Motivic cohomology in the arithmetic of function fields

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Cohomologie motivique en arithmétique des corps de fonctions

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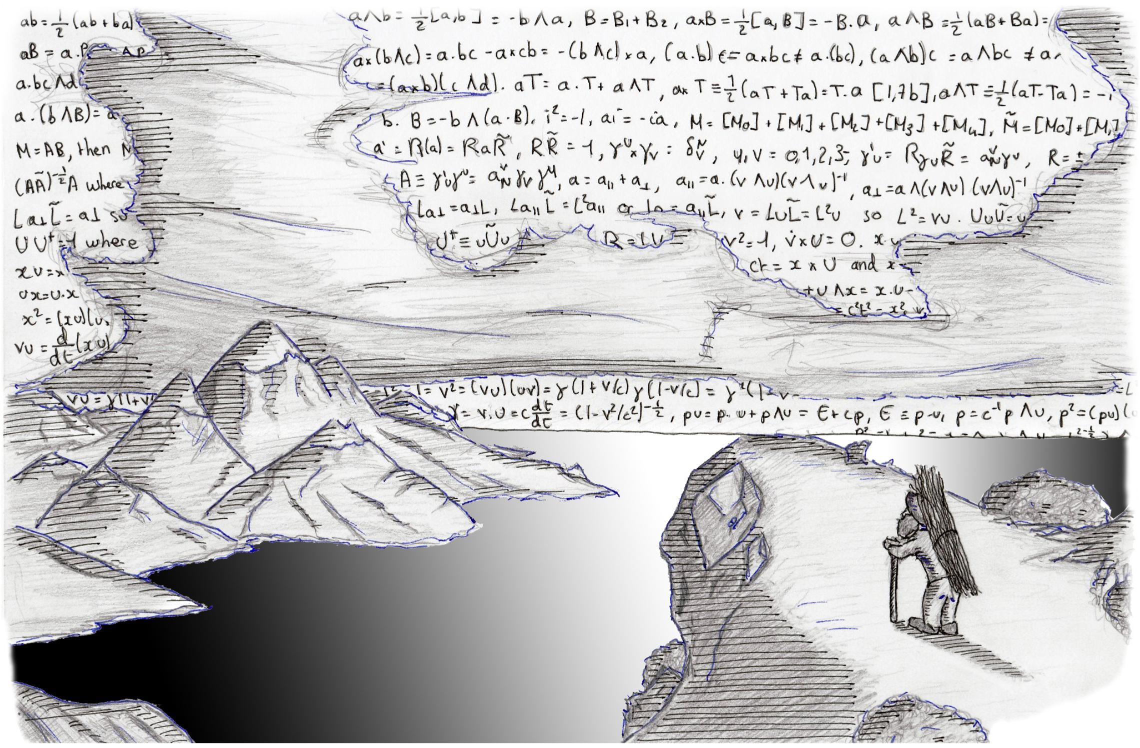
Cohomologie motivique en arithmétique des corps de fonctions

(Motivic Cohomology in Function Fields Arithmetic)



Quentin Gazda

Thèse de doctorat

Université Jean Monnet Saint-Étienne

"Ce sont les optimistes qui démontrent les théorèmes."
André Weil[1]

[1] Rapporté par Jean-Pierre Serre.

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Chapter 0

Introduction

0.1 The number field picture

The idea of *mixed motives* and *motivic cohomology* has been gradually formulated by Deligne, Beilinson and Lichtenbaum and aims to extend Grothendieck's philosophy of pure motives. Before discussing the function fields side, subject of this thesis, let us first present the classical setting.

The theory, mostly conjectural, starts with a number field F. The hypothetical landscape portrays a $\mathbb{Q}$-linear Tannakian category $\mathcal{MM}_F$ of *mixed motives* over F, equipped with several *realization functors* having $\mathcal{MM}_F$ as source (see [Del, §1]). Among them[1], the *Betti realization functor* Λ_v, at an infinite place v of F, takes values in the category of $\mathbb{Q}$-vector spaces and the *ℓ-adic realization functor* V_ℓ, for a prime number ℓ, takes values in the category of continuous ℓ-adic representation of the absolute Galois group G_F of F.

It is expected that *reasonable* cohomology theories factor through the category $\mathcal{MM}_F$: for all integer i, one foresees the existence of a functor h^i, from the category of algebraic varieties over F to $\mathcal{MM}_F$, making the following diagram of categories commute:

$$\begin{tikzcd}
 & & \{\text{Varieties}/F\} \arrow[ddll, bend right, "X\mapsto H^i_{\text{ét}}(X\times_F F^s{,}\mathbb{Q}_\ell)"'] \arrow[d, "h^i"] \arrow[ddrr, bend left, "X\mapsto H^i((X\times_{F,v}\mathbb{C})(\mathbb{C}){,}\mathbb{Q})"] & & \\
 & & \mathcal{MM}_F \arrow[dll, "V_\ell"'] \arrow[dl, dashed] \arrow[d, "\mathcal{H}^+"] \arrow[dr, dashed] \arrow[drr, "\Lambda_v"] & & \\
\mathrm{Rep}_{\mathbb{Q}_\ell}(G_F) & \cdots & \mathcal{MH}^+_{\mathbb{R}} & \cdots & \mathrm{Rep}_{\mathbb{Q}}(\mathrm{Gal}(\mathbb{C}|\mathbb{R}))
\end{tikzcd}$$

The dots hide other cohomology theories not discussed in this text (De Rham, crystaline, etc).

[1] There are other realization functors which we do not discuss in this text.

A *mixed Hodge structure* (abridged MHS) is a triple $(H, W_\bullet, F^\bullet)$ where H is a finite dimensional $\mathbb{R}$-vector space, $W_\bullet$ is an increasing filtration of H (the *weight filtration*) and $F^\bullet$ is a decreasing filtration of $H \otimes_{\mathbb{R}} \mathbb{C}$ (the *Hodge filtration*). The weight and Hodge filtrations are subject to a certain condition which guarantees that the category of MHS with morphisms preserving filtrations is abelian ([DelII]). Given an infinite place $v : F \hookrightarrow \mathbb{C}$, to a variety X over F one associates a MHS whose underlying $\mathbb{R}$-vector space is its Betti-realization at v:

$$H := \Lambda_v(h^i(X)) \otimes_{\mathbb{Q}} \mathbb{R} = H^i((X \times_v \mathbb{C})(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}. \tag{0.1}$$

Note that the complex conjugation c acting on the $\mathbb{C}$-points defines an involution on H, denoted by ϕ_∞, preserving $W_\bullet$ and such that $\phi_\infty \otimes c$ preserves $F^\bullet$. Such an involution is usually referred to as an *infinite Frobenius* (e.g. [Nek]). The category $\mathcal{MH}_{\mathbb{R}}^+$ appearing in the above diagram is the category of pairs $(\underline{H}, \phi_\infty)$ where $\underline{H}$ is a MHS and where ϕ_∞ is an infinite Frobenius. As presented in the diagram, the existence of an exact functor

$$\mathcal{H}^+ : \mathcal{MM}_F \longrightarrow \mathcal{MH}_{\mathbb{R}}^+$$

is expected, so that the association $X \mapsto (H, W_\bullet, F^\bullet)$ factors through $\mathcal{H}^+$. According to Deligne [Del, §1.3], the category $\mathcal{MM}_F$ should admit a *weight filtration* in the sense of Jannsen in [Jan1, Def 6.3], which would coincide with the filtration $W_\bullet$ on (0.1). The *weights* of a mixed motive M would then be defined as the breaks of its weight filtration.

From the Tannakian formalism, $\mathcal{MM}_F$ admits a tensor operation, extending the fiber product on varieties, and we fix $\mathbb{1}$ a neutral object. According to Beilinson [Bei1, §0.3] (see also [Andr2, Def 17.2.11]), the *motivic cohomology of M* is defined as the complex

$$\mathrm{RHom}_{\mathcal{MM}_F}(\mathbb{1}, M)$$

in the derived category of $\mathbb{Q}$-vector spaces. Its ith cohomology is the $\mathbb{Q}$-vector space $\mathrm{Ext}^i_{\mathcal{MM}_F}(\mathbb{1}, M)$, the ith Yoneda extension space of $\mathbb{1}$ by M in $\mathcal{MM}_F$. We quote from [Sch, §2] and [Del, §1.3] respectively:

Conjecture. We expect that:

(C1) for $i \notin \{0, 1\}$, $\mathrm{Ext}^i_{\mathcal{MM}_F}(\mathbb{1}, M) = 0$,

(C2) if the weights of M are positive, $\mathrm{Ext}^1_{\mathcal{MM}_F}(\mathbb{1}, M) = 0$.

Admitting those two conjectures, the interesting part of $\mathrm{RHom}_{\mathcal{MM}_F}(\mathbb{1}, M)$ is then concentrated in degree 1. From now on, we focus on the first Yoneda extension space $\mathrm{Ext}^1_{\mathcal{MM}_F}(\mathbb{1}, M)$.

A subspace thereof of fundamental importance is the space of *extension having everywhere good reduction.* Given a prime number ℓ, one predicts that the ℓ-adic realization V_ℓ in the above diagram is exact. Given a mixed motive M over F, this allows to construct a $\mathbb{Q}$-linear morphism, called *the ℓ-adic realization map of M*,

$$r_{M,\ell} : \operatorname{Ext}^1_{\mathcal{MM}_F}(\mathbb{1}, M) \longrightarrow H^1(G_F, V_\ell M)$$

which maps the class of an exact sequence $[E] : 0 \to M \to E \to \mathbb{1} \to 0$ in $\mathcal{MM}_F$ to the class of the exact sequence $[V_\ell E] : 0 \to V_\ell M \to V_\ell E \to V_\ell \mathbb{1} \to 0$ in $\operatorname{Rep}_{\mathbb{Q}_\ell}(G_F)$.

Scholl in [Sch] defines the *integral part* of the motivic cohomology as follows. Given a finite place $\mathfrak{p}$ of F not above ℓ, it is said that $[E] \in \operatorname{Ext}^1_{\mathcal{MM}_F}(\mathbb{1}, M)$ has *good reduction at* $\mathfrak{p}$ if $r_{M,\ell}([E])$ splits as a representation of $I_\mathfrak{p}$, the inertia group of $\mathfrak{p}$ (that is, $[V_\ell E]$ is zero in $H^1(I_\mathfrak{p}, G_F)$). In [Sch, §2 Rmk], Scholl conjectures:

Conjecture. We expect that:

(C3) The property that $[\underline{E}]$ has good reduction at $\mathfrak{p}$ is independent of the chosen prime ℓ such that $\ell \nmid \mathfrak{p}$.

Admitting (C3), the extension $[E]$ is said to have *everywhere good reduction* if E has good reduction at $\mathfrak{p}$ for all finite places $\mathfrak{p}$ of F. The set $\operatorname{Ext}^1_{\mathcal{O}_F}(\mathbb{1}, M)$ of extensions having everywhere good reduction defines a natural $\mathbb{Q}$-subspace of $\operatorname{Ext}^1_{\mathcal{MM}_F}(\mathbb{1}, M)$ which is at the heart Beilinson's conjectures (see below).

The forecasted property that the Hodge realization $\mathcal{H}^+$ is exact would similarly induce a $\mathbb{Q}$-linear morphism

$$\operatorname{Reg}(M) : \operatorname{Ext}^1_{\mathcal{MM}_F}(\mathbb{1}, M) \longrightarrow \operatorname{Ext}^1_{\mathcal{MH}^+_\mathbb{R}}(\mathbb{1}, \mathcal{H}^+(M)). \tag{0.2}$$

This is, or rather should be, *Beilinson's regulator for M*. Because the category $\mathcal{MH}^+_\mathbb{R}$ is $\mathbb{R}$-linear and have finite dimensional extension spaces over $\mathbb{R}$, the right hand side should have finite dimension. It is a much more elementary object than the source $\operatorname{Ext}^1_{\mathcal{MM}_F}(\mathbb{1}, M)$. The first Beilinson's conjecture can be formulated as follow.

Conjecture (Beilinson)**.** We expect that

(*i*) The $\mathbb{Q}$-vector space $\operatorname{Ext}^1_{\mathcal{O}_F}(\mathbb{1}, M)$ has finite dimension.

(*ii*) If the weights of M are < -2, the regulator $\operatorname{Reg}(M)$ induces an isomorphism of $\mathbb{R}$-vector spaces

$$\operatorname{Ext}^1_{\mathcal{O}_F}(\mathbb{1}, M) \otimes_\mathbb{Q} \mathbb{R} \cong \operatorname{Ext}^1_{\mathcal{MH}^+_\mathbb{R}}(\mathbb{1}, \mathcal{H}^+(M)).$$

Even if these conjectures are surrounded by heavy hypothetical theory, they have down-to-earth applications. We present some of these in a nutshell[2]. Let

[2] I thank François Brunault who taught me these examples.

$\mathbb{Q}(1) := h^1(\mathbb{G}_{m,F})^\vee$ in $\mathcal{MM}_F$ be the *Tate twist* over F and, given M in $\mathcal{MM}_F$ and $n \geq 1$, denote by $M(n)$ the mixed motive $M \otimes \mathbb{Q}(1)^{\otimes n}$. For $M = h^0(\operatorname{Spec} F)$, it is expected that $\operatorname{Ext}^1_{\mathcal{MM}_F}(\mathbb{1}, h^0(\operatorname{Spec} F)(1))$ is isomorphic to $F^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ and that $\operatorname{Ext}^1_{\mathcal{O}_F}(\mathbb{1}, h^0(\operatorname{Spec} F)(1))$ corresponds to $\mathcal{O}_F^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ under this isomorphism. Conjecture (i) then implies that $\mathcal{O}_F^\times$ modulo torsion is a finitely generated group (weak version of Dirichlet's unit Theorem). Given an abelian variety A over F and for $M = h^1(A)(1)$, $\operatorname{Ext}^1_{\mathcal{MM}_F}(\mathbb{1}, h^1(A)(1))$ should equal $\operatorname{Ext}^1_{\mathcal{O}_F}(\mathbb{1}, h^1(A)(1))$ and be isomorphic to $A(F) \otimes_{\mathbb{Z}} \mathbb{Q}$. Conjecture (i) implies that $A(F)$ modulo torsion has finite rank (weak version of Mordell-Weil Theorem).

Remark. The second Beilinson's conjecture, not discussed in this text, relates the *determinant* of $\operatorname{Reg}(M)$, relative to certain $\mathbb{Q}$-structures, to the *special L-value of M*. We refer to [Nek] for a more detailed introduction on Beilinson's conjectures.

0.2 The function field picture

Despite its intrisic obscurities, Motivic cohomology remains a difficult subject also because its definition sits on a completely conjectural framework. The present thesis grew out as an attempt to understand the analogous picture in function fields arithmetic. There, the theory looks more promising using *Anderson A-motives*, instead of classical motives, whose definition is not conjectural. This parallele has been drawn by many authors and led to celebrated achievements. The analogue of the Tate conjecture [Tag] [Tam], of Grothendieck's periods conjecture [Pap] and of the Hodge conjecture [HarJu] are now theorems on the function fields side. The recent volume [tMo] records some of these feats. Counterparts of Beilinson's conjectures in function fields arithmetic have not been studied yet, although very recent works of Taelman, V. Lafforgue, Mornev and Anglès-Ngo Dac-Tavares Ribeiro on class formulas [Tae3], [LafV], [Mo1], [ANT] strongly suggest the pertinence of such a project.

A disclaimer:
In the present document, we do not pretend to explain how *Anderson A-motives* are analogous to *classical motives.* We rather relate to the surveys articles in [tMo]. However, we hope the results presented below will give more intuitions and guidance on the parallel drawn between A-motives and classical motives.

The setting

Let us first give the setting and notations for global fields in positive characteristic.

Let $\mathbb{F}$ be a finite field, q its number of elements, and let $(C, \mathcal{O}_C)$ be a geometrically irreducible smooth projective curve over $\mathbb{F}$. The curve C is determined up to $\mathbb{F}$-isomorphism by its function field $K := \mathbb{F}(C)$. The set of closed points p of C is in correspondence with the set of discrete valuation rings O_p in K with field of fraction K. We write $\mathcal{O}_p$ for the completion of O_p with respect to the valuation v_p of O_p.

Let ∞ be a closed point on C with associated valuation v_∞ and consider the ring

$$A := \{x \in K \mid v_p(x) \geq 0 \text{ for all closed point } p \text{ of } C \text{ distinct from } \infty\}.$$

Geometrically, A corresponds to $\Gamma(C \setminus \{\infty\}, \mathcal{O}_C)$ and the field of fractions of A identifies with K. We let K_∞ be its completion with respect to v_∞, $\mathbb{C}_\infty$ be the completion of an algebraic closure of K_∞ and K_∞^s be the separable closure of K_∞ inside $\mathbb{C}_\infty$. We let $G_\infty = \mathrm{Gal}(K_\infty^s|K_\infty)$ be the absolute Galois group of K_∞. The analogy with number fields that should guide us in this text is:

$$\begin{array}{lccccccccccc} \text{Number fields:} & \mathbb{Z} & \subset & \mathbb{Q} & \subset & \mathbb{R} & \subset & \mathbb{C} & = & \mathbb{C} & & \mathrm{Gal}(\mathbb{C}|\mathbb{R}) \\ & \wr & & \wr & & \wr & & \wr & & \wr & & \wr \\ \text{Function fields:} & A & \subset & K & \subset & K_\infty & \subset & K_\infty^s & \subset & \mathbb{C}_\infty & & G_\infty \end{array}$$

Giving F a finite extension of K is equivalent to give a smooth projective curve X over $\mathbb{F}$ together with a non-constant morphism $f : X \to C$ of algebraic curves. We partition the closed points of X into the *infinite points* being the one *above* ∞ via f, and the other ones, the *finite points*. The ring of integers $\mathcal{O}_F$ of F is defined as the integral closure of A in F, and is equivalently described by

$$\mathcal{O}_F = \{x \in K \mid v_p(x) \geq 0 \text{ for all finite places of } F\}.$$

The analogy with number fields disappears when one considers the fiber product $C \times X$, which is at the heart of the definition of Anderson A-motives (unlabeled fiber and tensor products are over $\mathbb{F}$). On the surface $C \times X$, we consider the morphism τ which acts as the identity on C and as the q-Frobenius on X. $C \times X$ admits $\mathrm{Spec}(A \otimes F)$ as an affine Dedekind subscheme, and we let $\mathfrak{j}$ be the maximal ideal of $A \otimes F$ generated by the set $\{a \otimes 1 - 1 \otimes a | a \in A\}$.

Following [And], an *Anderson A-motive $\underline{M}$ over F* is a pair (M, τ_M) where M designates a finite locally free $A \otimes F$-module of constant rank, and where $\tau_M : (\tau^* M)[\mathfrak{j}^{-1}] \to M[\mathfrak{j}^{-1}]$ is an $(A \otimes F)[\mathfrak{j}^{-1}]$-linear isomorphism (see Definition 1.2). We let $\mathcal{M}_F$ denote the category of Anderson A-motives with obvious morphisms. $\mathcal{M}_F$ is known to be A-linear, rigid monoidal, and is exact in the sense of Quillen but not abelian ([HarJu, §2.3] or Section 1.1). Let $\mathbb{1}$ in $\mathcal{M}_F$ be a neutral object for the tensor operation.

Extensions of A-motives

The category $\mathcal{M}_F$, or rather full subcategories of it, will play the role of the category of Grothendieck's motives. Guided by this, the next theorem already describes the analogue of motivic cohomology in an explicit manner, and is the starting point of our research (see Theorem 3.4). Let $\underline{M}$ be an A-motive over F.

Theorem A. *The complex* $\left[M \stackrel{\mathrm{id}-\tau_M}{\longrightarrow} M[\mathfrak{j}^{-1}]\right]$ *of* A*-modules placed in degree* 0 *and* 1 *represents the complex* $\mathrm{RHom}_{\mathcal{M}_F}(\mathbb{1}, \underline{M})$.

We immediately deduce that $\mathrm{Ext}^i_{\mathcal{M}_F}(\mathbb{1}, \underline{M})$ is zero for $i > 1$, revealing that the analogue of the number fields conjecture (C1) is true for function fields. For $i = 1$, one obtains an isomorphism

$$\iota : \frac{M[\mathfrak{j}^{-1}]}{(\mathrm{id}-\tau_M)(M)} \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{M}_F}(\mathbb{1}, \underline{M}) \tag{0.3}$$

which can be explicitely described by mapping a representative $m \in M[\mathfrak{j}^{-1}]$ to the class of the extension of $\mathbb{1}$ by $\underline{M}$ given by $[M \oplus (A \otimes F), \left(\begin{smallmatrix} \tau_M & m \\ 0 & 1 \end{smallmatrix}\right)]$ (Section 3.1).

Remark. Extension groups in the full subcategory of $\mathcal{M}_F$ consisting of *effective A-motives* (see Definition 1.4) were already determined in the existing literature (see e.g. [Tae4], [Tae5], [PapRa]). The novelty of Theorem A is to consider the whole category $\mathcal{M}_F$.

To pursue the analogy with number fields, we now present the notion of *mixedness* and *weights* for Anderson A-motives. In the case $A = \mathbb{F}[t]$ or $\deg(\infty) = 1$ with base field $\mathbb{C}_\infty$, the corresponding definitions were carried out respectively by Taelman [Tae1] and Hartl-Juschka [HarJu]. We completed this picture in the most general way (any global field F and without any restriction on $\deg(\infty)$).

To an Anderson A-motive $\underline{M}$ over F, we attach an *isocrystal* $\mathcal{I}_\infty(\underline{M})$ over F at ∞ (in the sense of [Mor2]). The term *isocrystal* is borrowed from p-adic Hodge theory, where the function field setting allows to apply the non-archimedean theory at the infinite point ∞ of C as well. In Section 1.2, we prove that the isocrystal $\mathcal{I}_\infty(\underline{M})$ carries a uniquely determined *slope filtration* (see Definition 1.18):

$$0 = \mathcal{I}_\infty(\underline{M})_{\mu_0} \subsetneq \mathcal{I}_\infty(\underline{M})_{\mu_1} \subsetneq \mathcal{I}_\infty(\underline{M})_{\mu_2} \subsetneq \cdots \subsetneq \mathcal{I}_\infty(\underline{M})_{\mu_s} = \mathcal{I}_\infty(\underline{M}) \tag{0.4}$$

for uniquely determined rational numbers $\mu_1 < ... < \mu_s$ called the *weights of* $\underline{M}$. We say that $\underline{M}$ is *mixed* if there exists an increasing filtration $(W_{\mu_i}\underline{M})_{1\leq i\leq s}$ of $\underline{M}$ by subobjects in $\mathcal{M}_F$ whose associated filtration $(\mathcal{I}_\infty(W_{\mu_i}\underline{M}))_{1\leq i\leq s}$ of $\mathcal{I}_\infty(\underline{M})$ by subisocrystals matches with (0.4) (Definition 1.30). We let $\mathcal{MM}_F$ be the full subcategory of $\mathcal{M}_F$ whose objects are mixed Anderson A-motives over F (Section 1.2).

In Section 3.2, we prove:

Theorem B. *Let* $\underline{M}$ *be an object of* $\mathcal{MM}_F$. *If all the weights of* $\underline{M}$ *are negative, then every extension of* $\mathbb{1}$ *by* $\underline{M}$ *is mixed, that is:*

$$\mathrm{Ext}^1_{\mathcal{MM}_F}(\mathbb{1}, \underline{M}) = \mathrm{Ext}^1_{\mathcal{M}_F}(\mathbb{1}, \underline{M}).$$

If all the weights of $\underline{M}$ *are positive, then an extension of* $\mathbb{1}$ *by* $\underline{M}$ *is mixed if and only if its class is torsion, that is:*

$$\mathrm{Ext}^1_{\mathcal{MM}_F}(\mathbb{1}, \underline{M}) = \mathrm{Ext}^1_{\mathcal{M}_F}(\mathbb{1}, \underline{M})^{tors}.$$

Remark. Although the category of classical mixed motives is expected to be $\mathbb{Q}$-linear, the category $\mathcal{MM}_F$ of mixed Anderson A-motives over F is only A-linear. To obtain a K-linear category, it might be convenient to introduce $\widetilde{\mathcal{MM}}_F$ whose objects are the ones of $\mathcal{MM}_F$ and whose Hom-spaces are given by $\mathrm{Hom}_{\mathcal{MM}_F}(-,-)\otimes_A K$. In the literature, $\widetilde{\mathcal{MM}}_F$ is called the category of mixed A-motives over F *up to isogenies* [Har2], [HarJu]. Theorem B implies that $\mathrm{Ext}^1_{\widetilde{\mathcal{MM}}_F}(\mathbb{1},\underline{M})=0$ if the weights of $\underline{M}$ are positive. This is the function fields' formulation of conjecture (C2).

Extensions having good reduction

Let F^s be a separable closure of F and let $G_F=\mathrm{Gal}(F^s|F)$ be the absolute Galois group of F equipped with the profinite topology. Given λ a closed point of C distinct from ∞, there is a λ-*adic realization functor* from $\mathcal{M}_F$ to the category of continuous $\mathcal{O}_\lambda$-linear representations of G_F. For $\underline{M}$ an object of $\mathcal{M}_F$, it is given by the $\mathcal{O}_\lambda$-module

$$T_\lambda\underline{M}=\varprojlim_n\{m\in(M/\mathfrak{m}_\lambda^nM)\otimes_F F^s \mid m=\tau_M(\tau^*m)\}$$

where $\mathfrak{m}_\lambda$ is the maximal ideal of A corresponding to the point λ, and where G_F acts on the right of the tensor $(M/\mathfrak{m}_\lambda^nM)\otimes_F F^s$ (Definition 1.42). We prove in Corollary 1.46 that T_λ is exact.

This paves the way for introducing *extensions with good reduction* in $\mathcal{M}_F$, as Scholl did in the number fields setting. For a finite point $\mathfrak{p}$ of X not above λ with inertia group $I_\mathfrak{p}$, we consider the λ-*adic realization map* restricted to $I_\mathfrak{p}$:

$$r_{\underline{M},\lambda}:\mathrm{Ext}^1_{\mathcal{M}_F}(\mathbb{1},\underline{M})\longrightarrow H^1(I_\mathfrak{p},T_\lambda\underline{M}) \tag{0.5}$$

(we refer to Section 3.3). Mimicking Scholl's approach, we say that an extension $[\underline{E}]$ of $\mathbb{1}$ by $\underline{M}$ has *good reduction at* $\mathfrak{p}$ if $[\underline{E}]$ lies in the kernel of (0.5), and we let $\mathrm{Ext}^1_{\mathcal{M}_F,\mathcal{O}_\mathfrak{p}}(\mathbb{1},\underline{M})$ denote the kernel of (0.5) (Section 3.3). As expected in the number fields setting, we prove (consequence of Theorem 3.15):

Theorem C. *The A-module $\mathrm{Ext}^1_{\mathcal{M}_F,\mathcal{O}_\mathfrak{p}}(\mathbb{1},\underline{M})$ is independent of the place λ.*

This is the function field analogue of Conjecture (C3). We say that $[\underline{E}]\in\mathrm{Ext}^1_{\mathcal{M}_F}(\mathbb{1},\underline{M})$ has *everywhere good reduction* if $[\underline{E}]$ has good reduction at $\mathfrak{p}$ for all finite points $\mathfrak{p}$ of X. We let $\mathrm{Ext}^1_{\mathcal{M}_F,\mathcal{O}_F}(\mathbb{1},\underline{M})$ be the A-module of extensions with everywhere good reduction.

To prove Theorem C, we developed the notion of *integral models of A-motives* (Chapter 2). They form the function field analogue of Néron models of abelian varieties, or more generally, of proper flat models over $\mathrm{Spec}\,\mathcal{O}_F$ of varieties over $\mathrm{Spec}\,F$. We found inspiration for the next definition in the work of Gardeyn [Gar2], where he introduced the eponymous notion in the context of τ-sheaves.

Definition (Definition 2.10). An *$\mathcal{O}_F$-model for $\underline{M}$* is a finite sub-$A\otimes\mathcal{O}_F$-module L of M which generates M over F, and such that $\tau_M(\tau^*L)\subset L[\mathfrak{j}^{-1}]$. We say that L is *maximal* if L is not strictly contained in any other $\mathcal{O}_F$-models for $\underline{M}$.

As opposed to [Gar2, Def 2.1 & 2.3], we do not ask for an $\mathcal{O}_F$-model to be locally free. We show that this is implicit for maximal ones using Bourbaki's flatness criterion (Proposition 2.32). Compared to Gardeyn, our exposition is therefore simplified and avoids the use of a technical lemma due to L. Lafforgue [Gar2, §2.2]. Our next result should be compared with [Gar2, Prop 2.13] (see Propositions 2.30, 2.32 in the text).

Proposition. *A maximal $\mathcal{O}_F$-model $M_{\mathcal{O}}$ for $\underline{M}$ exists and is unique. It is locally free over $A\otimes\mathcal{O}_F$.*

The next theorem determines explicitly the module $\operatorname{Ext}^1_{\mathcal{M}_F,\mathcal{O}_F}(\mathbb{1},\underline{M})$ in terms of $M_{\mathcal{O}}$.

Theorem D. *The morphism ι in* (0.3) *induces an isomorphism of A-modules*

$$\frac{M_{\mathcal{O}}[\mathfrak{j}^{-1}]}{(\mathrm{id}-\tau_M)(M_{\mathcal{O}})}\xrightarrow{\sim}\operatorname{Ext}^1_{\mathcal{M}_F,\mathcal{O}_F}(\mathbb{1},\underline{M}).$$

We observe that $\operatorname{Ext}^1_{\mathcal{M}_F,\mathcal{O}_F}(\mathbb{1},\underline{M})$ in general is *not* a finitely generated A-module. This prevents the naive analogue of Beilinson's conjecture (i) to hold. To understand precisely why $\operatorname{Ext}^1_{\mathcal{M}_F,\mathcal{O}_F}(\mathbb{1},\underline{M})$ is not the *right* analogue of number fields' $\operatorname{Ext}^1_{\mathcal{O}_F}(\mathbb{1},M)$ of Section 0.1 and present the natural submodule which would play this role, a discussion of extensions of *function fields Mixed Hodge Structures* is called for.

Hodge realization functor

We define a *mixed Hodge Structure* (MHS) to be a triple $(H,W_\bullet,F^\bullet)$ where H is a finite dimensional K_∞-vector space, $W_\bullet$, the *weight filtration*, is an increasing rational filtration of H and $F^\bullet$, the *Hodge filtration*, is a decreasing filtration of $H_{K_\infty^s}=H\otimes_{K_\infty}K_\infty^s$ such that the weight and Hodge filtrations are submitted to a *local semi-stability* condition (Definition 4.6). They are the analogue of the eponymous structure for number fields, where K_∞ is replaced by $\mathbb{R}$. The category of MHS is denoted $\mathcal{MH}_{K_\infty}$.

More relevant to function field arithmetic are *Mixed Hodge-Pink Structures* (MHPS), introduced and extensively studied in [Pin]. They form a finner version of MHS where the Hodge filtration is replaced by the refined data of the *Hodge-Pink lattice* $\mathfrak{q}$ (Subsection 4.3). We denote $\mathcal{MHP}_{K_\infty}$ the category of MHPS. To any MHPS $(H,W_\bullet,\mathfrak{q})$ is associated an induced Hodge filtration $F^\bullet$ (Subsection 4.3.2). However, the datum of $(H,W_\bullet,F^\bullet)$ does not necessarily define a MHS (see Subsection 4.3.4).

To accord with the classical analogy, one requires infinite Frobenius in addition to Pink's theory. In our setting, we suggest the following alternative definition (Section 4.2):

Definition (Definitions 4.8 and 4.21)**.** Let $\underline{H}$ be a MHS (resp. MHPS) with underlying space H. An *infinite Frobenius for* $\underline{H}$ is a continuous K_∞-linear representation

$$\phi : G_\infty \to \mathrm{End}_{K_\infty}(H)$$

such that, for all $\sigma \in G_\infty$, $\phi(\sigma)$ preserves the weight filtration $\phi(\sigma)\otimes\sigma$ preserves the Hodge filtration (resp. the Hodge-Pink lattice).

As we saw in the number fields setting (Section 0.1), Beilinson's regulator is defined via the Hodge realization functor. However, the naive replicate of what is done for number fields does not work in our setting, and the next discussion aims to understand the reasons why.

We require few additional notations. For the sake of the introduction, we simplify slightly our setting by assuming that all A-motives are over K (we treat the case of A-motives over a finite extension of K in the text). Let $\underline{M}$ be a mixed A-motive.

The norm on $\mathbb{C}_\infty$ extends canonically to a norm on $A \otimes \mathbb{C}_\infty$, called the *Gauss norm*, for which nonzero elements of $A \otimes 1$ have norm 1. We denote $\mathbb{C}_\infty\langle A\rangle$ the completion of $A \otimes \mathbb{C}_\infty$ with respect to the Gauss norm (it was denoted $\mathbb{T}$ in [GazMa, §2]). The *Betti realization* $\Lambda(\underline{M})$ of $\underline{M}$ is the A-module

$$\Lambda(\underline{M}) := \{\omega \in M \otimes_{A\otimes K} \mathbb{C}_\infty\langle A\rangle \mid \omega = \tau_M(\tau^*\omega)\}.$$

In Section 1.4, we make the new observation that $\Lambda(\underline{M})$ is naturally endowed with a continuous action of G_∞ and compute the H^1 of this action in Theorem 1.56.

Following Anderson [And, §2], we say that $\underline{M}$ is *rigid analytically trivial* if the rank of $\Lambda(\underline{M})$ over A equals the rank of $\underline{M}$ (Definition 1.50). We let $\mathcal{MM}_K^{\mathrm{rig}}$ denote the full subcategory of $\mathcal{MM}_K$ whose objects are rigid analytically trivial.

Announced in [HarJu] and proved in [HarPi], there is an exact functor

$$\mathscr{H}^+ : \mathcal{MM}_K^{\mathrm{rig}} \longrightarrow \mathcal{MHP}_{K_\infty}^+$$

(See Definition 5.3), where $\mathcal{MHP}_{K_\infty}^+$ is the category of MHPS enriched with infinite Frobenius. This is the *Hodge-Pink realization functor*. As a striking difference with the classical picture, given a rigid analytically trivial mixed motive $\underline{M}$ over K, the data of $\mathscr{H}^+(\underline{M})$ and its induced Hodge filtration does not necessarily define a MHS. In other terms, the category $\mathcal{MM}_K^{\mathrm{rig}}$ is *too large* to be the source of a Hodge realization functor targeting $\mathcal{MH}_{K_\infty}^+$.

Regulated A-motives

Let $\underline{H} = (H, W_\bullet, \mathfrak{q})$ be a MHPS and let $F^\bullet$ be its induced Hodge filtration (Definition 4.23). As explained above, the data of $\underline{H}^\# := (H, W_\bullet, F^\bullet)$ does not necessarily define a MHS. In Subsection 4.3.3, we show that there exists a largest full and exact subcategory of $\mathcal{MHP}_{K_\infty}$, denoted $\mathcal{MHP}^{hd}_{K_\infty}$, such that the functor

$$\# : \mathcal{MHP}^{\mathrm{hd}}_{K_\infty} \longrightarrow \mathcal{MH}_{K_\infty}, \quad \underline{H} \longmapsto \underline{H}^\# \tag{0.6}$$

is well-defined and exact. Objects of $\mathcal{MHP}^{hd}_{K_\infty}$ are said to have *Hodge descent* (Definition 4.41).

At the side of A-motives, we say that an object $\underline{M}$ of $\mathcal{MM}^{\mathrm{rig}}_K$ is *regulated* whenever $\mathscr{H}^+(\underline{M})$ has Hodge descent. We choose the naming "regulated" to refer to the upcoming *regulators*. On the full subcategory $\mathcal{MM}^{\mathrm{reg}}_K$ they form, we define the exact functor:

$$\mathcal{H}^+ : \mathcal{MM}^{\mathrm{reg}}_K \longrightarrow \mathcal{MH}^+_{K_\infty}$$

as the composition of $\mathscr{H}^+$ and (0.6). We call $\mathcal{H}^+$ the *Hodge realization functor* (Definition 5.14).

General and Special regulators

Let $\underline{M}$ be an object of $\mathcal{MM}^{\mathrm{reg}}_K$. The exactness of the Hodge realization functor induces an K_∞-linear morphism at the level of extensions

$$\mathrm{Reg}(\underline{M}) : \mathrm{Ext}^1_{\mathcal{MM}^{\mathrm{reg}}_K}(\mathbb{1}, \underline{M}) \otimes_A K_\infty \longrightarrow \mathrm{Ext}^1_{\mathcal{MHP}^{\mathrm{hd}}_{K_\infty}}(\mathbb{1}, \mathcal{H}^+(\underline{M}))$$

which we call the *special regulator* of $\underline{M}$.

For various reasons, working in the category $\mathcal{MM}^{\mathrm{reg}}_K$ might be too restrictive. For instance, it has way less objects than $\mathcal{MM}^{\mathrm{rig}}_K$. We figured it was interesting for future applications to remove the "regulated" assumption on $\underline{M}$ and to displace it on the side of extensions. For $\underline{M}$ an object of $\mathcal{MM}^{\mathrm{rig}}_K$, we define a natural sub-A-module

$$\mathrm{Ext}^{1,\mathrm{reg}}_{\mathcal{MM}^{\mathrm{rig}}_K}(\mathbb{1}, \underline{M})$$

of $\mathrm{Ext}^1_{\mathcal{MM}^{\mathrm{rig}}_K}(\mathbb{1}, \underline{M})$ which coincides with $\mathrm{Ext}^1_{\mathcal{MM}^{\mathrm{reg}}_K}(\mathbb{1}, \underline{M})$ whenever $\underline{M}$ is regulated (Definition 5.16). Similarly, given a mixed Hodge structure $\underline{H}$, there is a natural K_∞-subspace

$$\mathrm{Ext}^{1,\mathrm{ha}}_{\mathcal{MHP}_{K_\infty}}(\mathbb{1}, \underline{H})$$

of $\mathrm{Ext}^1_{\mathcal{MHP}_{K_\infty}}(\mathbb{1}, \underline{H})$, already considered by Pink [Pin, §8], which coincides with $\mathrm{Ext}^1_{\mathcal{MHP}^{\mathrm{hd}}_{K_\infty}}(\mathbb{1}, \underline{H})$ whenever $\underline{H}$ has Hodge descent (Definition 4.50). The supscript *ha* stands for *Hodge additive*.

We prove that the exactness of $\mathscr{H}^+$ induces an K_∞-linear morphism

$$\mathscr{Reg}(\underline{M}) : \mathrm{Ext}^{1,\mathrm{reg}}_{\mathcal{MM}^{\mathrm{rig}}_K}(\mathbb{1}, \underline{M}) \otimes_A K_\infty \longrightarrow \mathrm{Ext}^{1,\mathrm{ha}}_{\mathcal{MHP}_{K_\infty}}(\mathbb{1}, \mathscr{H}^+(\underline{M}))$$

which we call the *general regulator* of $\underline{M}$ (Definition 5.21).

Analytic reduction at ∞

However, because extension spaces in the categories $\mathcal{MH}^+_{K_\infty}$ and $\mathcal{MHP}^+_{K_\infty}$ are intertwined with the continuous cohomology of the profinite group G_∞, the targeted vector spaces of $\operatorname{Reg}(\underline{M})$ and $\mathscr{Reg}(\underline{M})$ do not have finite dimension over K_∞ in general (see Section 4.2). For a similar reason, $\operatorname{Ext}^{1,\mathrm{reg}}_{\mathcal{MM}^{\mathrm{rig}}_K}(\mathbb{1},\underline{M})$ is not a finitely generated A-module (Corollary 5.30).

Inspired by Taelman in [Tae2] in the context of Drinfeld modules, we consider in Subsection 5.2 the morphism of A-modules

$$r_{\underline{M},\infty}:\operatorname{Ext}^1_{\mathcal{MM}^{\mathrm{rig}}_K}(\mathbb{1},\underline{M})\longrightarrow H^1(G_\infty,\Lambda(\underline{M}))$$

induced by the exactness of the functor Λ ([HarJu, Lem. 2.3.25] or Corollary 1.61). We prove the following theorem, which in many aspects resembles to [Tae2, Thm. 1] (or [Mo1, Thm. 1.2]) for Drinfeld modules.

Theorem E. *The A-linear morphism*

$$r_{\underline{M},\infty}:\operatorname{Ext}^{1,reg}_{\mathcal{MM}^{rig}_K,A}(\mathbb{1},\underline{M})\to H^1(G_\infty,\Lambda(\underline{M})),$$

where the index "A" refers to "everywhere good reduction", has finitely generated kernel and cokernel. If, in addition, all the weights of $\underline{M}$ are negative, then $\operatorname{coker}(r_{\underline{M},\infty})$ *is finite.*

Definition (Definition 5.26)**.** We let $\operatorname{Ext}^{1,\mathrm{reg},\infty}_{\mathcal{MM}^{\mathrm{rig}}_K,A}(\mathbb{1},\underline{M})$ be the finitely generated A-module given by the kernel of $r_{\underline{M},\infty}$.

Remark. In the number field case, the analogue of the K-vector space

$$\operatorname{Ext}^{1,\mathrm{reg},\infty}_{\mathcal{MM}^{\mathrm{rig}}_K,A}(\mathbb{1},\underline{M})\otimes_A K$$

would be $\operatorname{Ext}^1_{\mathcal{MM}_{\mathbb{Q}},\mathbb{Z}}(\mathbb{1},M)$ because $H^1(\operatorname{Gal}(\mathbb{C}|\mathbb{R}),-)$ is torsion. To that extent, Theorem E is the analogue of Conjecture (i).

At the side of Hodge-Pink structures, and under the assumption that the weights of $\underline{M}$ are negative, we construct similarly a surjective morphism of K_∞-vector spaces

$$d_{\underline{M}}:\operatorname{Ext}^{1,\mathrm{ha}}_{\mathcal{MHP}^+_{K_\infty}}(\mathbb{1}^+,\mathcal{H}^+(\underline{M}))\longrightarrow H^1(G_\infty,\Lambda(\underline{M})\otimes_A K_\infty),$$

functorially attached to $\underline{M}$, whose kernel has finite dimension over K_∞ (Definition 4.53). We denote by $\operatorname{Ext}^{1,\mathrm{ha},\infty}_{\mathcal{MHP}^+_{K_\infty}}(\mathbb{1}^+,\mathcal{H}^+(\underline{M}))$ the kernel of $d_{\underline{M}}$ and give a name to its elements: the *extensions having analytic reduction at* ∞.

Remark. Similarly, classical extensions of mixed Hodge structures would all have "analytic reduction at ∞" in that sense.

Let $\underline{M}$ be an object of $\mathcal{MM}_K^{\mathrm{rig}}$ with negative weights. A key observation is that the general regulator of $\underline{M}$ induces a K_∞-linear morphism:

$$\mathscr{Reg}(\underline{M}) : \mathrm{Ext}^{1,\mathrm{reg},\infty}_{\mathcal{MM}_K^{\mathrm{rig}},A}(\mathbb{1},\underline{M}) \otimes_A K_\infty \longrightarrow \mathrm{Ext}^{1,\mathrm{ha},\infty}_{\mathcal{MHP}^+_{K_\infty}}(\mathbb{1}^+,\mathcal{H}^+(\underline{M}))$$

(this follows from Theorem 5.31). We prove (see Theorem 6.4 in the text):

Theorem F. *Let $\underline{M}$ be an object of $\mathcal{MM}_K^{rig}$ with negative weights. The rank of the A-module $\mathrm{Ext}^{1,reg,\infty}_{\mathcal{MM}_K^{rig},A}(\mathbb{1},\underline{M})$ equals the dimension of $\mathrm{Ext}^{1,ha,\infty}_{\mathcal{MHP}^+_{K_\infty}}(\mathbb{1},\mathcal{H}^+(\underline{M}))$ over K_∞.*

The proofs of Theorems E and F use methods close to Shtuka Cohomology as developed by Mornev in his thesis [Mo1]. To a mixed rigid analytically trivial A-motive $\underline{M} = (M,\tau_M)$ with negative weights, we associate *non-canonically* a *$C \times C$-shtuka model for $\underline{M}$* (Definition 6.9). The later is a triple $(\mathcal{M},\mathcal{N},\tau_{\mathcal{M}})$, where $\mathcal{N}$ is a coherent sheaf on $C\times C$, $\mathcal{M}$ is a subsheaf of $\mathcal{N}$ and $\tau_{\mathcal{M}} : \tau^*\mathcal{M} \to \mathcal{N}$ is a morphism of coherent sheaves, such that $\tau_{\mathcal{M}}$ restricts to

$$\tau_M : \tau^* M_A \longrightarrow M_A[\mathfrak{j}^{-1}] \cap (M + \tau_M(\tau^* M))$$

on $\mathrm{Spec}(A \otimes A)$ (here, M_A, the maximal A-model of $\underline{M}$ seen as an A-motive over K), which satifies some technical assumptions (see Definition 6.9). The (coherent and rigid) cohomology of a $C \times C$-shtuka model traces back the modules of extensions introduced earlier, in a way similar to the cohomology of *global models* of Drinfeld modules in [Mo1]. The cohomological tools we developped in Section 6.3 to prove Theorems E and F are inspired by computations of V. Lafforgue in [LafV].

Finally, let us mention that Theorem F reveals what should be the corresponding function fields' analogue of conjecture (*ii*). At last, we formulate:

Definition. Let $\underline{M}$ is a mixed rigid analytically trivial A-motive over K with negative weights. We say that $\underline{M}$ *satisfies Beilinson's conjecture* if the linear map

$$\mathscr{Reg}(\underline{M}) \otimes_A \mathrm{id}_{K_\infty} : \mathrm{Ext}^{1,\mathrm{reg},\infty}_{\mathcal{MM}_K^{\mathrm{rig}},A}(\mathbb{1},\underline{M}) \otimes_A K_\infty \longrightarrow \mathrm{Ext}^{1,\mathrm{ha},\infty}_{\mathcal{MHP}^+_{K_\infty}}(\mathbb{1},\mathcal{H}^+(\underline{M}))$$

is an isomorphism of K_∞-vector spaces.

Surprisingly, it will appear from Corollary 7.34 that not all A-motives satisfy Beilinson's conjecture. We leave the problem of characterizing A-motives not satisfying Beilinson's conjecture open, and hope to study it in near future works.

Motivic cohomology of the Carlitz tensor powers

In our last Chapter 7 we describe the module $\mathrm{Ext}^{1,\mathrm{reg},\infty}_{\mathcal{MM}_K^{\mathrm{rig}},A}(\mathbb{1},\underline{M})$ where $\underline{M}$ is the nth tensor power of the Carlitz motive over K, analog to the classical nth

Tate twist motive.

The setting is as follows. The curve C is the projective plane $\mathbb{P}^1_{\mathbb{F}}$ over $\mathbb{F}$ and ∞ is the point of coordinates $[0:1]$. The ring A is identified with $\mathbb{F}[t]$, where t^{-1} is a uniformizer in K of $\mathcal{O}_\infty$. The field K is identified with $\mathbb{F}(t)$ and K_∞ with $\mathbb{F}((t^{-1}))$. The valuation v_∞ at ∞ corresponds to minus the degree in t. We identify the tensor product $A \otimes K = \mathbb{F}[t] \otimes K$ with $\mathbb{F}(\theta)[t]$ where t corresponds to $t \otimes 1$ and θ to $1 \otimes t$. The ideal $\mathfrak{j}$ is principal, generated by $t - \theta$. The morphism τ maps a polynomial $p(t) \in \mathbb{F}(\theta)[t]$ to $p(t)^{(1)}$ whose coefficients in t have been raised to the qth power.

The nth tensor power of the *Carlitz motive* $\underline{C}^n$ is the A-motive whose underlying module is $K[t]$ and where the τ-action is given by $\tau^* p(t) \mapsto (t-\theta)^n p(t)^{(1)}$ over K. We rather denote by $\underline{A}(n)$ the dual of $\underline{C}^n$ to stress the analogy with the Tate twists $\mathbb{Z}(n)$.

As an application of the above theory, we prove (see Theorem 7.1):

Theorem G. *Let n be an integer. $\underline{A}(n)$ is an object in $\mathcal{MM}_K^{reg}$, and we have*

$$\mathrm{Ext}^{1,\infty}_{\mathcal{MM}_K^{reg},A}(\mathbb{1}, \underline{A}(n)) \cong \begin{cases} 0 & \text{if } n \leq 0, \\ \mathbb{F}[t]^n & \text{if } n > 0 \text{ and } q-1 \nmid n, \\ \mathbb{F}[t]^{n-1} \oplus \mathbb{F}[t]/(d_n(t)) & \text{if } n > 0 \text{ and } q-1 | n \end{cases}$$

where $d_n(t)$ is a certain monic polynomial. If $n = q^k(q-1)$ for some $k \geq 0$, then

$$d_n(t) = t^{q^{k+1}} - t^{q^k}.$$

The general determination of $d_n(t)$ remains a difficult task which we partially complete using linear relations among the Carlitz period and polylogarithms (Theorem 7.23). We leave open the question of determining those extension groups for tensor powers of the Carlitz motive over finite extensions of K.

Remark. Compared to the number fields situation, $\mathrm{Ext}^{1,\infty}_{\mathcal{MM}_K^{\mathrm{reg}},A}(\mathbb{1}, \underline{A}(n))$ is "too large by a $n-1$-rank": in [Del, §1.4] (see also [BeiDe, §1.11]), Deligne conjectures that for $n \geq 0$:

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{1}, \mathbb{Q}(n)) \cong K_{2n-1}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We deduce $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{1}, \mathbb{Q}(n))$ is zero if n is even and has dimension 1 if n is odd. In the function fields/number fields dictionnary, multiples of $q-1$ are the analogue of even integers.

For $n > 0$, explicit computations with the regulator morphism $\mathscr{R}eg(\underline{M})$ in the case where $\underline{M} = \underline{A}(n)$ makes Carlitz's polylogarithms naturally appear. This is highly reminiscent of computations by Deligne and Beilinson in [BeiDe].

For $\alpha \in \mathbb{C}_\infty$ with $v_\infty(\alpha) > -nq/(q-1)$, the nth Carlitz polylogarithm of α is defined the converging series

$$\mathrm{Li}_n(\alpha) := \alpha + \sum_{k=1}^{\infty} \frac{\alpha^{q^k}}{(\theta-\theta^q)^n(\theta-\theta^{q^2})^n\cdots(\theta-\theta^{q^k})^n}.$$

Let $\tilde{\pi} \in K_\infty^s$ be the Carlitz's period (function fields analogue of $2i\pi$). As an application of the ABP criterion [ABP] and results of Chang and Yu in [ChaYu] together with the above corollary, we obtain an algebraic independence result for the latter series (see Corollaries 7.28 and 7.30):

Theorem H. *Let* $(\alpha_1, ..., \alpha_n)$ *be a basis of* $\{f \in \mathbb{F}[\theta] \mid \deg f < n\}$. *The series* $\mathrm{Li}_n(\alpha_1)$, $\mathrm{Li}_n(\alpha_2)$, ..., $\mathrm{Li}_n(\alpha_n)$ *are algebraically independent over* K. *If further* $q-1 \nmid n$, *the series* $\tilde{\pi}$, $\mathrm{Li}_n(\alpha_1)$, $\mathrm{Li}_n(\alpha_2)$, ..., $\mathrm{Li}_n(\alpha_n)$ *are algebraically independent over* K.

Remark. In the case where $q-1|n$, there is a linear relation among the converging series $\tilde{\pi}^n$, $\mathrm{Li}_n(1)$, $\mathrm{Li}_n(\theta)$, ..., $\mathrm{Li}_n(\theta^{n-1})$. It can be deduced from the Euler-Carlitz formula for the Carlitz zeta value $\zeta_C(n)$ together with Anderson-Thakur's identity [AndT, Thm 3.8.3], which yields:

$$\zeta_C(n) \in \mathrm{Span}_{\mathbb{F}(\theta)}\left\{\mathrm{Li}_n(\theta^i) \mid 0 \le i < nq/(q-1)\right\}.$$

See Lemma 7.20 in the text.

0.3 Plan of the thesis

The aim of Chapter 1 is to review the usual set up (notations, definitions, basic properties) of A-motives over an arbitrary A-algebra or field. We shall use it as a range of useful results that we will refer to throughout the next chapters. We follow [HarJu] as a guideline, though *loc. cit.* is concerned with the particular choice of a closed point ∞ of degree one and over a complete algebraically closed field. Most of the results on A-motives extend without changes to our larger setting. An addition to the existing literature is Section 1.2 where our presentation of mixedness sensibly differs from [HarJu, §2.3.2]. In section 1.4, we define and study the action of G_∞ on the Betti realization of an A-motive. We also review the notion of rigid analytic triviality in a way which avoids the use of rigid analytic geometry.

In Chapter 2, we develop the notion of *maximal integral models* of A-motives over a local or global function field. It splits into three Sections. In Section 2.1, we present integral models of *Frobenius spaces* over local function fields. The theory is much easier than the one for A-motives, introduced over a local function field in Section 2.2 and over a global function field in Section 2.3. Although our definition of integral model is inspired by Gardeyn's work in the context of τ-sheaves [Gar2], our presentation is simpler as we removed the *locally free* assumption. That maximal integral models are locally free is automatic, as we show in Propositions 2.14 and 2.32. The chief aim of this chapter is to do the groundwork for Chapter 3, precisely Section 3.3, where integral models play a leading role in the determination of extensions having everywhere good reduction.

Motivic Cohomology for function fields is introduced in Chapter 3. We describe the extension modules in $\mathcal{M}_F$ in Section 3.1 and deduce Theorem A from Theorem 3.4. We focus on extension modules in the category $\mathcal{MM}_F$ in Section 3.2 where we deduce Theorem B from Propositions 3.8, 3.9. The main results of this chapter are gathered in Section 3.3, where we study the module of extensions having good reduction. Theorem C follows from Theorem 3.15 and Theorem D from Theorem 3.18. There, the full force of Chapter 2 is required.

In Chapter 4, after a recall of filtered spaces in Section 4.1, we develop the theory of mixed Hodge structures (Section 4.2) and review mixed Hodge-Pink structures (Section 4.3). This chapter owes much to Pink's unpublished monograph [Pin], although we present new notions as *infinite Frobenius* (Definitions 4.8 and 4.21) and *Hodge descent* (Definition 4.41). Its objective is to describe the extension modules in several categories and pave the way for Chapter 5, where the present results will serve to define and describe function fields regulators.

We attach regulators to rigid analytically trivial mixed A-motives in Chapter 5. In section 5.1, we introduce the *Hodge-Pink realization functor* $\mathscr{H}^+$ as presented in [HarJu]. As explained above, $\mathscr{H}^+$ does not induce a functor targeting $\mathcal{MH}^+_{K_\infty}$. For this reason, we exhibit in Subsection 5.1.2 a full and exact subcategory $\mathcal{MM}_F^{\mathrm{reg}}$ of $\mathcal{MM}_F^{\mathrm{rig}}$. This category is constructed so that one can define the *Hodge realization functor* $\mathcal{H}^+ : \mathcal{MM}_F^{\mathrm{reg}} \to \mathcal{MH}^+_{K_\infty}$ which factors through $\mathscr{H}^+$. We define *general* and *special* regulators in Subsection 5.1.4. In Section 5.2, we define extensions in $\mathcal{MM}_F^{\mathrm{rig}}$ *having analytic reduction at* ∞. This notion will appear to be the right one to obtain a finitely generated natural sub-A-module of $\mathrm{Ext}^1_{\mathcal{MM}_F^{\mathrm{reg}},\mathcal{O}_F}(\mathbb{1},\underline{M})$. In Section 5.3, we conclude by a description of the extension modules in $\mathcal{M}_F^{\mathrm{rig}}$ in terms of solutions of τ-difference equations. These formulas will be used to describe explicitly general and special regulators (see Theorem 5.34). These observations are at the origin of Chapter 7, where we relate special regulators of the Carlitz tensor powers to function fields polylogarithms.

In Chapter 6, we state and prove Theorems E and F (Theorems 6.2 and 6.4 in the text). The impetus of all the previous chapters is called for. Chapter 6 splits into two sections. In Section 6.2.2, we develop the theory of *shtuka models* attached to A-motives. This is highly reminiscent of Mornev's *global models* [Mo1, §12] in the context of Drinfeld modules, although not directly linked. A major step in this section is to relate shtuka models on $C \times C$ to certain extension modules of Hodge structures (Subsection 6.2.3). In Section 6.3, we develop cohomological tools for coherent cohomology of schemes covered by two affine subschemes which, applied to the (Zariski and rigid) cohomology of shtuka models, will achieve the proofs of Theorems E and F. Our proof is inspired by cohomological techniques of V. Lafforgue in [LafV].

In Chapter 7, we apply the theory presented in the previous chapters to compute $\mathrm{Ext}^{1,\infty}_{\mathcal{MM}_K^{\mathrm{reg}}}(\mathbb{1}, \underline{A}(n))$. This results in Theorem G above, proved in Section 7.2. We prove Theorem H in Section 7.3 using the ABP criterion as a main ingredient [ABP]. We end this text by a equivalent formulation of function fields Beilinson's conjecture in the case of $\underline{A}(n)$ ($n > 0$) in terms of *generalized* polylogarithms (Proposition 7.31). As a corollary, we prove that Beilinson's conjecture is true for $\underline{A}(1)$ (Corollary 7.33) but false for $\underline{A}(n)$ whenever n is a multiple of the characteristic p of $\mathbb{F}$ (Corollary 7.34).

