_{y=v} \cap U_x) + T_p(\Omega_{w=z} \cap U_x) = T_p(\mathcal{F}(\mathbb{R}) \cap U_x).$$

□

Lemma 9.4.5. *The fundamental mod 2 homology classes of the subvarieties defined in 9.4.1 satisfy*

$$[\Omega_w] = [\Omega_x] = [\Omega_y], \quad [\Omega_z] = [\Omega_u] = [\Omega_v], \quad [\Omega_{y=v}] = [\Omega_{w=z}] = [\Omega_{x=u}].$$

Moreover, one has

$$[\Omega_z] \cdot s_\alpha s_\beta = [\Omega_{y=v}] \cdot s_\beta = [\Omega_w]$$

and

$$[\Omega_w] = [\Omega_z] + [\Omega_{y=v}].$$

Proof. Start by noticing that

$$\Omega_w = \overline{\{s=0,\ xyt \neq 0\}} = \overline{\bigcup_{\epsilon,\eta\in\{\pm\}} e^s_{x^\epsilon,y^\eta}} = \overline{e_1^2 \cup e_1^2 s_\alpha \cup e_2^2 \cup e_2^2 s_\alpha},$$

the closure being understood in $\mathcal{F}(\mathbb{R})$ and with respect to the Zariski or Euclidean topology (they give the same closure, as may be checked on affine charts). Similarly, one has

$$\Omega_x = \overline{\{x=0,\ yus \neq 0\}} = \overline{\bigcup_{\epsilon,\eta\in\{\pm\}} e^x_{y^\epsilon,u^\eta}} = \overline{e_3^2 \cup e_3^2 s_\alpha \cup e_4^2 \cup e_4^2 s_\alpha}.$$

Hence, viewing $[\Omega_w]$ and $[\Omega_x]$ as (classes of) column vectors in $\mathbb{F}_2[\mathfrak{S}_3]^6 = (\mathcal{K}_{\mathfrak{S}_3}(\mathbb{R}) \otimes \mathbb{F}_2)_2$, we get

$$[\Omega_w] = \begin{pmatrix} 1+s_\alpha \\ 1+s_\alpha \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad [\Omega_x] = \begin{pmatrix} 0 \\ 0 \\ 1+s_\alpha \\ 1+s_\alpha \\ 0 \\ 0 \end{pmatrix}.$$

Using the same method, we compute

$$[\Omega_y] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1+s_\alpha \\ 1+s_\alpha \end{pmatrix}, \quad [\Omega_z] = \begin{pmatrix} s_\beta s_\alpha+w_0 \\ s_\beta s_\alpha+w_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad [\Omega_u] = \begin{pmatrix} 0 \\ 0 \\ s_\beta s_\alpha+w_0 \\ s_\beta s_\alpha+w_0 \\ 0 \\ 0 \end{pmatrix}, \quad [\Omega_v] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ s_\beta s_\alpha+w_0 \\ s_\beta s_\alpha+w_0 \end{pmatrix},$$

and

$$[\Omega_{y=v}] = \begin{pmatrix} s_\beta+s_\alpha s_\beta \\ s_\beta+s_\alpha s_\beta \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad [\Omega_{w=z}] = \begin{pmatrix} 0 \\ 0 \\ s_\beta+s_\alpha s_\beta \\ s_\beta+s_\alpha s_\beta \\ 0 \\ 0 \end{pmatrix}, \quad [\Omega_{x=u}] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ s_\beta+s_\alpha s_\beta \\ s_\beta+s_\alpha s_\beta \end{pmatrix}.$$

These expressions prove the second statement. The only thing left to be shown to prove the first statement is its first equation. But one checks that

$$[\Omega_w] - [\Omega_x] = {}^t(\,1\ 1\ 1\ 1\ 0\ 0\,)\cdot(1+s_\alpha) = (d_3 \otimes \mathbb{F}_2)\begin{pmatrix} 1+w_0 \\ 1+w_0 \\ s_\beta \\ s_\beta \end{pmatrix}\cdot(1+s_\alpha)$$

and

$$[\Omega_y] - [\Omega_x] = {}^t(\,0\ 0\ 1\ 1\ 1\ 1\,)\cdot(1+s_\alpha) = (d_3 \otimes \mathbb{F}_2)\begin{pmatrix} 1+s_\beta \\ 1 \\ 1+s_\beta \\ 1 \end{pmatrix}\cdot(1+s_\alpha),$$

as desired. Finally, to prove the third statement, just notice that

$$[\Omega_w] + [\Omega_z] + [\Omega_{y=v}] = {}^t(\,1\ 1\ 0\ 0\ 0\ 0\,)\cdot\sigma_+ = (d_3 \otimes \mathbb{F}_2)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\cdot(1+s_\alpha s_\beta + s_\beta s_\alpha).$$

□

We now come to the main result of this section:

Theorem 9.4.6. *The assignement $x \mapsto [\Omega_z]^*$, $y \mapsto [\Omega_{y=v}]^*$ induces an $\mathfrak{S}_3$-equivariant zero-graded isomorphism of $\mathbb{F}_2$-algebras*

$$\phi : \frac{\mathbb{F}_2[x,y]}{(x^3, y^3, x^2+y^2+xy)} \xrightarrow{\sim} H^*(\mathcal{F}(\mathbb{R}), \mathbb{F}_2),$$

where $\mathfrak{S}_3$ acts on the left-hand side as

$$\begin{cases} s_\alpha x = y, \\ s_\alpha y = x \end{cases} \quad \textit{and} \quad \begin{cases} s_\beta x = x, \\ s_\beta y = x+y. \end{cases}$$

Proof. It suffices to show that the desired isomorphism is a well-defined map. Indeed, if it is, then it is an $\mathfrak{S}_3$-equivariant morphism of $\mathbb{F}_2$-algebras using the fact that Poincaré duality is $\mathfrak{S}_3$-equivariant and the relations $[\Omega_w] = [\Omega_z] + [\Omega_{y=v}]$ and $[\Omega_z] \cdot s_\beta s_\alpha = [\Omega_{y=v}] \cdot s_\beta = [\Omega_w]$ from Lemma 9.4.5. More precisely, one writes

$$[\Omega_z] \cdot s_\alpha = [\Omega_{y=v}], \qquad [\Omega_{y=v}] \cdot s_\beta = [\Omega_w] = [\Omega_z] + [\Omega_{y=v}]$$

and

$$[\Omega_z] = [\Omega_w] \cdot s_\beta s_\alpha = [\Omega_z] \cdot s_\beta s_\alpha + [\Omega_{y=v}] \cdot s_\beta s_\alpha = [\Omega_w] \cdot (s_\alpha s_\beta + s_\alpha) \;\Rightarrow\; [\Omega_z] \cdot s_\beta = [\Omega_z].$$

Moreover, it will be easily seen to be one-to-one by looking at the dimensions of the summands of the two algebras and using Theorem 9.3.1.

To prove that the map is well-defined, the only thing to be checked is the following relations

$$c_0 := ([\Omega_z]^*)^2 + ([\Omega_{y=v}]^*)^2 + [\Omega_z]^*[\Omega_{y=v}]^* = 0, \; ([\Omega_z]^*)^3 = ([\Omega_{y=v}]^*)^3 = 0.$$

Using Lemma 9.4.5 again yields

$$\begin{aligned} s_\alpha c_0 &= (([\Omega_z]s_\alpha)^*)^2 + (([\Omega_{y=v}]s_\alpha)^*)^2 + ([\Omega_z]s_\alpha)^* \cup ([\Omega_{y=v}]s_\alpha)^* \\ &= ([\Omega_{y=v}]^*)^2 + ([\Omega_z]^*)^2 + [\Omega_{y=v}]^*[\Omega_z]^* = c_0 \end{aligned}$$

and similarly, one gets $s_\beta c_0 = c_0$. Hence $wc_0 = c_0$ for every $w \in \mathfrak{S}_3$, that is,

$$c_0 \in H^2(\mathcal{F}(\mathbb{R}), \mathbb{F}_2)^{\mathfrak{S}_3}.$$

But since $H^2(\mathcal{F}(\mathbb{R}), \mathbb{F}_2) = \overline{\mathbf{2}}$ is the unique 2-dimensional irreducible representation of $\mathfrak{S}_3$ on $\mathbb{F}_2$, the module $H^2(\mathcal{F}(\mathbb{R}), \mathbb{F}_2)$ has no $\mathfrak{S}_3$-fixed point and hence $c_0 = 0$.

Now, by Lemma 9.4.5, we have $[\Omega_z]^* = [\Omega_u]^* = [\Omega_v]^*$ and Lemma 9.4.4 says that $\Omega_z \pitchfork \Omega_u \pitchfork \Omega_v \pitchfork \Omega_z$ and since $\Omega_z \cap \Omega_u \cap \Omega_v = \emptyset$, we obtain by the Theorem 9.4.3,

$$([\Omega_z]^*)^3 = [\Omega_z]^* \cup [\Omega_u]^* \cup [\Omega_v]^* = [\Omega_z \cap \Omega_u \cap \Omega_v]^* = [\emptyset]^* = 0$$

and similarly, since $\Omega_{y=v} \cap \Omega_{w=z} \cap \Omega_{x=u} = \emptyset$, we have

$$([\Omega_{y=v}]^*)^3 = [\Omega_{y=v}]^* \cup [\Omega_{w=z}]^* \cup [\Omega_{x=u}]^* = [\Omega_{y=v} \cap \Omega_{w=z} \cap \Omega_{x=u}]^* = 0,$$

as required. □

Corollary 9.4.7. *Let $\mathfrak{S}_3$ act naturally by permutation on the polynomial algebra $\mathbb{F}_2[x, y, z]$, the simple reflection s_α (resp. s_β) acting on $\{x, y, z\}$ as the transposition* (12) *(resp.* (23)*). In this context, one may consider the* mod 2 coinvariant algebra

$$\mathbb{F}_2[x, y, z]_{\mathfrak{S}_3} = \mathbb{F}_2[x, y, z] \Big/ (\mathbb{F}_2[x, y, z]^{\mathfrak{S}_3}).$$

Then, the assignment $x \mapsto [\Omega_z]^*$, $y \mapsto [\Omega_{y=v}]^*$, $z \mapsto [\Omega_w]^*$ induces an $\mathfrak{S}_3$-equivariant isomorphism of graded $\mathbb{F}_2$-algebras

$$\widetilde{\phi} : \mathbb{F}_2[x, y, z]_{\mathfrak{S}_3} \xrightarrow{\sim} H^*(\mathcal{F}(\mathbb{R}), \mathbb{F}_2).$$

Remark 9.4.8. *By [Tym07, Theorem 4.5], we know that $H^*(SL_3(\mathbb{C})/B, \mathbb{Q}) = H^*(\mathcal{F}, \mathbb{Q})$ is the coinvariant algebra $\mathbb{Q}[x, y, z]_{\mathfrak{S}_3}$ of the Weyl group $\mathfrak{S}_3$. This is obviously no longer the case for $H^*(\mathcal{F}(\mathbb{R}), \mathbb{Q})$. The previous corollary retrieves the result over $\mathbb{F}_2$. Notice however that $H^*(\mathcal{F}(\mathbb{R}), \mathbb{F}_2) \simeq \mathbb{F}_2[x, y, z]_{\mathfrak{S}_3}$ is not the regular representation of $\mathfrak{S}_3$ over $\mathbb{F}_2$, as $\mathbb{F}_2[x, y, z]_{\mathfrak{S}_3}$ is semisimple, whereas $\mathbb{F}_2[\mathfrak{S}_3]$ is not. This gives a counter-example to the Proposition 3.32 of [LT09] in the modular case.*

Proof. For $1 \leq i \leq 3$, denote $\sigma_i(x, y, z)$ the i$^{\text{th}}$ elementary symmetric polynomial with variables x, y and z. By the fundamental theorem on symmetric polynomials, one has

$$\left(\mathbb{F}_2[x, y, z]^{\mathfrak{S}_3}\right) = (\sigma_1(x, y, z), \sigma_2(x, y, z), \sigma_3(x, y, z)) = (x + y + z, xy + yz + xz, xyz).$$

We thus have to prove that the map defined in the statement induces an isomorphism

$$\frac{\mathbb{F}_2[x, y, z]}{(x + y + z, xy + yz + xz, xyz)} \xrightarrow{\sim} H^*(\mathcal{F}(\mathbb{R}), \mathbb{F}_2).$$

Next, the map

$$\begin{array}{rcccc} r & : & \mathbb{F}_2[x, y, z] & \twoheadrightarrow & \mathbb{F}_2[\widetilde{x}, \widetilde{y}] \\ & & P(x, y, z) & \longmapsto & P(\widetilde{x}, \widetilde{y}, \widetilde{x} + \widetilde{y}) \end{array}$$

defines a retraction of the natural inclusion

$$\begin{array}{rcccc} \iota & : & \mathbb{F}_2[\widetilde{x}, \widetilde{y}] & \hookrightarrow & \mathbb{F}_2[x, y, z] \\ & & P(\widetilde{x}, \widetilde{y}) & \longmapsto & P(x, y) \end{array}$$

and one has

$$r((x + y + z, xy + yz + xz, xyz)) = (\widetilde{x}^3, \widetilde{y}^3, \widetilde{x}^2 + \widetilde{y}^2 + \widetilde{x}\widetilde{y}).$$

Indeed, one has $r(x + y + z) = 0$, $r(xy + yz + xz) = \widetilde{x}\widetilde{y} + \widetilde{y}(\widetilde{x} + \widetilde{y}) + \widetilde{x}(\widetilde{x} + \widetilde{y}) = \widetilde{x}\widetilde{y} + \widetilde{y}^2 + \widetilde{x}^2$ and $r(xyz) = \widetilde{x}\widetilde{y}(\widetilde{x} + \widetilde{y}) = \widetilde{y}(\widetilde{x}^2 + \widetilde{x}\widetilde{y}) = \widetilde{y}^3$.

Therefore, the retraction r induces an equivariant zero-graded isomorphism of $\mathbb{F}_2$-algebras

$$\overline{r} : \frac{\mathbb{F}_2[x, y, z]}{(\mathbb{F}_2[x, y, z]^{\mathfrak{S}_3})} \xrightarrow{\sim} \frac{\mathbb{F}_2[x, y]}{(x^3, y^3, x^2 + y^2 + xy)}$$

which, using the relation $[\Omega_w]^* = [\Omega_z]^* + [\Omega_{y=v}]^*$ from Lemma 9.4.5, fits in a commutative diagram

$$\begin{tikzcd} \frac{\mathbb{F}_2[x,y,z]}{\left(\mathbb{F}_2[x,y,z]^{\mathfrak{S}_3}\right)} \arrow[rr, "\widetilde{\phi}"] \arrow[dr, "\sim", "\overline{r}"'] & & H^*(\mathcal{F}(\mathbb{R}), \mathbb{F}_2) \\ & \frac{\mathbb{F}_2[x,y]}{(x^3,y^3,x^2+y^2+xy)} \arrow[ur, "\sim", "\phi"'] & \end{tikzcd}$$

□

This result (and Lemmas 9.4.5 and 9.4.4) may be used to realize every cohomology class in $H^*(\mathcal{F}(\mathbb{R}), \mathbb{F}_2)$ by closed subvarieties. First, we have obviously

$$H^0(\mathcal{F}(\mathbb{R}), \mathbb{F}_2) = \mathbb{F}_2 \langle [\mathcal{F}(\mathbb{R})]^* \rangle \quad \text{and} \quad H^3(\mathcal{F}(\mathbb{R}), \mathbb{F}_2) = \mathbb{F}_2 \langle [\mathrm{pt}]^* \rangle$$

as well as

$$H^1(\mathcal{F}(\mathbb{R}), \mathbb{F}_2) = \{0, [\Omega_z]^*, [\Omega_{y=v}]^*, [\Omega_w]^*\}.$$

Next, for two elements $w, w' \in \mathfrak{S}_3$ such that $w \leq w'$ for the Bruhat order, denote by $w \text{ — } w'$ the Zariski closure of the union of the two 1-cells joining w and w'. For instance, one defines

$$\begin{aligned} 1 \text{ — } s_\alpha \; &:= \overline{\bigcup_{\epsilon=\pm} e^{s,t}_{y,z,\epsilon}} = e^{s,t}_{y,z,+} \cup e^{s,t}_{y,z,-} \cup \{B\} \cup \{Bs_\alpha\} \\ &= \left\{ \left[\begin{smallmatrix} 0 & 0 & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{smallmatrix}\right] \in \mathcal{F}(\mathbb{R}) \right\} = \Omega_x \cap \Omega_w. \end{aligned}$$

Now, since $\Omega_x \pitchfork \Omega_w$, we obtain

$$[\, 1 \text{ — } s_\alpha \,]^* = [\Omega_x \cap \Omega_w]^* = [\Omega_x]^*[\Omega_w]^* = ([\Omega_z]^*)^2 + ([\Omega_{y=v}]^*)^2 = [\Omega_z]^*[\Omega_{y=v}]^* \in H^2(\mathcal{F}(\mathbb{R}), \mathbb{F}_2).$$

In the same fashion, we compute

$$[\, 1 \text{ — } s_\beta \,]^* = [\Omega_u \cap \Omega_v]^* = [\Omega_u]^*[\Omega_v]^* = ([\Omega_z]^*)^2$$

and

$$[\, 1 \text{ — } w_0 \,]^* = [\Omega_{y=v} \cap \Omega_{x=u}]^* = [\Omega_{y=v}]^*[\Omega_{x=u}]^* = ([\Omega_{y=v}]^*)^2.$$

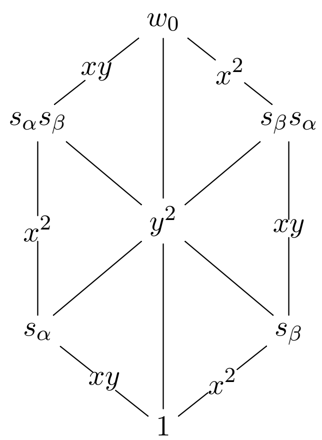
Hence,

$$H^2(\mathcal{F}(\mathbb{R}), \mathbb{F}_2) = \{0, [\, 1 \text{ — } s_\alpha \,]^*, [\, 1 \text{ — } s_\beta \,]^*, [\, 1 \text{ — } w_0 \,]^*\}.$$

We also have the following relations in $H_1(\mathcal{F}(\mathbb{R}), \mathbb{F}_2)$,

$$\begin{cases} [\, 1 \text{ — } s_\alpha \,] = [\, s_\beta \text{ — } s_\beta s_\alpha \,] = [\, s_\alpha s_\beta \text{ — } w_0 \,], \\ [\, 1 \text{ — } s_\beta \,] = [\, s_\alpha \text{ — } s_\alpha s_\beta \,] = [\, s_\beta s_\alpha \text{ — } w_0 \,], \\ [\, 1 \text{ — } w_0 \,] = [\, s_\alpha \text{ — } s_\beta s_\alpha \,] = [\, s_\beta \text{ — } s_\alpha s_\beta \,]. \end{cases}$$

Recalling the notation $x = [\Omega_z]^*$ and $y = [\Omega_{y=v}]^*$ we can represent this using the GKM graph of $\mathfrak{S}_3$:



where a label on an edge symbolizes the associated fundamental cohomology class and the relations in H^2 may be reformulated by saying that the sum of the classes of three parallel edges in the graph is zero.

Remark 9.4.9. *We have isomorphisms of real algebraic varieties*

$$\mathcal{F}(\mathbb{R}) \simeq SO(3)/S(O(1)^3) \simeq O(3)/O(1)^3 = \{(L_1, L_2, L_3) \in \mathbb{P}^2(\mathbb{R})^3 \;;\; L_1 \overset{\perp}{\oplus} L_2 \overset{\perp}{\oplus} L_3 = \mathbb{R}^3\}$$

and this allows us to see any element of $\mathcal{F}(\mathbb{R})$ *as a triple of orthogonal lines in* $\mathbb{R}^3$ *and this gives, for each* $1 \leq i \leq 3$*, an algebraic line bundle, called the* i^{th} universal line bundle *over* $\mathcal{F}(\mathbb{R})$,

$$\mathcal{L}_i := \left(\{((L_1, L_2, L_3), v) \in \mathcal{F}(\mathbb{R}) \times \mathbb{A}^3(\mathbb{R}) \ ; \ v \in L_i\} \overset{\mathrm{pr}_1}{\twoheadrightarrow} \mathcal{F}(\mathbb{R})\right).$$

The action of $\mathfrak{S}_3$ *on* $\mathcal{F}(\mathbb{R})$*, one can make* $\mathfrak{S}_3$ *acting on the universal line bundles as*

$$\begin{cases} \mathcal{L}_1 \cdot s_\alpha = \mathcal{L}_2, \\ \mathcal{L}_2 \cdot s_\alpha = \mathcal{L}_1, \end{cases} \qquad \begin{cases} \mathcal{L}_1 \cdot s_\beta = \mathcal{L}_1, \\ \mathcal{L}_2 \cdot s_\beta = \mathcal{L}_3 \end{cases}$$

and we have analoguous relations at the level of first Stiefel-Whitney classes. Now, if we let $\mathrm{triv}^3_{\mathcal{F}(\mathbb{R})}$ *be the trivial vector bundle of rank 3 on* $\mathcal{F}(\mathbb{R})$*, then we have immediately*

$$\mathrm{triv}^3_{\mathcal{F}(\mathbb{R})} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3.$$

Hence, denoting by $\mathbf{w}_{(i)} := \mathbf{w}(\mathcal{L}_i) = 1 + \mathbf{w}_1(\mathcal{L}_i) \in H^*(\mathcal{F}(\mathbb{R}), \mathbb{F}_2)$ *the total Stiefel-Whitney class of* $\mathcal{L}_i$*, the Whitney product formula (see for example [MS74, §4]) gives*

$$\mathbf{w}_{(1)}\mathbf{w}_{(2)}\mathbf{w}_{(3)} = w(\mathrm{triv}^3_{\mathcal{F}(\mathbb{R})}) = 1.$$

Expanding and using the degree on the cohomology algebra, we obtain

$$\mathbf{w}_{(1)}\mathbf{w}_{(2)}\mathbf{w}_{(3)} = 1 \ \Leftrightarrow \ \sigma_j(\mathbf{w}_1(\mathcal{L}_1), \mathbf{w}_1(\mathcal{L}_2), \mathbf{w}_1(\mathcal{L}_3)) = 0, \ \forall 1 \leq j \leq 3,$$

with σ_i *the* j^{th} *elementary symmetric polynomial in three variables. Using this and the formulae for the action of* $\mathfrak{S}_3$ *on the Stiefel-Whitney classes, we obtain*

$$\mathbf{w}_{(1)} = 1 + x = 1 + [\Omega_z]^*, \ \mathbf{w}_{(2)} = 1 + y = 1 + [\Omega_{y=v}]^*, \ \mathbf{w}_{(3)} = 1 + x + y = 1 + z = 1 + [\Omega_w]^*$$

and we retrieve the following isomorphism of rings, given in [Bor53a, Theorem 11.1] and in [He19, Theorem 4.1],

$$H^*(\mathcal{F}(\mathbb{R}), \mathbb{F}_2) = H^*(O(3)/O(1)^3, \mathbb{F}_2) \simeq \frac{\mathbb{F}_2[\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3]}{\left(1 + \prod_j (1 + \mathbf{w}_j)\right)}.$$

Part V

Construction of an $\mathfrak{S}_3$-equivariant cellular structure on the flag manifold of $SL_3(\mathbb{R})$ via binary spherical space forms

This part is taken from [CGS20] and [Gar20] and focuses on 3-dimensional spherical space forms that is, free quotients of the 3-sphere $\mathbb{S}^3$ by a finite group of isometries. In [CS17], the authors give a systematic method to build equivariant cellular structures on spheres, with respect to the free action of a finite group of isometries. Here, after some reminders on the method we study the *binary octahedral group* $\mathcal{O}$ and the *binary icosahedral group* $\mathcal{I}$. The octahedral case is of particular interest for us because we have a diffeomorphism $\mathcal{F}(\mathbb{R}) \simeq \mathbb{S}^3/\mathcal{Q}_8$, where $\mathcal{F}(\mathbb{R})$ is the real flag manifold of SL_3 and $\mathcal{Q}_8$ is the quaternion group of order 8. This diffeomorphism is $\mathfrak{S}_3$-equivariant and thus, the $\mathcal{O}$-equivariant cell structure on $\mathbb{S}^3$ yields an $\mathfrak{S}_3$-equivariant cellular structure on $\mathcal{F}(\mathbb{R})$, for which we describe the cellular chain complex.

10 Orbit polytopes

The following section gives the main tools for determining fundamental domains for finite groups acting isometrically on the sphere $\mathbb{S}^3$, by using their orbit polytopes. We recall results from [CS17]. For general properties of polytopes, the reader is referred to [Zie95].

10.1 Finite group acting freely on $\mathbb{S}^n$, orbit polytope and fundamental domains

Let $\Gamma \subset O(n)$ be a finite group acting freely on a sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ and such that any of its orbits span $\mathbb{R}^n$. Fix $v_0 \in \mathbb{S}^{n-1}$ and let $\mathscr{P} := \mathrm{conv}(\Gamma \cdot v_0)$ be the associated *orbit polytope*.

Recall that, if a group Γ acts on a topological space X, then a *fundamental domain* for the action of Γ on X is a subset $\mathcal{D}$ of X such that, for $\gamma \neq \gamma' \in \Gamma$, the set $\gamma\mathcal{D} \cap \gamma'\mathcal{D}$ has empty interior and the translates of $\mathcal{D}$ cover X, i.e. $X = \bigcup_{\gamma\in\Gamma} \gamma\mathcal{D}$.

Theorem 10.1.1 (*[CS17, §6.1-6.4]*)**.** *The following holds:*

i) If F and F' are distinct proper faces of $\mathscr{P}$ of the same dimension, then $F \cap \gamma F'$ has empty interior for every $1 \neq \gamma \in \Gamma$.

ii) The group Γ acts freely on the set $\mathscr{P}_d$ of d-dimensional faces of $\mathscr{P}$, for every $0 \leq d < \dim(\mathscr{P})$.

iii) Moreover, the origin 0 is an interior point of $\mathscr{P}$ and we have a Γ-equivariant homeomorphism

$$\begin{array}{ccc} \partial\mathscr{P} & \xrightarrow{\sim} & \mathbb{S}^{n-1} \\ x & \mapsto & x/|x| \end{array}$$

iv) *Given a system of representatives $F_1, \dots, F_r$ for the (free) action of Γ on the set of facets of $\mathscr{P}$ such that the union $\bigcup_i F_i$ is connected, then this union is a fundamental domain for the action of Γ on $\partial\mathscr{P}$. Furthermore, there exists such a system.*

We finish this section by giving a simple but useful fact.

Proposition 10.1.2. *Given distinct facets $F_1, \dots, F_r$ of $\mathscr{P}$, form their union $\mathcal{D} := \bigcup_{i=1}^r F_i$, consider the subset V of Γ defined by $\operatorname{vert}(\mathcal{D}) = V \cdot v_0$ and assume that $v_0 \in \bigcap_{i=1}^r \operatorname{vert}(F_i)$. If $V \cap V^{-1} = \{1\}$, then the F_i's belong to distinct Γ-orbits. If moreover $r|\Gamma| = |\mathscr{P}_{n-1}|$, then $\mathcal{D}$ is a fundamental domain for the action of Γ on $\partial\mathscr{P}$.*

Proof. Suppose that there are $1 \leq i \neq j \leq r$ and $\gamma \in \Gamma$ such that $F_j = \gamma F_i$. Since $v_0 \in \operatorname{vert}(F_i)$, we get $\gamma v_0 \in \gamma \operatorname{vert}(F_i) = \operatorname{vert}(\gamma F_i) = \operatorname{vert}(F_j)$, so $\gamma \in V$. On the other hand, $v_0 \in \operatorname{vert}(F_j) = \gamma \operatorname{vert}(F_i)$, hence $\gamma^{-1} v_0 \in \operatorname{vert}(F_i)$, that is $\gamma^{-1} \in V$. Therefore $\gamma \in V \cap V^{-1}$, so $\gamma = 1$ and thus $F_i = F_j$, a contradiction. Now, the equation $r|\Gamma| = |\mathscr{P}_{n-1}|$ ensures that $F_1, \dots, F_r$ is a system of representatives of facets and the condition $v_0 \in \bigcap_i \operatorname{vert}(F_i)$ shows that $\mathcal{D}$ is connected, hence the second statement follows from the theorem 10.1.1. $\square$

10.2 The curved join

Here, we shall define the notion of *curved join*, which allows one to describe the fundamental domain for $\partial\mathscr{P}$ as a subset of the sphere. It will also be used to reduce the higher dimensional cases $\mathbb{S}^{4n-1}$ to $\mathbb{S}^3$. For any detail, see [FGMS13, §2.4].

Given $W_1, W_2 \subset \mathbb{S}^{n-1} \subset \mathbb{R}^n$ such that $W_1 \cap (-W_2) = \emptyset$, we define their *curved join* $W_1 * W_2$ as the projection on $\mathbb{S}^{n-1}$ of $\operatorname{conv}(W_1 \cup W_2)$. For instance we have

$$\mathbb{S}^1 * \mathbb{S}^1 = \mathbb{S}^3.$$

This generalizes as follows: identifying $\mathbb{C}^m$ with $\mathbb{R}^{2m}$ and given the standard orthonormal basis $\{e_1, \dots, e_{2m}\}$ of $\mathbb{R}^{2m}$, for each $2 \leq r \leq 2m$, denote by Π_r the plane generated by $\{e_{r-1}, e_r\}$. Suppose $\Pi_{r_1} \cap \Pi_{r_2} = 0$ and let W_1 and W_2 be subsets of the unit circles of Π_{r_1} and Π_{r_2}, respectively. Then, one can define the curved join $W_1 * W_2$ as above. In particular, we denote by Σ_k the unit circle lying in the k^{th} copy of $\mathbb{C}$ in $\mathbb{C}^m$ and we have the following equality

$$\mathbb{S}^{2m-1} = \Sigma_1 * \Sigma_2 * \cdots * \Sigma_m.$$

Let $\Gamma \leq O(n)$ be a finite group acting freely on $\mathbb{S}^{n-1}$ and let $h \in \mathbb{N}^*$. Then, we can make Γ act diagonally on $\mathbb{R}^{hn}$. Under the identification $\mathbb{S}^{hn-1} = \mathbb{S}^{(h-1)n-1} * \mathbb{S}^{n-1}$, we have $\gamma \cdot (x * y) = \gamma x * \gamma y$.

To compute the boundaries, we shall need the following technical result:

Lemma 10.2.1 (*[FGMS13, Lemma 2.5]*)**.** *We have the following Leibniz formula for the oriented boundary of a curved join*

$$\partial(X * Y) = \partial X * Y - (-1)^{\dim X} X * \partial Y.$$

In fact, we will use the following general lemma, allowing to recursively determine a fundamental domain and an equivariant cellular decomposition on $\mathbb{S}^{hn-1}$, once we know one on $\mathbb{S}^{n-1}$. More precisely, let $\Gamma \leq O(n)$ be a finite group acting freely on $\mathbb{S}^{n-1}$. Assume that $\mathcal{D}$ is a fundamental domain for the action on $\mathbb{S}^{n-1}$ and that $\widetilde{L}$ is a cellular decomposition of $\mathcal{D}$. We obtain an equivariant cell decomposition $\widetilde{K} = \Gamma \cdot \widetilde{L}$ of $\mathbb{S}^{n-1}$ and $L = \widetilde{K}/\Gamma$ is a cellular decomposition of $\mathbb{S}^{n-1}/\Gamma$. Assume further that $\widetilde{Z}$ is a subcomplex of $\widetilde{L}$ that is a minimal decomposition of $\mathcal{D}$ by lifts of the cells of L.

Let $h \in \mathbb{N}^*$ and consider the diagonal action of Γ on $\mathbb{S}^{hn-1}$. Then, a fundamental domain for this action on $\mathbb{S}^{hn-1}$ is given by

$$\mathcal{D}' := \mathbb{S}^{(h-1)n-1} * \mathcal{D}.$$

Furthermore, we construct an equivariant cellular decomposition $\widetilde{K'}$ of $\mathbb{S}^{hn-1}$ and a minimal cellular decomposition $\widetilde{L}'$ of $\mathcal{D}'$ as follows:

- the $(h-1)n-1$-skeleton of $\widetilde{L}'$ is $\widetilde{L}'_{(h-1)n-1} = \widetilde{K}$;
- for the $(h-1)n$-skeleton of $\widetilde{L}'$, we attach $k_0 (h-1)n$-cells to $\widetilde{K}$, where k_0 is the number of 0-cells $\widetilde{e}^0_l$ of $\widetilde{Z}$ and the corresponding attaching map is given by the parametrization of the curved join $\widetilde{K} * \widetilde{e}^0_l$;
- for the $(h-1)n+1$-skeleton of $\widetilde{L}'$, we attach $k_1 (h-1)n+1$-cells to the $(h-1)n$-skeleton of $\widetilde{L}'$, where k_1 is the number of 1-cells $\widetilde{e}^1_l$ of $\widetilde{Z}$ and the attaching map is given by the parametrization of $\widetilde{L}'_{(h-1)n} * \widetilde{e}^1_l$;
- we carry on this procedure up to dimension $hn-1$.

We can summarize this in the following result:

Lemma 10.2.2 (*[FGMS13, Lemma 4.1]*)**.** *If $\Gamma \leq O(n)$ is a finite group acting freely on $\mathbb{S}^{n-1}$, if $\mathcal{D}$ is a fundamental domain for this action and if $\widetilde{L}$ is a cellular decomposition of $\mathcal{D}$, with associated Γ-equivariant cellular decomposition $\widetilde{K} = \Gamma \cdot \widetilde{L}$ of $\mathbb{S}^{n-1}$, then for every $h \in \mathbb{N}^*$, the subset*

$$\mathcal{D}' := \mathbb{S}^{(h-1)n-1} * \mathcal{D}$$

is a fundamental domain for the diagonal action of Γ on $\mathbb{S}^{hn-1}$ and the above construction gives a cell decomposition $\widetilde{L}'$ of $\mathcal{D}'$, with associated Γ-equivariant cell decomposition $\widetilde{K'} := \Gamma \cdot \widetilde{L}'$ of $\mathbb{S}^{hn-1}$.

11 The octahedral case

In the following two sections, we let both $\mathcal{O}$ and $\mathcal{I}$ act (freely) by (quaternion) multiplication on the left on $\mathbb{S}^3$.

We start with a brief reminder on binary poyhedral groups. The reader is referred to [LT09].

11.1 Binary polyhedral groups and spherical space forms

Consider the *quaternion group* $\mathcal{Q}_8 := \langle i, j \rangle = \{\pm 1, \pm i, \pm j, \pm k\}$, a finite subgroup of the sphere $\mathbb{S}^3$ of unit quaternions. The element $\varpi := \frac{1}{2}(-1 + i + j + k)$ has order 3 and normalizes $\mathcal{Q}_8$. Hence, the group

$$\mathcal{T} := \langle i, \varpi \rangle$$

has order 24, and the 16 elements of $\mathcal{T} \setminus \mathcal{Q}_8$ have the form $\frac{1}{2}(\pm 1 \pm i \pm j \pm k)$. The group $\mathcal{T}$ is the *binary tetrahedral group.*

Next, the element $\gamma := \frac{1}{\sqrt{2}}(1 + i)$ has order 8 and normalizes both $\mathcal{Q}_8$ and $\mathcal{T}$. Hence the group

$$\mathcal{O} := \langle \varpi, \gamma \rangle$$

is of order 48 (since $\gamma^2 = i$) and is called the *binary octahedral group* and we have $\mathcal{O} = \langle \varpi, \gamma \rangle$. The set $\mathcal{O} \setminus \mathcal{T}$ consists of the 24 elements $\frac{1}{\sqrt{2}}(\pm u \pm v)$ where $u \neq v \in \{1, i, j, k\}$.