Chapter 1

Anderson A-motives

Let $\mathbb{F}$ be a finite field of cardinality q. By convention, throughout this text unlabeled tensor products and fiber products are over $\mathbb{F}$. Let $(C, \mathcal{O}_C)$ be a geometrically irreducible smooth projective curve over $\mathbb{F}$, and fix a closed point ∞ on C. Let $A := \Gamma(C \setminus \{\infty\}, \mathcal{O}_C)$ be the ring of regular functions on $C \setminus \{\infty\}$ and let K be the function field of C.

We review here the theory of Anderson A-motives over any A-algebra and study their λ-adic and Betti realization functors. We shall use this first chapter as a range of useful results that we will refer to throughout the next chapters. One goal is to define the category $\mathcal{MM}_F^{\mathrm{rig}}$ which shall be our function field analogue of the category of mixed motives. We use [HarJu] as a guideline, though *loc. cit.* is concerned with the particular choice of a closed point ∞ of degree one and over a complete algebraically closed field. Most of the results on A-motives extend without changes to our larger setting, but we prefer to give full details when necessary.

In Section 1.1 we review the usual definitions of Anderson A-motives over an arbitrary A-algebra R and give the basic properties of the category $\mathcal{M}_R$ they form, ought to be known to specialists. We set our main notations for the rest of the text. We also introduce the category $\tilde{\mathcal{M}}_R$ of *A-motives up to isogenies*, and review the classical operations (tensor, dual, restriction of scalars,...)

In Section 1.2, we introduce *function fields isocrystals* over an A-field F following the recent work of Mornev in [Mor2]. We prove that isocrystals admit a uniquely determined *slope filtration* in Theorem 1.19. This was already proven in [Har1, Prop. 1.5.10] under a certain assumption, but we explain how the latter result implies ours. Given an object $\underline{M}$ in $\mathcal{M}_F$ and a closed point λ on C, we attach functorially an isocrystal $\mathcal{I}_\lambda(\underline{M})$ in Definition 1.28. We use the slope filtration of $\mathcal{I}_\infty(\underline{M})$ to define *mixedness* and *weights* in Definition 1.30.

In Section 1.3, we introduce the *λ-adic realization functor* for A-motives. We prove that it is exact in Corollary 1.46.

In Section 1.4, we introduce the Betti realization functor when the base

A-algebra is a finite extension of K. Compared to the existing litterature, where A-motives are considered over a subfield of $\mathbb{C}_\infty$, our definition involves the choice of a K-algebra morphism $v : F \to \mathbb{C}_\infty$ (Definition 1.48). We denote the Betti realization functor Λ_v. This allows us to define *v-rigid analytically triviality*, extending slightly Anderson's definition [And, §2]. For the study of the analogue of Beilinson's conjectures, as we do in the present text, we could have restricted ourselves to A-motives over K and the distinction of embeddings v could have been reduced to the inclusion $i : K \to \mathbb{C}_\infty$. We stick to this slightly more general setting for future references with incoming works.

We let F_v be the completion of F with respect to $|\cdot|_v := |v(\cdot)|$, we fix F_v^s a separable closure of F_v and we let G_v be the Galois group $\mathrm{Gal}(F_v^s|F_v)$. Given an A-motive $\underline{M}$ over F, a novelty of our approach is to consider the action of G_v on the v-Betti realization $\Lambda_v(\underline{M})$ of $\underline{M}$. Under the assumption that $\underline{M}$ is v-rigid analytically trivial, we show in Proposition 1.54 that this action is continuous, and we determine $H^1(G_v, \Lambda_v(\underline{M}))$ in Theorem 1.56. We end this chapter by interpreting elements of $\Lambda_v(\underline{M})$ as analytic functions on affinoid subdomains of $(C \times \mathrm{Spec}\,\mathbb{C}_\infty)^{\mathrm{rig}}$ and discuss their analytic continuations to larger subdomains. This is inspired by [HarJu, §2.3.3].

1.1 Definitions of A-motives

Let R be a commutative $\mathbb{F}$-algebra and let $\kappa : A \to R$ be an $\mathbb{F}$-algebra morphism. R will be refered to as *the base algebra* and κ as the *characteristic morphism.* The kernel of κ is called *the characteristic of* (R, κ). We consider the ideal $\mathfrak{j} = \mathfrak{j}_\kappa$ of $A \otimes R$ generated by the set $\{a \otimes 1 - 1 \otimes \kappa(a) | a \in A\}$; $\mathfrak{j}$ is equivalently defined as the kernel of $A \otimes R \to R$, $a \otimes f \mapsto \kappa(a)f$. The ideal $\mathfrak{j}$ is maximal if and only if R is a field, and is a prime ideal if and only if R is a domain.

Lemma 1.1. *Let $a \in A$ be a non constant element. Then, $a \otimes 1 - 1 \otimes \kappa(a)$ is a non-zero-divisor in $A \otimes R$.*

Proof. Let $\{f_1, ..., f_d\}$ be a lift in A of a basis of $A/(a)$ over $\mathbb{F}$. Then

$$\{a^n f_1, ..., a^n f_d\}_{n \geq 0}$$

forms a basis of A over $\mathbb{F}$. It follows that $\{a^n f_1 \otimes 1, ..., a^n f_d \otimes 1\}_{n \geq 0}$ forms a basis of $A \otimes R$ over R. In this basis, and given $x \in A \otimes R$, $(a \otimes 1)x$ and $(1 \otimes \kappa(a))x$ do not have the same coordinates unless $x = 0$. It follows that $a \otimes 1 - 1 \otimes \kappa(a)$ is regular. $\square$

Let $\mathrm{Quot}(A \otimes R)$ be the localization of $A \otimes R$ at its non-zero-divisors (if $A \otimes R$ is an integral domain, $\mathrm{Quot}(A \otimes R)$ is the field of fractions of $A \otimes R$).

Let M be an $A \otimes R$-module. For $n \in \mathbb{Z}$, we denote $\mathfrak{j}^{-n}M$ the submodule of $M \otimes_{A \otimes R} \mathrm{Quot}(A \otimes R)$ consisting of elements m for which $(a \otimes 1 - 1 \otimes \kappa(a))^n m \in$

M for all $a \in A \setminus \mathbb{F}$. We then set

$$M[\mathfrak{j}^{-1}] := \bigcup_{n \geq 0} \mathfrak{j}^{-n} M.$$

Let $\tau : A \otimes R \to A \otimes R$ be the A-linear morphism given by $a \otimes r \mapsto a \otimes r^q$ on elementary tensors. Let $\tau^* M$ denotes the pull-back of M by τ ([Bou, A.II.§5]). That is, $\tau^* M$ is the $A \otimes R$-module

$$(A \otimes R) \otimes_{\tau, A \otimes R} M$$

where the subscript τ signifies that the relation $(a \otimes_\tau bm) = (a\tau(b) \otimes_\tau m)$ holds for $a, b \in A \otimes R$, and where the $A \otimes R$-module structure on $\tau^* M$ corresponds to $b \cdot (a \otimes_\tau m) := (ba \otimes_\tau m)$. We let $\mathbf{1} : \tau^*(A \otimes R) \to A \otimes R$ be the $A \otimes R$-linear morphism which maps $(a \otimes r) \otimes_\tau (b \otimes s) \in \tau^*(A \otimes R) := (A \otimes R)_{\tau, A \otimes R}(A \otimes R)$ to $ab \otimes rs^q \in A \otimes R$.

The next definition takes its roots in the work of Anderson [And], though this version is borrowed from [Har2, Def. 2.1].

Definition 1.2. An *Anderson A-motive $\underline{M}$ (over R)* is a pair (M, τ_M) where M is a locally free $A \otimes R$-module of finite constant rank and where $\tau_M : (\tau^* M)[\mathfrak{j}^{-1}] \to M[\mathfrak{j}^{-1}]$ is an isomorphism of $(A \otimes R)[\mathfrak{j}^{-1}]$-modules.
In all the following, we shall more simply write *A-motive* instead of *Anderson A-motive*. The *rank of $\underline{M}$* is the (constant) rank of M over $A \otimes R$.
A morphism $(M, \tau_M) \to (N, \tau_N)$ of A-motives (over R) is an $A \otimes R$-linear morphism $f : M \to N$ such that $f \circ \tau_M = \tau_N \circ \tau^* f$. We let $\mathcal{M}_R$ be the A-linear category of A-motives over R.

Remark 1.3. A-motives as in Definition 1.2 are called *abelian A-motives* by several authors (see e.g. [BroPa]). The word *abelian* refers to the assumption that the underlying $A \otimes R$-module is finite locally free. Dropping this assumption is not a good strategy in our work, as too many analogies with number fields motives would fail to hold.

Definition 1.4. An A-motive $\underline{M} = (M, \tau_M)$ (over R) is called *effective* if $\tau_M(\tau^* M) \subset M$. We let $\mathcal{M}_R^{\mathrm{eff}}$ be the full subcategory of $\mathcal{M}_R$ whose objects are effective A-motives.

Let $\mathbb{1}$ be the *unit A-motive over R* defined as $(A \otimes R, \mathbf{1})$. The biproduct of two A-motives $\underline{M}$ and $\underline{N}$, denoted $\underline{M} \oplus \underline{N}$, is defined to be the A-motive whose underlying $A \otimes R$-module is $M \oplus N$ and whose τ-linear morphism is $\tau_M \oplus \tau_N$. Their tensor product, denoted $\underline{M} \otimes \underline{N}$, is defined to be $(M \otimes_{A \otimes R} N, \tau_M \otimes \tau_N)$. The tensor operation admits $\mathbb{1}$ as a neutral object. The dual of $\underline{M}$ is defined to be the A-motive whose underlying $A \otimes R$-module is $M^\vee := \operatorname{Hom}_{A \otimes R}(M, A \otimes R)$ and where $\tau_{M^\vee}$ is defined as

$$\tau_{M^\vee} : (\tau^* M^\vee)[\mathfrak{j}^{-1}] = (\tau^* M)^\vee[\mathfrak{j}^{-1}] \xrightarrow{\sim} M^\vee[\mathfrak{j}^{-1}], \quad h \longmapsto h \circ \tau_M^{-1}$$

(we refer to [HarJu, §.2.3] for more details). Given S an R-algebra, there is a *base-change* functor $\mathcal{M}_R \to \mathcal{M}_S$ mapping $\underline{M} = (M, \tau_M)$ to $\underline{M}_S := (M \otimes_R S, \tau_M \otimes_R \mathrm{id}_S)$. The *restriction functor* $\mathrm{Res}_{S/R} : \mathcal{M}_S \to \mathcal{M}_R$ maps an A-motive $\underline{M}$ over S to $\underline{M}$ seen as an A-motive over R. Given two A-motives $\underline{M}$ and $\underline{N}$ over R and S respectively, we have

$$\mathrm{Hom}_{\mathcal{M}_R}(\underline{M}, \mathrm{Res}_{S/R}\underline{N}) = \mathrm{Hom}_{\mathcal{M}_S}(\underline{M}_S, \underline{N}).$$

In other words, the base-change functor is left-adjoint to the restriction functor.

Example 1.5 (Carlitz's motive)**.** Let $C = \mathbb{P}^1_{\mathbb{F}}$ be the projective line over $\mathbb{F}$ and let ∞ be the closed point of coordinates $[0 : 1]$. If t is any element in $\Gamma(\mathbb{P}^1_{\mathbb{F}} \setminus \{\infty\}, \mathcal{O}_{\mathbb{P}^1})$ whose order of vanishing at ∞ is 1, we have an identification $A = \mathbb{F}[t]$. For an $\mathbb{F}$-algebra R, the tensor product $A \otimes R$ is identified with $R[t]$. The morphism τ acts on $p(t) \in R[t]$ by raising its coefficients to the qth-power. It is rather common to denote by $p(t)^{(1)}$ the polynomial $\tau(p(t))$. Let $\kappa : A \to R$ be an injective $\mathbb{F}$-algebra morphism and let $\theta = \kappa(t)$. The ideal $\mathfrak{j} \subset R[t]$ is principal, generated by $(t - \theta)$.

The Carlitz $\mathbb{F}[t]$-motive $\underline{C}$ over R is defined by the couple $(R[t], \tau_{\mathsf{C}})$ where τ_C maps $\tau^* p(t)$ to $(t-\theta)p(t)^{(1)}$. Its nth tensor power $\underline{C}^n := \underline{C}^{\otimes n}$ is isomorphic to the $\mathbb{F}[t]$-motive whose underlying module is $R[t]$ and where τ_{C^n} maps $\tau^* p(t)$ to $(t-\theta)^n p(t)^{(1)}$. We let $\underline{A}(n) := \underline{C}^{-n} = (\underline{C}^n)^\vee$.

For $A = \mathbb{F}[t]$, $\underline{A}(1)$ plays the role of the number fields' Tate motive $\mathbb{Z}(1)$ and, more generally, $\underline{A}(n)$ plays the role of $\mathbb{Z}(n)$. We discuss in more details the Carlitz motive and its tensor powers in Chapter 7.

The category $\mathcal{M}_R$ of A-motives over R is generally *not* abelian, even if $R = F$ is a field. This comes from the fact that a morphism in $\mathcal{M}_F$ might not admit a cokernel. However, there is a notion of exact sequences in the category $\mathcal{M}_R$ which we borrow from [HarJu, Rmk. 2.3.5(b)]:

Definition 1.6. We say that a sequence $0 \to \underline{M}' \to \underline{M} \to \underline{M}'' \to 0$ in $\mathcal{M}_R$ is *exact* if its underlying sequence of $A \otimes R$-modules is exact.

The next proposition appears and is discussed in [HarJu, Rmk. 2.3.5(b)] and will allow us to consider extension modules (Chapter 3). Although stated in the case where R is a particular A-algebra and $\deg(\infty) = 1$, it extends without changes to our setting:

Proposition 1.7. *The category $\mathcal{M}_R$ together with the notion of exact sequences as in Definition 1.6 is exact in the sense of Quillen [Qui1, §2].*

In the rest of this section, we define the category $\tilde{\mathcal{M}}_R$ of *A-motives up to isogeny (over R)* (see Definition 1.10) which is abelian when $R = F$ is a field. We first discuss the notion of *saturation*.

Definition 1.8. Let $\underline{M} = (M, \tau_M)$ be an Anderson A-motive over R. A *submotive* of $\underline{M}$ is an A-motive $\underline{N} = (N, \tau_N)$ such that $N \subset M$ and $\tau_N = \tau_M|_{\tau^* N[\mathfrak{j}^{-1}]}$.

We set $\underline{N}^{\text{sat}}$ to be the submotive of $\underline{M}$ whose underlying $A \otimes R$-module is

$$N^{\text{sat}} := \{n \in M \mid \exists a \in A \otimes R,\ an \in N\}$$

and call it the *saturation of* $\underline{N}$ *in* $\underline{M}$. We say that $\underline{N}$ is *saturated in* $\underline{M}$ if $\underline{N} = \underline{N}^{\text{sat}}$.

Following [Har2, Def. 5.5, Thm. 5.12], we have the next:

Definition 1.9. A morphism $f : \underline{M} \to \underline{N}$ in $\mathcal{M}_R$ is an *isogeny* if one of the following equivalent conditions is satisfied.

(a) f is injective and $\operatorname{coker}(f : M \to N)$ is a finite locally free R-module,

(b) M and N have the same rank and $\operatorname{coker} f$ is finite locally free over R,

(c) M and N have the same rank and f is injective,

(d) there exists $0 \neq a \in A$ such that f induces an isomorphism of $(A \otimes R)[a^{-1}]$-modules $M[a^{-1}] \xrightarrow{\sim} N[a^{-1}]$,

(e) there exists $0 \neq a \in A$ and $g : \underline{N} \to \underline{M}$ in $\mathcal{M}_R$ such that $f \circ g = a\,\mathrm{id}_N$ and $g \circ f = a\,\mathrm{id}_M$.

If an isogeny between $\underline{M}$ and $\underline{N}$ exists, $\underline{M}$ and $\underline{N}$ are said *isogenous.*

As a consequence of those equivalent definitions, a submotive of an A-motive $\underline{M}$ is isogenous to its saturation in $\underline{M}$. This motivates the definition of the category of A*-motives up to isogeny* (see [HarJu, Def. 2.3.1]).

Definition 1.10. Let $\tilde{\mathcal{M}}_R$ be the K-linear category whose objects are those of $\mathcal{M}_R$ and where the hom-sets of two objects $\underline{M}$ and $\underline{N}$ is given by the K-vector space

$$\operatorname{Hom}_{\tilde{\mathcal{M}}_R}(\underline{M}, \underline{N}) := \operatorname{Hom}_{\mathcal{M}_R}(\underline{M}, \underline{N}) \otimes_A K.$$

We call the objects of $\tilde{\mathcal{M}}_R$ the A*-motives over* R *up to isogeny.*

An isogeny in $\mathcal{M}_R$ then becomes an isomorphism in $\tilde{\mathcal{M}}_R$. According to [HarJu, Prop. 2.3.4], the category $\tilde{\mathcal{M}}_F$ is abelian. We also claim:

Proposition 1.11. *Any object of* $\tilde{\mathcal{M}}_R$ *has finite length.*

Proof. Any subobject of a rank 1 object in $\tilde{\mathcal{M}}_R$ has either rank 0 or 1. If it has rank 1, then it is isomorphic in $\tilde{\mathcal{M}}_R$ to the whole object by Definition 1.9(c). It follows that any rank 1 object in $\tilde{\mathcal{M}}_R$ is simple. As a consequence, for any increasing sequence

$$0 = \underline{M}_0 \subsetneq \underline{M}_1 \subsetneq \cdots \subsetneq \underline{M}_n = \underline{M}$$

in $\mathcal{M}_R$, n is bounded by the rank of $\underline{M}$. This concludes. $\square$

1.2 Mixed A-motives

We discuss here the notion of weights for Anderson A-motives in our setting. In the $A = \mathbb{F}[t]$-case, the definition of *pure* A-motives is traced back to the work of Anderson [And, 1.9], but the definition of *mixed* A-motives appeared only some decades later in the work of Taelman [Tae1] in the case $A = \mathbb{F}[t]$. It was extended to more general coefficients ring A by Hartl and Juschka in [HarJu, §3], under the assumption $\deg(\infty) = 1$ and over $\mathbb{C}_\infty$. Our presentation deals with the case of general A (that is, without assumption on $\deg(\infty)$) and $R = F$ a field. We require some pieces of the theory of *function fields isocrystals* to begin.

1.2.1 Isocrystals over a field

We introduce function fields isocrystals following [Mor2]. Our objective is to prove existence and uniqueness of the slope filtration with pure subquotients. The general theory has been developed in [Andr2], and the results of interest for us appear in [Har1]. The new account of this subsection is the adaptation of [Har1, Prop 1.5.10] to our more general setting (see Theorem 1.19). This result will allow us to define mixedness and weights in Subsection 1.2.2.

Let R be a Noetherian $\mathbb{F}$-algebra. Let k be a finite field extension of $\mathbb{F}$ of degree δ. Let E be the field of Laurent series over k in the formal variable π, $\mathcal{O}$ the subring of E consisting of power series over k and $\mathfrak{m}$ the maximal ideal of $\mathcal{O}$. Explicitely $E = k(\!(\pi)\!)$, $\mathcal{O} = k[\![\pi]\!]$ and $\mathfrak{m} = \pi\mathcal{O}$.

We let $\mathcal{A}(R)$ be the completion of the ring $\mathcal{O} \otimes R$ at its ideal $\mathfrak{m} \otimes R$, that is

$$\mathcal{A}(R) = \varprojlim_n (\mathcal{O} \otimes R)/(\mathfrak{m}^n \otimes R)$$

and we let $\mathcal{B}(R)$ be the tensor product $E \otimes_{\mathcal{O}} \mathcal{A}(R)$. Throughout the previous identifications, we readily check that $\mathcal{A}(R) = (k \otimes R)[\![\pi]\!]$ and $\mathcal{B}(R) = (k \otimes R)(\!(\pi)\!)$. Since $\mathcal{O} \otimes R$ is Noetherian, $\mathcal{A}(R)$ is flat over $\mathcal{O} \otimes R$ ([Bou, AC.III.§4,Thm 3(iii)]). It follows that $\mathcal{B}(R)$ is flat over $K \otimes R$.

Let $\tau : \mathcal{O} \otimes R \to \mathcal{O} \otimes R$, be the $\mathcal{O}$-linear map induced by $a \otimes r \mapsto a \otimes r^q$. We shall denote by τ also its continuous extension to $\mathcal{A}(R)$ or $\mathcal{B}(R)$. Similarly, we denote by $\mathbf{1}$ the canonical $A \otimes R$-linear morphisms $\tau^*\mathcal{A}(R) \to \mathcal{A}(R)$ and $\tau^*\mathcal{B}(R) \to \mathcal{B}(R)$.

We assume that $R = F$ is a field.

Definition 1.12. An *isocrystal $\underline{D}$ over F* is a pair (D, φ_D) where D is a free $\mathcal{B}(F)$-module of finite rank and $\varphi_D : \tau^*D \to D$ is a $\mathcal{B}(F)$-linear isomorphism. When understood by the context, we simply refer to $\underline{D}$ as an *isocrystal.*
A morphism $(D, \varphi_D) \to (C, \varphi_C)$ of isocrystals is a $\mathcal{B}(F)$-linear morphism of

the underlying modules $f : B \to C$ such that $f \circ \varphi_D = \varphi_C \circ \tau^* f$. We let $\mathcal{I}_F$ be the category of isocrystals over F.

From [Mor2, Prop. 4.1.1], we deduce that the category $\mathcal{I}_F$ is abelian.

Let $\underline{D} = (D, \varphi)$ be an isocrystal over F. In $\mathcal{I}_F$, a subobject of $\underline{D}$, or *subisocrystal of* $\underline{D}$, is an isocrystal $\underline{G} = (G, \varphi_G)$ for which $G \subset D$, $\varphi_G = \varphi_D|_{\tau^* G}$. The *quotient* of $\underline{D}$ by $\underline{G}$ is the pair $(D/G, \varphi_D)$ (this is indeed an isocrystal by [Mor2, Prop. 4.1.1]).

We define the *rank* $\operatorname{rk} \underline{D}$ *of* $\underline{D}$ to be the rank of D over $\mathcal{B}(F)$. If $\underline{D}$ is nonzero, let $\mathbf{b}$ be a basis of D and let U denote the matrix of φ expressed in $\tau^*\mathbf{b}$ and $\mathbf{b}$. A different choice of basis $\mathbf{b}'$ leads to a matrix U' such that $U = \tau(P)U'P^{-1}$ for an invertible matrix P with coefficients in $\mathcal{B}(F)$. As such, the valuation of $\det U$ in π is independent of $\mathbf{b}$. We denote it by $\deg \underline{D}$ and we name it the *degree of* $\underline{D}$. We define the *slope of* $\underline{D}$ to be the rational number $\mu(\underline{D}) = \delta \deg \underline{D} / \operatorname{rk} \underline{D}$, where δ is the degree of k over $\mathbb{F}$.

The degree and rank are additive in short exact sequences over the abelian category of isocrystals, and the association $\underline{D} \mapsto \mu(\underline{D})$ defines a slope function for $\mathcal{I}_F$ in the sense of [Andr2, Def. 1.3.1]. The next definition should be compared with [Andr2, Def. 1.3.6]:

Definition 1.13. The isocrystal $\underline{D}$ is *semistable* (resp. *isoclinic*) if, for any nonzero subisocrystal $\underline{D}'$ of $\underline{D}$, $\mu(\underline{D}') \leq \mu(\underline{D})$ (resp. $\mu(\underline{D}') = \mu(\underline{D})$).

These notions are related to the notion of *purity*, borrowed from [Mor2, Def. 3.4.6]. We first introduce $\mathcal{A}(F)$*-lattices*:

Definition 1.14. Let D be a free $\mathcal{B}(F)$-module of finite rank. An $\mathcal{A}(F)$*-lattice in* D is a sub-$\mathcal{A}(F)$-module of finite type of D which generates D over E.

Let L be an $\mathcal{A}(F)$-lattice in D. Because D is a free $\mathcal{B}(F)$-module and $\mathcal{A}(F)$ is a finite product of principal ideal domains, L is free as an $\mathcal{A}(F)$-module and has the same rank as D. We denote by $\langle \varphi_D L \rangle$ the sub-$\mathcal{A}(F)$-module $\varphi_D(\tau^* L)$ in D. Because φ_D is an isomorphism, $\langle \varphi_D L \rangle$ is a $\mathcal{A}(F)$-lattice in D. By recursion, we define $\langle \varphi_D^n L \rangle$ the $\mathcal{A}(F)$-lattice $\langle \varphi_D \langle \varphi_D^{n-1} L \rangle \rangle$. We set $\langle \varphi_D^0 L \rangle = L$.

Definition 1.15. A nonzero isocrystal (D, φ_D) over F is said to be *pure of slope* μ if there exist an $\mathcal{A}(F)$-lattice L in D and integers s and $r > 0$ such that $\langle \varphi_D^{r\delta} L \rangle = \pi^s L$ and $\mu = s/r$. By convention, the zero isocrystal is pure with *no slope*.

Example 1.16. Let D be the free $\mathcal{B}(F)$-module of rank $s \geq 1$ with basis $\{e_0, ..., e_{s-1}\}$ and let $\varphi_D : \tau^* D \to D$ be the unique linear map such that $\varphi_D(\tau^* e_{i-1}) = e_i$ for $1 \leq i < s$ and $\varphi_D(\tau^* e_{s-1}) = \pi^r e_0$. Then (D, φ_D) is a pure isocrystal of slope $r\delta/s$ with $\mathcal{A}(F)e_0 \oplus \cdots \oplus \mathcal{A}(F)e_{s-1}$ for $\mathcal{A}(F)$-lattice.

The following lemma relates the definition of slopes from purity and from slope functions:

Lemma 1.17. *If $\underline{D}$ is a pure isocrystal of slope μ, then $\mu(\underline{D}')=\mu$ for any nonzero sub-isocrystal $\underline{D}'$ of $\underline{D}$. In particular, $\underline{D}$ is isoclinic (hence semistable).*

Proof. Assume there exists an $\mathcal{A}(F)$-lattice T in D such that $\langle\varphi^{r\delta}T\rangle=\mathfrak{m}^sT$ for integers $r>0$ and d such that $\mu=s/r$. If $\underline{D}'=(D',\varphi)$ is a nonzero subisocrystal of $\underline{D}$, then $T'=T\cap D'$ is an $\mathcal{A}(F)$-lattice in D' such that $\langle\varphi^{r\delta}T'\rangle=\mathfrak{m}^sT'$. As T' is nonzero, let $\{t_1,...,t_\ell\}$ be a basis of T' over $\mathcal{A}(F)$. We have

$$(\det\varphi)^{r\delta}(t_1\wedge\cdots\wedge t_\ell)=\mathfrak{m}^{s\ell}(t_1\wedge\cdots\wedge t_\ell)\quad\text{in }\bigwedge^{\ell}T'.$$

Hence $r\delta\deg\underline{D}'=s\operatorname{rk}\underline{D}'$, which yields $\mu(\underline{D}')=\mu$. □

Definition 1.18. A *slope filtration for $\underline{D}$* is an increasing sequence of sub-isocrystals of $\underline{D}$

$$0=\underline{D}_0\subsetneq\underline{D}_1\subsetneq\cdots\subsetneq\underline{D}_s=\underline{D},$$

satisfying:

(*i*) $\forall i\in\{1,...,s\}$, $\underline{D}_i/\underline{D}_{i-1}$ is semi-stable,

(*ii*) we have $\mu(\underline{D}_1)>\mu(\underline{D}_2/\underline{D}_1)>\cdots\mu(\underline{D}_s/\underline{D}_{s-1})$.

It follows from [Andr2, Thm 1.4.7] applied to the slope function $\underline{D}\mapsto\mu(\underline{D})$ on the abelian category $\mathcal{I}_F$, that a slope filtration for $\underline{D}$ exists and is unique (up to unique isomorphism). A much stronger result holds: the quotients in (*i*) are pure. This is the next theorem.

Theorem 1.19. *Let $\underline{D}$ be an isocrystal over F. In the slope filtration for $\underline{D}$*

$$0=\underline{D}_0\subsetneq\underline{D}_1\subsetneq\cdots\subsetneq\underline{D}_s=\underline{D},\tag{1.1}$$

for all $i\in\{1,...,s\}$, the quotients $\underline{D}_i/\underline{D}_{i-1}$ are pure isocrystals.

Remark 1.20. It would be relevant to have a lemma stating the equivalence between semi-stable and isoclinic, so that Theorem 1.19 would follow from André's theory. Yet, the only proof I know already uses [Har1, Prop. 1.5.10] and follows from Theorem 1.19.

Proof of Theorem 1.19. If $\delta=1$, then $\mathcal{A}(F)$ is identified with $F[\![\pi]\!]$ and Theorem 1.19 is proved in [Har1, Prop. 1.5.10]. We now explain how the general case follows from the above. Let $\mathbb{G}$ be the finite field extension of $\mathbb{F}$ corresponding to

$$\mathbb{G}:=\{f\in\bar{\mathbb{F}}\cap F\mid f^{q^\delta}=f\}.$$

Let $\phi:\mathbb{G}\to F$ denote the inclusion. Note this defines an embedding of $\mathbb{G}$ in k. Let $\mathcal{A}_\phi(F)$ be the completion of $\mathcal{O}\otimes_{\mathbb{G}}F$ at the ideal $\mathfrak{m}\otimes_{\mathbb{G}}F$. In the theory of isocrystals over F with $\mathbb{G}$ in place of $\mathbb{F}$, $\mathcal{A}_\phi(F)$ appears in place of $\mathcal{A}(F)$ and $\delta=1$. In [Mor2, §4.2], Mornev defines a functor

$$[\phi]^*:(\mathcal{A}(F)-\text{isocrystals})\longrightarrow(\mathcal{A}_\phi(F)-\text{isocrystals})$$

By [Mor2, Prop 4.2.2] (see also [BorHa, Prop 8.5]), the functor $[\phi]^*$ defines an equivalence of categories such that $[\phi]^*(\underline{D})$ is a pure isocrystals of slope μ if $\underline{D}$ is. Let

$$[\phi]_* : (\mathcal{A}_\phi(F) - \text{isocrystals}) \longrightarrow (\mathcal{A}(F) - \text{isocrystals})$$

be a quasi-inverse of $[\phi]^*$ and let $\ell : [\phi]_*[\phi]^* \xrightarrow{\sim} \mathrm{id}$ be a natural transformation.

Let $\underline{D}$ be an $\mathcal{A}(F)$-isocrystal. We only need to prove existence of (1.1) with pure subquotients since uniqueness follows from [Andr2, Thm 1.4.7]. By [Har1, Prop. 1.5.10], there exists an increasing sequence of sub-$\mathcal{A}_\phi(F)$-isocrystals of $[\phi]^*\underline{D}$:

$$0 = \underline{G}_0 \subsetneq \underline{G}_1 \subsetneq \underline{G}_2 \subsetneq \cdots \subsetneq \underline{G}_s = [\phi]^*\underline{D}$$

the subquotients $\underline{G}_i/\underline{G}_{i-1}$ being pure of slopes μ_i with $\mu_1 > \cdots > \mu_s$. Applying $[\phi]_*$ and then ℓ, we obtain

$$0 = \underline{D}_0 \subsetneq \underline{D}_1 \subsetneq \underline{D}_2 \subsetneq \cdots \subsetneq \underline{D}_s = \underline{D} \tag{1.2}$$

with $\underline{D}_i := \ell([\phi]_*[\phi]^*\underline{D}_i)$ for all $i \in \{0, 1, ..., s\}$. We claim that the isocrystals $\underline{D}_i/\underline{D}_{i-1}$ are pure of slope μ_i. Indeed, we have

$$\underline{D}_i/\underline{D}_{i-1} \cong [\phi]_*\underline{G}_i/[\phi]_*\underline{G}_{i-1} \cong [\phi]_*(\underline{G}_i/\underline{G}_{i-1})$$

where the last isomorphism comes from the fact that $[\phi]_*$ is an exact functor (any equivalence of categories is exact). Because $\underline{G}_i/\underline{G}_{i-1}$ is pure of slope μ_i, $\underline{D}_i/\underline{D}_{i-1}$ is also pure of slope μ_i. We conclude that (1.2) is the slope filtration for $\underline{D}$ and satisfies the assumption of the theorem. □

We introduce the next definition:

Definition 1.21. Let $\underline{D}$ be an isocrystal over F and let $(\underline{D}_i)_{i\in\{0,...,s\}}$ be its slope filtration. The elements of the set $\{-\mu(\underline{D}_i/\underline{D}_{i-1}) \mid 1 \le i \le s\}$ are called the *weights* of $\underline{D}$. We call the *weight filtration of* $\underline{D}$ the increasing filtration $(\underline{D}_\lambda)_{\lambda\in\mathbb{Q}}$ of $\underline{D}$ defined by

$$\underline{D}_\lambda := \bigcup_{\mu_j \ge -\lambda} \underline{D}_j.$$

For $\lambda \in \mathbb{Q}$, we let $\mathrm{Gr}_\lambda \underline{D} := \underline{D}_\lambda / \bigcup_{\lambda'<\lambda} \underline{D}_{\lambda'}$.

Remark 1.22. The *breaks* of the weight filtration of $\underline{D}$ are the rational numbers λ such that $\mathrm{Gr}_\lambda \underline{D} \neq 0$. By definition, the set of breaks equals the set of weights.

It follows from Theorem 1.19 that any semi-stable isocrystal is pure, and using Lemma 1.17, that any semi-stable isocrystal is isoclinic. Restating [Andr2, Thm 1.5.9] in our setting, we obtain:

Corollary 1.23. *For all $\lambda \in \mathbb{Q}$, the assignment $\mathcal{I}_F \to \mathcal{I}_F$, $\underline{D} \mapsto \underline{D}_\lambda$ defines an exact functor. Equivalently, any morphism $f : \underline{D} \to \underline{C}$ of isocrystals over F is strict with respect to the weight filtration, that is:*

$$\forall \lambda \in \mathbb{Q}, \quad f(D_\lambda) = f(D) \cap C_\lambda.$$

The weight filtration is not split in general. However it splits when the ground field F is perfect.

Theorem 1.24. *If F is perfect, the weight filtration of $\underline{D}$ splits, i.e. $\underline{D}$ decomposes along a direct sum*

$$\underline{D} \cong \bigoplus_{\lambda \in \mathbb{Q}} \mathrm{Gr}_\lambda(\underline{D}).$$

Remark 1.25. The proof is similar to the argument given for Theorem 1.19: the corresponding result for $\delta = 1$ is proven in [Har1, Prop 1.5.10] and the general δ-case is easily deduced from [Mor2, Prop 4.2.2].

Remark 1.26. The above theorem is the *Dieudonné-Manin decomposition for isocrystals*. When F is algebraically closed, given $\mu \in \mathbb{Q}$ there exists a unique (up to isomorphisms) simple and pure isocrystal $\underline{S}_\mu$ of slope μ (see [Mor2, Prop 4.3.4]). Any pure isocrystal of slope μ decomposes as a direct sum of $\underline{S}_\mu$ (see [Mor2, Prop 4.3.7]) and together with Theorem 1.24 yields the *Dieudonné-Manin classification* (see [Lau]). It does not hold for any F, even separably closed, as noticed by Mornev in [Mor2, Rmk 4.3.5].

1.2.2 Isocrystals attached to A-motives

Let R be a Noetherian $\mathbb{F}$-algebra and let $\kappa : A \to R$ be an $\mathbb{F}$-algebra morphism.

We chose the rings $\mathcal{A}(R)$ and $\mathcal{B}(R)$ of subsections 1.2.1 in the following way. Given a closed point λ on C, we let $O_\lambda \subset K$ be the associated discrete valuation ring of maximal ideal $\mathfrak{m}_\lambda$. We denote $\mathcal{O}_\lambda$ the completion of O_λ and K_λ the completion of K. We let $\mathbb{F}_\lambda$ denote the residue field of λ (of finite dimension over $\mathbb{F}$, its dimension being the degree of λ). We let $\mathcal{A}_\lambda(R)$ and $\mathcal{B}_\lambda(R)$ be the completions of $\mathcal{O}_\lambda \otimes R$ and $K_\lambda \otimes R$ for the $\mathfrak{m}_\lambda$-adic topology.

Recall that $\mathfrak{j}_\kappa$ is the ideal of $A \otimes R$ generated by $\{a \otimes 1 - 1 \otimes \kappa(a) \mid a \in A\}$.

Lemma 1.27. *We have $\mathfrak{j}_\kappa \mathcal{B}_\infty(R) = \mathcal{B}_\infty(R)$. For λ a closed point of C distinct from ∞ such that $\kappa(\mathfrak{m}_\lambda)R = R$, then $\mathfrak{j}_\kappa \mathcal{A}_\lambda(R) = \mathcal{A}_\lambda(R)$.*

Proof. We prove the first assertion. let a be a non constant element of A so that $a^{-1} \in \mathfrak{m}_\infty$. Then $a \otimes 1 - 1 \otimes \kappa(a) \in \mathfrak{j}$ is invertible with $-\sum_{n \geq 0} a^{-(n+1)} \otimes \kappa(a)^n$ as inverse, where the infinite sum converges in $\mathcal{A}_\infty(R) \subset \mathcal{B}_\infty(R)$.

To prove the second assertion, let $\ell \in \mathfrak{m}_\lambda$ be such that $\kappa(\ell)$ is invertible in R. Then $\ell \otimes 1 - 1 \otimes \kappa(\ell) \in \mathfrak{j}$ is invertible with $-\sum_{n \geq 0} \ell^n \otimes \kappa(\ell)^{-(n+1)}$ as inverse, where the infinite sum converges in $\mathcal{A}_\lambda(R)$. □

We assume that $R = F$ is a field.

Definition 1.28. Let $\underline{M} = (M, \tau_M)$ be an A-motive over F and let λ be a closed point of C. We let $\mathcal{I}_\lambda(M)$ be the $\mathcal{B}_\lambda(F)$-module $M \otimes_{A \otimes R} \mathcal{B}_\lambda(F)$. Let $\mathcal{I}_\lambda(\underline{M})$ be the pair $(\mathcal{I}_\lambda(M), \tau_M \otimes \mathbf{1})$.

Proposition 1.29. *Let $\underline{M} = (M, \tau_M)$ be a nonzero A-motive over F. Let λ be a closed point of C distinct from* $\ker \kappa$.

(*i*) $\mathcal{I}_\lambda(\underline{M})$ *is an isocrystal over F.*

(*ii*) *If $\lambda \neq \infty$, $\mathcal{I}_\lambda(\underline{M})$ is pure of slope* 0.

Proof. Because M is locally free of constant rank and $\mathcal{B}_\lambda(F)$ is a finite product of fields, $\mathcal{I}_\lambda(M)$ is a free $\mathcal{B}_\lambda(F)$-module. Thus, point (i) follows from Lemma 1.27. To prove (ii), it suffices to note that $L = M \otimes_{A\otimes F} \mathcal{A}_\lambda(F)$ is an $\mathcal{A}_\lambda(F)$-lattice in $M \otimes_{A\otimes F} \mathcal{B}_\lambda(F)$ such that $\langle \tau_M L \rangle = L$. □

We now choose $\lambda = \infty$. Let $\underline{M}$ be an A-motive over F. The isocrystal $\mathcal{I}_\infty(\underline{M})$ admits a weight filtration $(\mathcal{I}_\infty(\underline{M})_\mu)_{\mu\in\mathbb{Q}}$:

$$0 = \mathcal{I}_\infty(\underline{M})_{\mu_0} \subsetneq \mathcal{I}_\infty(\underline{M})_{\mu_1} \subsetneq \mathcal{I}_\infty(\underline{M})_{\mu_2} \subsetneq \cdots \subsetneq \mathcal{I}_\infty(\underline{M})_{\mu_s} = \mathcal{I}_\infty(\underline{M})$$

where $\mu_1 < \mu_2 < ... < \mu_s$ are the rational numbers such that $\mathcal{I}_\infty(\underline{M})_{\mu_i}/\mathcal{I}_\infty(\underline{M})_{\mu_{i-1}}$ is a pure isocrystal of slope $-\mu_i$. For $\mu \in \mathbb{Q}$, we write $\mathcal{I}_\infty(\underline{M})_\mu = (\mathcal{I}_\infty(M)_\mu, \tau_M)$.

Definition 1.30. Let $\underline{M}$ be a nonzero A-motive over F.

(*a*) The elements of the set $w(\underline{M}) := \{\mu_1, ..., \mu_s\}$ are called the *weights* of $\underline{M}$. We agree that $w(\underline{0})$ is the empty set. We say that $\underline{M}$ pure of weight w if $\{\mu_1, ..., \mu_s\} = \{w\}$.

(*b*) We say that $\underline{M}$ is *mixed* if there exists an increasing filtration $(W_{\mu_i}\underline{M})_{i\in\mathbb{Q}}$ of $\underline{M}$ by sub-A-motives such that $(\mathcal{I}_\infty(W_{\mu_i}\underline{M}))_{i\in\mathbb{Q}}$ coincides with the weight filtration of $\mathcal{I}_\infty(\underline{M})$. In particular, a pure A-motive is mixed.

Remark 1.31. If $\underline{M}$ is an A-motive over F of weights $\{\mu_1, ..., \mu_s\}$ and F' is a field extension of F, then $\underline{M}_{F'}$ also has weights $\{\mu_1, ..., \mu_s\}$. This follows from the uniqueness of the slope filtration (Theorem 1.19). If $\underline{M}$ is mixed, then so is $\underline{M}_{F'}$.

Remark 1.32. In [HarJu, Ex. 2.3.13], the authors constructed an A-motive which is not mixed. The latter is constructed starting from an extension of a pure Anderson A-motive of weight 2 by another pure of weight 1.

If a filtration as in (b) exists, it might not be unique. However, if we impose that the filtration is composed with saturated submotives of $\underline{M}$, then it is unique. This follows from the next lemma.

Lemma 1.33. *Let $\underline{M}$ be an A-motive over F and let $\underline{P}$ be a submotive of $\underline{M}$. Then $\mathcal{I}_\infty(\underline{P}) = \mathcal{I}_\infty(\underline{P}^{sat})$. If $\underline{Q}$ is a submotive of $\underline{M}$ such that $\mathcal{I}_\infty(\underline{P}) = \mathcal{I}_\infty(\underline{Q})$ inside $\mathcal{I}_\infty(\underline{M})$, then $\underline{P}^{sat} = \underline{Q}^{sat}$.*

Proof. The inclusion $\underline{P} \subset \underline{P}^{\text{sat}}$ is an isogeny and therefore its cokernel is A-torsion (see [Har2, Thm. 5.12]). Consequently, $\mathcal{I}_\infty(\underline{P}) = \mathcal{I}_\infty(\underline{P}^{\text{sat}})$.
We prove the second part. The $A\otimes F$-modules P, P^{sat}, Q, Q^{sat} and $(P\cap Q)^{\text{sat}} = P^{\text{sat}} \cap Q^{\text{sat}}$ are locally-free of the same rank, as they become equal once $\mathcal{I}_\infty$

is applied. Note that $(P\cap Q)^{\rm sat}=P^{\rm sat}\cap Q^{\rm sat}$ is again endowed with an A-motive structure and hence so is the quotient $A\otimes F$-module $P^{\rm sat}/(P\cap Q)^{\rm sat}$. The underlying $A\otimes F$-module is locally-free and has rank 0; hence it is zero. The inclusion $P^{\rm sat}\cap Q^{\rm sat}\to P^{\rm sat}$ is therefore an isomorphism which yields $P^{\rm sat}\subset Q^{\rm sat}$. We conclude by exchanging the roles of P and Q in the above argument to obtain the converse inclusion. □

We deduce at once:

Proposition-Definition 1.34. Any mixed A-motive $\underline{M}$ over F admits a unique increasing filtration by saturated sub-Anderson A-motives $(W_{\mu_i}\underline{M})_{i\in\{1,\ldots,s\}}$ such that $(\mathcal{I}_\infty(W_{\mu_i}\underline{M}))_i$ coincides with the weight filtration of $\mathcal{I}_\infty(\underline{M})$ (Definition 1.21).

(i) We call $(W_{\mu_i}\underline{M})_i$ the *weight filtration of* $\underline{M}$.

(ii) For all $i\in\{1,...,s\}$, we let $W_{\mu_i}M$ be the underlying module of $W_{\mu_i}\underline{M}$.

(iii) For all $\mu\in\mathbb{Q}$, we set

$$W_\mu M:=\bigcup_{\mu_i\leq\mu}W_{\mu_i}M,\quad W_\mu\underline{M}:=(W_\mu M,\tau_M),$$
$$W_{<\mu} M:=\bigcup_{\mu_i<\mu}W_{\mu_i}M,\quad W_{<\mu}\underline{M}:=(W_{<\mu} M,\tau_M),$$

and $\operatorname{Gr}_\mu\underline{M}:=W_\mu\underline{M}/W_{<\mu}\underline{M}$. Both $\operatorname{Gr}_\mu\underline{M}$ and $W_\mu\underline{M}$, as well as $W_{<\mu}\underline{M}$, define mixed A-motives over F for all $\mu\in\mathbb{Q}$.

(iv) The *semi-simplification* $\underline{M}^{\rm ss}$ of $\underline{M}$ is the mixed A-motive over F given by

$$\underline{M}^{\rm ss}:=\bigoplus_{\mu\in\mathbb{Q}}\operatorname{Gr}_\mu\underline{M}.$$

(v) We let $\mathcal{MM}_F$ (resp. $\widetilde{\mathcal{MM}}_F$) be the full subcategory of $\mathcal{M}_F$ (resp. $\tilde{\mathcal{M}}_F$) whose objects are mixed.

The next lemma follows closely [HarJu, Prop 2.3.11(c)]

Lemma 1.35. *Any submotive $\underline{M}'\hookrightarrow\underline{M}$ and any quotient A-motive $\underline{M}\twoheadrightarrow\underline{M}''$ of mixed A-motives $\underline{M}$ is itself mixed.*

Proof. Let f denotes the morphism $\underline{M}\twoheadrightarrow\underline{M}''$ in $\mathcal{M}_F$. For $\mu\in\mathbb{Q}$, let $W_\mu M:=W_\mu M\cap M'$ and $W_\mu M'':=f(W_\mu M)^{\rm sat}\cap M''$. Both are saturated modules in M' and M'' respectively. They are also canonically endowed with an A-motive structure. Because $\mathcal{B}(F)$ is flat over $A\otimes F$ (as the composition of the flat morphisms $A\otimes F\subset K_\infty\otimes F\subset\mathcal{B}_\infty(F)$), [Bou, §I.2, Prop. 6] implies that $-\otimes_{A\otimes F}\mathcal{B}_\infty(F)$ commutes with finite intersections and we have

$$\begin{aligned}W_\mu M'\otimes_{A\otimes F}\mathcal{B}_\infty(F)&=(W_\mu M\otimes_{A\otimes F}\mathcal{B}_\infty(F))\cap\mathcal{I}_\infty(M')=\mathcal{I}_\infty(M)_\mu\cap\mathcal{I}_\infty(M')\\&=\mathcal{I}_\infty(M')_\mu\end{aligned}$$

where the last equality follows from Theorem 1.19. Similarly,

$$\begin{aligned} W_\mu M'' \otimes_{A\otimes F} \mathcal{B}_\infty(F) &= (f(W_\mu M)^{\mathrm{sat}} \otimes_{A\otimes F} \mathcal{B}_\infty(F)) \cap \mathcal{I}_\infty(M'') \\ &= f(\mathcal{I}_\infty(M)_\mu) \cap \mathcal{I}_\infty(M'') = \mathcal{I}_\infty(M'')_\mu. \end{aligned}$$

This shows that $\underline{M}'$ and $\underline{M}''$ are both mixed with respective weight filtrations $(W_\mu \underline{M}')_{\mu\in\mathbb{Q}}$ and $(W_\mu \underline{M}'')_{\mu\in\mathbb{Q}}$. □

As a consequence, we record:

Proposition 1.36. *The category $\mathcal{MM}_F$ is an exact subcategory of $\mathcal{M}_F$.*

Proof. This follows from Lemma 1.35 and [HarJu, Rmk 2.3.12]. □

Proposition 1.37. *Any morphism of mixed A-motives preserves the weight filtration, that is, given a morphism $f : \underline{M} \to \underline{N}$ in $\mathcal{M}_F$,*

$$\forall \mu \in \mathbb{Q}, \quad f(W_\mu M) \subset W_\mu N.$$

In particular, for all $\mu \in \mathbb{Q}$, the assignation $\underline{M} \mapsto W_\mu \underline{M}$ is functorial over $\mathcal{MM}_F$. Over $\widetilde{\mathcal{MM}}_F$, this assignation defines an exact functor, that is, the inclusion

$$\forall \mu \in \mathbb{Q}, \quad f(W_\mu M) \subset W_\mu N \cap f(M).$$

becomes an isogeny at the level of A-motives.

Proof. Let $f : \underline{M} \to \underline{N}$ be a morphism of mixed Anderson A-motives over F. In the category of isocrystals, f defines a morphism from $\mathcal{I}_\infty(\underline{M})$ to $\mathcal{I}_\infty(\underline{N})$. We have

$$\mathcal{I}_\infty(f(W_\mu \underline{M})) = f(\mathcal{I}_\infty(W_\mu \underline{M})) = f(\mathcal{I}_\infty(\underline{M})_\mu)$$

and, by flatness of $\mathcal{B}_\infty(F)$ over $A \otimes F$,

$$\begin{aligned} \mathcal{I}_\infty(f(\tau^* M) \cap W_\mu N) &= \mathcal{I}_\infty(f(\tau^* M)) \cap \mathcal{I}_\infty(W_\mu N) = f(\tau^* \mathcal{I}_\infty(M)) \cap \mathcal{I}_\infty(N)_\mu \\ &= f(\mathcal{I}_\infty(M)_\mu) \end{aligned} \tag{1.3}$$

where the last equality follows from Theorem 1.19. By Lemma 1.33 applied to (1.3),

$$f(W_\mu M) \subset f(W_\mu M)^{\mathrm{sat}} = f(\tau^* M)^{\mathrm{sat}} \cap W_\mu N \subset W_\mu N. \tag{1.4}$$

To conclude that W_μ is exact over $\widetilde{\mathcal{MM}}_F$, it suffices to note that all inclusions in (1.4) are isogenies of A-motives over F. □

Remark 1.38. Here is a description on how weights behave under linear algebra type operations. First note that $\mathbb{1}$ is a pure A-motive over F of weight 0. Given two mixed A-motives $\underline{M}$ and $\underline{N}$, their biproduct $\underline{M} \oplus \underline{N}$ is again mixed with weight filtration $W_\mu(\underline{M} \oplus \underline{N}) = W_\mu \underline{M} \oplus W_\mu \underline{N}$ ($\mu \in \mathbb{Q}$). Their tensor product $\underline{M} \otimes \underline{N}$ is also mixed, with λ-part of its weight filtration being:

$$W_\lambda(\underline{M} \otimes \underline{N}) = \left(\sum_{\mu+\nu=\lambda} W_\mu \underline{M} \otimes W_\nu \underline{N} \right)^{\mathrm{sat}}.$$

We took the saturation A-motive to ensure that the above is a saturated sub-A-motive of $\underline{M}\otimes\underline{N}$. The dual $\underline{M}^{\vee}$ is mixed, and the μ-part of its weight filtration $W_{\mu}\underline{M}$ has for underlying module $W_{\mu}M^{\vee}=\{m\in M^{\vee}|\forall\lambda<-\mu:\ m(W_{\lambda}M)=0\}^{\text{sat}}$. In general, given $\underline{M}$ and $\underline{N}$ two A-motives over F (without regarding whether $\underline{M}$ or $\underline{N}$ are mixed) and an exact sequence $0\to\underline{M}'\to\underline{M}\to\underline{M}''\to 0$ in $\mathcal{M}_F$, we have

$$\begin{array}{lcl} w(\underline{0}) & = & \emptyset \\ w(\underline{M}^{\vee}) & = & -w(\underline{M}) \\ w(\underline{M}\oplus\underline{N}) & = & w(\underline{M})\cup w(\underline{N}) \\ w(\underline{M}) & = & w(\underline{M}')\cup w(\underline{M}'') \\ w(\underline{M}\otimes\underline{N}) & = & \{w+v \mid w\in w(\underline{M}),\ v\in w(\underline{N})\} \end{array}$$

See [HarJu, Prop. 2.3.11].

1.2.3 Non-positively weighted A-motives

The content of this subsection will be used only in Chapter 6 where we will attach $C\times C$-*shtuka models* to A-motives with negative weights (Theorem 6.11). This construction is at the heart of the proof of Theorem F (Chapter 0).

The next lemma is the reason why A-motives with non-positive weights are peculiar.

Lemma 1.39. *Let $\underline{M}$ be an A-motive over F whose weights are all non-positive. Then $\mathcal{I}_{\infty}(M)$ contains an $\mathcal{A}_{\infty}(F)$-lattice stable by τ_M.*

Proof. We first treat the case where $\underline{M}$ is pure. In this case there is an $\mathcal{A}_{\infty}(F)$-lattice T in $\mathcal{I}_{\infty}(M)$ such that $\langle\tau_M^sT\rangle=\mathfrak{m}_{\infty}^rT$ for two integers $s>0$ and $r\geq 0$. The $\mathcal{A}_{\infty}(F)$-module generated by T, $\langle\tau_MT\rangle$, ..., $\langle\tau_M^{s-1}T\rangle$ defines an $\mathcal{A}_{\infty}(F)$-lattice T' stable by τ_M.

We now treat the general case. Let F' be a perfect field containing F. The A-motive $\underline{M}_{F'}$, obtained from $\underline{M}$ by base-change, has the same weights as $\underline{M}$ (Remark 1.31). The Dieudonné-Manin Theorem 1.24 states that $\mathcal{I}_{\infty}(M_{F'})$ decomposes as a direct sum:

$$\mathcal{I}_{\infty}(M_{F'})=M\otimes_{A\otimes F}\mathcal{B}_{\infty}(F')=\bigoplus_{i=1}^{s}D_i$$

where, for all i, D_i is a submodule of $\mathcal{I}_{\infty}(M_{F'})$ stable by τ_M and (D_i,τ_M) defines a pure isocrystal over F' of non-negative slope. As such, D_i contains an $\mathcal{A}_{\infty}(F')$-lattice T_i' stable by τ_M. We let $T':=\bigoplus_i T_i'$.

Let T be the $\mathcal{A}_{\infty}(F)$-module given by the intersection of the $\mathcal{A}_{\infty}(F')$-module T' and the $\mathcal{B}_{\infty}(F)$-module $\mathcal{I}_{\infty}(M)$. We claim that T is an $\mathcal{A}_{\infty}(F)$-lattice stable by τ_M. Stability by τ_M is clear, so we prove that T is an $\mathcal{A}_{\infty}(F)$-lattice. First of all, we have

$$\begin{aligned} T\otimes_{\mathcal{O}_{\infty}}K_{\infty} &= (T'\otimes_{\mathcal{O}_{\infty}}K_{\infty})\cap(M\otimes_{A\otimes F}\mathcal{B}_{\infty}(F)\otimes_{\mathcal{O}_{\infty}}K_{\infty})=M\otimes_{A\otimes F}\mathcal{B}_{\infty}(F) \\ &=\mathcal{I}_{\infty}(M) \end{aligned}$$

(for the first equality, we used that the inclusion $\mathcal{O}_\infty \to K_\infty$ is flat, and thus that $-\otimes_{\mathcal{O}_\infty} K_\infty$ commutes with finite intersections [Bou, §I.2, Prop. 6]). It follows that T generates $\mathcal{I}_\infty(M)$ over K_∞.