Setting $\varphi := \frac{1}{2}(1 + \sqrt{5})$, the element $\sigma := \frac{1}{2}(\varphi^{-1} + i + \varphi j)$ is of order 5 hence the *binary icosahedral group*

$$\mathcal{I} := \langle i, \sigma \rangle$$

has order 120 and we have $\mathcal{T} \leq \mathcal{I}$.

The universal covering map $\mathbb{S}^3 = SU(2) \twoheadrightarrow SO(3)$ can be interpreted as the action of unit quaternions on the space of purely imaginary quaternions

$$\mathrm{B} : \mathbb{S}^3 \to SO_3(\mathbb{R}).$$

The respective images of $\mathcal{T}$, $\mathcal{O}$ and $\mathcal{I}$ are the rotation groups $\mathfrak{A}_4$, $\mathfrak{S}_4$ and $\mathfrak{A}_5$ of a regular tetrahedron, octahedron and icosahedron respectively, hence the names.

It has been observed by Coxeter and Moser in [CM72, §6.4] that finite subgroups of $\mathbb{S}^3$ have nice presentation. Namely, denoting

$$\langle \ell, m, n \rangle := \left\langle r, s, t \mid r^\ell = s^m = t^n = rst \right\rangle,$$

we have isomorphisms

$$\langle 2, 3, 3 \rangle \simeq \mathcal{T}, \quad \langle 2, 3, 4 \rangle \simeq \mathcal{O}, \quad \langle 2, 3, 5 \rangle \simeq \mathcal{I}.$$

Finally, for $n \in \mathbb{N}^*$ and $\mathcal{G} \in \{\mathcal{T}, \mathcal{O}, \mathcal{I}\}$, we define the *polyhedral spherical space form*

$$\mathsf{P}^{4n-1}_{\mathcal{G}} := \mathbb{S}^{4n-1}/\mathcal{G}.$$

11.2 Fundamental domain

We use Theorem 10.1.1 to find a fundamental domain for $\mathcal{O}$ on $\mathbb{S}^3$. To this end, we first introduce the *orbit polytope* in $\mathbb{R}^4$

$$\mathscr{P} := \mathrm{conv}(\mathcal{O}).$$

Then, we know that $\mathcal{O}$ acts freely on the set $\mathscr{P}_3$ of facets of $\mathscr{P}$ and by Theorem 10.1.1, it suffices to find a set of representatives in $\mathscr{P}_3$ such that their union is connected; this will be a fundamental domain for the action on $\partial\mathscr{P}$, which we can transport to the sphere $\mathbb{S}^3$ using the equivariant homeomorphism $\partial\mathscr{P} \to \mathbb{S}^3$, $x \mapsto x/|x|$.

The 4-polytope $\mathscr{P}$ has 48 vertices, 336 edges, 576 faces and 288 facets and is known as the *disphenoidal 288-cell*; it is dual to the bitruncated cube. Since $\mathcal{O}$ acts freely on $\mathscr{P}_3$, there must be exactly six orbits in $\mathscr{P}_3$. We introduce the following elements of $\mathcal{O}$, also expressed in terms of the generators s and t in the Coxeter-Moser presentation:

$$\begin{cases} \omega_0 := \frac{1+i+j+k}{2} = s, \\ \omega_i := \frac{1-i+j+k}{2} = t^{-1}st^{-1}, \\ \omega_j := \frac{1+i-j+k}{2} = s^{-1}t^2, \\ \omega_k := \frac{1+i+j-k}{2} = t^{-1}st. \end{cases} \quad \text{and} \quad \begin{cases} \tau_i := \frac{1+i}{\sqrt{2}} = t, \\ \tau_j := \frac{1+j}{\sqrt{2}} = t^{-1}s, \\ \tau_k := \frac{1+k}{\sqrt{2}} = st^{-1}. \end{cases}$$

Next, we may find explicit representatives for the $\mathcal{O}$-orbits of $\mathscr{P}_3$.

Proposition 11.2.1. *The following tetrahedra (in $\mathbb{R}^4$)*

$$\Delta_1 := [1, \tau_i, \tau_j, \omega_0],\ \Delta_2 := [1, \tau_j, \tau_k, \omega_0],\ \Delta_3 := [1, \tau_k, \tau_i, \omega_0],$$

$$\Delta_4 := [1, \tau_i, \omega_k, \tau_j],\ \Delta_5 := [1, \tau_j, \omega_i, \tau_k],\ \Delta_6 := [1, \tau_i, \omega_j, \tau_k]$$

form a system of representatives of $\mathcal{O}$-orbits of facets of $\mathscr{P}$. Furthermore, the subset of $\mathscr{P}$ defined by

$$\mathscr{D} := \bigcup_{i=1}^{6} \Delta_i$$

is a (connected) polytopal complex and is a fundamental domain for the action of $\mathcal{O}$ on $\partial\mathscr{P}$.

Proof. First, we have to find the facets of $\mathscr{P}$ by giving the defining inequalities. To do this, we make the group $\{\pm 1\}^4 \rtimes \mathfrak{S}_4$ act on $\mathbb{R}^4$ by signed permutations of coordinates. Let

$$v_1 := \begin{pmatrix} 3-2\sqrt{2} \\ \sqrt{2}-1 \\ \sqrt{2}-1 \\ 1 \end{pmatrix}, \quad v_2 := \begin{pmatrix} 2-\sqrt{2} \\ 2-\sqrt{2} \\ 2\sqrt{2}-2 \\ 0 \end{pmatrix}.$$

By invariance of $\mathscr{P}$, to prove that the 288 inequalities $\langle v, x\rangle \leq 1$, with $v \in (\{\pm 1\}^4 \rtimes \mathfrak{A}_4) \cdot \{v_1, v_2\}$, are valid for $\mathscr{P}$, it suffices to check the two inequalities $\langle v_i, x\rangle \leq 1$, for $i = 1, 2$. As there are indeed 288 conditions, we have in fact all of them, hence the facets are given by the equalities $\langle v, x\rangle = 1$ and we find their vertices by looking at vertices of $\mathscr{P}$ that satisfy these equalities. We find

$$\operatorname{vert}(\mathscr{D}) = \{1, \tau_i, \tau_j, \tau_k, \omega_i, \omega_j, \omega_k, \omega_0\}.$$

Now, since $\mathbb{R}^4 = \operatorname{span}(\mathcal{O})$ and $\operatorname{vert}(\mathscr{D}) \cap \operatorname{vert}(\mathscr{D})^{-1} = \{1\}$, Proposition 10.1.2 ensures that $\mathscr{D}$ is indeed a fundamental domain for $\partial\mathscr{P}$. $\square$

Remark 11.2.2. *The recipe used to find these tetrahedra is quite simple. First, choose Δ_1 in some $\mathcal{O}$-orbit of $\partial\mathscr{P}_3$ and containing 1 as a vertex. Then, we arbitrarily choose another orbit and look at the dimensions of the intersections of Δ_1 with the facets of this second orbit. There is exactly one facet (namely Δ_2) for which the intersection has dimension 2 and we continue further until we obtain representatives for the six orbits. Hence, a lot of different fundamental domains can be produced in this way. The calculations can be done using the Maple package "Convex" (see [Fra]) and quaternionic multiplication, as in [GAP4].*

It should be noted that all the figures displayed in the sequel only reflect the combinatorics of the polytopes we consider, not the metric they carry as subsets of $\mathbb{S}^3$.

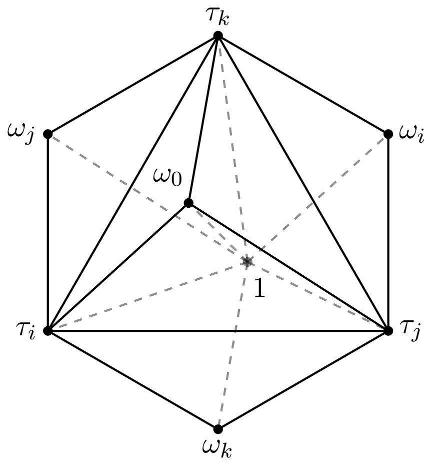


Figure 10: The six tetrahedra inside $\mathscr{D}$.

11.3 Associated $\mathcal{O}$-equivariant cellular decomposition of $\partial\mathscr{P}$

We shall now examine the combinatorics of the polytopal complex $\mathscr{D}$ constructed in the previous subsection to obtain a cellular decomposition of it. Since $\mathscr{D}$ is a fundamental domain for $\mathcal{O}$ on $\partial\mathscr{P}$, translating the cells will give an equivariant decomposition of $\partial\mathscr{P}$ and projecting to $\mathbb{S}^3$ will give the desired equivariant cellular structure on the sphere.

The facets of $\mathscr{D}$ are the ones of the six tetrahedra Δ_i, except those that are contained in some intersection $\Delta_i \cap \Delta_j$. We obtain the following facets

$$\mathscr{D}_2 = \{[1,\tau_j,\omega_i],[1,\omega_i,\tau_k],[1,\tau_k,\omega_j],[1,\omega_j,\tau_i],[1,\tau_i,\omega_k],[1,\omega_k,\tau_j],[\tau_j,\omega_i,\tau_k],$$
$$[\tau_k,\omega_j,\tau_i],[\tau_i,\omega_k,\tau_j],[\tau_i,\tau_j,\omega_0],[\tau_j,\tau_k,\omega_0],[\tau_k,\tau_i,\omega_0]\}.$$

We notice the following relations

$$\begin{cases} \tau_i \cdot [1,\tau_j,\omega_i] = [\tau_i,\omega_0,\tau_k], \\ \tau_i \cdot [1,\omega_i,\tau_k] = [\tau_i,\tau_k,\omega_j], \end{cases} \quad \begin{cases} \tau_j \cdot [1,\tau_i,\omega_j] = [\tau_j,\omega_k,\tau_i], \\ \tau_j \cdot [1,\omega_j,\tau_k] = [\tau_j,\tau_i,\omega_0], \end{cases} \quad \begin{cases} \tau_k \cdot [1,\tau_j,\omega_k] = [\tau_k,\omega_i,\tau_j], \\ \tau_k \cdot [1,\omega_k,\tau_i] = [\tau_k,\tau_j,\omega_0]. \end{cases}$$

These are the only relations linking facets, hence, we may gather facets two by two and define the following 2-cells and 1-cells, respectively

$$e_1^2 :=]\tau_j,1,\omega_i[\cup]1,\omega_i[\cup]1,\omega_i,\tau_k[,\ e_2^2 :=]\tau_i,1,\omega_j[\cup]1,\omega_j[\cup]1,\omega_j,\tau_k[,\ e_3^2 :=]\tau_i,1,\omega_k[\cup]1,\omega_k[\cup]1,\omega_k,\tau_j[,$$
$$e_1^1 :=]1,\tau_i[,\ e_2^1 :=]1,\tau_j[,\ e_3^1 :=]1,\tau_k[,$$

recalling that, for a polytope $[v_1,\ldots,v_n] := \mathrm{conv}(v_1,\ldots,v_n)$, we denote by $]v_1,\ldots,v_n[$ its *interior*, namely its maximal face.

If we add vertices of $\mathscr{D}$ and its interior, which is formed by only one cell e^3 by construction, then we may cover all of $\mathscr{D}$ with these cells and some of their translates. Thus, we have obtained the

Lemma 11.3.1. *Consider the following sets of cells in $\mathscr{D}$*

$$\begin{cases} E^0_{\mathscr{D}} := \{1,\ \tau_i,\ \tau_j,\ \tau_k,\ \omega_i,\ \omega_j,\ \omega_k\}, \\ E^1_{\mathscr{D}} := \{e^1_1,\ \tau_j e^1_1,\ \tau_k e^1_1,\ \omega_i e^1_1,\ e^1_2,\ \tau_i e^1_2,\ \tau_k e^1_2,\ \omega_j e^1_2,\ e^1_3,\ \tau_i e^1_3,\ \tau_j e^1_3,\ \omega_k e^1_3\}, \\ E^2_{\mathscr{D}} := \{e^2_1,\ \tau_i e^2_1,\ e^2_2,\ \tau_j e^2_2,\ e^2_3,\ \tau_k e^2_3\}, \\ E^3_{\mathscr{D}} := \{e^3\} \end{cases}$$

Then, one has the following cellular decomposition of the fundamental domain

$$\mathscr{D} = \coprod_{\substack{0\le j\le 3 \\ e\in E^j_{\mathscr{D}}}} e.$$

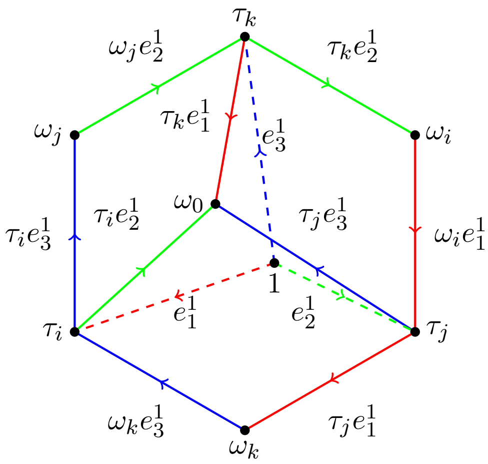


Figure 11: The 1-skeleton of $\mathscr{D}$.

Then, combining Proposition 11.2.1 and Lemma 11.3.1, yields the following result:

Proposition 11.3.2. *Letting $E^0 := \{1\}$, $E^1 := \{e^1_i,\ i = 1,2,3\}$, $E^2 := \{e^2_i,\ i = 1,2,3\}$ and $E^3 := \{e^3\}$ with the above notations, we have the following $\mathcal{O}$-equivariant cellular decomposition of $\partial\mathscr{P}$*

$$\partial\mathscr{P} = \coprod_{\substack{0\le j\le 3 \\ e\in E^j, g\in\mathcal{O}}} ge.$$

As a consequence, using the homeomorphism $\phi : \partial\mathscr{P} \xrightarrow{\sim} \mathbb{S}^3$ given by $x \mapsto x/|x|$, we obtain the following $\mathcal{O}$-equivariant cellular decomposition of the sphere

$$\mathbb{S}^3 = \coprod_{\substack{0\le j\le 3 \\ e\in E^j, g\in\mathcal{O}}} g\phi(e).$$

We now have to compute the boundaries of the cells and the resulting cellular homology chain complex. We choose to orient the 3-cell e^3 directly, and the 2-cells undirectly. The induced orientations seen in $\mathscr{D}$ can be visualized in Figure 12.

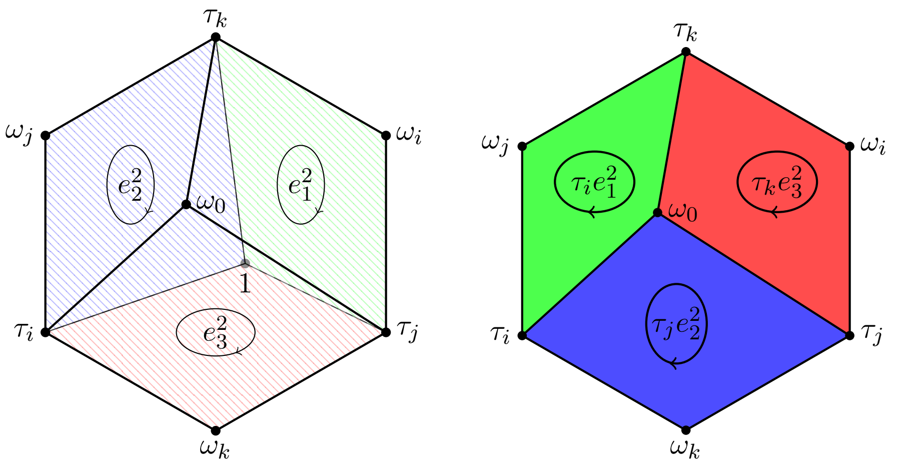


Figure 12: The fundamental domain with its 2-cells (back and front).

Proposition 11.3.3. *The cellular homology complex of $\partial\mathscr{P}$ associated to the cellular structure given in Proposition 11.3.2 is the chain complex of left $\mathbb{Z}[\mathcal{O}]$-modules*

$$\mathcal{K}_{\mathcal{O}} := \left(\mathbb{Z}[\mathcal{O}] \xrightarrow{\partial_3} \mathbb{Z}[\mathcal{O}]^3 \xrightarrow{\partial_2} \mathbb{Z}[\mathcal{O}]^3 \xrightarrow{\partial_1} \mathbb{Z}[\mathcal{O}] \right),$$

where

$$\partial_1 = \begin{pmatrix} \tau_i - 1 \\ \tau_j - 1 \\ \tau_k - 1 \end{pmatrix}, \quad \partial_2 = \begin{pmatrix} \omega_i & \tau_k - 1 & 1 \\ 1 & \omega_j & \tau_i - 1 \\ \tau_j - 1 & 1 & \omega_k \end{pmatrix}, \quad \partial_3 = \begin{pmatrix} 1-\tau_i & 1-\tau_j & 1-\tau_k \end{pmatrix}.$$

To conclude this section, we show in Figure 13 a tetrahedron in $\mathscr{P}_3$ containing 1 as a vertex. In this picture, we put the points $\omega_h^{\pm}$ (with $h = 0, i, j, k$) at the centers of the facets of the octahedron. The tetrahedra in question are constructed in the following way: one chooses an edge of the octahedron and the center of a face which is adjacent to this edge. The resulting four vertices (including 1) are vertices of the corresponding tetrahedron. This representation will be useful when we study the application to the flag manifold of $SL_3(\mathbb{R})$.

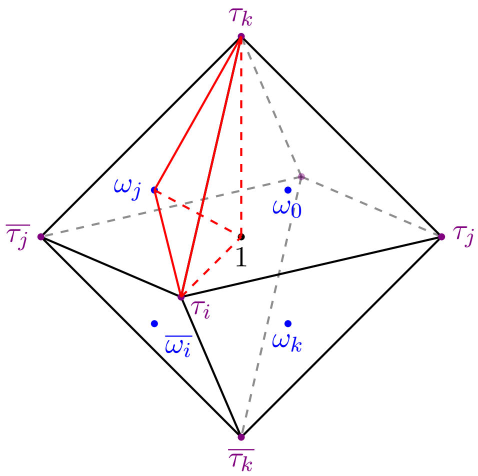


Figure 13: One of the twenty-four facets of $\mathscr{P}$ containing 1.

11.4 The case of spheres and free resolution of the trivial $\mathcal{O}$-module

Using Theorem 10.1.1, we derive a fundamental domain for $\mathcal{O}$ acting on $\mathbb{S}^3$ and thus obtain an $\mathcal{O}$-equivariant cellular decomposition of $\mathbb{S}^3$.

Theorem 11.4.1. *The following subset of $\mathbb{S}^3$ is a fundamental domain for the action of $\mathcal{O}$*

$$\begin{aligned}\mathscr{F}_3 :=&(\omega_i * 1 * \tau_j * \tau_k) \cup (1 * \tau_j * \tau_k * \omega_0) \cup (\omega_j * 1 * \tau_k * \tau_i)\\ &\cup (1 * \tau_k * \tau_i * \omega_0) \cup (\omega_k * 1 * \tau_i * \tau_j) \cup (1 * \tau_i * \tau_j * \omega_0).\end{aligned}$$

As a consequence, the sphere $\mathbb{S}^3$ admits an $\mathcal{O}$-equivariant cellular decomposition with the following cells as orbit representatives, where relint *denotes the relative interior,*

$$\widetilde{e}^0 := 1 * \emptyset = \{1\},\ \widetilde{e}^1_1 := \mathrm{relint}(1 * \tau_i),\ \widetilde{e}^1_2 := \mathrm{relint}(1 * \tau_j),\ \widetilde{e}^1_3 := \mathrm{relint}(1 * \tau_k),$$

$$\widetilde{e}^2_1 := \mathrm{relint}((1 * \omega_i * \tau_j) \cup (1 * \omega_i * \tau_k)),\ \widetilde{e}^2_2 := \mathrm{relint}((1 * \omega_j * \tau_k) \cup (1 * \omega_j * \tau_i)),$$

$$\widetilde{e}^2_3 := \mathrm{relint}((1 * \omega_k * \tau_i) \cup (1 * \omega_k * \tau_j)),\ \widetilde{e}^3 := \mathring{\mathscr{F}}_3$$

Furthermore, the associated cellular homology complex is the chain complex $\mathcal{K}_{\mathcal{O}}$ from the Proposition 11.3.3.

The relative interior of a simplex in $\partial\mathscr{P}$ is sent to the Riemannian relative interior in $\mathbb{S}^3$, that is, the subset of points of the geodesic simplex which do not belong to any geodesic sub-simplex of smaller dimension.

From this, we immediately deduce the following result, which gives a periodic free resolution of the constant module over $\mathbb{Z}[\mathcal{O}]$.

Corollary 11.4.2. *The following complex is a 4-periodic resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathcal{O}]$*

$$\ldots \longrightarrow \mathbb{Z}[\mathcal{O}]^3 \xrightarrow{\partial_{4q-3}} \mathbb{Z}[\mathcal{O}] \xrightarrow{\partial_{4q-4}} \ldots \longrightarrow \mathbb{Z}[\mathcal{O}]^3 \xrightarrow{\partial_2} \mathbb{Z}[\mathcal{O}]^3 \xrightarrow{\partial_1} \mathbb{Z}[\mathcal{O}] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0\,,$$

where, for $q \geq 1$,

$$\partial_{4q-3} = \begin{pmatrix} \tau_i - 1 \\ \tau_j - 1 \\ \tau_k - 1 \end{pmatrix}, \qquad \partial_{4q-2} = \begin{pmatrix} \omega_i & \tau_k - 1 & 1 \\ 1 & \omega_j & \tau_i - 1 \\ \tau_j - 1 & 1 & \omega_k \end{pmatrix},$$

$$\partial_{4q-1} = \begin{pmatrix} 1 - \tau_i & 1 - \tau_j & 1 - \tau_k \end{pmatrix}, \qquad \partial_{4q} = \left(\textstyle\sum_{g\in\mathcal{O}} g\right).$$

Recall the augmentation map $\varepsilon : \mathbb{Z}[\mathcal{O}] \to \mathbb{Z}$ defined by $\varepsilon\left(\sum_{g\in\mathcal{O}} a_g g\right) := \sum_{g\in\mathcal{O}} a_g$. We can now compute the group cohomology of $\mathcal{O}$ using the previous Corollary. But first, let us recall the following basic fact:

Lemma 11.4.3. *Let G be a finite group acting freely and cellularly on a CW-complex X.*

1. *If $\mathcal{K}$ denotes the cellular homology chain complex of X (a complex of free $\mathbb{Z}[G]$-modules), then the induced cellular homology complex of X/G is $\mathcal{K} \otimes_{\mathbb{Z}[G]} \mathbb{Z}$.*

2. *If $f : \mathbb{Z}[G]^m \to \mathbb{Z}[G]^n$ is a homomorphism of left $\mathbb{Z}[G]$-modules, identified with its matrix in the canonical bases, then the matrix of the induced homomorphism $f \otimes_{\mathbb{Z}[G]} id_{\mathbb{Z}} : \mathbb{Z}^m \to \mathbb{Z}^n$ is given by the matrix $\varepsilon(f)$, computed term by term.*

Corollary 11.4.4. *The group cohomology of $\mathcal{O}$ with integer coefficients is given as follows:*

$$\forall q \geq 1, \quad \begin{cases} H^q(\mathcal{O}, \mathbb{Z}) = \mathbb{Z} & \text{if } q = 0, \\ H^q(\mathcal{O}, \mathbb{Z}) = \mathbb{Z}/48\mathbb{Z} & \text{if } q \equiv 0 \pmod 4, \\ H^q(\mathcal{O}, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} & \text{if } q \equiv 2 \pmod 4, \\ H^q(\mathcal{O}, \mathbb{Z}) = 0 & \text{otherwise.} \end{cases}$$

Proof. In view of Lemma 11.4.3, is suffices to compute $\mathcal{C}(\mathsf{P}^{\infty}_{\mathcal{O}}, \mathbb{Z}[\mathcal{O}]) \otimes_{\mathbb{Z}[\mathcal{O}]} \mathbb{Z}$, with $\mathcal{C}(\mathsf{P}^{\infty}_{\mathcal{O}}, \mathbb{Z}[\mathcal{O}])$ the complex given in Corollary 11.4.2. The notation will become clear later (see Theorem 11.4.7). Computing the matrices $\varepsilon(\partial_i)$ and dualizing the result leads to the following cochain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}^3 \xrightarrow{\left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{smallmatrix}\right)} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 48} \mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \xrightarrow{\times 48} \mathbb{Z} \xrightarrow{0} \mathbb{Z}^3 \longrightarrow \cdots$$

and computing the elementary divisors of the only non-trivial matrix allows to conclude. □

Remark 11.4.5. *In [TZ08, Proposition 4.7], Tomoda and Zvengrowski give an explicit resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathcal{O}]$. They use the following presentation*

$$\mathcal{O} = \langle T, U \mid TU^2T = U^2,\ TUT = UTU \rangle$$

from [CM72]. As we would like to work with presentations, we use the isomorphism

$$\langle T, U \mid TU^2T = U^2,\ TUT = UTU \rangle \xrightarrow{\sim} \mathcal{O}$$

sending T to $\frac{1}{\sqrt{2}}(1+i)$ and U to $\frac{1}{\sqrt{2}}(1+j)$. Then, the Tomoda-Zvengrowski complex reads

$$\mathcal{K}^{TZ}_{\mathcal{O}} = \left(\mathbb{Z}[\mathcal{O}] \xrightarrow{\delta_3} \mathbb{Z}[\mathcal{O}]^2 \xrightarrow{\delta_2} \mathbb{Z}[\mathcal{O}]^2 \xrightarrow{\delta_1} \mathbb{Z}[\mathcal{O}] \right),$$

with

$$\delta_1 = \begin{pmatrix} T-1 \\ U-1 \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} 1+TU-U & T-1-UT \\ 1+TU^2 & T-U-1+TU \end{pmatrix}, \quad \delta_3 = \begin{pmatrix} 1-TU & U-1 \end{pmatrix}.$$

On the other hand, the differentials ∂_i of the complex $\mathcal{K}_{\mathcal{O}}$ from Proposition 11.3.3 are given, through the above presentation, by

$$\partial_1 = \begin{pmatrix} T-1 \\ U-1 \\ TUT^{-1}-1 \end{pmatrix}, \ \partial_2 = \begin{pmatrix} UT^{-1} & TUT^{-1}-1 & 1 \\ 1 & U^{-1}T & T-1 \\ U-1 & 1 & UT \end{pmatrix}, \ \partial_3 = \begin{pmatrix} 1-T & 1-U & 1-TUT^{-1} \end{pmatrix}.$$

We claim that the complexes $\mathcal{K}_{\mathcal{O}}$ and $\mathcal{K}^{TZ}_{\mathcal{O}}$ are homotopy equivalent. This observation relies on elementary operations on matrix rows and columns. Write $Z := U^4 = T^4$ for the only non trivial element of $Z(\mathcal{O})$. Consider

$$P := \begin{pmatrix} -Z & 0 & 0 \\ Z(1-T) & TUT & -U^2 \\ -U^{-3}T & -TUT & 0 \end{pmatrix}, \quad Q := \begin{pmatrix} 0 & -TUT & 0 \\ -TUT & 0 & 0 \\ U^2-TUT & U^2T & 1 \end{pmatrix},$$

then $P, Q \in GL_3(\mathbb{Z}[\mathcal{O}])$ *and*

$$P^{-1} = \begin{pmatrix} -Z & 0 & 0 \\ U^{-1} & 0 & -(TUT)^{-1} \\ U^{-2}(T-1)+U^{-1}T & -U^{-2} & -U^{-2} \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} 0 & -(TUT)^{-1} & 0 \\ -(TUT)^{-1} & 0 & 0 \\ UT^{-1} & TUT^{-1}-1 & 1 \end{pmatrix}.$$

Now, we have the following relations

$$-Q^{-1}d_1TUT = \begin{pmatrix} T-1 \\ U-1 \\ 0 \end{pmatrix}, \quad P^{-1}d_2Q = \begin{pmatrix} 0 & 0 & -Z \\ 1+TU-U & T-1-UT & 0 \\ 1+TU^2 & T-U-1+TU & 0 \end{pmatrix},$$

$$U^{-2}d_3P = \begin{pmatrix} 0 & 1-TU & U-1 \end{pmatrix}.$$

Hence, the isomorphism

$$\mathcal{K}_{\mathcal{O}} \simeq \mathcal{K}_{\mathcal{O}}^{TZ} \oplus \left(0 \longrightarrow \mathbb{Z}[\mathcal{O}] \xrightarrow{1} \mathbb{Z}[\mathcal{O}] \longrightarrow 0 \right),$$

confirms that $\mathcal{K}_{\mathcal{O}}$ *is indeed homotopy equivalent to* $\mathcal{K}_{\mathcal{O}}^{TZ}$.

In fact, the complex from the Corollary 11.4.2 carries geometric information.

Proposition 11.4.6. *The following subset of* $\mathbb{S}^{4n-1}$ *is a fundamental domain for the action*

$$\mathscr{F}_{4n-1} := \Sigma_1 * \Sigma_2 * \cdots * \Sigma_{2(n-1)} * \mathscr{F}_3,$$

with $\mathscr{F}_3$ *inside* $\Sigma_{2n-1} * \Sigma_{2n}$ *the fundamental domain from Theorem 11.4.1.*

We can now describe the resulting equivariant cellular decomposition on $\mathbb{S}^{4n-1}$ using Lemma 10.2.2 and Theorem 11.4.1. It only remains to consider the boundary of the cells $\widetilde{e}^{4q}$ for $q > 0$. But it follows from the fact that $\widetilde{e}^{4q} = \mathbb{S}^{4q-1} * \widetilde{e}^{4q-1}$, hence its boundary is given by all the cells in $\mathbb{S}^{4q-1}$, that is, all the orbits under $\mathcal{O}$. This gives the following result, which we prefer to state using the vocabulary of universal covering spaces. We denote by $C(\widetilde{K}, \mathbb{Z}[G])$ the chain complex of finitely generated free (left) $\mathbb{Z}[G]$-modules given by the cellular homology complex of the universal covering space $\widetilde{K}$ of a finite CW-complex K with the fundamental group G acting by covering transformations.

Theorem 11.4.7. *The chain complex* $\mathcal{C}(\mathsf{P}_{\mathcal{O}}^{4n-1}, \mathbb{Z}[\mathcal{O}])$ *of the universal covering space of the octahedral space forms* $\mathsf{P}_{\mathcal{O}}^{4n-1}$ *with the fundamental group acting by covering transformations is the following complex of left* $\mathbb{Z}[\mathcal{O}]$*-modules:*

$$0 \longrightarrow \mathbb{Z}[\mathcal{O}] \xrightarrow{\partial_{4n-1}} \mathbb{Z}[\mathcal{O}]^3 \longrightarrow \ldots \longrightarrow \mathbb{Z}[\mathcal{O}]^3 \xrightarrow{\partial_2} \mathbb{Z}[\mathcal{O}]^3 \xrightarrow{\partial_1} \mathbb{Z}[\mathcal{O}] \longrightarrow 0,$$

where, the boundaries are as in Corollary 11.4.2.

In particular, the complex is exact in middle terms, i.e.

$$\forall 0 < i < 4n-1, \; H_i(\mathcal{C}(\mathsf{P}_{\mathcal{O}}^{4n-1}, \mathbb{Z}[\mathcal{O}])) = 0$$

and we have

$$H_0(\mathcal{C}(\mathsf{P}_{\mathcal{O}}^{4n-1}, \mathbb{Z}[\mathcal{O}])) = H_{4n-1}(\mathcal{C}(\mathsf{P}_{\mathcal{O}}^{4n-1}, \mathbb{Z}[\mathcal{O}])) = \mathbb{Z}.$$

Proof. The computation of the complex follows from lemma 10.2.2 and the previous discussion. The claims on its homology follow, $\mathbb{S}^{4n-1}$ being the universal covering space of $\mathsf{P}_{\mathcal{O}}^{4n-1}$. □

12 The icosahedral case

12.1 Fundamental domain

We shall use for the binary icosahedral group $\mathcal{I}$ of order 120 exactly the same method as for $\mathcal{O}$. First, we are looking for a fundamental domain for $\mathcal{I}$ in $\mathbb{S}^3$. To do this, we consider the orbit polytope in $\mathbb{R}^4$

$$\mathscr{P} := \operatorname{conv}(\mathcal{I}).$$

This polytope has 120 vertices, 720 edges, 1200 faces and 600 facets and is known as the *600-cell* (or the *hexacosichoron*, or even the *tetraplex*). Since $\mathcal{I}$ acts freely on $\mathscr{P}_3$, there must be exactly five orbits in $\mathscr{P}_3$. Here again, we consider some elements of $\mathcal{I}$, also expressed in terms of the Coxeter-Moser generators s and t and with $\varphi := (1+\sqrt{5})/2$:

$$\begin{cases} \sigma_i^+ := \frac{\varphi+\varphi^{-1}i+j}{2} = t, \\ \sigma_i^- := \frac{\varphi+\varphi^{-1}i-j}{2} = st^{-2}, \end{cases} \quad \begin{cases} \sigma_j^+ := \frac{\varphi+\varphi^{-1}j-k}{2} = ts^{-1}t, \\ \sigma_j^- := \frac{\varphi-\varphi^{-1}j-k}{2} = s^{-1}t, \end{cases} \quad \begin{cases} \sigma_k^+ := \frac{\varphi+i+\varphi^{-1}k}{2} = st^{-1}, \\ \sigma_k^- := \frac{\varphi+i-\varphi^{-1}k}{2} = s^{-1}t^2. \end{cases}$$

As for $\mathcal{O}$, we may find explicit representatives for the $\mathcal{I}$-orbits of $\mathscr{P}_3$:

Proposition 12.1.1. *The following tetrahedra (in $\mathbb{R}^4$)*

$$\Delta_1 := [1, \sigma_k^-, \sigma_k^+, \sigma_i^+],\ \Delta_2 := [1, \sigma_k^-, \sigma_i^+, \sigma_j^+],\ \Delta_3 := [1, \sigma_k^-, \sigma_j^+, \sigma_j^-],$$

$$\Delta_4 := [1, \sigma_k^-, \sigma_j^-, \sigma_i^-],\ \Delta_5 := [1, \sigma_k^-, \sigma_i^-, \sigma_k^+]$$

form a system of representatives of $\mathcal{I}$-orbits of facets of $\mathscr{P}$. Furthermore, the subset of $\mathscr{P}$ defined by

$$\mathscr{D} := \bigcup_{i=1}^{5} \Delta_i$$

is a (connected) polytopal complex and is a fundamental domain for the action of $\mathcal{I}$ on $\partial\mathscr{P}$.