Secondly, we show that T is finitely generated over $\mathcal{A}_\infty(F)$. Since $\mathcal{A}_\infty(F)$ is a Noetherian ring, it suffices to find a finitely generated $\mathcal{A}_\infty(F)$-module which contains T. If $\underline{M}$ has rank r, the $\mathcal{B}_\infty(F)$-module $\mathcal{I}_\infty(M)$ is free of rank r by [Mor2, Cor. 3.5.2]. We fix $\mathbf{b}$ a basis of $\mathcal{I}_\infty(M)$. Then $\mathbf{b}$ induces a basis of the $\mathcal{B}_\infty(F')$-module $\mathcal{I}_\infty(M_{F'}) = \mathcal{I}_\infty(M) \otimes_{\mathcal{B}_\infty(F)} \mathcal{B}_\infty(F')$. We let L' be the free $\mathcal{A}_\infty(F')$-module generated by $\mathbf{b}$. Because T' is finitely generated over $\mathcal{A}_\infty(F')$, there is a large enough integer k such that $T' \subset \mathfrak{m}_\infty^{-k} L'$. Therefore,

$$T = T' \cap \mathcal{I}_\infty(M) \subset (\mathfrak{m}_\infty^{-k} L') \cap \mathcal{I}_\infty(M) = \mathfrak{m}_\infty^{-k}(L' \cap \mathcal{I}_\infty(M)).$$

Now, $L' \cap \mathcal{I}_\infty(M)$ equals the $\mathcal{A}_\infty(F)$-module L generated by $\mathbf{b}$. As desired, $T \subset \mathfrak{m}_\infty^{-k} L$ and T is finitely generated. □

Compare to Lemma 1.39, if the weights are negative there is further:

Lemma 1.40. *Let $\underline{M}$ be an A-motive over F whose weights are all negative. There exist an $\mathcal{A}_\infty(F)$-lattice T in $\mathcal{I}_\infty(M)$ and two positive integers d and h such that $\langle \tau_M^h T\rangle \subset \mathfrak{m}_\infty^d T$.*

Proof. If $\underline{M}$ is pure, this follows from the definition of purity. For the general case, we proceed as in the proof of Lemma 1.39. For F' a perfect field containing F, the $\mathcal{A}_\infty(F')$-module $\mathcal{I}_\infty(M_{F'})$ decomposes as a direct sum:

$$\mathcal{I}_\infty(M_{F'}) = M \otimes_{A\otimes F} \mathcal{B}_\infty(F') = \bigoplus_{i=1}^{s} D_i$$

where, for all i, (D_i, τ_M) defines a pure isocrystal over F' of negative slope. As such, D_i contains an $\mathcal{A}_\infty(F')$-lattice T_i' such that $\langle \tau_M^{h_i} T_i\rangle = \mathfrak{m}_\infty^{d_i} T_i$ for integers $h_i, d_i > 0$. We let $T' := \bigoplus_i T_i'$, so that $\langle \tau_M^h T'\rangle \subset \mathfrak{m}_\infty^d T'$ for $h = \max h_i$ and $d = \min d_i$. We let T be the $\mathcal{A}_\infty(F)$-module given by the intersection of the $\mathcal{A}_\infty(F')$-module T' and the $\mathcal{B}_\infty(F)$-module $\mathcal{I}_\infty(M)$. We prove that T is an $\mathcal{A}_\infty(F)$-lattice satisfying $\langle \tau_M^h T\rangle \subset \mathfrak{m}_\infty^d T$ as in the proof of Lemma 1.39. □

Conversely, we record:

Lemma 1.41. *Let $\underline{M}$ be an A-motive over F such that $\mathcal{I}_\infty(M)$ contains an $\mathcal{A}_\infty(F)$-lattice T such that $\langle \tau_M T\rangle \subset T$. Then, all the weights of $\underline{M}$ are non-positive.*

Proof. As in the proof of Lemma 1.39, up to a base-change, one can assume that F is a perfect field. By Theorem 1.24, the $\mathcal{B}_\infty(F)$-module $\mathcal{I}_\infty(M)$ decomposes as a direct sum

$$\mathcal{I}_\infty(M) = \bigoplus_{i=1}^{s} D_i$$

where, for all $i \in \{1, ..., s\}$, $\underline{D}_i := (D_i, \tau_M)$ defines a pure subisocrystal of $\mathcal{I}_\infty(\underline{M})$. For all $i \in \{1, ..., s\}$, there exists an $\mathcal{A}_\infty(F)$-lattice $L_i \subset D_i$ and integers $r_i > 0$, s_i such that $\langle \tau_M^{r_i} L_i \rangle = \mathfrak{m}_\infty^{s_i} L_i$. Taking determinants over $\mathcal{A}_\infty(F)$ yields

$$r_i \cdot \deg(\underline{D}_i) = s_i$$

On the other-hand, if $\mathcal{I}_\infty(M)$ contains an $\mathcal{A}_\infty(F)$-lattice T stable by τ_M, then $T \cap D_i$ defines an $\mathcal{A}_\infty(F)$-lattice T_i in D_i such that $\langle \tau_M T_i \rangle \subset T_i$. It follows that

$$\deg(\underline{D}_i) \geq 0.$$

Hence, $s_i \geq 0$ for all $i \in \{1, ..., s\}$ which implies that the slopes of $\mathcal{I}_\infty(\underline{M})$ are all non-negative. Hence, the weights of $\underline{M}$ are all non-positive. $\square$

1.3 The λ-adic realization functor

We now introduce the function field analogue of the ℓ-adic realization functor (Definition 1.42), and show that it is exact (Proposition 1.45). It will allow us to define *extensions with good reduction* in Chapter 3.

For the rest of this section, λ is a closed point of C distinct from ∞. We denote by $\mathfrak{m} = \mathfrak{m}_\lambda$ the maximal ideal of A associated to λ and by $\mathcal{O}_\lambda$ the completion of A at $\mathfrak{m}$. We let F be a field containing K and let $\kappa : A \to F$ be the inclusion, so that $\ker \kappa = (0)$ (*generic characteristic*). Let F^s be a separable closure of F and denote by $G_F = \mathrm{Gal}(F^s|F)$ the absolute Galois group of F equipped with the profinite topology.

Let $\underline{M} = (M, \tau_M)$ be an A-motive over F of rank r. Let $\underline{M}_{F^s} = (M_{F^s}, \tau_M)$ be the A-motive over F^s obtained from $\underline{M}$ by the base-change functor. By Proposition 1.29, $\mathcal{I}_\lambda(\underline{M}_{F^s})$ defines an isocrystal over F^s. Given $\sigma \in G_F$, σ acts on $\mathcal{A}_\lambda(F^s)$ via $\mathrm{id}_{\mathcal{O}_\lambda} \otimes \sigma$. This action extends to the $\mathcal{O}_\lambda$-module

$$\mathcal{J}_\lambda(M_{F^s}) := M \otimes_{A \otimes F} \mathcal{A}_\lambda(F^s)$$

which leaves the left-hand side of the tensor invariant. Following [HarJu, §2.3.5], we define:

Definition 1.42. We define the λ-*adic realization* $T_\lambda \underline{M}$ of $\underline{M}$ to be the $\mathcal{O}_\lambda$-module

$$T_\lambda \underline{M} := \{m \in \mathcal{J}_\lambda(M_{F^s}) \mid m = \tau_M(\tau^* m)\}$$

given with the compatible action of G_F it inherits as a submodule of $\mathcal{J}_\lambda(M_{F^s})$.

Remark 1.43. In [Mor2], Mornev extended this construction to the situation where λ is the closed point ∞.

The next lemma is well-known in the case of τ-sheaves (e.g. [Gar3, Prop.3.3] or [TagWa, Prop.6.1]).

Lemma 1.44. *The map* $T_\lambda \underline{M} \otimes_{\mathcal{O}_\lambda} \mathcal{A}_\lambda(F^s) \to \mathcal{J}_\lambda(M_{F^s})$, $\omega \otimes f \mapsto \omega \cdot f$ *is an isomorphism of* $\mathcal{A}_\lambda(F^s)$*-modules. In particular, the* $\mathcal{O}_\lambda$*-module* $T_\lambda \underline{M}$ *is free of rank* r *and the action of* G_F *on* $T_\lambda \underline{M}$ *is continuous.*

Proof. Let $n \geq 1$. The ideal $\mathfrak{j}$ is invertible in $\mathcal{A}_\lambda(F^s)$ by Lemma 1.27 and it follows that $m + \mathfrak{m}^n \mapsto \tau_M(\tau^* m) + \mathfrak{m}^n$ induces a well-defined A-linear automorphism of

$$M \otimes_F F^s / \mathfrak{m}^n (M \otimes_F F^s) = \mathcal{J}_\lambda(M_{F^s}) / \mathfrak{m}^n \mathcal{J}_\lambda(M_{F^s}).$$

The above is a finite dimensional F^s-vector space, and $m + \mathfrak{m}^n \mapsto \tau_M(\tau^* m) + \mathfrak{m}^n$ is *q-linear*[1] in the sense of [Kat1, §1]. By [Kat1, Prop. 1.1], the multiplication map

$$\{m \in \mathcal{J}_\lambda(M_{F^s})/\mathfrak{m}^n \mathcal{J}_\lambda(M_{F^s}) \mid \tau_M(\tau^* m) = m\} \otimes_{\mathbb{F}} F^s \to \mathcal{J}_\lambda(M_{F^s})/\mathfrak{m}^n \mathcal{J}_\lambda(M_{F^s}) \tag{1.5}$$

is an isomorphism. Taking the inverse limit over all n yields the desired isomorphism. Because $\mathcal{J}_\lambda(M_{F^s})$ is a free module of constant rank r over $\mathcal{A}_\lambda(F^s)$, the same is true for $\mathcal{J}_\lambda(M_{F^s})/\mathfrak{m}^n \mathcal{J}_\lambda(M_{F^s})$ over $(A/\mathfrak{m}^n) \otimes F^s$. The isomorphism (1.5) implies that the $A/\mathfrak{m}^n$-module

$$[\mathcal{J}_\lambda(M_{F^s})/\mathfrak{m}^n \mathcal{J}_\lambda(M_{F^s})]^{\tau_M = 1} := \{m \in \mathcal{J}_\lambda(M_{F^s}) \mid \tau_M(\tau^* m) = m\}$$

is free of rank r over $A/\mathfrak{m}^n$. The projective limit over n:

$$T_\lambda \underline{M} = \varprojlim_n [\mathcal{J}_\lambda(M_{F^s})/\mathfrak{m}^n \mathcal{J}_\lambda(M_{F^s})]^{\tau_M = 1}$$

is then a free $\mathcal{O}_\lambda$-module of rank r.

By definition, the action of G_F on $T_\lambda \underline{M}$ is continuous if, and only if, for all $n \geq 1$, the induced action of G_F on $T_\lambda \underline{M}/\mathfrak{m}^n T_\lambda \underline{M}$ factors through a finite quotient. For $n \geq 1$, let $\mathbf{t} = \{t_1, ..., t_s\}$ be a basis of the finite dimensional F-vector space $\mathcal{J}_\lambda(M)/\mathfrak{m}^n \mathcal{J}_\lambda(M)$. Let F_M be the matrix of τ_M written in the basis $\tau^* \mathbf{t}$ and $\mathbf{t}$. Let $\boldsymbol{\omega} = \{\omega_1, ..., \omega_s\}$ be a basis of $T_\lambda \underline{M}/\mathfrak{m}^n T_\lambda \underline{M}$ over $\mathbb{F}$.

By (1.5), $\boldsymbol{\omega}$ is a basis of $\mathcal{J}_\lambda(M_{F^s})/\mathfrak{m}^n \mathcal{J}_\lambda(M_{F^s})$ over F^s, and we let $w_{ij} \in F^s$ be the coefficients of $\boldsymbol{\omega}$ expressed in $\mathbf{t}$, that is, for $i \in \{1, ..., s\}$, $\omega_i = \sum w_{ij} t_j$. We let E_n denote the Galois closure of the finite separable extension $F(w_{ij} | (i,j) \in \{1, ..., s\}^2)$ of F in F^s. We have

$$T_\lambda \underline{M}/\mathfrak{m}^n T_\lambda \underline{M} = \{m \in (M \otimes_F E_n)/\mathfrak{m}^n (M \otimes_F E_n) \mid \tau_M(\tau^* m) = m\}.$$

As such, the action of G_F factors through $\operatorname{Gal}(E_n|F) = G_F / \operatorname{Gal}(F^s|E_n)$. We conclude that the action of G_F is continuous. ☐

Proposition 1.45. *The following sequence of* $\mathcal{O}_\lambda[G_F]$*-modules is exact*

$$0 \to T_\lambda \underline{M} \longrightarrow M \otimes_{A \otimes F} \mathcal{A}_\lambda(F^s) \stackrel{\mathrm{id} - \tau_M}{\longrightarrow} M \otimes_{A \otimes F} \mathcal{A}_\lambda(F^s) \longrightarrow 0.$$

[1] For k a field containing $\mathbb{F}$ and V a k-vector space, an $\mathbb{F}$-linear endomorphism f of V is *q-linear* if $f(rv) = r^q f(v)$ for all $r \in k$ and $v \in V$.

Proof. Everything is clear but the surjectivity of $\mathrm{id} - \tau_M$. Let π be a uniformizer of $\mathcal{O}_\lambda$ and let $\mathbb{F}_\lambda$ be its residue field. Let $f = \sum_{n\geq 0} a_n \pi^n$ be a series in $\mathcal{A}_\lambda(F^s) = (\mathbb{F}_\lambda \otimes F^s)[\![\pi]\!]$. Let $b_n \in \mathbb{F}_\lambda \otimes F^s$ be such that $[\mathrm{id}_{\mathbb{F}_\lambda} \otimes (\mathrm{id} - \mathrm{Frob}_q)](b_n) = a_n$ (which exists as F^s is separably closed), and let g be the series $\sum_{n\geq 0} b_n \pi^n$ in $\mathcal{A}_\lambda(F^s)$. For $\omega \in T_\lambda \underline{M}$, we have

$$(\mathrm{id} - \tau_M)(\omega \cdot g) = \omega \cdot f.$$

It follows that any element in $M \otimes_{A\otimes F} \mathcal{A}_\lambda(F^s)$ of the form $\omega \cdot f$ is in the image of $\mathrm{id} - \tau_M$. By the first part of Lemma 1.44, those elements generates $M \otimes_{A\otimes F} \mathcal{A}_\lambda(F^s)$. We conclude that $\mathrm{id} - \tau_M$ is surjective. $\square$

We obtain the main result of this section:

Corollary 1.46. *The functor $\underline{M} \mapsto T_\lambda \underline{M}$, from $\mathcal{M}_F$ to the category of continuous $\mathcal{O}_\lambda$-linear G_F-representations, is exact.*

Proof. Let $S : 0 \to \underline{M}' \to \underline{M} \to \underline{M}'' \to 0$ be an exact sequence in $\mathcal{M}_F$. The underlying sequence of $A \otimes F$-modules is exact, and because $\mathcal{A}_\lambda(F^s)$ is flat over $A \otimes F$, the sequence of $\mathcal{A}_\lambda(S)$-modules $\mathcal{J}_\lambda(S)$ is exact. In particular, the next commutative diagram of $\mathcal{A}_\lambda(F^s)$-modules has exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{J}_\lambda(M') \arrow[r] \arrow[d, "\mathrm{id} - \tau_{M'}"] & \mathcal{J}_\lambda(M) \arrow[r] \arrow[d, "\mathrm{id} - \tau_M"] & \mathcal{J}_\lambda(M'') \arrow[r] \arrow[d, "\mathrm{id} - \tau_{M''}"] & 0 \\
0 \arrow[r] & \mathcal{J}_\lambda(M') \arrow[r] & \mathcal{J}_\lambda(M) \arrow[r] & \mathcal{J}_\lambda(M'') \arrow[r] & 0
\end{tikzcd}$$

and the Snake Lemma together with Proposition 1.45 yields that $T_\lambda S$ is exact. $\square$

1.4 The Betti realization functor

Here, we introduce the *Betti realization* of an A-motive (Definition 1.48) and discuss *rigid analytically triviality* (Definition 1.50). One chief aim is to define the full subcategory $\mathcal{MM}_F^{\mathrm{rig}}$ of $\mathcal{MM}_F$ consisting of *rigid analytically trivial mixed A-motives*, which shall be the source of the Hodge realization functor to be defined in Chapter 5.

The notion of rigid analytic triviality dates back to Anderson [And, §2] (see also [HarJu, §2.3.3]). But a novelty that we add here is the definition of a natural action of the absolute Galois group of K_∞ on the Betti realization A-module, similar to the action of $\mathrm{Gal}(\mathbb{C}|\mathbb{R})$ on the Betti cohomology groups[2] $H^i(X(\mathbb{C}), \mathbb{Z})$ of an algebraic variety X over $\mathbb{Q}$.

We recall that K_∞ is the completion of K at the place ∞, and that $\mathcal{O}_\infty$ denotes its valuation ring. We fix K_∞^s a separable closure of K_∞ and we let

[2] This action is induced by functoriality of $X \mapsto H^i(X(\mathbb{C}), \mathbb{Z})$ on the action of $\mathrm{Gal}(\mathbb{C}|\mathbb{R})$ on the $\mathbb{C}$-points of X.

$\mathbb{C}_\infty$ be the completion of K_∞^s. The field $\mathbb{C}_\infty$ is algebraically closed by Krasner's Lemma (see [FonOu, Cor.3.2]). We let $|\cdot|$ be the unique extension of the norm on K_∞ to $\mathbb{C}_\infty$. Let G_∞ be the absolute Galois group $\mathrm{Gal}(K_\infty^s|K_\infty)$. By continuity, the action of G_∞ extends to $\mathbb{C}_\infty$.

Let L be a complete subfield of $\mathbb{C}_\infty$ containing K_∞. We present three equivalent constructions of the affinoid algebra $L\langle A\rangle$.

1. Consider the non-archimedean norm $|\cdot|$ on L it inherits as a subfield of $\mathbb{C}_\infty$, and define a norm on $A\otimes L$ by
$$\|x\| := \inf\left(\max_i |l_i|\right) \quad \text{for } x\in A\otimes L$$
where the infimum is taken over all the representations of x of the form $\sum_i (a_i\otimes l_i)$. We define $L\langle A\rangle$ to be the completion $A\hat{\otimes}L$ of $A\otimes L$ with respect to the norm $\|\cdot\|$. Given any basis $(t_i)_{i\geq 0}$ of A over $\mathbb{F}$, it is proved in [GazMa, Prop.2.2] that
$$L\langle A\rangle = \left\{ \sum_{i=0}^{\infty} t_i\otimes l_i \;\middle|\; l_i\in L,\; \lim_{n\to\infty} a_n \to 0 \right\}. \tag{1.6}$$

2. Let $\mathcal{O}_L$ be the valuation ring of L with maximal ideal $\mathfrak{m}_L$. We denote by $\mathcal{O}_L\langle A\rangle$ the completion of $A\otimes\mathcal{O}_L$ with respect to $A\otimes\mathfrak{m}_L$, that is $\mathcal{O}_L\langle A\rangle = \varprojlim_n (A\otimes\mathcal{O}_L/A\otimes\mathfrak{m}_L^n)$. Then, $L\langle A\rangle$ is isomorphic to the L-algebra $L\otimes_{\mathcal{O}_L}\mathcal{O}_L\langle A\rangle$.

3. Let $t\in A$ be a non constant element. The inclusion $\mathbb{F}[t]\to A$ makes A into a finite flat A-module. We let
$$L\langle t\rangle = \left\{ \sum_{n=0}^{\infty} a_n t^n \;\middle|\; a_n\in L;\; \lim_{n\to\infty} a_n\to 0 \right\}.$$
Then, $A\otimes_{\mathbb{F}[t]} L\langle t\rangle$ is isomorphic to $L\langle A\rangle$.

If L is fixed under the action of G_∞, the latter action extends to $L\langle A\rangle$ by leaving A invariant. Under the description 1, for elements in the form of (1.6), $\sigma\in G_\infty$ acts on $L\langle A\rangle$ as follows:
$$\left(\sum_{i=0}^{\infty} t_i\otimes l_i\right)^{\sigma} = \left(\sum_{n=0}^{\infty} t_i\otimes\sigma(l_i)\right).$$
This definition is independent of the chosen basis $(t_i)_{i\geq 0}$.

The following preliminary lemma will be used next, in the definition of the Betti realization functor.

Lemma 1.47. *Let $\kappa: A\to L$ be an $\mathbb{F}$-algebra morphism with discrete image. We have $\mathfrak{j}_\kappa L\langle A\rangle = L\langle A\rangle$.*

Proof. Because $\kappa(A)$ is discrete in L, it contains an element α of norm $|\alpha| > 1$. Let $a \in A$ be such that $\alpha = \kappa(a)$. Then, $\kappa(a)^{-1} \in \mathfrak{m}_L$ and the series

$$-\sum_{n\geq 0} a^n \otimes \kappa(a)^{-(n+1)}$$

converges in $\mathcal{O}_L\langle A\rangle$ to the inverse of $(a \otimes 1 - 1 \otimes \kappa(a))$. □

1.4.1 Definition

Let F be a finite extension of K and let $v : F \to \mathbb{C}_\infty$ be a K-algebra morphism. Here, $\kappa : K \to F$ is the inclusion of fields. The assignation $|x|_v := |v(x)|$ for $x \in F$ defines a norm on F, and we denote by F_v the corresponding completion of F. We let F_v^s be a separable closure of F_v. Let $G_v = \mathrm{Gal}(F_v^s|F_v)$ be the absolute Galois group of F_v. By continuity, G_v acts on $\mathbb{C}_\infty$.

Let $\underline{M} = (M, \tau_M)$ be an A-motive over F. By Lemma 1.47, the ideal $\mathfrak{j}$ of $A \otimes F$ is invertible in $\mathbb{C}_\infty\langle A\rangle$ and τ_M induces an isomorphism of modules over $\mathbb{C}_\infty\langle A\rangle$

$$\tau^*(M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle A\rangle) \xrightarrow{\sim} M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle A\rangle \tag{1.7}$$

which commutes with the action of G_v on $M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle A\rangle$, inherited from the right-hand side of the tensor. We still denote by τ_M the isomorphism (1.7).

Definition 1.48. The *v-adic Betti realization* of $\underline{M}$ is the A-module

$$\Lambda_v(\underline{M}) := \{\omega \in M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle A\rangle \mid \omega = \tau_M(\tau^*\omega)\}$$

endowed with the compatible action of G_v it inherits as a submodule of $M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle A\rangle$. Let $\Lambda_v(\underline{M})^+$ be the sub-A-module of $\Lambda_v(\underline{M})$ of elements fixed by the action of G_v. Similarly, the *Betti realization of $\underline{M}$* is the A-module

$$\Lambda(\underline{M}) := \{\omega \in M \otimes_{A\otimes K} \mathbb{C}_\infty\langle A\rangle \mid \omega = \tau_M(\tau^*\omega)\}.$$

endowed with the compatible action of G_∞. We let $\Lambda(\underline{M})^+$ be the sub-A-module of $\Lambda(\underline{M})$ of elements fixed by G_∞.

Remark 1.49. The distinction between *v-Betti* and *Betti realization* seems to be new. It does not appear in [HarJu] as A-motives are all considered over $\mathbb{C}_\infty$. This distinction is already made in the number fields case where, given a variety X over F and a positive integer i, the Betti realizations of the hypothetical mixed motive $M := h^i(X)$ are given by

$$M_B := H^i(X(\mathbb{C}), \mathbb{Z}), \quad M_{B,v} := H^i((X \times_{F,v} \mathbb{C})(\mathbb{C}), \mathbb{Z})$$

for $v : F \to \mathbb{C}$ an embedding. Admitting the philosophy of mixed motives, we have a direct sum decomposition $M_B = \bigoplus_{v|\infty} M_{B,v}$. In function fields arithmetic, there is no such a decompostion property for arbitrary extensions F/K. Indeed, let $K \subset E \subset F$ be such that $K \subset E$ is separable and $E \subset F$ is purely inseparable. Then $M \otimes_K \mathbb{C}_\infty$ decomposes as the direct sum $\bigoplus_v (M \otimes_E \mathbb{C}_\infty)$ indexed over K-linear embeddings $v : E \to \mathbb{C}_\infty$. In particular, if F is separable over K, we have

$$\Lambda(\underline{M}) \cong \bigoplus_v \Lambda_v(\underline{M})$$

where the sum is indexed over K-algebra morphisms $v : F \to \mathbb{C}_\infty$. Yet, if F/K is inseparable, this is no longer true.

An A-motive $\underline{M} = (M, \tau_M)$ over F of rank r induces an A-motive $\underline{M}'$ over K of rank $r[F : K]$ by seeing M as an $A \otimes K$-module. Let $\mathrm{Res}_{F/K} : \mathcal{M}_F \to \mathcal{M}_K$, $\underline{M} \mapsto \underline{M}'$ be the *restriction of scalars* functor. If $i : K \to \mathbb{C}_\infty$ denotes the inclusion of fields, we have by definition

$$\Lambda(\underline{M}) = \Lambda_i(\mathrm{Res}_{F/K} \underline{M}). \tag{1.8}$$

As the Betti realization is contained in the case $F = K$, it is enough to study the v-Betti realization.

Definition 1.50. The A-motive $\underline{M}$ is called *rigid analytically trivial* if the $\mathbb{C}_\infty\langle A\rangle$-linear morphism $\Lambda(\underline{M}) \otimes_A \mathbb{C}_\infty\langle A\rangle \to M \otimes_{A\otimes K} \mathbb{C}_\infty\langle A\rangle$ given by the multiplication is an isomorphism. For a K-algebra morphism $v : F \to \mathbb{C}_\infty$, $\underline{M}$ is called *v-rigid analytically trivial* if $\Lambda_v(\underline{M}) \otimes_A \mathbb{C}_\infty\langle A\rangle \to M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle A\rangle$ is an isomorphism.

Remark 1.51. Not every A-motive is rigid analytically trivial. An example of A-motive which is not rigid analytically trivial is given in [And, 2.2].

The following Proposition rephrases [BöcHa, Cor.4.3]:

Proposition 1.52. *Let $\underline{M}$ be an A-motive over F of rank r and let $v : F \to \mathbb{C}_\infty$ be a K-algebra morphism. Then $\Lambda_v(\underline{M})$ is a finite projective A-module of rank r' satisfying $r' \leq r$ with equality if and only if $\underline{M}$ v-rigid analytically trivial.*

The next proposition assembles the definitions of 1.50.

Proposition 1.53. *Let $\underline{M}$ be an A-motive over F. The following are equivalent:*

(i) *$\underline{M}$ is rigid analytically trivial.*

(ii) *There exists a K-algebra morphism $v : F \to \mathbb{C}_\infty$ such that $\underline{M}$ is v-rigid analytically trivial.*

(iii) *$\underline{M}$ is v-rigid analytically trivial for all K-algebra morphisms $v : F \to \mathbb{C}_\infty$.*

Proof. We have that *(iii)* implies *(ii)*. Conversely, assume *(ii)* and let $v' : F \to \mathbb{C}_\infty$ be a K-algebra morphism. The image of v and v' both land in the algebraic closure $\bar{K}$ of K in $\mathbb{C}_\infty$. Let $\sigma \in \mathrm{Aut}_K(\bar{K})$ be such that $v' = \sigma \circ v$. By continuity, σ extends to $\mathbb{C}_\infty$ and then A-linearly to $\mathbb{C}_\infty\langle A\rangle$. We have

$$(M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle A\rangle) \otimes_{\mathbb{C}_\infty\langle A\rangle,\sigma} \mathbb{C}_\infty\langle A\rangle \cong M \otimes_{A\otimes F,v'} \mathbb{C}_\infty\langle A\rangle$$

The above maps isomorphically $\Lambda_v(\underline{M})$ to $\Lambda_{v'}(\underline{M})$. We obtain (iii).

It remains to show that (i) is equivalent to (iii). Let $K \subset E \subset F$ be such that E is a separable extension of K and F is a purely inseparable extension of E. Let $s = [E : K]$ and $m = [F : E]$. We decompose $\Lambda(\underline{M})$ as follows:

$$\begin{aligned}
\Lambda(\underline{M}) &= \{\omega \in M \otimes_{A\otimes K} \mathbb{C}_\infty\langle A\rangle \mid \omega = \tau_M(\tau^*\omega)\} \\
&\cong \bigoplus_{v:E\to\mathbb{C}_\infty} \{\omega \in M \otimes_{A\otimes E,v} \mathbb{C}_\infty\langle A\rangle \mid \omega = \tau_M(\tau^*\omega)\} \\
&\cong \bigoplus_{v:E\to\mathbb{C}_\infty} \{\omega \in M \otimes_{A\otimes F} ((A\otimes F) \otimes_{A\otimes E,v} \mathbb{C}_\infty\langle A\rangle) \mid \omega = \tau_M(\tau^*\omega)\}
\end{aligned} \tag{1.9}$$

where the sum is indexed over the embeddings $v : E \to \mathbb{C}_\infty$. Let us denote by $\Lambda_v^*(\underline{M})$ the summand of (1.9) so that $\Lambda(\underline{M}) \cong \bigoplus_v \Lambda_v^*(\underline{M})$. Let $\mathbf{e} = (e_1, ..., e_m)$ be a basis of $F \otimes_E \mathbb{C}_\infty$ over $\mathbb{C}_\infty$. The A-linear map

$$\mathbf{e}_v^\vee : \Lambda_v(\underline{M})^m \longrightarrow \Lambda_v^*(\underline{M}), \quad (\omega_i)_i \longmapsto \sum_i \omega_i e_i$$

is an isomorphism, and thus $\mathrm{rk}_A \Lambda_v^*(\underline{M}) = m\, \mathrm{rk}_A \Lambda_v(\underline{M})$. Hence, we find

$$\mathrm{rk}_A \Lambda(\underline{M}) = \sum_{v:E\to\mathbb{C}_\infty} \mathrm{rk}_A \Lambda_v^*(\underline{M}) = m \sum_{v:E\to\mathbb{C}_\infty} \mathrm{rk}_A \Lambda_v(\underline{M}).$$

Because $\#\mathrm{Hom}_K(E, \mathbb{C}_\infty) = s$ and $ms = [F : K]$, the proof is ended by Proposition 1.52. □

In virtue of Proposition 1.53, we now voluntarily forget the notion of v-rigid analytic triviality. When $\underline{M}$ is rigid analytically trivial, in Definition 1.50 the field $\mathbb{C}_\infty$ can be replaced by a much smaller field. This is the next proposition.

Proposition 1.54. *Let $\underline{M}$ be a rigid analytically trivial A-motive over F and let $v : F \to \mathbb{C}_\infty$ be a K-algebra morphism. There exists a (complete) finite separable field extension L of F_v contained in $\mathbb{C}_\infty$ such that $\Lambda_v(\underline{M})$ is contained in $M \otimes_{A\otimes F,v} L\langle A\rangle$. In particular, the action of G_v equipped with the profinite topology, on $\Lambda_v(\underline{M})$ equipped with the discrete topology, is continuous.*

Proof. Let t be a nonconstant element of A. The inclusion $\mathbb{F}[t] \subset A$ makes A into a finite flat A-module, therefore $\underline{M}$ defines an $\mathbb{F}[t]$-motive of rank $\deg(t)\,\mathrm{rank}\,\underline{M}$ over F. Using the identification $\mathbb{F}[t]\otimes F = F[t]$, we rather write t for $t\otimes 1$ and θ for $1\otimes\kappa(t)$. Let $n > 0$ be an integer so that $(t-\theta)^n\tau_M(\tau^*M) \subset M$. Let $\underline{N}$ be the $\mathbb{F}[t]$-motive over F whose underlying module is $N = F[t]$ and where τ_N is the multiplication by $(t-\theta)^n$. If $\sqrt[q-1]{-\theta}$ denotes a $q-1$-root of $-\theta$ in $\mathbb{C}_\infty$, we have

$$\Lambda_v(\underline{N}) = (\sqrt[q-1]{-\theta})^{-n}\prod_{i=0}^{\infty}\left(1-\frac{t}{\theta^{q^i}}\right)^n\cdot\mathbb{F}[t]\subset K_\infty\left(\sqrt[q-1]{-\theta}\right)\langle t\rangle.$$

The $\mathbb{F}[t]$-motive $\underline{N}$ has been chosen so that $\underline{M}\otimes\underline{N}$ is effective (see Definition 1.2). By [And, Thm 4], there exists a finite extension H of F_v in $\mathbb{C}_\infty$ such that

$$\Lambda_v(\underline{M})\otimes_{\mathbb{F}[t]}\Lambda_v(\underline{N}) = \Lambda_v(\underline{M}\otimes\underline{N})\subset(M\otimes_{F[t]}N)\otimes_{F[t],v}H\langle t\rangle = M\otimes_{F[t],v}H\langle t\rangle.$$

It follows that there exists a finite extension L' of F_v such that $\Lambda_v(\underline{M})\subset M\otimes_{F[t],v}L'\langle t\rangle$ (take for instance $L':=H(\sqrt[q-1]{-\theta})$).

We now show that one can choose L' separable over F_v (the argument ressambles to the proof of Lemma 1.44). Note that $M\otimes_{F[t],v}F_v^s\langle t\rangle$ is free of finite rank over $F_v^s\langle t\rangle$. Therefore, $(M\otimes_{F[t],v}F_v^s\langle t\rangle)/(t^n)$ is a finite dimensional F_v^s-vector space for all positive integers n. By [Kat1, Prop. 1.1], the multiplication map

$$\left\{m\in(M\otimes_{F[t],v}F_v^s\langle t\rangle)/(t^n)\mid m=\tau_M(\tau^*m)\right\}\otimes F_v^s\to(M\otimes_{F[t],v}F_v^s\langle t\rangle)/(t^n)$$

is an isomorphism. In particular, the inclusion of

$$\left\{m\in(M\otimes_{F[t],v}F_v\langle t\rangle)/(t^n)|m=\tau_M(\tau^*m)\right\}$$

in $\left\{m\in(M\otimes_{F[t],v}\mathbb{C}_\infty\langle t\rangle)/(t^n)|m=\tau_M(\tau^*m)\right\}$ is an equality. It shows that $\Lambda_v(\underline{M})$ is both a submodule of $M\otimes_{F[t],v}F_v^s\langle t\rangle$ and $M\otimes_{F[t],v}L'\langle t\rangle$. Because M is free over $F[t]$, it follows that $\Lambda_v(\underline{M})\subset M\otimes_{F[t],v}L\langle t\rangle$ where L is the finite separable extension $L'\cap F_v^s$ of F_v in $\mathbb{C}_\infty$. Since $(A\otimes F)\otimes_{F[t],v}L\langle t\rangle$ is isomorphic to $L\langle A\rangle$, we deduce that $\Lambda_v(\underline{M})\subset M\otimes_{A\otimes F}L\langle A\rangle$. □

By the faithful flatness of $L\langle A\rangle\to\mathbb{C}_\infty\langle A\rangle$ ([Bou, AC I§3.5 Prop. 9]), we have:

Proposition 1.55. *Let $v:F\to\mathbb{C}_\infty$ be a K-algebra morphism and let $\underline{M}$ be a rigid analytically trivial A-motive over F. Let L be as in Proposition 1.54. The multiplication map*

$$\Lambda_v(\underline{M})\otimes_A L\langle A\rangle\longrightarrow M\otimes_{A\otimes F,v}L\langle A\rangle$$

is an isomorphism of $L\langle A\rangle$-modules.

The next result is inspired by [BöcHa, Prop. 6.1]. We have adapted it to descend from $\mathbb{C}_\infty$ to F_v^s in order to compute the module $H^1(G_\infty,\Lambda_v(\underline{M}))$ of continuous Galois cohomology.

Theorem 1.56. *Let $v:F\to\mathbb{C}_\infty$ be a K-algebra morphism, and let $\underline{M}$ be a rigid analytically trivial A-motive. There is an exact sequence of $A[G_v]$-modules:*

$$0\longrightarrow\Lambda_v(\underline{M})\longrightarrow M\otimes_{A\otimes F,v}\mathbb{C}_\infty\langle A\rangle\xrightarrow{\mathrm{id}-\tau_M}M\otimes_{A\otimes F,v}\mathbb{C}_\infty\langle A\rangle\longrightarrow 0.\quad(1.10)$$

Furthermore, it induces a long exact sequence of A-modules

$$0 \to \Lambda_v(\underline{M})^+ \to M \otimes_{A\otimes F,v} F_v\langle A\rangle \stackrel{\mathrm{id}-\tau_M}{\to} M \otimes_{A\otimes F,v} F_v\langle A\rangle \stackrel{\delta_v}{\to} H^1(G_v, \Lambda_v(\underline{M})) \to 0. \tag{1.11}$$

Remark 1.57. The fact that (1.10) implies (1.11) has nothing immediate. We have to descend from the completion $\widehat{F_v^p}$ of the perfection of F_v, which corresponds to the fixed subfield of $\mathbb{C}_\infty$ by G_v by the Ax-Sen-Tate Theorem, to F_v.

Remark 1.58. The morphism δ_v is defined using the Snake Lemma. It can be described as follows: for $m \in M \otimes_{A\otimes F,v} F_v\langle A\rangle$, let $\xi_m \in M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle A\rangle$ be such that

$$\xi_m - \tau_M(\tau^*\xi_m) = m.$$

For $\sigma \in G_v$, $\xi_m^\sigma - \xi_m$ is an element of $\Lambda_v(\underline{M})$. The cocycle

$$\delta_v(m) \in H^1(G_v, \Lambda_v(\underline{M}))$$

then corresponds to $\sigma \mapsto \xi_m^\sigma - \xi_m$, and the Theorem 1.56 states that any continuous cocycle of $G_v \to \Lambda_v(\underline{M})$ is of this form.

Proof of Theorem 1.56. Let $\mathbb{F}[t] \to A$ be a nonconstant morphism of rings (A is then a finite and flat $\mathbb{F}[t]$-module). We have $\mathbb{C}_\infty\langle A\rangle = A \otimes_{\mathbb{F}[t]} \mathbb{C}_\infty\langle t\rangle$ where $\mathbb{C}_\infty\langle t\rangle$ is the Tate algebra over $\mathbb{C}_\infty$ in the variable t.

The exactness of (1.10) follows from [BöcHa, Prop. 6.1]. We shall use the same argument as in *loc. cit.* to show that the sequence

$$0 \longrightarrow \Lambda_v(\underline{M}) \longrightarrow M \otimes_{F[t],v} F_v^s\langle t\rangle \stackrel{\mathrm{id}-\tau_M}{\longrightarrow} M \otimes_{F[t],v} F_v^s\langle t\rangle \longrightarrow 0, \tag{1.12}$$

where the first inclusion is well-defined by Proposition 1.54, is exact. It suffices to show the surjectivity of $\mathrm{id}-\tau_M$ on $M \otimes_{F[t],v} F_v^s\langle t\rangle$. The argument ressambles to the proof of Proposition 1.46. Let $f \in M \otimes_{F[t],v} F_v^s\langle t\rangle$. Since $\underline{M}$ is v-rigid analytically trivial, without loss of generality we can assume that $f = c\cdot\omega$ for $c = \sum_{n\geq 0} c_n t^n \in F_v^s\langle t\rangle$ and $\omega \in \Lambda_v(\underline{M})$. There exists a solution b_n in F_v^s of $x - x^q = c_n$ for all $n \geq 0$. The condition $|c_n| \to 0$ implies $|b_n| \to 0$. Hence, the element

$$g := \left(\sum_{n=0}^{\infty} b_n t^n\right)\cdot\omega$$

belongs to $M \otimes_{F[t],v} F_v^s\langle t\rangle$ and satisfies $(\mathrm{id}-\tau_M)(g) = f$. It follows that (1.12) is exact. By Proposition 1.54, G_v acts continuously on (1.12) and taking invariants yields a long exact sequence of A-modules:

$$\Lambda_v(\underline{M})^+ \hookrightarrow M \otimes_{A\otimes F,v} F_v\langle A\rangle \stackrel{\mathrm{id}-\tau_M}{\longrightarrow} M \otimes_{A\otimes F} F_v\langle A\rangle \longrightarrow H^1(G_v, \Lambda_v(\underline{M})) \longrightarrow \cdots$$

To conclude, it remains to prove that the module

$$H^1(G_v, M \otimes_{F[t],v} F_v^s\langle t\rangle) \cong M \otimes_{F[t],v} H^1(G_v, F_v^s\langle t\rangle)$$

is zero. We claim that $H^1(G_v, F_v^s\langle t\rangle)$ vanishes. By continuity, it suffices to show that $H^1(G_v, L\langle t\rangle)$ vanishes for any $L \subset F_v^s$ finite Galois extension of F_v, with Galois group H. By the additive version of Hilbert's 90 Theorem [Ser1, x.§1, Prop. 1], $H^1(G_v, L) = H^1(H, L)$ vanishes and it follows that $H^1(G_v, L[\![t]\!])$ is zero. Therefore, we have a long exact sequence

$$0 \to F_v\langle t\rangle \to F_v[\![t]\!] \to (L[\![t]\!]/L\langle t\rangle)^{G_v} \to H^1(G_v, L\langle t\rangle) \to 0.$$

In particular, for any cocycle $c : G_v \to L\langle t\rangle$, there exists $f \in L[\![t]\!]$ such that,

$$\forall \sigma \in G_v : \quad c(\sigma) = f^\sigma - f.$$

Because L is separable over F_v its trace forme is non-degenerated, that is, there exists $\alpha \in L$ such that $\eta := \sum_{\sigma\in H} \alpha^\sigma \in F_v$ is nonzero. Thus, f can be written as

$$f = \left(\eta^{-1}\sum_{\sigma\in H}\alpha^\sigma f^\sigma\right) - \left(\eta^{-1}\sum_{\sigma\in H}\alpha^\sigma c(\sigma)\right) \in F_v[\![t]\!] + L\langle t\rangle.$$

It follows that c is trivial, and that $H^1(G_v, L\langle t\rangle) = 0$. This concludes the proof. □

We introduce the category mentioned in the introduction.

Definition 1.59. We let $\mathcal{M}_F^{\mathrm{rig}}$ (resp. $\mathcal{MM}_F^{\mathrm{rig}}$) be the full subcategory of $\mathcal{M}_F$ (resp. $\mathcal{MM}_F$) whose objects are rigid analytically trivial.

The next proposition, which ensures us that extension modules in the category $\mathcal{MM}_F^{\mathrm{rig}}$ are well-defined, is borrowed from [HarJu, Lem. 2.3.25].

Proposition 1.60. *Let $0 \to \underline{M}' \to \underline{M} \to \underline{M}'' \to 0$ be an exact sequence in $\mathcal{M}_F$. Then $\underline{M}$ is rigid analytically trivial if and only if $\underline{M}'$ and $\underline{M}''$ are. In particular, the category $\mathcal{M}_F^{rig}$ (resp. $\mathcal{MM}_F^{rig}$) is exact.*

We finally record that Betti realization functors having $\mathcal{MM}_F^{\mathrm{rig}}$ as its source are exact.

Corollary 1.61. *The functors $\underline{M} \mapsto \Lambda(\underline{M})$ and $\underline{M} \mapsto \Lambda_v(\underline{M})$ from $\mathcal{M}_F^{rig}$ (resp. $\mathcal{MM}_F^{rig}$) to the category $\mathrm{Rep}_A(G_v)$, of continuous A-linear representations of G_v, is exact.*

Proof. The proof is similar to the one of Corollary 1.46: this follows from Theorem 1.56 together with the Snake Lemma. □

1.4.2 Analytic continuation

We end this subsection by showing that elements of $\Lambda(\underline{M})$ – which can be seen as *functions* over the affinoid subdomain $\mathrm{Spm}\,\mathbb{C}_\infty\langle A\rangle$ *with values in* $M \otimes_K \mathbb{C}_\infty$ – can be *meromorphically* continuated to the whole rigid analytification of the affine curve $\mathrm{Spec}\, A\otimes\mathbb{C}_\infty$, with their only poles supported at the closed support of $\mathfrak{j}$ and its iterates $\tau^*\mathfrak{j}$, $\tau^{*2}\mathfrak{j}$, ... In the $\deg(\infty) = 1$-case, this is treated in [HarJu, §2.3.4]. The results of this subsection will be needed in Chapter 5 where we attach a *mixed Hodge-Pink structure* to a mixed rigid analytically trivial A-motive.

Fix $v : F \to \mathbb{C}_\infty$ a K-algebra morphism and let L be any complete subfield of $\mathbb{C}_\infty$ that contains F_v. Let $|\cdot|$ be the norm on L it inherits as a subfield of $\mathbb{C}_\infty$. In what follows, we construct two sub-L-algebras $L\langle\langle A\rangle\rangle$ and $L\langle\langle A\rangle\rangle_{\mathfrak{j}}$ (Definition 1.63) of $L\langle A\rangle$. Our aim is to show that $\Lambda_v(\underline{M}) \subset M \otimes_{A\otimes F,v} L\langle\langle A\rangle\rangle_{\mathfrak{j}}$ for any rigid analytically trivial A-motive $\underline{M}$ over F (Theorem 1.66).

By the so-called *rigid analytic GAGA functor* [Bos, §I.5.4], we associate to $C \times \operatorname{Spec} L$ its rigid analytification $(C \times \operatorname{Spec} L)^{\mathrm{rig}}$. It contains the rigid analytification $\mathfrak{A}_L$ of $\operatorname{Spec}(A \otimes L)$ as an affinoid subdomain. We recall briefly its construction. Let $t \in A$ be a non-constant element and fix $c \in L$ whose norm satisfies $|c| > 1$. We define:

$$L\left\langle \frac{t}{c} \right\rangle := \left\{ \sum_{n=0}^{\infty} a_n t^n \;\middle|\; a_n \in L;\ \lim_{n\to\infty} a_n c^n = 0 \right\}, \quad L\left\langle \frac{A}{c} \right\rangle := A \otimes_{\mathbb{F}[t]} L\left\langle \frac{t}{c} \right\rangle.$$

The inclusions

$$L\langle A\rangle \supset L\left\langle \frac{A}{c} \right\rangle \supset L\left\langle \frac{A}{c^2} \right\rangle \supset \cdots \supset A \otimes L$$

give rise to inclusions of affinoid subdomains

$$\operatorname{Spm} L\langle A\rangle \subset \operatorname{Spm} L\left\langle \frac{A}{c} \right\rangle \subset \operatorname{Spm} L\left\langle \frac{A}{c^2} \right\rangle \subset \cdots$$

where $\operatorname{Spm} L\left\langle \frac{A}{c^i} \right\rangle$ can be interpreted as the scale of coefficient $|c|^i$ of $\operatorname{Spm} L\langle A\rangle$. The union of all these domains can be constructed using a glueing process, resulting in the rigid analytic space $\mathfrak{A}_L = (\operatorname{Spec} A \otimes L)^{\mathrm{rig}}$ equipped with the admissible covering $\bigcup_{i=0}^{\infty} \operatorname{Spm} L\left\langle \frac{A}{c^i} \right\rangle$. This construction is independent of the choice of t and c (we refer to [Bos, §I.5.4] for details). We recall that, as sets, $\mathfrak{A}_L$ and $\operatorname{Spm} A \otimes L$ coincide. Given an ideal $\mathfrak{a}$ of $A \otimes L$, we let $V(\mathfrak{a})$ be the finite subset $\{\mathfrak{m} \in \operatorname{Spm} A \otimes L \mid \mathfrak{a} \subseteq \mathfrak{m}\}$ of $\mathfrak{A}_L$. We denote by $L\langle\langle A\rangle\rangle$ the ring of global sections of $\mathfrak{A}_L$.

We again denote by τ the scheme endomorphism of $C \times \operatorname{Spec} L$ which acts as the identity on C and as the q-Frobenius on $\operatorname{Spec} L$. τ extends to $(C \times \operatorname{Spec} L)^{\mathrm{rig}}$, and stabilizes both $\mathfrak{A}_L$ and $\operatorname{Spm} L\langle A\rangle$. For $\mathfrak{a}$ a nonzero ideal of $A \otimes L$ and $i \geq 0$, we let $\mathfrak{a}^{(i)}$ be the ideal of $A \otimes L$ generated by the image of $\tau^i(\mathfrak{a})$. As $A \otimes L$-modules, $\mathfrak{a}^{(i)}$ is isomorphic to $\tau^{*i}\mathfrak{a}$. For instance, $\mathfrak{j}^{(i)} = \mathfrak{j}_\kappa^{(i)}$ is the maximal ideal of $A \otimes L$ generated by the set $\{a \otimes 1 - 1 \otimes \kappa(a)^{q^i} | a \in A\}$.

Example 1.62. Let $C = \mathbb{P}^1_{\mathbb{F}}$ and let ∞ be the point $[0 : 1]$. We identify A with $\mathbb{F}[t]$ and the tensor product $A \otimes F$ with $F[t]$. We let $\theta \in F$ denote $\kappa(t)$

so that $\tau^{*i}\mathfrak{j}$ corresponds to the ideal $(t-\theta^{q^i})$ of $F[t]$. Let $v : F \to \mathbb{C}_\infty$ be a K-algebra morphism and let L be a complete field in $\mathbb{C}_\infty$ containing F. We have

$$L\langle A\rangle = L\langle t\rangle = \left\{ \sum_{n=0}^{\infty} a_n t^n \;\middle|\; a_n \in L;\ \lim_{n\to\infty} a_n \to 0 \right\},$$

$$L\langle\langle A\rangle\rangle = L\langle\langle t\rangle\rangle = \left\{ \sum_{n=0}^{\infty} a_n t^n \;\middle|\; a_n \in L;\ \forall \rho>1:\ \lim_{n\to\infty} a_n\rho^n \to 0 \right\}. \quad (1.13)$$

The ring $L\langle A\rangle$ corresponds to series converging in the *closed* unit disc, whereas $L\langle\langle A\rangle\rangle$ consists of entire series. The morphism τ acts on both rings by mapping $f=\sum_{n\geq 0} a_n t^n$ to $f^{(1)}=\sum_{n\geq 0} a_n^q t^n$.

Let $u\in A$ be a *separating element*, that is, such that K is a separable extension of $\mathbb{F}(u)$. Let $L\langle\langle u\rangle\rangle$ denote the subring of $L\langle u\rangle$ defined by (1.13). The multiplication map

$$A\otimes_{\mathbb{F}[u]} L\langle u\rangle \to L\langle A\rangle, \quad A\otimes_{\mathbb{F}[u]} L\langle\langle u\rangle\rangle \to L\langle\langle A\rangle\rangle$$

are isomorphisms. For $i\geq 0$, the converging product

$$\Pi_u^{(i)} := \prod_{j=i}^{\infty}\left(1-\frac{u\otimes 1}{1\otimes \kappa(u)^{q^i}}\right)$$

defines an element in $L\langle\langle A\rangle\rangle$ whose only zeros in $\mathfrak{A}_L$ are supported at

$$\bigcup_{j\geq i} V(u\otimes 1 - 1\otimes\kappa(u)^{q^j}).$$

We set $\Pi_u := \Pi_u^{(0)}$.

Definition 1.63. We let $L\langle\langle A\rangle\rangle_{\mathfrak{j}^{(i)}}$ be the subring of $\operatorname{Quot} L\langle\langle A\rangle\rangle$ consisting of elements f for which there exists $n\geq 0$ such that $(\Pi_u^{(i)})^n f\in L\langle\langle A\rangle\rangle$ for all separating element $u\in A$.

Remark 1.64. The ring $\mathbb{C}_\infty\langle\langle A\rangle\rangle_{\mathfrak{j}^{(i)}}$ could have been defined as the subring of $\operatorname{Quot}\mathbb{C}_\infty\langle\langle A\rangle\rangle$ consisting of elements f which are meromorphic on $\mathfrak{A}_{\mathbb{C}_\infty}$ and whose poles are supported at $V(\mathfrak{j}^{(i)})$, $V(\mathfrak{j}^{(i+1)})$,... with bounded orders. Definition 1.63 has the small advantage of not requiring much of rigid analytic geometry. The next lemma[3] is a bridge between both definitions:

Lemma 1.65. *Let $\mathfrak{m}$ be a maximal ideal of $A\otimes\mathbb{C}_\infty$ distinct from $\mathfrak{j}$, $\mathfrak{j}^{(1)}$, $\mathfrak{j}^{(2)}$, ... There exists a separating element u such that, for all non-negative integer i, $u\otimes 1-1\otimes\kappa(u)^{q^i}$ does not belong to $\mathfrak{m}$. In particular,*

$$\bigcap_u \left(\bigcup_{i=0}^{\infty} V(u\otimes 1-1\otimes\kappa(u)^{q^i})\right) = \bigcup_{i=0}^{\infty} V(\mathfrak{j}^{(i)})$$

where the intersection is indexed over separating elements $u\in A$.

[3]I thank Andreas Maurischat who gave me permission to include this lemma, which originally was part of an unpublished collaborative work.

Proof. Let t be a separating element. We first compute the prime ideal decomposition of $(t\otimes 1-1\otimes\kappa(t))$ in the Dedekind domain $A\otimes\mathbb{C}_\infty$. The inclusion of Dedekind ring $\mathbb{F}[t]\otimes\mathbb{C}_\infty\subset A\otimes\mathbb{C}_\infty$ makes $A\otimes\mathbb{C}_\infty$ a free $\mathbb{F}[t]\otimes\mathbb{C}_\infty$-module of rank $[K:\mathbb{F}(t)]$. In particular, there are at most $[K:\mathbb{F}(t)]$ prime divisors of $(t\otimes 1-1\otimes\kappa(t))$. For $\sigma:\kappa(K)\to\mathbb{C}_\infty$ an $\mathbb{F}(t)$-algebra morphism, the ideal $\mathfrak{j}^\sigma$ of $A\otimes\mathbb{C}_\infty$ generated by the set $\{a\otimes 1-1\otimes\sigma(\kappa(a))|a\in A\}$ is maximal and divides the principal ideal $(t\otimes 1-1\otimes\kappa(t))$. There are $\#\operatorname{Hom}_{\mathbb{F}(t)}(\kappa(K),\mathbb{C}_\infty)=[K:\mathbb{F}(t)]$ such ideals, hence

$$(t\otimes 1-1\otimes\kappa(t))=\prod_\sigma \mathfrak{j}^\sigma$$

where the product runs over $\sigma\in\operatorname{Hom}_{\mathbb{F}(t)}(\kappa(K),\mathbb{C}_\infty)$.

We turn to the proof of the lemma. Assume the converse, that is, for all separating element v there exists $j\geq 0$ such that $v\otimes 1-1\otimes\kappa(v)^{q^j}\in\mathfrak{m}$. This means that there exists a non-negative integer i for which $\mathfrak{m}\supset(t\otimes 1-1\otimes\kappa(t)^{q^i})=\prod_\sigma(\mathfrak{j}^\sigma)^{(i)}$. By uniqueness of the prime ideal decomposition, there exists $\sigma\in\operatorname{Hom}_{\kappa(\mathbb{F}(t))}(\kappa(K),\mathbb{C}_\infty)$ such that $\mathfrak{m}=(\mathfrak{j}^\sigma)^{(i)}$. Because $\mathfrak{m}$ is distinct from $\mathfrak{j}$, $\mathfrak{j}^{(1)}$, $\mathfrak{j}^{(2)}$, ..., the morphism σ is not the inclusion $\kappa(K)\subset\mathbb{C}_\infty$. Because K is generated by separating elements over $\mathbb{F}$, there exists a separating element u such that $\sigma(\kappa(u))\neq\kappa(u)$. From our converse assumption, there exists a non-negative integer j such that $u\otimes 1-1\otimes\kappa(u)^{q^j}\in\mathfrak{m}=(\mathfrak{j}^\sigma)^{(i)}$. Hence, both $u\otimes 1-1\otimes\kappa(u)^{q^j}$ and $u\otimes 1-1\otimes\sigma(\kappa(u))^{q^i}$ are in $\mathfrak{m}$. Since $\mathfrak{m}\neq A\otimes\mathbb{C}_\infty$, this implies $\sigma(\kappa(u))^{q^i}=\kappa(u)^{q^j}$.

This is a contradiction. Indeed, $\kappa(u)^{q^i}$ and $\sigma(\kappa(u))^{q^i}=\kappa(u)^{q^j}$ have the same minimal polynomial over $\kappa(\mathbb{F}(t))$ so that either the latter polynomial has coefficients in $\mathbb{F}$ or $i=j$. The first option is impossible as it would imply $\kappa(u)\in\bar{\mathbb{F}}\cap\kappa(A)=\mathbb{F}$. The second option is also impossible as we choose u such that $\sigma(\kappa(u))\neq\kappa(u)$. □

We are now in position to prove the main result of this subsection (compare with [HarJu, Prop. 2.3.30]).