Proof. We argue as in the proof of Proposition 11.2.1. Let $\varphi := (1+\sqrt{5})/2$. By invariance of $\mathscr{P}$, to verify that the following 600 inequalities

$$\langle v, x\rangle \leq 1,$$

with $v \in (\{\pm 1\}^4 \rtimes \mathfrak{A}_4) \cdot U$ and

$$U := \left\{ \begin{pmatrix} 4-2\varphi \\ 4-2\varphi \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2-\varphi \\ 2-\frac{3}{\varphi} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2\varphi-3 \\ \frac{3}{\varphi}-1 \\ \varphi-1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2\varphi-3 \\ 2\varphi-3 \\ 2\varphi-3 \\ 1 \end{pmatrix}, \begin{pmatrix} \varphi-1 \\ \varphi-1 \\ \varphi-1 \\ 2-\frac{3}{\varphi} \end{pmatrix}, \begin{pmatrix} 2-\varphi \\ 2-\varphi \\ 2-\varphi \\ \frac{3}{\varphi}-1 \end{pmatrix}, \begin{pmatrix} 2\varphi-3 \\ 2-\varphi \\ \varphi-1 \\ 4-2\varphi \end{pmatrix} \right\},$$

are valid for $\mathscr{P}$, it is enough to check those for $v \in U$ and this is straightforward. Then, the facets are given by the equalities $\langle v, x\rangle = 1$ and we find their vertices:

$$\operatorname{vert}(\mathscr{D}) = \{1, \sigma_i^{\pm}, \sigma_j^{\pm}, \sigma_k^{\pm}\}$$

and since $\operatorname{vert}(\mathscr{D}) \cap \operatorname{vert}(\mathscr{D})^{-1} = \{1\}$, the Proposition 10.1.2 finishes the proof. □

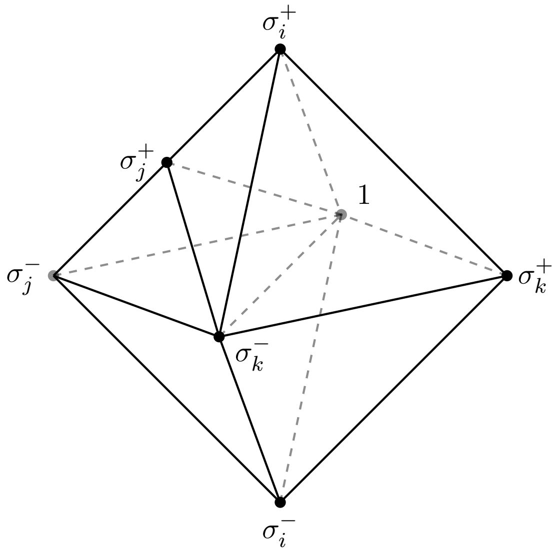


Figure 14: The five tetrahedra inside $\mathscr{D}$.

12.2 Associated $\mathcal{I}$-cellular decomposition of $\partial\mathscr{P}$

Here also, we investigate the combinatorics of the polytopal fundamental domain $\mathscr{D}$ constructed above to obtain a cellular decomposition of it. This will give a cellular structure on $\partial\mathscr{P}$ and projecting to $\mathbb{S}^3$ gives the desired cellular structure.

The facets of $\mathscr{D}$ are the ones of the five tetrahedra Δ_i, except the ones that are contained in some intersection $\Delta_i \cap \Delta_j$. We obtain the following facets

$$\mathscr{D}_2 = \{[1,\sigma_i^-,\sigma_k^+],[1,\sigma_k^+,\sigma_i^+],[1,\sigma_i^+,\sigma_j^+],[1,\sigma_j^+,\sigma_j^-],[1,\sigma_j^-,\sigma_i^-],$$

$$[\sigma_k^-,\sigma_i^-,\sigma_k^+],[\sigma_k^-,\sigma_k^+,\sigma_i^+],[\sigma_k^-,\sigma_i^+,\sigma_j^+],[\sigma_k^-,\sigma_j^+,\sigma_j^-],[\sigma_k^-,\sigma_j^-,\sigma_i^-]\}.$$

We remark the following relations among them

$$\sigma_j^+\cdot[1,\sigma_i^-,\sigma_k^+]=[\sigma_j^+,\sigma_j^-,\sigma_k^-],\ \sigma_j^-\cdot[1,\sigma_k^+,\sigma_i^+]=[\sigma_j^-,\sigma_i^-,\sigma_k^-],\ \sigma_i^-\cdot[1,\sigma_i^+,\sigma_j^+]=[\sigma_i^-,\sigma_k^+,\sigma_k^-],$$

and

$$\sigma_k^+\cdot[1,\sigma_j^+,\sigma_j^-]=[\sigma_k^+,\sigma_i^+,\sigma_k^-],\ \sigma_i^+\cdot[1,\sigma_j^-,\sigma_i^-]=[\sigma_i^+,\sigma_j^+,\sigma_k^-].$$

These are the only relations linking facets, hence we may define the following 2-cells

$$e_1^2 :=]1,\sigma_j^-,\sigma_i^-[,\ e_2^2 :=]1,\sigma_i^-,\sigma_k^+[,\ e_3^2 :=]1,\sigma_k^+,\sigma_i^+[,\ e_4^2 :=]1,\sigma_i^+,\sigma_j^+[,\ e_5^2 :=]1,\sigma_j^+,\sigma_j^-[.$$

Now, define the following 1-cells

$$e_1^1 :=]1,\sigma_k^+[,\quad e_2^1 :=]1,\sigma_i^+[,\quad e_3^1 :=]1,\sigma_j^+[,\quad e_4^1 :=]1,\sigma_j^-[,\quad e_5^1 :=]1,\sigma_i^-[.$$

If we add to this the vertices of $\mathscr{D}$ and its interior, which is formed by only one cell e^3 by construction, then we may cover all of $\mathscr{D}$ with these cells and some of their translates. Thus, we have obtained the following result:

Proposition 12.2.1. *Letting $E^0 := \{1\}$, $E^1 := \{e_i^1,\ 1 \le i \le 5\}$, $E^2 := \{e_i^2,\ 1 \le i \le 5\}$ and $E^3 := \{e^3\}$ with the above notations, we have the following $\mathcal{I}$-equivariant cellular decomposition of the sphere*

$$\mathbb{S}^3 = \coprod_{\substack{0\le j\le 3\\ e\in E^j, g\in\mathcal{I}}} g\phi(e),$$

where $p : \partial\mathscr{P} \xrightarrow{\sim} \mathbb{S}^3$ is the $\mathcal{I}$-homeomorphism given by projection.

The 1-skeleton of $\mathscr{D}$ is displayed in figure 15.

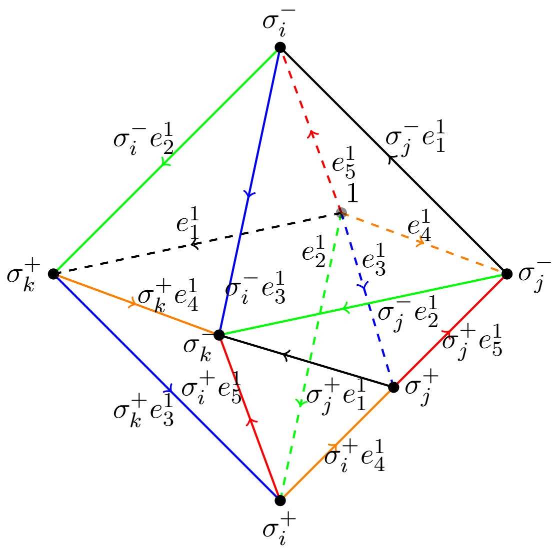


Figure 15: The oriented 1-skeleton of $\mathscr{D}$.

We now have to compute the boundaries of the cells and the resulting cellular homology chain complex. We choose to orient the 3-cell e^3 undirectly, and the 2-cells directly.

Proposition 12.2.2. *The cellular homology complex of $\partial\mathscr{P}$ associated to the cellular structure given in Proposition 12.2.1 is the chain complex of free left $\mathbb{Z}[\mathcal{I}]$-modules*

$$\mathcal{K}_{\mathcal{I}} := \left(\mathbb{Z}[\mathcal{I}] \xrightarrow{\partial_3} \mathbb{Z}[\mathcal{I}]^5 \xrightarrow{\partial_2} \mathbb{Z}[\mathcal{I}]^5 \xrightarrow{\partial_1} \mathbb{Z}[\mathcal{I}] \right),$$

where

$$\partial_1 = \begin{pmatrix} \sigma_k^+ - 1 \\ \sigma_i^+ - 1 \\ \sigma_j^+ - 1 \\ \sigma_j^- - 1 \\ \sigma_i^- - 1 \end{pmatrix}, \quad \partial_2 = \begin{pmatrix} \sigma_j^- & 0 & 0 & 1 & -1 \\ -1 & \sigma_i^- & 0 & 0 & 1 \\ 1 & -1 & \sigma_k^+ & 0 & 0 \\ 0 & 1 & -1 & \sigma_i^+ & 0 \\ 0 & 0 & 1 & -1 & \sigma_j^+ \end{pmatrix},$$

$$\partial_3 = \begin{pmatrix} \sigma_i^+ - 1 & \sigma_j^+ - 1 & \sigma_j^- - 1 & \sigma_i^- - 1 & \sigma_k^+ - 1 \end{pmatrix}.$$

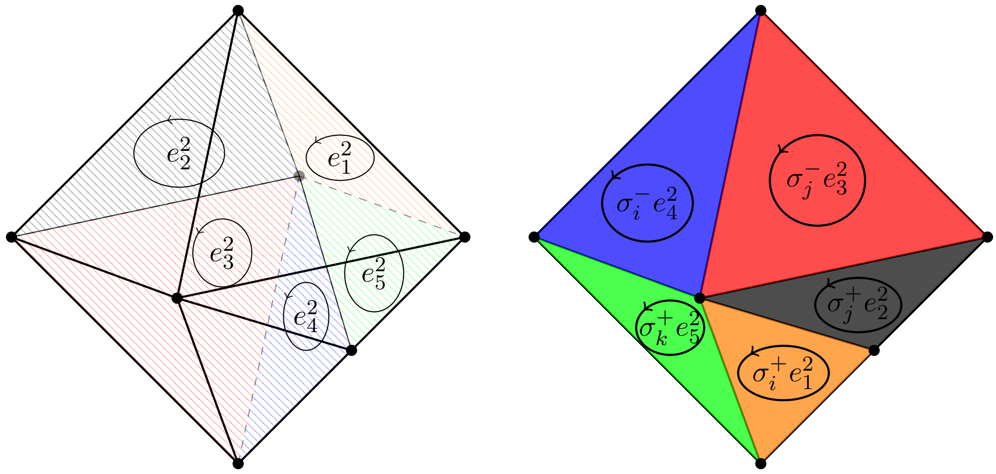


Figure 16: The oriented 2-skeleton of $\mathscr{D}$ (back and front).

12.3 The case of spheres and free resolution of the trivial $\mathcal{I}$-module

Here again, we shall describe the fundamental domain obtained above in $\mathbb{S}^3$ in terms of curved join and give a fundamental domain on $\mathbb{S}^{4n-1}$ and the equivariant cellular structure on that goes with it. We finish by giving a 4-periodic free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathcal{I}]$.

Theorem 12.3.1. *The following subset of $\mathbb{S}^3$ is a fundamental domain for the action of $\mathcal{I}$*

$$\begin{aligned}\mathscr{F}_3 :=& (1 * \sigma_k^- * \sigma_i^+ * \sigma_j^+) \cup (1 * \sigma_k^- * \sigma_j^+ * \sigma_j^-) \cup (1 * \sigma_k^- * \sigma_j^- * \sigma_i^-)\\ &\cup (1 * \sigma_k^- * \sigma_i^- * \sigma_k^+) \cup (1 * \sigma_k^- * \sigma_k^+ * \sigma_i^+).\end{aligned}$$

Therefore, the sphere $\mathbb{S}^3$ admits a $\mathcal{I}$-equivariant cellular decomposition with the following cells as orbit representatives

$$\widetilde{e}^0 := 1 * \emptyset = \{1\},$$

$$\widetilde{e}_1^1 := \mathrm{relint}(1 * \sigma_k^+),\ \widetilde{e}_2^1 := \mathrm{relint}(1 * \sigma_i^+),\ \widetilde{e}_3^1 := \mathrm{relint}(1 * \sigma_j^+),\ \widetilde{e}_4^1 := \mathrm{relint}(1 * \sigma_j^-),\ \widetilde{e}_5^1 := \mathrm{relint}(1 * \sigma_i^-),$$

$$\widetilde{e}_1^2 := \mathrm{relint}(1 * \sigma_j^- * \sigma_i^-),\ \widetilde{e}_2^2 := \mathrm{relint}(1 * \sigma_i^- * \sigma_k^+),\ \widetilde{e}_3^2 := \mathrm{relint}(1 * \sigma_k^+ * \sigma_i^+),$$

$$\widetilde{e}_4^2 := \mathrm{relint}(1 * \sigma_i^+ * \sigma_j^+),\ \widetilde{e}_5^2 := \mathrm{relint}(1 * \sigma_j^+ * \sigma_j^-),\ \widetilde{e}^3 := \mathring{\mathscr{F}}_3\ .$$

Furthermore, the associated cellular homology complex is the chain complex $\mathcal{K}_\mathcal{I}$ from the Proposition 12.2.2.

Remark 12.3.2. *Using the augmentation map $\varepsilon : \mathbb{Z}[\mathcal{I}] \twoheadrightarrow \mathbb{Z}$, we can compute the complex $\mathcal{K}_\mathcal{I} \otimes_{\mathbb{Z}[\mathcal{I}]} \mathbb{Z}$ and since we have*

$$\det(\partial_2 \otimes \mathbb{Z}) = \det \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} = 1,$$

we find that $\mathbb{S}^3/\mathcal{I}$ is a homology sphere, but it is not a sphere. That is, one has $H_*(\mathbb{S}^3/\mathcal{I}, \mathbb{Z}) = H_*(\mathbb{S}^3, \mathbb{Z})$, and however $\mathbb{S}^3/\mathcal{I}$ is not homeomorphic to $\mathbb{S}^3$, since $\pi_1(\mathbb{S}^3/\mathcal{I}) = \mathcal{I} \neq 1 = \pi_1(\mathbb{S}^3)$.

This space has a long story, it is called the Poincaré homology sphere. *It can also be constructed as the link of the simple singularity of type E_8 of the complex affine variety*

$\{(x, y, z) \in \mathbb{C}^3 \; ; \; x^2 + y^3 + z^5 = 0\}$ *near the origin, as the* Seifert bundle *or as the* dodecahedral space. *This last one corresponds to the original construction of Poincaré. For a detailed expository paper on the Poincaré homology sphere, we refer the reader to [KS79].*

Corollary 12.3.3. *The following complex is a 4-periodic resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathcal{I}]$*

$$\ldots \longrightarrow \mathbb{Z}[\mathcal{I}]^5 \xrightarrow{\partial_{4q-3}} \mathbb{Z}[\mathcal{I}] \xrightarrow{\partial_{4q-4}} \ldots \longrightarrow \mathbb{Z}[\mathcal{I}]^5 \xrightarrow{\partial_2} \mathbb{Z}[\mathcal{I}]^5 \xrightarrow{\partial_1} \mathbb{Z}[\mathcal{I}] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0\,,$$

where, for $q \geq 1$,

$$\partial_{4q-3} = \begin{pmatrix} \sigma_k^+ - 1 \\ \sigma_i^+ - 1 \\ \sigma_j^+ - 1 \\ \sigma_j^- - 1 \\ \sigma_i^- - 1 \end{pmatrix}, \qquad \partial_{4q-2} = \begin{pmatrix} \sigma_j^- & 0 & 0 & 1 & -1 \\ -1 & \sigma_i^- & 0 & 0 & 1 \\ 1 & -1 & \sigma_k^+ & 0 & 0 \\ 0 & 1 & -1 & \sigma_i^+ & 0 \\ 0 & 0 & 1 & -1 & \sigma_j^+ \end{pmatrix},$$

$$\partial_{4q-1} = \begin{pmatrix} \sigma_i^+ - 1 & \sigma_j^+ - 1 & \sigma_j^- - 1 & \sigma_i^- - 1 & \sigma_k^+ - 1 \end{pmatrix}, \qquad \partial_{4q} = \left(\textstyle\sum_{g \in \mathcal{I}} g \right).$$

We are now able to compute the group cohomology of $\mathcal{I}$ using this result.

Corollary 12.3.4. *The group cohomology of $\mathcal{I}$ with integer coefficients is given as follows:*

$$\forall q \in \mathbb{N}, \begin{cases} H^0(\mathcal{I}, \mathbb{Z}) = \mathbb{Z} & \textit{if} \;\; q = 0, \\ H^q(\mathcal{I}, \mathbb{Z}) = \mathbb{Z}/120\mathbb{Z} & \textit{if} \;\; q \equiv 0 \pmod 4, \\ H^q(\mathcal{I}, \mathbb{Z}) = 0 & \textit{otherwise.} \end{cases}$$

Proof. In view of Lemma 11.4.3, it is suffices to compute $\mathcal{C}(\mathsf{P}^\infty_{\mathcal{I}}, \mathbb{Z}[\mathcal{I}]) \otimes_{\mathbb{Z}[\mathcal{I}]} \mathbb{Z}$, with $\mathcal{C}(\mathsf{P}^\infty_{\mathcal{I}}, \mathbb{Z}[\mathcal{I}])$ the complex given in Theorem 12.3.6. Computing the matrices $\varepsilon(\partial_i)$ leads to the following complex

$$\ldots \longrightarrow \mathbb{Z}^5 \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 120} \mathbb{Z} \longrightarrow \ldots \longrightarrow \mathbb{Z} \xrightarrow{\times 120} \mathbb{Z} \xrightarrow{0} \mathbb{Z}^5 \xrightarrow{\partial} \mathbb{Z}^5 \xrightarrow{0} \mathbb{Z} \longrightarrow 0\,,$$

where $\partial = \partial_2 \otimes \mathbb{Z}$ is the matrix given in Remark 12.3.2. $\square$

Remark 12.3.5. *The Corollary 12.3.4 agrees with the previously known result on the cohomology of $\mathcal{I}$, see [TZ08, Theorem 4.16].*

Theorem 12.3.6. *The chain complex $\mathcal{C}(\mathsf{P}^{4n-1}_{\mathcal{I}}, \mathbb{Z}[\mathcal{I}])$ of the universal covering space of the icosahedral space forms $\mathsf{P}^{4n-1}_{\mathcal{I}}$ with the fundamental group acting by covering transformations is the following complex of left $\mathbb{Z}[\mathcal{I}]$-modules:*

$$0 \longrightarrow \mathbb{Z}[\mathcal{I}] \xrightarrow{\partial_{4n-1}} \mathbb{Z}[\mathcal{I}]^5 \longrightarrow \ldots \longrightarrow \mathbb{Z}[\mathcal{I}]^5 \xrightarrow{\partial_2} \mathbb{Z}[\mathcal{I}]^5 \xrightarrow{\partial_1} \mathbb{Z}[\mathcal{I}] \longrightarrow 0\,,$$

where the boundaries are as in Corollary 12.3.3.

In particular, the complex is exact in middle terms, i.e.

$$\forall 0 < i < 4n-1,\;\; H_i(\mathcal{C}(\mathsf{P}^{4n-1}_{\mathcal{I}}, \mathbb{Z}[\mathcal{I}])) = 0$$

and we have

$$H_0(\mathcal{C}(\mathsf{P}^{4n-1}_{\mathcal{I}}, \mathbb{Z}[\mathcal{I}])) = H_{4n-1}(\mathcal{C}(\mathsf{P}^{4n-1}_{\mathcal{I}}, \mathbb{Z}[\mathcal{I}])) = \mathbb{Z}.$$

13 The tetrahedral case

Even if the case of $\mathcal{T}$ has already been treated in [FGMS16], we can recover it by applying the above methods to this case. Note that all the groups in the tetrahedral family are studied in [CS17], but there $\mathcal{T}$ is excluded since, while it is the simplest one of the family, it is somehow different from all the other ones. Since it's always the same arguments and the case is solved, we omit the proofs.

13.1 Fundamental domain

We consider the orbit polytope in $\mathbb{R}^4$

$$\mathscr{P} := \operatorname{conv}(\mathcal{T}).$$

This polytope has 24 vertices, 96 edges, 96 faces and 24 facets and is known as the *24-cells* (or the *icositetrachoron*, or even the *octaplex*). Since $\mathcal{T}$ acts freely on $\mathscr{P}_3$, there must be exactly one orbit in $\mathscr{P}_3$. We keep the notations of the Section 11 and define

$$\begin{cases} \omega_i = \frac{1-i+j+k}{2} = t^{-1}s, \\ \omega_j = \frac{1+i-j+k}{2} = st^{-1}, \\ \omega_k = \frac{1+i+j-k}{2} = t \end{cases} \quad \text{and} \quad \begin{cases} \omega_0 = \frac{1+i+j+k}{2} = s, \\ \omega_{ij} := \frac{1-i-j+k}{2} = t^{-1}. \end{cases}$$

Proposition 13.1.1. *The subset of $\mathscr{P}$ defined by*

$$\mathscr{D} := [1, \omega_0, \omega_j, \omega_i, \omega_{ij}, k]$$

is a (connected) polytopal complex and is a fundamental domain for the action of $\mathcal{T}$ on $\partial\mathscr{P}$.

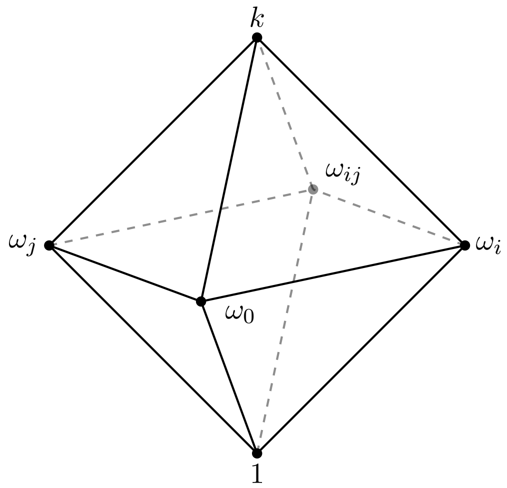


Figure 17: The tetrahedron $\mathscr{D}$.

13.2 Associated $\mathcal{T}$-cellular decomposition of $\partial\mathscr{P}$

The facets of $\mathscr{D}$ are the following

$$\mathscr{D}_2 = \{[1,\omega_j,\omega_0],[1,\omega_0,\omega_i],[1,\omega_i,\omega_{ij}],[1,\omega_{ij},\omega_j],\\ [k,\omega_j,\omega_0],[k,\omega_0,\omega_i],[k,\omega_i,\omega_{ij}],[k,\omega_{ij},\omega_j]\}.$$

We remark the following relations among them

$$\omega_{ij}\cdot[1,\omega_j,\omega_0] = [\omega_{ij},k,\omega_i],\ \omega_j\cdot[1,\omega_0,\omega_i] = [\omega_j,k,\omega_{ij}],$$

and

$$\omega_0\cdot[1,\omega_i,\omega_{ij}] = [\omega_0,k,\omega_j],\ \omega_i\cdot[1,\omega_{ij},\omega_j] = [\omega_i,k,\omega_0].$$

These are the only relations linking facets, hence we may define the following 2-cells

$$e^2_1 :=]1,\omega_j,\omega_0[,\ e^2_2 :=]1,\omega_0,\omega_i[,\ e^2_3 :=]1,\omega_i,\omega_{ij}[,\ e^2_4 :=]1,\omega_{ij},\omega_j[.$$

Now, define the following 1-cells

$$e^1_1 :=]1,\omega_{ij}[,\quad e^1_2 :=]1,\omega_j[,\quad e^1_3 :=]1,\omega_0[,\quad e^1_4 :=]1,\omega_i[.$$

If we add to this the vertices of $\mathscr{D}$ and its interior, which is formed by only one cell e^3 by construction, then we may cover all of $\mathscr{D}$ with these cells and some of their translates. The 1-skeleton of $\mathscr{D}$ is displayed in Figure 18.

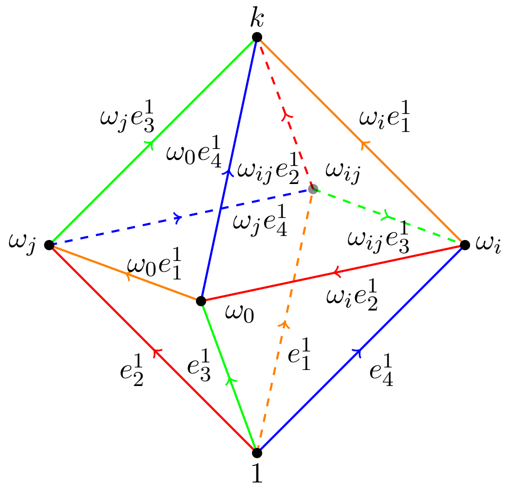


Figure 18: The oriented 1-skeleton of $\mathscr{D}$.

Proposition 13.2.1. *Letting* $E^0 := \{1\}$, $E^1 := \{e^1_i,\ 1 \le i \le 4\}$, $E^2 := \{e^2_i,\ 1 \le i \le 4\}$ *and* $E^3 := \{e^3\}$ *with the above notations and denoting by* $p : \partial\mathscr{P} \xrightarrow{\sim} \mathbb{S}^3$ *the* $\mathcal{T}$*-homeomorphism, we obtain the following* $\mathcal{T}$*-equivariant cellular decomposition of the sphere*

$$\mathbb{S}^3 = \coprod_{\substack{0\le j\le 3\\ e\in E^j, g\in\mathcal{T}}} g\phi(e).$$

We now compute the boundaries of the cells and the resulting cellular homology chain complex. We choose to orient the 3-cell e^3 directly, and the 2-cells undirectly.

Proposition 13.2.2. *The cellular homology complex of $\partial\mathscr{P}$ associated to the cellular structure given in Proposition 13.2.1 is the chain complex of free left $\mathbb{Z}[\mathcal{T}]$-modules*

$$\mathcal{K}_{\mathcal{T}} := \left(\mathbb{Z}[\mathcal{T}] \xrightarrow{\partial_3} \mathbb{Z}[\mathcal{T}]^4 \xrightarrow{\partial_2} \mathbb{Z}[\mathcal{T}]^4 \xrightarrow{\partial_1} \mathbb{Z}[\mathcal{T}] \right),$$

where

$$\partial_1 = \begin{pmatrix} \omega_{ij}-1 \\ \omega_j - 1 \\ \omega_0 - 1 \\ \omega_i - 1 \end{pmatrix}, \quad \partial_2 = \begin{pmatrix} \omega_0 & -1 & 1 & 0 \\ 0 & \omega_i & -1 & 1 \\ 1 & 0 & \omega_{ij} & -1 \\ -1 & 1 & 0 & \omega_j \end{pmatrix},$$

$$\partial_3 = \begin{pmatrix} 1-\omega_{ij} & 1-\omega_j & 1-\omega_0 & 1-\omega_i \end{pmatrix}.$$

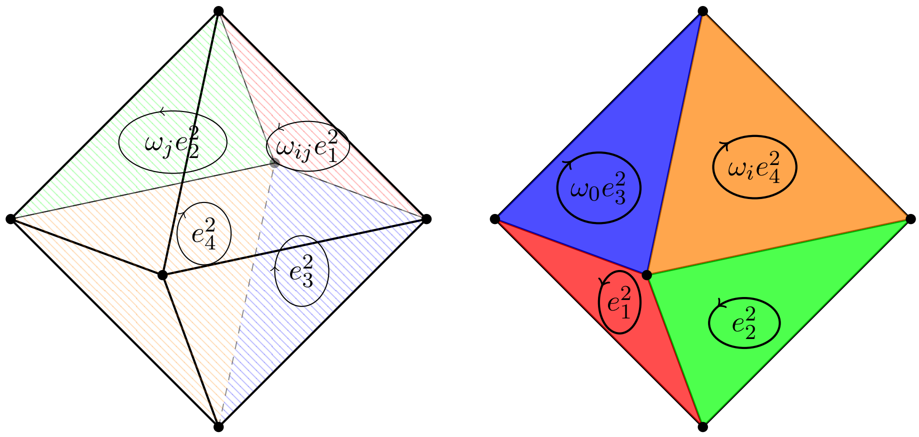


Figure 19: The oriented 2-skeleton of $\mathscr{D}$.

13.3 The case of spheres and free resolution of the trivial $\mathcal{T}$-module

Here again, we shall describe the fundamental domain obtained above in $\mathbb{S}^3$ in terms of curved join and give a fundamental domain on $\mathbb{S}^{4n-1}$ and the equivariant cellular structure on that goes with it. We finish by giving a 4-periodic free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathcal{T}]$.

Theorem 13.3.1. *The following subset of $\mathbb{S}^3$ is a fundamental domain for the action of $\mathcal{T}$*

$$\mathscr{F}_3 := (1 * \omega_{ij} * \omega_i * \omega_0 * \omega_j) \cup (\omega_{ij} * \omega_i * \omega_0 * \omega_j * k).$$

In particular, the sphere $\mathbb{S}^3$ admits a $\mathcal{T}$-equivariant cellular decomposition with the following cells as orbit representatives

$$\widetilde{e}^0 := 1 * \emptyset = \{1\},$$

$$\widetilde{e}^1_1 := \mathrm{relint}(1 * \omega_{ij}),\ \widetilde{e}^1_2 := \mathrm{relint}(1 * \omega_j),\ \widetilde{e}^1_3 := \mathrm{relint}(1 * \omega_0),\ \widetilde{e}^1_4 := \mathrm{relint}(1 * \omega_i),$$

$$\widetilde{e}^2_1 := \mathrm{relint}(1*\omega_j*\omega_0),\ \widetilde{e}^2_2 := \mathrm{relint}(1*\omega_0*\omega_i),\ \widetilde{e}^2_3 := \mathrm{relint}(1*\omega_i*\omega_{ij}),\ \widetilde{e}^2_4 := \mathrm{relint}(1*\omega_{ij}*\omega_j),$$

$$\widetilde{e}^3 := \mathring{\mathscr{F}}_3\,.$$

Furthermore, the associated cellular homology complex is the chain complex $\mathcal{K}_{\mathcal{T}}$ from the Proposition 13.2.2.

Corollary 13.3.2. *The following chain complex is a 4-periodic free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathcal{T}]$*

$$\dots \longrightarrow \mathbb{Z}[\mathcal{T}]^4 \xrightarrow{\partial_{4q-3}} \mathbb{Z}[\mathcal{T}] \xrightarrow{\partial_{4q-4}} \dots \longrightarrow \mathbb{Z}[\mathcal{T}]^4 \xrightarrow{\partial_2} \mathbb{Z}[\mathcal{T}]^4 \xrightarrow{\partial_1} \mathbb{Z}[\mathcal{T}] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0\,,$$

where

$$\partial_{4q-3} = \begin{pmatrix} \omega_{ij}-1 \\ \omega_j - 1 \\ \omega_0 - 1 \\ \omega_i - 1 \end{pmatrix}, \quad \partial_{4q-2} = \begin{pmatrix} \omega_0 & -1 & 1 & 0 \\ 0 & \omega_i & -1 & 1 \\ 1 & 0 & \omega_{ij} & -1 \\ -1 & 1 & 0 & \omega_j \end{pmatrix},$$

$$\partial_{4q-1} = \begin{pmatrix} 1-\omega_{ij} & 1-\omega_j & 1-\omega_0 & 1-\omega_i \end{pmatrix}, \quad \partial_{4q} = \left(\textstyle\sum_{g\in\mathcal{T}} g\right).$$

Corollary 13.3.3. *The group cohomology of $\mathcal{T}$ with integer coefficients is given as follows:*

$$\forall q \geq 1, \quad \begin{cases} H^q(\mathcal{T},\mathbb{Z}) = \mathbb{Z} & \text{if } q = 0, \\ H^q(\mathcal{T},\mathbb{Z}) = \mathbb{Z}/24\mathbb{Z} & \text{if } q \equiv 0 \pmod 4, \\ H^q(\mathcal{T},\mathbb{Z}) = \mathbb{Z}/3\mathbb{Z} & \text{if } q \equiv 2 \pmod 4, \\ H^q(\mathcal{T},\mathbb{Z}) = 0 & \text{otherwise.} \end{cases}$$

Theorem 13.3.4. *The chain complex $\mathcal{C}(\mathsf{P}_{\mathcal{T}}^{4n-1}, \mathbb{Z}[\mathcal{T}])$ of the universal covering space of the tetrahedral space forms $\mathsf{P}_{\mathcal{T}}^{4n-1}$ with the fundamental group acting by covering transformations is the following complex of left $\mathbb{Z}[\mathcal{T}]$-modules:*

$$0 \longrightarrow \mathbb{Z}[\mathcal{T}] \xrightarrow{\partial_{4n-1}} \mathbb{Z}[\mathcal{T}]^4 \longrightarrow \dots \longrightarrow \mathbb{Z}[\mathcal{T}]^4 \xrightarrow{\partial_2} \mathbb{Z}[\mathcal{T}]^4 \xrightarrow{\partial_1} \mathbb{Z}[\mathcal{T}] \longrightarrow 0\,,$$

where the boundaries are as in Corollary 13.3.2.

In particular, the complex is exact in middle terms, i.e.

$$\forall 0 < i < 4n-1, \; H_i(\mathcal{C}(\mathsf{P}_{\mathcal{T}}^{4n-1}, \mathbb{Z}[\mathcal{T}])) = 0$$

and we have

$$H_0(\mathcal{C}(\mathsf{P}_{\mathcal{T}}^{4n-1}, \mathbb{Z}[\mathcal{T}])) = H_{4n-1}(\mathcal{C}(\mathsf{P}_{\mathcal{T}}^{4n-1}, \mathbb{Z}[\mathcal{T}])) = \mathbb{Z}.$$

13.4 Simplicial structure and minimal resolution

Since we have chosen polytopal fundamental domains for $\mathcal{T}$, $\mathcal{O}$ and $\mathcal{I}$, it is clear that we can refine our cellular decompositions to equivariant simplicial decompositions of $\mathbb{S}^3$. We will just investigate the case of $\mathcal{T}$, since the other ones can be treated in a similar way. The method is trivial: just take each one of the facets Δ_i of $\mathscr{P}$ as the 3-cells and their boundary (up to multiplication) as 2-cells.

For instance, here, take as 3-cells the following open curved joins:

$$c_1^3 :=]\omega_0, 1, \omega_{ij}, \omega_j[, \; c_2^3 :=]\omega_0, 1, \omega_i, \omega_{ij}[, \; c_3^3 :=]\omega_0, \omega_{ij}, k, \omega_j[, \; c_4^3 :=]k, \omega_0, \omega_i, \omega_{ij}[$$

and as 2-cells the following open triangles:

$$\forall 1 \leq i \leq 4, \; c_i^2 := e_i^2$$

and

$$c_5^2 :=]\omega_0, 1, \omega_{ij}[,\ c_6^2 :=]\omega_{ij}, \omega_j, \omega_0[,\ c_7^2 :=]\omega_0, \omega_i, \omega_{ij}[,\ c_8^2 :=]\omega_0, k, \omega_{ij}[$$

and we may keep the 1-cells as they are, i.e. $c_i^1 := e_i^1$ for $1 \leq i \leq 4$. Then, the resulting simplicial homology complex is easily computed (for example, by orienting the 3-cells directly), just as we did above. One shall find of course a complex that is homotopy equivalent to the complex $\mathcal{K}_{\mathcal{T}}$ defined in Theorem 13.3.1. We omit the details.

We conclude by discussing the minimal resolution. Group resolution and group cohomology are purely algebraic invariants of the given group G. Under this point of view, Swan [Swa65] proved the existence of a minimal periodic free resolution of $\mathbb{Z}$ over G, for a family of finite groups containing the spherical space form groups. This means a resolution with minimal $\mathbb{Z}[G]$ module's ranks. He also gave a bound for these ranks. This point has been discussed in [CS17] for the resolution over the groups $P'_{8\cdot 3^s}$ of the tetrahedral family. Here, we show how to "reduce" our resolution for $\mathcal{T}$ to the minimal one, that has ranks 1-2-2-1, compare [CS17, p. 10.6]. (We note that in [CS17, p. 10.5] there is a misprint: one should read $f_h(F^\bullet)$ instead of $\mu_h(G)$ in the statement of the proposition.) We first describe the underlying geometric idea, and next we give an explicit chain homotopy.

Geometrically, the construction is as follows: start with the cellular decomposition from Theorem 13.3.1. As seen in Figure 19, the four upper triangles are sent by different group elements to the four lower triangles. It is clear that there is no way of collecting two triangles in one single 2-cell but we may proceed as follows. Pick up one triangle, say e_1^2, and one of its neighbours, say $\omega_0 e_3^2$ and set a_1 to be the union of these two triangles, namely

$$a_1 := e_1^2 + e_2^2.$$

Then, we have that $\omega_{ij} a_1 = \omega_{ij} e_1^2 + \omega_{ij} e_2^2$ and $y := \omega_{ij} e_2^2$ does not belong to the boundary of the fundamental domain $\mathscr{F}_{\mathcal{T},3}$. However, we may find another pair of coherent triangles such that one of them is mapped to y by some group element, while the other one is mapped to some triangle in the boundary of $\mathscr{F}_{\mathcal{T},3}$. For example, take

$$a_2 := \omega_0 e_3^2 + \omega_j e_2^2.$$

Then, we have $\omega_0^{-1} a_2 = e_3^2 + y$. As a consequence,

$$\omega_0^{-1} a_2 - \omega_{ij} a_1 = e_3^2 - \omega_{ij} e_1^2$$

and this means that we can use the three 2-cells a_1, a_2 and e_4^2 to cover all the boundary of $\mathscr{F}_{\mathcal{T},3}$. We would like to add one more triangle to the first two 2-cells in order to reduce the total number to two, but we easily see that the same procedure fails. However, we may proceed in the following "dual" way. Let x be a triangle such that $\omega_0^{-1} x = e_4^2$ and $\omega_{ij} x = \omega_i e_4^2$. We can take $x :=]i, \omega_j, \omega_0[$ and then we define

$$b_1 := a_1 + x = e_1^2 + e_2^2 + x$$

and

$$b_2 := a_2 + x = \omega_0 e_3^2 + \omega_j e_2^2 + x.$$

Then, after a simple calculation, we find that

$$b_1 - b_2 + \omega_0^{-1} b_2 - \omega_{ij} b_1 = a_1 - a_2 + \omega_0^{-1} a_2 - \omega_{ij} a_1 + \omega_0^{-1} x - \omega_{ij} x$$

$$= (1 - \omega_{ij}) e_1^2 + (1 - \omega_j) e_2^2 + (1 - \omega_0) e_3^2 + (1 - \omega_i) e_4^2 = d_3(e^3),$$

that is, the whole boundary of $\mathscr{F}_{\mathcal{T},3}$ is obtained using only the two 2-chains b_1 and b_2.

We can then give the reduced complex. It is given by the following

$$\mathcal{K}'_{\mathcal{T}} := \left(0 \longrightarrow K'_3 \xrightarrow{\partial'_1} K'_2 \xrightarrow{\partial'_2} K'_1 \xrightarrow{\partial'_1} K'_0 \longrightarrow 0 \right),$$

where $K'_0 = \mathbb{Z}[\mathcal{T}]\left\langle f^0 \right\rangle$, $K'_3 = \mathbb{Z}[\mathcal{T}]\left\langle f^3 \right\rangle$, $K'_1 = \mathbb{Z}[\mathcal{T}]\left\langle f^1_1, f^1_2 \right\rangle$ and $K'_2 = \mathbb{Z}[\mathcal{T}]\left\langle f^2_1, f^2_2 \right\rangle$ and

$$\begin{cases} \partial'_3(f^3) = (1-\omega_{ij})f^2_1 + (1-\omega_0)f^2_2, \\ \partial'_2(f^2_1) = (\omega_0+\omega_i-1)f^1_1 + (i+1)f^1_2, \\ \partial'_2(f^2_2) = (1+(-i))f^1_1 + (\omega_j - 1 + \omega_{ij})f^1_2, \\ \partial'_1(f^1_1) = (\omega_j - 1)f^0, \\ \partial'_1(f^1_2) = (\omega_i - 1)f^0, \end{cases}$$

i.e. are given in the canonical bases by right multiplication by the following matrices

$$\partial'_1 = \begin{pmatrix} \omega_j - 1 \\ \omega_i - 1 \end{pmatrix}, \quad \partial'_2 = \begin{pmatrix} \omega_0+\omega_i-1 & 1+i \\ 1+(-i) & \omega_j - 1 + \omega_{ij} \end{pmatrix}, \quad \partial'_3 = \begin{pmatrix} 1-\omega_{ij} & 1-\omega_0 \end{pmatrix}.$$

We finish by giving explicit homotopy equivalences $\varphi : \mathcal{K}_{\mathcal{T}} \to \mathcal{K}'_{\mathcal{T}}$ and $\varphi' : \mathcal{K}'_{\mathcal{T}} \to \mathcal{K}_{\mathcal{T}}$. We define $\varphi(e^i) := f^i$ and $\varphi'(f^i) := e^i$ for $i = 0, 3$ as well as

$$\begin{cases} \varphi_2(e^2_1) := f^2_1, \\ \varphi_2(e^2_2) = \varphi_2(e^2_4) := 0, \\ \varphi_2(e^2_3) := f^2_2, \end{cases} \quad \text{and} \quad \begin{cases} \varphi'_2(f^2_1) := e^2_1 + e^2_2 + \omega_0 e^2_4, \\ \varphi'_2(f^2_2) := \omega_{ij} e^2_2 + e^2_3 + e^2_4 \end{cases}$$

also

$$\begin{cases} \varphi_1(e^1_1) := f^1_1 + \omega_j f^1_2, \\ \varphi_1(e^1_2) := f^1_1, \\ \varphi_1(e^1_3) := f^1_2 + \omega_i f^1_1, \\ \varphi_1(e^1_4) := f^1_2, \end{cases} \quad \text{and} \quad \begin{cases} \varphi'_1(f^1_1) := e^1_2, \\ \varphi'_1(f^1_2) := e^1_4. \end{cases}$$

We immediately check that $\varphi \circ \varphi' = id_{\mathcal{K}'_{\mathcal{T}}}$ and we just have to show that the other composition is homotopic to $id_{\mathcal{K}_{\mathcal{T}}}$. If we define $H : \mathcal{K}_* \to \mathcal{K}_{*+1}$ by $H_0 = H_2 = 0$, $H_1(e^1_2) = H_1(e^1_4) := 0$ and $H_1(e^1_1) := e^2_4$, $H_1(e^1_3) := e^2_2$, then we have $\varphi'_1\varphi_1 = id + \partial_2 H_1 + H_0\partial_1$ and $\varphi'_2\varphi_2 = id + \partial_3 H_2 + H_1\partial_2$, i.e.

$$\varphi' \circ \varphi = id_{\mathcal{K}_{\mathcal{T}}} + \partial H + H\partial$$

and φ is indeed a homotopy equivalence, with homotopy inverse φ'. Thus, we have proved that the complex $\mathcal{K}_{\mathcal{T}}$ from the Theorem 13.3.1 is homotopy equivalent to the complex

$$\mathcal{K}'_{\mathcal{T}} = \left(0 \longrightarrow \mathbb{Z}[\mathcal{T}] \xrightarrow{\partial'_3} \mathbb{Z}[\mathcal{T}]^2 \xrightarrow{\partial'_2} \mathbb{Z}[\mathcal{T}]^2 \xrightarrow{\partial'_1} \mathbb{Z}[\mathcal{T}] \longrightarrow 0 \right)$$

defined above.

Remark 13.4.1. *Observe that this process works for the group $\mathcal{T}$ but fails for the other two groups, $\mathcal{O}$ and $\mathcal{I}$. This is not unexpected, since the resolutions determined in the present work are characterised by their geometric feature, i.e. constructed through particular orthogonal representations of the groups, and it is not likely that this characterisation would produce a minimal resolution, that in general may not be induced by a representation. Indeed, it would be interesting to investigate the possible bounds for the ranks of a free periodic resolution induced by a linear representation.*

14 Application of the octahedral case to the flag manifold $\mathcal{F}(\mathbb{R})$ of $SL_3(\mathbb{R})$

The $\mathcal{O}$-equivariant cellular structure of $\mathbb{S}^3$ may be used to obtain a cellular decomposition of the real points of the flag manifold $SU_3(\mathbb{C})/T$ of type A_2.

First of all, we have to identify spaces and actions. We begin with a trivial lemma.

Lemma 14.0.1. *Let P be a finite group acting freely by diffeomorphisms on a manifold X and $Q \trianglelefteq P$ be a normal subgroup of P. Then, P/Q acts freely on the quotient manifold X/Q and the projection $X \twoheadrightarrow X/P$ induces a natural diffeomorphism*

$$(X/Q)\Big/_{(P/Q)} \xrightarrow{\sim} X/P.$$

We will apply this lemma to $P = \mathcal{O}$, $Q = \mathcal{Q}_8$ and $X = \mathbb{S}^3$. One has to be careful at this point: we let $\mathcal{O}$ act on $\mathbb{S}^3$ on the *left*, whereas $W = \mathfrak{S}_3$ naturally acts on $\mathcal{F}(\mathbb{R})$ on the *right*. Hence we let $\mathcal{O}$ act on the right on $\mathbb{S}^3$ by multiplication. It is straightforward to adapt our results to this case. For instance, we replace $\Delta_i =: \mathrm{conv}(q_1, q_2, q_3, q_4)$ by $\widehat{\Delta_i} := \mathrm{conv}(q_1^{-1}, q_2^{-1}, q_3^{-1}, q_4^{-1})$ and $\mathscr{F}_3$ by $\widehat{\mathscr{F}_3} := \mathrm{pr}(\widehat{\mathscr{D}})$ where $\mathrm{pr}(x) = x/|x|$ is the usual projection and $\widehat{\mathscr{D}} := \bigcup_i \widehat{\Delta_i}$ and we can do the same for the cells in $\mathbb{S}^3$. Briefly, we just have to replace every quaternion appearing in sections 11.2, 11.3 and 11.4 by its inverse and left multiplications by right multiplications.

Now, denoting by $\mathcal{F} := SU_3(\mathbb{C})/T \simeq SL_3(\mathbb{C})/B$ the flag manifold, we have a diffeomorphism

$$\mathcal{F}(\mathbb{R}) \simeq SO(3)/S(O(1)^3).$$

Recall the surjective homomorphism $\mathrm{B} : \mathbb{S}^3 \twoheadrightarrow SO(3)$, with kernel $\{\pm 1\}$. We have a surjective homomorphism

$$\phi : \mathbb{S}^3 \overset{\mathrm{B}}{\twoheadrightarrow} SO(3) \twoheadrightarrow SO(3)/S(O(1)^3) \simeq \mathcal{F}(\mathbb{R}).$$

Now, it is clear that $\mathrm{B}^{-1}(S(O(1)^3)) = \{\pm 1, \pm i, \pm j, \pm k\} = \mathcal{Q}_8$. The lemma 14.0.1 applied to $G = \mathcal{Q}_8$, $N := \{\pm 1\} = Z(\mathcal{Q}_8)$ and $X = \mathbb{S}^3$ leads to the following result:

Lemma 14.0.2. *Denoting by $\mathcal{F} := SU_3(\mathbb{C})/T$ the flag manifold of type A_2, the above defined map ϕ induces a diffeomorphism*

$$\psi : \mathbb{S}^3/\mathcal{Q}_8 \xrightarrow{\sim} \mathcal{F}(\mathbb{R}).$$

Now, one has $W = \mathfrak{S}_3 = \left\langle s_\alpha, s_\beta \mid s_\alpha^2 = s_\beta^2 = 1,\ s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta \right\rangle$ (the notation $s_\alpha,\ s_\beta$ makes reference to the simple roots α and β of the root system of type A_2). The reflections s_α ans s_β can be represented in $SO(3)$ by the following matrices

$$\dot{s_\alpha} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \dot{s_\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

These matrices may be obtained from $\mathbb{S}^3$ using B:

$$\dot{s_\alpha} = \mathrm{B}\left(\frac{1+k}{\sqrt{2}}\right),\ \dot{s_\beta} = \mathrm{B}\left(\frac{1+i}{\sqrt{2}}\right),$$

and this induces a well-defined isomorphism

$$\begin{array}{rcccc} \sigma & : & \mathcal{O}/\mathcal{Q}_8 & \xrightarrow{\sim} & \mathfrak{S}_3 \\ & & (1+i)/\sqrt{2} & \longmapsto & s_\beta \\ & & (1+k)/\sqrt{2} & \longmapsto & s_\alpha \end{array}$$

Therefore, recalling that $\mathfrak{S}_3 = N_{SU_3(\mathbb{C})}(T)/T = (N_{SO(3)}(SO(3) \cap T))/(SO(3) \cap T)$ acts on $\mathcal{F}(\mathbb{R})$ by multiplication on the right by a representative matrix, one obtains the following relation

$$\forall (x, g) \in \mathbb{S}^3 \times \mathcal{O},\ \psi(B(x)) \cdot \sigma(g) = \phi(xg).$$

Henceforth, using the lemma 14.0.1, one obtains the following result:

Proposition 14.0.3. *The diffeomorphism ψ from the Lemma 14.0.2 is $\mathfrak{S}_3$-equivariant. In particular, the $\mathcal{O}$-equivariant cellular structure on $\mathbb{S}^3$ defined in Theorem 11.4.1 induces an $\mathfrak{S}_3$-equivariant cellular structure on the real flag manifold $\mathcal{F}(\mathbb{R})$.*

Corollary 14.0.4. *The fundamental groups of the real flag manifold $\mathcal{F}(\mathbb{R})$ and of its quotient space by $\mathfrak{S}_3$ are given by*

$$\pi_1(\mathcal{F}(\mathbb{R}), *) = \mathcal{Q}_8 \quad \text{and} \quad \pi_1(\mathcal{F}(\mathbb{R})/\mathfrak{S}_3, *) = \mathcal{O}.$$

We are now in a position to state and prove the principal result of this section:

Theorem 14.0.5. *The real flag manifold $\mathcal{F}(\mathbb{R}) = SO(3)/S(O(1)^3)$ admits an $\mathfrak{S}_3$-equivariant cellular decomposition with orbit representatives cells given by*

$$\mathfrak{e}^i_j := \psi\left(\pi_{\mathcal{Q}_8}\left((e^i_j)^{-1}\right)\right),$$

where $\pi_{\mathcal{Q}_8} : \mathbb{S}^3 \to \mathbb{S}^3/\mathcal{Q}_8$ is the natural projection, $\psi : \mathbb{S}^3/\mathcal{Q}_8 \to \mathcal{F}(\mathbb{R})$ is the $\mathfrak{S}_3$-equivariant diffeomorphism from the Proposition 14.0.2 and e^i_j are the cells of the $\mathcal{O}$-equivariant cellular decomposition from the Theorem 11.4.1.

Furthermore, the associated cellular homology complex is the chain complex of free right $\mathbb{Z}[\mathfrak{S}_3]$-modules

$$\mathcal{K}_{\mathfrak{S}_3} := \left(\mathbb{Z}[\mathfrak{S}_3] \xrightarrow{\partial_3} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{\partial_2} \mathbb{Z}[\mathfrak{S}_3]^3 \xrightarrow{\partial_1} \mathbb{Z}[\mathfrak{S}_3] \right),$$

where

$$\partial_1 = \begin{pmatrix} 1 - s_\beta & 1 - w_0 & 1 - s_\alpha \end{pmatrix}, \quad \partial_2 = \begin{pmatrix} s_\alpha s_\beta & 1 & w_0 - 1 \\ s_\alpha - 1 & s_\alpha s_\beta & 1 \\ 1 & s_\beta - 1 & s_\alpha s_\beta \end{pmatrix}, \quad \partial_3 = \begin{pmatrix} 1 - s_\beta \\ 1 - w_0 \\ 1 - s_\alpha \end{pmatrix}.$$

Proof. This only relies on Proposition 14.0.3 and the fact that $((e^i_j)^{-1})_{i,j}$ is an $\mathcal{O}$-equivariant cell decomposition of $\mathbb{S}^3$, the group $\mathcal{O}$ acting by right multiplication on the sphere. Next, we have to determine the images of the points of $\mathcal{O}$ we used to construct $\widehat{\mathscr{F}_{\mathcal{O},3}}$ under the projection

$$\pi^{\mathcal{O}} : \mathcal{O} \twoheadrightarrow \mathcal{O}/\mathcal{Q}_8 \overset{\sigma}{\simeq} \mathfrak{S}_3.$$

Recall that, denoting by s_α and s_β the simple reflections in the Weyl group $W = \mathfrak{S}_3$, we have

$$\mathfrak{S}_3 = \left\langle s_\alpha, s_\beta \mid s_\alpha^2 = s_\beta^2 = 1,\ s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta \right\rangle = \{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, s_\alpha s_\beta s_\alpha\}$$

and we denote by $w_0 := s_\alpha s_\beta s_\alpha$ the *longest element* of $\mathfrak{S}_3$. We compute

$$\tau_i \mapsto s_\beta,\ \tau_j \mapsto w_0,\ \tau_k \mapsto s_\alpha,\ \omega_i, \omega_j, \omega_k \mapsto s_\beta s_\alpha,\ \omega_0 \mapsto s_\alpha s_\beta.$$

Thus, the resulting cellular homology chain complex can be computed from the one in Theorem 11.4.1, replacing each coefficient $q \in \mathcal{O}$ in ∂_i by $\pi^{\mathcal{O}}(q^{-1})$ and transposing the matrices. $\square$

Finally, using Figure 13, we can describe the 3-cells in a more combinatorial way. More precisely, one can describe all the curved tetrahedra having a given element $w \in \mathfrak{S}_3$ in its boundary. By right multiplication by w^{-1}, we may assume that $w = 1$. First consider the octahedron as in Figure 13, with vertices (and centers of faces) given by the images of the ones of 13 under the projection $\pi^{\mathcal{O}} : \mathcal{O} \twoheadrightarrow \mathfrak{S}_3$ as in Figure 20. A curved tetrahedron containing 1 can be described in the following way:

1. Choose a face F of the octahedron,
2. Choose an edge of F,
3. The curved tetrahedron has its vertices given by the center of F, the two vertices of the chosen edge of F and 1.

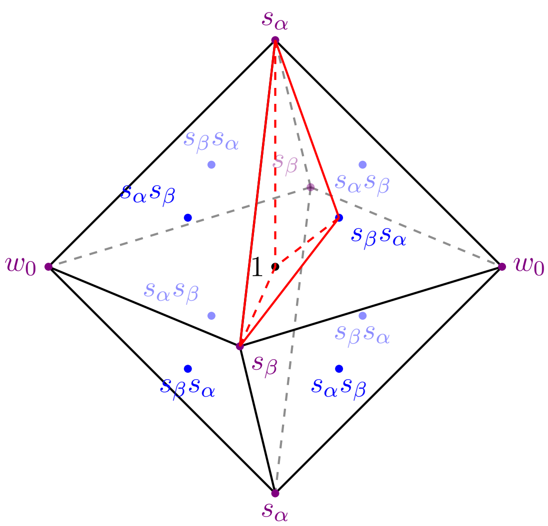


Figure 20: A curved tetrahedron in $\mathcal{F}(\mathbb{R})$ containing 1 in its boundary.

Remark 14.0.6. *Note that in this representation, many different cells can have the same vertices. For instance, the 1-cell formed by the edge linking 1 to the w_0 on the right, and then from the other copy of w_0 on the left, back to one is not a trivial path in $\mathcal{F}(\mathbb{R})$. In fact, it corresponds to the element j of the group $\mathcal{Q}_8 \simeq \pi_1(\mathcal{F}(\mathbb{R}), 1)$.*

Part VI

Normal homogeneous metrics on flag manifolds and Dirichlet-Voronoi fundamental domain

In this final part, we adopt a Riemannian point of view on flag manifolds. As usual, we let K be a semisimple compact Lie group, $T < K$ be a maximal torus, $W = N_K(T)/T$ is the Weyl group and $\mathcal{F} = K/T$ is the flag manifold, whose real points are denoted by $\mathcal{F}(\mathbb{R})$. We hope to generalize the construction of equivariant cellular structures on flag manifolds using a *Dirichlet-Voronoi fundamental domain* for W acting on $\mathcal{F}$. To do this we need a Riemannian metric on $\mathcal{F}$, for which the group W acts by isometries. A natural class of metrics to consider on $\mathcal{F}$ are the *normal homogeneous* metrics, that is, the ones coming from bi-invariant metrics on the compact Lie group K.

After recalling the basics of Riemannian geometry on Lie groups and homogeneous spaces, we prove that the natural metric on the real flag manifold $SO(3)/S(O(1)^3) \simeq \mathbb{S}^3/\mathcal{Q}_8$ is normal homogeneous and we study the geodesic properties of the cell structure of the previous part. In particular, we prove that the 1-cells are geodesics.

The last section introduces Dirichlet-Voronoi domains for a discrete group of isometries acting on a connected complete Riemannian manifold. We study some properties and prove in particular that such domains are indeed fundamental domains. Next, we focus on the case of W acting on $\mathcal{F}$, equipped with a normal homogeneous metric. Under a condition on the *injectivity radius* of $\mathcal{F}$, it can be seen that the interior of the Dirichlet-Voronoi domain is a cell and its boundary is a sphere, which is a first step toward a construction of an equivariant cellular structure on $\mathcal{F}(\mathbb{R})$, from a cell structure on a Dirichlet-Voronoi domain, which is compatible with the partial action of W. As we shall see, this is reasonable only for real flag manifolds. As a preliminary result for a future project, we compute the injectivity radius of the flag manifolds $SO(n)/S(O(1)^n)$. We finish by proving that this method indeed gives an $\mathfrak{S}_3$-equivariant cellular structure on $SO(3)/S(O(1)^3)$ and we exhibit the associated equivariant chain complex.

15 The Riemannian structure on $\mathcal{F}(\mathbb{R})$ inherited from the round metric on the sphere $\mathbb{S}^3$

15.1 Lightning introduction to Riemannian geometry

We start by giving some reminders on Riemannian manifolds and in particular on bi-invariant metrics on flag manifolds. For more details on Riemannian manifolds, the reader is invited to have a look at [Lee] or [GHL04].

Recall that a *Riemannian manifold* is a pair (M, g), where M is a smooth manifold and g is a symmetric positive-definite $(2, 0)$-tensor on M. To simplify the notation, we use freely the *Einstein convention* for repeated indices. For instance, we simply write $x_i e^i$ to mean

$\sum_i x_i e^i$. Denote by $\mathcal{X}(M) := \Gamma(TM)$ the set of vector fields on M (i.e. the sections of the tangent bundle TM of M, identified with derivations of the algebra $\mathcal{C}^\infty(M, \mathbb{R})$) and recall that an *affine connection* on M is a bilinear map $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ denoted by $(X, Y) \mapsto \nabla_X Y$, which is $\mathcal{C}^\infty(M, \mathbb{R})$-linear on the left and which satisfies the Leibniz rule on the right, i.e. such that

$$\forall f \in \mathcal{C}^\infty(M, \mathbb{R}),\ \forall X, Y \in \mathcal{X}(M),\ \begin{cases} \nabla_{fX} Y = f \nabla_X Y, \\ \nabla_X (fY) = \mathrm{d}f(X) Y + f \nabla_X Y. \end{cases}$$

If (M, g) is a Riemannian manifold, then there exists a unique affine connection on M such that for all vector fields $X, Y, Z \in \mathcal{X}(M)$ we have

$$\begin{cases} Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y), \\ \nabla_X Y - \nabla_Y X = [X, Y]. \end{cases}$$

This is called *Levi-Civita connection* on M. It may be implicitly defined by the Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &\quad + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X). \end{aligned} \tag{K}$$

It may be useful to express these objects in local charts. If $p \in M$, take $(x^1, \dots, x^n)$ a local system of coordinates around p and define the *metric coefficients*

$$g_{ij} := g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

as well as (g^{ij}) the inverse matrix of (g_{ij}). If ∇ is the Levi-Civita connection on M, we define the *Christoffel symbols* Γ^k_{ij} by

$$\nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial}{\partial x^j}\right) =: \Gamma^k_{ij} \frac{\partial}{\partial x^k},$$

i.e.

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right).$$

Now, a *geodesic* is a curve $\gamma :]a, b[\to M$ such that the covariant derivative of $\gamma' =: \dot\gamma$ vanishes, that is

$$\nabla_{\dot\gamma} \dot\gamma = 0, \quad \text{where} \quad \nabla_{\dot\gamma} := (\gamma^* \nabla)_{\frac{d}{dt}}.$$

Here, the connection $\gamma^* \nabla$ is defined as the only connection on $\gamma^*(TM)$ such that, for $x \in]a, b[$, $v \in \mathbb{R} = T_x(]a, b[)$ and for a vector field $X \in \mathcal{X}(M)$, one has

$$(\gamma^* \nabla)_v (\gamma^* X) = \gamma^* \left(\nabla_{d_x \gamma(v)}(X)\right). \tag{$\triangle$}$$

Using the Christoffel symbols, this can be rephrased as the following differential system of $\dim M$ equations:

$$\frac{d^2 \gamma^k}{dt^2} + \Gamma^k_{ij} \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} = 0.$$

Using the Picard-Lindelöf theorem[21], given $p_0 \in M$, there exists an open neighborhood $U \subset M$ of p_0 and $\varepsilon > 0$ such that, for $p \in U$ and $v \in T_p M$ with $|v| < \varepsilon$, there is a unique

[21] also called the Cauchy-Lipschitz theorem

geodesic $c_v :]-2,2[\to M$ such that $c_v(0) = p$ and $c_v'(0) = v$. With this notation, we consider the *exponential map*

$$\begin{array}{cccc} \mathrm{Exp}_m & : & B_{T_pM}(0,\varepsilon) & \longrightarrow & M \\ & & v & \longmapsto & c_v(1) \end{array}$$

Like the maximal solutions of an ordinary differential equation, the maximal geodesics need not to be defined for all $t \in \mathbb{R}$. When they are, the manifold M is said to be *geodesically complete*. It can be shown that, in this case, the exponential map Exp_p is defined on the whole tangent space T_pM, for every $p \in M$.

Next, we define the *length* of a curve $\gamma : [a,b] \to M$ by the integral

$$\mathrm{L}(\gamma) := \int_a^b \sqrt{g(\dot\gamma(t),\dot\gamma(t))}\mathrm{d}t = \int_a^b \|\dot\gamma(t)\|_g \mathrm{d}t.$$

For $p, q \in M$ two points on a connected Riemannian manifold (M,g), denote by $\mathcal{C}(p,q)$ the set of piecewise smooth curves $\gamma : [0,1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$ and define the quantity

$$d_g(p,q) := \inf_{\gamma\in\mathcal{C}(p,q)} \mathrm{L}(\gamma).$$

This is well-defined since $\mathcal{C}(p,q) \neq \emptyset$ (see [Lee, Proposition 2.50]) and it is easy to see that the function $d_g : M \times M \to \mathbb{R}_+$ is a distance on M, making (M, d_g) into a metric space. Furthermore, we say that a geodesic γ between two points p and q of M is *minimal* if $\mathrm{L}(\gamma) = d_g(p,q)$. Moreover, the topology induced by this distance is the original topology of M (see [GHL04, Definition-Proposition 2.91]). In fact, according to the Theorem 6.15 from [Lee], every geodesic is locally minimal and every minimal curve is a geodesic, when it is given the unit-speed parametrization ([Lee, Theorem 6.4]). However, it is typically not true that any two points of M can be joined by a minimal geodesic. For example, the points $(-1,0)$ and $(1,0)$ cannot be joined by a segment in $\mathbb{R}^2 \setminus \{0\}$. Even if so, there can be several minimal geodesics between two given points, for instance, two antipodal points on a circle can be joined by two different minimal geodesics. A manifold in which any two points can be joined by a (minimal) geodesic is called *complete*. Recall the Hopf-Rinow theorem:

Theorem 15.1.1 (*Hopf-Rinow, [Lee, Theorem 6.19]*)**.** *Metric and geodesic completeness are equivalent in a connected Riemannian manifold. Moreover, if the manifold is complete, then any two points can be joined by a minimal geodesic.*

A *local isometry* between Riemannian manifolds (M,g) and (N,h) is a smooth map $f : M \to N$ such that

$$\forall p \in M,\ \forall u,v \in T_pM,\ h_{f(p)}(\mathrm{d}_pf(u), \mathrm{d}_pf(v)) = g_p(u,v).$$

Note that this condition implies that f is a local diffeomorphism, by the inverse function theorem. Moreover, a local isometry is called an *isometry* if it is a diffeomorphism. Note that an isometry preserves Riemannian distances between points.

Let $\pi : (\widetilde{M}, \widetilde{g}) \to (M,g)$ be a smooth submersion between Riemannian manifolds. For $x \in \widetilde{M}$, we define the following subspaces of $T_x\widetilde{M}$

$$V_x := \ker(\mathrm{d}_x\pi) = T_x(\pi^{-1}(\pi(x))) \quad \text{and} \quad H_x := {V_x}^{\perp},$$

the orthogonal being taken with respect to the inner product $\widetilde{g}_x$. These are respectively called the *vertical* and *horizontal* tangent spaces. We say that π is a *Riemannian submersion* if $\mathrm{d}_x\pi$ restricts to a linear isometry from H_x to $T_{\pi(x)}M$, i.e.

$$\forall u,v \in H_x,\ \widetilde{g}_x(u,v) = g_{\pi(x)}(\mathrm{d}_x\pi(u), \mathrm{d}_x\pi(v)).$$

We say that π is a *Riemannian covering map* if it is a covering map, that is also a Riemannian submersion.

Proposition 15.1.2 (*[GHL04, Proposition 2.81]*)**.** *If $\pi : \widetilde{M} \to M$ is a Riemannian covering map, then the geodesics of M are the projections of the geodesics of $\widetilde{M}$ and conversely, every geodesic of M lifts to a geodesic of $\widetilde{M}$.*

We shall need the following fundamental result:

Theorem 15.1.3. *Let (M, g) be a connected Riemannian manifold and G be a Lie group acting freely, properly and isometrically on M. Then, there exists a unique Riemannian metric $\overline{g}$ on M/G such that the projection $\pi : M \twoheadrightarrow M/G$ is a Riemannian submersion.*

If moreover M and M/G are geodesically complete (which is the case for instance if M is compact and G is finite, by the Hopf-Rinow theorem), then the geodesic distance on M/G is given by

$$\forall x, y \in M,\ d_{\overline{g}}(\pi(x), \pi(y)) = \inf_{h \in G} d_g(x, hy).$$

Proof. The existence and uniqueness of $\overline{g}$ is a standard fact and can be found for instance in [Lee], Corollary 2.29 or in [Bes87, §9.12]. Only the statement about the distance remains to be proved. Take $x, y \in M$ and $\overline{x} := \pi(x)$, $\overline{y} := \pi(y)$. Fix some $h \in G$. Since M is complete, there exists a minimizing geodesic arc $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = hy$. Then $\pi \circ \gamma$ is a geodesic linking $\overline{x}$ and $\overline{y}$ and since π is a Riemannian submersion, it is a local isometry so one has

$$\mathrm{L}(\gamma) \overset{\mathrm{df}}{=} \int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\mathrm{d}t = \mathrm{L}(\pi \circ \gamma).$$

Hence, $\pi \circ \gamma$ is a geodesic between $\overline{x}$ and $\overline{y}$ of the same length as γ, so by definition of the geodesic distance, one gets $d_{\overline{g}}(\overline{x}, \overline{y}) \leq d_g(x, hy)$. Because $h \in G$ is arbitrary, we get $d_{\overline{g}}(\overline{x}, \overline{y}) \leq \inf_h d_g(x, hy)$. We have to prove the converse inequality to conclude. Consider then a minimizing geodesic arc $\widetilde{\gamma} : [0, 1] \to M/G$ between $\overline{x}$ and $\overline{y}$. Using again the Proposition 15.1.2, there exists a geodesic arc $\gamma : [0, 1] \to M$ such that $\pi \circ \gamma = \widetilde{\gamma}$ and we have $\mathrm{L}(\gamma) = \mathrm{L}(\widetilde{\gamma}) = d_{\overline{g}}(\overline{x}, \overline{y})$. By construction, there exist $h_0, h_1 \in G$ such that $\gamma(0) = h_0 x$ and $\gamma(1) = h_1 y$ we have $d_g(x, h_0^{-1} h_1 y) = d_g(h_0 x, h_1 y) \leq \mathrm{L}(\gamma) = d_{\overline{g}}(\overline{x}, \overline{y})$. ☐

Remark 15.1.4. *If we take $M = \mathbb{S}^{2n+1}$ endowed with its natural round metric and $G = \mathbb{Z}/2\mathbb{Z}$ acting on $\mathbb{S}^{2n+1}$ as the antipode, then $M/G = \mathbb{P}^n(\mathbb{C})$ and there is a unique metric on $\mathbb{P}^n(\mathbb{C})$ making the projection $\mathbb{S}^{2n+1} \twoheadrightarrow \mathbb{P}^n(\mathbb{C})$ into a Riemannian submersion. This metric is called the* Fubini-Study metric*. Using the previous Theorem, we can easily see that the induced distance d_{FS} on $\mathbb{P}^n(\mathbb{C})$ is given by the following*

$$\forall p, q \in \mathbb{P}^n(\mathbb{C}),\ d_{FS}(p, q) = \arccos \frac{|\langle p, q \rangle|}{\|p\| \|q\|}.$$

15.2 Bi-invariant metrics on Lie groups and homogeneous spaces

We now review some basic facts about invariant Riemannian metrics on Lie groups and their flag manifolds. Let G be a Lie group and $\mathfrak{g} := T_1 G$ be its Lie algebra. For an element $p \in G$, denote by $L_p : G \to G$ and $R_p : G \to G$ left multiplication maps $q \mapsto pq$ and $q \mapsto qp$, respectively. Of course, there may exist many Riemannian metrics on G, but a natural restriction is to look for *invariant* metrics. More precisely, a Riemannian metric g on G is said to be *left-invariant* if

$$\forall p \in G,\ \forall X, Y \in \mathfrak{g},\ g_p(\mathrm{d}_1 L_p(X), \mathrm{d}_1 L_p(Y)) = g_1(X, Y).$$

Denoting by $\mathcal{X}_\ell(G)$ the set of *left-invariant* vector fields on G (i.e. the set of all $X \in \mathcal{X}(G)$ such that $\mathrm{d}_q L_p(X_q) = X_{pq}$ for all $p, q \in G$) and using the bijection $\mathcal{X}_\ell(G) \to \mathfrak{g}$ defined by $X \mapsto X_1$, we see that g is left-invariant if and only if the following is true:

$$\forall p \in G,\ \forall X, Y \in \mathcal{X}_\ell(G),\ g_p(X_p, Y_p) = g_1(X_1, Y_1).$$

Analogously, g is *right-invariant* if the above condition is verified for right-invariant vector fields on G. Finally, the metric g is *bi-invariant* if it is both left and right-invariant.

Lemma 15.2.1. *Let G be a Lie group. The map $g \mapsto g_1$ is a bijective correspondence between the set of left-invariant (resp. right-invariant) metrics on G and the set of inner products on $\mathfrak{g}$.*

Furthermore, the same map restricts to a bijective correspondence between the set bi-invariant metrics on G and the set of ad*-invariant inner products on $\mathfrak{g}$.*

In particular, if G is compact then the Killing form $\kappa(X, Y) := \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$ on $\mathfrak{g}$ is negative definite ([Bes87, Lemma 7.36]) and thus there exists a bi-invariant metric on G.

A first convenient fact about bi-invariant metrics is that the associated Levi-Civita connection is easily computed on invariant vector fields.