Theorem 1.66. *Let $v:F\to\mathbb{C}_\infty$ be a K-algebra morphism and let $\underline{M}$ be a rigid analytically trivial A-motive over F. There exists a finite separable extension L of F_v such that $\Lambda_v(\underline{M})\subset M\otimes_{A\otimes F,v}L\langle\langle A\rangle\rangle_{\mathfrak{j}}$.*

Let us start with a lemma:

Lemma 1.67. *Let $n\geq 0$ be such that $\mathfrak{j}^n\tau_M(\tau^*M)\subset M$. Let u be a separating element in A. Then $\Pi_u^n\cdot\Lambda_v(\underline{M})\subset M\otimes_{A\otimes F,v}\mathbb{C}_\infty\langle\langle A\rangle\rangle$.*

Proof. Let $\sqrt[q-1]{-\kappa(u)}$ be a $q-1$-root of $-\kappa(u)$ in $\mathbb{C}_\infty$. Let

$$\omega_u:=\sqrt[q-1]{-\kappa(u)}\prod_{i=0}^{\infty}\left(1-\frac{u\otimes 1}{1\otimes\kappa(u)^{q^i}}\right)^{-1}=\sqrt[q-1]{-\kappa(u)}\cdot\Pi_u^{-1}\in\operatorname{Quot}\mathbb{C}_\infty\langle\langle A\rangle\rangle.$$

As in the proof of Proposition 1.54, let $\underline{N}$ be the $\mathbb{F}[u]$-motive over F whose underlying module is $N=F[u]$ and where τ_N is the multiplication by $(u\otimes$

$1 - 1 \otimes \kappa(u))^n$. We have $\Lambda_v(\underline{N}) = \omega_u^{-n} \cdot \mathbb{F}[u]$. The $\mathbb{F}[u]$-motive $\underline{N}$ has been chosen so that $\underline{M} \otimes \underline{N}$ is effective. Using [BöcHa, Prop. 3.4], we deduce that $\Lambda_v(\underline{M} \otimes \underline{N}) \subset M \otimes_{F[u],v} \mathbb{C}_\infty\langle\!\langle u \rangle\!\rangle$, and hence $\omega_u^{-n} \cdot \Lambda_v(\underline{M}) \subset M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle\!\langle A \rangle\!\rangle$. The Lemma follows. □

Proof of Theorem 1.66. Because M is projective over $A \otimes F$, there exists $t \geq 1$ and an $A \otimes F$-module M' such that $M \oplus M' \cong (A \otimes F)^t$. We let p_M denote the projection from $(A \otimes F)^t$ onto M. By Lemma 1.67, there exists $n \geq 0$ such that, for all u separating element of A, $\Pi_u^n \cdot \Lambda_v(\underline{M}) \subset M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle\!\langle A \rangle\!\rangle$. This yields

$$\Lambda_v(\underline{M}) \subset p_M \left(\bigcap_u \Pi_u^{-n} \cdot \mathbb{C}_\infty\langle\!\langle A \rangle\!\rangle^t \right)$$

where the inner intersection is over separating elements $u \in A$. The right-hand side is $M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle\!\langle A \rangle\!\rangle_{\mathrm{j}}$ by definition of $\mathbb{C}_\infty\langle\!\langle A \rangle\!\rangle_{\mathrm{j}}$. It follows at once that:

$$\Lambda_v(\underline{M}) \subset (M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle\!\langle A \rangle\!\rangle_{\mathrm{j}}) \cap (M \otimes_{A\otimes F,v} L\langle A \rangle).$$

To conclude that the right-hand side is $M \otimes_{A\otimes F,v} L\langle\!\langle A \rangle\!\rangle_{\mathrm{j}}$, we use the equality

$$L\langle\!\langle A \rangle\!\rangle_{\mathrm{j}} = \mathbb{C}_\infty\langle\!\langle A \rangle\!\rangle_{\mathrm{j}} \cap L\langle A \rangle$$

together with the flatness of M over $A \otimes F$, which by [Bou, §.I.2, Prop.6] implies that $M \otimes_{A\otimes F} -$ commutes with finite intersections. □

We end this subsection by the following consequence of Theorem 1.66:

Corollary 1.68. *Let $v : F \to \mathbb{C}_\infty$ be a K-algebra morphism and let $\underline{M}$ be a rigid analytically trivial A-motive over F. Let $m \in M[\mathrm{j}^{-1}]$ and let $\xi \in M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle A \rangle$ be such that $\xi - \tau_M(\tau^*\xi) = m$ (ξ exists by Theorem 1.56). There exists a separable field extension L of F_v such that $\xi \in M \otimes_{A\otimes F,v} L\langle\!\langle A \rangle\!\rangle_{\mathrm{j}}$.*

Proof. Let $\underline{E} = (E, \tau_E)$ be the A-motive over F whose underlying module is $E := M \oplus (A \otimes F)$ and where τ_E acts by the matrix $\left(\begin{smallmatrix} \tau_M & m \\ 0 & 1 \end{smallmatrix}\right)$ (namely, $\underline{E}$ is an extension of $\mathbb{1}$ by $\underline{M}$). The A-module $\Lambda_v(\underline{E})$ is described by the couples (ψ, a) where $\psi \in M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle A \rangle$ and $a \in \mathbb{C}_\infty\langle A \rangle$, satisfying

$$\begin{pmatrix} \tau_M & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau^*\psi \\ \tau^* a \end{pmatrix} = \begin{pmatrix} \psi \\ a \end{pmatrix} \tag{1.14}$$

The bottom rom equation yields that $a \in A$ whereas the top row yields $\tau_M(\tau^*\psi) + am = \psi$. It follows that

$$\Lambda(\underline{E}) = \{(\omega + a\xi, a) \mid \omega \in \Lambda(\underline{M}),\ a \in A\}.$$

The corollary follows from Theorem 1.66 applied to $\underline{E}$. □

Chapter 2

Integral models of Anderson A-motives

In this chapter we illustrate the notion of *maximal integral models*. For A-motives, maximal integral models are understood as an analogue of Néron models of abelian varieties. The notion dates back to Gardeyn's work on *models of τ-sheaves* [Gar2] and their reduction [Gar1], where he proved a Néron-Ogg-Shafarevich type criterion. However our setting differs by the fact that, in opposition to τ-sheaves, A-motives might not be *effective*. We also removed Gardeyn's assumption for an *integral model* to be locally free. We will show in Propositions 2.14 and 2.32 that this is implicit for maximal ones over local and global function fields. Our presentation thus allows to avoid the use of a technical lemma due to Lafforgue in Gardeyn's exposition [Gar2, §2]. In that sense, the content of this chapter is original.

In practice, to make maximal integral models of A-motives explicit is a difficult task. In section 2.1, we consider the easier problem of finding maximal integral models of *Frobenius spaces*. Those are pairs (V, φ) where V is a finite dimensional vector space over a local field E containing $\mathbb{F}$ and φ is a q-linear endomorphism of V. We show in Proposition 2.2 that there exists a unique $\mathcal{O}_E$-lattice in V stable by φ and which is maximal for this property. In the remaining of the section, we study the *type* of an $\mathcal{O}_E$-lattice stable by φ and provide numerical criteria of maximality (see Propositions 2.6, 2.8).

In Sections 2.2 and 2.3, we shall be concerned with integral models of A-motives. Given $R \subset S$ an inclusion of $\mathbb{F}$-algebras and an A-motive $\underline{M} = (M, \tau_M)$ over S, an R-model for $\underline{M}$ is a finite sub-$A \otimes R$-module of M *stable by* τ_M (Definition 2.10).

We study the case where $\underline{M}$ is an A-motive over a local function field S and where R is its valuation ring in Section 2.2. In Proposition 2.12, we prove existence and uniqueness of R-models which are maximal for the inclusion, and we prove that they are locally free in Proposition 2.14. We show that, given a well-chosen maximal ideal $\mathfrak{m} \subset A$ and a positive integer n, the data of $(M/\mathfrak{m}^n M, \tau_M)$ defines a Frobenius space over S. Theorem 2.19, our main result of this section, describes how to recover the maximal integral model of $\underline{M}$ in terms of the data of the maximal integral model of $(M/\mathfrak{m}^n M, \tau_M)$ for all n. It permits to obtain a good reduction criterion for A-motives in Proposition 2.28.

In Section 2.3, we treat the case where $\underline{M}$ is an A-motive over a global function field S and R is a Dedekind domain whose fraction field is S. If $\mathfrak{p}$ is a nonzero prime ideal of R, we obtain an A-motive $\underline{M}_{S_\mathfrak{p}}$ by the base field extension from S to $S_\mathfrak{p}$. Our Proposition 2.30 explain how to recover the maximal integral model of $\underline{M}$ from the data of the maximal integral models of $\underline{M}_{S_\mathfrak{p}}$ for all $\mathfrak{p}$.

We end this chapter by our Remark 2.40 which explains how our notion of maximal integral models matches Gardeyn's.

2.1 Integral models of Frobenius spaces

In this subsection we work with notations that are more general to what we need in the sequel. We let k be a field containing $\mathbb{F}$ and let $E = k(\!(\varpi)\!)$ be the field of Laurent series over $\mathbb{F}$ in the the variable ϖ. We let $\sigma : E \to E$ denote the q-Frobenius Frob_q on E (it fixes $\mathbb{F}$), v_E be the valuation of E, $\mathcal{O}_E = k[\![\varpi]\!]$ be its valuation ring with maximal ideal $\mathfrak{m} = \mathfrak{m}_E = (\varpi)$.

Given a matrix M with coefficients in $\bar{E}$, we let M^σ be the matrix whose entries have been raised to the power q. If M has coefficients in E, we let $v_E(M)$ be the minimum of the valuations of the entries of M. We have $v_E(M^\sigma) = qv_E(M)$ and $v_E(MN) \geq v_E(M) + v_E(N)$ for any two matrices M and N whose product makes sense.

Our object of study are pairs (V, φ) where V is a finite dimensional E-vector space and $\varphi : \sigma^* V \to V$ is an E-linear isomorphism. In the existing litterature, there are generally referred to as *étale finite* $\mathbb{F}$*-shtukas over* E (e.g. [Har2, §4]). We prefer here the shorter name *Frobenius spaces*. By an $\mathcal{O}_E$*-lattice in* V we mean a finitely generated sub-$\mathcal{O}_E$-module L of V which generates V over E. A sub-$\mathcal{O}_E$-module L *is stable by* φ if $\varphi(\sigma^* L) \subset L$.

Definition 2.1. We say that L is an *integral model for* (V, φ) if L is an $\mathcal{O}_E$-lattice in V stable by φ. We say that L is *maximal* if it contains all the integral models for (V, φ).

Proposition 2.2. *There exists a unique maximal integral model.*

Proof. If it exists, a maximal integral model is clearly unique. We prove existence by first showing that there exists an $\mathcal{O}_E$-lattice in V stable by φ. Let $\mathbf{v} := (v_1, ..., v_\ell)$ be a basis of V and let T' be the $\mathcal{O}_E$-module generated by $\mathbf{v}$. There exists a positive integer k such that $\varphi(\sigma^* T') \subset \varpi^{-k} T'$. Inspired by the proof of [Gar2, Prop. 2.2], we let $T := \varpi^k T'$ so that

$$\varphi(\sigma^* T) = \varpi^{qk} \varphi(\sigma^* T') \subset \varpi^{(q-1)k} T' = \varpi^{(q-2)k} T \subset T.$$

Hence, the $\mathcal{O}_E$-module T is an $\mathcal{O}_E$-lattice in V stable by φ.

We turn to the existence of the maximal integral model. Let L be the union of all the $\mathcal{O}_E$-lattices in V stable by φ. The union L is non empty (it contains T') and hence generates V over E. It suffices to show that L is finitely generated. Let U be an $\mathcal{O}_E$-lattice stable by φ. As $\mathcal{O}_E$ is a discrete valuation ring, U is free over $\mathcal{O}_E$. Let $\mathbf{u}$ be a basis of U and let $A \in \mathrm{GL}_\ell(E)$ be the base-change matrix expressing $\mathbf{u}$ in $\mathbf{v}$. Let F be the matrix of φ expressed in the bases $\sigma^*\mathbf{v}$ and $\mathbf{v}$. As U is stable by φ, we have $FA^\sigma = AM$ for a certain matrix M with coefficients in $\mathcal{O}_E$. Therefore,

$$qv_E(A) = v_E(A^\sigma) = v_E(F^{-1}AM) \geq v_E(F^{-1}) + v_E(A)$$

and hence $v_E(A) \geq v_E(F^{-1})/(q-1)$. This implies $U \subset \varpi^{v_E(F^{-1})/(q-1)}T'$ and it follows that $L \subset \varpi^{v_E(F^{-1})/(q-1)}T'$. As $\mathcal{O}_E$ is Noetherian, it yields that L is finitely generated. $\square$

Example 2.3. Suppose $V := E$, $f \in \mathcal{O}_E$ a nonzero element and φ is the morphism corresponding to $x \mapsto fx^q$. Write $f = u\varpi^k h^{q-1}$ where $u \in \mathcal{O}_E^\times$, $0 \leq k < q-1$ is an integer and $h \in \mathcal{O}_E$. Then, the maximal integral model of (V, φ) is given by $h^{-1}\mathcal{O}_E$. This follows from Proposition 2.6 below.

Let T be an integral model for (V, φ) and let r be its rank as a free $\mathcal{O}_E$-module. The cokernel of the inclusion $\varphi(\sigma^*T) \subset T$ is a torsion $\mathcal{O}_E$-module of finite type and there exists elements g_1,...,g_r in $\mathcal{O}_E$ with $v_E(g_i) \leq v_E(g_{i+1})$ such that

$$T/\varphi(\sigma^*T) \cong \mathcal{O}_E/(g_1) \oplus \mathcal{O}_E/(g_2) \oplus \cdots \oplus \mathcal{O}_E/(g_r).$$

Equivalently, there exists a basis $(v_1, ..., v_r)$ of T over $\mathcal{O}_E$ such that

$$\varphi(\sigma^*T) = (g_1)v_1 \oplus (g_2)v_2 \oplus \cdots \oplus (g_r)v_r.$$

The elements g_1,...,g_r are unique up to multiplication by units and are called *the elementary divisors relative to the inclusion of $\mathcal{O}_E$-lattices $\varphi(\sigma^*T) \subset T$.*

Lemma 2.4. *Let $\mathbf{t}$ be a basis of T over $\mathcal{O}_E$ and let F be the matrix of φ written in the bases $\sigma^*\mathbf{t}$ and $\mathbf{t}$. The elementary divisors relative to the inclusion $\varphi(\sigma^*T) \subset T$ are the elementary divisors of the matrix F, up to units in $\mathcal{O}_E$.*

Proof. If $(f_1, ..., f_r)$ denotes the elementary divisors of F, the Smith's normal form Theorem implies that there exists $U, V \in \mathrm{GL}_r(\mathcal{O}_E)$ such that $UF = \mathrm{diag}(f_1, ..., f_r)V$. If we let $\mathbf{v} = (v_1, ..., v_r)$ be the basis of T corresponding to $V \cdot \mathbf{t}$, this relation reads

$$\varphi(\sigma^*T) = (f_1)v_1 \oplus (f_2)v_2 \oplus \cdots \oplus (f_r)v_r.$$

By uniqueness of the ideals $(g_1), ..., (g_r)$, we conclude that $(f_i) = (g_i)$ for all $i \in \{1, ..., r\}$. $\square$

Definition 2.5. We let the *type of* T be the sequence $(e_1, ..., e_r)$ of the valuations of the elementary divisors relative to the inclusion $\varphi(\sigma^* T) \subset T$ ordered such that $e_1 \leq e_2 \leq ... \leq e_r$. We define *the range* r_T *of* T to be the integer e_r.

Proposition 2.6. *Let T be an integral model for (V, φ).*

1. *If T is the maximal integral model of (V, φ), then $e_1 < q - 1$.*
2. *If $r_T < q - 1$, then T is the maximal integral model of (V, φ).*

Proof. Let $g_1, ..., g_r$ be the elementary divisors in $\mathcal{O}_E$ relative to the inclusion $\varphi(\sigma^* T) \subset T$. There exists a basis $\mathbf{v} = (v_1, ..., v_r)$ of T over $\mathcal{O}_E$ such that

$$\varphi(\sigma^* T) = (g_1)v_1 \oplus (g_2)v_2 \oplus ... \oplus (g_r)v_r.$$

We prove 1. If $e_1 \geq q - 1$, then $e_i \geq q - 1$ for all $i \in \{1, ..., r\}$ and hence $\varphi(\sigma^* T) \subset \varpi^{q-1} T$. If we set $T' := \varpi^{-1} T$, then

$$\varphi(\sigma^* T') = \varpi^{-q} \varphi(\sigma^* T) \subset \varpi^{-1} T = T',$$

and T' is an integral model for (V, φ) which contains T strictly. Hence T is not maximal.
We prove 2. Because $r_T < q - 1$, we have $0 \leq e_i < q - 1$ for all $i \in \{1, ..., r\}$. Let L be the maximal integral model of (V, φ), let $\mathbf{u} = (u_1, ..., u_r)$ be a basis of L over $\mathcal{O}_E$ and let $M \in \mathcal{M}_r(\mathcal{O}_E)$, invertible over E, such that $\mathbf{v} = M \cdot \mathbf{u}$. To conclude that $T = L$, we have to prove that $M \in \mathrm{GL}_r(\mathcal{O}_E)$.

Let F be the matrix of φ expressed in $\sigma^* \mathbf{v}$ and $\mathbf{v}$. We have

$$\varphi(\sigma^* L) = \mathcal{M}_r(\mathcal{O}_E) \cdot \varphi(\sigma^* \mathbf{u}) = \mathcal{M}_r(\mathcal{O}_E) F (M^{-1})^\sigma \cdot \mathbf{v} = \mathcal{M}_r(\mathcal{O}_E) F (M^{-1})^\sigma M \cdot \mathbf{u}.$$

The inclusion $\varphi(\sigma^* L) \subset L$ implies that $F(M^{-1})^\sigma M$ has coefficients in $\mathcal{O}_E$. Therefore, we have $q v_E(M^{-1}) \geq v_E(F^{-1}) + v_E(M^{-1})$ and hence $v_E(M^{-1}) \geq v_E(F^{-1})/(q-1)$. The elementary divisors of F^{-1} are $(g_r^{-1}, ..., g_1^{-1})$ and our assumption reads $v_E(F^{-1}) > -(q-1)$, which amounts to $v_E(M^{-1}) \geq 0$. This concludes. $\square$

Remark 2.7 (Extension of the base field). Let E' be a finite field extension of E with ring of integers $\mathcal{O}_{E'}$. We let V' be $V \otimes_E E'$ and φ' be the extension of φ to V'. Let L and L' be the maximal integral models of (V, φ) and (V', φ'). We have $L \otimes_{\mathcal{O}_E} \mathcal{O}_{E'} \subset L'$, but we cannot always claim equality. Indeed, if $(e_1, ..., e_r)$ is the type of L and e is the ramification index of E'/E, then the type of $L \otimes_{\mathcal{O}_E} \mathcal{O}_{E'}$ is $(ee_1, ..., ee_r)$. In particular, if $ee_1 \geq q - 1$ then $L \otimes_{\mathcal{O}_E} \mathcal{O}_{E'}$ is not maximal by Proposition 2.6.

The following proposition enables us to say *how far* an integral lattice is from being maximal given its range.

Proposition 2.8. *Let T be an $\mathcal{O}_E$-lattice in V stable by φ and let L be the maximal integral model of (V, φ). Let s be a non-negative integer. If the range of T satifies $r_T \leq s(q-1)$, then $L \subset \varpi^{-s} T$.*

We start by a lemma:

Lemma 2.9. *Let U be an $\mathcal{O}_E$-lattice in V such that $U \subset \varphi(\sigma^* U)$. Then $L \subset U$.*

Proof. For $n \geq 0$, we let $\sigma^{n*} := (\sigma^n)^*$ and denote by $\varphi^n : \sigma^{n*}V \to V$ the E-linear morphism given by the composition

$$\sigma^{n*}V \xrightarrow{\sigma^{(n-1)*}\varphi} \sigma^{(n-1)^*}V \longrightarrow \cdots \longrightarrow \sigma^* V \xrightarrow{\varphi} V.$$

We consider the following sub-$\mathcal{O}_E$-module of V:

$$L \cap \left(\bigcup_{n=0}^{\infty} \varphi^n(\sigma^{n*}U) \right). \tag{2.1}$$

It is stable by φ, finitely generated because contained in L, and generates V over E because contains the $\mathcal{O}_E$-lattice $L \cap U$. By maximality, (2.1) equals L and we deduce that there exists a non-negative integer m such that $L \subset \varphi^m(\sigma^{m*}U)$. Because $\varphi(\sigma^* L) \subset L$, we have $\sigma^* L \subset \varphi^{-1}(L)$ and by immediate recursion one gets $\sigma^{m*}L \subset \varphi^{-m}(L) \subset \sigma^{m*}U$. We conclude that $L \subset U$ because $\sigma : \mathcal{O}_E \to \mathcal{O}_E$ is faithfully flat. □

Proof of Proposition 2.8. Let $(e_1, ..., e_r)$ be the type of T. Recall that $\mathfrak{m} = \mathfrak{m}_E$ denotes the maximal ideal of $\mathcal{O}_E$. There exists a basis $(t_1, ..., t_r)$ of T such that $\varphi(\sigma^* T) = \mathfrak{m}^{e_1} t_1 \oplus \mathfrak{m}^{e_2} t_2 \oplus \cdots \oplus \mathfrak{m}^{e_r} t_r$. By assumption, $e_1, ..., e_r \leq s(q-1)$ and thus

$$\begin{aligned}
\varpi^{-s} T &\subset \varpi^{-s} \left(\mathfrak{m}^{e_1 - s(q-1)} t_1 \oplus \mathfrak{m}^{e_2 - s(q-1)} t_2 \oplus \cdots \oplus \mathfrak{m}^{e_r - s(q-1)} t_r \right) \\
&= \mathfrak{m}^{e_1 - sq} t_1 \oplus \mathfrak{m}^{e_2 - sq} t_2 \oplus \cdots \oplus \mathfrak{m}^{e_r - sq} t_r \\
&= \varpi^{-qs} \varphi(\sigma^* T) \\
&= \varphi(\sigma^*(\varpi^{-s} T)).
\end{aligned}$$

Hence, $U := \varpi^{-s} T$ satisfies $U \subset \varphi(\sigma^* U)$ and we deduce that $L \subset U$ by Lemma 2.9. □

2.2 Integral models of A-motives over a local field

Let R be a commutative $\mathbb{F}$-algebra given together with an $\mathbb{F}$-algebra morphism $\kappa : A \to R$. Let S be a commutative $\mathbb{F}$-algebra containing R. Let $\underline{M} = (M, \tau_M)$ be an A-motive over S (with characteristic morphism $\kappa : A \to S$).

Definition 2.10. We define an *R-model L for $\underline{M}$* to be a sub-$A \otimes R$-module of M of finite type such that

(*i*) L generates M over $A \otimes S$, (*ii*) $\tau_M(\tau^* L) \subset L[\mathfrak{j}^{-1}]$.

We say that L is maximal if it contains all the R-models of $\underline{M}$.

The next proposition is inspired by [Gar2, Prop. 2.2]:

Proposition 2.11. *If S is obtained from R by localization, an R-model for $\underline{M}$ exists.*

Proof. Let $(m_1, ..., m_s)$ be generators of M as an $A \otimes S$-module, and let L_0 be the sub-$A \otimes R$-module of M generated by $(m_1, ..., m_s)$. Let $d \in R$ be such that $\tau_M(\tau^* L_0) \subset d^{-1} L_0[\mathfrak{j}^{-1}]$, and set $L := dL_0$. We have

$$\tau_M(\tau^* L) = d^q \tau_M(\tau^* L_0) \subset d^{q-1} L_0[\mathfrak{j}^{-1}] = d^{q-2} L[\mathfrak{j}^{-1}] \subset L[\mathfrak{j}^{-1}].$$

Thus L is an R-model. □

2.2.1 Existence and first properties

Let E be a local field containing $\mathbb{F}$, let $\mathcal{O} = \mathcal{O}_E$ be its ring of integers and let $k = k_E$ be its residue field. In this subsection, we shall be concerned with the case where $S = E$ and $R = \mathcal{O}_E$, where *the characteristic morphism* $\kappa : A \to \mathcal{O}_E$ is an $\mathbb{F}$-linear morphism. Let $\underline{M}$ be an A-motive over E of characteristic κ.

Proposition 2.12. *A maximal $\mathcal{O}_E$-model for $\underline{M}$ exists and is unique.*

Proof. A maximal $\mathcal{O}_E$-model, if it exists, is necessarily unique. We show existence. Let U be the $A \otimes \mathcal{O}_E$-module given by the union of all the $\mathcal{O}_E$-models for $\underline{M}$. We claim that U is the maximal $\mathcal{O}_E$-model of $\underline{M}$. As U is non-empty by Proposition 2.11, it generates M over E. We also have $\tau_M(\tau^* U) \subset U[\mathfrak{j}^{-1}]$. So our task is to show that U is finitely generated.

Let T be an $\mathcal{O}_E$-model for $\underline{M}$ and let $\mathbf{t} = \{t_1, ..., t_s\}$ be a set of generators of T over $A \otimes \mathcal{O}_E$. Let $\mathbf{m}$ be a basis of $M \otimes_{A \otimes E} \operatorname{Quot}(A \otimes E)$ as a vector space over $\operatorname{Quot}(A \otimes E)$, and let $F_M \in \mathrm{GL}_r(\operatorname{Quot}(A \otimes E))$ be the matrix of τ_M written in the bases $\tau^* \mathbf{m}$ and $\mathbf{m}$. Let $P \in \mathcal{M}_{s,r}(A \otimes E)$ be the matrix expressing $\mathbf{t}$ in $\mathbf{m}$. Because of points (*ii*) in Definition 2.10, there exists $N \in \mathcal{M}_s(A \otimes \mathcal{O}_E[\mathfrak{j}^{-1}])$ such that $P^{(1)} F_M = NP$. If v denotes the valuation in $\operatorname{Quot}(A \otimes E)$ at the special fiber $C \times \operatorname{Spec} k_E$ of $C \times \operatorname{Spec} \mathcal{O}_E$, then $v(N) \geq 0$ and

$$qv(P) = v(P^{(1)}) = v(NPF_M^{-1}) \geq v(N) + v(P) + v(F_M^{-1}) \geq v(P) + v(F_M^{-1}).$$

Hence, $v(P) \geq v(F_M^{-1})/(q-1)$. We conclude that T is contained in the $A \otimes \mathcal{O}_E$-module

$$U_0 := \{a_1 m_1 \oplus ... \oplus a_r m_r \mid \forall\, i \in \{1, ..., r\},\ a_i \in A \otimes E,\ (q-1) v(a_i) \geq v(F_M^{-1})\}. \tag{2.2}$$

In particular, U is contained in U_0. The latter being a finitely generated module over the Noetherian ring $A \otimes \mathcal{O}_E$, the former is finitely generated. □

Definition 2.13. We denote by $M_{\mathcal{O}}$ the unique maximal $\mathcal{O}_E$-model of $\underline{M}$.

We have the next:

Proposition 2.14. *The maximal $\mathcal{O}_E$-model $M_{\mathcal{O}}$ of $\underline{M}$ is locally free over $A\otimes\mathcal{O}_E$.*

We start with a useful lemma.

Lemma 2.15. *Let $\mathfrak{a}\subset A$ be an ideal. Then $M_{\mathcal{O}}\cap\mathfrak{a}M=\mathfrak{a}M_{\mathcal{O}}$.*

Proof. The inclusion $\supset$ is clear. We assume $\mathfrak{a}\neq 0$ and consider the sub-A-motive $(\mathfrak{a}M,\tau_M)$ of $\underline{M}$. If T is an $\mathcal{O}_E$-model for $(\mathfrak{a}M,\tau_M)$, then $\mathfrak{a}^{-1}T$ is an $\mathcal{O}_E$-model for $\underline{M}$ and we have $\mathfrak{a}^{-1}T\subset M_{\mathcal{O}}$. This implies that $\mathfrak{a}M_{\mathcal{O}}$ is the maximal $\mathcal{O}_E$-model of $(\mathfrak{a}M,\tau_M)$ so that $(\mathfrak{a}M)_{\mathcal{O}}=\mathfrak{a}(M_{\mathcal{O}})$. Therefore, the inclusion $M_{\mathcal{O}}\cap\mathfrak{a}M\subset\mathfrak{a}M_{\mathcal{O}}$ follows from the fact that $M_{\mathcal{O}}\cap\mathfrak{a}M$ is an $\mathcal{O}_E$-model for $(\mathfrak{a}M,\tau_M)$. □

Proof of Proposition 2.14. Because $A\otimes\mathcal{O}_E$ is a Noetherian domain and $M_{\mathcal{O}}$ is finitely generated, it is enough to show that $M_{\mathcal{O}}$ is flat. We use Bourbaki's local criterion of flatness. Let $\mathfrak{m}\subset A$ be a maximal ideal and let $\mathbb{F}_{\mathfrak{m}}$ be its residue field. Note that

$$\operatorname{Tor}_1^{A\otimes\mathcal{O}_E}(A/\mathfrak{m}\otimes\mathcal{O}_E,M_{\mathcal{O}})=\{m\in M_{\mathcal{O}}\mid \forall r\in\mathfrak{m},\ (r\otimes 1)m=0\}=0.$$

Hence, by [Bou, AC §III.5.2 Thm. 1], the flatness of $M_{\mathcal{O}}$ over $A\otimes\mathcal{O}_E$ is equivalent to that of $M_{\mathcal{O}}/\mathfrak{m}M_{\mathcal{O}}$ over $\mathbb{F}_{\mathfrak{m}}\otimes\mathcal{O}_E$. The ring $\mathbb{F}_{\mathfrak{m}}\otimes\mathcal{O}_E$ is a product of discrete valuation rings and thus $M_{\mathcal{O}}/\mathfrak{m}M_{\mathcal{O}}$ is flat (and then locally free) if and only if it is $\mathcal{O}_E$-torsion free. The latter condition is easily seen to be equivalent to the equality:

$$\mathfrak{m}M_{\mathcal{O}}=M_{\mathcal{O}}\cap\mathfrak{m}M$$

which follows from Lemma 2.31. □

Remark 2.16. Let $\underline{M}$ and $\underline{N}$ be two A-motives over E, and let $M_{\mathcal{O}}$ and $N_{\mathcal{O}}$ be their respective integral models. While the maximal integral model of $\underline{M}\oplus\underline{N}$ is easily shown to be $M_{\mathcal{O}}\oplus N_{\mathcal{O}}$, it is not true in general that the maximal integral model of $\underline{M}\otimes\underline{N}$ is the image of $M_{\mathcal{O}}\otimes_{A\otimes\mathcal{O}_E}N_{\mathcal{O}}$ in $M\otimes_{A\otimes E}N$. To find a counter-example, we assume $q>2$ and consider $\varpi\in\mathcal{O}_E$ a uniformizer. We consider the A-motive $\underline{M}$ over E where $M=A\otimes E$ and where $\tau_M=\varpi\cdot\mathbf{1}$. The maximal integral model of $\underline{M}$ is $M_{\mathcal{O}}=A\otimes\mathcal{O}_E$. However, $\underline{M}^{\otimes(q-1)}$ has $\varpi^{-1}M_{\mathcal{O}}^{\otimes(q-1)}$ for maximal integral model.

2.2.2 Comparison with Frobenius spaces

As in Section 2.1, let $E=k(\!(\varpi)\!)$ for a field k containing $\mathbb{F}$, let $\mathcal{O}_E=k[\![\varpi]\!]$ be its valuation ring and let $\mathfrak{m}_E=(\varpi)$ be the maximal ideal of $\mathcal{O}_E$. Let $\mathfrak{m}$ be a maximal ideal of A. Note that $\mathfrak{j}(A/\mathfrak{m}^n\otimes E)=A/\mathfrak{m}^n\otimes E$ for all positive integers n.

Let $\underline{M}$ be an A-motive over E. We have canonical isomorphisms

$$\forall n \geq 1 : \quad M/\mathfrak{m}^n M \cong M[\mathfrak{j}^{-1}]/\mathfrak{m}^n M[\mathfrak{j}^{-1}]. \tag{2.3}$$

In particular, for all $n \geq 1$, τ_M defines an $A \otimes E$-linear morphism $\tau^*(M/\mathfrak{m}^n M) \to M/\mathfrak{m}^n M$ through the composition

$$\tau^*(M/\mathfrak{m}^n M) \xrightarrow{\tau_M} M[\mathfrak{j}^{-1}]/\mathfrak{m}^n M[\mathfrak{j}^{-1}] \xrightarrow{(2.3)} M/\mathfrak{m}^n M$$

which we still denote by τ_M. The pair $(M/\mathfrak{m}^n M, \tau_M)$ defines a Frobenius space over E in the sense of Section 2.1. Let $L_n \subset M/\mathfrak{m}^n M$ be its maximal integral model.

Remark 2.17. In general, we cannot claim equality between $(M_{\mathcal{O}} + \mathfrak{m}^n M)/\mathfrak{m}^n M$ and L_n. Here is a counter-example.

Suppose that $A = \mathbb{F}[t]$, so that $A \otimes \mathcal{O}_E$ is identified with $\mathcal{O}_E[t]$, and let $\mathfrak{m} = (t)$. Let $\kappa : A \to \mathcal{O}_E$ be the $\mathbb{F}$-algebra morphism which maps t to ϖ. In this setting, $\mathfrak{j}$ is the principal ideal of $\mathcal{O}_E[t]$ generated by $(t - \varpi)$. Consider the A-motive $\underline{M} := (E[t], f \cdot \mathbf{1})$ over E where $f = \varpi^{q-1} - \varpi^{q-2} t$. We claim that the maximal integral model of $\underline{M}$ is $\mathcal{O}_E[t]$. Clearly, $\mathcal{O}_E[t]$ is an integral model for $\underline{M}$ so that $\mathcal{O}_E[t] \subset M_{\mathcal{O}}$. Conversely, by [Qui2, Thm. 4], $M_{\mathcal{O}}$ is free of rank one over $\mathcal{O}_E[t]$. If h generates $M_{\mathcal{O}}$, there exists $b \in \mathcal{O}_E[t]$ such that $f h^{(1)} = bh$. For $p \in E[t]$, let $v(p)$ be the infinimum of the valuations of the coefficients of p. We have

$$v(h) \geq -\frac{v(f)}{q-1} = -\frac{q-2}{q-1} > -1$$

and $h \in \mathcal{O}_E[t]$. We get $M_{\mathcal{O}} \subset \mathcal{O}_E[t]$.

On the other-hand, the Frobenius space $(M/\mathfrak{m}M, \tau_M)$ is isomorphic to $(\mathcal{O}_E, \varpi^{q-1}\mathbf{1})$ whose maximal integral model is $\varpi^{-1}\mathcal{O}_E$, not $\mathcal{O}_E$.

If one wants to compare $M_{\mathcal{O}}$ with $(L_n)_{n \geq 1}$, then one wishes that $(M_{\mathcal{O}} + \mathfrak{m}^n M)/\mathfrak{m}^n M$ defines an integral model for $(M/\mathfrak{m}^n M, \tau_M)$ for all $n \geq 1$. This is the case in Remark 2.17, although it is not maximal, because the considered A-motive $\underline{M}$ is effective. In general, this is not true[1]. From now on, we assume

$(C_{\mathfrak{m}})$ The ideal $\mathfrak{m} \subset A$ is such that $\kappa(\mathfrak{m})$ contains a unit in $\mathcal{O}_E$, that is,

$$\kappa(\mathfrak{m})\mathcal{O}_E = \mathcal{O}_E.$$

The above assumption ensures that $\mathfrak{j}(A/\mathfrak{m}^n \otimes \mathcal{O}_E) = A/\mathfrak{m}^n \otimes \mathcal{O}_E$ for all $n \geq 1$ (e.g. the proof Proposition 1.29), and thus that $(M_{\mathcal{O}} + \mathfrak{m}^n M)/\mathfrak{m}^n M$ is an integral model for $(M/\mathfrak{m}^n M, \tau_M)$.

[1]For instance, consider the t-motive $(E[t], (t-\varpi)^{-1}\mathbf{1})$ over E, whose maximal $\mathcal{O}_E$-model is $\mathcal{O}_E[t]$, together with $\mathfrak{m} = (t)$.

Remark 2.18. Note that there always exists a maximal ideal $\mathfrak{m}$ in A satisfying $(C_{\mathfrak{m}})$: it suffices to take a maximal ideal $\mathfrak{m}$ in A coprime to $\kappa^{-1}(\mathfrak{m}_E)$.

Even though we cannot claim always equality between $(M_{\mathcal{O}}+\mathfrak{m}^n M)/\mathfrak{m}^n M$ and L_n, the data of L_n for all $n\geq 1$ is enough to recover $M_{\mathcal{O}}$ as we show in the next theorem[2].

Theorem 2.19. *Let L_n be the maximal integral model of the Frobenius space $(M/\mathfrak{m}^n M, \tau_M)$. Let $m\in M$. Then $m\in M_{\mathcal{O}}$ if and only if $m+\mathfrak{m}^n M\in L_n$ for all positive integers n large enough.*

We start with some lemmas:

Lemma 2.20. *The $\mathcal{O}_E$-module L_n is an $A/\mathfrak{m}^n\otimes\mathcal{O}_E$-module.*

Proof. For an elementary tensor $r\otimes f$ in $A/\mathfrak{m}^n\otimes\mathcal{O}_E$, the $\mathcal{O}_E$-module $(r\otimes f)L_n$ is stable by τ_M. Indeed, we have $\tau_M(\tau^*(r\otimes f)L_n)=(r\otimes f^q)\tau_M(\tau^*L_n)\subset (r\otimes f)L_n$. By maximality of L_n, we have $(r\otimes f)L_n\subset L_n$. □

Lemma 2.21. *Let r_n be the range of the $\mathcal{O}_E$-lattice $(M_{\mathcal{O}}+\mathfrak{m}^n M)/\mathfrak{m}^n M$ in $M/\mathfrak{m}^n M$. Then $(r_n)_{n\geq 1}$ is bounded.*

Proof. Note that $M_{\mathcal{O}}$ is a finite projective $A\otimes\mathcal{O}_E$-module by Proposition 2.14. Let P be a finitely generated $A\otimes\mathcal{O}_E$-module such that $N:=M_{\mathcal{O}}\oplus P$ is free of finite rank. Let r' be the rank of N and let $\mathbf{n}$ be a basis of N. Let also $\tau_N:\tau^*N[\mathfrak{j}^{-1}]\to N[\mathfrak{j}^{-1}]$ be the morphism $\tau_M\oplus 0$, and denote by

$$F_N=(b_{ij})_{ij}\in\mathcal{M}_{r'}(A\otimes\mathcal{O}_E[\mathfrak{j}^{-1}])$$

the matrix of τ_N written in the bases $\tau^*\mathbf{n}$ and $\mathbf{n}$.

For $n\geq 1$, let $\mathbf{t}_n$ be a basis of $A/\mathfrak{m}^n$ over $\mathbb{F}$. For $i,j\in\{1,...,r'\}$, let B_{ij}^n be the matrix with coefficients in $\mathcal{O}_E$ representing the multiplication by b_{ij} on $A/\mathfrak{m}^n\otimes\mathcal{O}_E$ in the basis $\mathbf{t}_n\otimes 1$. Then, the matrix of $\tau_N:\tau^*(N/\mathfrak{m}^n N)\to N/\mathfrak{m}^n N$, seen as an $\mathcal{O}_E$-linear map, and written in the bases $\tau^*(\mathbf{t}_n\otimes\mathbf{n})$ and $\mathbf{t}_n\otimes\mathbf{n}$, takes the form of the block matrix:

$$F_N^n:=(B_{ij}^n)_{ij}\in\mathcal{M}_{r'd_n}(\mathcal{O}_E)$$

where d_n is the dimension of $A/\mathfrak{m}^n$ over $\mathbb{F}$. One verifies that $v(B_{ij}^n)$ equals the infinimum of the valuation of the coefficients of b_{ij} (mod $\mathfrak{m}^n$) in $\mathcal{O}_E$ written in $\mathbf{t}_n$. Thus, for large values of n, we have

$$\forall\ i,j\in\{1,...,r'\}:\quad v(B_{ij}^n)=v(b_{ij})\quad (n \text{ large enough}). \tag{2.4}$$

For all $n\geq 1$, note that $(M_{\mathcal{O}}+\mathfrak{m}^n M)/\mathfrak{m}^n M\cong M_{\mathcal{O}}/\mathfrak{m}^n M_{\mathcal{O}}$ by Lemma 2.15. Because $M_{\mathcal{O}}/\mathfrak{m}^n M_{\mathcal{O}}$ is a direct factor in $N/\mathfrak{m}^n N$, the range of $(M_{\mathcal{O}}+\mathfrak{m}^n M)/\mathfrak{m}^n M$

[2]In Theorem 2.19, one could weaken assumption $(C_{\mathfrak{m}})$ to "$(M_{\mathcal{O}}+\mathfrak{m}^n M)/\mathfrak{m}^n M$ is an integral model for all $n\geq 1$". It would then include the setting of Remark 2.17. We do not need, however, this degree of generality in the sequel.

equals the maximal valuation of the (nonzero) elementary divisors relative to the inclusion of $\mathcal{O}_E$-modules

$$\tau_N(\tau^*(N/\mathfrak{m}^n N)) \subset N/\mathfrak{m}^n N. \tag{2.5}$$

The elementary divisors relative to (2.5) coincide, up to units of $\mathcal{O}_E$, to those appearing in the Smith normal form of the matrix $F_N^n \in \mathcal{M}_{r'd_n}(\mathcal{O}_E)$.

By (2.4), the valuations of the coefficients of F_N^n are stationary. The range of $(M_{\mathcal{O}} + \mathfrak{m}^n M)/\mathfrak{m}^n M$ in $M/\mathfrak{m}^n M$ is thus stationary and hence bounded. □

For $n \geq 0$, let $\tilde{L}_n$ be the inverse image in M of $L_n \subset M/\mathfrak{m}^n M$.

Proof of Theorem 2.19. The statement is equivalent to the equality

$$M_{\mathcal{O}} = \bigcap_{n=D}^{\infty} (\tilde{L}_n + \mathfrak{m}^n M)$$

for all positive integer $D \geq 1$. The sequence of subsets $(\tilde{L}_n + \mathfrak{m}^n M)_{n\geq 1}$ decreases for the inclusion: for $n \geq 1$, we have $\tilde{L}_{n+1} + \mathfrak{m}^{n+1} M \subset \tilde{L}_{n+1} + \mathfrak{m}^n M$ and, because $(\tilde{L}_{n+1} + \mathfrak{m}^n M)/\mathfrak{m}^n M$ defines an integral model for $(M/\mathfrak{m}^n M, \tau_M)$, we also have $\tilde{L}_{n+1} + \mathfrak{m}^n M \subset \tilde{L}_n + \mathfrak{m}^n M$. Consequently, it suffices to treat the case $D = 1$.

Consider

$$L := \bigcap_{n=1}^{\infty} (\tilde{L}_n + \mathfrak{m}^n M).$$

By Lemma 2.20, L is an $A \otimes \mathcal{O}_E$-module. The inclusion $M_{\mathcal{O}} \subset L$ follows from the fact that, for all n, $(M_{\mathcal{O}} + \mathfrak{m}^n M)/\mathfrak{m}^n M$ is an integral model for $(M/\mathfrak{m}^n M, \tau_M)$. To prove the converse inclusion, we show that L is an integral model for $\underline{M}$. From $M_{\mathcal{O}} \subset L$, one deduces that L generates M over E. Because $\tau_M(\tau^*(\tilde{L}_n + \mathfrak{m}^n M)) \subset \tilde{L}_n + \mathfrak{m}^n M[\mathfrak{j}^{-1}]$, we also have $\tau_M(\tau^* L) \subset L[\mathfrak{j}^{-1}]$. The theorem follows once we have proved that L is finitely generated.

Assume that L is not finitely generated. From the Noetherianity of $A \otimes \mathcal{O}_E$, for all $s \geq 0$, it follows that $L \not\subset \varpi^{-s} M_{\mathcal{O}}$. Equivalently, there exists an unbounded increasing sequence $(s_n)_{n\geq 0}$ of non-negative integers such that $\varpi^{s_n} L_n \not\subset (M_{\mathcal{O}} + \mathfrak{m}^n M)/\mathfrak{m}^n M$. By Proposition 2.8, the range of $(M_{\mathcal{O}} + \mathfrak{m}^n M)/\mathfrak{m}^n M$ is $> s_n(q-1)$. But this contradicts Lemma 2.21. □

For the sequel, it is also useful to have the next statement which is easily deduced from Theorem 2.19.

Corollary 2.22. *For all positive integers n, let L_n be the maximal integral model of $(M[\mathfrak{j}^{-1}]/\mathfrak{m}^n M[\mathfrak{j}^{-1}], \tau_M)$. Let $m \in M[\mathfrak{j}^{-1}]$. Then $m \in M_{\mathcal{O}}[\mathfrak{j}^{-1}]$ if and only if $m + \mathfrak{m}^n M[\mathfrak{j}^{-1}] \in L_n$ for all positive integers n large enough.*

For the next chapter, we shall not only be interested in how to recover $M_{\mathcal{O}}$ from L_n, but also in how to recover $M_{\mathcal{O}} + (\mathrm{id} - \tau_M)(M)$. We continue with some technical lemmas.

Even if we do not have equality between $\tilde{L}_n + \mathfrak{m}^n M$ and $M_{\mathcal{O}} + \mathfrak{m}^n M$, the former is a good approximation of the latter as we show in the next two lemmas.

Lemma 2.23. *Let $n \geq 1$. The sequence $(\tilde{L}_m + \mathfrak{m}^n M)_{m \geq n}$ is decreasing for the inclusion, stationary and converges to $M_{\mathcal{O}} + \mathfrak{m}^n M$.*

Proof. Let $m \geq 1$. $(\tilde{L}_{m+1} + \mathfrak{m}^m M)/\mathfrak{m}^m M$ is an $\mathcal{O}_E$-lattice stable by τ_M in $M/\mathfrak{m}^m M$ so that $\tilde{L}_{m+1} + \mathfrak{m}^m M \subset \tilde{L}_m + \mathfrak{m}^m M$. If $m \geq n$, we have $\tilde{L}_{m+1} + \mathfrak{m}^n M \subset \tilde{L}_m + \mathfrak{m}^n M$ which shows that $(\tilde{L}_m + \mathfrak{m}^m)_{m \geq n}$ decreases. Similarly, $M_{\mathcal{O}} + \mathfrak{m}^n M \subset \tilde{L}_m + \mathfrak{m}^n M$ for all $m \geq n$. Because the set of $\mathcal{O}_E$-lattices Λ such that $M_{\mathcal{O}} + \mathfrak{m}^n M \subseteq \Lambda \subseteq \tilde{L}_n + \mathfrak{m}^n M$ is finite, the sequence $(\tilde{L}_m + \mathfrak{m}^n M)_{m \geq n}$ is stationary. We denote by $\mathscr{L}_n$ its limit. By Theorem 2.19, we have

$$\mathscr{L}_n = \bigcap_{m=n}^{\infty} (\tilde{L}_m + \mathfrak{m}^n M) = \bigcap_{m=n}^{\infty} (\tilde{L}_m + \mathfrak{m}^m M) + \mathfrak{m}^n M = M_{\mathcal{O}} + \mathfrak{m}^n M.$$

This concludes the proof. □

Lemma 2.24. *There exists an unbounded and increasing sequence $(k_n)_{n \geq 1}$ of non-negative integers such that, $\tilde{L}_n + \mathfrak{m}^n M \subset M_{\mathcal{O}} + \mathfrak{m}^{k_n} M$ (typically, $k_n \leq n$ for all n).*

Proof. For $m \geq 1$, let I_m be the set of non-negative integers k such that $\tilde{L}_m + \mathfrak{m}^m M \subset M_{\mathcal{O}} + \mathfrak{m}^k M$. I_m is nonempty as it contains 0. I_m is further bounded: otherwise we would have

$$\tilde{L}_m + \mathfrak{m}^m M \subset \bigcap_k (M_{\mathcal{O}} + \mathfrak{m}^k M) = M_{\mathcal{O}} \tag{2.6}$$

which is impossible ($\tilde{L}_m + \mathfrak{m}^m M$ is an $A \otimes \mathcal{O}_E$-module which is not of finite type). Hence I_m has a maximal element, which we denote by k_m. Because $\tilde{L}_{m+1} + \mathfrak{m}^{m+1} M \subset \tilde{L}_m + \mathfrak{m}^m M$, we have $k_{m+1} \geq k_m$. This shows that $(k_m)_{m \geq 1}$ increases. We show that it is unbounded. Let $n \geq 1$. By Lemma 2.23, there exists $m \geq n$ such that $M_{\mathcal{O}} + \mathfrak{m}^n = \tilde{L}_m + \mathfrak{m}^n M$. Thus $\tilde{L}_m + \mathfrak{m}^m M \subset M_{\mathcal{O}} + \mathfrak{m}^n M$. In particular, there exists $m \geq n$ such that $k_m \geq n$. □

Proposition 2.25. *Let $m \in M[\mathfrak{j}^{-1}]$. We have $m \in M_{\mathcal{O}} + (\mathrm{id} - \tau_M)(M)$ if and only, for all positive integers $n \geq 0$, the image of m in $M[\mathfrak{j}^{-1}]/\mathfrak{m}^n M$ belongs to $L_n + (\mathrm{id} - \tau_M)(M) + \mathfrak{m}^n M$.*

Proof. By Theorem 2.19, the inclusion

$$M_{\mathcal{O}} + (\mathrm{id} - \tau_M)(M) \subset \bigcap_{n=1}^{\infty} \left[\tilde{L}_n + (\mathrm{id} - \tau_M)(M) + \mathfrak{m}^n M\right]$$

holds as subsets of $M[\mathfrak{j}^{-1}]$. The converse inclusion follows from Lemma 2.24:

$$\begin{aligned} \bigcap_{n=1}^{\infty} \left[\tilde{L}_n + (\mathrm{id} - \tau_M)(M) + \mathfrak{m}^n M\right] &\subset \bigcap_{n=1}^{\infty} \left[M_{\mathcal{O}} + (\mathrm{id} - \tau_M)(M) + \mathfrak{m}^{k_n} M\right] \\ &= M_{\mathcal{O}} + (\mathrm{id} - \tau_M)(M). \end{aligned}$$

□

Similarly, we rewrite Proposition 2.25 in view of (2.3).

Corollary 2.26. *For all positive integers n, let L_n be the maximal integral model of $(M[\mathfrak{j}^{-1}]/\mathfrak{m}^n M[\mathfrak{j}^{-1}], \tau_M)$. Let $m \in M[\mathfrak{j}^{-1}]$. We have $m \in M_{\mathcal{O}}[\mathfrak{j}^{-1}] + (\mathrm{id} - \tau_M)(M)$ if and only if, for all $n \geq 0$, the image of m in $M[\mathfrak{j}^{-1}]/\mathfrak{m}^n M[\mathfrak{j}^{-1}]$ belongs to $L_n + (\mathrm{id} - \tau_M)(M) + \mathfrak{m}^n M[\mathfrak{j}^{-1}]$.*

2.2.3 Good reduction criterion

In this subsection we prove a good reduction criterion, similar to Gardeyn's slogan *"a τ-sheaf has good reduction if and only if it admits a good model"* (see Proposition [Gar2, Prop.2.13(ii)]). The results of this subsection are not needed in the remaining of the text, although they are useful in examples to compute maximal models.

We begin with the following key lemma:

Lemma 2.27. *If there exists an $\mathcal{O}_E$-model L for $\underline{M}$ such that $\tau_M(\tau^* L)[\mathfrak{j}^{-1}] = L[\mathfrak{j}^{-1}]$, then $L = M_{\mathcal{O}}$.*

Proof. From Remark 2.18, let $\mathfrak{m}$ be a maximal ideal of A such that $(C_{\mathfrak{m}})$ holds. For all positive integers n, the $\mathcal{O}_E$-lattice $L_n := (L + \mathfrak{m}^n M)/\mathfrak{m}^n M$ in $M/\mathfrak{m}^n M$ defines an integral model of $(M/\mathfrak{m}^n M, \tau_M)$ satisfying $\tau_M(\tau^* L_n) = L_n$. By Proposition 2.6, L_n is maximal. We conclude that $L = M_{\mathcal{O}}$ by Theorem 2.19. $\square$

The next proposition follows by Lemma 2.27 and Proposition 2.14.

Proposition 2.28. *The following statements are equivalent:*

(i) $\tau_M(\tau^* M_{\mathcal{O}})[\mathfrak{j}^{-1}] = M_{\mathcal{O}}[\mathfrak{j}^{-1}]$,

(ii) there exists an $\mathcal{O}_E$-model N for $\underline{M}$ such that $\tau_M(\tau^* N)[\mathfrak{j}^{-1}] = N[\mathfrak{j}^{-1}]$,

(iii) the data $(M_{\mathcal{O}}, \tau_M)$ forms an A-motive over $\mathcal{O}_E$,

(iv) there exists an A-motive $\underline{N}$ over $\mathcal{O}_E$ such that $\underline{N}_E$ is isomorphic to $\underline{M}$.

Definition 2.29. We say that $\underline{M}$ has *good reduction* if one of the equivalent points of Proposition 2.28 is satisfied.

2.3 Integral models of A-motives over a global field

We go back to Definition 2.10. Let $S = F$ be a global function field that contains $\mathbb{F}$, and let R be a sub-$\mathbb{F}$-algebra of F which, as a ring, is a Dedekind domain whose fraction field is F. Given a maximal ideal $\mathfrak{p}$ in R, we denote by $R_\mathfrak{p}$ the completion of R at $\mathfrak{p}$ and we let $F_\mathfrak{p}$ be the fraction field of $R_\mathfrak{p}$. Let κ :

Let $\underline{M} = (M, \tau_M)$ be an A-motive over F of rank r, and let $\underline{M}_\mathfrak{p} = \underline{M}_{F_\mathfrak{p}}$ be the A-motive over $F_\mathfrak{p}$ of rank r obtained from $\underline{M}$ by base extension from F to $F_\mathfrak{p}$. We let $M_{R_\mathfrak{p}}$ denote the maximal $R_\mathfrak{p}$-model of $\underline{M}_\mathfrak{p}$.

Proposition 2.30. *There exists a unique maximal R-model for $\underline{M}$. It equals the intersection $\bigcap_\mathfrak{p}(M \cap M_{R_\mathfrak{p}})$ for $\mathfrak{p}$ running over the maximal ideals of R. We denote it M_R.*

Proof. The uniqueness of a maximal R-model, when it exists, is clear. We focus on existence.

We begin by the *maximality.* Let N be an R-model for $\underline{M}$ (whose existence is ensured by Proposition 2.11). For any maximal ideal $\mathfrak{p}$ of R, we have $N \subset N \otimes_R R_\mathfrak{p} \subset M_{R_\mathfrak{p}}$ by maximality of $M_{R_\mathfrak{p}}$. Therefore $N \subset \bigcap_\mathfrak{p}(M \cap M_{R_\mathfrak{p}})$.

It remains to show that $\bigcap_\mathfrak{p}(M \cap M_{R_\mathfrak{p}})$ is an R-model. First note that it is a sub-$A \otimes R$-module of M which, as it contains N, generates M over F. To show stability by τ_M, let $e \geq 0$ be such that $\tau_M(\tau^* M) \subset \mathfrak{j}^{-e} M$. Then,

$$\tau_M\left(\tau^* \bigcap_\mathfrak{p}(M \cap M_{R_\mathfrak{p}})\right) \subset \bigcap_\mathfrak{p} \mathfrak{j}^{-e}(M \cap M_{R_\mathfrak{p}}) \subset \left(\bigcap_\mathfrak{p} M \cap M_{R_\mathfrak{p}}\right)[\mathfrak{j}^{-1}].$$

It then suffices to show that $\bigcap_\mathfrak{p}(M \cap M_{R_\mathfrak{p}})$ is finitely generated over $A \otimes R$. We use a similar argument than that of the proof of Proposition 2.12. Let $\mathbf{m} := (m_1, ..., m_r)$ be a family of elements in $M \otimes_{A\otimes F} \operatorname{Quot}(A \otimes F)$ and let $\mathfrak{a} \subset A \otimes F$ be a nonzero ideal such that $M = (A \otimes F)m_1 \oplus \cdots \oplus (A \otimes F)m_{r-1} \oplus \mathfrak{a} m_r$. Let $F_M \in \mathrm{GL}_r(\operatorname{Quot}(A \otimes F))$ be the matrix of τ_M written in $\tau^*\mathbf{m}$ and $\mathbf{m}$ and let $U \in \mathcal{M}_{s,r}(\operatorname{Quot}(A \otimes F))$ be a matrix expressing generators $(u_1, ..., u_s)$ of N in $\mathbf{m}$. Because $\tau_M(\tau^* N) \subset N[\mathfrak{j}^{-1}]$, there exists $P \in \mathcal{M}_s(A \otimes R[\mathfrak{j}^{-1}])$ such that $U^{(1)} F_M = PU$. In particular, for all nonzero prime ideal $\mathfrak{p}$ of R of valuation $v_\mathfrak{p}$,

$$q v_\mathfrak{p}(U) = v_\mathfrak{p}(U^{(1)}) = v_\mathfrak{p}(PUF_M^{-1}) \geq v_\mathfrak{p}(U) + v_\mathfrak{p}(F_M^{-1}).$$

Hence, $v_\mathfrak{p}(U) \geq v_\mathfrak{p}(F_M^{-1})/(q-1)$. The $A \otimes R$-module:

$$\{a_1 m_1 + ... + a_r m_r | \forall i, \forall \mathfrak{p} \in \operatorname{Spm} R : a_i \in A \otimes F, (q-1)v_\mathfrak{p}(a_i) \geq v_\mathfrak{p}(F_M^{-1})\}$$

contains $\bigcap_\mathfrak{p}(M \cap M_{R_\mathfrak{p}})$ and is finitely generated (compare with (2.2)). Because $A \otimes R$ is Noetherian, $\bigcap_\mathfrak{p}(M \cap M_{R_\mathfrak{p}})$ is finitely generated. □

We now state the global version of Lemma 2.15 and Proposition 2.14 (with R in place of $R_\mathfrak{p}$). The argument is similar, so we omit proofs.

Lemma 2.31. *Let $\mathfrak{a} \subset A$ be an ideal. Then*

$$M_R \cap \mathfrak{a} M = \mathfrak{a} M_R.$$

Proposition 2.32. *The maximal R-model of $\underline{M}$ is locally-free over $A \otimes R$.*

Remark 2.33. An $\mathcal{O}_F$-model for $\underline{M}$, when not maximal, is not necessarily locally-free. For instance, the $\mathbb{F}[t]$-motive $\mathbb{1} = (\mathbb{F}[t](\theta), \mathbf{1})$ over $\mathbb{F}(\theta)$ admits $L := t\mathbb{F}[t,\theta] + \theta\mathbb{F}[t,\theta]$ as $\mathbb{F}[\theta]$-model. But it is well-known that L is not a flat $\mathbb{F}[t,\theta]$-module. A short way to see this consists in considering the element $\Delta := (t \otimes \theta - \theta \otimes t) \in L \otimes_{\mathbb{F}[t,\theta]} L$. Δ is nonzero in $L \otimes_{\mathbb{F}[t,\theta]} L$, but

$$\theta \cdot \Delta = (\theta t) \otimes \theta - \theta \otimes (\theta t) = (\theta t) \otimes \theta - (\theta t) \otimes \theta = 0.$$

Then L is not flat because $L \otimes_{\mathbb{F}[t,\theta]} L$ has non trivial torsion.