Lemma 15.2.2. *If g is a bi-invariant Riemannian metric on a Lie group G and if ∇ is the associated Levi-Civita connection, then*

$$\forall X, Y \in \mathcal{X}_\ell(G),\ \nabla_X Y = \frac{1}{2}[X, Y].$$

Proof. Let $Z \in \mathcal{X}_\ell(G)$ and note that, since g is bi-invariant, the function $g(X, Y)$ is constant on G and hence $Z(g(X, Y)) = 0$. Also, since g_1 is ad-invariant on $\mathfrak{g}$, we have $g(X, [Y, Z]) = g([X, Y], Z)$. Hence, the Koszul formula (K) reads

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) + g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X) \\ &= g([X, Y], Z) - g([X, Z], Y) + g(X, [Z, Y]) = g([X, Y], Z). \end{aligned}$$

Since this is true for arbitrary Z, the result follows. □

We now come to the following important result:

Theorem 15.2.3 (*[YWL19, Theorem 2.5]*)**.** *If G is a compact Lie group endowed with a bi-invariant metric g and $H \leq G$ is a closed subgroup then the orbit space G/H, endowed with the induced Riemannian metric given by the Theorem 15.1.3, is a* geodesic orbit space, *meaning that every geodesic on G/H is the orbit of a one-parameter subgroup of G.*

Proof. Using [GHL04, Proposition 2.81], we only have to prove that G is a geodesic orbit space. Denote by $\mathfrak{g}$ the Lie algebra of G. For $p \in G$ and $X \in \mathfrak{g}$, let γ be the curve defined on $\mathbb{R}$ by $\gamma : t \mapsto pe^{tX}$. Then γ is an X-integral curve, i.e. $\gamma' = \widetilde{X} \circ \gamma$, where $\widetilde{X} \in \mathcal{X}_\ell(G)$ is the left-invariant vector field associated to $X \in \mathfrak{g}$. Then, one calculates

$$\forall t \in \mathbb{R},\ \nabla_{\dot\gamma}\dot\gamma(t) \overset{\mathrm{df}}{=} (\gamma^*\nabla)_{\frac{d}{dt}}(\gamma^*\widetilde{X})(t) \overset{(\triangle)}{=} \left(\gamma^*\left(\nabla_{\dot\gamma(t)}\widetilde{X}\right)\right)(t) = \left(\nabla_{\widetilde{X}}\widetilde{X}\right)(\gamma(t)) = 0,$$

since $\nabla_{\widetilde{X}}(\widetilde{X}) = \frac{1}{2}[\widetilde{X}, \widetilde{X}] = 0$ by Lemma 15.2.2, we get that $\nabla_{\dot\gamma}\dot\gamma = 0$ and hence γ is a geodesic and by the Picard-Lindelöf theorem, this is the only geodesic on G such that $\gamma(0) = p$ and $\gamma'(0) = pX = d_1L_p(X)$. Then, we have proved that any geodesic on G is of the form $t \mapsto pe^{tX}$ for some $p \in G$ and $X \in \mathfrak{g}$ and hence is a one-parameter subgroup of G. $\square$

15.3 The quaternionic bi-invariant Riemannian metric on the flag manifold of $SL_3(\mathbb{R})$

We shall now equip the manifold $\mathcal{F}(\mathbb{R})$ with a bi-invariant Riemannian metric. An SU_3-invariant Riemannian metric on $\mathcal{F} = SU_3(\mathbb{C})/T$ is easily seen to be determined by its value on the tangent space $T_1\mathcal{F}$ (this is a general fact about homogeneous spaces which relies on Lemma 15.2.1). Now,

$$\mathfrak{sl}_3(\mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{\delta\in\Phi^+} (\mathbb{C}e_\delta \oplus \mathbb{C}f_\delta)$$

is the root spaces decomposition of $\mathfrak{sl}_3$, with $(e_\delta, f_\delta, h_\delta)_{\delta\in\Phi^+}$ the Serre basis of $\mathfrak{sl}_3$, and $\Phi^+ = \{\alpha, \beta, \alpha+\beta\}$ is the set of positive roots, then one has the Cartan decomposition

$$\mathfrak{su}_3(\mathbb{C}) = \mathfrak{t} \oplus \bigoplus_{\delta\in\Phi^+} \mathfrak{p}_\delta, \quad \text{with } \mathfrak{p}_\delta := \mathbb{R}\underbrace{(e_\delta - f_\delta)}_{u_\delta} \oplus \mathbb{R}\,\underbrace{i(e_\delta + f_\delta)}_{v_\delta} \quad \text{and } \mathfrak{t} = \bigoplus_{\delta\in\Phi^+} \mathbb{R}ih_\delta.$$

Now, one has $T_1\mathcal{F} \simeq \bigoplus_{\delta\in\Phi^+}\mathfrak{p}_\delta =: \mathfrak{p}$ and recalling that the Killing form $\kappa(X,Y) := 6\,\mathrm{tr}(\mathrm{ad}(X)\circ\mathrm{ad}(Y)) = 6\,\mathrm{tr}(XY)$ on $\mathfrak{su}_3(\mathbb{C})$ is a negative-definite symmetric bilinear form (since SU_3 is compact, see [Bes87, Lemma 7.36]), any left SU_3-invariant metric g on $\mathcal{F}$ may be written as

$$g = -\sum_{\delta\in\Phi^+} x_\delta \cdot \kappa(\cdot,\cdot)_{|\mathfrak{p}_\delta}, \quad \text{with } x_\delta \in \mathbb{R}^+,\ \forall\delta\in\Phi^+$$

and this metric is bi-invariant if and only if $x_\delta = x_{\delta'}$ for all $\delta, \delta' \in \Phi^+$. Thus there is only one bi-invariant metric on $\mathcal{F}$, up to scalar. These standard considerations can be found in [Sak99] or [PS97]. Then, we take the *quaternionic bi-invariant metric*

$$g^8 := -\frac{1}{48}(\kappa_{|\mathfrak{p}_\alpha} + \kappa_{|\mathfrak{p}_\beta} + \kappa_{|\mathfrak{p}_{\alpha+\beta}}) = -\frac{1}{48}\kappa_{|\mathfrak{p}}$$

on $\mathcal{F}$, and restrict it to $\mathcal{F}(\mathbb{R})$. The reason of taking such a normalization will appear soon. Notice that this metric is *Einstein*, meaning that the Ricci tensor is a scalar multiple of the metric tensor, i.e. there exists a function λ such that $\mathrm{Ric}_{g^8} = \lambda g^8$ everywhere.

Proposition 15.3.1. *The metric g^8 on $SO(3)$ defined above induces a Riemannian metric $\overline{g}^8$ on $\mathcal{F}(\mathbb{R})$ making $(\mathcal{F}(\mathbb{R}), \overline{g}^8)$ into a geodesic orbit space. Moreover, for $p \in SO(3)$ and $X \in \mathfrak{so}(3) := \mathfrak{so}_3(\mathbb{R})$, the arc-length of the geodesic $\gamma : s \mapsto pe^{sX}\cdot S(O(1)^3)$ is given by*

$$\forall t \geq 0,\ \mathrm{L}(\gamma_{|[0,t]}) = \frac{t\|X\|_F}{2\sqrt{2}},$$

where $\|\cdot\|_F$ is the Frobenius norm, defined by $\|(a_{i,j})_{i,j}\|_F = \sqrt{\sum_{i,j}|a_{i,j}|^2}$.

Proof. The first statement is just a particular case of the Theorem 15.2.3. For the second statement we just calculate, for $t \in \mathbb{R}_+$,

$$\mathrm{L}(\gamma_{|[0,t]}) \overset{\text{def}}{=} \int_0^t \sqrt{g^8_{\gamma(s)}(\gamma'(s), \gamma'(s))}\mathrm{d}s = \int_0^t \sqrt{g^8_{\gamma(s)}(pX\exp(sX), pX\exp(sX))}\mathrm{d}s$$

$$= \int_0^t \sqrt{g_1^8(X,X)}\mathrm{d}s = t\sqrt{g_1^8(X,X)} = t\sqrt{\frac{\mathrm{tr}({}^tXX)}{8}} = \frac{t\|X\|_F}{2\sqrt{2}}.$$

□

15.4 Isometry between $\mathcal{F}(\mathbb{R})$ and the quaternionic spherical space form $\mathbb{S}^3/\mathcal{Q}_8$

Recall the isomorphism

$$\begin{array}{cccc} \sigma & : & \mathcal{O}/\mathcal{Q}_8 & \xrightarrow{\sim} & \mathfrak{S}_3 \\ & & \frac{1+i}{\sqrt{2}} & \longmapsto & s_\beta \\ & & \frac{1+k}{\sqrt{2}} & \longmapsto & s_\alpha \end{array}$$

We equip $\mathbb{S}^3/\mathcal{Q}_8$ with the quotient metric $q_{\mathcal{Q}_8}$ induced by the standard round metric on $\mathbb{S}^3$ and we shall prove that the diffeomorphism ψ from Lemma 14.0.2 is in fact an isometry. For this, we need the following lemma:

Lemma 15.4.1. *The map*

$$\begin{array}{ccc} \mathbb{S}^3 & \xrightarrow{\mathrm{B}} & SO(3) \\ q & \longmapsto & \mathrm{Mat}_{(i,j,k)}(L(q)R(q)) \end{array}$$

is smooth and we have

$$\begin{array}{ccc} \mathbb{R}^3 & \xrightarrow{\mathrm{d}_1\mathrm{B}} & \mathfrak{so}(3) \\ (x,y,z) & \longmapsto & 2\begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix} \end{array}$$

In particular, if $\mathbb{S}^3$ is equipped with the standard round metric induced from $\mathbb{R}^4$ and $SO(3)$ with the bi-invariant metric g^8 defined above, then we have an isometry

$$\overline{B} : \mathbb{S}^3/\{\pm 1\} \xrightarrow{\sim} SO(3).$$

Proof. Recall the space $V := \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ of pure quaternions. For $u, h \in V \simeq \mathbb{R}^3$, we simply compute

$$\mathrm{d}_1\mathrm{B}(u)\cdot h = \frac{d}{dt}\,\mathrm{B}(1+tu)(h)|_{t=0} = \frac{d}{dt}\,(1+tu)h\overline{(1+tu)}\Big|_{t=0}$$

$$= \frac{d}{dt}\,(h + th\overline{u} + tuh + t^2uh\overline{u})\big|_{t=0} = h\overline{u} + uh = uh - hu = [u,h].$$

Hence, by computing the matrix of $\mathrm{d}_1B(u)$ with respect to the canonical basis (i,j,k) of V, one obtains the matrix from the first statement.

Now, since $\{\pm 1\}$ acts freely and isometrically on $\mathbb{S}^3$, the map $\mathbb{S}^3 \twoheadrightarrow \mathbb{S}^3/\{\pm 1\}$ is a Riemannian covering, hence a local isometry (in particular, a local diffeomorphism). Therefore, if we prove that B is a local isometry, then $\overline{\mathrm{B}}$ will be a bijective local isometry, hence an isometry by the inverse function theorem, as required. But since B is a homomorphism of Lie groups, it suffices to show that $\mathrm{d}_1\mathrm{B}$ is a linear isometry. This is where the normalization by $\frac{1}{48}$ comes into the game. Since we have endowed $\mathbb{S}^3$ with the round metric, we can compute for $u := (x,y,z) \in \mathbb{R}^3 = T_1\mathbb{S}^3$,

$$g_1^8(\mathrm{d}_1\mathrm{B}(u), \mathrm{d}_1\mathrm{B}(u)) \overset{\mathrm{df}}{=} -\frac{6\,\mathrm{tr}(\mathrm{d}_1\mathrm{B}(u)^2)}{48} = -\frac{4}{8}\,\mathrm{tr}\left(\begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}^2\right)$$

$$= -\frac{1}{2}\operatorname{tr}\begin{pmatrix} -y^2-z^2 & xy & xz \\ xy & -x^2-z^2 & yz \\ xz & yz & -x^2-y^2 \end{pmatrix} = x^2+y^2+z^2 = g_1^{\mathbb{S}^3}(u,u).$$

$\square$

Proposition 15.4.2. *If we endow respectively $\mathbb{S}^3/\mathcal{Q}_8$ and $\mathcal{F}(\mathbb{R})$ with the metrics $g_{\mathcal{Q}_8}$ and $\overline{g}^8$, then the map ψ of Lemma 14.0.2 is an isometry.*

Proof. The quotient map

$$\mathbb{S}^3/\{\pm 1\} \twoheadrightarrow (\mathbb{S}^3/\{\pm 1\})/K_4 \simeq \mathbb{S}^3/\mathcal{Q}_8$$

is a Riemannian covering, hence a local isometry. On the other hand, the map

$$SO(3) \twoheadrightarrow SO(3)/S(O(1)^3) = \mathcal{F}(\mathbb{R})$$

is a Riemannian covering too. Now, since $\overline{\mathrm{B}} : \mathbb{S}^3/\{\pm 1\} \to SO(3)$ is an isometry by the previous Lemma and since the following diagram commutes

$$\begin{tikzcd} \mathbb{S}^3/\{\pm 1\} \arrow[r, "\overline{\mathrm{B}}", "\sim"'] \arrow[d, two heads] & SO(3) \arrow[d, two heads] \\ \mathbb{S}^3/\mathcal{Q}_8 \arrow[r, "\sim", "\psi"'] & \mathcal{F}(\mathbb{R}) \end{tikzcd}$$

one concludes that ψ is a bijective local isometry, hence a global isometry. $\square$

In particular, combining Proposition 15.4.2 and Theorem 15.1.3 yields the following corollary:

Corollary 15.4.3. *For $q := a + bi + cj + dk \in \mathbb{S}^3$, one has*

$$d_{g^8}(1, \mathrm{B}(q)) = \min_{\varepsilon=\pm 1} d_{\mathbb{S}^3}(1, \varepsilon q) = \arccos|a|$$

and

$$d_{\overline{g}^8}(1, \phi(q)) = \min_{g\in\mathcal{Q}_8} d_{\mathbb{S}^3}(1, gq) = \min_{x=\pm a,\pm b,\pm c,\pm d}(\arccos(x)) = \min_{x=a,b,c,d}(\arccos|x|).$$

16 Interpretation of the quaternionic cellular structure on $\mathcal{F}(\mathbb{R})$ in terms of geodesics

16.1 Geodesics in $\mathcal{F}(\mathbb{R})$ as projections of geodesics in $\mathbb{S}^3$

Now that we know what geodesics look like and that we can compute the distance between two flags, we can start describing the cells. But before that, we have to adapt the curved join construction to $\mathcal{F}(\mathbb{R})$. This is not as easy as in the case of $\mathbb{S}^3$, since there can exist many minimizing geodesics between two points in $\mathcal{F}(\mathbb{R})$ (as for two antipodal points in $\mathbb{S}^3$). Since $SO(3)$ acts transitively by isometries on $\mathcal{F}(\mathbb{R})$, it suffices to look at geodesics starting at 1 and translate them. It turns out that, if a matrix in $SO(3)$, seen as a rotation, has angle different from π, then there will be a unique minimizing geodesic linking it to 1. For this, we shall use the matrix logarithm.

Recall that, given $X \in \mathfrak{so}(3)$ and $\theta \in [0, 2\pi]$, we have the *Rodrigues formula* (see[CL10, §2])

$$e^{\theta X} = I_3 + \sin(\theta)X + (1 - \cos(\theta))X^2,$$

hence we obtain $\sin(\theta)X = \frac{e^{\theta X} - {}^t(e^{\theta X})}{2}$ and if $\theta \neq 0, \pi$, then

$$X = \frac{1}{2\sin(\theta)}(e^{\theta X} - e^{-\theta X}).$$

Thus, if $R \in SO(3)$ is a rotation with $\mathrm{tr}(R) \neq -1, 3$, then there is a unique $X \in \mathfrak{so}(3)$ such that $e^X = R$ and X is given by

$$X = \frac{\theta}{2\sin(\theta)}(R - {}^tR), \ \theta = \arccos\left(\frac{\mathrm{tr}(R) - 1}{2}\right).$$

We shall denote $X := \log(R)$. This is uniquely defined as soon as $\theta \neq 0, \pi$. If $\theta = 0$, we can just take $\log(R) = 0$. With this notion, we see that the curve $\gamma_R : t \mapsto e^{t\log(R)}$ is a geodesic from 1 to R in $SO(3)$ and hence its projection $\overline{\gamma_R} : t \mapsto e^{t\log(R)}S(O(1)^3)$ is a geodesic from 1 to $R \cdot S(O(1)^3)$ in $\mathcal{F}(\mathbb{R})$.

Now, we have to prove that the images of the geodesics we used in $\mathbb{S}^3$ to construct our $\mathcal{O}$-cellular decomposition go to geodesics in $\mathcal{F}(\mathbb{R})$. Denote by $\pi_{\mathcal{Q}_8} : \mathbb{S}^3 \twoheadrightarrow \mathbb{S}^3/\mathcal{Q}_8$ the natural projection and recall the isometry $\psi : \mathbb{S}^3/\mathcal{Q}_8 \to \mathcal{F}(\mathbb{R})$. We have the following result:

Proposition 16.1.1. *Let*

$$q := (\cos\omega, \sin\omega\cos\varphi, \sin\omega\sin\varphi\cos\theta, \sin\omega\sin\varphi\sin\theta) \in \mathbb{S}^3$$

be a point expressed in spherical coordinates, with $0 \leq \omega, \varphi \leq \pi$ *and* $0 \leq \theta \leq 2\pi$*. Suppose* $0 < \omega < \frac{\pi}{2}$ *and denote by* $\widetilde{\gamma}_q$ *the unique minimizing geodesic such that* $\widetilde{\gamma}_q(0) = 1$ *and* $\widetilde{\gamma}_q(1) = q$*. Then one has*

$$\forall 0 \leq t \leq 1, \ (\psi \circ \pi_{\mathcal{Q}_8})\widetilde{\gamma}_q(t) = \exp(tX_q) \cdot S(O(1)^3) =: \gamma_q(t),$$

where

$$X_q := 2\omega\begin{pmatrix} 0 & -\sin(\varphi)\sin(\theta) & \sin(\varphi)\cos(\theta) \\ \sin(\varphi)\sin(\theta) & 0 & -\cos(\varphi) \\ -\sin(\varphi)\cos(\theta) & \cos(\varphi) & 0 \end{pmatrix} \in \mathfrak{so}(3).$$

In particular, one has

$$\mathrm{L}(\gamma_q) = \mathrm{L}(\widetilde{\gamma}_q) = \omega.$$

Moreover, $B \circ \widetilde{\gamma}_q$ *is the only geodesic (up to reparametrization) in* $SO(3)$ *from* 1 *to* $\mathrm{B}(q)$*.*

Proof. The round metric on $\mathbb{S}^3$ is given in spherical coordinates (around 1) by the matrix (g_{ij}) where $g_{ij} = 0$ for $i \neq j$ and

$$g_{\omega\omega} = 1, \quad g_{\varphi\varphi} = \sin^2\omega, \quad g_{\theta\theta} = \sin^2\omega\sin^2\varphi.$$

hence, the Christoffel symbols Γ_{ij}^k are easily computed and the geodesic equations $\ddot{\gamma}^k + \Gamma_{ij}^k\dot{\gamma}^i\dot{\gamma}^j = 0$ for a curve $t \mapsto (\omega(t), \varphi(t), \theta(t))$ are given by the system

$$\begin{cases} \ddot{\omega} - \sin(\omega)\cos(\omega)(\dot{\varphi}^2 + \sin^2(\varphi)\dot{\theta}^2) = 0, \\ \ddot{\varphi} + \cot(\omega)\dot{\varphi}\dot{\omega} - \sin(\varphi)\cos(\varphi)\dot{\theta}^2 = 0, \\ \ddot{\theta} + \dot{\theta}(\cot(\varphi)\dot{\varphi} + \cot(\omega)\dot{\omega}) = 0. \end{cases}$$

Hence, the curve

$$\widetilde{\gamma_q} : t \mapsto (\cos(t\omega), \sin(t\omega)\cos\varphi, \sin(t\omega)\sin\varphi\cos\theta, \sin(t\omega)\sin\varphi\sin\theta)$$

is a geodesic, with $\widetilde{\gamma_q}(0) = (1,0,0,0)$ and $\widetilde{\gamma_q}(1) = q$. Moreover, it is minimizing since

$$\mathrm{L}(\widetilde{\gamma_q}) = \int_0^1 \sqrt{g^{\mathbb{S}^3}_{\widetilde{\gamma_q}(t)}(\dot{\widetilde{\gamma_q}}(t), \dot{\widetilde{\gamma_q}}(t))}dt = \int_0^1 \sqrt{\dot{\omega}(t)^2 + \sin^2\omega(t)\underbrace{\left(\dot{\varphi}(t)^2 + \sin^2\varphi(t)\dot{\theta}(t)^2\right)}_{=0}}dt$$

$$= \int_0^1 \dot{\omega}(t)dt = \omega = d_{\mathbb{S}^3}(1,q).$$

Now, since $0 < \omega < \frac{\pi}{2}$, we have $2t\omega < \pi$ and hence, we can compute

$$\mathrm{tr}(\mathrm{B}(\widetilde{\gamma_q}(t))) = 2\cos^2(t\omega) - 1 = \cos(2t\omega) \neq -1.$$

Thus, the logarithm $\log(\mathrm{B}(\widetilde{\gamma_q}(t)))$ is well-defined and the Rodrigues formula yields

$$\log(\mathrm{B}(\widetilde{\gamma_q}(t))) = 2t\omega \begin{pmatrix} 0 & -\sin(\varphi)\sin(\theta) & \sin(\varphi)\cos(\theta) \\ \sin(\varphi)\sin(\theta) & 0 & -\cos(\varphi) \\ -\sin(\varphi)\cos(\theta) & \cos(\varphi) & 0 \end{pmatrix} \overset{\text{def}}{=} tX_q,$$

so that $\mathrm{B}(\widetilde{\gamma_q}(t)) = e^{tX_q}$. Finally, since $\psi \circ \pi_{\mathcal{Q}_8} = \pi \circ \mathrm{B}$ where $\pi : SO(3) \twoheadrightarrow \mathcal{F}(\mathbb{R})$, we have the result. The statement about uniqueness follows immediately from the fact that $\log(\mathrm{B}(q))$ is uniquely defined and that $SO(3)$ is a geodesic-orbit space. $\square$

Recall that in the Section 14, we have denoted $\widehat{\mathscr{D}_\mathcal{O}} := \bigcup_i \widehat{\Delta_i}$ and $\widehat{\mathscr{F}_\mathcal{O}} := \mathrm{pr}(\widehat{\mathscr{D}_\mathcal{O}})$.

Corollary 16.1.2. *For every $q \in \widehat{\mathscr{F}_\mathcal{O}}$, the logarithm $\log(\mathrm{B}(q)) \in \mathfrak{so}(3)$ is well-defined and the curve $t \mapsto \exp(t\log\mathrm{B}(q))$ is the only minimal geodesic in $SO(3)$ from 1 to $\mathrm{B}(q)$. Furthermore, its projection γ_q is a geodesic in $\mathcal{F}(\mathbb{R})$.*

Proof. In view of Proposition 16.1.1, we only have to prove that $\Re(q) > 0$, because in this case we will have $\omega_q = \arccos(\Re(q)) < \frac{\pi}{2}$. Hence, we have to prove that for $1 \leq i \leq 6$ and for $x = (x_1, x_2, x_3, x_4) \in \widehat{\Delta_i}$, we have $x_1 > 0$; given that the $\widehat{\Delta_i}$'s are defined as convex hulls, it suffices to show that their vertices have positive first coordinates. But since these vertices are among

$$\left\{\frac{1}{2}(1,\pm1,\pm1,\pm1), \frac{1}{\sqrt{2}}(1,-1,0,0), \frac{1}{\sqrt{2}}(1,0,-1,0), \frac{1}{\sqrt{2}}(1,0,0,-1)\right\},$$

the result is now clear. $\square$

16.2 The cells of the $\mathfrak{S}_3$-equivariant cellular structure of $\mathcal{F}(\mathbb{R})$ as unions of open geodesics

We shall now describe the cells in $\mathcal{F}(\mathbb{R})$ from Theorem 3.4.6 of [CGS20] as unions of images of geodesics in $\mathcal{F}(\mathbb{R})$, with respect to the quaternionic metric $\overline{g}^8$. First, we briefly recall the curved join construction. Given two points x and $y \neq -x$ in $\mathbb{S}^3$, we write $x * y$ to denote the image $\gamma_{x,y}([0,1])$ of the unique minimal geodesic $\gamma_{x,y} : [0,1] \to \mathbb{S}^3$ joining them. The resulting curve is called the *curved join* of x and y. Also, $x \mathring{*} y$ denotes the image $\gamma_{x,y}(]0,1[)$ that is, the image of the geodesic $\gamma_{x,y}$ with endpoints removed. We can extend the curved join to subsets of $\mathbb{S}^3$: if $U, V \subset \mathbb{S}^3$ are such that $U \cap (-V) = \emptyset$, then we can define

$$U * V := \bigcup_{\substack{u \in U \\ v \in V}} u * v.$$

This is easily seen to be associative on subsets. We may also define $U \mathring{*} V := \bigcup_{u,v} u \mathring{*} v$.

We introduce some notation. If $q \in \mathbb{S}^3$ with $\operatorname{tr}(q) > 0$, recall the unique geodesic $\widetilde{\gamma_q}$ from 1 to q on $\mathbb{S}^3$ and its image $\gamma_q := \psi \circ \pi_{\mathcal{Q}_8} \circ \widetilde{\gamma_q}$ on $\mathcal{F}(\mathbb{R})$ defined by $\gamma_q(t) = \exp(t \log(\mathrm{B}(q)))S(O(1)^3)$. We shall denote by $\Gamma_q := \gamma_q\,(]0,1[)$ the image of the *open* geodesic $(\gamma_q)_{|]0,1[}$. Next, for $u \neq v \in \{i,j,k\}$, let

$$e_v^u := \bigcup_{q \in \tau_u \mathring{*} \omega_v} \widetilde{\gamma_{q^{-1}}}(]0,1[) \quad \text{and} \quad e^{uv} := \bigcup_{q \in \tau_u \mathring{*} \tau_v} \widetilde{\gamma_{q^{-1}}}(]0,1[),$$

as well as

$$\underline{e}_v^u := \psi(\pi_{\mathcal{Q}_8}(e_v^u)) = \bigcup_{q \in \tau_u \mathring{*} \omega_v} \Gamma_{q^{-1}} \quad \text{and} \quad \underline{e}^{uv} := \psi(\pi_{\mathcal{Q}_8}(e^{uv})) = \bigcup_{q \in \tau_u \mathring{*} \tau_v} \Gamma_{q^{-1}}.$$

Note that we may of course define also, for $u \in \{i,j,k\}$,

$$e_0^u := \bigcup_{q \in \tau_u \mathring{*} \omega_0} \widetilde{\gamma_{q^{-1}}}(]0,1[) \quad \text{and} \quad \underline{e}_0^u := \psi(\pi_{\mathcal{Q}_8}(e_0^u)) = \bigcup_{q \in \tau_u \mathring{*} \omega_0} \Gamma_{q^{-1}}.$$

With this notation we can determine the images $\widetilde{\Delta_i} := \psi(\pi_{\mathcal{Q}_8}(\widehat{\Delta_i}))$ as

$$\begin{cases} \widetilde{\Delta_1} = \bigcup_{q \in \underline{e}_j^k} \Gamma_{q\tau_j}, \\ \widetilde{\Delta_2} = \bigcup_{q \in \underline{e}_k^i} \Gamma_{q\tau_k}, \\ \widetilde{\Delta_3} = \bigcup_{q \in \underline{e}_i^j} \Gamma_{q\tau_i}, \end{cases} \quad \text{and} \quad \begin{cases} \widetilde{\Delta_4} = \bigcup_{q \in \underline{e}_j^i} \Gamma_{q\tau_j}, \\ \widetilde{\Delta_5} = \bigcup_{q \in \underline{e}_k^j} \Gamma_{q\tau_k}, \\ \widetilde{\Delta_6} = \bigcup_{q \in \underline{e}_i^k} \Gamma_{q\tau_i}. \end{cases}$$

Remark 16.2.1. *We have used quaternions to define these subsets, however, it should be remarked that one can write them using only the exponential. For instance, one has*

$$\underline{e}_j^i = \bigcup_{q \in \tau_i \mathring{*} \omega_j} \Gamma_{q^{-1}}$$

$$= \left\{ \exp\left(\frac{2s \arccos\left(\frac{\cos\frac{t\pi}{4} + \sin\frac{t\pi}{4}}{2}\right)}{\sqrt{3 - \sin\frac{t\pi}{2}}} \begin{pmatrix} 0 & \sqrt{1 - \sin\frac{t\pi}{2}} & -\cos\frac{t\pi}{4} - \sin\frac{t\pi}{4} \\ -\sqrt{1 - \sin\frac{t\pi}{2}} & 0 & \sin\frac{t\pi}{4} - \cos\frac{t\pi}{4} \\ \cos\frac{t\pi}{4} + \sin\frac{t\pi}{4} & \cos\frac{t\pi}{4} - \sin\frac{t\pi}{4} & 0 \end{pmatrix} \right) S(O(1)^3),\ 0 < s,t < 1 \right\}$$

To see this, first notice that

$$\underline{e}_j^i = \bigcup_{q \in \tau_j \mathring{*} \omega_i} \Gamma_{q^{-1}} = \bigcup_{q \in \tau_j^{-1} \mathring{*} \omega_i^{-1}} \Gamma_q = \bigcup_{q \in (1 \mathring{*} \tau_i^{-1}) \cdot \omega_i^{-1}} \Gamma_q = \bigcup_{q \in \operatorname{im} \mathring{\widetilde{\gamma_{\tau_i^{-1}}}}} \Gamma_{q\omega_i^{-1}}.$$

But, one has that $q \in \operatorname{im}(\mathring{\widetilde{\gamma_{\tau_i^{-1}}}})$ *if there exists* $0 < t < 1$ *such that* $q = \cos\frac{t\pi}{4} - i \sin\frac{t\pi}{4}$. *To simplify notations, denote* $c_t := \cos\frac{t\pi}{4}$ *and* $s_t := \sin\frac{t\pi}{4}$. *Then, one has*

$$q\omega_i^{-1} = \frac{(c_t + s_t) + i(c_t - s_t) - j(c_t + s_t) - k(c_t - s_t)}{2}$$

$$= \cos(\omega_t) + i\sin(\omega_t)\cos(\varphi_t) + j\sin(\omega_t)\sin(\varphi_t)\cos(\theta_t) + k\sin(\omega_t)\sin(\varphi_t)\sin(\theta_t),$$

where

$$\omega_t = \arccos\left(\frac{c_t + s_t}{2}\right), \quad \varphi_t = \arccos\left(\frac{c_t - s_t}{\sqrt{3 - \sin\frac{t\pi}{2}}}\right), \quad \theta_t = \arccos\left(\frac{c_t + s_t}{\sqrt{2}}\right) - \pi.$$

Now, we have that $z \in \Gamma_{q\omega_i^{-1}} = \operatorname{im}\ \overset{\circ}{\gamma}_{q\omega_i^{-1}}$ *if there exists* $0 < s < 1$ *such that* $z = e^{sX_{q\omega_i^{-1}}} \cdot S(O(1)^3)$ *and since we have*

$$X_{q\omega_i^{-1}} = 2\omega_t \begin{pmatrix} 0 & -\sin(\varphi_t)\sin(\theta_t) & \sin(\varphi_t)\cos(\theta_t) \\ \sin(\varphi_t)\sin(\theta_t) & 0 & -\cos(\varphi_t) \\ -\sin(\varphi_t)\cos(\theta_t) & \cos(\varphi_t) & 0 \end{pmatrix}$$

$$= \frac{2\arccos\left(\frac{c_t+s_t}{2}\right)}{\sqrt{3 - \sin\frac{t\pi}{2}}} \begin{pmatrix} 0 & \sqrt{1 - \sin\frac{t\pi}{2}} & -c_t - s_t \\ -\sqrt{1 - \sin\frac{t\pi}{2}} & 0 & s_t - c_t \\ c_t + s_t & c_t - s_t & 0 \end{pmatrix}$$

and we find indeed the announced description.

We are now in a position to state the main result:

Theorem 16.2.2. *With the above notation, the real flag manifold* $\mathcal{F}(\mathbb{R}) = SO(3)/S(O(1)^3)$ *admits an* $\mathfrak{S}_3$*-equivariant cellular decomposition with orbit representatives cells given by*

$$\mathfrak{e}^0 := \{1 \cdot S(O(1)^3)\},$$

$$\mathfrak{e}^1_1 := \Gamma_{\tau_i^{-1}}, \quad \mathfrak{e}^1_2 := \Gamma_{\tau_j^{-1}}, \quad \mathfrak{e}^1_3 := \Gamma_{\tau_k^{-1}},$$

and

$$\mathfrak{e}^2_1 := \bigcup_{q \in (\tau_i \overset{\circ}{*} \omega_k) \cup (\omega_k \overset{\circ}{*} \tau_j)} \Gamma_{q^{-1}}, \quad \mathfrak{e}^2_2 := \bigcup_{q \in (\tau_j \overset{\circ}{*} \omega_i) \cup (\omega_i \overset{\circ}{*} \tau_k)} \Gamma_{q^{-1}}, \quad \mathfrak{e}^2_3 := \bigcup_{q \in (\tau_k \overset{\circ}{*} \omega_j) \cup (\omega_j \overset{\circ}{*} \tau_i)} \Gamma_{q^{-1}},$$

as well as

$$\mathfrak{e}^3 := \widetilde{\Delta_4} \cup \underline{e}^{ij} \cup \widetilde{\Delta_1} \cup \underline{e}^j_0 \cup \widetilde{\Delta_2} \cup \underline{e}^{jk} \cup \widetilde{\Delta_5} \cup \underline{e}^k_0 \cup \widetilde{\Delta_3} \cup \underline{e}^{ki} \cup \widetilde{\Delta_6} \cup \underline{e}^i_0.$$

Moreover, the closures of the 1-cells $\mathfrak{e}^1_j$ *are minimal geodesics from* $\mathfrak{e}^0$ *to* $\mathfrak{e}^0 \cdot s_\beta$, $\mathfrak{e}^0 \cdot w_0$ *and* $\mathfrak{e}^0 \cdot s_\alpha$*, respectively.*

Proof. We just have to check that, if $\widehat{e}^i_j$ is a cell of the analogue of the cellular decomposition provided by the Theorem 11.4.1 for the action of $\mathcal{O}$ on $\mathbb{S}^3$ by multiplication on the *right*, then one has $\psi \circ \pi_{\mathcal{Q}_8}(\widehat{e}^i_j) = \mathfrak{e}^i_j$, in other words,

$$\mathfrak{e}^i_j = \psi\left(\pi_{\mathcal{Q}_8}\left((e^i_j)^{-1}\right)\right),$$

but we have defined the cells $\mathfrak{e}^i_j$ in this way.

Next, take for instance the closure $\overline{\mathfrak{e}^1_1} = \gamma_{\tau_i^{-1}}([0,1])$, the other two being treated in the same way. By the Corollary 16.1.2, $\gamma_{\tau_i^{-1}}$ is a geodesic in $\mathcal{F}(\mathbb{R})$ and by the Corollary 15.4.3, we have $d_{\overline{g}^8}(1, \psi(\overline{\tau_i^{-1}})) = \min\left(\arccos\left(\pm\frac{\sqrt{2}}{2}\right), \arccos(0)\right) = \frac{\pi}{4}$. Thus, we have to show that $\mathrm{L}(\gamma_{\tau_i^{-1}}) = \frac{\pi}{4}$. But since $\gamma_{\tau_i^{-1}}(1) = \pi_{\mathcal{Q}_8}(\tau_i^{-1}) = s_\beta$ and

$$\log(\dot{s_\beta}) = \frac{\pi}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

by the Proposition 15.3.1, we get $\mathrm{L}(\gamma_{\tau_i^{-1}}) = \frac{1}{2\sqrt{2}}\|\log(\dot{s_\beta})\|_F = \frac{\pi}{4} = d_{\overline{g}^8}(1, s_\beta)$, as required. $\square$

Remark 16.2.3. *We can also describe more explicitly the* 1*-cells as*

$$\mathfrak{e}_1^1 = \left\{\exp\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{t\pi}{2} \\ 0 & \frac{t\pi}{2} & 0 \end{pmatrix} S(O(1)^3),\ 0 < t < 1\right\} = \left\{e^{\frac{t\pi}{2}u_\beta} \cdot S(O(1)^3),\ 0 < t < 1\right\},$$

$$\mathfrak{e}_2^1 = \left\{\exp\begin{pmatrix} 0 & 0 & -\frac{t\pi}{2} \\ 0 & 0 & 0 \\ \frac{t\pi}{2} & 0 & 0 \end{pmatrix} S(O(1)^3),\ 0 < t < 1\right\} = \left\{e^{\frac{t\pi}{2}u_{\alpha+\beta}} \cdot S(O(1)^3),\ 0 < t < 1\right\},$$

$$\mathfrak{e}_3^1 = \left\{\exp\begin{pmatrix} 0 & -\frac{t\pi}{2} & 0 \\ \frac{t\pi}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} S(O(1)^3),\ 0 < t < 1\right\} = \left\{e^{\frac{t\pi}{2}u_\alpha} \cdot S(O(1)^3),\ 0 < t < 1\right\}.$$

Notice that the closure $\overline{\mathfrak{e}^3}$ is a fundamental domain for $\mathfrak{S}_3$ acting on $\mathcal{F}(\mathbb{R})$.

17 Dirichlet-Voronoi domain for a normal homogeneous metric on K/T

17.1 Definition and general properties

In this section, we fix (M, g) a connected complete Riemannian manifold, with geodesic distance d, and $W \leq \mathrm{Isom}(M)$ a discrete subgroup of the isometry group of (M, g). By a classical result (see [AKLM07, Lemma 2.1] for instance), this means that each W-orbit is discrete as a subset of M. We fix also $x_0 \in M$ a *regular point*, i.e. a point with trivial stabilizer.

Inspired by the study of Fuchsian groups, we may consider the Dirichlet-Voronoi domain of W acting on $\mathcal{F}$:

Definition 17.1.1. *Let $x_0 \in M$ be a regular point.*

- *The* Dirichlet-Voronoi domain *centered at x_0 is the following subset of M:*

$$\mathcal{DV} := \{x \in M\ ;\ \forall w \in W,\ d(x_0, x) \leq d(wx_0, x)\}.$$

- *For $w \in W$, we denote by H_w the* dissecting hypersurface

$$H_w := \{x \in M\ ;\ d(x_0, x) = d(wx_0, x)\}.$$

If the action if free and considering the orbit space M/W equipped with the quotient metric g/W and the associated distance $d_{M/W}$, the set $\mathcal{DV}$ can be interpreted as the set of elements $x \in M$ realizing the distance of their orbit: $d(x_0, x) = d_{M/W}(Wx_0, Wx)$.

Remark 17.1.2. *As mentioned earlier, the domains $\mathcal{DV}$ as defined above are mainly studied for hyperbolic manifolds (see [Bow93]) or more generally manifolds with constant sectional curvature (see [Rat06, §6.6]). This is because we want $\mathcal{DV}$ to be a* fundamental polyhedron *for W acting on M and in particular, geodesically convex. In the case of flag manifolds, the curvature is no longer constant and one has to be careful with the meaning of convexity, because minimal geodesics are not unique in general. A relevant notion to introduce regarding this matter is the* injectivity radius $\mathrm{inj}_{x_0}(M)$ *of M at x_0 (see [Lee, §6.2] or [GHL04, Definition 2.116]).*

It follows immediately from the above definition that for $w \in W$, the subset $M \setminus H_w$ is the disjoint union of the two open subsets $\{x \in M \ ; \ d(x_0, x) < d(wx_0, x)\}$ and $\{x \in M \ ; \ d(x_0, x) > d(wx_0, x)\}$ and moreover, the interior of $\mathcal{DV}$ is the connected component of $M \setminus \bigcup_{1 \neq w \in W} H_w$ containing x_0.

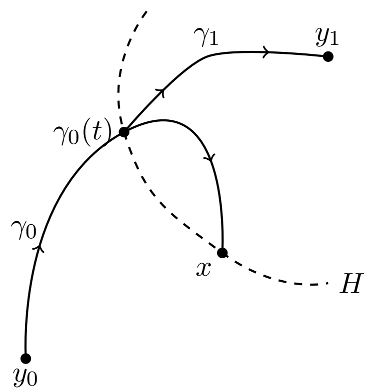
It is reasonable to expect $\mathcal{DV}$ to be a fundamental domain for W acting on M. This is indeed the case and we shall need a technical preliminary result on the behaviour of the hypersurfaces H_w with respect to minimal geodesics. This has been done in [AKLM07], a result which we reproduce here for the sake of self-containment. One should be careful with the terminology: though we call the H_w's "hypersurfaces", they are not submanifolds of M a priori.

Lemma 17.1.3 (*[AKLM07, Lemma 2.2]*)**.** *Let $y_0, y_1 \in M$ be distinct points of M and consider the hypersurface $H := H_{y_0,y_1} = \{x \in M \ ; \ d(x, y_0) = d(x, y_1)\}$. If $x \in H$, then every minimal geodesic from y_0 to x meets H only at x.*

Proof. Let γ_0 be a minimal geodesic parametrized by arc-length such that $\gamma_0(0) = y_0$ and $\gamma_0(\ell) = x$, where $\ell = d(y_0, x)$ and suppose for contradiction that $\gamma_0(t) \in H$ for some $t < \ell$. We compute

$$d(x, y_1) = d(x, y_0) = d(y_0, \gamma_0(t)) + d(\gamma_0(t), x) = d(y_1, \gamma_0(t)) + d(\gamma_0(t), x),$$

and so we are in the case of equality in the triangular inequality. Let γ_1 be a minimal geodesic from $\gamma_0(t)$ to y_1 and let $\widetilde{\gamma_0}$ be the curve $s \mapsto \gamma_0(\ell - s)$ for $0 \leq s \leq \ell - t$. The situation can be visualized as follows:



Then, the concatenation $\gamma_2 := \gamma_1 * \widetilde{\gamma_0}$ is a piecewise smooth curve from x to y_1 satisfying $\mathrm{L}(\gamma_2) = \mathrm{L}(\gamma_1) + \mathrm{L}(\widetilde{\gamma_0}) = \ell = d(x, y_1)$. By [Car92, Chapter 3, Corollary 3.9], this implies that γ_2 is in fact a (smooth) minimal geodesic from x to y_1, which coincides with the geodesic $s \mapsto \gamma_0(\ell - s)$ on a non-empty interval and by Picard-Lindelöf, this implies that $\gamma_2(s) = \gamma_0(\ell - s)$ for $0 \leq s \leq \ell$ and thus $y_0 = \gamma_0(0) = \gamma_2(\ell) = y_1$, a contradiction. $\square$

Another interesting feature of $\mathcal{DV}$ is that it is path-connected. More precisely, we have the following result:

Lemma 17.1.4. *The domain $\mathcal{DV}$ is* geodesically star-shaped *with respect to x_0, meaning that for every $x \in \mathcal{DV}$ and any minimal geodesic $\gamma : [0,1] \to M$ from x_0 to x, we have $\gamma(t) \in \mathcal{DV}$ for every $t \in [0,1]$. In particular, $\mathcal{DV}$ is path-connected.*

Proof. Let $t \in [0,1]$ and $w \in W$. We write

$$\begin{aligned} d(x_0, w\gamma(t)) &= d(x_0, w\gamma(t)) + d(w\gamma(t), wx) - d(\gamma(t), x) && (w \text{ is an isometry}) \\ &\geq d(x_0, wx) - d(\gamma(t), x) && (\text{triangular inequality}) \\ &\geq d(x_0, x) - d(\gamma(t), x) && (x_0 \in \mathcal{DV}) \\ &= d(x_0, \gamma(t)) && (\gamma \text{ is minimal}) \end{aligned}$$

and therefore we have $\gamma(t) \in \mathcal{DV}$, as required. $\square$

Proposition 17.1.5. *The Dirichlet-Voronoi domain $\mathcal{DV}$ is a geodesically star-shaped fundamental domain for W acting on M.*

Proof. Obviously we have $M = \bigcup_{w \in W} w\mathcal{DV}$. On the other hand, if $x \in \mathcal{DV} \cap w\mathcal{DV}$ for some $w \in W \setminus \{1\}$ and if $B = B(x, \delta)$ is a small (geodesic) ball centered at x with radius $\delta > 0$ included in $\mathcal{DV} \cap w\mathcal{DV}$, then $B \subset H_w$. However, if we denote by γ a minimal geodesic from x_0 to x parametrized by arc-length and if $\ell := d(x_0, x) = \mathrm{L}(\gamma)$, then $d(\gamma(t), x) = \ell - t < \delta$ for $t > \ell - \delta$ and thus $\gamma(t) \in H_w$ for $\ell - \delta < t \leq \ell$, contradicting Lemma 17.1.3. Therefore, $\mathcal{DV} \cap w\mathcal{DV}$ has empty interior. $\square$

We intend to use the domain $\mathcal{DV}$ to build a W-equivariant CW-structure on M. However, this is too much to ask in the general setting, as the *walls* of $\mathcal{DV}$, i.e. the subsets of the form $H_w \cap \mathcal{DV}$ containing a non-empty open subset of H_w, are not necessarily cells. For example, letting the cyclic group $C_2 = \{1, s\}$ act on $\mathbb{S}^2$ via the antipode, we have that H_s is a circle. However, we see that if we take again a Dirichlet-Voronoi domain for the induced action of C_2 on H_s, we finally obtain indeed a C_2-CW-structure on $\mathbb{S}^2$. This gives a hope of a general method for constructing a W-equivariant cell structure on flag manifolds. The first feature to ask is that the interior of $\mathcal{DV}$ should itself be a cell and to ensure this, we have to control the size of $\mathcal{DV}$.

Before stating the result, we introduce some notation: for any $x \in B(x_0, \mathrm{inj}_{x_0}(M))$ we denote by γ_x the *unique* minimal geodesic from x_0 to x, extended to $\mathbb{R}$ by completeness of M. The geodesic γ_x is defined by $\gamma_x(s) = \mathrm{Exp}_{x_0}(su/\|u\|)$ for any $s \in \mathbb{R}$, where $u := \dot{\gamma_x}(0) \in T_{x_0}M$.

Proposition 17.1.6. *If the domain $\mathcal{DV}$ satisfies*

$$\mathcal{DV} \subset B(x_0, \rho) \text{ for some } 0 < \rho < \mathrm{inj}_{x_0}(M),$$

then the open domain $\mathring{\mathcal{DV}}$ is a $\dim(M)$-cell.

Proof. Let $\delta := \max_{x\in\mathcal{DV}} d(x_0, x) < \rho$. This maximum exists as $\mathcal{DV}$ is closed in the compact subset $\overline{B(x_0,\rho)} \simeq \overline{B_{T_{x_0}M}(0,\rho)}$ of M. By Lemma 17.1.3, for $y \in B_{T_{x_0}M}(0,\delta) \setminus \{0\}$, there is a unique $0 < \delta_y \leq \delta$ such that $\gamma_y(\delta_y) \in \partial\mathcal{DV}$ and in fact we have $\delta_y = \mathrm{dist}(\mathrm{Exp}(y), \partial\mathcal{DV})$. Thus, the element $\gamma_y(\frac{\|y\|\delta_y}{\delta})$ is in the interior of $\mathcal{DV}$ and the assignment

$$\begin{array}{rcccc} \Phi & : & B_{T_{x_0}M}(0,\delta) & \longrightarrow & \overset{\circ}{\mathcal{DV}} \\ & & y & \longmapsto & \left\{ \begin{array}{cl} 1 & \text{if } y = 0, \\ \gamma_y(\|y\|\delta_y/\delta) & \text{otherwise.} \end{array} \right. \end{array}$$

defines a continuous map. Conversely, if $x \in \overset{\circ}{\mathcal{DV}} \setminus \{1\}$, then there is a unique $0 < \delta_x \leq \delta$ such that $\gamma_x(\delta_x) \in \partial\mathcal{DV}$ and if $\ell_x := d(1, x)$, then the assignment

$$\begin{array}{rcccc} \Psi & : & \overset{\circ}{\mathcal{DV}} & \longrightarrow & B_{T_{x_0}M}(0,\delta) \\ & & x & \longmapsto & \left\{ \begin{array}{cl} 0 & \text{if } x = 1, \\ \mathrm{Exp}^{-1}(\gamma_x(\delta\ell_x/\delta_x)) & \text{otherwise.} \end{array} \right. \end{array}$$

is continuous. It is routine to check that it defines an inverse to Φ. $\square$

We finish this section by a technical lemma that helps finding a bound on $\delta > 0$ such that $\mathcal{DV} \subset \overline{B(x_0,\delta)}$ when the acting group W is finite and under some injectivity radius condition:

Lemma 17.1.7 ("No antenna lemma"). *Let $W \leq \mathrm{Isom}(M)$ be finite, with associated Dirichlet-Voronoi domain $\mathcal{DV}$ and assume that $\mathcal{DV} \subset B(x_0,\rho)$ for some $0 < \rho < \mathrm{inj}_{x_0}(M)$. If $0 < \delta < \rho$ is such that the intersection $\mathcal{DV} \cap S(x_0,\delta)$ of $\mathcal{DV}$ and the sphere of radius δ consists of isolated points, then $\mathcal{DV} \subset \overline{B(x_0,\delta)}$.*

Proof. If there is some $z \in \mathcal{DV}$ such that $d(x_0, z) > \delta$, then $x := \gamma_z(\delta) \in \mathcal{DV} \cap S(x_0,\delta)$ and thus for any $0 < \varepsilon < d(x,z)$, the element $\gamma_z(\delta + \varepsilon/2)$ is in $\mathcal{DV} \cap {}^{\complement}B(x_0,\delta) \cap B(x,\varepsilon)$. We will prove however that this set is empty for $\varepsilon > 0$ sufficiently small.

As $\mathcal{DV} \cap S(x_0,\delta)$ consists of isolated points, we may choose $0 < \varepsilon < \frac{\rho-\delta}{2}$ such that

$$\mathcal{DV} \cap S(x_0,\delta) \cap B(x, 2\varepsilon) = \{x\}.$$

Suppose for contradiction that $y \in \mathcal{DV} \cap {}^{\complement}B(x_0,\delta) \cap B(x,\varepsilon)$. We denote $d_y := d(x_0, y)$ and compute

$$d(x,\gamma_y(\delta)) \leq d(x,y) + d(y,\gamma_y(\delta)) < \varepsilon + d(\gamma_y(d_y),\gamma_y(\delta)) = \varepsilon + d_y - \delta \leq \varepsilon + d(x,y) < 2\varepsilon,$$

so $\gamma_y(\delta) \in \mathcal{DV} \cap S(x_0,\delta) \cap B(x,2\varepsilon)$ and so $\gamma_y(\delta) = x = \gamma_z(\delta)$ and therefore $\gamma_y = \gamma_z$ as there is only one minimal geodesic from x_0 to x, since $x \in \mathcal{DV} \subset B(x_0, \mathrm{inj}_{x_0} M)$. This proves that

$$\mathcal{DV} \cap {}^{\complement}B(x_0,\delta) \cap B(x,\varepsilon) = \gamma_z([\delta, \delta+\varepsilon[).$$

The situation (which we are to prove is impossible) is depicted in Figure 21, giving its name to the lemma.

On the other hand, by Lemma 17.1.3, if we have $d(x_0,\gamma_z(t)) = d(wx_0,\gamma_z(t))$ for some $1 \neq w \in W$ and some $\delta < t < d_y$, then $t =: t_w$ is unique (we set $t_w := (\delta + d_y)/2$ in other cases) and therefore, if $\delta < s < t_w$, then $d(x_0,\gamma_z(s)) < d(wx_0,\gamma_z(s))$. Since W is finite, we

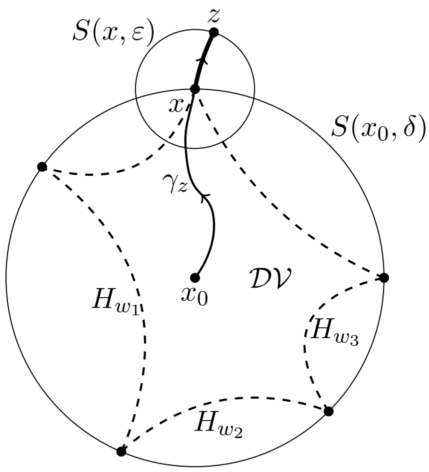


Figure 21: The forbidden antenna $\gamma_z([\delta,\delta+\varepsilon[)$.

may choose t_0 such that $\delta < t_0 < \min_{1\neq w\in W} t_w < d_y$ with $d(x_0,\gamma_z(t_0)) < d(wx_0,\gamma_z(t_0))$ for all $1\neq w\in W$.

For a unit vector $v \in T_{x_0}M$, we let $\gamma^v : \mathbb{R} \to M$ be the geodesic $s \mapsto \mathrm{Exp}_{x_0}(sv)$. The following set

$$\begin{aligned}&\{v \in S_{T_{x_0}M}(0,1)\ ;\ \gamma^v(t_0) \in \overset{\circ}{\mathcal{DV}}\}\\ &= \{v \in S_{T_{x_0}M}(0,1)\ ;\ d(x_0,\gamma^v(t_0)) < d(wx_0,\gamma^v(t_0)),\ \forall 1\neq w\in W\}\end{aligned}$$

is an open neighborhood of $u := \dot{\gamma_z}(0)$ in $S_{T_{x_0}M}(0,1)$. Hence, we may choose $0<\eta<1$ such that

$$\forall v \in S_{T_{x_0}M}(0,1),\ \|u-v\| < \eta \implies \gamma^v(t_0) \in \overset{\circ}{\mathcal{DV}}\ .$$

Since the map Exp_{x_0} is continuous, by shrinking η if needed and as $\gamma_z(t_0) \in B(x,\varepsilon)$, we may assume that $\gamma^v(t_0) \in B(x,\varepsilon)$ for $\|u-v\|<\eta$. As $d(x_0,\gamma^v(t_0)) = t_0 > \delta$, we obtain

$$\forall v \in S_{T_{x_0}M}(0,1),\ \|u-v\| < \eta \implies \gamma^v(t_0) \in \overset{\circ}{\mathcal{DV}} \cap \complement B(x_0,\delta) \cap B(x,\varepsilon) = \gamma_z([\delta,\delta+\varepsilon[).$$

Thus, we can find some $v \neq \pm u$ such that $\mathrm{Exp}_{x_0}(t_0 v) \overset{\mathrm{df}}{=} \gamma^v(t_0) = \gamma_z(s) \overset{\mathrm{df}}{=} \mathrm{Exp}_{x_0}(su)$ for some $\delta \le s < \delta+\varepsilon$. But since $t_0, s \le \delta+\varepsilon < \frac{1}{2}(\rho+\delta) < \rho < \mathrm{inj}_{x_0}(M)$, this implies that $t_0 v = su$ and so u and v are colinear and on the same sphere, a contradiction. □

Remark 17.1.8. *In the case where* $\mathrm{inj}_{x_0}(M) < \infty$*, the existence of some* $0<\rho<\mathrm{inj}_{x_0}(M)$ *such that* $\mathcal{DV} \subset B(x_0,\rho)$ *is equivalent to the assumption* $\mathcal{DV} \subset B(x_0,\mathrm{inj}_{x_0}(M))$*.*

17.2 Some conjectures on a potential method to build a W-cell structure on K/T

Back to the case of flag manifolds, we take as usual K a semisimple compact Lie group and $T < K$ a maximal torus in K, with their Lie algebras $\mathfrak{k}$ and $\mathfrak{t}$ and Weyl group $W = N_K(T)/T$. We fix, once and for all, a *normal homogeneous* metric g on the flag manifold $\mathcal{F} := K/T$, i.e. a metric coming from a bi-invariant metric on K. The analysis given at the beginning of Section 15.3 is still valid in the general case and we see that such a normal homogeneous metric g is unique up to a scalar. The geodesic distance induced by g is simply denoted by

d. Note that we still have the Cartan decomposition $\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{p}$ as in Section 15.3 and we identify $T_1\mathcal{F} \simeq \mathfrak{p}$.

Since the action of W is free, any point is regular and in particular, denoting abusively by 1 the class of 1 in $\mathcal{F}$, we consider the Dirichlet-Voronoi domain

$$\mathcal{DV} := \{x \in \mathcal{F} \ ; \ d(1,x) \leq d(w,x), \ \forall w \in W\}.$$

By the preceding section, $\mathcal{DV}$ is a geodesically star-shaped fundamental domain for W in $\mathcal{F}$. Moreover, as the normal homogeneous metric g on $\mathcal{F}$ is unique up to scalar, the domain $\mathcal{DV}$ doesn't depend on the chosen normal homogeneous metric.

In order to prove that the (intersections of the) walls of $\mathcal{DV}$ do form cells, we pull the situation back to the tangent space $T_1\mathcal{F} = \mathfrak{p}$ via the Riemannian exponential map $\mathrm{Exp} : \mathfrak{p} \to \mathcal{F}$. This last map fits in a commutative square

$$\begin{tikzcd}
\mathfrak{k} \arrow[r, "\exp_K = \mathrm{Exp}_K"] & K \arrow[d, two heads] \\
\mathfrak{p} = T_1\mathcal{F} \arrow[u, hook] \arrow[r, "\mathrm{Exp}"'] & \mathcal{F}
\end{tikzcd}$$

where $\exp_K$ is the Lie group exponential map, which coincides with the Riemannian exponential map Exp_K since the metric on K is bi-invariant.

Of course, in order not to lose information doing this, the exponential map should be injective on $\mathcal{DV}$. As $\mathcal{DV}$ is centered at 1, a natural sufficient condition for this to hold is to have

$$\sup_{x \in \mathcal{DV}} d(1,x) < \mathrm{inj}(\mathcal{F}), \tag{†}$$

with $\mathrm{inj}(\mathcal{F})$ the injectivity radius of $\mathcal{F}$ (which is the same at all points, since $\mathcal{F}$ is a homogeneous space). As we will see, this condition somehow solves the problem with convexity mentioned in Remark 17.1.2.

If this is true, then we can find $0 < \delta < \mathrm{inj}(\mathcal{F})$ such that $\mathcal{DV} \subset \overline{B(1,\delta)}$ and Exp realizes a homeomorphism $\overline{B(1,\delta)} \simeq \overline{B_{\mathfrak{p}}(0,\delta)} = \mathbb{B}^{2N}$, where $\mathbb{B}^{2N}$ is the Euclidean $2N$-ball, with $N := \dim_{\mathbb{C}} \mathcal{F} = |\Phi^+|$ is the number of reflections in W. Thus, we can project the "cells" onto the bounding sphere $S(1,\delta) = \partial\overline{B(1,\delta)} \simeq S_{\mathfrak{p}}(0,\delta)$, just as we did for binary polyhedral groups. More precisely, we consider the map

$$\begin{array}{cccc} \pi_\delta & : & \overline{B(1,\delta)} \setminus \{1\} & \twoheadrightarrow & S_{\mathfrak{p}}(0,\delta) \simeq \mathbb{S}^{2N-1} \\ & & x & \longmapsto & \delta \frac{\mathrm{Exp}^{-1}(x)}{\|\mathrm{Exp}^{-1}(x)\|} \end{array}$$

Geometrically, this can also be defined using geodesics: for $x \in \overline{B(1,\delta)}$, as $d(1,x) < \mathrm{inj}(\mathcal{F})$, there is a unique minimal geodesic γ from 1 to x, which we extend until it meets the sphere $S(1,\delta)$ and the preimage of this point under Exp is $\pi_\delta(x)$. Now, if we have a wall $P_w := \mathcal{DV} \cap H_w$ and if $y = \pi_\delta(x)$ for $x \in P_w$, then the unique minimal geodesic from 1 to $\mathrm{Exp}(y)$ intersects P_w in at least one point and in fact in a unique point by Lemma 17.1.3 so that $x = \pi_\delta^{-1}(y)$ is well-defined and thus π_δ restricts to a homeomorphism $P_w \xrightarrow{\sim} \pi_\delta(P_w) \subset \mathbb{S}^{2N-1}$. We may glue these homeomorphisms together to obtain a homeomorphism

$$\pi_\delta : \partial\mathcal{DV} \xrightarrow{\sim} \mathbb{S}^{2N-1},$$

which in turn restricts to a homeomorphism

$$\partial\mathcal{DV} \cap \mathcal{F}(\mathbb{R}) \xrightarrow{\sim} \mathbb{S}^{N-1}.$$

This approach may also be used to prove that the interior of $\mathcal{DV}$ is a cell (see Proposition 17.1.6 above).

We would like to find a parametrization of a wall using this homeomorphism and thus to prove that the intersections of walls are cells and how to compute their oriented boundary. However, one should be aware that "walls" are not connected in general: they are rather finite unions of cells. We shall see this while we investigate the case of $SL_3(\mathbb{R})$.

However, this is expectable only in the case of the real flag manifold $\mathcal{F}(\mathbb{R})$ as the walls are not unions of cells in the complex case. We have seen above that for the flag manifold $\mathbb{S}^2$ of type A_1, the curve H_s is an equatorial line. On the other hand, if we restrict our attention to the real points, then we indeed obtain cells and an equivariant cell structure. To obtain a structure on the whole manifold $\mathbb{S}^2$, we can take any 0-cell of the structure on the real points and consider again a Dirichlet-Voronoi domain centered at this 0-cell, intersected with H_s to further decompose the wall. We obtain a cell structure on the wall with two 0-cells and two 1-cells and this, together with the 2-cell given by the interior of $\mathcal{DV}$, leads to the well-known C_2-cell structure on $\mathbb{S}^2$.

A similar trick could work in general: for any wall H_w, we take a point in $H_w \cap \mathcal{F}(\mathbb{R})$ and consider a Dirichlet-Voronoi domain in H_w centered at this point, for the action of the subgroup $\mathrm{Stab}_W(H_w)$. The problem is that it is not clear how to choose the centers of the various new domains in the walls. In doing so, we should try to get the least possible number of walls.

We may summarize the above discussion in the following conjectures:

Conjecture 17.2.1. *The Dirichlet-Voronoi domain $\mathcal{DV}$ satisfies the condition (†), i.e.*

$$\mathcal{DV} \subset B(1, \mathrm{inj}(\mathcal{F})).$$

By Proposition 17.1.6, this would imply that $\mathcal{DV}$ is a $2N$-cell. Concerning a cell structure on the flag manifold, we focus on the real points to give a precise statement:

Conjecture 17.2.2. *If Conjecture 17.2.1 is true and if $I \subset W$ is any subset of W, then the relative interior of $\mathcal{DV} \cap \bigcap_{w\in I} H_w$ is a (possibly empty) union of $(N - |I|)$-cells*

Finally, inspired by the case of $SU(n)/T$, which we investigate below, we conjecture that the injectivity radius has the following Lie theoretic meaning:

Conjecture 17.2.3. *The injectivity radius of $\mathcal{F} = K/T$ is the minimal distance between two elements of the Weyl group W. Moreover, it is equal to $d(1, r)$, for some reflection r of W.*

17.3 The case of $SU(n)$ and the link with the Fubini-Study metric

In the case $K = SU(n)$ and $T = S(U(1)^n)$ the subgroup of diagonal matrices in $SU(n)$, the flag manifold is $\mathcal{F}_n := SU(n)/T$ and the Killing form is given by

$$\forall X, Y \in \mathfrak{su}(n),\ \kappa(X, Y) = 2n\,\mathrm{tr}(XY)$$

and the induced norm on $\mathfrak{su}(n)$ is $\sqrt{2n}$ times the Frobenius norm $\|\cdot\|_F$ and we equip $\mathcal{F}_n$ with the associated normal homogeneous metric g_n. First some good news: we can compute $\mathrm{inj}(\mathcal{F}_n(\mathbb{R}))$!

Proposition 17.3.1. *For $n \geq 2$, we have*

$$\mathrm{inj}(\mathcal{F}_n(\mathbb{R}), g_n) = \pi\sqrt{n}.$$

Moreover, this is the distance between 1 and any reflection in $W = \mathfrak{S}_n$.

Proof. We use Klingenberg's Lemma [Car92, Chapter 13, Proposition 2.13] (or [Kli82, Proposition 2.6.8]) to estimate the injectivity radius. For this, we have to bound the length of a closed geodesic and the sectional curvature. More precisely, if the sectional curvature $K_{\mathcal{F}_n(\mathbb{R})}$ of $\mathcal{F}_n(\mathbb{R})$ satisfies $K_{\mathcal{F}_n(\mathbb{R})} \leq K_{\max}$ and if ℓ is the minimal length of a closed geodesic in $\mathcal{F}_n(\mathbb{R})$, then we have

$$\mathrm{inj}(\mathcal{F}_n(\mathbb{R})) \geq \min\left(\frac{\pi}{\sqrt{K_{\max}}}, \frac{\ell}{2}\right).$$

- For $Z \in \mathfrak{su}_n = \mathfrak{t} \oplus \mathfrak{p}$, we denote by $Z^{\mathfrak{t}} \in \mathfrak{t}$ and $Z^{\mathfrak{p}} \in \mathfrak{p}$ be the only two elements such that $Z = Z^{\mathfrak{t}} + Z^{\mathfrak{p}}$. By O'Neill's formula [GHL04, Theorem 3.61] and using the fact that the metric on $SU(n)$ is bi-invariant, the sectional curvature of $\mathcal{F}_n$ (see also [GHL04, Theorem 3.65]) is

  $$K_{\mathcal{F}_n}(X,Y) = \frac{1}{4}\|[X,Y]^{\mathfrak{p}}\|^2 + \|[X,Y]^{\mathfrak{t}}\|^2 = \frac{1}{4}\|[X,Y]\|^2 + \frac{3}{4}\|[X,Y]^{\mathfrak{t}}\|^2,$$

  where (X,Y) is a pair of orthonormal vectors in $\mathfrak{p} = T_1\mathcal{F} = \mathfrak{t}^{\perp}$ and $\|\cdot\| = \sqrt{2n}\|\cdot\|_F$ is the norm on $\mathfrak{p}$. First notice that we have $K_{\mathcal{F}_n}(X,Y) \leq \frac{1}{4}\|[X,Y]\|^2 + \frac{3}{4}\|[X,Y]\|^2 = \|[X,Y]\|^2$ for any such pair (X,Y). On the other hand, there is a sharp estimate of the Frobenius norm of a commutator of matrices proved in [BW08, Theorem 2.2]:

  $$\forall A, B \in \mathcal{M}_n(\mathbb{C}),\ \|[A,B]\|_F \leq \sqrt{2}\|A\|_F\|B\|_F.$$

  Thus we obtain

  $$0 \leq K_{\mathcal{F}_n}(X,Y) \leq \|[X,Y]\|^2 = 2n\|[X,Y]\|_F^2 \leq \frac{1}{n}.$$

  The same argument works for $\mathcal{F}_n(\mathbb{R})$ and we have $0 \leq K_{\mathcal{F}_n(\mathbb{R})} \leq \frac{1}{n}$.

- First notice that if $\widetilde{\gamma} : t \mapsto e^{tX}$ is a closed geodesic in $SU(n)$ such that $e^X = 1$ and $X \neq 0$, then there is at least one eigenvalue of X with module at least 2π and since the norm $\|X\|_F$ is the 2-norm of the vector of eigenvalues of X (with multiplicities), we get $\|X\|_F \geq 2\pi$. This implies that $\mathrm{L}_{SU(n)}(\widetilde{\gamma}) \geq 2\pi\sqrt{2n} > 2\pi\sqrt{n}$. As the submanifold $SO(n) \subset SU(n)$ is totally geodesic, this also holds for $SO(n)$. Now we have

  $$S(O(1)^n) = \{\mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)\ ;\ \epsilon_i = \pm 1,\ \epsilon_1\epsilon_2\cdots\epsilon_n = 1\}$$

  and

  $$\min_{1 \neq t \in S(O(1)^3)} d_{SU(n)}(1,t) = d_{SU(n)}(1, \mathrm{diag}(-1,-1,1,\ldots,1))$$
  $$= \sqrt{2n}\left\| \begin{pmatrix} \begin{smallmatrix} 0 & \pi \\ -\pi & 0 \end{smallmatrix} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \right\|_F = 2\pi\sqrt{n}$$

  and therefore, if $\gamma : [0,1] \to \mathcal{F}_n(\mathbb{R})$ is a closed geodesic such that $\gamma(0) = \gamma(1) = 1$, then $\mathrm{L}(\gamma) \geq 2\pi\sqrt{n}$. This, together with the above observation on closed geodesics of $SU(n)$, implies that any closed geodesic in $\mathcal{F}_n(\mathbb{R})$ has length at least $2\pi\sqrt{n}$.

Combining the two estimates above and applying Klingenberg's Lemma, we obtain

$$\mathrm{inj}(\mathcal{F}_n(\mathbb{R})) \geq \pi\sqrt{n}.$$

Besides, let $X := \mathrm{diag}\left(\left(\begin{smallmatrix} 0 & \pi \\ -\pi & 0 \end{smallmatrix}\right), 0, \dots, 0\right)$ be the above matrix and $\gamma : t \mapsto e^{tX} \cdot S(O(1)^3)$. Then $t \mapsto \gamma(t/2)$ and $t \mapsto \gamma(1 - t/2)$ are two distinct minimal geodesics from 1 to $\dot{s_{\alpha_1}}$ of length $\pi\sqrt{n}$ and thus $\mathrm{inj}(\mathcal{F}_n(\mathbb{R})) \leq \pi\sqrt{n}$. $\square$

Remark 17.3.2. *Notice that this agrees with [Püt04]. Using the fact that $\mathcal{F}_3$ is of positive curvature, he also was able to give the injectivity radius of $\mathcal{F}_3(\mathbb{C})$. However, this approach cannot be generalized as the other flag manifolds (apart from types A_1 and A_2) only have non-negative curvature, rather than positive curvature, see [Wal72]. See also [WZ18] for a complete classification of simply-connected compact homogeneous spaces of positive curvature.*

On another hand, we notice that the above proof can be extended to the complex case as soon as the following elementary conjecture holds:

Conjecture 17.3.3. *For $X \in \mathfrak{su}(n)$ such that $\|X\|_F < \sqrt{2}\pi$, if $e^X \in T$ then $X \in \mathfrak{t}$. In particular, this implies that* $\mathrm{inj}(\mathcal{F}_n, g_n) = \pi\sqrt{n}$.

Unfortunately, we were not able to prove it so far. We just have numerically verified it using Maple [Map19]. Still, we have the following weaker result:

Lemma 17.3.4. *For $n \geq 2$, the injectivity radius of $\mathcal{F}_n$ satisfies*

$$\mathrm{inj}(\mathcal{F}_n, g_n) \geq \pi\sqrt{\frac{n}{2}}.$$

Proof. Indeed, since we have already seen that the curvature satisfies $0 \leq K_{\mathcal{F}_n} \leq \frac{1}{n}$, it suffices to prove that the length of a closed geodesic (based in 1, say) is at least π. Indeed, letting ρ denote the *spectral radius* of a matrix, by [LM19, Example 5.13], the (Lie group) exponential map of $SU(n)$ is injective on the Ad-invariant subset $\{X \in \mathfrak{su}(n) \ ; \ \rho(X) < \pi\}$ and thus if $X \in \mathfrak{p} \subset \mathfrak{su}(n)$ is such that $e^X \in T$, say $e^X = e^{t_0}$ with $t_0 \in \mathfrak{t}$ and if $\rho(X) < \pi$, then $\rho(t_0) < \pi$ too and thus $X = t_0 \in \mathfrak{t}$, so $X \in \mathfrak{t} \cap \mathfrak{p}$ and $X = 0$. Therefore, if $t \mapsto e^{tX} \cdot T$ is a closed geodesic such that $\gamma(0) = \gamma(1) = 1$, then $\rho(X) \geq \pi$ and so $\|X\|_F = \sqrt{\sum_{\lambda \in \mathrm{Sp}(X)} |\lambda|^2} \geq \rho(X) \geq \pi$, as required. $\square$

The Conjecture 17.2.1 seems really hard to decide here, because the expression of the distance is very unhandy. We will see that there is another distance on $\mathcal{F}_n$ induced by a (product of) Fubini-Study metric, which seems to be a good "approximation" of the intrinsic distance d on $\mathcal{F}_n$.