Here is a useful consequence of Lemma 2.31:

Proposition 2.34. *If R equals the integral closure of $\kappa(A)$ in F, then*

$$M_R[\mathfrak{j}^{-1}] \cap M = M_R.$$

Proof. The inclusion $\supset$ is clear. Because R is the integral closure of $\kappa(A)$ in F, for $m \in M$ there exists $a \in A$ such that $(1 \otimes \kappa(a))m \in M_R$. If $m \in M_R[\mathfrak{j}^{-1}]$, for k large enough we have $(a \otimes 1 - 1 \otimes \kappa(a))^{p^k} m \in M_R$. Hence, $(a \otimes 1)^{p^k} m \in M_R$. Therefore $(a \otimes 1)^{p^k} m$ belongs to $M_R \cap (a \otimes 1)^{p^k} M$. We conclude by Lemma 2.31 that $(a \otimes 1)^{p^k} m \in (a \otimes 1)^{p^k} M_R$. $\square$

Definition 2.35. We say that $\underline{M}$ has *good reduction at* $\mathfrak{p}$ if $\underline{M}_{\mathfrak{p}}$ has good reduction. We say that $\underline{M}$ has *everywhere good reduction* if $\underline{M}$ has good reduction at $\mathfrak{p}$ for all maximal ideals $\mathfrak{p}$ of R.

The good reduction criterion of Lemma 2.27 can be extended to the global situation:

Proposition 2.36. *Let L be an R-model for $\underline{M}$ such that $(\tau^* L)[\mathfrak{j}^{-1}] = L[\mathfrak{j}^{-1}]$. Then $L = M_R$ and $\underline{M}$ has everywhere good reduction.*

Proof. Let us first show that L is a flat $A \otimes R$-module. Let $\mathfrak{m}$ be a maximal ideal in A. By Bourbaki's local criterion for flatness, L is flat over $A \otimes R$ if and only if $L/\mathfrak{m}L$ is a flat $A/\mathfrak{m} \otimes R$-module. Because $A/\mathfrak{m} \otimes R$ is a finite product of finite flat R-algebras, $L/\mathfrak{m}L$ is flat if and only if it is R-torsion free. Therefore, we reduce the problem to showing the equality:

$$\mathfrak{m}M \cap L = \mathfrak{m}L$$

as subsets of $\mathfrak{m}M$. Clearly, we have an inclusion $\mathfrak{m}L \subset \mathfrak{m}M \cap L$ whose cokernel is R-torsion. For all nonzero prime ideal $\mathfrak{p} \subset R$, $L \otimes_R R_{\mathfrak{p}}$ is an R-model for $\underline{M}_{\mathfrak{p}}$ which satisfies the equivalent points of Proposition 2.28. As such, $L \otimes_R R_{\mathfrak{p}}$ is maximal and $\underline{M}$ has good reduction at $\mathfrak{p}$. Hence, $\underline{M}$ has everywhere good reduction and it follows that

$$\begin{aligned}(\mathfrak{m}L) \otimes_R R_{\mathfrak{p}} = \mathfrak{m}(L \otimes_R R_{\mathfrak{p}}) &= \mathfrak{m}(M \otimes_R R_{\mathfrak{p}}) \cap (L \otimes_R R_{\mathfrak{p}}) \quad \text{(by Proposition 2.15)} \\ &= (\mathfrak{m}M \cap L) \otimes_R R_{\mathfrak{p}}.\end{aligned}$$

Because this equality holds for all $\mathfrak{p}$, we conclude that $\mathfrak{m}L = \mathfrak{m}M \cap L$. Thus, L is flat over $A \otimes R$.

We now show that $L = M_R$. Let $m \in M_R$, and let $d \in R$ be such that $m \in L[d^{-1}]$. If $\mathfrak{p}_1$, $\mathfrak{p}_2$, ..., $\mathfrak{p}_s$ are the prime ideal divisor in R of (d), then

$$\bigcap_{i=1}^{s}(R_{\mathfrak{p}_i} \cap R[d^{-1}]) = R.$$

Because L is flat over $A \otimes R$, we have

$$\bigcap_{i=1}^{s}(L_{\mathfrak{p}_i} \cap L[d^{-1}]) = L.$$

Because m is a member of the left-hand side, we deduce that $m \in L$. □

We continue this section by recording additional properties of maximal R-models. Those will eventually by useful in Chapter 3 for the computation of extensions groups with good reduction in the category of A-motives.

Proposition 2.37. *Let N be a finitely generated sub-$A \otimes R$-module of M such that $\tau_M(\tau^* N) \subset N[\mathfrak{j}^{-1}]$. Then, $N \subset M_R$. In particular, any element $m \in M$ such that $\tau_M(\tau^* m) = m$ belongs to M_R.*

Proof. It suffices to notice that the module L generated by M_R and N over $A \otimes R$ is an R-model for $\underline{M}$, and hence $N \subset L \subset M_R$. □

Corollary 2.38. *We have* $(\mathrm{id} - \tau_M)(M_R) = (\mathrm{id} - \tau_M)(M) \cap M_R[\mathfrak{j}^{-1}]$.

Proof. The inclusion $(\mathrm{id} - \tau_M)(M_R) \subset (\mathrm{id} - \tau_M)(M) \cap M_R[\mathfrak{j}^{-1}]$ is clear. Conversely, let $m \in M_R[\mathfrak{j}^{-1}]$ and let $n \in M$ be such that $m = n - \tau_M(\tau^* n)$. The sub-$A \otimes R$-module $\langle M_R, n\rangle$ of M generated by elements of M_R together with n over $A \otimes R$ is an R-model for $\underline{M}$. In particular, $\langle M_R, n\rangle \subset M_R$ and $n \in M_R$. □

We end this chapter with a remark on the assignment $\underline{M} \mapsto M_R$, and explain how our notion of maximal integral model matches [Gar2, Def. 2.3].

Corollary 2.39. *Let $f : \underline{M} \to \underline{N}$ be a morphism in $\mathcal{M}_F$. Then $f(M_R) \subset N_R$. In particular, the assignation $\underline{M} \mapsto M_R$ is functorial.*

Remark 2.40. Let $\mathcal{M}'_R$ be the category whose objects are pairs $\underline{N} = (N, \tau_N)$ where N is a finite locally-free $A \otimes R$-module and $\tau_N : (\tau^* N)[\mathfrak{j}^{-1}] \to N[\mathfrak{j}^{-1}]$ is an injective $A \otimes R$-linear morphism whose cokernel is R-torsion. We have a functor $\mathcal{M}'_R \to \mathcal{M}_F$ which assigns to $\underline{N} = (N, \tau_N)$ the A-motive $\underline{N}_F := (N \otimes_R F, \tau_M \otimes_R \mathbf{1})$.

Given an A-motive $\underline{M}$ over F, $\underline{N}$ an object in $\mathcal{M}'_R$ and $h : \underline{N}_F \xrightarrow{\sim} \underline{M}$ an isomorphism in $\mathcal{M}_F$, we call $(\underline{N}, h)$ a *Gardeyn's model* for $\underline{M}$. We call $(\underline{N}, h)$ *maximal* if it satisfies the following universal property: given an object $\underline{N}'$ in

$\mathcal{M}'_R$ and a morphism $h' : \underline{N}'_F \to \underline{M}$ in $\mathcal{M}_F$, there exists a unique morphism $f : \underline{N}' \to \underline{N}$ in $\mathcal{M}'_R$ such that the following diagram commutes

$$\begin{tikzcd} \underline{N}'_F \arrow[r, "h'"] \arrow[dr, "f_F"'] & \underline{M} \\ & \underline{N}_F \arrow[u, "h"'] \end{tikzcd}$$

It is formal to check that if a maximal Gardeyn's model exists, it is necessarily unique up to unique isomorphism. If $M_{\mathcal{O}}$ is the maximal integral model of $\underline{M}$, we check easily that $(\underline{M}_{\mathcal{O}}, \underline{M}_{\mathcal{O}} \otimes_{\mathcal{O}_F} F \cong \underline{M})$ is a maximal Gardeyn's model of $\underline{M}$, where $\underline{M}_{\mathcal{O}} := (M_{\mathcal{O}}, \tau_M)$. This proves existence.

In other words, the covariant functor

$$\mathcal{M}'_R \longrightarrow \mathsf{Set}, \quad \underline{N}' \longmapsto \operatorname{Hom}_{\mathcal{M}_F}(\underline{M}, \underline{N}'_F)$$

is representable. Equivalently, the functor $\underline{M} \mapsto \underline{M}_{\mathcal{O}}$ is left-adjoint to $\underline{N}' \mapsto \underline{N}'_F$.

This universal property satisfied by maximal integral models is closed to the Néron mapping property for schemes (cf. [BLR]).

Chapter 3

A-Motivic Cohomology

Let R be a commutative A-algebra whose A-algebra structure is given through a morphism $\kappa : A \to R$.

As we showed in Proposition 1.7, the category $\mathcal{M}_R$ is A-linear and exact. The Yoneda extension modules $\mathrm{Ext}^i_{\mathcal{M}_R}$ do make sense and this section is devoted to their computation. There have already been researches related to these computations in the particular case of extensions in the the full subcategory $\mathcal{M}_R^{\mathrm{eff}}$ of $\mathcal{M}_R$ (see Definition 1.4). We refer for instance to [PapRa], [Tae1]. To the extent of my knowledge, the general case of $\mathcal{M}_R$ has not been studied yet.

Section 3.1 is devoted to the computation of $\mathrm{Ext}^i_{\mathcal{M}_R}$ for $i \geq 0$. To achieve the determination of these extension groups, we rather offer the computation *by hand* of $\mathrm{Ext}^1_{\mathcal{M}_R}$. Given an A-motive $\underline{M}$ over R, we show in Theorem 3.4 that there is an A-linear isomorphism

$$\iota : \frac{M[\mathfrak{j}^{-1}]}{(\mathrm{id} - \tau_M)(M)} \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{M}_R}(\mathbb{1}, \underline{M}).$$

This explicit description will allow us to conclude that $\mathrm{Ext}^i_{\mathcal{M}_R}$ vanishes for $i > 1$ when R is Noetherian. This is conjectured for classical mixed motives over a number field [Nek, §4].

In Subsection 3.2, we restrict our attention to extensions in the subcategory $\mathcal{MM}_F$ of mixed A-motives over $R = F$ a field. In the case where $\underline{M}$ is pure of weight μ over F, we prove in Corollary 3.12 that

$$\mathrm{Ext}^1_{\mathcal{MM}_F}(\mathbb{1}, \underline{M}) = \begin{cases} \mathrm{Ext}^1_{\mathcal{M}_F}(\mathbb{1}, \underline{M}) & \text{if } \mu \leq 0 \\ \mathrm{Ext}^1_{\mathcal{M}_F}(\mathbb{1}, \underline{M})^{\mathrm{tors}} & \text{if } \mu > 0. \end{cases}$$

This solves the analogue of a number field conjecture which states that every extension of the unit motive $\mathbb{1}$ by a pure motive of positive weight, in the category of mixed motives over a number field, is split ([Del, §1.3]).

In the case where F is a local or a global function field, and R is its *ring of integers*, following Scholl [Sch] in the number field case we consider, in Subsection 3.3, the A-module $\mathrm{Ext}^1_{\mathcal{M}_F,R}(\mathbb{1}, \underline{M})$ of *extensions having everywhere* *good reduction.* We present our main results in this chapter, Theorems 3.15 and 3.18, which state that ι induces an isomorphism

$$\frac{M_R[\mathfrak{j}^{-1}]}{(\mathrm{id}-\tau_M)(M_R)} \stackrel{\sim}{\longrightarrow} \mathrm{Ext}^1_{\mathcal{M}_F,R}(\mathbb{1},\underline{M})$$

where M_R denotes the maximal R-model of $\underline{M}$. This answers positively to the analogue of a number field's conjecture due to Scholl.

3.1 Extension modules in $\mathcal{M}_R$

Let R be an $\mathbb{F}$-algebra and let $\underline{M}$ and $\underline{N}$ be two A-motives over R. The morphisms from $\underline{N}$ to $\underline{M}$ in $\mathcal{M}_R$ are precisely the $A\otimes R$-linear map of the underlying modules $f:N\to M$ such that $\tau_M\circ\tau^*f=f\circ\tau_N$. Because 0th extension group is given by the homomorphisms, we have:

$$\mathrm{Ext}^0_{\mathcal{M}_R}(\underline{N},\underline{M})=\mathrm{Hom}_{\mathcal{M}_R}(\underline{N},\underline{M})=\{f\in\mathrm{Hom}_{A\otimes R}(N,M)\mid \tau_M\circ\tau^*f=f\circ\tau_N\}.$$

As we saw in Proposition 1.7, $\mathcal{M}_R$ possesses exact sequences in the sense of Quillen which turns it into an A-linear exact category. It allows us to consider higher Yoneda extension A-modules $\mathrm{Ext}^n_{\mathcal{M}_R}(\underline{N},\underline{M})$ (for $n\geq 1$) of two A-motives $\underline{M}$ and $\underline{N}$. The next proposition computes the first extension group.

Proposition 3.1. *Let $\underline{M}$ and $\underline{N}$ be A-motives over R. There is a canonical isomorphism of A-modules*

$$\frac{\mathrm{Hom}_{A\otimes R}(\tau^*N,M)[\mathfrak{j}^{-1}]}{\{f\circ\tau_N-\tau_M\circ\tau^*f\mid f\in\mathrm{Hom}_{A\otimes R}(N,M)\}} \stackrel{\sim}{\longrightarrow} \mathrm{Ext}^1_{\mathcal{M}_R}(\underline{N},\underline{M}),$$

which maps the class of a morphism $u\in\mathrm{Hom}_{A\otimes R}(\tau^*N,M)[\mathfrak{j}^{-1}]$ to the class of the extension $[M\oplus N,\left(\begin{smallmatrix}\tau_M & u\\ 0 & \tau_N\end{smallmatrix}\right)]$ in $\mathrm{Ext}^1_{\mathcal{M}_R}(\underline{N},\underline{M})$.

Proof. Let $[\underline{E}]:0\to\underline{M}\xrightarrow{\iota}\underline{E}\xrightarrow{\pi}\underline{N}\to 0$ be an exact sequence in $\mathcal{M}_R$, that is an exact sequence of the underlying $A\otimes R$-modules with commuting τ-action. Because N is a projective module, there exists $s:N\to E$ a section of the underlying short exact sequence of $A\otimes R$-modules. We let $\xi:=\iota\oplus s: M\oplus N\to E$. We have a congruence:

$$\begin{tikzcd}
0 \arrow[r] & \underline{M} \arrow[r, "\iota"] & \underline{E} \arrow[r, "\pi"] & \underline{N} \arrow[r] & 0 \\
0 \arrow[r] & \underline{M} \arrow[r] \arrow[u, "\mathrm{id}"] & (M\oplus N,\xi^{-1}\circ\tau_E\circ\xi) \arrow[r] \arrow[u, "\xi"] & \underline{N} \arrow[r] \arrow[u, "\mathrm{id}"] & 0
\end{tikzcd}$$

Because $\xi^{-1}\circ\tau_M\circ\xi$ is an isomorphism from $\tau^*M[\mathfrak{j}^{-1}]\oplus\tau^*N[\mathfrak{j}^{-1}]$ to $M[\mathfrak{j}^{-1}]\oplus N[\mathfrak{j}^{-1}]$ which restricts to τ_M on the left and to τ_N on the right, there exists $u\in\mathrm{Hom}_{A\otimes R}(\tau^*N[\mathfrak{j}^{-1}],M[\mathfrak{j}^{-1}])=\mathrm{Hom}_{A\otimes R}(\tau^*N,M)[\mathfrak{j}^{-1}]$ such that $\xi^{-1}\circ\tau_E\circ\xi=\left(\begin{smallmatrix}\tau_M & u\\ 0 & \tau_N\end{smallmatrix}\right)$. We have just shown that the map

$$\iota:\mathrm{Hom}_{A\otimes R}(\tau^*N,M)[\mathfrak{j}^{-1}]\to\mathrm{Ext}^1_{\mathcal{M}_R}(\underline{N},\underline{M}),\quad u\mapsto[M\oplus N,\left(\begin{smallmatrix}\tau_M & u\\ 0 & \tau_N\end{smallmatrix}\right)]$$

is onto. Note that $\iota(0)$ corresponds to the class of the split extension. Further, $\iota(u+v)$ corresponds to the Baer sum of $\iota(u)$ and $\iota(v)$. In addition, given the exact sequence $[\underline{E}]$ and $a \in A$, the pullback of multiplication by a on N and π gives another extension which defines $a \cdot [\underline{E}]$. If $[\underline{E}] = \iota(u)$ then it is formal to check that $a \cdot [\underline{E}] = \iota(au)$. As such, ι is a surjective A-module morphism. To find its kernel, it suffices to determine when $\iota(u)$ is congruent to the split extension. The extension $\iota(u)$ is congruent to the split extension if and only if there is a commutative diagram in $\mathcal{M}_R$ of the form

$$\begin{tikzcd}
0 \arrow[r] & \underline{M} \arrow[r] \arrow[d, "\mathrm{id}_M"] & \underline{M} \oplus \underline{N} \arrow[r] \arrow[d, "h"] & \underline{N} \arrow[r] \arrow[d, "\mathrm{id}_N"] & 0 \\
0 \arrow[r] & \underline{M} \arrow[r] & {[M \oplus N, \left(\begin{smallmatrix} \tau_M & u \\ 0 & \tau_N \end{smallmatrix}\right)]} \arrow[r] & \underline{N} \arrow[r] & 0
\end{tikzcd}$$

where h is a morphism in $\mathcal{M}_R$. Since the diagram commutes in the category of $A \otimes R$-modules, it follows that h is of the form $\left(\begin{smallmatrix} \mathrm{id}_M & f \\ 0 & \mathrm{id}_N \end{smallmatrix}\right)$ for an $A \otimes R$-linear map $f : N \to M$. Because it is a diagram in $\mathcal{M}_R$, it further requires commuting τ-action which translates to

$$\begin{pmatrix} \tau_M & u \\ 0 & \tau_N \end{pmatrix} \tau^* \begin{pmatrix} \mathrm{id}_M & f \\ 0 & \mathrm{id}_N \end{pmatrix} = \begin{pmatrix} \mathrm{id}_M & f \\ 0 & \mathrm{id}_N \end{pmatrix} \begin{pmatrix} \tau_M & 0 \\ 0 & \tau_N \end{pmatrix}.$$

The above equation amounts to $u = f \circ \tau_N - \tau_M \circ \tau^* f$, and hence

$$\ker(\iota) = \{ f \circ \tau_N - \tau_M \circ \tau^* f \mid f \in \mathrm{Hom}_{A \otimes R}(N, M) \}.$$

This concludes. $\square$

Corollary 3.2. *Suppose that R is Noetherian. Let $\underline{N}$ be an A-motive over R and let $f : \underline{M} \to \underline{M}''$ be a surjective morphism in $\mathcal{M}_R$. Then, the induced map $\mathrm{Ext}^1_{\mathcal{M}_R}(\underline{N}, \underline{M}) \to \mathrm{Ext}^1_{\mathcal{M}_R}(\underline{N}, \underline{M}'')$ is onto.*

Proof. As R is Noetherian, so is $A \otimes R$. Because $\tau^* N$ is finite locally-free over $A \otimes R$, it is projective. The induced morphism $\mathrm{Hom}_{A\otimes R}(\tau^* N, M)[\mathfrak{j}^{-1}] \to \mathrm{Hom}_{A\otimes R}(\tau^* N, M'')[\mathfrak{j}^{-1}]$ is thus surjective. We conclude by Proposition 3.1. $\square$

Let $\underline{N}$ be an A-motive over R. The functor $\mathrm{Hom}_{\mathcal{M}_R}(\underline{N}, -)$ from the category $\mathcal{M}_R$ to the category Mod_A of A-modules is left-exact and therefore right-derivable. Because $\mathcal{M}_R$ is an exact category, the higher extensions modules $\mathrm{Ext}^i_{\mathcal{M}_R}(\underline{N}, \underline{M})$ are computed by the cohomology of $\mathrm{RHom}_{\mathcal{M}_R}(\underline{N}, \underline{M})$. This implies that, given a short exact sequence in $\mathcal{M}_R$

$$0 \longrightarrow \underline{M}' \longrightarrow \underline{M} \longrightarrow \underline{M}'' \longrightarrow 0,$$

we have a long-exact sequence of A-modules given by its cohomology

$$\mathrm{Hom}_{\mathcal{M}_R}(\underline{N}, \underline{M}') \hookrightarrow \mathrm{Hom}_{\mathcal{M}_R}(\underline{N}, \underline{M}) \to \mathrm{Hom}_{\mathcal{M}_R}(\underline{N}, \underline{M}'') \to \mathrm{Ext}^1_{\mathcal{M}_R}(\underline{N}, \underline{M}') \to \ldots$$

We deduce the following from Corollary 3.2.

Proposition 3.3. *Suppose that R is Noetherian. The modules $\mathrm{Ext}^i_{\mathcal{M}_R}(\underline{N},\underline{M})$ vanish for $i>1$. In particular, the cohomology of $\mathrm{RHom}_{\mathcal{M}_R}(\underline{N},\underline{M})$ is represented by the complex of A-modules*

$$\left[\mathrm{Hom}_{A\otimes R}(N,M) \overset{\tau_N^\vee-\tau_M}{\longrightarrow} \mathrm{Hom}_{A\otimes R}(\tau^*N,M)[\mathfrak{j}^{-1}]\right] \tag{3.1}$$

placed in degree 0 and 1.

Proof. Let C be the complex (3.1). We have

$$\begin{aligned}\mathrm{Ext}^0_{\mathcal{M}_R}(\underline{N},\underline{M}) = \mathrm{Hom}_{\mathcal{M}_R}(\underline{N},\underline{M}) &= \{f\in \mathrm{End}_{A\otimes R}(M,N)\mid f\circ\tau_N=\tau_M\circ\tau^*f\}\\ &= \ker(\tau_N^\vee-\tau_M)=H^0(C).\end{aligned}$$

By Proposition 3.1, $\mathrm{Ext}^1_{\mathcal{M}_R}(\underline{N},\underline{M})\cong H^1(C)$. By Corollary 3.2, the functor $\mathrm{Ext}^1_{\mathcal{M}_R}(N,-)$ is right-exact. By [PetSt][Lem. A.33], we deduce that $\mathrm{Ext}^i_{\mathcal{M}_R}(\underline{N},\underline{M})\cong H^i(C)=0$ for $i\geq 2$. □

Let $\underline{N}$ be an nonzero A-motive over R. The canonical morphism of A-motives

$$\mathbf{1}_N:\mathbb{1}\to \underline{N}\otimes\underline{N}^\vee=\mathrm{Hom}(\underline{N},\underline{N}),\quad a\longmapsto a\cdot \mathrm{id}_{\underline{N}}$$

induces functorial isomorphisms for all $i\geq 0$:

$$\mathrm{Ext}^i_{\mathcal{M}_R}(\underline{N},\underline{M})\xrightarrow{\sim}\mathrm{Ext}^i_{\mathcal{M}_R}(\mathbb{1},\underline{M}\otimes\underline{N}^\vee). \tag{3.2}$$

In particular, there is no loss of generality in considering extension modules of the form $\mathrm{Ext}^i_{\mathcal{M}_R}(\mathbb{1},\underline{M})$. From now on, we will be interested mainly in extension modules of the latter form. We shall restate the main results of this section in this case (repeated from Theorem A in Chapter 0).

Theorem 3.4. *Suppose that R is Noetherian, and let $\underline{M}$ be an A-motive over R. The cohomology of $\mathrm{RHom}_{\mathcal{M}_R}(\mathbb{1},\underline{M})$ is computed by the cohomology of the complex of A-modules*

$$\left[M\overset{\mathrm{id}-\tau_M}{\longrightarrow} M[\mathfrak{j}^{-1}]\right]$$

placed in degree 0 and 1. The induced A-module isomorphism

$$\iota:\frac{M[\mathfrak{j}^{-1}]}{(\mathrm{id}-\tau_M)(M)}\xrightarrow{\sim}\mathrm{Ext}^1_{\mathcal{M}_R}(\mathbb{1},\underline{M})$$

is given explicitely by mapping the class of $m\in M[\mathfrak{j}^{-1}]$ to the class of the extension

$$0\to\underline{M}\to[M\oplus(A\otimes R),\left(\begin{smallmatrix}\tau_M & m\\ 0 & 1\end{smallmatrix}\right)]\to\mathbb{1}\to 0.$$

We end this subsection by some remarks.

Remark 3.5. There is a harmless abuse of notations in the above theorem, where, in the matrix $\left(\begin{smallmatrix}\tau_M & m\\ 0 & 1\end{smallmatrix}\right)$, 1 rather designates the $A\otimes R$-linear map $\varepsilon:\tau^*(A\otimes R)\to A\otimes R$ which maps $a\otimes_\tau b\in(A\otimes R)\otimes_{\tau,A\otimes R}(A\otimes R)$ to $a\tau(b)$, and m rather designates $m\cdot\varepsilon$.

Remark 3.6. Let $\tilde{\mathcal{M}}_R$ be the category of motives over R up to isogenies (see Definition 1.10). From the identity $\operatorname{Hom}_{\tilde{\mathcal{M}}_R}(-,-) = \operatorname{Hom}_{\mathcal{M}_R}(-,-) \otimes_A K$, we deduce that the extension groups of $\mathbb{1}$ by $\underline{M}$ in the K-linear category $\tilde{\mathcal{M}}_R$ are computed by the complex

$$\left[M \otimes_A K \stackrel{\mathrm{id}-\tau_M}{\longrightarrow} M[\mathfrak{j}^{-1}] \otimes_A K \right].$$

Remark 3.7. Let S be a Noetherian R-algebra, and consider the *restriction of scalar* functor $\operatorname{Res}_{S/R} : \mathcal{M}_S \to \mathcal{M}_R$. Given $\underline{M}$ an A-motive over S, then $\operatorname{Ext}^1_{\mathcal{M}_S}(\mathbb{1}_S, \underline{M})$ is naturally isomorphic to $\operatorname{Ext}^1_{\mathcal{M}_R}(\mathbb{1}_R, \operatorname{Res}_{S/R} \underline{M})$.

3.2 Extension modules in $\mathcal{MM}_F$

Let F be a field containing $\mathbb{F}$ and consider an $\mathbb{F}$-algebra morphism $\kappa : A \to F$. As we showed in Proposition 1.36, the category $\mathcal{MM}_F$ of mixed A-motives over F is exact, and we can consider the extension groups $\operatorname{Ext}^i_{\mathcal{MM}_F}(\mathbb{1}, -)$. Given a mixed A-motive $\underline{M}$ and $i > 0$, the module $\operatorname{Ext}^i_{\mathcal{MM}_F}(\mathbb{1}, \underline{M})$ is a submodule of $\operatorname{Ext}^i_{\mathcal{M}_F}(\mathbb{1}, \underline{M})$. In particular, $\operatorname{Ext}^i_{\mathcal{MM}_F}(\mathbb{1}, \underline{M}) = 0$ for $i \notin \{0, 1\}$. The next propostion can be used to spot some cases of equality of the inclusion $\operatorname{Ext}^1_{\mathcal{MM}_F}(\mathbb{1}, \underline{M}) \subset \operatorname{Ext}^i_{\mathcal{M}_F}(\mathbb{1}, \underline{M})$.

Proposition 3.8. *Let* $0 \to \underline{M}' \stackrel{i}{\to} \underline{M} \stackrel{p}{\to} \underline{M}'' \to 0$ *be an exact sequence of A-motives in $\mathcal{M}_F$.*

1. *If $\underline{M}$ is mixed, so are $\underline{M}'$ and $\underline{M}''$.*
2. *If $\underline{M}'$ and $\underline{M}''$ are mixed, and if the smallest weight of $\underline{M}''$ is strictly bigger than the biggest weight of $\underline{M}'$, then $\underline{M}$ is mixed.*

Proof. Point 1 is a reformulation of Lemma 1.35.
We move to point 2. Let $\{\mu_1, ..., \mu_s\}$ be the union of the sets of weights for $\underline{M}$, $\underline{M}'$ and $\underline{M}''$ sorted by increasing order. Let $i \in \{1, ..., s-1\}$. By Proposition 1.37, the sequence of $K \otimes F$-modules

$$0 \longrightarrow (W_{\mu_i} M') \otimes_A K \longrightarrow (W_{\mu_i} M) \otimes_A K \longrightarrow (W_{\mu_i} M'') \otimes_A K \longrightarrow 0$$

is exact. Let $s_i : (W_{\mu_i} M'') \otimes_A K \to (W_{\mu_i} M) \otimes_A K$ be a section of the above. We have decompositions of $K \otimes F$-modules

$$W_{\mu_{i+1}} M \cong \operatorname{Gr}_{\mu_{i+1}} M \oplus W_{\mu_i} M, \quad \text{and} \quad W_{\mu_{i+1}} M'' \cong \operatorname{Gr}_{\mu_{i+1}} M'' \oplus W_{\mu_i} M''.$$

Let $s_{i+1} : W_{\mu_{i+1}} M'' \to W_{\mu_{i+1}} M$ be the the linear morphism $g_{i+1} \oplus s_i$ where $g_{i+1} : \operatorname{Gr}_{\mu_{i+1}} M'' \to \operatorname{Gr}_{\mu_{i+1}} M$ is a splitting of

$$0 \longrightarrow (\operatorname{Gr}_{\mu_{i+1}} M') \otimes_A K \longrightarrow (\operatorname{Gr}_{\mu_{i+1}} M) \otimes_A K \longrightarrow (\operatorname{Gr}_{\mu_{i+1}} M'') \otimes_A K \longrightarrow 0.$$

By induction on i, we construct a splitting $s : M'' \otimes_A K \to M \otimes_A K$ of

$$0 \longrightarrow M' \otimes_A K \longrightarrow M \otimes_A K \longrightarrow M'' \otimes_A K \longrightarrow 0$$

preserving the weight filtration.

Extending scalars from $K \otimes F \to \mathcal{B}_\infty(F)$, we obtain a splitting of $\mathcal{B}_\infty(F)$-modules

$$(\mathcal{I}_\mu(M))_\mu = (i(\mathcal{I}_\mu(M')) \oplus s(\mathcal{I}_\mu(M'')))_\mu.$$

The filtration $(i(W_\mu M') \oplus s(W_\mu M''))_{\mu\in\mathbb{Q}}$ seems a good candidate to be the weight filtration on $\underline{M}$. It only remains to check that elements of this family indeed define A-motives. The action of τ_M on $i(M') \oplus s(M'')$ decomposes as $\left(\begin{smallmatrix}\tau_{M'} & v \\ 0 & \tau_{M''}\end{smallmatrix}\right)$ for a certain morphism $v : \tau^*M''[\mathfrak{j}^{-1}] \to M'[\mathfrak{j}^{-1}]$ of $A \otimes F[\mathfrak{j}^{-1}]$-modules. From the weight assumption, the fact that v preserves the weight filtration on $\underline{M}''$ and $\underline{M}'$ is automatic. Therefore $(i(W_\mu M')\oplus s(W_\mu M''), \left(\begin{smallmatrix}\tau_{M'} & v \\ 0 & \tau_{M''}\end{smallmatrix}\right))$ defines a sub-A-motive of $\underline{M}$ for all $\mu \in \mathbb{Q}$. $\square$

Proposition 3.8 implies that $\operatorname{Ext}^1_{\mathcal{M}_F}(\underline{M}'', \underline{M}') = \operatorname{Ext}^1_{\mathcal{MM}_F}(\underline{M}'', \underline{M}')$ when the weights of $\underline{M}''$ are bigger than the biggest weight of $\underline{M}'$. In general this is not true. In this direction we record:

Proposition 3.9. *Let* $0 \to \underline{M}' \to \underline{M} \to \underline{M}'' \to 0$ *be an exact sequence of A-motives where $\underline{M}'$ and $\underline{M}''$ are mixed. We assume that all the weights of $\underline{M}''$ are strictly smaller than the smallest weight of $\underline{M}'$. Then, the sequence is torsion in* $\operatorname{Ext}^1_{\mathcal{M}_F}(\underline{M}'', \underline{M}')$ *if, and only if, $\underline{M}$ is mixed.*

Proof of Proposition 3.9. Taking $\underline{N} := \underline{M}' \otimes (\underline{M}'')^\vee$, we can assume that the exact sequence is of the form $(S) : \ 0 \to \underline{N} \to \underline{E} \to \mathbb{1} \to 0$, $\underline{N}$ having positive weights In view of Theorem 3.4, we may assume that $\underline{E}$ is of the form $[M \oplus (A \otimes F), \left(\begin{smallmatrix}\tau_M & u \\ 0 & 1\end{smallmatrix}\right)]$ for some $u \in M[\mathfrak{j}^{-1}]$. Note that 0 is a weight of $\underline{E}$, the smallest.

If $\underline{E}$ is mixed, $\underline{E}$ contains a sub-A-motive $\underline{L} = (L, \tau_M)$ of weight 0 which is isomorphic to $\mathbb{1}$. Let $(m \oplus a) \in M \oplus (A \otimes F)$ be a generator of L over $A \otimes F$. We have

$$\begin{pmatrix}\tau_M & u \\ 0 & 1\end{pmatrix}\begin{pmatrix}\tau^*m \\ \tau^*a\end{pmatrix} = \begin{pmatrix} m \\ a\end{pmatrix}.$$

This amounts to $a \in A$ and $au \in \operatorname{im}(\mathrm{id} - \tau_M)$, and then that $a[\underline{E}] = 0$ in $\operatorname{Ext}^1_{\mathcal{M}_F}(\mathbb{1}, \underline{N})$. Conversely, if there exists a nonzero $a \in A$ such that $a[\underline{E}]$ is split, Theorem 3.4 implies that there exists $m \in N$ such that $au = m - \tau_M(\tau^*m)$. The nonzero $A \otimes F$-module L generated by $m \oplus a$ together with τ_M defines a sub-A-module of $\underline{E}$ isomorphic to $\mathbb{1}$. For all $\mu \in \mathbb{Q}$, we define the $A \otimes F$-module

$$W_\mu E := W_\mu M + \mathbf{1}_{\mu\geq 0} L$$

where $(W_\mu \underline{M})_{\mu\in\mathbb{Q}}$ is the weight filtration of $\underline{M}$. It is easy to see that $W_\mu \underline{E} := (W_\mu E, \tau_M)$ defines a sub-A-motive of $\underline{E}$ and that $(\mathcal{I}_\infty(W_\mu \underline{M}))_{\mu\in\mathbb{Q}}$ coincides with the slope filtration of $\mathcal{I}_\infty(\underline{E})$. Hence, $\underline{E}$ is mixed. $\square$

Remark 3.10. Under the same hypothesis, Proposition 3.9 can be rephrased into

$$\operatorname{Ext}^1_{\mathcal{M}_F}(\underline{M}'', \underline{M}')^{\mathrm{tors}} = \operatorname{Ext}^1_{\mathcal{MM}_F}(\underline{M}'', \underline{M}').$$

In particular, the K-vector space $\mathrm{Ext}^1_{\widetilde{\mathcal{MM}_F}}(\underline{M}'', \underline{M}')$ vanishes. The latter is only conjectured to be true in the number fields setting ([Del, §1.3]).

According to the latter reference, it is believed that the extension group of two classical pure motives motives of the same weight vanishes. This is not expectable for mixed Anderson A-motives by the next proposition:

Proposition 3.11. *Let $\underline{M}$ be a pure A-motive of weight* 0 *over F. We have*

$$\mathrm{Ext}^1_{\mathcal{MM}_F}(\mathbb{1}, \underline{M}) = \mathrm{Ext}^1_{\mathcal{M}_F}(\mathbb{1}, \underline{M}).$$

Proof. As the weights remain unchanged by base change, we can assume that F is a perfect field. Let $[\underline{E}] \in \mathrm{Ext}^1_{\mathcal{M}_F}(\mathbb{1}, \underline{M})$. By the Dieudonné-Manin Theorem 1.24, 0 is the only weight of $\underline{E}$. In particular, $\mathcal{I}_\infty(\underline{E})$ is pure of slope 0. $\square$

As a consequence of Propositions 3.8, 3.9 and 3.11, we record:

Corollary 3.12. *Let $\underline{M}$ be a pure A-motive over F of weight μ. We have:*

$$\mathrm{Ext}^1_{\mathcal{MM}_F}(\mathbb{1}, \underline{M}) = \begin{cases} \mathrm{Ext}^1_{\mathcal{M}_F}(\mathbb{1}, \underline{M}) & \text{if } \mu \leq 0 \\ \mathrm{Ext}^1_{\mathcal{M}_F}(\mathbb{1}, \underline{M})^{tors} & \text{if } \mu > 0. \end{cases}$$

3.3 Extensions with good reduction

In this section, we introduce and study extensions with good reduction for A-motives over local and global function fields. Our philosophy follows Scholl's ones in the number fields setting [Sch].

We first describe the local situation. Let E be a local field containing $\mathbb{F}$ and let $\mathcal{O}_E$ be its valuation ring. We let k be the residue field of E (it is a finite extension of $\mathbb{F}$). For a uniformizer $\varpi \in E$, we have the identifications $E = k(\!(\varpi)\!)$ and $\mathcal{O}_E = k[\![\varpi]\!]$. Let E^s be an algebraic closure of E and let E^{ur} be the maximal unramified extension of E in E^s. We have $E^{\mathrm{ur}} = \bar{k}(\!(\varpi)\!)$ where $\bar{k}$ is the algebraic closure of k in E^s. Finally, we let $G_E = \mathrm{Gal}(E^s|E)$ be the absolute Galois group of E and let $I_E = \mathrm{Gal}(E^s|E^{\mathrm{ur}})$ be the inertia group of E.

Let $\kappa : A \to \mathcal{O}_E$ be an $\mathbb{F}$-algebra morphism, and let $\mathfrak{m} \subset A$ be a maximal ideal such that $\kappa(\mathfrak{m})\mathcal{O}_E = \mathcal{O}_E$ (hence, condition $(C_\mathfrak{m})$ is satisfied and the results of Subsection 2.2.2 apply). Let $\mathcal{O}_\mathfrak{m}$ be the completion of A at $\mathfrak{m}$ endowed with the trivial action of G_E. Let $\underline{M}$ be an A-motive over E.

Given an extension $[\underline{E}] :\ 0 \to \underline{M} \to \underline{E} \to \mathbb{1} \to 0$ in $\mathrm{Ext}^1_{\mathcal{M}_E}(\mathbb{1}, \underline{M})$, the sequence of $\mathcal{O}_\mathfrak{m}[G_E]$-modules $[T_\mathfrak{m}\underline{E}] :\ 0 \to T_\mathfrak{m}\underline{M} \to T_\mathfrak{m}\underline{E} \to \mathcal{O}_\mathfrak{m} \to 0$ is exact by Corollary 1.46. In the category of $\mathcal{O}_\mathfrak{m}$-modules, the latter sequence splits and the choice of a splitting yields an $\mathcal{O}_\mathfrak{m}$-module isomorphism $T_\mathfrak{m}\underline{E} \cong T_\mathfrak{m}\underline{M} \oplus \mathcal{O}_\mathfrak{m}$ on which $\sigma \in G_E$ acts by a matrix of the form $\left(\begin{smallmatrix} \sigma & c \\ 0 & 1 \end{smallmatrix}\right)$. The mapping $\sigma \mapsto c$ defines a cocycle $c_{\underline{E}} : G_E \to T_\mathfrak{m}\underline{M}$ which, up to principal cocycles, does not depend on the choice of a splitting. The association $[\underline{E}] \mapsto c_{\underline{E}}$ defines an A-linear map

$$\operatorname{Ext}^1_{\mathcal{M}_E}(\mathbb{1}, \underline{M}) \longrightarrow H^1(G_E, T_{\mathfrak{m}}\underline{M}).$$

We call the above the $\mathfrak{m}$-*adic realization map of* $\underline{M}$. We shall rather be interested in the restriction of the $\mathfrak{m}$-adic realization map to the inertia subgroup:

$$r_{\mathfrak{m}} : \operatorname{Ext}^1_{\mathcal{M}_E}(\mathbb{1}, \underline{M}) \longrightarrow H^1(I_E, T_{\mathfrak{m}}\underline{M}). \tag{3.3}$$

Definition 3.13. We let $\operatorname{Ext}^1_{\mathcal{M}_E,\mathfrak{m}_E}(\mathbb{1}, \underline{M})$ be the kernel of $r_{\mathfrak{m}}$. We say that the extension $[\underline{E}]$ has *good reduction* if $[\underline{E}]$ lies in $\operatorname{Ext}^1_{\mathcal{M}_E,\mathfrak{m}_E}(\mathbb{1}, \underline{M})$.

Remark 3.14. This definition is the analogue of Scholl's notion of *extensions of mixed motives over* $\mathbb{Z}$ in the number fields setting (see [Sch, §III]). It will follow from our Theorem 3.15 below that $\operatorname{Ext}^1_{\mathcal{M}_E,\mathfrak{m}_E}(\mathbb{1}, \underline{M})$ does not depend on the choice of $\mathfrak{m}$. The analogous result for number fields is still a conjecture ([Sch, §III,Rmk.(i)]).

The next result implies Theorem C of Chapter 0.

Theorem 3.15. *Let* $\underline{M}$ *be an* A*-motive over* E *and let* $M_{\mathcal{O}}$ *be its maximal* $\mathcal{O}_E$*-model. The morphism* ι *of Theorem 3.4 induces an isomorphism of* A*-modules*

$$\frac{M_{\mathcal{O}}[\mathfrak{j}^{-1}]}{(\operatorname{id} - \tau_M)(M_{\mathcal{O}})} \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{M}_E,\mathfrak{m}_E}(\mathbb{1}, \underline{M}).$$

We now present a similar result for the *global* situation. Let F be a finite extension of K and let $\mathcal{O}_F$ be the integral closure of A in F. We let $\kappa : A \to \mathcal{O}_F$ denote the inclusion. We fix S to be a finite set of nonzero prime ideals of $\mathcal{O}_F$ and consider the Dedekind subring $R := \mathcal{O}_F[S^{-1}]$ of F whose spectrum is $(\operatorname{Spec} \mathcal{O}_F) \setminus S$.

We let F^s be a separable closure of F and let $G_F = \operatorname{Gal}(F^s|F)$ be the absolute Galois group of F. We let R^s denote the integral closure of R in F^s. For $\mathfrak{p}$ a maximal ideal in R, we let $R_{\mathfrak{p}}$ be the completion of R at $\mathfrak{p}$ and we let $F_{\mathfrak{p}}$ be its fraction field. If $\mathfrak{P}$ denotes a maximal ideal in R^s above $\mathfrak{p}$, we let

$$G_{\mathfrak{P}} := \{\rho \in G_F \mid \rho\mathfrak{P} = \mathfrak{P}\}$$

be the decomposition group of $\mathfrak{P}$. Any automorphism ρ in $G_{\mathfrak{P}}$ induces an automorphism $\bar{\rho}$ of $\mathbb{F}_{\mathfrak{P}} := R^s/\mathfrak{P}$ leaving $\mathbb{F}_{\mathfrak{p}} := R/\mathfrak{p}$ invariant. Note that $\mathbb{F}_{\mathfrak{P}}$ is an algebraic closure of $\mathbb{F}_{\mathfrak{p}}$. Further, it is well-known that the group morphism

$$G_{\mathfrak{P}} \longrightarrow \operatorname{Gal}(\mathbb{F}_{\mathfrak{P}}|\mathbb{F}_{\mathfrak{p}}), \quad \rho \longmapsto \bar{\rho}$$

is surjective ([Neu, §I,Prop.9.2]) and that its kernel is $I_{\mathfrak{P}}$ the inertia group of $\mathfrak{P}$ ([Neu, §I,Def.9.5]). If $\mathfrak{P}'$ is a maximal ideal of R^s above $\mathfrak{p}$, then $I_{\mathfrak{P}}$ and $I_{\mathfrak{P}'}$ are conjugated.

For any maximal ideal $\mathfrak{m} \subset A$ such that $\mathfrak{p} \cap A \neq \mathfrak{m}$ (we say that $\mathfrak{p}$ *is not above* $\mathfrak{m}$), we have $\mathfrak{m}R_{\mathfrak{p}} = R_{\mathfrak{p}}$. The morphism (3.3) in the local function field situation induces

$$r_{\mathfrak{P},\mathfrak{m}} : \operatorname{Ext}^1_{\mathcal{M}_F}(\mathbb{1}, \underline{M}) \longrightarrow H^1(I_{\mathfrak{P}}, T_{\mathfrak{m}}\underline{M}). \tag{3.4}$$

We denote by $\operatorname{Ext}^1_{\mathcal{M}_F,\mathfrak{p}}(\mathbb{1}, \underline{M})$ the kernel of $r_{\mathfrak{P},\mathfrak{m}}$. It does not depend on the choice of $\mathfrak{P}$ above $\mathfrak{p}$, and by Theorem 3.15, it does not depend on the choice of $\mathfrak{m}$.

Definition 3.16. We say that $[\underline{E}]$ has *good reduction at the maximal ideal* $\mathfrak{p}$ if $[\underline{E}]$ lies in $\operatorname{Ext}^1_{\mathcal{M}_F,\mathfrak{p}}(\mathbb{1}, \underline{M})$.

In other terms, $[\underline{E}]$ has good reduction at $\mathfrak{p}$ if and only if $[\underline{E}_{F_{\mathfrak{p}}}]$ has good reduction in the sense of Definition 3.13.

Definition 3.17. We say that $[\underline{E}] \in \operatorname{Ext}^1_{\mathcal{M}_F}(\mathbb{1}, \underline{M})$ has *everywhere good reduction* if $[\underline{E}]$ has good reduction at $\mathfrak{p}$ for all maximal ideals $\mathfrak{p}$ of R. We let $\operatorname{Ext}^1_{\mathcal{M}_F,R}(\mathbb{1}, \underline{M})$ be the sub-A-module consisting of extensions having everywhere good reduction, that is,

$$\operatorname{Ext}^1_{\mathcal{M}_F,R}(\mathbb{1}, \underline{M}) = \bigcap_{\mathfrak{p} \subset R} \operatorname{Ext}^1_{\mathcal{M}_F,\mathfrak{p}}(\mathbb{1}, \underline{M})$$

where the intersection is indexed over nonzero prime ideals of R.

Repeated from Theorem D, we state:

Theorem 3.18. *Let $\underline{M}$ be an A-motive over F and let M_R be its maximal R-model. The morphism ι of Theorem 3.4 induces an isomorphism of A-modules*

$$\frac{M_R[\mathfrak{j}^{-1}]}{(\mathrm{id} - \tau_M)(M_R)} \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{M}_F,R}(\mathbb{1}, \underline{M}).$$

The next sections are devoted to the proof of Theorems 3.15 and 3.18.

3.3.1 Preliminaries for Theorem 3.15

We recall the notations used in the context of Frobenius spaces (Subsection 2.1): E is a complete local field containing $\mathbb{F}$ whose residual field is denoted by k, v_E is its valuation, $\mathcal{O}_E$ is its ring of integers, $\mathfrak{m}_E$ is its maximal ideal, ϖ is a uniformizer. We let $\bar{E}$ be an algebraic closure of E, E^s the separable closure of E in $\bar{E}$ and E^{ur} the subfield of $\bar{E}$ given by the union of the unramified field extensions of E in $\bar{E}$.

We recall some notations from section 1.2. Given $\mathfrak{m}$ a maximal ideal of A, we let $\mathcal{O}_{\mathfrak{m}}$ be the completion of A at $\mathfrak{m}$. For L a subfield of $\bar{E}$, we let $\mathcal{A}_{\mathfrak{m}}(L)$ denote the completion of $A \otimes L$ at $\mathfrak{m} \otimes L$.

Let $\underline{M}$ be an A-motive over E and let $M_{\mathcal{O}}$ be its maximal $\mathcal{O}_E$-model. By Theorem 3.4, every extension $[\underline{E}]$ of $\mathbb{1}$ by $\underline{M}$ in $\mathcal{M}_E$ is of the form $\iota(m) = [M \oplus (A \otimes E), (\begin{smallmatrix} \tau_M & m \\ 0 & 1 \end{smallmatrix})]$ for some $m \in M[\mathfrak{j}^{-1}]$. Let us precise the morphism (3.3) under this explicit description.

The $\mathfrak{m}$-adic realization $T_{\mathfrak{m}}\underline{E}$ of $\underline{E} = \iota(m)$ is the $\mathcal{O}_{\mathfrak{m}}[G_E]$-module consisting of solutions $\xi \oplus a \in M \otimes_{A\otimes E} \mathcal{A}_{\mathfrak{m}}(E^s) \oplus \mathcal{A}_{\mathfrak{m}}(E^s)$ of the equation

$$\begin{pmatrix} \tau_M & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau^*\xi \\ \tau^* a \end{pmatrix} = \begin{pmatrix} \xi \\ a \end{pmatrix}$$

(see Definition 1.42). It follows that $a \in \mathcal{O}_{\mathfrak{m}}$ and that ξ satisfies $\xi - \tau_M(\tau^*\xi) = am$. A splitting of $[T_{\mathfrak{m}}\underline{E}]$ as a sequence of $\mathcal{O}_{\mathfrak{m}}$-modules corresponds to the choice of a particular solution $\xi_m \in M \otimes_{A\otimes E} \mathcal{A}_{\mathfrak{m}}(E^s)$ of $\xi - \tau_M(\tau^*\xi) = m$ (whose existence is provided by Proposition 1.45). We then have

$$T_{\mathfrak{m}}\underline{M} \oplus \mathcal{O}_{\mathfrak{m}} \xrightarrow{\sim} T_{\mathfrak{m}}\underline{E}, \quad (\omega, a) \longmapsto (\omega + a\xi_m, a).$$

It follows that the morphism (3.3) maps $\iota(m)$ for $m \in M[\mathfrak{j}^{-1}]$ to the cocycle $(\sigma \mapsto \xi_m^{\sigma} - \xi_m)$, where ξ_m is any solution in $M \otimes_{A\otimes E} \mathcal{A}_{\mathfrak{m}}(E^s)$ of the equation $\xi - \tau_M(\tau^*\xi) = m$.

Therefore, an extension $[M \oplus (A \otimes E), (\begin{smallmatrix} \tau_M & m \\ 0 & 1 \end{smallmatrix})]$ has good reduction if and only if, given a solution $\xi_m \in M \otimes_{A\otimes E} \mathcal{A}_{\mathfrak{m}}(E^s)$ of $\xi - \tau_M(\tau^*\xi) = m$, the cocycle $(\sigma \mapsto \xi_m^{\sigma} - \xi_m)$ is trivial in $H^1(I_E, T_{\mathfrak{m}}\underline{M})$. This amounts to say that there exists $\omega \in T_{\mathfrak{m}}\underline{M}$ for which $(\xi_m - \omega)^{\sigma} = \xi_m - \omega$ for all $\sigma \in I_E$. This can be rephrased into the following:

Proposition 3.19. *The following are equivalent:*

(a) The extension $\iota(m) = [M \oplus (A \otimes E), (\begin{smallmatrix} \tau_M & m \\ 0 & 1 \end{smallmatrix})]$ for $m \in M[\mathfrak{j}^{-1}]$ has good reduction,

(b) There exists a separable unramified extension E' of E such that the equation $\xi - \tau_M(\tau^*\xi) = m$ admits a solution ξ in $M \otimes_{A\otimes E} \mathcal{A}_{\mathfrak{m}}(E')$.

This motivates the study of the equation $\xi - \tau_M(\tau^*\xi) = m$, and the search of solutions ξ inside $M \otimes_{A\otimes E} \mathcal{A}_{\mathfrak{m}}(E^{\mathrm{ur}})$ rather than inside $M \otimes_{A\otimes E} \mathcal{A}_{\mathfrak{m}}(E^s)$. We discuss this problem in the setting of Frobenius spaces in the next subsection.

3.3.2 Artin-Schreier type equations in Frobenius modules

Let (V, φ) be a Frobenius space over E and let L be its maximal integral model. The next proposition is an essential ingredient in the proof of Theorem 3.15.

Proposition 3.20. *Let $x \in V$. The following are equivalent:*

(i) there exists $y \in V \otimes_E E^{ur}$ such that $y - \varphi(\sigma^* y) = x$,

(ii) $x \in L + (\mathrm{id}_V - \varphi)(V)$.

Proposition 3.20 is a useful criterion to determine the existence of solutions y over E^{ur} of $y - \varphi(\sigma^* y) = x$. This subsection is devoted to its proof. We begin by some preliminary results Lemmas 3.21, 3.22 and 3.23, the first one being a well-known particular case of what we aim to prove.

Lemma 3.21. *Let $x \in E$ and $y \in \bar{E}$ be such that $y - y^q = x$. Then, the following are equivalent:*

(a) *the extension $E(y)/E$ is unramified,*

(b) $x \in \mathcal{O}_E + (\mathrm{id} - \mathrm{Frob}_q)(E)$.

Proof. Let us show that (a) implies (b) Assume that x does not belong to $\mathcal{O}_E + (\mathrm{id} - \mathrm{Frob}_q)(E)$. Let e be the ramification index of the extension $E(y)/E$ so that $v_{E(y)} = e v_E$ over E. Up to the substitution of x by $x + k - k^q$ and y by $y + k$ for $k \in E$ (this does not change $E(y)$), we can assume that $v_E(x) < 0$ and that $v_E(x)$ is not divisible by q. As $y - y^q = x$, we have $v_{E(y)}(y) < 0$ and hence $q v_L(y) = v_L(x) = e v_E(x)$. As q does not divide $v_E(x)$, we have $e > 1$ and the extension $E(y)/E$ is ramified.

We now show that (b) implies (a). Assume that x belongs to $\mathcal{O}_E + (\mathrm{id} - \mathrm{Frob}_q)(E)$. Up to replacing x by $x + k - k^q$ for some $k \in E$ we can assume that $v_E(x) \geq 0$. Let $P(X)$ be the minimal polynomial of y over E. It has integral coefficients. By Hensel's Lemma, if k' is a finite extension of k in which $P(X)$ splits modulo (ϖ), then $P(X)$ splits in $k'(\!(\varpi)\!)$ ($P(X)$ and $P'(X)$ are coprime since $P(X)$ divides $X^q - X + x$). We then have $k(\!(\pi)\!) \subset E(y) \subset k'(\!(\varpi)\!)$, where $k'(\!(\pi)\!)$ is a unramified extension of $k(\!(\varpi)\!)$. Therefore, $E(y)/E$ is unramified. $\square$

The next lemma is a generalization of Lang's isogeny Theorem for prescribed ranks.

Lemma 3.22. *Let k be a field containing $\mathbb{F}$, and let $M \in \mathcal{M}_\ell(k)$ be a matrix with rank r. There exists an finite extension k' of k and $G \in \mathrm{GL}_\ell(k')$ such that*

$$M = G^{-1} I_{\ell,r} G^{\sigma}$$

where $I_{\ell,r}$ denotes the diagonal matrix of rank r with diagonal $(1, ..., 1, 0, ..., 0)$.

Proof. Our proof is inspired by [Kat1, Cor 1.1.2]. Denote by $\mathcal{M}_\ell(k, r)$ the set of square matrices of size $\ell \times \ell$ of rank r over the field k. $\mathcal{M}_\ell(k, r)$ is a Zariski open subset of

$$\{A \in \mathcal{M}_\ell(k) \mid \det(U) = 0 \text{ for all square submatrices } U \text{ of } A \text{ of size } r+1\} = \bigcup_{r' \leq r} \mathcal{M}_\ell(k, r')$$

and hence is an algebraic variety. If $\bar{k}$ is an algebraic closure of k, $\mathcal{M}_\ell(\bar{k}, r)$ is also irreducible. Indeed, $\mathrm{GL}_\ell(\bar{k})^2$ is irreducible and we have a surjective regular morphism

$$\mathrm{GL}_\ell(\bar{k})^2 \longrightarrow \mathcal{M}_\ell(\bar{k}, r), \quad (P, Q) \longmapsto P I_{\ell,r} Q^{-1}.$$

The morphism

$$\varphi_k : \mathrm{GL}_\ell(k) \longrightarrow \mathcal{M}_\ell(k,r), \quad G \longmapsto G^{-1} I_{\ell,r} G^\sigma$$

is étale as one can see by computing is tangent map. Therefore its image is open. As $\mathcal{M}_\ell(\bar{k}, r)$ is irreducible, $\varphi_{\bar{k}}$ is surjective. □

The next technical lemma will help to use a *Galois descent argument*, reducing the proof of Proposition 3.20 to the case of Lemma 3.21.