More precisely, if $k \in SU(n)$, then the columns $[k_{i,1}]_i, \dots, [k_{i,n}]_i$ define a family of pairwise orthogonal lines in $\mathbb{C}^n$ and the resulting map $SU(n) \to (\mathbb{CP}^{n-1})^n$ induces an embedding

$$\iota : \mathcal{F}_n \hookrightarrow (\mathbb{CP}^{n-1})^n.$$

Furthermore, the space $(\mathbb{CP}^{n-1})^n$ is endowed with a natural metric: the product metric of the Fubini-Study metric on each copy of $\mathbb{CP}^{n-1}$. It turns out that this metric is compatible with the metric on $\mathcal{F}_n$. More precisely, we have the following result:

Lemma 17.3.5. *If g_{FS} denotes the product metric on $(\mathbb{CP}^{n-1})^n$ induced by the Fubini-Study metric on each factor, then the restriction of g_{FS} to $\mathcal{F}_n \overset{\iota}{\hookrightarrow} (\mathbb{CP}^{n-1})^n$ is proportional to the normal homogeneous metric g_n on $\mathcal{F}_n$. More precisely,*

$$g_n = 2n\ \iota^* g_{FS}.$$

Proof. Denote by $\langle \cdot, \cdot \rangle$ the usual Hermitian product on $\mathbb{C}^n$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}} := \Re(\langle \cdot, \cdot \rangle)$ the associated dot product. Denote also by $(e_i)_{i=1,\dots,n}$ the canonical basis of $\mathbb{C}^n$. We have

$$T_{\mathrm{Id}}((\mathbb{CP}^{n-1})^n) = \bigoplus_{i=1}^{n} \mathrm{span}(e_i)^{\perp} \simeq \left\{ \begin{pmatrix} 0 & & (*) \\ & \ddots & \\ (*) & & 0 \end{pmatrix} \right\},$$

that is, $T_{\mathrm{Id}}((\mathbb{CP}^{n-1})^n)$ can be identified with the space of matrices with zero diagonal and the subspace $T_1\mathcal{F}_n = \mathfrak{p}$ is given by the skew-Hermitian matrices with zero diagonal. By definition of the product metric, we have

$$\forall X = (x_{i,j}), Y = (y_{i,j}) \in T_{\mathrm{Id}}((\mathbb{CP}^{n-1})^n),\ g_{FS}(X, Y) = \sum_{j=1}^{n} \langle (x_{i,j})_i, (y_{i,j})_i \rangle_{\mathbb{R}} = \sum_{1 \leq i,j \leq n} \Re(x_{i,j}\overline{y_{i,j}}).$$

Now, if X and Y are skew-Hermitian (i.e. if $\overline{x_{i,j}} = -x_{j,i}$ and similarly for Y), then we have

$$\begin{aligned} g_{FS}(X, Y) &= \sum_{1 \leq i,j \leq n} \Re(x_{i,j}\overline{y_{i,j}}) = -\sum_{i,j} \Re(x_{i,j}y_{j,i}) = -\Re(\mathrm{tr}(XY)) = -\tfrac{1}{2}(\mathrm{tr}(XY) + \mathrm{tr}(\overline{XY})) \\ &= -\tfrac{1}{2}(\mathrm{tr}(XY) + \mathrm{tr}({}^tX{}^tY)) = -\mathrm{tr}(XY) = \frac{\kappa(X, Y)}{2n} = \frac{g_n(X, Y)}{2n}. \end{aligned}$$

□

Corollary 17.3.6. *If d_{FS} denotes the geodesic distance on $(\mathbb{CP}^{n-1})^n$ induced by g_{FS}, then*

$$\forall x, y \in \mathcal{F}_n,\ d_{FS}(x, y) \leq d(x, y).$$

By [Mic19], it is known that the two distances d and d_{FS} are equivalent on $\mathcal{F}_n$. We have a candidate for a bound for the case of real points and more precisely, we formulate the next conjecture:

Conjecture 17.3.7. *The following holds*

$$\forall x, y \in \mathcal{F}_n(\mathbb{R}),\ d_{FS}(x, y) \leq d(x, y) \leq \sqrt{2} d_{FS}(x, y).$$

This would suggest that the distances are close enough to consider the following domain instead of the classical Dirichlet-Voronoi domain associated with g_n:

$$\mathcal{DV}_{FS} := \{x \in \mathcal{F}_n(\mathbb{R})\ ;\ d_{FS}(1, x) \leq d_{FS}(w, x),\ \forall w \in \mathfrak{S}_n\}.$$

Though we haven't found a proof of it yet, it should be easier to prove that $\max_{x \in \mathcal{DV}_{FS}} d_{FS}(1, x)$ is smaller than $\mathrm{inj}(\mathcal{F}_n(\mathbb{R}), g_{FS})$ than with the distance d. Indeed, we have the following simple formulas:

Lemma 17.3.8. *The injectivity radius of the metric g_{FS} on $(\mathbb{CP}^{n-1})^n$ is given by*

$$\operatorname{inj}((\mathbb{CP}^{n-1})^n, g_{FS}) = \frac{\pi}{2}.$$

Moreover, if $x = AT \in \mathcal{F}_n$ with $A = (a_{i,j})_{i,j} \in SU(n)$, then

$$d_{FS}(1, x) = \sqrt{\sum_{i=1}^{n} \arccos(|a_{i,i}|)^2}.$$

In particular, for $w \in \mathfrak{S}_n$ we have

$$d_{FS}(w, x) = \sqrt{\sum_{w_{i,j} \neq 0} \arccos(|a_{i,j}|)^2} = \sqrt{\sum_{i=1}^{n} \arccos(|a_{w(i),i}|)^2}.$$

Proof. It is clear that the injectivity radius of a product metric is the minimum of the injectivity radii of the factors. On the other hand, by O'Neill's formula, the curvature of $\mathbb{CP}^{n-1}$ is given by

$$K_{\mathbb{CP}^{n-1}}(X, Y) = 1 + 3 \left\langle \overline{X}, Y \right\rangle_{\mathbb{R}}^2 \in [1, 4]$$

and since $\mathbb{CP}^{n-1}$ is simply-connected, Klingenberg's special estimate ([Kli82, Theorem 2.6.A.1]) implies that $\operatorname{inj}(\mathbb{CP}^{n-1}) = \frac{\pi}{2}$, hence the first statement. For the second statement, recall that the Fubini-Study distance d_{FS} on $\mathbb{CP}^{n-1}$ satisfies

$$d_{FS}(\mathbb{C}p, \mathbb{C}q) = \arccos \frac{|\langle p, q \rangle|}{|p||q|}.$$

Therefore, we obtain

$$d_{FS}(1, AT)^2 = \sum_{j=1}^{n} d_{FS}(e_j, (a_{i,j})_i)^2 = \sum_{j=1}^{n} \arccos(|\langle e_j, (a_{i,j})_i \rangle|)^2 = \sum_{j=1}^{n} \arccos(|a_{j,j}|)^2.$$

$\square$

The equality $g_n = 2n\ \iota^* g_{FS}$ in Lemma 17.3.5 relates two natural metrics to consider on $\mathcal{F}_n$. However, the calculations involving d_{FS}, though a bit more manageable than the ones with d, are still hard to handle and even if the injectivity radius condition on $\mathcal{DV}_{FS}$ is true, it remains to compare the distances d and d_{FS} precisely as in the later Conjecture above. Hence, it is not clear if working with d_{FS} is efficient enough for our purpose.

17.4 The particular case of $\mathcal{F}_3(\mathbb{R}) \simeq SO(3)/S(O(1)^3)$

In this last section, we use the Dirichlet-Voronoi domain $\mathcal{DV}$ for $\mathcal{F}_3(\mathbb{R})$ to construct a third $\mathfrak{S}_3$-equivariant cell structure on it. We use the metric induced by $\mathbb{S}^3$ on $\mathcal{F}_3(\mathbb{R})$ to do the calculations, as the associated distance is far easier to handle. Throughout this section, we denote by $\mathcal{DV}_3$ the Dirichlet-Voronoi domain for $\mathfrak{S}_3$ acting on $\mathcal{F}_3(\mathbb{R})$. We first determine the maximal value of the function $d(1, -)$ on $\mathcal{DV}_3$. Recall that we have the projection $\phi : \mathbb{S}^3 \twoheadrightarrow \mathcal{F}_3(\mathbb{R}) = \mathbb{S}^3/\mathcal{Q}_8$ from Lemma 14.0.2.

Lemma 17.4.1. *The radius*

$$\max_{x\in\mathcal{DV}_3} d(1,x) = 4\sqrt{3}\arccos\left(\frac{1}{2}+\frac{\sqrt{2}}{4}\right) =: \delta_0$$

of $\mathcal{DV}_3$ is smaller than the injectivity radius $\pi\sqrt{3}$, i.e. Conjecture 17.2.1 holds and in particular the interior $\overset{\circ}{\mathcal{DV}_3}$ is a 3-cell. Moreover, this maximum is attained by the following twenty-four points

$$\phi\left(\tfrac{1}{2}+\tfrac{\sqrt{2}}{4}+bi+cj+dk\right),$$

where (b,c,d) is any permutation of $\left(\pm\frac{\sqrt{2}}{4},\pm\frac{\sqrt{2}}{4},\pm\left(\frac{1}{2}-\frac{\sqrt{2}}{4}\right)\right)$.

Proof. First, we have to prove that $\mathcal{DV}_3 \subset B(1,\mathrm{inj}(\mathcal{F}_3(\mathbb{R}))) = B(1,\pi\sqrt{3})$. Recall from Corollary 15.4.3 that (after normalization) the distance from 1 to an element $\phi(q)\in\mathcal{F}_3(\mathbb{R})$, where $q=a+bi+cj+dk\in\mathbb{S}^3$, with respect to the metric $g_3 = 48\overline{g}^8$ is given by

$$d(1,\phi(q)) = 4\sqrt{3}\min_{x=a,b,c,d}(\arccos(|x|)).$$

On the other hand, the projection ϕ takes the elements $\frac{1}{\sqrt{2}}(1+k)$, $\frac{1}{\sqrt{2}}(1+i)$ and $\frac{1}{\sqrt{2}}(1+j)$ to the reflections s_α, s_β and $w_0 = s_{\alpha+\beta}$ of $\mathfrak{S}_3$ respectively and thus, for $q=a+bi+cj+dk$, a direct computation shows that

$$\begin{cases} d(s_\alpha,\phi(q)) = \min_{x=a\pm d,b\pm c}(\arccos(|x|/\sqrt{2})),\\ d(s_\beta,\phi(q)) = \min_{x=a\pm b,c\pm d}(\arccos(|x|/\sqrt{2})),\\ d(s_{\alpha+\beta},\phi(q)) = \min_{x=a\pm c,b\pm d}(\arccos(|x|/\sqrt{2})).\end{cases}$$

Take $q=a+bi+cj+dk\in\mathbb{S}^3$ such that $\phi(q)\in\mathcal{DV}_3$. In particular, the inequalities $d(1,\phi(q))\le d(s,\phi(q))$ are valid for any reflection $s\in\mathfrak{S}_3$. Up to multiplication by an element of $\mathcal{Q}_8=\{\pm1,\pm i,\pm j,\pm k\}$, we may assume that $|a|\ge|b|,|c|,|d|$ and that $a>0$ in such a way that $d(1,\phi(q)) = 4\sqrt{3}\arccos(a)$. Therefore, taking the cosine of the inequalities associated to the reflections yields among other inequalities the following system

$$\begin{cases} a\ge(a\pm b)/\sqrt{2},\\ a\ge(a\pm c)/\sqrt{2},\\ a\ge(a\pm d)/\sqrt{2},\end{cases} \iff \begin{cases} a(\sqrt{2}-1)\ge\pm b,\\ a(\sqrt{2}-1)\ge\pm c,\\ a(\sqrt{2}-1)\ge\pm d.\end{cases}$$

Hence, we obtain

$$1-a^2 = b^2+c^2+d^2 \le 3a^2(\sqrt{2}-1)^2 \implies 1\le a^2(10-6\sqrt{2})$$

and thus

$$d(1,\phi(q)) = 4\sqrt{3}\arccos(a) \le 4\sqrt{3}\arccos\left(\frac{1}{\sqrt{10-6\sqrt{2}}}\right) \approx 4.31166 < \pi\sqrt{3}\approx 5.4414,$$

as required.

Now that we know that $\mathcal{DV}_3\subset B(1,\mathrm{inj}(\mathcal{F}_3(\mathbb{R})))$, we want to apply Lemma 17.1.7. To do this, it is enough to show that there is at most a finite number of $x\in\mathcal{DV}_3$ such that $d(1,x)$ is equal to δ_0 and to find them. As above, if $\phi(q) \in \mathcal{DV}_3$ for $q = a+bi+cj+dk \in \mathbb{S}^3$ for which we assume that $a \geq |b|, |c|, |d|$, we have $a(\sqrt{2}-1) \geq |b|, |c|, |d|$ and similar computations for $s_\alpha s_\beta$ and $s_\beta s_\alpha$ show that we also have

$$2a \geq \max_{x=a\pm b\pm c\pm d} |x|.$$

Specializing at $a = 1/2 + \sqrt{2}/4$ (so that $d(1,x) = \delta_0$), we obtain the following system of inequalities

$$\begin{cases} |b|, |c|, |d| \leq \sqrt{2}/4, \\ |b \pm c \pm d| \leq 1/2 + \sqrt{2}/4, \\ b^2 + c^2 + d^2 = \frac{1}{8}(5 - 2\sqrt{2}) \end{cases}$$

and this defines a full truncated cube, intersected with a sphere, as depicted in Figure 22.

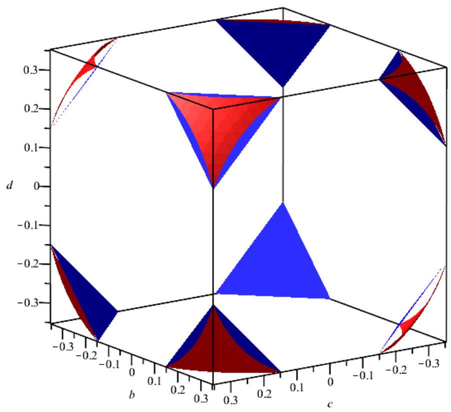


Figure 22: The blue triangles are faces of the truncated cube and the red parts describe the intersection of the sphere with the full cube.

This full truncated cube is the convex hull of its twenty-four vertices listed in the statement, and these lie on the sphere. As the Euclidean norm on $\mathbb{R}^3$ is uniformly convex, any point in this truncated cube which is not a vertex belongs to the open ball of radius $\frac{1}{8}(5 - 2\sqrt{2})$, hence is not on the sphere. Therefore, the considered intersection consists exactly of the twenty-four points of the statement. $\square$

Theorem 17.4.2. *For each $I \subset \mathfrak{S}_3$, the relative interior of $\mathcal{DV}_3 \cap \bigcap_{w\in I} H_w$ is either empty or a union of $(3-|I|)$-cells. In other words, Conjecture 17.2.2 holds for $\mathcal{F}_3(\mathbb{R})$. The resulting cellular decomposition of $\mathcal{DV}_3$ induces an $\mathfrak{S}_3$-equivariant cellular structure on the real flag manifold $\mathcal{F}_3(\mathbb{R})$, whose cellular homology chain complex*

$$\mathbb{Z}[\mathfrak{S}_3] \xrightarrow{\partial_3} \mathbb{Z}[\mathfrak{S}_3]^7 \xrightarrow{\partial_2} \mathbb{Z}[\mathfrak{S}_3]^{12} \xrightarrow{\partial_1} \mathbb{Z}[\mathfrak{S}_3]^6$$

has boundaries given by (left) multiplication by the following matrices

$$\partial_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & s_\beta & -s_\beta & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & s_\beta s_\alpha & 0 & 0 & -1 \\ -w_0 & 0 & 0 & 0 & 0 & 0 & 0 & w_0 & 0 & s_\beta & -w_0 & 0 \\ s_\beta s_\alpha & -s_\beta s_\alpha & 0 & 0 & s_\alpha & -s_\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & s_\beta s_\alpha & -s_\beta s_\alpha & 0 & 0 & 0 & 0 & 0 & -w_0 & 0 & w_0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & s_\beta & -s_\beta & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\partial_2 = \begin{pmatrix} 1 & 0 & w_0 & 0 & 0 & 0 & -w_0 \\ 1 & -s_\alpha s_\beta & 0 & 0 & -1 & 0 & 0 \\ 1 & s_\beta & 0 & -s_\beta & 0 & 0 & 0 \\ 1 & 0 & s_\alpha & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & -w_0 & 0 & 0 \\ 1 & s_\alpha & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -s_\beta & 0 \\ 1 & 0 & -s_\beta s_\alpha & -1 & 0 & 0 & 0 \\ 0 & -1 & -w_0 & 0 & -s_\beta & 0 & 0 \\ 0 & s_\beta & 1 & 0 & 0 & 0 & -s_\beta \\ 0 & w_0 & -w_0 & -1 & 0 & 0 & 0 \\ 0 & -s_\beta s_\alpha & 1 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad \partial_3 := \begin{pmatrix} 1-s_\alpha \\ 1-s_\beta \\ 1-w_0 \\ 1-s_\beta s_\alpha \\ 1-s_\alpha s_\beta \\ 1-s_\beta s_\alpha \\ 1-s_\alpha s_\beta \end{pmatrix}.$$

Proof. First observe that, for $p := (x, y, z) \in \mathbb{R}^3$ such that $r^2 := x^2 + y^2 + z^2 < \pi$, we have

$$\exp\left(t\begin{pmatrix} 0 & x & z \\ -x & 0 & y \\ -z & -y & 0 \end{pmatrix}\right) = \frac{1}{r^2}\begin{pmatrix} (x^2+z^2)\cos(tr)+y^2 & yz(\cos(tr)-1)+xr\sin(tr) & xy(1-\cos(tr))+zr\sin(tr) \\ yz(\cos(tr)-1)-xr\sin(tr) & (x^2+y^2)\cos(tr)+z^2 & xz(\cos(tr)-1)+yr\sin(tr) \\ xy(1-\cos(tr))-zr\sin(tr) & xz(\cos(tr)-1)-yr\sin(tr) & (y^2+z^2)\cos(tr)+x^2 \end{pmatrix}$$

projects modulo $S(O(1)^3)$ to the minimal geodesic $\gamma_{(x,y,z)}$ from 1 to $\exp\begin{pmatrix} 0 & x & z \\ -x & 0 & y \\ -z & -y & 0 \end{pmatrix}$ inside $\mathcal{F}_3(\mathbb{R})$. Moreover, the map $\mathbb{S}^3 \to SO(3)$ is a diffeomorphism on a neighbourhood of $(1, 0, 0, 0)$ to a neighbourhood of the identity in $SO(3)$ and composing the above geodesic with the inverse of this local diffeomorphism gives a map

$$\begin{array}{cccc} \Psi_q & : &]0,1] & \longrightarrow & \mathbb{S}^3 \\ & & t & \longmapsto & \frac{\sin(tr)}{r\sqrt{2\cos(tr)+2}}\left(\frac{r(1+\cos(tr))}{\sin(tr)}, y, z, x\right) \end{array}$$

Notice that if we have a vertex $\frac{1}{2} + \frac{\sqrt{2}}{4} + bi + cj + dk$ as in Lemma 17.4.1, we can compute its image in $SO(3)$ and take the logarithm of the result (using the Rodrigues formula for instance). The resulting twenty-four points of $\operatorname{Exp}^{-1}(\mathcal{DV}_3) \subset \mathbb{R}^3$ are denoted by $p_{1,2,3}^{\pm,\pm,\pm}$, according to which one of the coefficients (b, c, d) is $\pm\left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right)$ and to their signs. For example, the vertex $\frac{1}{2} + \frac{\sqrt{2}}{4} - \left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right)i + \frac{\sqrt{2}}{4}j - \frac{\sqrt{2}}{4}k$ is sent to $p_1^{-,+,-}$. Then, the points $p_{1,2,3}^{\pm,\pm,\pm}$ are the vertices of a *truncated cube* (see Figure 24).

We now prove that the walls $H_w \cap \mathcal{DV}_3$ are unions of cells. To do this we choose a triangulation of each of the octagonal faces of this truncated cube using six triangles, like in Figure 23. This leads to a triangulation of the boundary of the truncated cube $\partial\operatorname{conv}(p_{1,2,3}^{\pm,\pm,\pm})$

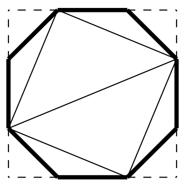


Figure 23: Triangulation of an octagon using six triangles.

since the other faces already are triangles. Fix a triangle of this triangulation, with vertices p, p' and p'', say. Take also a point $p := \lambda p + \mu p' + (1 - \lambda - \mu)p''$ in the open triangle (i.e. $0 < \lambda, \mu, \lambda + \mu < 1$). We project it to the point $p_0 := \delta_0 p/|p|_2 \in S(0, \delta_0)$ and consider the minimal geodesic γ_{p_0} such that $\gamma_{p_0}(0) = 1$ and $\gamma_{p_0}(1) = p_0$. There is a unique $0 < t_0 \leq 1$ such that $\gamma_{p_0}(t_0) \in \partial\mathcal{DV}_3$ and there is a unique $w \in \mathfrak{S}_3$ such that $\gamma_{p_0}(t_0) \in H_w$. This can be seen as follows: from the formula defining the map Ψ_{p_0} and the equations for the distances $d(1, \Psi_{p_0}(t)\omega)$, where $\omega \in \{\frac{1}{2}(1 \pm i + j + k), \frac{1}{\sqrt{2}}(1+i), \frac{1}{\sqrt{2}}(1+j), \frac{1}{\sqrt{2}}(1+k)\}$ describes a system of representatives of elements of $\mathfrak{S}_3$ in $\mathbb{S}^3$. Now, if $\gamma_{p_0}(t) \in H_w$ for some w and some $0 < t < 1$, then t is unique by Lemma 17.1.3 and may be explicitly determined, as a function depending on the triplet (λ, μ, w). Then, t_0 is given by

$$t_0 = \min\{0 < t < 1 \ ; \ \exists w \in \mathfrak{S}_3 \ ; \ \gamma_{p_0}(t) \in H_w\}.$$

After elementary but tedious computations we will omit, we find that there is a unique $w \in \mathfrak{S}_3$ such that $\gamma_{p_0}(t_0) \in H_w$ for each p in the open triangle with vertices p, p' and p'' and that this holds for any open triangle of the triangulation of $\mathrm{conv}(p_{1,2,3}^{\pm,\pm,\pm})$. Therefore, the relative interior of each facet of $\mathrm{conv}(p_{1,2,3}^{\pm,\pm,\pm})$ projects to a 2-cell in $S(0,\delta_0) \simeq \mathbb{S}^2$ and the image of this 2-cell under the inverse homeomorphism $\pi_{\delta_0}^{-1} : S(0,\delta_0) \xrightarrow{\sim} \partial\mathcal{DV}_3$ lands in the relative interior of a unique wall $H_w \cap \mathcal{DV}_3$. As the obtained 2-cells glue together to form a cell decomposition of $\mathbb{S}^2$, we obtain that for each $w \in \mathfrak{S}_3$ the relative interior of the wall $H_w \cap \mathcal{DV}_3 \simeq \pi_{\delta_0}(H_w \cap \mathcal{DV}_3)$ is a disjoint union of two 2-cells in $\mathbb{S}^2$ if w is a reflection, and of four 2-cells if w has length 2. By taking the closures of these cells we obtain the 1-cells and the stated $\mathfrak{S}_3$-equivariant cellular structure on $\mathcal{F}_3(\mathbb{R})$.

The computation of the orbits and boundaries is routine, using the twenty-four vertices of the fundamental domain $\mathcal{DV}_3$. The detailed combinatorics of the cellular decomposition of $\mathcal{DV}_3$ can be found in Appendix C. $\square$

The domain $\mathcal{DV}_3$ can be represented by $\mathrm{Exp}^{-1}(\mathcal{DV}_3) \subset \mathbb{R}^3$ as in the following picture:

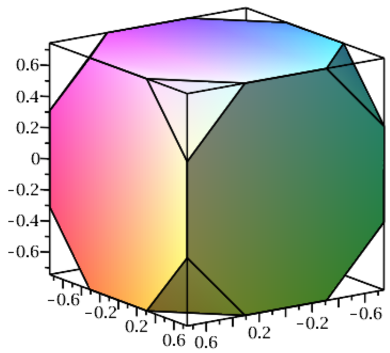


Figure 24: The truncated cube $\mathrm{conv}(p_{1,2,3}^{\pm,\pm,\pm}) \simeq \mathrm{Exp}^{-1}(\mathcal{DV}_3) \subset \mathbb{R}^3$.

Remark 17.4.3. *We finish with the following observations:*

- *In the truncated cube in Figure 24, each face corresponds to a connected component of some H_w for $w \in \mathfrak{S}_3$ and we have observed experimentally that the edges are minimal geodesics in $\mathcal{F}_3(\mathbb{R})$.*
- *It could also be interesting to look for some kind of "dual" of this decomposition; something like a* Delaunay triangulation *in the sense of [LL00]. The combinatorics of the resulting chain complex could be nicer but it is not clear at all how to achieve this.*

A Hyperbolic extensions of finite irreducible Weyl groups

As the irreducible hyperbolic Coxeter groups have rank ≤ 10 and are all classified (see [Che69]), we can check each reflection of each irreducible finite Weyl group to see which one of them give hyperbolic extensions. There may be other possible reflections and extensions, but the resulting Coxeter diagram must appear in the following table. The computations were made using [GAP4]. Of course, for the case of $G_2 = I_2(6)$, we find in particular the diagram corresponding to $\widehat{I_2(6)}$ defined above.

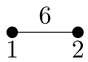
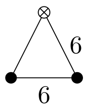
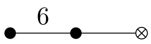
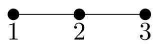
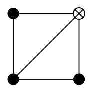
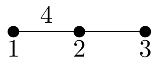
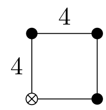
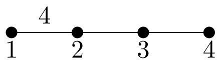
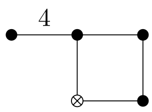
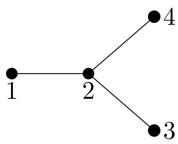
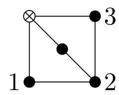
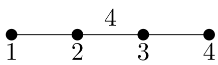
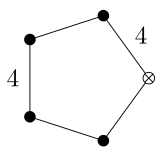
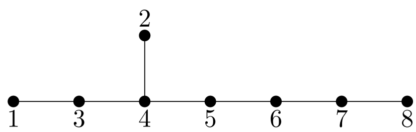
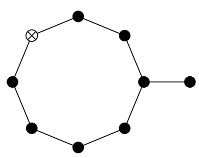
Type	Dynkin diagram	Hyperbolic diagram	Reflection	Compact?
G_2		 and 	$s_1^{s_2}$ and $s_1^{s_2 s_1}$	both
A_3			$s_3^{s_2}$	no
C_3			$s_2^{s_1}$	no
C_4			$s_3^{s_1^{s_2}} = s_3^{s_2 s_1 s_2}$	no
D_4			$s_4^{s_2 s_3}$	no
F_4			$s_3^{s_2 s_1}$	no
E_8			$s_7^{(s_3 s_2)^{s_4 s_5 s_6} (s_5 s_2)^{s_4 s_3 s_1}}$	no

Table 6: Hyperbolic extensions of finite irreducible Weyl groups.

B Presentations of $\pi_1(\mathbf{T}(H_3))$ and $\pi_1(\mathbf{T}(H_4))$

In this appendix, we make the presentation of $\pi_1(\mathbf{T}(W))$ from Theorem 5.4.4 more explicit in the cases of H_3 and H_4.

We write $\Psi_3 =: \{f_i,\ 1 \leq i \leq 30\}$ for the set of (abstract) generators of $\pi_1(\mathbf{T}(H_3))$. We obtain the presentation

$$\pi_1(\mathbf{T}(H_3)) = \langle \Psi_3 \mid R_3 \rangle ,$$

where

$$\begin{gathered}
R_3 := \{f_{22}f_{25}^{-}f_{4}^{-},\ f_{10}^{-}f_{12}^{-}f_{11},\ f_{1}f_{25}f_{20},\ f_{2}f_{26}f_{21},\ f_{1}f_{29}f_{18},\ f_{3}f_{28}f_{16},\ f_{2}f_{30}f_{19},\ f_{3}f_{27}f_{17},\ f_{5}f_{26}f_{23}^{-},\ f_{5}f_{6}^{-}f_{12}, \\
f_{15}f_{13}f_{14}^{-},\ f_{4}f_{7}^{-}f_{11},\ f_{16}f_{14}^{-}f_{21}^{-},\ f_{17}f_{13}^{-}f_{20}^{-},\ f_{10}f_{8}f_{9}^{-},\ f_{28}^{-}f_{9}^{-}f_{23},\ f_{29}f_{24}^{-}f_{7},\ f_{27}^{-}f_{8}^{-}f_{22},\ f_{6}^{-}f_{24}f_{30}^{-},\ f_{15}f_{18}^{-}f_{19}, \\
f_{23}f_{19}^{-}f_{12}f_{21}f_{24}^{-},\ f_{22}f_{16}^{-}f_{10}f_{17}f_{23}^{-},\ f_{28}^{-}f_{20}f_{21}^{-}f_{27}f_{15}^{-},\ f_{1}f_{8}f_{13}^{-}f_{4}^{-}f_{3}^{-},\ f_{2}f_{9}f_{14}^{-}f_{5}^{-}f_{3}^{-},\ f_{4}f_{30}^{-}f_{10}f_{29}f_{5}^{-}, \\
f_{27}^{-}f_{12}^{-}f_{25}f_{9}^{-}f_{7},\ f_{22}f_{18}^{-}f_{11}f_{20}f_{24}^{-},\ f_{28}^{-}f_{11}^{-}f_{26}f_{8}^{-}f_{6},\ f_{17}f_{19}^{-}f_{25}f_{14}^{-}f_{29}^{-},\ f_{1}f_{6}f_{15}f_{7}^{-}f_{2}^{-},\ f_{16}f_{18}^{-}f_{26}f_{13}^{-}f_{30}^{-}\}.
\end{gathered}$$

We verify the following relations among generators:

$$\begin{gathered}
f_1 = f_{13}f_{3}f_{8}^{-}f_{4},\ f_5 = f_{10}f_{8}f_{3}^{-}f_{14}^{-}f_{2},\ f_6 = f_{11}f_{8}f_{3}^{-}f_{14}^{-}f_{2},\ f_7 = f_{11}f_{4},\ f_9 = f_{10}f_{8},\ f_{12} = f_{11}f_{10}^{-},\ f_{15} = f_{14}f_{13}^{-}, \\
f_{16} = f_{21}f_{14},\ f_{18} = f_{19}f_{14}f_{13}^{-},\ f_{20} = f_{17}f_{13}^{-},\ f_{22} = f_{8}f_{3}^{-}f_{17}^{-},\ f_{23} = f_{10}f_{8}f_{3}^{-}f_{14}^{-}f_{21}^{-},\ f_{24} = f_{11}f_{8}f_{3}^{-}f_{14}^{-}f_{19}^{-}, \\
f_{25} = f_{4}^{-}f_{8}f_{3}^{-}f_{17}^{-},\ f_{26} = f_{2}^{-}f_{21}^{-},\ f_{27} = f_{3}^{-}f_{17}^{-},\ f_{28} = f_{3}^{-}f_{14}^{-}f_{21}^{-},\ f_{29} = f_{4}^{-}f_{8}f_{3}^{-}f_{14}^{-}f_{19}^{-},\ f_{30} = f_{2}^{-}f_{19}^{-}.
\end{gathered}$$

Thus, replacing the generators on the left hand sides by the words on the right hand side yields a presentation $\pi_1(\mathbf{T}(H_3)) = \langle \Psi_3' \mid R_3' \rangle$ with

$$\Psi_3' = \{f_i,\ i \in \underbrace{\{2, 3, 4, 8, 10, 11, 13, 14, 17, 19, 21\}}_{=:I_{11}}\},$$

and

$$\begin{gathered}
R_3' := \{f_{13}f_{3}f_{8}^{-}f_{4}f_{8}f_{13}^{-}f_{4}^{-}f_{3}^{-},\ f_{2}f_{10}f_{8}f_{14}^{-}f_{2}^{-}f_{14}f_{3}f_{8}^{-}f_{10}^{-}f_{3}^{-},\ f_{21}f_{14}f_{3}f_{17}f_{13}^{-}f_{21}^{-}f_{3}^{-}f_{17}^{-}f_{13}f_{14}^{-},\ f_{21}f_{14}f_{13}f_{14}^{-}f_{19}^{-}f_{2}^{-}f_{21}^{-}f_{13}^{-}f_{19}f_{2}, \\
f_{8}f_{3}^{-}f_{17}^{-}f_{14}^{-}f_{21}^{-}f_{10}f_{17}f_{21}f_{14}f_{3}f_{8}^{-}f_{10}^{-},\ f_{21}f_{14}f_{3}f_{11}^{-}f_{2}^{-}f_{21}^{-}f_{8}^{-}f_{11}f_{8}f_{3}^{-}f_{14}^{-}f_{2},\ f_{17}f_{19}^{-}f_{4}^{-}f_{8}f_{3}^{-}f_{17}^{-}f_{14}^{-}f_{19}f_{14}f_{3}f_{8}^{-}f_{4}, \\
f_{17}f_{3}f_{10}f_{11}^{-}f_{4}^{-}f_{8}f_{3}^{-}f_{17}^{-}f_{8}^{-}f_{10}^{-}f_{11}f_{4},\ f_{4}f_{19}f_{2}f_{10}f_{4}^{-}f_{8}f_{3}^{-}f_{14}^{-}f_{19}^{-}f_{14}f_{8}^{-}f_{10}^{-}f_{2}^{-}f_{3},\ f_{3}f_{4}f_{13}f_{8}^{-}f_{11}f_{10}^{-}f_{3}^{-}f_{2}f_{10}f_{8}f_{13}^{-}f_{4}^{-}f_{11}^{-}f_{2}^{-}, \\
f_{10}f_{8}f_{3}^{-}f_{14}^{-}f_{21}^{-}f_{19}^{-}f_{11}f_{10}^{-}f_{21}f_{19}f_{14}f_{3}f_{8}^{-}f_{11}^{-},\ f_{8}f_{3}^{-}f_{17}^{-}f_{13}f_{14}^{-}f_{19}^{-}f_{11}f_{17}f_{13}^{-}f_{19}f_{14}f_{3}f_{8}^{-}f_{11}^{-}\}.
\end{gathered}$$

We notice that any of the above relation becomes trivial once abelianized and indeed, $\pi_1(\mathbf{T}(H_3))^{\mathrm{ab}} = \mathbb{Z}^{11}$.