Lemma 3.23. *Let E' be a tamely ramified finite Galois extension of E and let (V', φ') be the Frobenius module obtained from (V, φ) by base-change to E'. Let L' be the maximal integral model of (V', φ') and let $x \in V$. If $x \in L' + (\mathrm{id}_{V'} - \varphi')(V')$, then $x \in L + (\mathrm{id}_V - \varphi)(V)$.*

Proof. The group $\mathrm{Gal}(E'|E)$ acts on $V' = V \otimes_E E'$ by $\mathrm{id}_V \otimes \rho$ for $\rho \in \mathrm{Gal}(E'|E)$. We claim that L' is stable under this action. Indeed, for $\rho \in \mathrm{Gal}(E'|E)$ and $\ell \in L'$, $\varphi(\sigma^* \ell^\rho) = \varphi(\sigma^* x)^\rho$ so that $\{\ell^\rho | \ell \in L'\}$ is stable by φ. It is also an $\mathcal{O}_{E'}$-lattice in V', and $\{\ell^\rho | \ell \in L'\} \subset L'$ by maximality of L'.

We show that $H^1(\mathrm{Gal}(E'|E), L') = 0$. If k' denotes the residue field of E', then k'/k is a finite separable extension and there exists $\alpha \in k'$ such that $\mathrm{Tr}_{k'|k}(\alpha) \neq 0$. If e denotes the ramification degree of E'/E, then

$$\mathrm{Tr}_{E'|E}(\alpha) = e \, \mathrm{Tr}_{k'|k}(\alpha) \in k^\times$$

as E'/E is tamely ramified.

Let $c \in H^1(\mathrm{Gal}(E'|E), L')$. By the additive version of Hilbert's 90 Theorem (see [Ser1, §X.1, Prop 1]), $H^1(\mathrm{Gal}(E'|E), V')$ is zero, and there exists $w \in V'$ such that $c(\rho) = w - w^\rho$ for all $\rho \in \mathrm{Gal}(E'|E)$. We have

$$w = \left(\frac{1}{\mathrm{Tr}_{E'|E}(\alpha)} \sum_{\rho \in \mathrm{Gal}(E'|E)} \rho(\alpha) w^\rho \right) - \left(\frac{1}{\mathrm{Tr}_{E'|E}(\alpha)} \sum_{\rho \in \mathrm{Gal}(E'|E)} \rho(\alpha) c(\rho) \right) \in V + L'.$$

In particular, there exists $\ell \in L'$ such that $c(\rho) = w - w^\rho = \ell - \ell^\rho$ for all $\rho \in \mathrm{Gal}(E'|E)$, and c is trivial.

We return to the proof of the lemma. Assume that $x \in L' + (\mathrm{id} - \varphi')(V')$. There exists $a \in L'$, $w \in V'$ such that $x = a + w - \varphi(\sigma^* w)$. From $x \in V$, we have $x^\rho = x$ for all $\rho \in \mathrm{Gal}(E'|E)$. This reads

$$\varphi'(\sigma^*(w - w^\rho)) = (a - a^\rho) + (w - w^\rho).$$

In particular, the $\mathcal{O}_{E'}$-module $L' + (w - w^\rho) \cdot \mathcal{O}_{E'}$ is stable by φ' and defines an integral model for (V', φ'). This implies $L' + \mathcal{O}_{E'}(w - w^\rho) \subset L'$ by maximality and thus $w - w^\rho \in L'$. The map

$$\mathrm{Gal}(E'|E) \longrightarrow L', \quad \rho \longmapsto w - w^\rho$$

defines a cocycle which is trivial by $H^1(\mathrm{Gal}(E'|E),L')=0$. Therefore, there exists $v\in V$ and $b\in L'$ such that $w=v+b$. We thus have

$$x=\ell+v-\varphi(\sigma^*v)$$

where $\ell=a+b-\varphi'(\sigma^*b)\in L'$. Since $\ell^\rho=\ell$ for all $\rho\in\mathrm{Gal}(E'|E)$, we also have $\ell\in V\cap L'\subset L$. It follows that $x\in L+(\mathrm{id}-\varphi)(V)$. □

Proof of Proposition 3.20. Let $\mathbf{v}=(v_1,...,v_\ell)$ be a basis of L and let $F\in\mathrm{GL}_\ell(E)$ with coefficients in $\mathcal{O}_E$ be the matrix of φ written in $\tau^*\mathbf{v}$ and $\mathbf{v}$. By the Lang-Steinberg Theorem for GL_ℓ [Ste, Thm 10.1], there exists a finite Galois extension E'/E and $G\in\mathrm{GL}_\ell(E')$ such that $F=G^{-1}G^\sigma$. Without loss, E' can be chosen so that E'/E is tamely ramified. Indeed, for s big enough the matrix G^{σ^s} has coefficients in a tamely ramified extension $E^t\subset E'$ of E, and we have

$$G=G^{\sigma^s}(FF^\sigma\cdots F^{\sigma^{s-1}})^{-1}\in\mathrm{GL}_\ell(E^t).$$

The relation $F=G^{-1}G^\sigma$ also implies that G has coefficients in $\mathcal{O}_{E'}$.

Let $X\in\mathcal{M}_{\ell1}(E)$ be the column vector expressing the coefficients of x in the basis $\mathbf{v}$. If (i) is satisfied, there exists $Y\in\mathcal{M}_{\ell1}(E^{\mathrm{ur}})$ such that $Y-FY^\sigma=X$ and hence $GY-(GY)^\sigma=GX$. The extension $E'(Y)/E'$ is unramified and Lemma 3.21 implies that $GX\in\mathcal{O}_{E'}^\ell+(\mathrm{id}_\ell-\mathrm{Frob}_q)(E')^\ell$ which is equivalent to $X\in G^{-1}\mathcal{O}_{E'}^\ell+(\mathrm{id}_\ell-F\,\mathrm{Frob}_q)((E')^\ell)$. The lattice $G^{-1}\mathcal{O}_{E'}^\ell$ corresponds to an $\mathcal{O}_{E'}$-lattice in $V\otimes_E E'$ stable by φ'. Because E'/E is tamely ramified, Lemma 3.23 implies that $x\in L+(\mathrm{id}-\varphi)(V)$.

Conversely, if (ii) is satisfied, we write $X=A+W-FW^\sigma$ for some $A\in\mathcal{O}_E^\ell$ and $W\in E^\ell$. It suffices to show that the equation

$$Y-FY^\sigma=A \tag{3.5}$$

admits a solution Y in $(E^{\mathrm{ur}})^\ell$. Let $\bar{A}$ and $\bar{F}$ denote the reduction modulo $\mathfrak{m}_E$ in k of A and F. Let k^s be the separable closure of k given by the residue field of E^{ur}. By Lemma 3.22, there exists $G_0\in\mathrm{GL}_\ell(k^s)$ such that $\bar{F}=G_0^{-1}I_{\ell,r}G_0^\sigma$ for some integer $r\leq\ell$ corresponding to the rank of $\bar{F}$. Let $Y_1\in(k^s)^\ell$ be a solution of $Y_1-I_{\ell,r}Y_1^\sigma=G_0\bar{A}$. Then $Y_0:=G_0^{-1}Y_1$ is a solution of (3.5) modulo $\mathfrak{m}_E$. By the multivariate Hensel's Lemma [Con], a solution of (3.5) exists in $k^s(\!(\pi)\!)=E^{\mathrm{ur}}$. □

3.3.3 Proof of Theorem 3.15

Let $\underline{M}=(M,\tau_M)$ be an A-motive over E and let $M_{\mathcal{O}}$ be its maximal $\mathcal{O}_E$-model. Let $\mathfrak{m}$ be a maximal ideal of A such that $\kappa(\mathfrak{m})\mathcal{O}_E=\mathcal{O}_E$.

Proof of Theorem 3.15. We aim to prove that the morphism ι of Theorem 3.4 induces an isomorphism of A-modules

$$\frac{M_{\mathcal{O}}[\mathfrak{j}^{-1}]}{(\mathrm{id}-\tau_M)(M_{\mathcal{O}})}\xrightarrow{\sim}\mathrm{Ext}^1_{\mathcal{M}_E,\mathcal{O}_E}(\mathbb{1},\underline{M}).$$

By Corollary 2.38, we have

$$\frac{M_{\mathcal{O}}[\mathfrak{j}^{-1}]}{(\mathrm{id}-\tau_M)(M_{\mathcal{O}})} = \frac{M_{\mathcal{O}}[\mathfrak{j}^{-1}]+(\mathrm{id}-\tau_M)(M)}{(\mathrm{id}-\tau_M)(M)} \subset \frac{M[\mathfrak{j}^{-1}]}{(\mathrm{id}-\tau_M)(M)}.$$

Thus, in view of Proposition 3.19, we need to show that the following are equivalent:

(*i*) there exists $\xi \in M \otimes_{A\otimes E} \mathcal{A}_{\mathfrak{m}}(E^{\mathrm{ur}})$ such that $\xi - \tau_M(\tau^*\xi) = m$,

(*ii*) $m \in M_{\mathcal{O}}[\mathfrak{j}^{-1}] + (\mathrm{id}-\tau_M)(M)$.

For a positive integer n, we denote (V_n, φ_n) the Frobenius module

$$(M[\mathfrak{j}^{-1}]/\mathfrak{m}^n M[\mathfrak{j}^{-1}], \tau_M)$$

and let L_n be its maximal $\mathcal{O}_E$-model.

Let $n \geq 1$. If (*i*) is satisfied, then $\xi_n := \xi \pmod{\mathfrak{m}^n}$ belongs to $V_n \otimes_E E^{\mathrm{ur}}$ and satisfies $\xi_n - \varphi_n(\tau^*\xi_n) = m \pmod{\mathfrak{m}^n}$. By Proposition 3.20, m belongs to $L_n + (\mathrm{id}-\tau_M)(M) + \mathfrak{m}^n M[\mathfrak{j}^{-1}]$ and by Corollary 2.26, we have $m \in M_{\mathcal{O}}[\mathfrak{j}^{-1}] + (\mathrm{id}-\tau_M)(M)$.

Conversely, if (*ii*) is satisfied, Corollary 2.26 implies $m \in L_n + (\mathrm{id}-\tau_M)(M) + \mathfrak{m}^n M[\mathfrak{j}^{-1}]$ for all positive integers n. By Proposition 3.20, there exists $\xi_n \in V_n \otimes_E E^{\mathrm{ur}}$ such that

$$\xi_n - \tau_M(\tau^*\xi_n) = m \pmod{\mathfrak{m}^n}. \tag{3.6}$$

Note that, for each n, there are only finitely many such ξ_n. To obtain (*i*), we need to show that we can choose *compatible* ξ_n for all n (that is $\xi_{n+1} \equiv \xi_n \pmod{\mathfrak{m}^n}$). To this end, let us define a tree T indexed by $n \geq 1$ whose nodes at the height n are the solutions ξ_n of (3.6) in $V_n \otimes_E E^{\mathrm{ur}}$. There is an edge between z_n and z_{n+1} if and only if z_{n+1} coincides with z_n modulo $\mathfrak{m}^n M[\mathfrak{j}^{-1}]$. The tree has finitely many nodes at each height and it is infinite from the fact that a solution of (3.6) exists for all n. By König's Lemma, there exists an infinite branch on T. This branch corresponds to a converging sequence $(\xi_n)_{n\geq 1}$ whose limit ξ in $M \otimes_{A\otimes E} \mathcal{A}_{\mathfrak{m}}(E^{\mathrm{ur}})$ satisfies $m = \xi - \tau_M(\tau^*\xi)$. $\square$

3.3.4 Proof of Theorem 3.18

Let F be a finite field extension of K and let $\mathcal{O}_F$ be the integral closure of A in F. We let $\kappa : A \to \mathcal{O}_F$ denote the inclusion. We fix S to be a set of nonzero prime ideals of $\mathcal{O}_F$ and consider the subring $R := \mathcal{O}_F[S^{-1}]$ of F. The ring R is a Dedekind domain whose fraction field is F.

Let $\underline{M} = (M, \tau_M)$ be an Anderson A-motive over F. Given a maximal ideal $\mathfrak{p} \subset R$, let $\underline{M}_{\mathfrak{p}}$ be the A-motive over $F_{\mathfrak{p}}$ obtained from $\underline{M}$ by base-change from F to $F_{\mathfrak{p}}$. Let $M_{R_{\mathfrak{p}}}$ be the integral model of $\underline{M}_{\mathfrak{p}}$.

Lemma 3.24. *We have*

$$M[\mathfrak{j}^{-1}] \cap \left(M_{R_{\mathfrak{p}}}[\mathfrak{j}^{-1}] + (\mathrm{id}-\tau_M)(M_{\mathfrak{p}})\right) = M[\mathfrak{j}^{-1}] \cap M_{R_{\mathfrak{p}}}[\mathfrak{j}^{-1}] + (\mathrm{id}-\tau_M)(M).$$

Proof. The inclusion $\supset$ is clear. Since M is generated over F by elements in $M \cap M_{R_\mathfrak{p}}$ and $F_\mathfrak{p} = F + R_\mathfrak{p}$, we have $M_\mathfrak{p} = M + M_{R_\mathfrak{p}}$. Let $m \in M[\mathfrak{j}^{-1}] \cap \big(M_{R_\mathfrak{p}}[\mathfrak{j}^{-1}] + (\mathrm{id} - \tau_M)(M_\mathfrak{p})\big)$. We can write m as $m_\mathfrak{p} + n_\mathfrak{p} - \tau_M(\tau^* n_\mathfrak{p}) + n - \tau_M(\tau^* n)$ where $m_\mathfrak{p} \in M_{R_\mathfrak{p}}[\mathfrak{j}^{-1}]$, $n_\mathfrak{p} \in M_{R_\mathfrak{p}}$ and $n \in M$. In particular, $m_\mathfrak{p} + n_\mathfrak{p} - \tau_M(\tau^* n_\mathfrak{p})$ belongs to $M[\mathfrak{j}^{-1}] \cap M_{R_\mathfrak{p}}[\mathfrak{j}^{-1}]$ which implies that $m \in M[\mathfrak{j}^{-1}] \cap M_{R_\mathfrak{p}}[\mathfrak{j}^{-1}] + (\mathrm{id} - \tau_M)(M)$. □

Lemma 3.25. *Let $m \in M$. Then $m \in M_{\mathcal{O}_\mathfrak{p}}$ for almost all prime ideal $\mathfrak{p}$ of R.*

Proof. There exists a nonzero element $d \in R$ such that $dm \in M_R$. Let $\{\mathfrak{q}_1, ..., \mathfrak{q}_s\}$ be the finite set of maximal ideals in R that contain (d). By Proposition 2.30, $m \in M_{R_\mathfrak{p}}$ for all $\mathfrak{p}$ not in $\{\mathfrak{q}_1, ..., \mathfrak{q}_s\}$. □

Let N be a finite dimensional vector space over F (resp. $F_\mathfrak{p}$). By a *lattice in* N we mean a finitely generated module over R (resp. $R_\mathfrak{p}$) in N that contains a basis of N.

Lemma 3.26 (Strong approximation)**.** *Let N be a finite dimensional F-vector space and, for all maximal ideals $\mathfrak{p}$ of R, let $N_{R_\mathfrak{p}}$ be an $R_\mathfrak{p}$-lattice in $N_\mathfrak{p} := N \otimes_R F_\mathfrak{p}$ such that the intersection $\bigcap_\mathfrak{p} \big(N \cap N_{R_\mathfrak{p}}\big)$, over all maximal ideals $\mathfrak{p}$ of R, is an R-lattice in N. Let T be a finite set of maximal ideals in R and, for $\mathfrak{q} \in T$, let $n_\mathfrak{q} \in N_\mathfrak{q}$. Then, there exists $n \in N$ such that $n - n_\mathfrak{q} \in N_{R_\mathfrak{q}}$ for all $\mathfrak{q} \in T$ and $n \in N_{R_\mathfrak{p}}$ for all $\mathfrak{p}$ not in T.*

Proof. Let N_R denote the intersection $\bigcap_\mathfrak{p} \big(N \cap N_{R_\mathfrak{p}}\big)$ over all maximal ideals $\mathfrak{p}$ of R. By the structure Theorem for finitely generated modules over the Dedekind domain R, there exists a nonzero ideal $\mathfrak{a} \subset R$ and elements $\{b_1, ..., b_r\} \subset M$ such that

$$N_R = Rb_1 \oplus \cdots \oplus Rb_{r-1} \oplus \mathfrak{a} b_r.$$

Because $N_R \otimes_R R_\mathfrak{p} \subset N_{R_\mathfrak{p}}$ for $\mathfrak{p} \subset R$, we have $R_\mathfrak{p} b_1 \oplus \cdots \oplus \mathfrak{p}^{v_\mathfrak{p}(\mathfrak{a})} R_\mathfrak{p} b_r \subset N_{R_\mathfrak{p}}$. For $\mathfrak{q} \in T$, let us write $n_\mathfrak{q} = \sum_i f_{\mathfrak{q},i} b_i$ with $f_{\mathfrak{q},i} \in F_\mathfrak{q}$. By the strong approximation Theorem [Ros, Thm. 6.13], for all $i \in \{1, ..., r\}$, there exists $f_i \in F$ such that

1. for $\mathfrak{q} \in T$ and $i \in \{1, ..., r-1\}$, $v_\mathfrak{q}(f_i - f_{\mathfrak{q},i}) \geq 0$,
2. for $\mathfrak{q} \in T$, $v_\mathfrak{q}(f_r - f_{\mathfrak{q},r}) \geq v_\mathfrak{q}(\mathfrak{a})$,
3. for $\mathfrak{p} \notin T$ and $i \in \{1, ..., r-1\}$, $v_\mathfrak{p}(f_i) \geq 0$,
4. for $\mathfrak{p} \notin T$, $v_\mathfrak{p}(f_r) \geq v_\mathfrak{p}(\mathfrak{a})$.

The element $n = \sum_i f_i b_i \in N$ satisfies the assumption of the lemma. □

Lemma 3.27. *We have*

$$\bigcap_{\mathfrak{p} \subset R} \big(M[\mathfrak{j}^{-1}] \cap M_{R_\mathfrak{p}}[\mathfrak{j}^{-1}] + (\mathrm{id} - \tau_M)(M)\big) = M_R[\mathfrak{j}^{-1}] + (\mathrm{id} - \tau_M)(M)$$

where the intersection is indexed over the maximal ideals of R.

Proof. The inclusion $\supset$ follows from Proposition 2.30. Conversely, let m be an element of $\bigcap_{\mathfrak{p}\subset R}\left(M[\mathfrak{j}^{-1}]\cap M_{R_{\mathfrak{p}}}[\mathfrak{j}^{-1}]+(\mathrm{id}-\tau_M)(M)\right)$. By Lemma 3.25, there exists a finite subset T of maximal ideals of R such that $m\in M_{R_{\mathfrak{p}}}[\mathfrak{j}^{-1}]$ for $\mathfrak{p}\notin T$. For $\mathfrak{q}\in T$, there exists $n_{\mathfrak{q}}\in M$ and $m_{\mathfrak{q}}\in M[\mathfrak{j}^{-1}]\cap M_{R_{\mathfrak{q}}}[\mathfrak{j}^{-1}]$ such that $m=m_{\mathfrak{q}}+n_{\mathfrak{q}}-\tau_M(\tau^*n_{\mathfrak{q}})$.

Let N be a finite dimensional sub-F-vector space of M that contains m and $n_{\mathfrak{q}}$ for all $\mathfrak{q}\in T$. For a maximal ideal $\mathfrak{p}$ of R, let $N_{R_{\mathfrak{p}}}:=M_{R_{\mathfrak{p}}}\cap(N\otimes_F F_{\mathfrak{p}})$. We have $N_R:=\bigcap_{\mathfrak{p}}(N\cap N_{R_{\mathfrak{p}}})=N\cap M_R$. The latter is an R-lattice in N and hence we are in the situation of Lemma 3.26. Therefore, there exists $n\in N$ such that $n-n_{\mathfrak{q}}\in N_{R_{\mathfrak{q}}}$ for all $\mathfrak{q}\in T$ and $n\in N_{R_{\mathfrak{p}}}$ for all $\mathfrak{p}$ not in T. Then $m+n-\tau_M(\tau^*n)\in N_R\subset M_R$, which ends the proof. $\square$

Proof of Theorem 3.18. Let $[\underline{E}]$ be an extension of $\mathbb{1}$ by $\underline{M}$ in $\mathcal{M}_F$, and let $m\in M[\mathfrak{j}^{-1}]$ be such that $[\underline{E}]=\iota(m)$ where ι is the isomorphism of Theorem 3.4. Let $\operatorname{Spm}R$ be the set of maximal ideals of R. We have the following equivalences

$$
\begin{array}{lll}
 & [\underline{E}] \text{ has everywhere good reduction,} & \\
\Longleftrightarrow & \forall\mathfrak{p}\in\operatorname{Spm}R\colon\ [\underline{E}_{F_{\mathfrak{p}}}] \text{ has good reduction,} & \text{(Definition 3.16)}\\
\Longleftrightarrow & \forall\mathfrak{p}\in\operatorname{Spm}R\colon\ m\in M[\mathfrak{j}^{-1}]\cap[M_{R_{\mathfrak{p}}}[\mathfrak{j}^{-1}]+(\mathrm{id}-\tau_M)(M_{\mathfrak{p}})], & \text{(Theorem 3.15)}\\
\Longleftrightarrow & \forall\mathfrak{p}\in\operatorname{Spm}R\colon\ m\in M[\mathfrak{j}^{-1}]\cap M_{R_{\mathfrak{p}}}[\mathfrak{j}^{-1}]+(\mathrm{id}-\tau_M)(M), & \text{(Lemma 3.24)}\\
\Longleftrightarrow & m\in M_R[\mathfrak{j}^{-1}]+(\mathrm{id}-\tau_M)(M), & \text{(Lemma 3.27).}
\end{array}
$$

We obtain that ι induces an isomorphism of A-modules

$$\frac{M_R[\mathfrak{j}^{-1}]+(\mathrm{id}-\tau_M)(M)}{(\mathrm{id}-\tau_M)(M)}\longrightarrow\operatorname{Ext}^1_{\mathcal{M}_F,R}(\mathbb{1},\underline{M}).$$

We conclude by Corollary 2.38. $\square$

Chapter 4

Mixed Hodge Structures in equal characteristic

To define function fields regulators and state the analogue of Beilinson's first conjecture, we require the notion of mixed Hodge structures (abridged MHS) in positive characteristic. In this chapter, we provide the definitions and main properties related to MHS. Our ultimate reference is Pink's unpublished monograph [Pin], from which we adapt the definitions to allow slightly more general coefficient rings R. The main novelty of this section is the introduction of function fields' *infinite Frobenius*, which allows us to define the categories $\mathcal{MH}^+_{K_\infty}$ and $\mathcal{MHP}^+_{K_\infty}$ evoked in the introduction (Chapter 0). The objective is to describe the extension spaces in both categories, a task we achieve under mild conditions (Propositions 4.14 and 4.55). In Chapter 5, these computations will be used to express *regulator morphisms* in a very explicit manner (see Theorem 5.34).

After a review of filtered spaces in Section 4.1, we develop the theory of MHS in Section 4.2, we construct the abelian categories $\mathcal{MH}_R$ and $\mathcal{MH}^+_R$ over a certain *coefficient ring* R and compute the respective extension modules.

More relevant to function field arithmetic are *Mixed Hodge-Pink Structures* (MHPS) which form higher versions of MHS, where the Hodge filtration is replaced by the refined data of the *Hodge-Pink lattice.* In Section 4.3, following closely [Pin], we define the categories $\mathcal{MHP}_R$ and $\mathcal{MHP}^+_R$ and compute the extension modules similarly.

To an MHPS, one associates an Hodge filtration. Yet, this does not necessarily define a MHS. To circumvent this obstruction, we introduce a category $\mathcal{MHP}^{\mathrm{hd}}_{K_\infty}$ in Subsection 4.3.4, the supscript *hd* standing for *Hodge descent.* We prove that there is an exact functor:

$$\# : \mathcal{MHP}^{\mathrm{hd}}_R \longrightarrow \mathcal{MH}_R$$

(see Theorem 4.47). This constitutes an improvement of Pink's theory. The functor # will be used in Chapter 5 to define *regulated* objects of $\mathcal{MM}^{\mathrm{rig}}_F$.

4.1 Filtered spaces and classical mixed Hodge structures

4.1.1 Filtered spaces

We review filtrations of vector spaces following [DaOrR, §I.1]. Let V be a finite dimensional vector space over a field k. An *increasing filtration* F on V is an increasing function of ordered sets

$$(\mathbb{R}, \leq) \longrightarrow (\{\text{subspaces of } V\}, \subset), \quad x \longmapsto F_x V,$$

that is, $F_x V \subset F_y V$ if $x \leq y$. The filtration F is *separated* if $F_x V = (0)$ for $x \ll 0$ and *exhaustive* if $F_x V = V$ for $x \gg 0$. For $x \in \mathbb{R}$, the *xth graded piece of F* is the k-vector space

$$\mathrm{Gr}_x^F(V) := F_x V / \bigcup_{y<x} F_y V.$$

A *break* of F is an element $x \in \mathbb{R}$ such that $\mathrm{Gr}_x^F(V)$ is nonzero. Because V is finite dimensional, the number of breaks is finite. For S a subset of $\mathbb{R}$, we say that F is *S-graded* if the all the breaks of F are in S. The *degree of F* is the real number given by the finite sum

$$\deg^F(V) = \sum_{x \in \mathbb{R}} x \dim_k \mathrm{Gr}_x^F(V).$$

If $V' \subset V$ is a subspace, the *induced filtration by F on V'* is the increasing filtration

$$(\mathbb{R}, \leq) \longrightarrow (\{\text{subspaces of } V'\}, \subseteq), \quad x \longmapsto F_x V \cap V'.$$

The *induced filtration by F on V/V'* is the increasing filtration

$$(\mathbb{R}, \leq) \longrightarrow (\{\text{subspaces of } V/V'\}, \subseteq), \quad x \longmapsto (F_x V + V')/V'.$$

Let W be a finite dimensional vector space equipped with an increasing filtration G. A k-linear map $f : V \to W$ is said to *preserve filtrations*, to be *compatible with the filtrations*, or is a *morphism of filtrated spaces from (V, F) to (W, G)* if, for all $x \in \mathbb{R}$, $f(F_x V) \subset G_x W$. f is said to be *strict* if, for all $x \in \mathbb{R}$, $f(F_x V) = f(V) \cap G_x W$. In other words, f is strict if the filtration $x \mapsto f(F_x V)$ on W equals the induced filtration by G on $f(V)$.

Lemma 4.1. *Let $S : 0 \to V' \to V \to V'' \to 0$ be an exact sequence of k-vector spaces equipped with filtrations which are preserved by the morphisms of the sequence. The following properties hold:*

(i) The morphisms of S preserve the filtration strictly if and only if the sequence $F_x S : 0 \to F_x V' \to F_x V \to F_x V'' \to 0$ is exact for all $x \in \mathbb{R}$.

(*ii*) *If* (*i*) *is satisfied, there exists a section* $s : V'' \to V$ *which preserves the filtration strictly.*

Proof. (*i*) is obvious. We prove (*ii*). Let $\{x_1, ..., x_\ell\}$ be the union of the set of breaks for the filtrations on V', V and V'' sorted by increasing order. For $1 \leq i < \ell$, fix a section $s_i : F_{x_i}V'' \to F_{x_i}V$ of $F_{x_i}S$ which preserves the induced filtration. The following sequence is exact by (*i*):

$$\mathrm{Gr}^F_{x_{i+1}} S: \quad 0 \longrightarrow \mathrm{Gr}^F_{x_{i+1}} V' \longrightarrow \mathrm{Gr}^F_{x_{i+1}} V \longrightarrow \mathrm{Gr}^F_{x_{i+1}} V'' \longrightarrow 0.$$

Let $t_{i+1} : \mathrm{Gr}^F_{x_{i+1}} V'' \to \mathrm{Gr}^F_{x_{i+1}} V$ be a section of the above. We have the decompositions

$$F_{x_{i+1}}V'' \cong \mathrm{Gr}^F_{x_{i+1}} V'' \oplus F_{x_i}V'', \text{ and } F_{x_{i+1}}V \cong \mathrm{Gr}^F_{x_{i+1}} V \oplus F_{x_i}V,$$

so that $s_{i+1} := t_{i+1} \oplus s_i$ defines a section of $F_{x_{i+1}}S$ preserving the filtration. By induction, we deduce the existence of a section $s : V'' \to V$ of S which preserves the filtration. One verifies that s preserves the filtrations strictly because $p : V \to V''$ is strict and $p \circ s = \mathrm{id}_{V''}$. $\square$

Let n be a positive integer. By a *polygon of length* n, we mean the graph in $\mathbb{R}^2$ of a piecewise linear convex function $[0, n] \to \mathbb{R}$, mapping 0 to 0, and such that the length of the subinterval on which the function has a given slope $x \in \mathbb{R}$ is an integer, called the *multiplicity of* x. Convexity means that the slopes increase.

To a filtration F on V one can attach a unique polygon P_F of length the dimension of V, such that for all $x \in \mathbb{R}$ the multiplicity of x equals the dimension of $\mathrm{Gr}^F_x(V)$. We call P_F *the polygon of* F.

Remark 4.2. We define *decreasing filtrations* and the notions attached similarly. For a decreasing filtration F, we switch upperscripts and subscript, hence writing: $\deg_F$, F^x, Gr^x_F, etc. All the results can be turned in corresponding statements for decreasing filtrations by considering $x \mapsto F_{-x}V$.

4.1.2 Classical mixed Hodge structures

Let us review the classical definition of mixed Hodge Structures in the number fields setting. According to Deligne (we refer to [DelI, Def. 1.1] for the precise definition), a mixed Hodge structure (over $\mathbb{Z}$) consists of

(*a*) A $\mathbb{Z}$-module H of finite type (the *integral lattice*),

(*b*) A $\mathbb{Z}$-graded increasing filtration W on $H_{\mathbb{Q}} := H \otimes_{\mathbb{Z}} \mathbb{Q}$ by $R \otimes_{\mathbb{Z}} \mathbb{Q}$-subspaces (the *weight filtration*),

(*c*) A $\mathbb{Z}$-graded decreasing filtration F on $H_{\mathbb{C}} := H \otimes_{\mathbb{Z}} \mathbb{C}$ by $\mathbb{C}$-subspaces (the *Hodge filtration*).

Those datas are subject to the following condition:
Condition (Deligne): For all $n \in \mathbb{Z}$, there exists on $\mathrm{Gr}_W^n(H_\mathbb{C})$ a bigraded filtration by subspaces $H^{p,q}$ such that

(*i*) $\mathrm{Gr}_n^W(H_\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}$,

(*ii*) the filtration F induces on $\mathrm{Gr}_n^W(H_\mathbb{C})$ the filtration given for $p \in \mathbb{Z}$ by

$$F^p \,\mathrm{Gr}_n^W(H_\mathbb{C}) = \bigoplus_{\substack{p'\geq p \\ p'+q'=n}} H^{p',q'},$$

(*iii*) $(\mathrm{id}_H \otimes_\mathbb{Z} c)(H^{p,q}) = H^{q,p}$ where $c : \mathbb{C} \to \mathbb{C}$ is the complex conjugation.

We form the category $\mathcal{MH}_\mathbb{Z}$ whose objects are mixed Hodge structures and whose morphisms are morphisms of the underlying $\mathbb{Z}$-modules which preserve the weight (resp. Hodge) filtration once scalars are extended to $\mathbb{Q}$ (resp. $\mathbb{C}$).

An equivalent formulation of points (*i*)-(*iii*) was found by Pink in [Pin, Prop. 2.4]:
Condition (Pink): For every $\mathbb{R}$-subspace $H'_\mathbb{R}$ of $H_\mathbb{R} := H \otimes_\mathbb{Z} \mathbb{R}$, we have

$$\deg_F(H'_\mathbb{C}) \leq \frac{1}{2} \deg^W(H'_\mathbb{R})$$

with equality whenever $H'_\mathbb{R} = W_n H_\mathbb{R}$ for $n \in \mathbb{Z}$.

Remark 4.3. Pink's condition is also called the *local semistability condition over* $\mathbb{R}$. The factor $1/2$ reflects the degree of $\mathbb{C}/\mathbb{R}$ and can be removed by renormalizing the weight filtration using half-integers. As there is no analogue of the complex conjugation in the function fields setting, hence no clear analogue of Deligne's condition, it is better suggested to follow Pink's reformulation.

According to Nekovar [Nek, (2.4)] and Deligne [Del, §1.4 (M7)], an *infinite Frobenius* ϕ_∞ for a mixed Hodge structure (H, W, F) is an involution of the $\mathbb{Z}$-module H which is compatible with the weight filtration once scalars have been extended to $\mathbb{Q}$ and such that $\phi_\infty \otimes c$ preserves the Hodge filtration. MHS coming from the singular cohomology groups of a variety X over a number field are naturally equipped with an infinite Frobenius coming from the action of the complex conjugation on the complex points $X(\mathbb{C})$.

We let $\mathcal{MH}_\mathbb{Z}^+$ be the category whose objects are pairs $(\underline{H}, \phi_\infty)$ where $\underline{H}$ is a mixed Hodge structures and ϕ_∞ is an infinite Frobenius for $\underline{H}$. Morphisms in $\mathcal{MH}_\mathbb{Z}^+$ are the morphisms in $\mathcal{MH}_\mathbb{Z}$ which commute to infinite Frobenius.

4.2 Function fields' mixed Hodge structures

We now discuss the function fields situation. To the extent of my knowledge, the *ad hoc* definition of mixed Hodge structures over function fields was not defined yet, although Deligne's formalism seems to apply without changes using Pink's condition.

Let L be a complete subfield of $\mathbb{C}_\infty$ containing K, and fix L^s a separable closure of L. In the next chapters, L will be F_v for F be a finite extension of K, $v : F \to \mathbb{C}_\infty$ be a K-algebra morphism and F_v be the completion of F with respect to $|x|_v := |v(x)|$. Let R be a Noetherian subring of K_∞ containing A such that $R \otimes_A K$ is a field (in all the following, R will be A, K or K_∞). Because L contains K_∞, the field $R \otimes_A K$ identifies canonically with a subfield of L.

4.2.1 The categories $\mathcal{MH}$ and $\mathcal{MH}^+$

Definition 4.4. A *pre-mixed Hodge structure* $\underline{H}$ *(with base field* L*, coefficients ring* R*)* consists of a triple (H, W, F) where

- H is a finitely generated R-module,
- W is a $\mathbb{Q}$-graded increasing filtration on $H_K = H \otimes_R (R \otimes_A K)$ by $R \otimes_A K$-subspaces which is exhaustive, separated,
- F is a $\mathbb{Z}$-graded decreasing filtration of $H_{L^s} := H \otimes_R L^s$ by L^s-subspaces which is exhaustive, separated.

We call W the *weight filtration of* $\underline{H}$ and F the *Hodge filtration of* $\underline{H}$. The *weights* of $\underline{H}$ are the breaks of its weight filtration.

Let $\underline{H}$ and $\underline{H}'$ be two pre-mixed Hodge structures. A morphism $f : \underline{H} \to \underline{H}'$ is an R-linear morphism $f : H \to H'$ such that $f_K = f \otimes \mathrm{id}_K : H_K \to H'_K$ and $f_{L^s} = f \otimes \mathrm{id}_{L^s} : H_{L^s} \to H'_{L^s}$ preserve W and F respectively. The morphism f is said to be *strict* if f_K and f_{L^s} are strictly compatible with the weight and Hodge filtrations respectively.

Remark 4.5. The category of pre-mixed Hodge structures is *too large* to be an abelian category. Indeed, given a morphism f between two objects in this category, the natural map $\operatorname{coim} f \to \operatorname{im} f$ needs not to be an isomorphism (it is bijective on the underlying vector spaces, but its inverse does not necessarily respect both filtrations). It is an isomorphism if and only if f is strict.

Let $\underline{H}$ be a pre-mixed Hodge structure. A sub-R-module $H' \subset H$ defines a subobject of $\underline{H}' = (H', W', F')$ of $\underline{H}$ with weight filtration W' (resp. Hodge filtration F') the induced filtration on H' by W (resp. on H'_{L^s} by F). By definition, the inclusion $\underline{H}' \hookrightarrow \underline{H}$ is strict. Similarly, the quotient $\underline{H}/\underline{H}' = (H/H', W'', F'')$ is defined so that W'' (resp. F'') is the induced filtration on H/H' by W (resp. on H_{L^s}/H'_{L^s} by F). The quotient map $\underline{H} \to \underline{H}/\underline{H}'$ is also strict.

The *weight* and *Hodge degrees* of $\underline{H}$ are defined as follows:

$$\deg^W(\underline{H}) := \sum_{\mu\in\mathbb{Q}} \mu \dim_K \mathrm{Gr}_\mu^W(H_K), \quad \deg_F(\underline{H}) := \sum_{p\in\mathbb{Z}} p \dim_{L^s} \mathrm{Gr}_F^p(H_{L^s}).$$

The next definition is directly inspired by Pink's condition.

Definition 4.6. Let $\underline{H}$ be a pre-mixed Hodge structure with coefficients ring K_∞. We call $\underline{H}$ *locally semistable* if for each K_∞-subspace $H' \subset H$, we have

$$\deg_F(\underline{H}') \leq \deg^W(\underline{H}'),$$

with equality whenever $H'_K = W_\mu H_K$ for some $\mu \in \mathbb{Q}$.
If $\underline{H} = (H, W, K)$ is a pre-mixed Hodge structure over R, we say that $\underline{H}$ a *mixed Hodge structure* if $\underline{H}_\infty$ is *locally semistable*, where $\underline{H}_\infty$ is the pre-mixed Hodge structure with coefficient ring K_∞ given by $(H\otimes_R K_\infty, W\otimes_R K_\infty, F)$. We form the category $\mathcal{MH}_R$ as the full subcategory of the category of pre-mixed Hodge structures whose objects are mixed Hodge structures.

The following proposition, inspired by Deligne's [DelII, Thm. 2.3.5], is due to Pink (see [Pin, Thm. 4.15]) in the case of mixed Hodge-Pink structures. Because Pink's proof applies almost without changes to our situation, we omit the proof.

Proposition 4.7. *Every morphism of mixed Hodge structures is strict. The category $\mathcal{MH}_R$ is R-linear abelian.*

We fix L^s a separable closure of L and let $G_L = \mathrm{Gal}(L^s|L)$ be the absolute Galois group of L. One novelty of our account is the notion of *function fields infinite Frobenius* akin to the eponymous notion for number fields:

Definition 4.8. Let $\underline{H}$ be an object of $\mathcal{MH}_R^+$. An *infinite Frobenius for* $\underline{H} = (H, W, F)$ is an R-linear continuous representation $\phi_{\underline{H}} : G_L \to \mathrm{End}_R(H)$, G_L carrying the profinite topology and H the discrete topology, such that, for all $\sigma \in G_L$,

1. $\phi_{\underline{H}}(\sigma)\otimes_A \mathrm{id}_K$ preserves the weight filtration, that is, $(\phi_{\underline{H}}(\sigma)\otimes_A \mathrm{id}_K)(W_\mu H_K) \subset W_\mu H_K$ for all $\mu \in \mathbb{Q}$,

2. $\phi_{\underline{H}}(\sigma)\otimes_R \sigma$ preserves the Hodge filtration, that is, $(\phi_{\underline{H}}(\sigma)\otimes\sigma)(F^p H_{L^s}) \subset F^p H_{L^s}$ for all $p \in \mathbb{Z}$.

Remark 4.9. Let $\mathbb{1}$ be the triplet (R, W, F) where $W = \mathbf{1}_{\mu\geq 0}(R\otimes_A K)$ and $F = \mathbf{1}_{p\leq 0}(R\otimes_A L^s)$. One easily verifies that $\mathbb{1}$ is a mixed Hodge structure and we call it the *unit mixed Hodge structure with coefficient ring R*. The map $\phi_{\mathbb{1}} : G_L \to R$, $\sigma \mapsto \mathrm{id}_R$ defines an infinite Frobenius for $\mathbb{1}$.

We let $\mathcal{MH}_R^+$ be the category whose objects are pairs $(\underline{H}, \phi_{\underline{H}})$ where $\underline{H}$ is a mixed Hodge structure and where $\phi_{\underline{H}}$ is an infinite Frobenius for $\underline{H}$. A morphism $(\underline{H}, \phi) \to (\underline{H}', \phi')$ in $\mathcal{MH}_R^+$ is a morphism $f : \underline{H} \to \underline{H}'$ in $\mathcal{MH}_R$ such that $f\circ\phi(\sigma) = \phi'(\sigma)\circ f$ for all $\sigma \in G_L$. We let $\mathbb{1}^+$ be the object $(\mathbb{1}, \phi_{\mathbb{1}})$ in $\mathcal{MH}_R^+$.

The next proposition is a formal consequence of Proposition 4.7 so we omit the proof.

Proposition 4.10. *The category $\mathcal{MH}_R^+$ is R-linear abelian.*

4.2.2 Extensions of mixed Hodge structures

By Propositions 4.7 and 4.10, the extension R-modules in the categories $\mathcal{MH}_R$ and $\mathcal{MH}_R^+$ are well-defined, and this subsection is devoted to their description.

Comparison with the classical theory

In the classical setting, extension modules in the category $\mathcal{MH}_{\mathbb{R}}$ of mixed Hodge structures *with real coefficients* are well known. Given $\underline{H}$ an object of $\mathcal{MH}_{\mathbb{R}}$, the complex of $\mathbb{R}$-vector spaces

$$\left[W_0H \oplus F^0W_0H_{\mathbb{C}} \stackrel{(x,y)\mapsto x-y}{\longrightarrow} W_0H_{\mathbb{C}}\right]$$

represents the cohomology of $\mathrm{RHom}_{\mathcal{MH}_{\mathbb{R}}}(\mathbb{1},\underline{H})$ (e.g. [Bei1, §1], [Carls, Prop. 2], [PetSt, Thm. 3.31]). We obtain an $\mathbb{R}$-linear morphism

$$\frac{W_0H_{\mathbb{C}}}{W_0H + F^0W_0H_{\mathbb{C}}} \stackrel{\sim}{\longrightarrow} \mathrm{Ext}^1_{\mathcal{MH}_{\mathbb{R}}}(\mathbb{1},\underline{H}). \tag{4.1}$$

If now $\underline{H}^+$ denotes an object in the category $\mathcal{MH}_{\mathbb{R}}^+$ with infinite Frobenius ϕ_∞, the complex $\mathrm{RHom}_{\mathcal{MH}_{\mathbb{R}}^+}(\mathbb{1}^+,\underline{H}^+)$ is rather represented by

$$\left[(W_0H)^+ \oplus (F^0W_0H_{\mathbb{C}})^+ \stackrel{(x,y)\mapsto x-y}{\longrightarrow} (W_0H_{\mathbb{C}})^+\right]$$

where the subscript $+$ means the corresponding $\mathbb{R}$-subspace fixed by $\phi_\infty \otimes c$ (e.g. [Bei1, §1], [Nek, (2.5)]). We obtain an $\mathbb{R}$-linear morphism

$$\frac{(W_0H_{\mathbb{C}})^+}{(W_0H)^+ + (F^0W_0H_{\mathbb{C}})^+} \stackrel{\sim}{\longrightarrow} \mathrm{Ext}^1_{\mathcal{MH}_{\mathbb{R}}^+}(\mathbb{1},\underline{H}). \tag{4.2}$$

In this subsection, we proceed to the analogue constructions for MHS and MHPS in the function fields setting. While the ingenous analogue of (4.1) holds in $\mathcal{MH}_R^+$ (Proposition 4.11), a description similar as (4.2) does not hold in our setting as G_L is an infinite group. For $\underline{H}^+ = (\underline{H},\phi_H)$ an object in $\mathcal{MH}_R^+$ (resp. $\mathcal{MHP}_R^+$), the extension space $\mathrm{Ext}^1_{\mathcal{MH}_R^+}(\mathbb{1}^+,\underline{H}^+)$ is intertwined with the Galois cohomology of G_L, preventing an isomorphism as simple as (4.2) to exist. In fact, the latter extension space has generally infinite dimension. In order to clarify how Galois cohomology interferes with the computation of extension spaces, we introduce an R-linear morphism $d_{\underline{H}^+}$ (Definition 4.13), functorial in $\underline{H}^+$, which inserts in a non-necessarily exact sequence of R-modules:

$$0 \to \frac{(W_0H_{L^s})^+}{(W_0H)^+ + (F^0W_0H_{L^s})^+} \longrightarrow \mathrm{Ext}^1_{\mathcal{MH}_R^+}(\mathbb{1}^+,\underline{H}^+) \stackrel{d_{\underline{H}^+}}{\longrightarrow} H^1(G_\infty,H) \to 0$$

where the superscript $+$ now designates the submodule of elements fixed by $\phi_H(\sigma)\otimes\sigma$ for all $\sigma\in G_L$. We prove in Proposition 4.14 that, under the condition that $H^1(G_L, H_{L^s})$ and $H^1(G_L, F^0H_{L^s})$ vanish, the above sequence is exact (this condition will hold for MHS coming from *regulated* objects in $\mathcal{MM}_F^{\mathrm{rig}}$, see Theorem 5.31). The kernel of $d_{\underline{H}^+}$ is then the correct analogue of the right-hand side of (4.2) (its elements will be said to have *analytic reduction*).

Extensions in $\mathcal{MH}_R$

We now turn to the computation of extension modules in the category $\mathcal{MH}_R$. The theory is similar to the number field situation. For instance, the number fields version of the next proposition is due to Carlson [Carls]. We give a proof adapted to our different setting.

Proposition 4.11. *Let $\underline{B}$ and $\underline{C}$ be two mixed Hodge structures such that there exists $\mu\in\mathbb{Q}$ for which $W_\mu C_K = C_K$ and $W_\mu B_K = 0$. Assume further that the underlying module B of $\underline{B}$ is projective. There is a canonical isomorphism of R-modules*

$$\frac{\mathrm{Hom}_{L^s}(B_{L^s}, C_{L^s})}{\mathrm{Hom}_R(B,C)+\mathrm{Hom}^F_{L^s}(B_{L^s}, C_{L^s})} \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{MH}_R}(\underline{B},\underline{C}) \tag{4.3}$$

where $\mathrm{Hom}^F_{L^s}(B_{L^s}, C_{L^s})$ designates the L^s-vector space of L^s-linear map from B_{L^s} to C_{L^s} which preserve the Hodge filtrations. (4.3) *is given explicitely by mapping $h\in\mathrm{Hom}_{L^s}(B_{L^s}, C_{L^s})$ to the class of the extension*

$$[h] := \left[C\oplus B, (W_\mu C_K\oplus W_\mu B_K)_{\mu\in\mathbb{Q}}, \left(\begin{pmatrix}\mathrm{id}_C & h\\ 0 & \mathrm{id}_B\end{pmatrix} F^pC_{L^s}\oplus F^pB_{L^s}\right)_{p\in\mathbb{Z}}\right]$$

in $\mathrm{Ext}^1_{\mathcal{MH}_R}(\underline{B},\underline{C})$.

Proof of Proposition 4.11. Let $h: B_{L^s}\to C_{L^s}$ be an L^s-linear morphism. To see that the pre-mixed Hodge structure $[h]$ is a mixed Hodge structure, it suffices to note that the canonical morphisms α and β appearing in the sequence $0\to\underline{C}\xrightarrow{\beta}[h]\xrightarrow{\alpha}\underline{B}\to 0$ are strict. Because $W_\mu C_K = C_K$ and $W_\mu B_K = 0$ for some $\mu\in\mathbb{Q}$, $[h]$ is locally semistable by [Pin, Prop. 4.11].

Conversely, let $0\to\underline{C}\xrightarrow{\beta}\underline{H}\xrightarrow{\alpha}\underline{B}\to 0$ be a short exact sequence in $\mathcal{MH}_R$. Since the underlying R-module B of $\underline{B}$ is projective, the exact sequence

$$0\longrightarrow C\longrightarrow H\longrightarrow B\longrightarrow 0$$

splits. We fix $s: B\to H$ a section. From Proposition 4.7, the $R\otimes_A K$-linear maps α_K and β_K are strict with respect to the weight filtration. As the highest weight of $\underline{C}$ is lower than the lowest weight of $\underline{C}$, $s_K: B_K\to H_K$ automatically preserves the weight filtration strictly. The next diagram of R-modules

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "\beta"] & H \arrow[r, "\alpha"] & B \arrow[r] & 0 \\ 0 \arrow[r] & C \arrow[r] \arrow[u, "\mathrm{id}"] & C\oplus B \arrow[r] \arrow[u, "\beta\oplus s"] & B \arrow[r] \arrow[u, "\mathrm{id}"] & 0. \end{tikzcd}$$

defines a congruence between the class of the extension $[\underline{H}]$ in $\mathrm{Ext}^1_{\mathcal{MH}_R}(\underline{B},\underline{C})$ with the class of the extension

$$\left[C\oplus B,(W_\mu C_K\oplus W_\mu B_K)_{\mu\in\mathbb{Q}},(\beta\oplus s)^{-1}(F^pH_{L^s})_{p\in\mathbb{Z}}\right].$$

By Proposition 4.7, β_{L^s} and α_{L^s} are strict with respect to the Hodge filtration and induce, for all $p\in\mathbb{Z}$, an exact sequence $0\to F^pC_{L^s}\stackrel{\beta}{\to}F^pH_{L^s}\stackrel{\alpha}{\to}F^pB_{L^s}\to 0$ of L^s-vector spaces. By Lemma 4.1(*ii*), the latter sequence possesses its own splitting $\sigma: B_{L^s}\to C_{L^s}$ preserving the Hodge filtration strictly. Setting $h:=\sigma-s_{L^s}$, we obtain that $(\beta\oplus s)^{-1}(F^pH_{L^s})$ has the form $\left(\begin{smallmatrix}\mathrm{id}_C & h\\ 0 & \mathrm{id}_B\end{smallmatrix}\right)(F^pC_{L^s}\oplus F^pB_{L^s})$ for all integers p. Therefore $[\underline{H}]$ is congruent to $[h]$ as desired.

Note that if $h'\in\mathrm{Hom}^F_{L^s}(B_{L^s},C_{L^s})$, then $[h+h']=[h]$ as their Hodge filtrations coincide. We have proved that there is a surjective application

$$\frac{\mathrm{Hom}_{L^s}(B_{L^s},C_{L^s})}{\mathrm{Hom}^F_{L^s}(B_{L^s},C_{L^s})}\longrightarrow \mathrm{Ext}^1_{\mathcal{MH}_R}(\underline{B},\underline{C}),\quad h\longmapsto [h]. \tag{4.4}$$

Note that $[0]$ corresponds to the class of the split extension. Further, $[h+k]$ corresponds to the Baer sum of $[h]$ and $[k]$. In addition, given the exact sequence $[\underline{H}]$ and $a\in R$, the pullback of the extension $[\underline{H}]$ by the multiplication by a on $\underline{B}$ gives another extension which defines $a\cdot[\underline{H}]$. If $[\underline{H}]=[h]$ then it is formal to check that $a\cdot[\underline{H}]=[ah]$. As such, (4.4) is R-linear. The extension $[h]$ is congruent to the split extension if and only if there is a commutative diagram in $\mathcal{MH}_R$ of the form

$$\begin{tikzcd}
0 \arrow[r] & \underline{C} \arrow[r]\arrow[d, "\mathrm{id}_C"] & \underline{C}\oplus\underline{B} \arrow[r]\arrow[d, "j"] & \underline{B} \arrow[r]\arrow[d, "\mathrm{id}_B"] & 0\\
0 \arrow[r] & \underline{C} \arrow[r] & {[h]} \arrow[r] & \underline{B} \arrow[r] & 0
\end{tikzcd}$$

where j is a morphism in $\mathcal{MH}_R$. The fact that j is an automorphism of $C\oplus B$ which restricts to the identity on C and B implies that there exists $g\in\mathrm{Hom}_R(B,C)$ such that $j=\left(\begin{smallmatrix}\mathrm{id}_C & g\\ 0 & \mathrm{id}_B\end{smallmatrix}\right)$. The property that j stabilizes the Hodge filtration is therefore equivalent to $g-h\in\mathrm{Hom}^F_{L^s}(B_{L^s},C_{L^s})$. We conclude that the kernel of (4.4) is $\mathrm{Hom}^F_{L^s}(B_{L^s},C_{L^s})+\mathrm{Hom}_R(B,C)$ as desired. $\square$

The following corollary is a reformulation of Proposition 4.11 in the special case of $\underline{B}=\mathbb{1}$.

Corollary 4.12. *Let $\underline{H}$ be an object in $\mathcal{MH}_R$ whose weights are negative and whose underlying R-module is projective. We have an R-linear isomorphism*

$$\frac{H_{L^s}}{H+F^0H_{L^s}}\xrightarrow{\sim}\mathrm{Ext}^1_{\mathcal{MH}_R}(\mathbb{1},\underline{H})$$

which maps the class of $h\in H_{L^s}$ to the class of the extension

$$\left[H\oplus R,(W_\mu H_K\oplus \mathbf{1}_{\mu\geq 0}K)_{\mu\in\mathbb{Q}},\left(\begin{pmatrix}\mathrm{id}_H & h\\ 0 & 1\end{pmatrix}F^pH_{L^s}\oplus\mathbf{1}_{p\leq 0}L^s\right)_{p\in\mathbb{Z}}\right]$$

in $\mathrm{Ext}^1_{\mathcal{MH}_R}(\mathbb{1},\underline{H})$.

Extensions in $\mathcal{MH}_R^+$

Extension modules in the category $\mathcal{MH}_R^+$ are more involved than the ones in $\mathcal{MH}_R$ as they involve the cohomology of the profinite group G_L. We are able to compute them under some vanishing assumptions on Galois cohomology groups (see Proposition 4.14 below). In the number fields case, the situation is much easier as $\mathrm{Gal}(\mathbb{C}|\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ (e.g. [Nek, (2.4)]).

Given an object $\underline{H}^+ = (\underline{H}, \phi_{\underline{H}})$ of $\mathcal{MH}_R^+$, the R-module H is endowed with a continuous action of G_L via $\phi_{\underline{H}}$. We suppose that the weights of $\underline{H}$ are negative and that its underlying R-module is projective.

We next define a canonical R-linear map

$$d_{\underline{H}^+} : \mathrm{Ext}^1_{\mathcal{MH}_R^+}(\mathbb{1}^+, \underline{H}^+) \longrightarrow H^1(G_L, H) \tag{4.5}$$

as follows. An extension in $\mathrm{Ext}^1_{\mathcal{MH}_R^+}(\mathbb{1}^+, \underline{H})$ is of the form $[\underline{E}, \phi_{\underline{E}}]$ where, by Corollary 4.12, $[\underline{E}]$ is congruent to an extension in the form

$$[h] = \left[H \oplus R, (W_\mu B_K \oplus \mathbf{1}_{\mu \geq 0} K)_{\mu \in \mathbb{Q}}, \left(\begin{pmatrix} \mathrm{id}_H & h \\ 0 & 1 \end{pmatrix} F^p H_{L^s} \oplus \mathbf{1}_{p \leq 0} L^s \right)_{p \in \mathbb{Z}} \right].$$

Under the above description, and for $\sigma \in G_L$, the action of $\phi_{\underline{E}}(\sigma)$ on the underlying R-module is given by $\left(\begin{smallmatrix} \phi_{\underline{H}}(\sigma) & c(\sigma) \\ 0 & 1 \end{smallmatrix}\right)$ for a certain cocycle $c : G_L \to H$. We define $d_{\underline{H}}([\underline{E}])$ to be the image of the cocycle c in $H^1(G_L, H)$. It is well-defined as if $[h] = [h']$, there exists $m \in H$ such that $h' - h \in m + F^0 H_{L^s}$. It follows that $d_{\underline{H}}([h'])(\sigma) = c(\sigma) + m - \phi_{\underline{H}}(\sigma)(m)$, and thus that $d_{\underline{H}}([h])$ and $d_{\underline{H}}([h'])$ are equivalent.

Definition 4.13. We call $d_{\underline{H}}$ the *adic realization map of* $\underline{H}$.

Proposition 4.14. *Suppose that* $H^1(G_L, H_{L^s}) = H^1(G_L, F^0 H_{L^s}) = 0$. *We have an exact sequence of R-modules*

$$0 \longrightarrow \frac{(H_{L^s})^+}{H^+ + (F^0 H_{L^s})^+} \longrightarrow \mathrm{Ext}^1_{\mathcal{MH}_R^+}(\mathbb{1}^+, \underline{H}^+) \xrightarrow{d_{\underline{H}^+}} H^1(G_L, H) \longrightarrow 0$$

where the second arrow maps the class of $h \in (H_{L^s})^+$ to the class of the extension

$$\left([h], \phi_{[h]} : \sigma \mapsto \begin{pmatrix} \phi_{\underline{H}}(\sigma) & 0 \\ 0 & 1 \end{pmatrix} \right)$$

in $\mathrm{Ext}^1_{\mathcal{MH}_R^+}(\mathbb{1}^+, \underline{H}^+)$.

Proof. For a cocycle $c : G_L \to H$, we denote by $[c]$ the R-linear G_L-representation of $H \oplus R$ given by

$$[c] : G_L \longrightarrow \mathrm{End}_R(H \oplus R), \quad \sigma \longmapsto \begin{pmatrix} \phi_{\underline{H}}(\sigma) & c(\sigma) \\ 0 & 1 \end{pmatrix}.$$

We first show that $d_{\underline{H}^+}$ is surjective. Let $c : G_L \to H$ be a cocycle. Because $H^1(G_L, H_{L^s}) = 0$, there exists $h \in H_{L^s}$ such that $c(\sigma) = h - (\varphi_{\underline{H}}(\sigma) \otimes \sigma)(h)$. It is formal to check that $[c]$ defines an infinite Frobenius for $[h]$ and that the extension given by the pair $([h], [c])$ is an element of $\operatorname{Ext}^1_{\mathcal{MH}_R^+}(\mathbb{1}^+, \underline{H}^+)$. Its image through $d_{\underline{H}^+}$ is c, as desired.

Before computing the kernel of $d_{\underline{H}^+}$, we begin with an observation. Let c be a cocycle $G_L \to H$ such that $([h], [c])$ defines an extension of $\mathbb{1}^+$ by $\underline{H}^+$ in $\mathcal{MH}_R^+$. For $m \in H$, the diagram

$$\begin{tikzcd}
0 \arrow[r] & \underline{H}^+ \arrow[r] \arrow[d, "\mathrm{id}_{\underline{H}}"] & ([h],[c]) \arrow[r] \arrow[d, "\begin{pmatrix} \mathrm{id}_H & m \\ 0 & 1 \end{pmatrix}"] & \mathbb{1}^+ \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & \underline{H}^+ \arrow[r] & ([h+m], [\sigma \mapsto c(\sigma) + m - \phi_{\underline{H}}(\sigma)(m)]) \arrow[r] & \mathbb{1}^+ \arrow[r] & 0
\end{tikzcd}$$

defines a congruence in $\mathcal{MH}_R^+$ between the extensions:

$$([h], [c]) \quad \text{and} \quad ([h+m], [\sigma \mapsto c(\sigma) + m - \phi_{\underline{H}}(\sigma)(m)]) . \tag{4.6}$$

Let us compute the kernel of $d_{\underline{H}}$. If $[\underline{E}]$ is an element of $\ker d_{\underline{H}}$ there exists $h' \in H_{L^s}$ and $m \in H$ such that $[\underline{E}]$ is congruent to an extension of the form $([h'], [\sigma \mapsto m - \phi_{\underline{H}}(\sigma)(m)])$. By our computation (4.6), we can assume without loss of generality that $[\underline{E}]$ is of the form $([h], [0])$. The condition that the infinite Frobenius of $\underline{E}$ preserves the Hodge filtration reads

$$\forall \sigma \in G_L, \quad (\phi_{\underline{H}}(\sigma) \otimes \sigma)(h) - h \in F^0 H_{L^s}.$$

In particular, $h + F^0 H_{L^s}$ is invariant under G_L in $(H_{L^s}/F^0 H_{L^s})^+$. Because $H^1(G_L, F^0 H_{L^s})$ vanishes, we have $(H_{L^s}/F^0 H_{L^s})^+ = (H_{L^s})^+/(F^0 H_{L^s})^+$. To conclude, it suffices to note that $[\underline{E}] = ([h], [0])$ is congruent to $([k], [0])$ if and only if $h - k \in H^+$. $\square$

Remark 4.15. We present a cohomological definition of $d_{\underline{H}^+}$, although less explicit, which provides an alternative proof of Proposition 4.14. Given two objects $(\underline{X}, \phi_{\underline{X}})$ and $(\underline{Y}, \phi_{\underline{Y}})$ in $\mathcal{MH}_R^+$, the R-module

$$\operatorname{Hom}_{\mathcal{MH}_R}(\underline{X}, \underline{Y})$$

is naturally endowed with a continuous action of G_L: to $\sigma \in G_L$ and a morphism $f : \underline{X} \to \underline{Y}$ in $\mathcal{MH}_R$, we define f^σ to be $\phi_{\underline{Y}}(\sigma) \circ f \circ \phi_{\underline{X}}(\sigma)^{-1}$. By definition of morphisms in $\mathcal{MH}_R^+$, we have

$$\operatorname{Hom}_{\mathcal{MH}_R}(\underline{X}, \underline{Y})^+ = \operatorname{Hom}_{\mathcal{MH}_R^+}((\underline{X}, \phi_{\underline{X}}), (\underline{Y}, \phi_{\underline{Y}}))$$

where the superscript $+$ designates the submodule of elements fixed by G_L. That is to say, the functor $\Gamma^+ : \mathcal{MH}_R^+ \to \mathsf{Mod}_R$, which associates $\operatorname{Hom}_{\mathcal{MH}_R^+}(\mathbb{1}^+, \underline{H}^+)$ to $\underline{H}^+ = (\underline{H}, \phi_{\underline{H}})$, factors through

$$\begin{tikzcd}
\mathcal{MH}_R^+ \arrow[r, "\Gamma"] \arrow[dr, "\Gamma^+"'] & \operatorname{Rep}_R(G_L) \arrow[d, "\Gamma_{G_L}"] \\
& \mathsf{Mod}_R
\end{tikzcd}$$

where Γ assignates $\mathrm{Hom}_{\mathcal{MH}_R}(\mathbb{1}, \underline{H})$ equipped with its continuous action of G_L to $\underline{H}^+$ and Γ_{G_L} takes the invariant under the action of G_L. Now, if the weights of $\underline{H}$ are negative, Corollary 4.12 implies that there is a distinguished triangle in the derived category of $R[G_L]$-modules:

$$H[-1] \oplus F^0 H_{L^s}[-1] \longrightarrow H_{L^s}[-1] \longrightarrow \mathrm{R}\Gamma(\underline{H}) \longrightarrow [1].$$

By the composition Theorem for right-derived functor, there is a canonical isomorphism $\mathrm{R}\Gamma_{G_L} \circ \mathrm{R}\Gamma \cong \mathrm{R}\Gamma^+$. Applying $\mathrm{R}\Gamma_{G_L}$ to the above triangle yields another a distinguished triangle

$$\mathrm{R}\Gamma_{G_L}(H)[-1] \oplus \mathrm{R}\Gamma_{G_L}(F^0 H_{L^s})[-1] \longrightarrow \mathrm{R}\Gamma_{G_L}(H_{L^s})[-1] \longrightarrow \mathrm{R}\Gamma^+(\underline{H}) \longrightarrow [1].$$

We obtain a long exact sequence of R-modules

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^0_{\mathcal{MH}_R^+}(\mathbb{1}^+, \underline{H}^+) \arrow[r] & H^+ \oplus (F^0 H_{L^s})^+ \arrow[r] & (H_{L^s})^+ \arrow[dll, rounded corners] \\
& \mathrm{Ext}^1_{\mathcal{MH}_R^+}(\mathbb{1}^+, \underline{H}^+) \arrow[r] & H^1(G_L, H) \oplus H^1(G_L, F^0 H_{L^s}) \arrow[r] & H^1(G_L, H_{L^s})
\end{tikzcd}$$

The first morphism on the bottom row recovers $d_{\underline{H}^+}$ as the induced morphism

$$d_{\underline{H}^+} : \mathrm{Ext}^1_{\mathcal{MH}_R^+}(\mathbb{1}^+, \underline{H}^+) \longrightarrow H^1(G_L, H).$$

If $H^1(G_L, F^0 H_{L^s})$ and $H^1(G_L, H_{L^s})$ vanishes, we obtain the exact sequence of Proposition 4.14.

4.3 Mixed Hodge-Pink structures

In this section, we discuss mixed Hodge-Pink structures (MHPS) as in [Pin] and compare them with MHS. The main innovation here is the study of *Hodge descent* (Subsection 4.3.4). This notion will be used in Chapter 5 to define the Hodge realization functor.

4.3.1 Completion along the diagonal

We begin with some preliminaries on the ring $L[\![\mathfrak{j}]\!]$ for a complete subfield L of $\mathbb{C}_\infty$ containing K, generalizing Pink's $\mathbb{C}_\infty[\![z-\zeta]\!]$. Let L be an A-field. We let $L[\![\mathfrak{j}]\!]$ be the completion of $A \otimes L$ at the ideal $\mathfrak{j} = \mathfrak{j}_\kappa$ generated by the set $\{a \otimes 1 - 1 \otimes \kappa(a) \mid a \in A\}$:

$$L[\![\mathfrak{j}]\!] = \varprojlim_n A \otimes L/\mathfrak{j}^n.$$

This is a discrete valuation ring with maximal ideal $\mathfrak{j}$, residue field L, and we let $L(\!(\mathfrak{j})\!)$ denote its fraction field.

From now on we assume that L is a complete subfield of $\mathbb{C}_\infty$ that contains K. Before introducing MHPS, we begin by some prelimary results on the ring $L[\![\mathfrak{j}]\!]$, where $\kappa : A \to L$ is the inclusion. The next lemma extends [Pin, Prop. 3.1].

Lemma 4.16. *The map $\nu : A \to A \otimes L$, $a \mapsto a \otimes 1$ defines a ring homomorphism $A \to L[\![\mathfrak{j}]\!]$ which extends uniquely to a ring homomorphism $\nu : K_\infty \to L[\![\mathfrak{j}]\!]$ such that the composition of ν followed by reduction modulo $\mathfrak{j}$ coincide with the canonical inclusion $K_\infty \hookrightarrow L \cong L[\![\mathfrak{j}]\!]/\mathfrak{j}$.*

Proof. Uniqueness is clear. We proceed in three steps for the existence. The first step is to extend ν to K. Let $a \in A$. We have $a \otimes 1 \cong 1 \otimes a \pmod{\mathfrak{j}}$. Additionally, $L[\![\mathfrak{j}]\!]$ is a discrete valuation ring with maximal ideal $\mathfrak{j}$ and residue field L. Hence, if a is nonzero, $a \otimes 1$ is invertible because a is invertible in L. This extends ν to K.

Let $\pi_\infty \in K$ be a uniformizing parameter for K_∞ and let a, $b \neq 0$ be elements of A such that $\pi_\infty = a/b$. We have the identification $K_\infty = \mathbb{F}_\infty(\!(\pi_\infty)\!)$ where $\mathbb{F}_\infty$ is the residue field of K_∞. Let E be the subfield $E := \mathbb{F}(\!(\pi_\infty)\!)$ of K_∞. Our second step is to extend ν to E. Following Pink's observation [Pin, Prop. 3.1], we unfold the formal computation

$$\begin{aligned} E \ni \sum_k (f_k \pi_\infty^k \otimes 1) &= \sum_k f_k (1 \otimes \pi_\infty + \pi_\infty \otimes 1 - 1 \otimes \pi_\infty)^k \\ &= \sum_k f_k \left(1 \otimes \pi_\infty + \frac{a \otimes b - b \otimes a}{b \otimes b}\right)^k \\ &= \sum_k f_k \sum_{\ell \geq 0} \binom{k}{l} \left(\frac{a \otimes b - b \otimes a}{b \otimes b}\right)^\ell (1 \otimes \pi_\infty)^{k-\ell} \\ &= \sum_{\ell \geq 0} \left(1 \otimes \sum_k f_k \binom{l}{k} \pi_\infty^{k-\ell}\right) \left(\frac{a \otimes b - b \otimes a}{b \otimes b}\right)^\ell \end{aligned}$$

where the inner sum converges in K_∞. We then set

$$\nu\left(\sum_k f_k \pi_\infty^k\right) := \sum_{\ell \geq 0} \left(1 \otimes \sum_k f_k \binom{l}{k} \pi_\infty^{k-\ell}\right) \left(\frac{a \otimes b - b \otimes a}{b \otimes b}\right)^\ell \in L[\![\mathfrak{j}]\!].$$

It is formal to check that this defines a ring homomorphism $E \to L[\![\mathfrak{j}]\!]$ which extends ν.

Finally, we extend ν to K_∞. Let $\alpha \in K_\infty$ and let $p_\alpha(X)$ be the minimal polynomial of α over E. As K_∞/E is a separable extension, $p_\alpha(X) \in E[X]$ is separable. We consider $p_\alpha(X)$ as a polynomial in $L[\![\mathfrak{j}]\!][X]$ via $\nu : E \to L[\![\mathfrak{j}]\!]$. It admits the image of α through $K_\infty \hookrightarrow L \cong L[\![\mathfrak{j}]\!]/\mathfrak{j}$ as a root modulo $\mathfrak{j}$. By Hensel's Lemma, $p_\alpha(X)$ admits a unique root $\tilde{\alpha}$ in $L[\![\mathfrak{j}]\!]$ which lifts α. Setting $\nu(\alpha) = \tilde{\alpha}$ extends ν to $K_\infty \to L[\![\mathfrak{j}]\!]$ in a morphism which verifies the assumption of the lemma. $\square$

The next lemma follows from the above construction:

Lemma 4.17. *The kernel* $\mathfrak{v}$ *of* $\nu\otimes\mathrm{id}: K_\infty\otimes L\to L[\![\mathfrak{j}]\!]$ *is the ideal of* $K_\infty\otimes L$ *generated by the set* $\{f\otimes 1-1\otimes f \mid f\in\mathbb{F}_\infty\}$.

Proof. Let $d_\infty := [\mathbb{F}_\infty:\mathbb{F}]$. For $i\in\mathbb{Z}/d_\infty\mathbb{Z}$, we consider the ideal of $K_\infty\otimes L$ given by

$$\mathfrak{d}^{(i)}=\langle\{f\otimes 1-1\otimes f^{q^i}\mid f\in\mathbb{F}_\infty\}\rangle.$$

It is the kernel of the map $K_\infty\otimes L\to L$, $a\otimes b\mapsto ab^{q^i}$, hence is a maximal ideal. For $f\in\mathbb{F}_\infty$, the polynomial $\prod_{i\in\mathbb{Z}/d_\infty\mathbb{Z}}(x-f^{q^i})$ belongs to $\mathbb{F}[x]$, and thus the product of the $\mathfrak{d}^{(i)}$ is zero. By the chinese remainders Theorem, we have

$$K_\infty\otimes L=K_\infty\otimes L/\mathfrak{d}^{(0)}\mathfrak{d}^{(1)}\cdots\mathfrak{d}^{(d_\infty-1)}=\prod_{i\in\mathbb{Z}/d_\infty\mathbb{Z}}K_\infty\otimes L/\mathfrak{d}^{(i)}$$

which is a product of d_∞ fields. Because $\mathfrak{v}$ is a prime ideal of $K_\infty\otimes L$, we have $\mathfrak{v}=\mathfrak{d}^{(i)}$ for some i. If $f\in\mathbb{F}_\infty$, then $f\otimes 1-1\otimes f$ belongs to $\mathfrak{v}$ by definition. We deduce that $i=0$. ☐

4.3.2 The categories $\mathcal{MHP}$ and $\mathcal{MHP}^+$

The next definitions are inspired by [Pin, Def. 3.2]. We slightly generalize Pink's setting to handle a general coefficient ring R and A not necessarily a polynomial ring.

Definitions of mixed Hodge-Pink structures

Let R be a Noetherian subring of K_∞ that contains A and such that $R\otimes_A K$ is a field. As in the previous ssubsection, let L be a complete subfield of $\mathbb{C}_\infty$ that contains K.

Let $\nu: K_\infty\to L[\![\mathfrak{j}]\!]$ be the A-algebra morphism of Lemma 4.16. Given a R-module H, we obtain a $L[\![\mathfrak{j}]\!]$-module $H\otimes_{R,\nu}L[\![\mathfrak{j}]\!]$ by tensoring H with $L[\![\mathfrak{j}]\!]$ seen as an R-algebra via ν.

Compared to Definition 4.4, we obtain MHPS by replacing the data of the Hodge filtration by the data of an $L^s[\![\mathfrak{j}]\!]$-lattice:

Definition 4.18. A *pre-mixed Hodge-Pink structure* $\underline{H}$ (*with base field* L, *coefficients ring* R) consists of a triple $(H,W,\mathfrak{q})$ where

- H is a finitely generated R-module,
- W is a $\mathbb{Q}$-graded increasing filtration of $H_K=H\otimes_A K$ by sub-$(R\otimes_A K)$-vector spaces which is exhaustive, separated,
- $\mathfrak{q}$ is a $L^s[\![\mathfrak{j}]\!]$-lattice in the $L^s(\!(\mathfrak{j})\!)$-vector space $H\otimes_{R,\nu}L^s(\!(\mathfrak{j})\!)$, that is, $\mathfrak{q}$ is a finitely generated $L^s[\![\mathfrak{j}]\!]$-module in $H\otimes_{R,\nu}L^s(\!(\mathfrak{j})\!)$ that contains a basis.

We call W the *weight filtration of* $\underline{H}$ and $\mathfrak{q}$ the *Hodge-Pink lattice of* $\underline{H}$. We define the *tautological lattice of* $\underline{H}$ to be $\mathfrak{p} := H \otimes_{R,\nu} L^s[\![\mathfrak{j}]\!]$.

As for filtrations, we have a notion of degree for lattices. For any $L^s[\![\mathfrak{j}]\!]$-lattice $\mathfrak{l}$ contained both in $\mathfrak{q}$ and $\mathfrak{p}$, the quotients $\mathfrak{q}/\mathfrak{l}$ and $\mathfrak{p}/\mathfrak{l}$ are finite dimensional L^s-vector spaces. Under this observation, we let the *Hodge-Pink degree of* $\underline{H}$ be the integer

$$\deg_{\mathfrak{q}}(\underline{H}) := \dim_{L^s}\left(\frac{\mathfrak{q}}{\mathfrak{p}\cap\mathfrak{q}}\right) - \dim_{L^s}\left(\frac{\mathfrak{p}}{\mathfrak{p}\cap\mathfrak{q}}\right).$$

Let $\underline{H} = (H, W, \mathfrak{q})$ and $\underline{H}' = (H', W', \mathfrak{q}')$ be two pre-mixed Hodge-Pink structures. The next definitions are borrowed from [Pin, Def. 3.7]:

(a) A morphism $f : \underline{H} \to \underline{H}'$ is an R-linear morphism $f : H \to H'$ such that $f_K = f \otimes_A \mathrm{id}_K : H_K \to H'_K$ preserves the weight filtration and $f_{L^s(\!(\mathfrak{j})\!)} = f \otimes_{R,\nu} \mathrm{id}_{L^s(\!(\mathfrak{j})\!)} : H \otimes_{R,\nu} L^s(\!(\mathfrak{j})\!) \to H' \otimes_{R,\nu} L^s(\!(\mathfrak{j})\!)$ satisfies $f_{L^s(\!(\mathfrak{j})\!)}(\mathfrak{q}) \subset \mathfrak{q}'$.

(b) A morphism $f : \underline{H} \to \underline{H}'$ is *strict* if f_K is strictly compatible with the weight filtrations, and if $f_{L^s(\!(\mathfrak{j})\!)}$ satisfies

$$f_{L^s(\!(\mathfrak{j})\!)}(\mathfrak{q}) = \mathfrak{q}' \cap f_{L^s(\!(\mathfrak{j})\!)}(H \otimes_{R,\nu} L^s(\!(\mathfrak{j})\!)).$$

As for $\deg^W$, $\deg_{\mathfrak{q}}$ is additive in strict short exact sequences.

We form the category of pre-mixed Hodge-Pink structures with morphism as in (a).

Let $\underline{H} = (H, W, \mathfrak{q})$ be a pre-mixed Hodge-Pink structure. A sub-R-module $H' \subset H$ defines a subobject $\underline{H}' = (H', W', \mathfrak{q}')$ of $\underline{H}$ by taking for W' the induced filtration on H' by W, and for $\mathfrak{q}'$ the lattice $\mathfrak{q} \cap (H' \otimes_{R,\nu} L^s(\!(\mathfrak{j})\!))$. The canonical morphism $\underline{H}' \hookrightarrow \underline{H}$ is strict. Similarly, the quotient $\underline{H}/\underline{H}'$ is defined so that the underlying module is H/H', its weight filtration W'' is the filtration on H_K/H'_K induced by W, and its Hodge-Pink lattice $\mathfrak{q}''$ is $\mathfrak{q}/\mathfrak{q}'$. The canonical morphism $\underline{H} \to \underline{H}/\underline{H}'$ is strict.

The next definition is inspired by [Pin, Def. 4.5].

Definition 4.19. Let $\underline{H} = (H, W, \mathfrak{q})$ be a pre-mixed Hodge-Pink structure with coefficients ring R.

- If $R = K_\infty$, we call $\underline{H}$ *locally semistable* if for each K_∞-subspace $H' \subset H$, we have
$$\deg_{\mathfrak{q}}(\underline{H}') \leq \deg^W(\underline{H}'),$$
with equality whenever $H'_K = W_\mu H_K$ for some $\mu \in \mathbb{Q}$.

- For general coefficients R, we call $\underline{H} = (H, W, \mathfrak{q})$ a *mixed Hodge-Pink structure* if $\underline{H}_{K_\infty}$ is *locally semistable*, where $\underline{H}_{K_\infty}$ is the pre-mixed Hodge structure with coefficients in K_∞ given by $(H \otimes_R K_\infty, W \otimes_R K_\infty, \mathfrak{q})$.

We form the category $\mathcal{MHP}_R$ as the full subcategory of the category of pre-mixed Hodge-Pink structures whose objects are mixed Hodge-Pink structures.

The triplet $(R, W, \mathfrak{q})$ where $W := \mathbf{1}_{\mu\geq 0}(R \otimes_A K)$ and $\mathfrak{q} = L^s[\![\mathfrak{j}]\!]$ defines a mixed Hodge-Pink structure. It is the *unit mixed Hodge-Pink structure with coefficients ring* R, we denote it $\mathbb{1}$.

The following Proposition is due to Pink [Pin, Thm. 4.15]. Although our setting differs slightly, it is straighforward to adapt Pink's proof to our context.

Proposition 4.20. *Every morphism of mixed Hodge-Pink structures is strict. The category* $\mathcal{MHP}_R$ *is abelian.*

Definition 4.21. An *infinite Frobenius for* $\underline{H} = (H, W, \mathfrak{q})$ an object of $\mathcal{MHP}_R$ is an R-linear continuous representation $\phi_{\underline{H}} : G_L \to \operatorname{End}_R(H)$, H carrying the discrete topology, such that, for all $\sigma \in G_L$,

1. $\phi_{\underline{H}}(\sigma) \otimes_A \mathrm{id}_K : H_K \to H_K$ preserves the weight filtration,
2. $\phi_{\underline{H}}(\sigma) \otimes_R \sigma : H_{L^s(\!(\mathfrak{j})\!)} \to H_{L^s(\!(\mathfrak{j})\!)}$ preserves the Hodge-Pink lattice, that is, we have $(\phi_{\underline{H}}(\sigma) \otimes_R \sigma)(\mathfrak{q}) \subset \mathfrak{q}$.

We let $\mathcal{MHP}_R^+$ denote the category whose objects are pairs $(\underline{H}, \phi_{\underline{H}})$ where $\underline{H}$ is a mixed Hodge structure and where $\phi_{\underline{H}}$ is an infinite Frobenius for $\underline{H}$. Morphisms in $\mathcal{MHP}_R^+$ are morphisms in $\mathcal{MHP}_R$ compatible with the infinite Frobenius.

It follows easily from Proposition 4.20 that:

Proposition 4.22. *The category* $\mathcal{MHP}_R^+$ *is abelian.*

Induced Hodge filtration

We now discuss the relation between pre-MHPS and pre-MHS. Let $\underline{H} = (H, W, \mathfrak{q})$ be a pre-mixed Hodge-Pink structure over R. $\underline{H}$ induces a pre-mixed Hodge structure $\underline{H}^{\#}$ as follows (see [Pin, Def. 3.5]). By Lemma 4.16, the reduction modulo $\mathfrak{j}$:

$$\mathfrak{p} := H \otimes_{R,\nu} L^s[\![\mathfrak{j}]\!] \longrightarrow H \otimes_R L^s =: H_{L^s}$$

identifies $\mathfrak{p}/\mathfrak{j}\mathfrak{p}$ with H_{L^s}. We define a $\mathbb{Z}$-graded decreasing, exhaustive and separated filtration F on H_{L^s} by setting, for all $p \in \mathbb{Z}$, $F^p H_{L^s}$ to be the image of $\mathfrak{p} \cap \mathfrak{j}^p \mathfrak{q}$ in $\mathfrak{p}/\mathfrak{j}\mathfrak{p}$.

Definition 4.23. We define $\underline{H}^{\#}$ to be the pre-mixed Hodge structure (H, W, F), and we call F *the induced Hodge filtration.* We let $\deg_F(\underline{H})$ be the degree of the filtration F.

The next lemma is immediate:

Lemma 4.24 (Functoriality of the Hodge filtration)**.** *Let $f : \underline{H} \to \underline{H}'$ be a morphism of pre-mixed Hodge-Pink structures over R. Then, $f_{L^s} = f \otimes_R \mathrm{id}_{L^s}$ is compatible with the Hodge filtration, that is, for all integers p we have $f_{L^s}(F^p H_{L^s}) \subset F^p H'_{L^s}$. Furthermore, $\deg_{\mathfrak{q}}(\underline{H}) = \deg_F(\underline{H})$.*

Remark 4.25. Be aware that a strict morphism $\underline{H}' \to \underline{H}$ of pre-mixed Hodge-Pink structures does not necessarily induce a strict morphism $\underline{H}'^{\#} \to \underline{H}^{\#}$ of pre-mixed Hodge structures. As a consequence, if $\underline{H}' \to \underline{H}$ is an inclusion of subobject, the inequality (following from Lemma 4.24)

$$\deg_{\mathfrak{q}}(\underline{H}') \leq \deg_{F|H'}(\underline{H}')$$

might not be an equality.

Remark 4.26. Related to the above, that $\underline{H}$ is a mixed Hodge-Pink structure does not necessarily imply that $\underline{H}^{\#}$ is a mixed Hodge structure. We use Pink's [Pin, Ex. 6.14] as a counter-example. Consider the pre-mixed Hodge-Pink structure $\underline{H}$ pure of weight 0 where $H = R^{\oplus 2}$ and with Hodge-Pink lattice

$$\mathfrak{q} = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} q_e \\ 1 \end{pmatrix} \right\rangle_{L^s[\![\mathfrak{j}]\!]}$$

where $q_e \in L^s(\!(\mathfrak{j})\!)$ is an element of valuation $-e$ for an integer $e > 0$. By [Pin, Cor. 4.12], $\underline{H}$ is locally semistable and hence defines an object in $\mathcal{MHP}_R$. Consider H' the R-submodule of H generated by $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. It defines a strict subobject of $\underline{H}$ whose Hodge-Pink lattice is

$$\mathfrak{q}' = \mathfrak{q} \cap (H' \otimes_{R,\nu} L^s(\!(\mathfrak{j})\!)) = H' \otimes_{R,\nu} L^s[\![\mathfrak{j}]\!] =: \mathfrak{p}'.$$

We then have

$$\deg_{\mathfrak{q}}(\underline{H}') = \dim_{L^s}\left(\frac{\mathfrak{q}'}{\mathfrak{p}' \cap \mathfrak{q}'}\right) - \dim_{L^s}\left(\frac{\mathfrak{p}'}{\mathfrak{p}' \cap \mathfrak{q}'}\right) = 0.$$

On the other-hand, the induced Hodge filtration on H_{L^s} turns out to be

$$F^p H_{L^s} = \begin{cases} 0 & \text{if } p > e \\ H'_{L^s} & \text{if } e \geq p > -e \\ H_{L^s} & \text{if } p \leq -e \end{cases}$$

As such, $\deg_F((\underline{H}')^{\#}) = e > 0$ although $\deg^W((\underline{H}')^{\#}) = 0$. Hence, $\underline{H}^{\#}$ is not locally semistable.

Let $\underline{H} = (H, W, \mathfrak{q})$ be an object in $\mathcal{MHP}_R$ and let r be the dimension of $H \otimes_A K$ over $R \otimes_A K$. Let $\mathfrak{p}$ be the tautological lattice of $\underline{H}$. By the relative version of the elementary divisors Theorem applied to the discrete valuation ring $L^s[\![\mathfrak{j}]\!]$, there exists a family of integers $(w_1, ..., w_r)$ sorted by ascending order such that, for any large enough integer e for which $\mathfrak{j}^e\mathfrak{p} \subset \mathfrak{q}$ and $\mathfrak{j}^e\mathfrak{q} \subset \mathfrak{p}$, we have

$$\mathfrak{q}/\mathfrak{j}^e\mathfrak{p} \cong \bigoplus_{i=1}^{r} L^s[\![\mathfrak{j}]\!]/\mathfrak{j}^{e+w_i}, \quad \mathfrak{p}/\mathfrak{j}^e\mathfrak{q} \cong \bigoplus_{i=1}^{r} L^s[\![\mathfrak{j}]\!]/\mathfrak{j}^{e-w_i}.$$

The next result is immediate from Definition 4.23 (see also [HarJu, Rmk. 2.2.4]).

Lemma 4.27. *For all $p \in \mathbb{Z}$, we have*

$$\dim_{L^s}(F^p H_{L^s}) = \#\{i \in \{1, ..., r\} \mid p = w_i\}$$

where F is the induced Hodge filtration. In particular, the elements of $\{w_1, ..., w_r\}$ are the breaks of F.

Definition 4.28. The *Hodge polygon of $\underline{H}$* is the polygon of length r whose multiplicity at any $x \in \mathbb{R}$ is $\#\{i \in \{1, ..., r\} | x = w_i\}$. We denote it by $\operatorname{HodPol}(\underline{H})$. By Lemma 4.27, the Hodge polygon of $\underline{H}$ is the polygon of the induced filtration F.

4.3.3 Hodge additivity

We review some materials of [Pin, §7]. In order to characterize the *Tannakian Hodge group* of a mixed Hodge-Pink structure, Pink introduced the notion of *Hodge additivity*. The very same notion will allow us to define a sub-abelian category $\mathcal{MHP}_R^{\text{ha}}$ of $\mathcal{MHP}_R$ on which the assignation $\underline{H} \mapsto \underline{H}^{\#}$ defines an exact functor

$$\mathcal{MHP}_R^{\text{ha}} \longrightarrow \mathcal{MH}_R.$$

We first need the concept of *semisimplification* in abelian categories with finite length. Let $\mathcal{A}$ be an abelian category and let X be an object of $\mathcal{A}$. We refer to [EGNO, Def. 1.5.3] for the next definition:

Definition 4.29. (*i*) X is called *simple* if it is nonzero and 0 and X are its only subobjects.

(*ii*) X is said to have *finite length* if there exists a sequence of inclusions

$$0 = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X \tag{4.7}$$

such that X_i/X_{i-1} is simple for all i. Such a filtration is called a *Jordan-Hölder series* of X. We will say that this Jordan-Hölder series contains a simple object Y with multiplicity m if the number of values of i for which X_i/X_{i-1} is isomorphic to Y is m.

We refer to [EGNO, Thm. 1.5.4] for the next lemma.

Lemma 4.30 (Jordan-Hölder)**.** *Suppose that X has finite length. Then any increasing sequence of inclusions*

$$0 = Z_0 \subset Z_1 \subset \cdots \subset Z_{s-1} \subset Z_s = X$$

can be completed into a Jordan-Hölder series of X, and any two Jordan-Hölder series of X contain any simple object with the same multiplicity, so in particular have the same length.

Definition 4.31. Fix a Jordan-Hölder series for X as in (4.7). The associated *semisimplification of* X, denoted X^{ss}, is the object of $\mathcal{A}$ given by

$$X^{\mathrm{ss}} := \bigoplus_{i=1}^{n} X_i/X_{i-1}.$$

By Jordan-Hölder (Lemma 4.30), X^{ss} does not depend on the Jordan-Hölder series of X, up to isomorphisms. We call n the *length* of $\underline{X}$.

We come back to the category $\mathcal{MHP}_R$. In the case where the coefficient ring R is K_∞, any object $\underline{H}$ of $\mathcal{MHP}_{K_\infty}$ has finite length (this can fail for R not a field). Indeed, any subobject $\underline{H}'$ of $\underline{H}$ is such that the inclusion $\underline{H}' \hookrightarrow \underline{H}$ is strict, and hence is determined by its underlying K_∞-vector space. A Jordan-Hölder series for $\underline{H}$ is then constructed by immediate induction. It yields the inequality:

$$\operatorname{length}(\underline{H}) \leq \dim_{K_\infty}(H). \tag{4.8}$$

Remark 4.32. Note however that (4.8) may not be an equality as not every K_∞-subspace H' of H equipped with the induced filtrations is locally semistable (we do not necessarily have $\deg_F(\underline{H}') = \deg^W(\underline{H}')$.)

By Jordan-Hölder (Lemma 4.30), the Hodge polygon (Definition 4.28) of a semisimplification $\underline{H}^{\mathrm{ss}}$ attached to a Jordan-Hölder series of an object $\underline{H}$ of $\mathcal{MHP}_{K_\infty}$ does not depend on the chosen Jordan-Hölder series. We denote it by $\operatorname{HodPol}(\underline{H}^{\mathrm{ss}})$. The next lemma is adapted from [Pin, Prop. 6.9] (see also [Kat2, Lem. 1.2.3]).

Lemma 4.33. *Consider an exact sequence* $0 \to \underline{H}' \to \underline{H} \to \underline{H}'' \to 0$ *in* $\mathcal{MHP}_{K_\infty}$. *The Hodge polygon of* $\underline{H}' \oplus \underline{H}''$ *is above that of* $\underline{H}$ *and has the same endpoints. In particular, the Hodge polygon of* $\underline{H}^{ss}$ *is above that of* $\underline{H}$.

Before proving Lemma 4.33, some notations are called for. For P and P' two polygons of length n and n', we write $P \sqcup P'$ for the polygon of length $n+n'$ given by the slope-by-slope concatenation of P and P': if P (resp. P') has multiplicity m (resp. m') at the slope $x \in \mathbb{R}$, then $P \sqcup P'$ has multiplicity $m+m'$ at x. Note that, for $\underline{H}$ and $\underline{H}'$ two objects in $\mathcal{MHP}_R$, we have

$$\operatorname{HodPol}(\underline{H} \oplus \underline{H}') = \operatorname{HodPol}(\underline{H}) \sqcup \operatorname{HodPol}(\underline{H}').$$

If $n = n'$, we write $P \leq P'$ if P' lies above P, that is, if for every point (x,y) of P, the point with the same abscissa (x,y') of P' satisfies $y \leq y'$.

Proof of Lemma 4.33. Only the second assertion does not already appear in [Pin, Prop. 6.9] (nor [Kat2, Lem. 1.2.3]). It follows by induction on the length n of $\underline{H}$. If $n=1$, $\underline{H}$ is simple and $\underline{H} = \underline{H}^{\mathrm{ss}}$. We fix $n \geq 2$ and assume the assertion proven for $n-1$. We consider a Jordan-Hölder series of $\underline{H}$:

$$0 = \underline{H}_0 \subsetneq \underline{H}_1 \subsetneq \cdots \subsetneq \underline{H}_n = \underline{H}.$$

By assumption, $\mathrm{HodPol}(\underline{H}_{n-1}) \leq \mathrm{HodPol}(\underline{H}_{n-1}^{\mathrm{ss}})$. The first part of the lemma applied to the exact sequence $0 \to \underline{H}_{n-1} \to \underline{H} \to \underline{H}/\underline{H}_{n-1} \to 0$ yields:

$$\begin{aligned}\mathrm{HodPol}(\underline{H}) &\leq \mathrm{HodPol}(\underline{H}_{n-1} \oplus \underline{H}/\underline{H}_{n-1}) \\ &= \mathrm{HodPol}(\underline{H}_{n-1}) \sqcup \mathrm{HodPol}(\underline{H}/\underline{H}_{n-1}) \\ &\leq \mathrm{HodPol}(\underline{H}_{n-1}^{\mathrm{ss}}) \sqcup \mathrm{HodPol}(\underline{H}/\underline{H}_{n-1}) \\ &= \mathrm{HodPol}(\underline{H}_{n-1}^{\mathrm{ss}} \oplus \underline{H}/\underline{H}_{n-1}) \\ &= \mathrm{HodPol}(\underline{H})^{\mathrm{ss}}\end{aligned}$$

as desired. $\square$

Following [Pin, Def. 7.1 (a)], we define *Hodge additivity* as follows:

Definition 4.34. Let $\underline{H}$ be an object of $\mathcal{MHP}_R$.

- If $R = K_\infty$, we say that $\underline{H}$ is *Hodge additive* if the Hodge polygons of $\underline{H}^{\mathrm{ss}}$ and $\underline{H}$ coincide.
- For general ring R, we say that $\underline{H} = (H, W, \mathfrak{q})$ in $\mathcal{MHP}_R$ is *Hodge additive* if $\underline{H}_{K_\infty} := (H \otimes_R K_\infty, W \otimes_R K_\infty, \mathfrak{q})$ is Hodge additive as an object of $\mathcal{MHP}_{K_\infty}$.

Remark 4.35. Let us use the notations of Remark 4.26. Assuming first $R = K_\infty$, we easily find out that $0 \subsetneq \underline{H}' \subsetneq \underline{H}$ is a Jordan-Hölder series of $\underline{H}$ with respective quotients isomorphic to $\mathbb{1}$. By Remark 4.26, the Hodge polygon of $\underline{H}^{\mathrm{ss}} \cong \mathbb{1}^{\oplus 2}$ and that of $\underline{H}$ do not coincide as q_e has valuation < 0. Hence, for general coefficient ring R, $\underline{H}$ is not Hodge additive. On the contrary, if q_e has non negative valuation, $\underline{H}$ is Hodge additive.

Our next objective is to define the category of Hodge additive objects, and show that it is abelian. We begin with a crucial observation.

Lemma 4.36. *Let $\underline{H}$ be Hodge additive. Any subobject or quotient of $\underline{H}$ is also Hodge additive. Any exact sequence $0 \to \underline{H}' \to \underline{H} \to \underline{H}'' \to 0$ in $\mathcal{MHP}_R$ is such that the Hodge polygons of $\underline{H}' \oplus \underline{H}''$ and $\underline{H}$ coincide.*

Proof. By Definition 4.34, we may assume $R = K_\infty$. In $\mathcal{MHP}_{K_\infty}$, consider an exact sequence $0 \to \underline{H}' \to \underline{H} \to \underline{H}'' \to 0$. By Lemma 4.33, we have

$$\mathrm{HodPol}(\underline{H}) \leq \mathrm{HodPol}(\underline{H}') \sqcup \mathrm{HodPol}(\underline{H}'').$$

By Jordan-Hölder (Lemma 4.30), there exists a Jordan-Hölder series $(\underline{H}_i)_{0\leq i\leq n}$ for $\underline{H}$ such that $(\underline{H}_i)_{0\leq i\leq m}$ and $(\underline{H}_i/\underline{H}')_{m\leq i\leq n}$ are Jordan-Hölder series for $\underline{H}'$ and $\underline{H}''$ respectively. Because $\underline{H}$ is Hodge additive, it follows that:

$$\mathrm{HodPol}(\underline{H}) = \mathrm{HodPol}(\underline{H}^{\mathrm{ss}}) = \mathrm{HodPol}(\underline{H}'^{\mathrm{ss}}) \sqcup \mathrm{HodPol}(\underline{H}''^{\mathrm{ss}}).$$

By Lemma 4.33 again, we obtain:

$$\mathrm{HodPol}(\underline{H}') \sqcup \mathrm{HodPol}(\underline{H}'') \leq \mathrm{HodPol}(\underline{H}).$$

This proves the second assertion. The first assertion follows since

$$\mathrm{HodPol}(\underline{H}') \sqcup \mathrm{HodPol}(\underline{H}'') \leq \mathrm{HodPol}(\underline{H}')^{\mathrm{ss}} \sqcup \mathrm{HodPol}(\underline{H}'')^{\mathrm{ss}}$$

is an equality. $\square$

Conversely, we have:

Lemma 4.37. *If $0 \to \underline{H}' \to \underline{H} \to \underline{H}'' \to 0$ is an exact sequence in $\mathcal{MHP}_R$ with $\underline{H}'$ and $\underline{H}''$ Hodge additive and such that the Hodge polygons of $\underline{H}' \oplus \underline{H}''$ and $\underline{H}$ coincide, then $\underline{H}$ is Hodge additive.*

Proof. We have

$$\begin{aligned} \mathrm{HodPol}(\underline{H})^{\mathrm{ss}} &= \mathrm{HodPol}(\underline{H}')^{\mathrm{ss}} \sqcup \mathrm{HodPol}(\underline{H}'')^{\mathrm{ss}} \\ &= \mathrm{HodPol}(\underline{H}') \sqcup \mathrm{HodPol}(\underline{H}'') \\ &= \mathrm{HodPol}(\underline{H}) \end{aligned}$$

as desired. $\square$

Definition 4.38. We let $\mathcal{MHP}_R^{\mathrm{ha}}$ be the full subcategory of $\mathcal{MHP}_R$ whose objects are Hodge additive.

The next proposition is an immediate consequence of Lemma 4.36 (compare with [Pin, Thm. 7.9]):

Proposition 4.39. *The category $\mathcal{MHP}_R^{ha}$ is R-linear abelian.*

An important feature of the category $\mathcal{MHP}_R^{\mathrm{ha}}$ is that it preserves the exactness of the induced filtration. We end this subsection by quoting the corresponding statement of Pink ([Pin, Prop. 6.12]):

Proposition 4.40. *Let $0 \to \underline{H}' \to \underline{H} \to \underline{H}'' \to 0$ be an exact sequence in $\mathcal{MHP}_R$. The following are equivalent:*

- (a) *the Hodge poylgons of $\underline{H}' \oplus \underline{H}''$ and $\underline{H}$ coincide,*
- (b) *For all integer p, the sequence $0 \to F^pH'_{L^s} \to F^pH_{L^s} \to F^pH''_{L^s} \to 0$ is exact in the category of L^s-vector spaces.*

Proof. Suppose (b). We obtain $\dim(F^pH_{L^s}) = \dim(F^pH'_{L^s}) + \dim(F^pH''_{L^s})$ for all integers p, and (a) follows.

Conversely, let j be a uniformizing parameter of the complete discrete valuation ring $L^s[\![\mathfrak{j}]\!]$. For an integer p, consider the commutative diagram of L^s-vector spaces with exact columns:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{p}' \cap \mathfrak{j}^{p-1}\mathfrak{q}' \arrow[r] \arrow[d, hook, "\times j"] & \mathfrak{p} \cap \mathfrak{j}^{p-1}\mathfrak{q} \arrow[r] \arrow[d, hook, "\times j"] & \mathfrak{p}'' \cap \mathfrak{j}^{p-1}\mathfrak{q}'' \arrow[r] \arrow[d, hook, "\times j"] & 0 \\ 0 \arrow[r] & \mathfrak{p}' \cap \mathfrak{j}^{p}\mathfrak{q}' \arrow[r, hook] \arrow[d, two heads] & \mathfrak{p} \cap \mathfrak{j}^{p}\mathfrak{q} \arrow[r] \arrow[d, two heads] & \mathfrak{p}'' \cap \mathfrak{j}^{p}\mathfrak{q}'' \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & F^pH'_{L^s} \arrow[r] & F^pH_{L^s} \arrow[r] & F^pH''_{L^s} \arrow[r] & 0 \end{tikzcd} \tag{4.9}$$

(the last row is well-defined by Lemma 4.24). Consider for induction hypothesis: the sequence

$$0 \to \mathfrak{p}' \cap \mathfrak{j}^p\mathfrak{q}' \to \mathfrak{p} \cap \mathfrak{j}^p\mathfrak{q} \to \mathfrak{p}'' \cap \mathfrak{j}^p\mathfrak{q}'' \to 0 \tag{H_p}$$

is exact. There exists an integer p_0 (large enough) such that for all $p \geq p_0$, the above equals $0 \to \mathfrak{j}^p\mathfrak{q}' \to \mathfrak{j}^p\mathfrak{q} \to \mathfrak{j}^p\mathfrak{q}'' \to 0$ which is exact by strictness of $0 \to \underline{H}' \to \underline{H} \to \underline{H}'' \to 0$. Hence (H_p) is exact for $p \geq p_0$. If (H_p) is exact for an integer p, a simple diagram chase shows that $F^pH_{L^s} \to F^pH''_{L^s}$ is surjective.

Suppose (a) and let $p \in \mathbb{Z}$ such that (H_p) is exact. We obtain $\dim(F^pH_{L^s}) = \dim(F^pH'_{L^s}) + \dim(F^pH''_{L^s})$ so that the bottom row is exact. The 3×3-Lemma on (4.9) then implies that the upper row is exact. That is, (H_{p-1}) is exact. By descending induction, (b) follows. □

4.3.4 Hodge descent

In the previous section, we have revealed a category $\mathcal{MHP}_R^{\mathrm{ha}}$ on which the hashtag functor # preserves the exactness of the Hodge filtration (Proposition 4.40). Yet, this does not guarantee us that the image of # lands in $\mathcal{MH}_R$ as Example 4.43 below shows. This motivates the next definition.

Definition 4.41. Let $\underline{H}$ be a mixed Hodge-Pink structure. We say that $\underline{H}$ has *Hodge descent* if it is Hodge additive and if $\underline{H}^{\#}$ is a mixed Hodge structure.

We let $\mathcal{MHP}_R^{hd}$ be the full subcategory of $\mathcal{MHP}_R^{\mathrm{ha}}$ whose objects have Hodge descent.

Remark 4.42. We would have preferred a less artificial Definition 4.41. We hope to come up with a less obvious version in a second form of this text.

Example 4.43. We exhibit a simple object $\underline{H}$ in the category of mixed Hodge-Pink structures such that $\underline{H}^{\#}$ is not a mixed Hodge structure. Hence, $\underline{H}$ is Hodge additive but does not have Hodge descent.

We assume $R = K_\infty$, and consider the pre-mixed Hodge structure $\underline{H}$, made pure of weight 0, whose underlying vector space is $H = K_\infty^{\oplus 2}$ and whose Hodge-Pink lattice is

$$\mathfrak{q}_{\underline{H}} := \left\langle \begin{pmatrix} \mathfrak{j}^e \\ 0 \end{pmatrix}, \begin{pmatrix} \mathfrak{j}^{-f} \\ \mathfrak{j}^{-e} \end{pmatrix} \right\rangle$$

for two distinct positive integers $f > e > 0$.

We claim that $\underline{H}$ is locally-semistable. First note that if $\mathfrak{p}$ denotes the tautological lattice of $\underline{H}$, the elementary divisor Theorem provide a basis (e_1, e_2) of $\mathfrak{p}$ over $L^s[\![\mathfrak{j}]\!]$ with respect to which we have

$$\mathfrak{q} = \left\langle \begin{pmatrix} \mathfrak{j}^{-f} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathfrak{j}^{f} \end{pmatrix} \right\rangle, \quad \mathfrak{p} \cap \mathfrak{q} = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathfrak{j}^{f} \end{pmatrix} \right\rangle.$$

It follows that

$$\deg_{\mathfrak{q}}(\underline{H}) = \dim_{L^s}\left(\frac{\mathfrak{q}}{\mathfrak{p} \cap \mathfrak{q}}\right) - \dim_{L^s}\left(\frac{\mathfrak{p}}{\mathfrak{p} \cap \mathfrak{q}}\right) = f - f = 0 = \deg^W(\underline{H}).$$

Let H' be a nonzero strict subspace of H. Then H' has dimension 1, and we fix a basis of it $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ $(a, b \in K_\infty)$. We compute its associated Hodge-Pink lattice $\mathfrak{q}'$:

$$\mathfrak{q}' = \underline{H}'_{L^s((\mathfrak{j}))} \cap \mathfrak{q} = \underline{H}'_{L^s((\mathfrak{j}))} \cap \left\{ x \begin{pmatrix} \mathfrak{j}^e \\ 0 \end{pmatrix} + y \begin{pmatrix} \mathfrak{j}^{-f} \\ \mathfrak{j}^{-e} \end{pmatrix} \;\middle|\; x, y \in L^s[\![\mathfrak{j}]\!] \right\}.$$

If $b \neq 0$, we find $\mathfrak{q}' = \mathfrak{j}^f \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ whose degree is $\deg_{\mathfrak{q}}(\underline{H}') = -f < 0$. If $b = 0$, then $\mathfrak{q}' = \mathfrak{j}^e \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ and $\deg_{\mathfrak{q}}(\underline{H}') = -e < 0$. We conclude that $\underline{H}$ is locally semistable, hence is a MHPS, and that $\underline{H}$ is a simple object in the category $\mathcal{MHP}_{K_\infty}$.

We claim that $\underline{H}^{\#}$, however, is not locally semistable. An easy computation shows that the induced Hodge filtration on $\underline{H}$ has the form:

$$F^p H_{L^s} = \begin{cases} 0 & \text{if } p > f \\ L^s \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) & \text{if } f \geq p > -f \\ H_{L^s} & \text{if } p \leq -f \end{cases}$$

In particular, the choice of $H' = K_\infty \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ provides a K_∞-subspace of H such that $\deg_F(\underline{H}') = f > \deg^W(\underline{H}')$.

The next Proposition is inspired by Pink [Pin, Prop. 4.11]

Proposition 4.44. *Let $0 \to \underline{H}' \to \underline{H} \to \underline{H}'' \to 0$ be an exact sequence in $\mathcal{MHP}_R^{ha}$ such that $\underline{H}'$ and $\underline{H}''$ have Hodge descent. Then $\underline{H}$ has Hodge descent.*

Proof. The sequence $0 \to \underline{H}'^{\#} \to \underline{H}^{\#} \to \underline{H}''^{\#} \to 0$ is strict by Proposition 4.40, and by assumption, both $\underline{H}'^{\#}$ and $\underline{H}''^{\#}$ are locally semistable. Our aim is to show that $\underline{H}^{\#}$ is locally semistable as well.

We assume $R = K_\infty$. Let G be a subspace of H, and let $\underline{\hat{G}}$ be the strict subobject[1] of $\underline{H}^{\#}$ whose underlying space is G. Let $\underline{\hat{G}}'$ be the strict subobject of $\underline{\hat{G}}$ whose underlying space is $G' := G \cap H'$. Let $\underline{\hat{G}}''$ be the strict quotient of $\underline{\hat{G}}$ by $\underline{\hat{G}}'$. We have a strict exact sequence of pre-mixed Hodge structures:

$$0 \longrightarrow \underline{\hat{G}}' \longrightarrow \underline{\hat{G}} \longrightarrow \underline{\hat{G}}'' \longrightarrow 0. \tag{4.10}$$

While $\underline{\hat{G}}' \hookrightarrow \underline{H}'^{\#}$ is strict, $\underline{\hat{G}}'' \hookrightarrow \underline{H}''^{\#}$ may not. Let $\underline{\hat{K}}''$ be the strict subobject of $\underline{H}''^{\#}$ whose underlying space is $G'' := G/G'$. We have an inclusion of pre-mixed Hodge structures

$$\underline{\hat{G}}'' \hookrightarrow \underline{\hat{K}}'' \tag{4.11}$$

which is an equality if and only if it is strict, if and only if $\underline{\hat{G}}'' \hookrightarrow \underline{H}''^{\#}$ is strict.

[1] We denoted it by $\underline{\hat{G}}$ to avoid confusions with $\underline{G}$ which would be here the strict sub-pre-mixed Hodge-Pink structure of $\underline{H}$ whose underlying space is G; note that we might not even have $\underline{G}^{\#} = \underline{\hat{G}}$.

We have:

$$\begin{aligned}\deg_F(\hat{\underline{G}}) &= \deg_F(\hat{\underline{G}}') + \deg_F(\hat{\underline{G}}'') \quad ((4.10) \text{ is strict exact})\\ &\leq \deg^W(\hat{\underline{G}}') + \deg_F(\hat{\underline{G}}'') \quad (\text{local semistability of } \underline{H}'^{\#})\\ &\leq \deg^W(\hat{\underline{G}}') + \deg_F(\hat{\underline{K}}'') \quad (\text{by } (4.11))\\ &\leq \deg^W(\hat{\underline{G}}') + \deg^W(\hat{\underline{K}}'') \quad (\text{local semistability of } \underline{H}''^{\#})\\ &\leq \deg^W(\hat{\underline{G}}') + \deg^W(\hat{\underline{G}}'') \quad (\text{by } (4.11) \text{ again})\\ &= \deg^W(\hat{\underline{G}}) \quad ((4.10) \text{ is strict exact}).\end{aligned} \tag{4.12}$$

This is the required inequality for the local semistability of $\underline{H}^{\#}$, and it remains to prove that this is an equality whenever $G = W_\nu H$ for some $\nu \in \mathbb{Q}$.

We have a commutative diagram with exact lines in the category $\mathcal{MHP}_R^{\mathrm{ha}}$:

$$\begin{tikzcd} 0 \arrow[r] & W_\nu\underline{H}' \arrow[r] \arrow[d, hook] & W_\nu\underline{H} \arrow[r] \arrow[d, hook] & W_\nu\underline{H}'' \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \underline{H}' \arrow[r] & \underline{H} \arrow[r] & \underline{H}'' \arrow[r] & 0 \end{tikzcd}$$

whose morphisms are all strict for the Hodge filtration (by Proposition 4.40). Hence for $G = W_\nu H$, then $\hat{\hat{\underline{G}}} = W_\nu \underline{H}^{\#}$, $\hat{\underline{G}}' = W_\nu \underline{H}'^{\#}$, $\hat{\underline{G}}'' = W_\nu \underline{H}''^{\#}$ and (4.11) is an equality. From the local semistability of $\underline{H}'^{\#}$ and $\underline{H}''^{\#}$, we conclude that (4.12) is an equality for $G = W_\nu H$. □

It follows from Proposition 4.44 that $\mathcal{MHP}_R^{\mathrm{hd}}$ is a Serre subcategory of $\mathcal{MHP}_R^{\mathrm{ha}}$ (in the sense of (02MO)). We obtain from (02MP):

Corollary 4.45. *The category $\mathcal{MHP}_R^{hd}$ is a strictly full abelian subcategory of $\mathcal{MPH}_R^{ha}$.*

We also record the following result, useful to spot when an Hodge additive MHPS has Hodge descent:

Corollary 4.46. *Let $\underline{H}$ be an object of $\mathcal{MHP}_R^{ha}$, and let*

$$0 = \underline{H}_0 \subsetneq \underline{H}_1 \subsetneq \underline{H}_2 \subsetneq \cdots \subsetneq \underline{H}_m = \underline{H}$$

be a Jordan-Hölder series for $\underline{H}$. Suppose that for all $i \in \{1, ..., m\}$, the K_∞-vector spaces $(H_i/H_{i-1}) \otimes_R K_\infty$ have dimension 1. Then $\underline{H}$ has Hodge descent.

Proof. If the underlying vector space of $\underline{H}$ has dimension 1, then $\underline{H}^{\#}$ is a mixed Hodge structure as $\deg_{\mathfrak{q}}(\underline{H}) = \deg_F(\underline{H})$. The general case follows by induction on the length of $\underline{H}$ using Proposition 4.44. □

We summarize the previous observations into a theorem, which constitutes the main innovation of this subsection.

Theorem 4.47. *The association $\underline{H} \mapsto \underline{H}^{\#}$ defines an exact functor*

$$\# : \mathcal{MHP}_R^{hd} \longrightarrow \mathcal{MH}_R$$

of abelian categories.

Remark 4.48. One shows that $\mathcal{MHP}_R^{\mathrm{hd}}$ is the *biggest* full subcategory of $\mathcal{MHP}_R$ on which $\underline{H} \mapsto \underline{H}^{\#}$ defines an exact functor with target $\mathcal{MH}_R$. This is almost tautological: if $\mathcal{A}$ is a full subcategory of $\mathcal{MHP}_R$ with the same property, then for $\underline{H}$ object of $\mathcal{A}$, $\underline{H}^{\#}$ must be a mixed Hodge structure. The exactness of $\#$ imposes all exact sequences in $\mathcal{A}$ to satisfy (a) of Proposition 4.40, and we deduce that $\mathcal{A} \subseteq \mathcal{MHP}_R^{\mathrm{ha}}$. Therefore, $\mathcal{A} \subseteq \mathcal{MHP}_R^{\mathrm{hd}}$.

4.3.5 Extensions of Hodge-Pink structures

Next we consider extensions modules in the categories $\mathcal{MHP}_R$, $\mathcal{MHP}_R^{\mathrm{ha}}$ and $\mathcal{MHP}_R^{+}$. This has been initiated by Pink in [Pin, §8]. Some arguments are similar enough to the ones discussed in Subsection 4.2.2 to be omitted.

Extensions in the category $\mathcal{MHP}_R$

In [Pin, §.8], Pink studied extension modules of Hodge-Pink structures. We restate Proposition 8.6 in *loc. cit.* adaptin it to our notations:

Proposition 4.49 (Prop. 8.6 *loc. cit.*)**.** *Let $\underline{B}$ and $\underline{C}$ be two mixed Hodge-Pink structures such that there exists $\mu \in \mathbb{Q}$ for which $W_\mu C_K = C_K$ and $W_\mu B_K = 0$. Assume further that the underlying module B of $\underline{B}$ is projective. There is a canonical isomorphism of R-modules*

$$\frac{\mathrm{Hom}_{L^s(\!(\mathfrak{j})\!)}(B_{L^s(\!(\mathfrak{j})\!)}, C_{L^s(\!(\mathfrak{j})\!)})}{\mathrm{Hom}_R(B,C) + \mathrm{Hom}_{L^s[\![\mathfrak{j}]\!]}(\mathfrak{q}_{\underline{B}}, \mathfrak{q}_{\underline{C}})} \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{MHP}_R}(\underline{B}, \underline{C}). \tag{4.13}$$

It is given explicitly by mapping $h \in \mathrm{Hom}_{L^s(\!(\mathfrak{j})\!)}(B_{L^s(\!(\mathfrak{j})\!)}, C_{L^s(\!(\mathfrak{j})\!)})$ to the class of the extension

$$[h] := \left[C \oplus B, (W_\mu C_K \oplus W_\mu B_K)_{\mu \in \mathbb{Q}}, \begin{pmatrix} \mathrm{id}_C & h \\ 0 & \mathrm{id}_B \end{pmatrix} \mathfrak{q}_{\underline{C}} \oplus \mathfrak{q}_{\underline{B}} \right]$$

in $\mathrm{Ext}^1_{\mathcal{MHP}_R}(\underline{B}, \underline{C})$.