In terms of H_3-conjugates of $q_0 := \widehat{s}_0 r_{H_3}$, the 11 generators $\{f_i,\ i \in I_{11}\}$ of $\pi_1(\mathbf{T}(H_3))$ given above may be written $f_i = q_0^{u_i}$ where the u_i's are given by

$$\{u_i\}_{i\in I_{11}} = \{xy(s_2s_1)^2,\ y^{s_2s_1}s_2s_1,\ s_1,\ s_2s_1,\ y^{-1}s_3,\ s_2y^{-1}s_3,\ (x^{-1}s_1)^2,\ s_3(x^{-1}s_1)^2,\ s_3ys_1,\ s_2y,\ y\},$$

with

$$x := s_3s_1^{s_2},\ y := ((s_1s_2)^2)^{s_3}.$$

We can do the same for H_4. We have a presentation

$$\pi_1(\mathbf{T}(H_4)) = \langle \Psi_4 \mid R_4 \rangle\,,$$

where $\Psi_4 =: \{f_i,\ 1 \le i \le 60\}$ and

$$\begin{gathered}
R_4 = \{f_{3}^{-} f_{9}^{-} f_{19}^{-} f_{10} f_{2},\ f_{1} f_{7} f_{17}^{-} f_{5}^{-} f_{2}^{-},\ f_{1} f_{8} f_{18}^{-} f_{6}^{-} f_{3}^{-},\ f_{2} f_{10} f_{19}^{-} f_{9}^{-} f_{3}^{-},\ f_{1} f_{13} f_{22} f_{16}^{-} f_{4}^{-},\ f_{2} f_{11} f_{20} f_{14}^{-} f_{4}^{-},\ f_{3} f_{12} f_{21} f_{15}^{-} f_{4}^{-},\ f_{6} f_{25}^{-} f_{17}^{-} f_{23} f_{9}^{-}, \\
f_{3}^{-} f_{58} f_{36}^{-} f_{29} f_{51},\ f_{52} f_{57}^{-} f_{30}^{-} f_{49} f_{9},\ f_{6} f_{28}^{-} f_{22} f_{33} f_{12}^{-},\ f_{9} f_{26}^{-} f_{20} f_{32} f_{12}^{-},\ f_{9}^{-} f_{45}^{-} f_{35}^{-} f_{13}^{-} f_{58}^{-},\ f_{17} f_{39}^{-} f_{32}^{-} f_{33} f_{36},\ f_{5} f_{25}^{-} f_{18}^{-} f_{24} f_{10}^{-},\ f_{6}^{-} f_{28} f_{22}^{-} f_{33}^{-} f_{12}, \\
f_{5} f_{29}^{-} f_{22} f_{34} f_{11}^{-},\ f_{10} f_{11}^{-} f_{32} f_{21} f_{27}^{-},\ f_{10} f_{27}^{-} f_{21} f_{32} f_{11}^{-},\ f_{17} f_{31}^{-} f_{14} f_{16}^{-} f_{29},\ f_{23}^{-} f_{7} f_{8}^{-} f_{24} f_{19}^{-},\ f_{53} f_{15} f_{50} f_{34}^{-} f_{59}^{-},\ f_{23}^{-} f_{36}^{-} f_{22} f_{42} f_{26},\ f_{1}^{-} f_{57} f_{43}^{-} f_{32} f_{54}, \\
f_{19} f_{27}^{-} f_{15} f_{14}^{-} f_{26},\ f_{59}^{-} f_{13}^{-} f_{36}^{-} f_{44}^{-} f_{10}^{-},\ f_{59}^{-} f_{34}^{-} f_{50} f_{15} f_{53},\ f_{2}^{-} f_{60} f_{40}^{-} f_{30} f_{53},\ f_{24}^{-} f_{35}^{-} f_{22} f_{41} f_{27},\ f_{45} f_{35} f_{13} f_{58} f_{9},\ f_{45} f_{28} f_{29}^{-} f_{44}^{-} f_{19}^{-},\ f_{45} f_{42}^{-} f_{50} f_{21} f_{49}^{-}, \\
f_{9}^{-} f_{52}^{-} f_{57} f_{30} f_{49}^{-},\ f_{8} f_{1} f_{3}^{-} f_{6}^{-} f_{18}^{-},\ f_{51} f_{29} f_{36}^{-} f_{58} f_{3}^{-},\ f_{18} f_{40}^{-} f_{32}^{-} f_{34} f_{35},\ f_{8} f_{57} f_{14} f_{42} f_{35},\ f_{3} f_{52}^{-} f_{31}^{-} f_{39} f_{60}^{-},\ f_{17} f_{36} f_{33} f_{32}^{-} f_{39}^{-},\ f_{8} f_{24}^{-} f_{19} f_{23} f_{7}^{-}, \\
f_{2} f_{56}^{-} f_{41} f_{33}^{-} f_{54}^{-},\ f_{8} f_{30}^{-} f_{21} f_{33} f_{13}^{-},\ f_{7} f_{31}^{-} f_{20} f_{34} f_{13}^{-},\ f_{9} f_{38} f_{41} f_{16} f_{55},\ f_{44} f_{38} f_{12} f_{60} f_{5},\ f_{18} f_{24}^{-} f_{10} f_{5}^{-} f_{25},\ f_{44} f_{41}^{-} f_{50} f_{20} f_{48}^{-},\ f_{18} f_{30}^{-} f_{15} f_{16}^{-} f_{28}, \\
f_{25}^{-} f_{39}^{-} f_{20} f_{42} f_{28},\ f_{47}^{-} f_{36} f_{38}^{-} f_{49} f_{18}^{-},\ f_{47}^{-} f_{13}^{-} f_{54}^{-} f_{60} f_{25},\ f_{17} f_{5} f_{2} f_{1}^{-} f_{7}^{-},\ f_{18} f_{45}^{-} f_{26}^{-} f_{31} f_{46},\ f_{1}^{-} f_{13}^{-} f_{22}^{-} f_{16} f_{4},\ f_{7} f_{57} f_{15} f_{41} f_{36},\ f_{47}^{-} f_{28} f_{55} f_{52}^{-} f_{7}^{-}, \\
f_{8} f_{59} f_{11} f_{39} f_{46},\ f_{17} f_{23}^{-} f_{9} f_{6}^{-} f_{25},\ f_{47}^{-} f_{35} f_{37}^{-} f_{48} f_{17}^{-},\ f_{24} f_{35} f_{22}^{-} f_{41}^{-} f_{27}^{-},\ f_{25}^{-} f_{40}^{-} f_{21} f_{41} f_{29},\ f_{59}^{-} f_{11}^{-} f_{39}^{-} f_{46}^{-} f_{8}^{-},\ f_{24} f_{45}^{-} f_{55} f_{4}^{-} f_{53},\ f_{19} f_{38}^{-} f_{33}^{-} f_{34} f_{37}, \\
f_{3}^{-} f_{55} f_{42}^{-} f_{34} f_{54},\ f_{2}^{-} f_{56} f_{41}^{-} f_{33} f_{54},\ f_{24} f_{59} f_{54}^{-} f_{12}^{-} f_{49}^{-},\ f_{18} f_{47} f_{36}^{-} f_{38} f_{49}^{-},\ f_{59}^{-} f_{37} f_{26}^{-} f_{52}^{-} f_{1},\ f_{6}^{-} f_{40}^{-} f_{43}^{-} f_{14}^{-} f_{55}^{-},\ f_{45} f_{37} f_{11} f_{60} f_{6},\ f_{9}^{-} f_{38}^{-} f_{41}^{-} f_{16}^{-} f_{55}^{-}, \\
f_{51} f_{56}^{-} f_{27}^{-} f_{49} f_{6},\ f_{51} f_{4}^{-} f_{57} f_{46}^{-} f_{25},\ f_{6}^{-} f_{45}^{-} f_{37}^{-} f_{11}^{-} f_{60}^{-},\ f_{24} f_{30}^{-} f_{43}^{-} f_{20}^{-} f_{37},\ f_{23} f_{58} f_{54}^{-} f_{11}^{-} f_{48}^{-},\ f_{10} f_{56} f_{16} f_{42} f_{37},\ f_{1}^{-} f_{59} f_{37}^{-} f_{26} f_{52},\ f_{7} f_{47} f_{28}^{-} f_{55}^{-} f_{52}, \\
f_{51} f_{55}^{-} f_{26}^{-} f_{48} f_{5},\ f_{23} f_{31}^{-} f_{43}^{-} f_{21}^{-} f_{38},\ f_{58}^{-} f_{54} f_{11} f_{48} f_{23}^{-},\ f_{23} f_{44}^{-} f_{56} f_{4}^{-} f_{52},\ f_{10} f_{59} f_{13} f_{36} f_{44},\ f_{44} f_{27} f_{30}^{-} f_{46}^{-} f_{17}^{-},\ f_{5}^{-} f_{51}^{-} f_{55} f_{26} f_{48}^{-},\ f_{5}^{-} f_{39}^{-} f_{43}^{-} f_{15}^{-} f_{56}^{-}, \\
f_{44} f_{56}^{-} f_{4} f_{52}^{-} f_{23}^{-},\ f_{23} f_{36} f_{22}^{-} f_{42}^{-} f_{26}^{-},\ f_{17} f_{29} f_{16}^{-} f_{14} f_{31}^{-},\ f_{3}^{-} f_{12}^{-} f_{21}^{-} f_{15} f_{4},\ f_{52} f_{14} f_{50} f_{33}^{-} f_{58}^{-},\ f_{2}^{-} f_{11}^{-} f_{20}^{-} f_{14} f_{4},\ f_{25}^{-} f_{51}^{-} f_{4} f_{57}^{-} f_{46},\ f_{9}^{-} f_{26} f_{20}^{-} f_{32}^{-} f_{12}, \\
f_{10} f_{48} f_{31}^{-} f_{57}^{-} f_{53},\ f_{49}^{-} f_{21} f_{50} f_{42}^{-} f_{45},\ f_{7} f_{58} f_{12} f_{40} f_{46},\ f_{6}^{-} f_{55}^{-} f_{14}^{-} f_{43}^{-} f_{40}^{-},\ f_{17} f_{44}^{-} f_{27}^{-} f_{30} f_{46},\ f_{59}^{-} f_{54} f_{12} f_{49} f_{24}^{-},\ f_{23} f_{38} f_{21}^{-} f_{43}^{-} f_{31}^{-},\ f_{25}^{-} f_{29} f_{41} f_{21} f_{40}^{-}, \\
f_{8} f_{47} f_{29}^{-} f_{56}^{-} f_{53},\ f_{59}^{-} f_{35} f_{28}^{-} f_{51}^{-} f_{2},\ f_{24} f_{37} f_{20}^{-} f_{43}^{-} f_{30}^{-},\ f_{18} f_{28} f_{16}^{-} f_{15} f_{30}^{-},\ f_{8} f_{13}^{-} f_{33} f_{21} f_{30}^{-},\ f_{48}^{-} f_{20} f_{50} f_{41}^{-} f_{44},\ f_{8} f_{35} f_{42} f_{14} f_{57},\ f_{7} f_{36} f_{41} f_{15} f_{57}, \\
f_{60}^{-} f_{40} f_{30}^{-} f_{53}^{-} f_{2},\ f_{18} f_{35} f_{34} f_{32}^{-} f_{40}^{-},\ f_{19} f_{48} f_{39}^{-} f_{40} f_{49}^{-},\ f_{2}^{-} f_{59} f_{35}^{-} f_{28} f_{51},\ f_{10} f_{37} f_{42} f_{16} f_{56},\ f_{51} f_{16} f_{50} f_{32}^{-} f_{60}^{-},\ f_{5}^{-} f_{56}^{-} f_{15}^{-} f_{43}^{-} f_{39}^{-},\ f_{47}^{-} f_{22} f_{50} f_{43}^{-} f_{46}, \\
f_{48}^{-} f_{39} f_{40}^{-} f_{49} f_{19}^{-},\ f_{48}^{-} f_{31} f_{57} f_{53}^{-} f_{10}^{-},\ f_{47}^{-} f_{29} f_{56} f_{53}^{-} f_{8}^{-},\ f_{19} f_{45}^{-} f_{28}^{-} f_{29} f_{44},\ f_{1}^{-} f_{58} f_{38}^{-} f_{27} f_{53},\ f_{19} f_{26} f_{14}^{-} f_{15} f_{27}^{-},\ f_{60}^{-} f_{39} f_{31}^{-} f_{52}^{-} f_{3},\ f_{60}^{-} f_{32}^{-} f_{50} f_{16} f_{51}, \\
f_{45} f_{55}^{-} f_{4} f_{53}^{-} f_{24}^{-},\ f_{45} f_{26} f_{31}^{-} f_{46}^{-} f_{18}^{-},\ f_{25}^{-} f_{47} f_{13} f_{54} f_{60}^{-},\ f_{19} f_{37} f_{34} f_{33}^{-} f_{38}^{-},\ f_{58}^{-} f_{33}^{-} f_{50} f_{14} f_{52},\ f_{46} f_{43}^{-} f_{50} f_{22} f_{47}^{-},\ f_{25}^{-} f_{28} f_{42} f_{20} f_{39}^{-},\ f_{6}^{-} f_{51}^{-} f_{56} f_{27} f_{49}^{-}, \\
f_{58}^{-} f_{38} f_{27}^{-} f_{53}^{-} f_{1},\ f_{58}^{-} f_{12}^{-} f_{40}^{-} f_{46}^{-} f_{7}^{-},\ f_{1}^{-} f_{54} f_{32} f_{43}^{-} f_{57},\ f_{17} f_{47} f_{35}^{-} f_{37} f_{48}^{-},\ f_{7} f_{13}^{-} f_{34} f_{20} f_{31}^{-},\ f_{5}^{-} f_{29} f_{22}^{-} f_{34}^{-} f_{11},\ f_{5}^{-} f_{44}^{-} f_{38}^{-} f_{12}^{-} f_{60}^{-},\ f_{3} f_{55}^{-} f_{42} f_{34}^{-} f_{54}^{-}\}.
\end{gathered}$$

We verify the following relations among generators:

$$f_1 = f_{52}f_{14}f_{50}f_{33}^{-}f_{38}^{-}f_{27}f_{53},\ f_5 = f_{17}^{-}f_{23}f_9^{-}f_3^{-}f_{52}f_{26}f_{37}^{-}f_{10}^{-}f_{44}^{-}f_{36}^{-}f_{33}^{-}f_{21}^{-}f_{30}f_{24}^{-}f_{10},\ f_6 = f_3^{-}f_{52}f_{31}f_{17}^{-}f_{36}^{-}f_{33}^{-}f_{50}f_{22}f_{36}^{-}f_{23}^{-}f_{26}f_{37}^{-}f_{10}^{-}f_{27}f_{49}^{-},$$

$$f_7 = f_{53}^{-}f_{27}^{-}f_{38}f_{33}f_{50}^{-}f_{14}^{-}f_{52}^{-}f_2f_{17}^{-}f_{23}f_9^{-}f_3^{-}f_{52}f_{26}f_{37}^{-}f_{10}^{-}f_{44}^{-}f_{36}^{-}f_{33}^{-}f_{21}^{-}f_{30}f_{24}^{-}f_{10}f_{17},$$

$$f_8 = f_{53}^{-}f_{27}^{-}f_{38}f_{33}f_{50}^{-}f_{14}^{-}f_{31}f_{17}^{-}f_{36}^{-}f_{33}^{-}f_{50}f_{22}f_{36}^{-}f_{23}^{-}f_{26}f_{37}^{-}f_{10}^{-}f_{27}f_{49}^{-}f_{24}f_{10}^{-}f_{17}^{-}f_{23}f_9^{-}f_3^{-}f_{52}f_{26}f_{37}^{-}$$
$$f_{10}^{-}f_{44}^{-}f_{36}^{-}f_{33}^{-}f_{21}^{-}f_{30}f_{24}^{-}f_{10}f_{17}f_{23}^{-}f_9f_{49}f_{27}^{-}f_{10}f_{37}f_{26}^{-}f_{23}f_{36}f_{22}^{-}f_{50}^{-}f_{33}f_{36}f_{17}f_{31}^{-}f_{52}^{-}f_3,$$

$$f_{11} = f_2^{-}f_4f_{14}f_{21}^{-}f_{38}f_{23}f_{31}^{-}f_{30}f_{24}^{-}f_{37}^{-},\ f_{12} = f_{21}^{-}f_{27}f_{10}^{-}f_2^{-}f_4f_{14}f_{26}^{-}f_9,\ f_{13} = f_{53}^{-}f_{27}^{-}f_{38}f_{33}f_{50}^{-}f_{14}^{-}f_{52}^{-}f_4f_{22}f_{36}^{-}f_{44}^{-}f_{10}^{-}f_{37}^{-}f_{26}f_{52}f_4^{-}f_{22}^{-},$$

$$f_{15} = f_4^{-}f_3f_{21}^{-}f_{27}f_{10}^{-}f_2^{-}f_4f_{14}f_{26}^{-}f_9f_{21},\ f_{16} = f_{22}f_{36}^{-}f_{44}^{-}f_{10}^{-}f_{37}^{-}f_{26}f_{52}f_4^{-},$$

$$f_{18} = f_{24}f_{10}^{-}f_{17}^{-}f_{23}f_9^{-}f_3^{-}f_{52}f_{26}f_{37}^{-}f_{10}^{-}f_{44}^{-}f_{36}^{-}f_{33}^{-}f_{21}^{-}f_{30}f_{24}^{-}f_{10}f_{17}f_{23}^{-}f_9f_{49}f_{27}^{-}f_{10}f_{37}f_{26}^{-}f_{23}f_{36}f_{22}^{-}f_{50}^{-}f_{33}f_{36}f_{17}f_{31}^{-}f_{52}^{-}f_3,\ f_{19} = f_{10}f_2f_3^{-}f_9^{-},$$

$$f_{20} = f_{37}f_{24}f_{30}^{-}f_{31}f_{23}^{-}f_{38}^{-}f_{21},\ f_{25} = f_{17}^{-}f_{23}f_9^{-}f_3^{-}f_{52}f_{31}f_{17}^{-}f_{36}^{-}f_{33}^{-}f_{50}f_{22}f_{36}^{-}f_{23}^{-}f_{26}f_{37}^{-}f_{10}^{-}f_{27}f_{49}^{-},$$

$$f_{28} = f_{22}f_{33}f_9^{-}f_{26}f_{14}^{-}f_4^{-}f_2f_{10}f_{27}^{-}f_{21}f_3^{-}f_{52}f_{31}f_{17}^{-}f_{36}^{-}f_{33}^{-}f_{50}f_{22}f_{36}^{-}f_{23}^{-}f_{26}f_{37}^{-}f_{10}^{-}f_{27}f_{49}^{-},f_{29} = f_{22}f_{36}^{-}f_{44}^{-}f_{10}^{-}f_{37}^{-}f_{26}f_{52}f_4^{-}f_{14}^{-}f_{31}f_{17}^{-},$$

$$f_{32} = f_{21}^{-}f_{38}f_{23}f_{31}^{-}f_{30}f_{24}^{-}f_{37}^{-}f_{26}f_9^{-}f_{21}^{-}f_{27}f_{10}^{-}f_2^{-}f_4f_{14}f_{26}^{-}f_9,\ f_{34} = f_{33}f_{38}f_9f_3f_2^{-}f_{10}^{-}f_{37}^{-},$$

$$f_{35} = f_{22}f_{44}f_{10}f_{53}f_{30}f_{49}^{-}f_9^{-}f_{52}^{-}f_{23}^{-}f_{38}^{-}f_{21}f_{30}^{-}f_{24}f_{37}f_{50}f_{27}f_{24}^{-},\ f_{39} = f_9^{-}f_{26}f_{14}^{-}f_4^{-}f_2f_{10}f_{27}^{-}f_{21}f_9f_{26}^{-}f_{37}f_{24}f_{30}^{-}f_{31}f_{23}^{-}f_{38}^{-}f_{21}f_{33}f_{36}f_{17},$$

$$f_{40} = f_{30}f_{53}f_2^{-}f_9^{-}f_{26}f_{14}^{-}f_4^{-}f_2f_{10}f_{27}^{-}f_{21}f_{38}^{-}f_{44}^{-}f_{10}^{-}f_{24}f_{30}^{-}f_{21}f_{33}f_{36}f_{44}f_{10}f_{37}f_{26}^{-}f_{52}^{-}f_3f_9f_{23}^{-}f_{17},$$

$$f_{41} = f_{44}f_{10}f_{53}f_{30}f_{49}^{-}f_9^{-}f_{52}^{-}f_{23}^{-}f_{38}^{-}f_{21}f_{30}^{-}f_{24}f_{37}f_{50},\ f_{42} = f_{22}^{-}f_{36}f_{23}f_{26}^{-},\ f_{43} = f_{21}^{-}f_{38}f_{23}f_{31}^{-},\ f_{45} = f_{22}^{-}f_{36}f_{23}f_{26}^{-}f_{50}^{-}f_{21}^{-}f_{49},$$

$$f_{46} = f_{31}^{-}f_{26}f_{22}^{-}f_{36}f_{23}f_{26}^{-}f_{50}^{-}f_{21}^{-}f_{49}f_3^{-}f_{52}f_{31}f_{17}^{-}f_{36}^{-}f_{33}^{-}f_{50}f_{22}f_{36}^{-}f_{23}^{-}f_{26}f_{37}^{-}f_{10}^{-}f_{27}f_{49}^{-}f_9^{-}f_{23}f_{17}^{-}f_{10}^{-}f_{24}f_{30}^{-}f_{21}f_{33}f_{36}f_{44}f_{10}f_{37}f_{26}^{-}f_{52}^{-}f_3f_9f_{23}^{-}f_{17}f_{10}f_{24}^{-},$$

$$f_{47} = f_{36}f_{38}^{-}f_{49}f_3^{-}f_{52}f_{31}f_{17}^{-}f_{36}^{-}f_{33}^{-}f_{50}f_{22}f_{36}^{-}f_{23}^{-}f_{26}f_{37}^{-}f_{10}^{-}f_{27}f_{49}^{-}f_9^{-}f_{23}f_{17}^{-}f_{10}^{-}f_{24}f_{30}^{-}f_{21}f_{33}f_{36}f_{44}f_{10}f_{37}f_{26}^{-}f_{52}^{-}f_3f_9f_{23}^{-}f_{17}f_{10}f_{24}^{-},$$

$$f_{48} = f_{44}f_{50}^{-}f_{37}^{-}f_{24}^{-}f_{30}f_{21}^{-}f_{38}f_{23}f_{52}f_9f_{49}f_{30}^{-}f_{53}^{-}f_{10}^{-}f_{44}^{-}f_{50}f_{37}f_{24}f_{30}^{-}f_{31}f_{23}^{-}f_{38}^{-}f_{21},\ f_{51} = f_4f_{52}^{-}f_{23}^{-}f_{44}f_{10}f_{37}f_{26}^{-}f_{23}f_{36}f_{22}^{-}f_{50}^{-}f_{33}f_{36}f_{17}f_{31}^{-}f_{52}^{-}f_3,$$

$$f_{54} = f_9^{-}f_{26}f_{14}^{-}f_4^{-}f_2f_{10}f_{27}^{-}f_{21}f_{49}^{-}f_{24}f_{37}f_{26}^{-}f_{14}f_{50}f_{33}^{-}f_{38}^{-}f_{27}f_{53},\ f_{55} = f_4f_{52}^{-}f_{26}^{-}f_{37}f_{10}f_{44}f_{36}f_{22}^{-}f_{50}^{-}f_{37}^{-}f_{24}^{-}f_{30}f_{21}^{-}f_{38}f_{23}f_{52}f_9f_{49}f_{30}^{-}f_{53}^{-}f_{10}^{-}f_{44}^{-}f_{38}^{-}f_9^{-},$$

$$f_{56} = f_{44}f_{10}f_{53}f_{30}f_{49}^{-}f_9^{-}f_{52}^{-}f_{23}^{-}f_{38}^{-}f_{21}f_{30}^{-}f_{24}f_{37}f_{50}f_{33}^{-}f_{53}^{-}f_{27}^{-}f_{38}f_{33}f_{50}^{-}f_{14}^{-}f_{26}f_{37}^{-}f_{24}^{-}f_{49}f_{21}^{-}f_{27}f_{10}^{-}f_2^{-}f_4f_{14}f_{26}^{-}f_9f_2,\ f_{57} = f_{30}^{-}f_{49}f_9f_{52},$$

$$f_{58} = f_{52}f_{31}f_{17}^{-}f_{36}^{-}f_{33}^{-}f_{50}f_{22}f_{36}^{-}f_{23}^{-}f_{26}f_{37}^{-}f_{10}^{-}f_{44}^{-}f_{23}f_{52}f_4^{-}f_{17}f_{31}^{-}f_{14}f_4f_{52}^{-}f_{26}^{-}f_{37}f_{10}f_{44}f_{36}f_{22}^{-}f_{36},$$

$$f_{59} = f_{53}f_4^{-}f_3f_{21}^{-}f_{27}f_{10}^{-}f_2^{-}f_4f_{14}f_{26}^{-}f_9f_{21}f_{50}f_{37}f_{10}f_2f_3^{-}f_9^{-}f_{38}^{-}f_{33}^{-},$$

$$f_{60} = f_9^{-}f_{26}f_{14}^{-}f_4^{-}f_2f_{10}f_{27}^{-}f_{21}f_{38}^{-}f_{44}^{-}f_{10}^{-}f_{24}f_{30}^{-}f_{21}f_{33}f_{36}f_{44}f_{10}f_{37}f_{26}^{-}f_{52}^{-}f_3f_9f_{23}^{-}f_{17}.$$

As for H_3, replacing the generators on the left hand sides by the words on the right hand side yields a presentation $\pi_1(\mathbf{T}(H_4)) = \langle \Psi'_4 \mid R'_4 \rangle$ where

$$\Psi'_4 = \{f_i,\ i \in \underbrace{\{2, 3, 4, 9, 10, 14, 17, 21, 22, 23, 24, 26, 27, 30, 31, 33, 36, 37, 38, 44, 49, 50, 52, 53\}}_{=:I_{24}}\},$$

and the relations of R'_4 become all trivial in the abelianization and thus $\pi_1(\mathbf{T}(H_4)) = \mathbb{Z}^{24}$.

In terms of H_4-conjugates of $q_0 := \widehat{s}_0 r_{H_4}$, the 24 generators $\{f_i,\ i \in I_{24}\}$ of $\pi_1(\mathbf{T}(H_4))$

given above may be written $f_i = q_0^{u_i}$ where the u_i's are given by

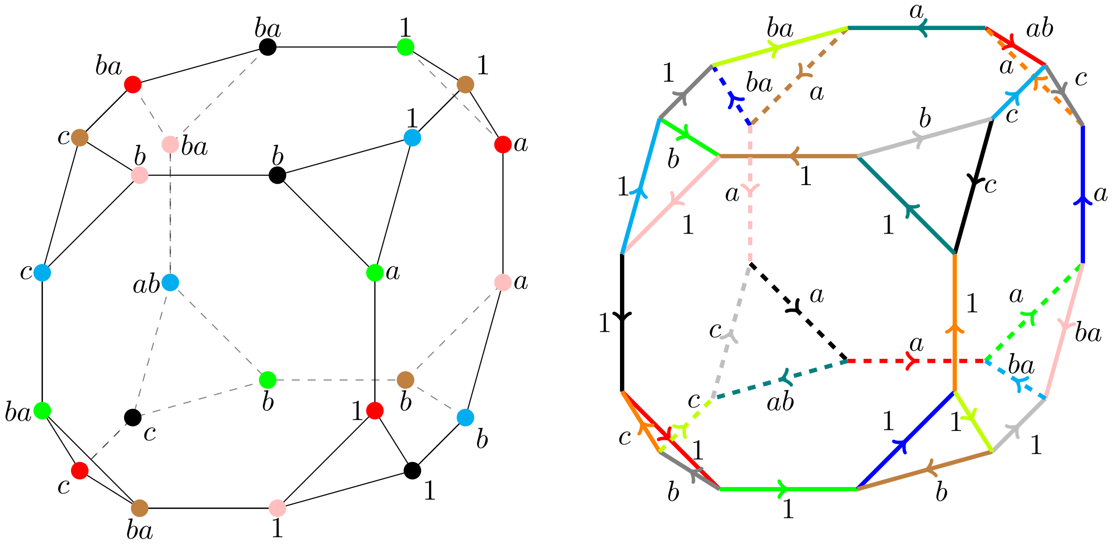
$$\{u_i\}_{i\in I_{24}} = \{(xys_4)^4,\ (s_4xy)^4s_4 = r_{H_4},\ (s_1(ys_4s_3)^{s_2})^3s_1s_2ys_4,\ xys_4,\ (xy)^{s_4},\ t^{s_2}ys_4,\ ts_2y(xy)^{s_4},$$
$$(s_2(s_4x)^{s_3}s_2^{s_1})^2s_2s_3s_4,\ s_2zy(xy)^{s_4},\ xzys_4,\ (xzy)^{s_4},\ xts_2ys_4,\ (xts_2y)^{s_4},\ (s_4xs_3)^2s_3ys_4,$$
$$xs_3(xy)^{s_4},\ s_1t^{s_2}ys_4,\ (s_4^xs_3s_1)^2x^{-1}s_4,\ xts_2y(xy)^{s_4},\ s_4xts_2y(xy)^{s_4}, xzy(xy)^{s_4},$$
$$(xzy)^{s_4}s_1^{s_2}ys_4,\ s_1s_2zys_4s_1^{s_2}ys_4,\ ((ys_4x)^{s_3})^3s_3ys_4,\ x_3ys_4(s_1^{s_2}ys_4s_3)^2s_1^{s_2}ys_4\},$$

with

$$x := s_3s_1^{s_2},\ y := ((s_1s_2)^2)^{s_3},\ z := s_3s_4s_1s_2,\ t := s_3s_4s_2s_1.$$

C Figures describing the combinatorics of the Dirichlet-Voronoi domain for $SO(3)/\{\pm 1\}^2$

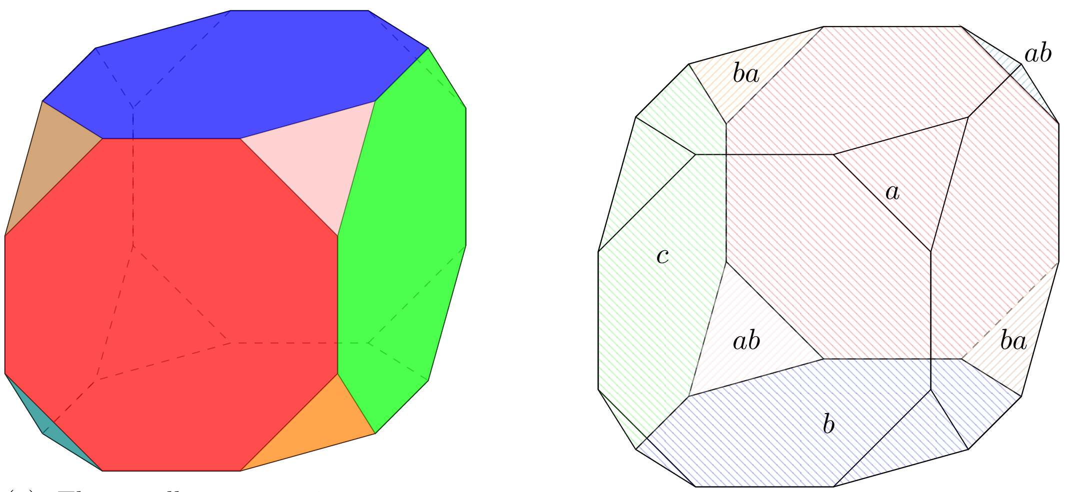
In this appendix we give some figures that help visualize how to obtain the chain complex from Theorem 17.4.2. We choose representative cells in each dimension and in each one of these figures, the cells that belong to the same $\mathfrak{S}_3$-orbit share the same color and the order on the colors given in the legends corresponds to the order chosen to build the matrices of the boundaries of the complex. Moreover, for simplicity we replace s_α, s_β and $w_0 = s_\alpha s_\beta s_\alpha$ respectively by a, b and c. Finally, we chose to orient $\mathfrak{p} \simeq \mathbb{R}^3$ directly and the 2-cells accordingly.



(a) The 0-cells representing orbits: black, red, cyan, green, brown, pink

(b) The 1-cells representing orbits: black, red, green, blue, orange, teal, brown, pink, gray, lightgray, cyan, lime

Figure 25: The 0-cells and 1-cells of $\mathcal{F}_3(\mathbb{R})$



(a) The 2-cells representing orbits: red, blue, green, brown, teal, orange, pink

(b) Some translates of the representing 2-cells

Figure 26: The 2-cells of $\mathcal{F}_3(\mathbb{R})$

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Résumé. Ce travail vise à construire des structures cellulaires explicites sur des espaces apparaissant en théorie de Lie, équivariantes pour l'action d'un groupe de Weyl W. En général, l'étude de telles structures sur un W-espace X a pour but d'exhiber un complexe bien défini dans la catégorie homotopique bornée $\mathcal{K}^b(\mathbb{Z}[W])$ des $\mathbb{Z}[W]$-modules, qui est un modèle pour le foncteur dérivé des sections globales $R\Gamma(X, \underline{\mathbb{Z}})$ dans la catégorie dérivée $\mathcal{D}^b(\mathbb{Z}[W])$.

Les deux classes d'espaces sur lesquelles nous nous concentrons sont les variétés de drapeaux et tores maximaux des groupes de Lie compacts. Plus spécifiquement, étant donné un groupe de Lie simple compact K et un tore maximal $T < K$, on donne une structure simpliciale générale sur T, équivariante pour l'action du groupe de Weyl $W := N_K(T)/T$, puis nous décrivons le W-anneau différentiel gradué associé, en fonction du réseau des caractères de T. Pour les groupes de Coxeter finis non cristallographiques, nous construisons des variétés hyperboliques compactes qui peuvent être vues comme des analogues des tores, en utilisant des extensions hyperboliques plutôt que des extensions affines. Dans le cas des groupes diédraux, nous obtenons des surfaces de Riemann arithmétiques.

Concernant les variétés de drapeaux, nous étudions trois décompositions cellulaires $\mathfrak{S}_3$-équivariantes distinctes de la variété de drapeaux réelle $\mathcal{F}_3(\mathbb{R})$ de $\mathbb{R}^3$, qui constitue le premier exemple non-trivial. La première utilise le graphe de Goresky-Kottwitz-MacPherson de $\mathfrak{S}_3$ et un plongement algébrique $\mathcal{F}_3(\mathbb{R}) \hookrightarrow \mathbb{RP}^7$, la deuxième utilise le fait que le revêtement universel de $\mathcal{F}_3(\mathbb{R})$ est la 3-sphère et fournit un complexe d'homologie cellulaire particulièrement joli et simple. La troisième est probablement la plus prometteuse, puisqu'elle repose sur un domaine de Dirichlet-Voronoi, défini uniquement à partir d'une métrique riemannienne normale homogène sur $\mathcal{F}_3(\mathbb{R})$. Ainsi, il est raisonnable d'attendre de cette méthode qu'elle se généralise aux autres variétés de drapeaux. Nous donnons quelques résultats préliminaires dans ce sens.

Mots-clefs : structures cellulaires et simpliciales équivariantes, complexes de chaînes cellulaires, variétés de drapeaux, groupes de Weyl, tores, groupes de Coxeter hyperboliques, métriques normales homogènes.

Abstract. This work aims to construct explicit cellular structures on spaces arising in Lie theory, that are equivariant with respect to the action of a Weyl group W. In general, the main purpose of studying such structures on a W-space X is to exhibit a well-defined complex in the bounded homotopy category $\mathcal{K}^b(\mathbb{Z}[W])$ of $\mathbb{Z}[W]$-modules, which is a model for the derived functor of global sections $R\Gamma(X, \underline{\mathbb{Z}})$ in the derived category $\mathcal{D}^b(\mathbb{Z}[W])$.

The two classes of spaces we focus on are flag manifolds and maximal tori of compact Lie groups. More precisely, given a simple compact Lie group K and a maximal torus $T < K$, we give a general explicit simplicial structure on T, equivariant with respect to the action of the Weyl group $W := N_K(T)/T$ and we describe the associated W-dg-ring, depending on the character lattice of T. For non-crystallographic finite Coxeter groups, we construct compact hyperbolic manifolds which may be seen as analogues of tori, using hyperbolic extensions rather than affine extensions. In the case of dihedral groups, these are arithmetic Riemann surfaces.

Concerning flag manifolds, we study three different $\mathfrak{S}_3$-equivariant cellular decompositions of the real flag manifold $\mathcal{F}_3(\mathbb{R})$ of $\mathbb{R}^3$, which is the first non-trivial example. The first one starts with the Goresky-Kottwitz-MacPherson graph of $\mathfrak{S}_3$ and an algebraic embedding $\mathcal{F}_3(\mathbb{R}) \hookrightarrow \mathbb{RP}^7$, the second one uses the fact that the universal cover of $\mathcal{F}_3(\mathbb{R})$ is the 3-sphere and yields a particularly nice and simple cellular homology chain complex. The third one is perhaps the most promising one, as it relies on a Dirichlet-Voronoi fundamental domain, defined using only a normal homogeneous Riemannian metric on $\mathcal{F}_3(\mathbb{R})$. Therefore, this method is expected to be generalizable to other flag manifolds. We give some preliminary results in this direction.

Keywords: equivariant cellular and simplicial structures, cellular chain complexes, flag manifolds, Weyl groups, tori, hyperbolic Coxeter groups, normal homogeneous metrics.

Mathematics Subject Classification (2020): 57R91, 14M15 (57M60, 20F55, 22F30).