Extensions in the category $\mathcal{MHP}_R^{\mathrm{ha}}$

Let $\underline{B}$ and $\underline{C}$ be two objects in $\mathcal{MHP}_R$. As one notices by Proposition 4.13, the module $\mathrm{Ext}^1_{\mathcal{MHP}_R}(\underline{B}, \underline{C})$ is not finitely generated over R. Handling finitely generated extension modules guarantees the possibility to construct appropriate regulator maps. In this regard, finite generation is necessary. As Pink already noticed ([Pin, Prop. 8.7]), this is solved by working with extension modules in the category $\mathcal{MHP}_R^{\mathrm{ha}}$. However, it might be too restrictive. For that reason we introduce *Hodge additive extensions*:

Definition 4.50. An extension $[\underline{H}]$ in $\mathrm{Ext}^1_{\mathcal{MHP}_R}(\underline{B},\underline{C})$ is said to be *Hodge additive* if the Hodge polygon of $\underline{H}$ coincides with that of $\underline{C}\oplus\underline{B}$. We denote by $\mathrm{Ext}^{1,\mathrm{ha}}_{\mathcal{MHP}_R}(\underline{B},\underline{C})$ the subset of $\mathrm{Ext}^1_{\mathcal{MHP}_R}(\underline{B},\underline{C})$ consisting of extensions that are congruent to an Hodge-additive extension.

The next proposition rephrases [Pin, Prop. 8.7].

Proposition 4.51. *Under the assumptions of Proposition 4.49, the map* (4.13) *induces an isomorphism*

$$\frac{\mathrm{Hom}_{L^s[\![\mathfrak{j}]\!]}(\mathfrak{p}_{\underline{B}},\mathfrak{p}_{\underline{C}})+\mathrm{Hom}_{L^s[\![\mathfrak{j}]\!]}(\mathfrak{q}_{\underline{B}},\mathfrak{q}_{\underline{C}})}{\mathrm{Hom}_R(B,C)+\mathrm{Hom}_{L^s[\![\mathfrak{j}]\!]}(\mathfrak{q}_{\underline{B}},\mathfrak{q}_{\underline{C}})}\xrightarrow{\sim}\mathrm{Ext}^{1,ha}_{\mathcal{MHP}_R}(\underline{B},\underline{C}).$$

If we suppose that $\underline{B}$ and $\underline{C}$ are Hodge additive, then $\mathrm{Ext}^1_{\mathcal{MHP}_R^{\mathrm{ha}}}(\underline{B},\underline{C})$ is well-defined and lemma 4.37 yields the equality

$$\mathrm{Ext}^{1,\mathrm{ha}}_{\mathcal{MHP}_R}(\underline{B},\underline{C})=\mathrm{Ext}^1_{\mathcal{MHP}_R^{\mathrm{ha}}}(\underline{B},\underline{C}).$$

In this case, Proposition 4.51 describes extension modules in the category $\mathcal{MHP}_R^{\mathrm{ha}}$ under the assumptions of Proposition 4.49. If further $\underline{B}$ and $\underline{C}$ have Hodge descent, then $\mathrm{Ext}^1_{\mathcal{MHP}_R^{\mathrm{hd}}}(\underline{B},\underline{C})$ is well-defined and Proposition 4.44 implies that

$$\mathrm{Ext}^{1,\mathrm{ha}}_{\mathcal{MHP}_R}(\underline{B},\underline{C})=\mathrm{Ext}^1_{\mathcal{MHP}_R^{\mathrm{hd}}}(\underline{B},\underline{C}).$$

Using the functor # (Definition 4.23), these modules can be compared with extension modules in the category $\mathcal{MH}_R$. By Theorem 4.47, # is exact and the assignation $[\underline{H}]\mapsto[\underline{H}^{\#}]$ defines an R-linear morphism

$$\mathrm{Ext}^1_{\mathcal{MHP}_R^{\mathrm{hd}}}(\underline{B},\underline{C})\longrightarrow\mathrm{Ext}_{\mathcal{MH}_R}(\underline{B}^{\#},\underline{C}^{\#}).$$

Under the explicit descriptions of Propositions 4.51 and 4.11, the above map specializes to "reduction modulo $\mathfrak{j}$":

Proposition 4.52. *Let $\underline{B}$ and $\underline{C}$ be two objects of $\mathcal{MHP}_R^{hd}$. Under the assumptions of Proposition 4.49, we have a commutative square of R-modules:*

$$\begin{tikzcd}
\dfrac{\mathrm{Hom}_{L^s[\![\mathfrak{j}]\!]}(\mathfrak{p}_{\underline{B}},\mathfrak{p}_{\underline{C}})}{\mathrm{Hom}_R(B,C)+\mathrm{Hom}_{L^s[\![\mathfrak{j}]\!]}(\mathfrak{q}_{\underline{B}},\mathfrak{q}_{\underline{C}})\cap\mathrm{Hom}_{L^s[\![\mathfrak{j}]\!]}(\mathfrak{p}_{\underline{B}},\mathfrak{p}_{\underline{C}})} \arrow[r, "4.51", "\sim"'] \arrow[d] & \mathrm{Ext}^1_{\mathcal{MHP}_R^{hd}}(\underline{B},\underline{C}) \arrow[d, "{[\underline{H}]\mapsto[\underline{H}^{\#}]}"] \\
\dfrac{\mathrm{Hom}_{L^s}(B_{L^s},C_{L^s})}{\mathrm{Hom}_R(B,C)+\mathrm{Hom}^F_{L^s}(B_{L^s},C_{L^s})} \arrow[r, "(4.3)", "\sim"'] & \mathrm{Ext}^1_{\mathcal{MH}_R}(\underline{B}^{\#},\underline{C}^{\#})
\end{tikzcd}$$

where the left vertical map is induced by $\mathrm{Hom}_{L^s[\![\mathfrak{j}]\!]}(\mathfrak{p}_{\underline{B}},\mathfrak{p}_{\underline{C}})\to\mathrm{Hom}_{L^s}(B_{L^s},C_{L^s})$*, mapping h to the composition* $B_{L^s}\hookrightarrow B_{L^s[\![\mathfrak{j}]\!]}\xrightarrow{h}C_{L^s[\![\mathfrak{j}]\!]}\xrightarrow{\text{mod }\mathfrak{j}}C_{L^s}$.

Proof. Given $[\underline{H}]$ in $\operatorname{Ext}^1_{\mathcal{MHP}_R^{\mathrm{hd}}}(\underline{B},\underline{C})$, we can assume by Proposition 4.51 that there exists $h \in \operatorname{Hom}_{L^s[\![\mathfrak{j}]\!]}(\mathfrak{p}_{\underline{B}},\mathfrak{p}_{\underline{C}})$ such that $[\underline{H}]=[h]$. The Hodge-Pink lattice of $\underline{H}$ is then $\left(\begin{smallmatrix}\mathrm{id}_B & h\\ 0 & \mathrm{id}_C\end{smallmatrix}\right)\mathfrak{q}_{\underline{B}}\oplus\mathfrak{q}_{\underline{C}}$. Let $j\in L^s[\![\mathfrak{j}]\!]$ be a uniformizer at $\mathfrak{j}$. The induced Hodge filtration is defines so that, for all integer p, the L^s-vector space $F^pH_{L^s}$ identifies with the quotient of

$$(\mathfrak{p}_{\underline{B}}\oplus\mathfrak{p}_{\underline{C}})\cap\mathfrak{j}^{p-1}\left(\begin{smallmatrix}\mathrm{id}_B & h\\ 0 & \mathrm{id}_C\end{smallmatrix}\right)(\mathfrak{q}_{\underline{B}}\oplus\mathfrak{q}_{\underline{C}})\xhookrightarrow{\times j}(\mathfrak{p}_{\underline{B}}\oplus\mathfrak{p}_{\underline{C}})\cap\mathfrak{j}^{p}\left(\begin{smallmatrix}\mathrm{id}_B & h\\ 0 & \mathrm{id}_C\end{smallmatrix}\right)(\mathfrak{q}_{\underline{B}}\oplus\mathfrak{q}_{\underline{C}}).$$

It follows that $F^pH=\begin{pmatrix}\mathrm{id}_B & \delta(h)\\ 0 & \mathrm{id}_C\end{pmatrix}(F^pB_{L^s}\oplus F^pC_{L^s})$, where $\delta:\operatorname{Hom}_{L^s[\![\mathfrak{j}]\!]}(\mathfrak{p}_{\underline{B}},\mathfrak{p}_{\underline{C}})\to\operatorname{Hom}_{L^s}(B_{L^s},C_{L^s})$ is the map appearing in the proposition. We conclude by Proposition 4.11. $\square$

We immediately reformulate Proposition 4.52 in the simpler case (the only one used later on) where $\underline{B}$ is the unit mixed Hodge-Pink structure $\mathbb{1}$ over R (it is an object of $\mathcal{MHP}_R^{\mathrm{ha}}$ by Corollary 4.46) and $\underline{C}=\underline{H}$ has Hodge descent, has negative weights, and has a projective underlying R-module. The diagram of Proposition 4.52 clarifies to

$$\begin{tikzcd}
\dfrac{\mathfrak{p}_{\underline{H}}}{H+\mathfrak{p}_{\underline{H}}\cap\mathfrak{q}_{\underline{H}}} \arrow[r, "\sim"] \arrow[d, "h\mapsto h(\mathrm{mod}\ \mathfrak{j})"'] & \operatorname{Ext}^1_{\mathcal{MHP}_R^{\mathrm{hd}}}(\mathbb{1},\underline{H}) \arrow[d, "{[\underline{E}]\mapsto[\underline{E}^{\#}]}"] \\
\dfrac{H_{L^s}}{H+F^0H_{L^s}} \arrow[r, "\sim"] & \operatorname{Ext}^1_{\mathcal{MH}_R}(\mathbb{1},\underline{H}^{\#})
\end{tikzcd}$$

Extensions in the category $\mathcal{MHP}_R^+$

We end this chapter by considering extension modules in the category $\mathcal{MHP}_R^+$. Let $\phi_{\mathbb{1}}:\sigma\mapsto\mathrm{id}_R$ be the trivial infinite Frobenius attached to $\mathbb{1}$ and let $\mathbb{1}^+$ denote the object $(\mathbb{1},\phi_{\mathbb{1}})$ of $\mathcal{MHP}_R^+$.

Let $\underline{H}^+=(\underline{H},\phi_{\underline{H}})$ be an object of $\mathcal{MHP}_R^+$. We suppose that the weights of $\underline{H}$ are negative and that H is projective over R. Under these assumptions, the R-module $\operatorname{Ext}^1_{\mathcal{MHP}_R}(\mathbb{1},\underline{H})$ is described by Proposition 4.11. As we did in Subsection 4.2.2, Definition 4.13, we now define an *adic realization map* for $\underline{H}^+$:

$$d_{\underline{H}^+}:\operatorname{Ext}^1_{\mathcal{MHP}_R^+}(\mathbb{1}^+,\underline{H}^+)\longrightarrow H^1(G_L,H).$$

We use the same process: by Proposition 4.49, any extension $[\underline{E}]$ of $\mathbb{1}$ by $\underline{H}$ in the category $\mathcal{MHP}_R$ is of the form

$$[h]:=\left[C\oplus B,(W_\mu C_K\oplus W_\mu B_K)_{\mu\in\mathbb{Q}},\begin{pmatrix}\mathrm{id}_C & h\\ 0 & \mathrm{id}_B\end{pmatrix}\mathfrak{q}_{\underline{C}}\oplus\mathfrak{q}_{\underline{B}}\right] \tag{4.14}$$

for some $h\in H\otimes_{R,\nu}L^s(\!(\mathfrak{j})\!)$, and the infinite Frobenius $\phi_{\underline{E}}$ acting on $C\oplus B$ takes the form $\left(\begin{smallmatrix}\phi_H & c\\ 0 & 1\end{smallmatrix}\right)$ for a certain cocycle $c:G_L\to H$. The choice of an other expression of the form (4.14) produces an equivalent cocycle, and we define $d_{\underline{H}}([\underline{E}^+])$ to be the well-defined class of c in $H^1(G_L,H)$.

Definition 4.53. We call $d_{\underline{H}^+}$ the *adic realization map of* $\underline{H}^+$.

Remark 4.54. When $\underline{H}$ has Hodge descent, note that $d_{\underline{H}^+}$ on $\operatorname{Ext}^{1,\mathrm{ha}}_{\mathcal{MHP}_R^+}(\mathbb{1}^+,\underline{H}^+)$ coincides with the composition:

$$\operatorname{Ext}^{1,\mathrm{ha}}_{\mathcal{MHP}_R^+}(\mathbb{1}^+,\underline{H}^+) \stackrel{[\underline{E}]\mapsto[\underline{E}^{\#}]}{\longrightarrow} \operatorname{Ext}^1_{\mathcal{MH}_R^+}(\mathbb{1}^+,(\underline{H}^+)^{\#}) \stackrel{d_{(\underline{H}^+)^{\#}}}{\longrightarrow} H^1(G_L,H)$$

where $d_{(\underline{H}^+)^{\#}}$ is the adic realization map of $(\underline{H}^+)^{\#}$ (Definition 4.13).

The next proposition synthesizes Proposition 4.14 and extends it to the Hodge-Pink context. It allows to compute the extension spaces under certain Galois cohomology vanishing assumptions.

Proposition 4.55. *Let $\underline{H}^+=(\underline{H},\phi_{\underline{H}})$ be an object in $\mathcal{MHP}_R^+$ whose weights are negative, whose underlying R-module is projective, and such that $\underline{H}$ has Hodge descent. Assume that $H^1(G_L,H_{L^s})=H^1(G_L,\mathfrak{p}_{\underline{H}}\cap\mathfrak{q}_{\underline{H}})=H^1(G_L,\mathfrak{q}_{\underline{H}})=0$. The following diagram of R-modules is commutative and exact on rows:*

$$\begin{tikzcd}
\dfrac{(H\otimes_{R,\nu}L^s(\!(\mathfrak{j})\!))^+}{H^+ + \mathfrak{q}_{\underline{H}}^+} \arrow[r, hook, "h\mapsto([h],[0])"] & \operatorname{Ext}^1_{\mathcal{MHP}_R^+}(\mathbb{1}^+,\underline{H}^+) \arrow[r, two heads, "d_{\underline{H}^+}"] & H^1(G_L,H) \\
\dfrac{\mathfrak{p}_{\underline{H}}^+}{H^+ + (\mathfrak{p}_{\underline{H}}\cap\mathfrak{q}_{\underline{H}})^+} \arrow[u] \arrow[r, hook] \arrow[d, "h\mapsto h(mod\ \mathfrak{j})"] & \operatorname{Ext}^{1,ha}_{\mathcal{MHP}_R^+}(\mathbb{1}^+,\underline{H}^+) \arrow[u, hook] \arrow[r, two heads, "d_{\underline{H}^+}"] \arrow[d, "{[\underline{E}]\mapsto[\underline{E}^{\#}]}"] & H^1(G_L,H) \arrow[u, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] \\
\dfrac{(H_{L^s})^+}{H^+ + (F^0H_{L^s})^+} \arrow[r, hook] & \operatorname{Ext}^1_{\mathcal{MH}_R^+}(\mathbb{1}^+,(\underline{H}^+)^{\#}) \arrow[r, two heads, "d_{(\underline{H}^+)^{\#}}"] & H^1(G_L,H)
\end{tikzcd}$$

Proof. We claim that $H^1(G_L,\mathfrak{p}_{\underline{H}}\cap\mathfrak{q}_{\underline{H}})=0$ implies $H^1(G_L,F^0H_{L^s})=0$. Indeed, a cocycle $c:G_L\to F^0H_{L^s}$ can be lifted to a cocycle $G_L\to\mathfrak{p}_{\underline{H}}\cap\mathfrak{q}_{\underline{H}}$ via the splitting

$$\mathfrak{p}_{\underline{H}}\cap\mathfrak{q}_{\underline{H}} = F^0H_{L^s}\oplus\mathfrak{j}(\mathfrak{p}_{\underline{H}}\cap\mathfrak{q}_{\underline{H}}) \tag{4.15}$$

and our assumption $H^1(G_L,\mathfrak{p}_{\underline{H}}\cap\mathfrak{q}_{\underline{H}})=0$ implies that there exists $h\in\mathfrak{p}_{\underline{H}}\cap\mathfrak{q}_{\underline{H}}$ such that $c(\sigma)=h-(\varphi_{\underline{H}}(\sigma)\otimes\sigma)(h)$ for all $\sigma\in G_L$. The fact that $c(\sigma)\in F^0H_{L^s}$ yields that the projection of h onto $\mathfrak{j}(\mathfrak{p}_{\underline{H}}\cap\mathfrak{q}_{\underline{H}})$ under the decomposition (4.15) is G_L-invariant. As such, $c(\sigma)=h_0-(\varphi_{\underline{H}}(\sigma)\otimes\sigma)(h_0)$ where h_0 denotes the projection of h onto $F^0H_{L^s}$. Hence, c is trivial.

By Proposition 4.14 and the previous claim, the last line is exact. The fact that $H^1(G_L,H)=0$ and $H^1(G_L,\mathfrak{p}_{\underline{H}}\cap\mathfrak{q}_{\underline{H}})=0$ (resp. $H^1(G_L,\mathfrak{q}_{\underline{H}})=0$) implies that the middle line (resp. the first line) is exact, yet the details of the proof are similar enough to the one of Proposition 4.14 to be skipped. $\square$

Chapter 5

Hodge realizations and regulators of A-motives

In the hypothetical landscape of classical mixed motives, Beilinson's regulators are constructed as follows. There should exist a Hodge realization functor $\mathcal{H}^+$ from the category of mixed motives over a number field F to the category of mixed Hodge structures over $\mathbb{R}$, equipped with infinite Frobenius, expected to be exact. In this hypothesis, given a mixed motive M over F, $\mathcal{H}^+$ would induce an $\mathbb{R}$-linear map at the level of extension modules:

$$\operatorname{Ext}^1_{\mathcal{MM}_F}(\mathbb{1}, M) \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow \operatorname{Ext}^1_{\mathcal{MH}^+_{\mathbb{R}}}(\mathbb{1}^+, \mathcal{H}^+(M)). \tag{5.1}$$

The above is the abstract construction of the Beilinson regulator for M ([Nek, (2.6.1)]). As any extension space in the category $\mathcal{MH}^+$, the right-hand side of (5.1) is expected to be finite dimensional (e.g. beginning of Subsection 4.2.2). Beilinson's first conjecture states that, under an assumption on the weights of M, (5.1) is an isomorphism once restricted to the subspace of extensions having everywhere good reduction (e.g. [Nek]).

In the first Section 5.1, we construct regulators in the function field situation. Let F be a finite extension of $K = \mathbb{F}(C)$, and let $\mathcal{MM}^{\mathrm{rig}}_F$ be the category of mixed rigid analytically trivial A-motives over F (Definition 1.59). Following Pink, we define a *Hodge-Pink realization functor* $\mathscr{H}^+$ from the category $\mathcal{MM}^{\mathrm{rig}}_F$ to the category $\mathcal{MHP}^+_{K_\infty}$ (Subsection 5.1.1). Given an object $\underline{M}$ of $\mathcal{MM}^{\mathrm{rig}}_F$, the exactness of $\mathscr{H}$ (Corollary 5.6) allows us to define the *general regulator of* $\underline{M}$ (Definition 5.21):

$$\mathscr{Reg}(\underline{M}) : \operatorname{Ext}^1_{\mathcal{MM}^{\mathrm{rig}}_F}(\mathbb{1}, \underline{M}) \otimes_A K_\infty \to \operatorname{Ext}^1_{\mathcal{MHP}^+_{K_\infty}}(\mathbb{1}^+, \mathscr{H}^+(\underline{M})).$$

However, because the image of $\mathscr{H}^+$ does not land in the subcategory of Hodge additive objects of $\mathcal{MHP}^+_{K_\infty}$, it does not induce a functor $\mathcal{MM}^{\mathrm{rig}}_F \to \mathcal{MH}^+_{K_\infty}$. This prevents the analogue of (5.1) to exist in the function field setting. To palliate this issue, we introduce in Subsection 5.1.2 the notion of *regulated objects* of $\mathcal{MM}^{\mathrm{rig}}_F$ and the subcategory $\mathcal{MM}^{\mathrm{reg}}_F$ they define. We will show that Pink's functor $\mathscr{H}^+$ induces an exact functor

$$\mathcal{H}^+ : \mathcal{MM}_F^{\mathrm{reg}} \longrightarrow \mathcal{MH}_{K_\infty}^+$$

which, by construction, factors through $\mathcal{MHP}_{K_\infty}^{\mathrm{ha}}$ (Proposition 5.15). In the case where $\underline{M}$ is regulated, we define the *special regulator of* $\underline{M}$ from the exactness of $\mathcal{H}^+$ (Definition 5.23):

$$\mathrm{Reg}(\underline{M}) : \mathrm{Ext}^1_{\mathcal{MM}_F^{\mathrm{reg}}}(\mathbb{1}, \underline{M}) \otimes_A K_\infty \to \mathrm{Ext}^1_{\mathcal{MH}^+_{K_\infty}}(\mathbb{1}^+, \mathscr{H}^+(\underline{M})).$$

Although special regulators are closer in analogy to (5.1), it will appear next (Chapter 6) that general regulators are more relevant in the study of Beilinson's first conjecture. Special regulators will play an important role in Chapter 7 to study algebraic relations among polylogarithm.

For an object of $\mathcal{MM}_F^{\mathrm{rig}}$, to be regulated appears to be a strong condition. As considering extension modules in the category $\mathcal{MM}_F^{\mathrm{reg}}$ may be too restrictive, we discuss in Subsection 5.1.3 *regulated extensions in* $\mathcal{MM}_F^{rig}$ and the associated modules $\mathrm{Ext}^{1,\mathrm{reg}}_{\mathcal{MM}_F^{\mathrm{rig}}}$. From Definition 5.16, the general regulator of $\underline{M}$, an object of $\mathcal{MM}_F^{\mathrm{rig}}$, induces

$$\mathscr{R}\!\mathit{eg}(\underline{M}) : \mathrm{Ext}^{1,\mathrm{reg}}_{\mathcal{MM}_F^{\mathrm{rig}}}(\mathbb{1}, \underline{M}) \otimes_A R \longrightarrow \mathrm{Ext}^{1,\mathrm{ha}}_{\mathcal{MHP}_R^+}(\mathbb{1}^+, \mathscr{H}_R^+(\underline{M})).$$

This will be required for Chapter 6.

The target spaces of general and special regulators are not finite dimensional over K_∞. In that respect, we showed in Chapter 4 that kernels of adic realization maps are more convenient to represent the analogue of the right-hand side of (5.1). In Section 5.2, we investigate the notion of *extensions in* $\mathcal{MM}_F^{rig}$ *having analytic reduction at* v, $v : F \to \mathbb{C}_\infty$ being a K-algebra morphism (Definition 5.26). Retrospectively, our definition shares similarities with Taelman's work [Tae2] in the context of Drinfeld modules. Analytic reduction at v will provide us a natural module $\mathrm{Ext}^{1,v}_{\mathcal{MM}_F^{\mathrm{rig}}}(\mathbb{1}, \underline{M})$ whose image through general and special regulators lands in the kernel of adic realization map. $\mathrm{Ext}^{1,\infty}_{\mathcal{MM}_K^{\mathrm{rig}}}(\mathbb{1}, \underline{M})$ (for $F = K$, v is the inclusion) plays a central role in our counterpart of Beilinson's conjecture (c.f. Chapter 6).

Finally, in Section 5.3, we give a description of the several extension groups in $\mathcal{MM}_F^{\mathrm{rig}}$ or $\mathcal{MM}_F^{\mathrm{reg}}$ we encountered in terms of modules of solutions of certain τ-difference equations (Proposition 5.33). These formulas will be needed for explicit computations with special and general regulators (Theorem 5.34), and widely used in the proof of Theorems E and F (Chapter 6).

5.1 Hodge structures associated to A-motives

In this section, we define the Hodge-Pink and Hodge realization functor, the general and special regulators.

Let F be a finite extension of K and let $v : F \to \mathbb{C}_\infty$ be a K-algebra morphism. We denote F_v the completion of F with respect to $|x|_v = |v(x)|$. We fix F_v^s a separable closure of F_v and let $G_v = \mathrm{Gal}(F_v^s|F_v)$ be the absolute Galois group of F_v. Let R be a Noetherian subring of K_∞ containing A such that $R \otimes_A K$ is a field (in all the following, R will be A, K or K_∞). We denote $\mathcal{MHP}_R$ (resp. $\mathcal{MHP}_R^+$, $\mathcal{MHP}_R^{\mathrm{ha}}$) the corresponding category with $L = F_v$ for base field.

5.1.1 The general Hodge realization functor

Let $\underline{M}$ be a mixed rigid analytically trivial A-motive over F (Definition 1.50). Let $\Lambda_v(\underline{M})$ be the v-Betti realization of $\underline{M}$ (Definition 1.48). By Theorem 1.66, there exists a finite separable extension L in $\mathbb{C}_\infty$ of F_v such that $\Lambda_v(\underline{M})$ identifies with the sub-A-module of $M \otimes_{A\otimes F,v} L\langle\langle A\rangle\rangle_{\mathfrak{j}}$ of elements satisfying $\omega = \tau_M(\tau^*\omega)$. Because $\underline{M}$ is rigid analytically trivial and because the inclusion $L\langle\langle A\rangle\rangle_{\mathfrak{j}} \to \mathbb{C}_\infty\langle A\rangle$ is faithfully flat, the multiplication map

$$\Lambda_v(\underline{M}) \otimes_A L\langle\langle A\rangle\rangle_{\mathfrak{j}} \longrightarrow M \otimes_{A\otimes F,v} L\langle\langle A\rangle\rangle_{\mathfrak{j}} \tag{5.2}$$

is an isomorphism of $L\langle\langle A\rangle\rangle_{\mathfrak{j}}$-modules. Localizing at $\mathfrak{j}$, the multiplication

$$\Lambda_v(\underline{M}) \otimes_{A,\nu} F_v^s(\!(\mathfrak{j})\!) \longrightarrow M \otimes_{A\otimes F,v} F_v^s(\!(\mathfrak{j})\!), \quad \omega \otimes f \longmapsto \omega f, \tag{5.3}$$

where $\nu : A \to F_v^s[\![\mathfrak{j}]\!]$, $a \mapsto a \otimes 1$, is the morphism of lemma 4.16, is an isomorphism of $F_v^s(\!(\mathfrak{j})\!)$-modules.

Definition 5.1. We denote by $\gamma_{\underline{M}}^v$ the isomorphism (5.3).

A trivial yet important remark is the following:

Lemma 5.2. *The morphism $\gamma_{\underline{M}}^v$ is G_v-equivariant, where $\sigma \in G_v$ acts on the right-hand side of* (5.3) *via $\sigma \otimes \sigma$ and on the left via* $\mathrm{id}_M \otimes \sigma$.

In the next definition, attributed to Pink, we attach a pre-Hodge-Pink structure to $\underline{M}$ following [HarJu, Def. 2.3.32].

Definition 5.3. We let $\mathscr{H}_R(\underline{M})$ be the mixed pre-Hodge-Pink structure (at v, with coefficients ring R)

- whose underlying R-module is $\Lambda_v(\underline{M}) \otimes_A R$,
- whose weight filtration is given, for all $\mu \in \mathbb{Q}$, by

$$\Lambda(W_\mu \underline{M})_K = \Lambda(W_\mu \underline{M}) \otimes_R (R \otimes_A K) \quad \text{(see Definition 1.34)},$$

- whose Hodge-Pink lattice is $\mathfrak{q}_{\underline{M}} = (\gamma_{\underline{M}}^v)^{-1}(M \otimes_{A\otimes F,v} F_v^s[\![\mathfrak{j}]\!])$.

The tautological lattice of $\mathscr{H}_R(\underline{M})$ is $\mathfrak{p}_{\underline{M}} = \Lambda_v(\underline{M}) \otimes_A F_v^s[\![\mathfrak{j}]\!]$. The action of G_v on $\Lambda_v(\underline{M})$ is continuous (1.54) and defines an infinite Frobenius $\phi_{\underline{M}}$ for $\mathscr{H}_R(\underline{M})$. We denote by $\mathscr{H}_R^+(\underline{M})$ the pair $(\mathscr{H}_R(\underline{M}), \phi_{\underline{M}})$.

The following Theorem is announced in [HarPi], and proved in [HarJu, Thm. 2.3.34] under the assumption $\deg(\infty) = 1$.

Theorem 5.4. *The pre-mixed Hodge-Pink structure $\mathscr{H}_R(\underline{M})$ is a mixed Hodge-Pink structure. The assignment $\underline{M} \mapsto \mathscr{H}_R(\underline{M})$ defines a fully faithfull exact functor $\mathscr{H}_R : \mathcal{MM}_F^{rig} \to \mathcal{MHP}_R$.*

Remark 5.5. Theorem 2.3.34 in *loc. cit.* is more elaborate than the subpart we quote, and states an analogue of the Hodge conjecture in function fields arithmetic.

As an immediate consequence of Theorem 5.4, we obtain the corresponding version for $\mathcal{MHP}_R^+$:

Corollary 5.6. *The datum of $\mathscr{H}_R^+(\underline{M})$ defines an obect in $\mathcal{MHP}_R^+$. The assignment $\underline{M} \mapsto \mathscr{H}_R^+(\underline{M})$ defines an exact functor $\mathscr{H}_R^+ : \mathcal{MM}_F^{rig} \to \mathcal{MHP}_R^+$.*

We now discuss the induced mixed Hodge structure $\mathscr{H}_R(\underline{M})^\#$ (Definition 4.23). The induced Hodge filtration is computed by the elementary divisors of $\mathfrak{p}_{\underline{M}}$ relative to $\mathfrak{q}_{\underline{M}}$. In this direction, the next key lemma precises $\mathfrak{p}_{\underline{M}}$ seen as a submodule of $M \otimes_{A\otimes F,v} F_v^s[\![\mathfrak{j}]\!]$.

Lemma 5.7. *We have $\gamma_{\underline{M}}^v(\mathfrak{p}_{\underline{M}}) = \tau_M(\tau^* M) \otimes_{A\otimes F,v} F_v^s[\![\mathfrak{j}]\!]$.*

Proof. If one take the pullback of (5.2) by $L\langle\!\langle A\rangle\!\rangle_{\mathfrak{j}} \to L\langle\!\langle A\rangle\!\rangle_{\mathfrak{j}^{(1)}}$, $f \mapsto \tau(f)$, one obtains an isomorphism of $L\langle\!\langle A\rangle\!\rangle_{\mathfrak{j}^{(1)}}$-modules:

$$\Lambda_v(\underline{M}) \otimes_A L\langle\!\langle A\rangle\!\rangle_{\mathfrak{j}^{(1)}} \xrightarrow{\sim} (\tau^* M) \otimes_{A\otimes F,v} L\langle\!\langle A\rangle\!\rangle_{\mathfrak{j}^{(1)}}.$$

The local ring of $L\langle\!\langle A\rangle\!\rangle_{\mathfrak{j}^{(1)}}$ at $\mathfrak{j}$ is canonically identified with $L[\![\mathfrak{j}]\!]$. It follows that the morphism of $F_v^s[\![\mathfrak{j}]\!]$-modules:

$$\delta_{\underline{M}}^v : \Lambda(\underline{M}) \otimes_A F_v^s[\![\mathfrak{j}]\!] \xrightarrow{\sim} (\tau^* M) \otimes_{A\otimes F,v} F_v^s[\![\mathfrak{j}]\!],$$

defined as the multiplication, is an isomorphism. It further inserts in a commutative diagram

$$\begin{tikzcd}
\Lambda(\underline{M}) \otimes_A F_v^s(\!(\mathfrak{j})\!) \arrow[r, "\delta_{\underline{M}}^v \otimes_{F_v^s[\![\mathfrak{j}]\!]} \mathrm{id}_{F_v^s(\!(\mathfrak{j})\!)}"] \arrow[dr, "\gamma_{\underline{M}}^v"'] & (\tau^* M) \otimes_{A\otimes F,v} F_v^s(\!(\mathfrak{j})\!) \arrow[d, "\tau_M \otimes \mathrm{id}_{F_v^s(\!(\mathfrak{j})\!)}"] \\
& M \otimes_{A\otimes F,v} F_v^s(\!(\mathfrak{j})\!)
\end{tikzcd}$$

Note that this already appears in [HarJu, Prop.2.3.30] under different notations. The equality $\gamma_{\underline{M}}(\mathfrak{p}_{\underline{M}}) = \tau_M(\tau^* M) \otimes_{A\otimes F,v} F_v[\![\mathfrak{j}]\!]$ follows from the commutativity of the above diagram together with the fact that $\delta_{\underline{M}}^v$ is an isomorphism. $\square$

Let e be a large enough integer so that we both have $\mathfrak{j}^e\tau_M(\tau^*M) \subset M$ and $\mathfrak{j}^e M \subset \tau_M(\tau^*M)$ as inclusions of $A\otimes F$-modules. The cokernels of those inclusions are annihilated by a power of $\mathfrak{j}$, and thus, are canonically endowed with a $F[\![\mathfrak{j}]\!]$-module structure. By the elementary divisors Theorem applied to the principal ideal domain $F[\![\mathfrak{j}]\!]$, we deduce that there is a family of integers $(w_1,...,w_r)$ sorted by ascending order, r being the rank of $\underline{M}$, such that

$$M/\mathfrak{j}^e\tau_M(\tau^*M) \cong \bigoplus_{i=1}^r A\otimes F/\mathfrak{j}^{e+w_i}, \quad \tau_M(\tau^*M)/\mathfrak{j}^e M \cong \bigoplus_{i=1}^r A\otimes F/\mathfrak{j}^{e-w_i}.$$

The family $(w_1,...,w_r)$ is independent of e, and we call it *the Hodge weights of* $\underline{M}$. In virtue of Definition 4.28 and Lemma 5.7, we deduce:

Proposition 5.8. *The breaks of the Hodge filtration on $\mathscr{H}_R(\underline{M})^\# = (H,W,F)$ are exactly the Hodge weights of $\underline{M}$. The multiplicity of w in the family $(w_1,...,w_r)$ equals the dimension of $\operatorname{Gr}_F^w(H_{F_v^s})$ over F_v^s. In particular, the Hodge Polygon of $\mathscr{H}_R(\underline{M})$ is independent of v.*

Remark 5.9. Note that this is in accordance with what is expected in the classical setting [Jan2, Principle 1.7].

5.1.2 Regulated A-motives

Let $\underline{M} = (M,\tau_M)$ be a mixed rigid analytically trivial A-motive over F. In this subsection, we answer to the following question: *when $\mathscr{H}_R(\underline{M})$ has Hodge descent?* We design the notion of *regulated* A-motives to address the above.

Definition 5.10. We say that $\underline{M}$ is *v-regulated* if the mixed Hodge-Pink structure $\mathscr{H}_R(\underline{M})$ has Hodge descent (Definition 4.41). We say that $\underline{M}$ is *regulated* if $\operatorname{Res}_{F/K}(\underline{M})$ is i-regulated, where $i: K\to\mathbb{C}_\infty$ is the inclusion. We let $\mathcal{MM}_F^{\mathrm{reg}}$ be the full subcategory of $\mathcal{MM}_F^{\mathrm{rig}}$ whose objects are regulated.

Example 5.11. Let e be an integer and consider the A-motive $\underline{M}$ over K whose underlying module is $(A\otimes K)^{\oplus 2}$, and where τ_M acts by

$$\begin{pmatrix}\tau^*a\\ \tau^*b\end{pmatrix} \mapsto \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix}\tau(a)\\ \tau(b)\end{pmatrix}$$

for $t\in(A\otimes K)[\mathfrak{j}^{-1}]$. It is an extension of $\mathbb{1}$ by $\mathbb{1}$ in the category $\mathcal{MM}_K^{\mathrm{rig}}$ and hence it is pure of weight 0 (Proposition 3.11). The mixed Hodge-Pink structure $\mathscr{H}_R(\underline{M})$ is isomorphic to the one presented in Remark 4.26 with e being the opposite of the $\mathfrak{j}$-valuation of t. By Corollary 4.46, $\underline{M}$ is regulated if and only if $e\leq 0$, that is, if and only if $t\in A\otimes K$.

Remark 5.12. The property of being regulated is independent on the choice of R. Let us check that the Hodge additivity of $\mathscr{H}_R(\underline{M})$ is independent on v. For that, we write $\mathscr{H}_v := \mathscr{H}_R$. For the choice of another K-algebra morphism $v': F\to\mathbb{C}_\infty$, we denote by $\mathscr{H}_{v'}$ the functor $\mathscr{H}_R$ which associates a mixed

Hodge-Pink structure with base field $F_{v'}$. Proposition 5.8 implies that the Hodge polygons of $\mathscr{H}_v(\underline{M})$ and $\mathscr{H}_{v'}(\underline{M})$ coincide, so that it remains to show that the Hodge polygons of $\mathscr{H}_v(\underline{M})^{\text{ss}}$ and $\mathscr{H}_{v'}(\underline{M})^{\text{ss}}$ are equal. By [HarPi] (or [HarJu, Thm. 2.3.34]), the essential image of $\mathscr{H}_R$ is closed under the formation of subquotients. This implies that there exists a sequence of inclusions in $\mathcal{MM}_F^{\text{rig}}$

$$0 = \underline{M}_0 \subsetneq \underline{M}_1 \subsetneq \cdots \subsetneq \underline{M}_n = \underline{M}$$

such that $(\mathscr{H}_v(\underline{M}_i))_i$ is a Jordan-Hölder series for $\mathscr{H}_v(\underline{M})$ (note that $n \leq \operatorname{rank}_A \Lambda_v(\underline{M}) = \operatorname{rank} \underline{M}$). In addition, the quotients $\underline{M}_i/\underline{M}_{i-1}$ are simple in $\widetilde{\mathcal{MM}}_F^{\text{rig}}$. It follows that $(\mathscr{H}_{v'}(\underline{M}_i))_i$ is a Jordan-Hölder series for $\mathscr{H}_{v'}(\underline{M})$. Still by Proposition 5.8, the Hodge polygons of $\mathscr{H}_v(\underline{M})^{\text{ss}}$ and $\mathscr{H}_{v'}(\underline{M})^{\text{ss}}$ coincide.

We leave open the question whether the property: "$\mathscr{H}_R(\underline{M})$ have Hodge descent" is independent on v.

From the exactness of $\mathscr{H}_R$ (Theorem 5.4), we record:

Proposition 5.13. *The category $\mathcal{MM}_F^{reg}$ is R-linear exact.*

Proof. That $\mathcal{MM}_F^{\text{reg}}$ is R-linear follows from the fact that $\mathcal{MM}_F^{\text{reg}}$ is a full subcategory of $\mathcal{MM}_F^{\text{rig}}$. We now show that $\mathcal{MM}_F^{\text{reg}}$ together with the notion of exact sequences inherited from $\mathcal{MM}_F^{\text{rig}}$ forms an exact category.

If $\underline{M}'$ and $\underline{M}''$ are two objects in $\mathcal{MM}_F^{\text{reg}}$, then $\mathscr{H}_R(\underline{M}' \oplus \underline{M}'') = \mathscr{H}_R(\underline{M}') \oplus \mathscr{H}_R(\underline{M}'')$ is Hodge additive by Lemma 4.37. By Proposition 4.44, $\mathscr{H}_R(\underline{M}' \oplus \underline{M}'')$ has Hodge descent. It follows that $\underline{M}' \oplus \underline{M}''$ is in $\mathcal{MM}_F^{\text{reg}}$ and that

$$0 \longrightarrow \underline{M}' \longrightarrow \underline{M}' \oplus \underline{M}'' \longrightarrow \underline{M}'' \longrightarrow 0$$

is exact in $\mathcal{MM}_F^{\text{reg}}$.

Secondly, we need to show that extensions in $\mathcal{MM}_F^{\text{reg}}$ are stable through pushouts and pullbacks. We denote by $\times$ and $\sqcup$ the fiber product and the amalgamated sum in $\mathcal{MM}_F^{\text{rig}}$ respectively (see [Stack, 001U, 04AN]). Let $0 \to \underline{M}' \xrightarrow{i} \underline{M} \xrightarrow{p} \underline{M}'' \to 0$ be an exact sequence in $\mathcal{MM}_F^{\text{reg}}$. Because $\mathscr{H}_R$ is exact, it preserves pushouts and pullbacks. That is, given a morphism $\underline{N} \to \underline{M}''$ in $\mathcal{MM}_F^{\text{reg}}$, we have

$$\mathscr{H}_R(\underline{M} \times_{\underline{M}''} \underline{N}) = \mathscr{H}_R(\underline{M}) \times_{\mathscr{H}_R(\underline{M}'')} \mathscr{H}_R(\underline{N}).$$

As a subobject of $\mathscr{H}_R(\underline{M}) \oplus \mathscr{H}_R(\underline{N})$, the above is Hodge additive (Lemma 4.36) and has Hodge descent (Proposition 4.44). We deduce that $\underline{M} \times_{\underline{M}''} \underline{N}$ is regulated. Dually, given a morphism $\underline{N} \to \underline{M}'$ in $\mathcal{MM}_F^{\text{reg}}$, we have

$$\mathscr{H}_R(\underline{M}' \sqcup_{\underline{M}'} \underline{N}) = \mathscr{H}_R(\underline{M}') \sqcup_{\mathscr{H}_R(\underline{M}')} \mathscr{H}_R(\underline{N}).$$

The above is a quotient of $\mathscr{H}_R(\underline{M}) \oplus \mathscr{H}_R(\underline{N})$ and hence is Hodge additive (and has Hodge descent by Proposition 4.44). Hence $\underline{M}' \sqcup_{\underline{M}'} \underline{N}$ is regulated. The sequences

$$0 \longrightarrow \underline{M}' \longrightarrow \underline{M} \times_{\underline{M}''} \underline{N} \longrightarrow \underline{N} \longrightarrow 0$$

$$0 \longrightarrow \underline{N} \longrightarrow \underline{M} \sqcup_{\underline{M}'} \underline{N} \longrightarrow \underline{M}'' \longrightarrow 0$$

are then exact in $\mathcal{MM}_F^{\mathrm{reg}}$, as desired.

That admissible monomorphisms (resp. epimorphisms) are kernels (resp. cokernels) of their corresponding admissible epimorphisms (resp. monomorphims), and that the composition of two admissible monomorphisms (resp. epimorphims) is admissible, is clear as exact sequences in $\mathcal{MM}_F^{\mathrm{reg}}$ are exact sequences in $\mathcal{MM}_F^{\mathrm{rig}}$. □

By Theorem 4.47, if $\underline{M}$ is an object of $\mathcal{MM}_F^{\mathrm{reg}}$ then $\mathscr{H}_R(\underline{M})^{\#}$ is a mixed Hodge structure. We are thus in position to define the Hodge realization functor.

Definition 5.14. We call *the Hodge realization functor* and denote it by $\mathcal{H}_R$ the functor $\mathcal{MM}_F^{\mathrm{reg}} \to \mathcal{MH}_R$ given by the composition of $\mathscr{H}_R$ and $\underline{H} \mapsto \underline{H}^{\#}$. We define $\mathcal{H}_R^+$ similarly, with $\mathscr{H}_R^+$ in place of $\mathscr{H}_R$.

The next statement makes sense by Proposition 5.13 and is evident from Theorem 5.4 (resp. Corollary 5.6) and Theorem 4.47.

Proposition 5.15. *The functors $\mathcal{H}_R : \mathcal{MM}_F^{reg} \to \mathcal{MH}_R$ and $\mathcal{H}_R^+ : \mathcal{MM}_F^{reg} \to \mathcal{MH}_R^+$ are exact.*

5.1.3 Regulated extensions

In Chapter 4, we introduced the concept of Hodge additive extensions in order to work with more general extension spaces than the ones in the category $\mathcal{MHP}_R^{\mathrm{ha}}$. Similarly, considering extension modules in the category $\mathcal{MM}_F^{\mathrm{reg}}$ may be too restrictive, and we discuss in the rest of this section the notion of *regulated* extensions.

Let $\underline{N}$ and $\underline{M}$ be two objects of $\mathcal{MM}_F^{\mathrm{rig}}$.

Definition 5.16. Let $[\underline{E}] \in \mathrm{Ext}^1_{\mathcal{MM}_F^{\mathrm{rig}}}(\underline{N}, \underline{M})$. We say that $[\underline{E}]$ is *regulated* if $[\mathscr{H}_R(\underline{E})]$ is an Hodge additive extension of $\mathscr{H}_R(\underline{N})$ by $\mathscr{H}_R(\underline{M})$ in $\mathcal{MHP}_R$ (Definition 4.50). We let $\mathrm{Ext}^{1,\mathrm{reg}}_{\mathcal{MM}_F^{\mathrm{rig}}}(\underline{N}, \underline{M})$ be the subset of extensions that are congruent in $\mathcal{MM}_F^{\mathrm{rig}}$ to a regulated extension.

Remark 5.17. As in Remark 5.12, to be *regulated* for an extension does not depend on R nor on v. This follows from Proposition 5.8.

Remark 5.18. Note that if both $\underline{N}$ and $\underline{M}$ are object of $\mathcal{MM}_F^{\mathrm{reg}}$, then

$$\mathrm{Ext}^{1,\mathrm{reg}}_{\mathcal{MM}_F^{\mathrm{rig}}}(\underline{N}, \underline{M}) = \mathrm{Ext}^1_{\mathcal{MM}_F^{\mathrm{reg}}}(\underline{N}, \underline{M}).$$

This follows immediately from Proposition 4.44. As expected, this shows that the bifunctor $\mathrm{Ext}^{1,\mathrm{reg}}_{\mathcal{MM}_F^{\mathrm{rig}}}$ on $\mathcal{MM}_F^{\mathrm{rig}}$ extends the bifunctor $\mathrm{Ext}^1_{\mathcal{MM}_F^{\mathrm{reg}}}$ on $\mathcal{MM}_F^{\mathrm{reg}}$.

We now assume that the smallest weight of $\underline{N}$ is bigger than the highest weight of $\underline{M}$. By Proposition 3.8, any extension of $\underline{N}$ by $\underline{M}$ is mixed, and by Proposition 3.1, we have an isomorphism of A-modules

$$\iota_{\underline{N},\underline{M}} : \frac{\operatorname{Hom}_{A\otimes F}(\tau^*N, M)[\mathfrak{j}^{-1}]}{\{f\circ\tau_N - \tau_M\circ\tau^*f \mid f\in \operatorname{Hom}_{A\otimes F}(N,M)\}} \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{MM}_F^{\mathrm{rig}}}(\underline{N},\underline{M}).$$

Proposition 5.19. *Let $u \in \operatorname{Hom}_{A\otimes F}(\tau^*N, M)[\mathfrak{j}^{-1}]$. The extension $\iota_{\underline{N},\underline{M}}(u)$ is regulated if and only if $u(\tau^*N)\subset \tau_M(\tau^*M)$.*

Proof. Let $[\underline{E}]$ be the extension $\iota_{\underline{N},\underline{M}}(u)$. By Proposition 3.1, $\underline{E}$ is the A-motive whose underlying module is $M\oplus N$ and where τ_E acts by $\left(\begin{smallmatrix}\tau_M & u\\ 0 & \tau_N\end{smallmatrix}\right)$. The extension of mixed Hodge-Pink structures $[\mathcal{H}^+(\underline{E})]$ is Hodge additive if and only if, for e large enough, the $F_v^s[\![\mathfrak{j}]\!]$-modules $\mathfrak{q}_{\underline{E}}/\mathfrak{j}^e\mathfrak{p}_{\underline{E}}$ and $(\mathfrak{q}_{\underline{M}}/\mathfrak{j}^e\mathfrak{p}_{\underline{M}})\oplus(\mathfrak{q}_{\underline{N}}/\mathfrak{j}^e\mathfrak{p}_{\underline{N}})$ have the same elementary divisors with the same multiplicities. By Lemma 5.7, they are respectively isomorphic to

$$M\oplus N/\mathfrak{j}^e\left\{(\tau_M(\tau^*m)+u(\tau^*n),\tau_N(\tau^*n)) \mid (m,n)\in M\oplus N\right\}$$

and $M/\mathfrak{j}^e\tau_M(\tau^*M)\oplus N/\mathfrak{j}^e\tau_N(\tau^*N)$. It follows that $[\mathcal{H}^+(\underline{E})]$ is Hodge additive if and only if $u(\tau^*N)\subset\tau_M(\tau^*M)$. $\square$

Corollary 5.20. *Let $\underline{M}$ be a mixed A-motive with negative weights. Then, the morphism ι of Theorem 3.4 induces an isomorphism*

$$\frac{M+\tau_M(\tau^*M)}{(\mathrm{id}-\tau_M)(M)} \xrightarrow{\sim} \operatorname{Ext}^{1,reg}_{\mathcal{MM}_F^{rig}}(\mathbb{1},\underline{M}).$$

In particular, $\operatorname{Ext}^{1,reg}_{\mathcal{MM}_F^{rig}}(\mathbb{1},\underline{M})$ *is an A-module.*

5.1.4 Regulators of A-motives

We are now in position to discuss regulators in the function fields setting. We define to types of them: the *general regulator* being associated with the functor $\mathscr{H}_R^+$, and the *special regulator* associated with $\mathcal{H}_R^+$.

General regulator

Let $\underline{M}$ be an object in $\mathcal{MM}_F^{\mathrm{rig}}$. We recall that $v: F\to\mathbb{C}_\infty$ is a K-algebra morphism and R is a Noetherian sub-A-algebra of K_∞. Corollary 5.6 makes the next definition consistent.

Definition 5.21. The *general v-regulator of $\underline{M}$ (with coefficient ring R)* is the R-linear morphism

$$\mathscr{R}eg_R^v(\underline{M}) : \operatorname{Ext}^1_{\mathcal{MM}_F^{\mathrm{rig}}}(\mathbb{1},\underline{M})\otimes_A R \longrightarrow \operatorname{Ext}^1_{\mathcal{MHP}_R^+}(\mathbb{1}^+,\mathscr{H}_R^+(\underline{M})),$$

which maps the class of $[\underline{E}]$ to the class of $[\mathscr{H}_R^+(\underline{E})]$.

In this subsection, we describe explicitly $\overset{v}{\mathscr{R}eg}_R(\underline{M})$ in the case where the weights of $\underline{M}$ are all negative. Let ι be the isomorphism of Theorem 3.4. From Proposition 3.82, ι induces an A-linear isomorphism

$$\iota : \frac{M[\mathfrak{j}^{-1}]}{(\mathrm{id} - \tau_M)(M)} \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{M}\mathcal{M}_F^{\mathrm{rig}}}(\mathbb{1}, \underline{M}).$$

Let $[\underline{E}] \in \mathrm{Ext}^1_{\mathcal{M}\mathcal{M}_F^{\mathrm{rig}}}(\mathbb{1}, \underline{M})$ and let $m \in M[\mathfrak{j}^{-1}]$ be such that $[\underline{E}] = \iota(m)$. From Proposition 4.49, the extension of mixed Hodge-Pink structure $\mathscr{H}_R([\underline{E}])$ associated to $[\underline{E}]$ is of the form

$$\left(\Lambda_v(\underline{M})_R \oplus \Lambda_v(\mathbb{1})_R, (\Lambda_v(W_\mu \underline{M})_K \oplus \mathbf{1}_{\mu \geq 0} K)_\mu, \begin{pmatrix} \mathrm{id} & h \\ 0 & \mathrm{id} \end{pmatrix} \mathfrak{q}_{\underline{M}} \oplus \mathfrak{q}_{\mathbb{1}} \right) \tag{5.4}$$

for a certain $h \in M \otimes_{A \otimes F, v} F_v^s((\mathfrak{j}))$. The infinite Frobenius $\phi_{\underline{E}}$ acts as

$$\sigma \longmapsto \begin{pmatrix} \phi_{\underline{M}}(\sigma) & c(\sigma) \\ 0 & 1 \end{pmatrix} \tag{5.5}$$

for a certain cocycle $c : G_v \to \Lambda_v(\underline{M})_R$. Our aim is to express h and c in terms of m.

Proposition 5.22. *Let $m \in M[\mathfrak{j}^{-1}]$ and let ξ be a solution of*

$$\xi - \tau_M(\tau^* \xi) = m$$

in $M \otimes_{A \otimes F, v} \mathbb{C}_\infty\langle A \rangle$ (By Corollary 1.68, ξ exists in $M \otimes_{A \otimes F, v} F_v^s((\mathfrak{j}))$). Then, the image of $\iota(m)$ through $\overset{v}{\mathscr{R}eg}_R(\underline{M})$ is congruent to an extension of the form (5.4) *with $h = -(\gamma^v_{\underline{M}})^{-1}(\xi)$ and with infinite Frobenius of the form* (5.5) *with $c(\sigma) = \xi^\sigma - \xi$ for all $\sigma \in G_L$.*

Proof. By definition, $[\underline{E}] = \iota(m)$ is of the form $[M \oplus A \otimes F, \left(\begin{smallmatrix} \tau_M & m \\ 0 & 1 \end{smallmatrix}\right)]$. Let us first compute its v-Betti realization. The A-module $\Lambda_v(\underline{E})$ is described by couples (ω, a) where $\omega \in M \otimes_{A \otimes F, v} \mathbb{C}_\infty\langle A \rangle$ and $a \in \mathbb{C}_\infty\langle A \rangle$, satisfying

$$\begin{pmatrix} \tau_M & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau^* \omega \\ \tau^* a \end{pmatrix} = \begin{pmatrix} \omega \\ a \end{pmatrix} \tag{5.6}$$

The bottom row equation yields that $a \in A$ whereas the top arrow yields $\tau_M(\tau^* \omega) + am = \omega$. Let ξ be any solution in $\mathbb{C}_\infty\langle A \rangle$ of the equation $\xi - \tau_M(\tau^* \xi) = m$. Then,

$$\Lambda_v(\underline{E}) = \{(\omega + a\xi, a) \mid \omega \in \Lambda_v(\underline{M}),\ a \in A\}.$$

The choice of ξ is equivalent to the choice of a splitting $\Lambda_v(\underline{M}) \oplus A \xrightarrow{\sim} \Lambda_v(\underline{E})$ which maps (ω, a) to $(\omega + a\xi, a)$. Because the weights of $\underline{M}$ are negative, this splitting preserves the weight filtration. In addition, the infinite Frobenius $\phi_{\underline{E}}$

acts on $\Lambda_v(\underline{E})$ by mapping, for $\sigma \in G_v$, $(\omega + a\xi, a)$ to $(\omega^\sigma + a\xi^\sigma, a)$. In terms of the above splitting, we get a commutative diagram:

$$\begin{tikzcd}
\Lambda_v(\underline{M}) \oplus A \arrow[r] \arrow[d, "\left(\begin{smallmatrix} \phi_{\underline{M}}(\sigma) & \xi^\sigma - \xi \\ 0 & 1 \end{smallmatrix}\right)"'] & \Lambda_v(\underline{E}) \arrow[d, "\phi_{\underline{E}}(\sigma)"] \\
\Lambda_v(\underline{M}) \oplus A \arrow[r] & \Lambda_v(\underline{E})
\end{tikzcd}$$

It follows that the cocycle $c : G_v \to \Lambda_v(\underline{E})_R$ is given by $\sigma \mapsto \xi^\sigma - \xi$.

We come back to the determination of h. By Corollary 1.68, there exists a finite separable extension L of K_∞ in $\mathbb{C}_\infty$ such that $\xi \in L\langle\langle A\rangle\rangle_{\mathfrak{j}}$. In particular, the image of ξ in $F_v^s(\!(\mathfrak{j})\!)$ is well-defined. The morphism $\gamma_{\underline{E}}$ is computed from that splitting through the following diagram of $F_v^s(\!(\mathfrak{j})\!)$-modules:

$$\begin{tikzcd}
\Lambda_v(\underline{E}) \otimes_A F_v^s(\!(\mathfrak{j})\!) \arrow[r, "\gamma_{\underline{E}}", "\sim"'] & E \otimes_{A\otimes F, v} F_v^s(\!(\mathfrak{j})\!) \\
(\Lambda_v(\underline{M}) \oplus A) \otimes_A F_v^s(\!(\mathfrak{j})\!) \arrow[u, "\wr"] \arrow[r, "\left(\begin{smallmatrix} \gamma_{\underline{M}} & \xi \\ 0 & 1 \end{smallmatrix}\right)"] & (M \oplus A \otimes F) \otimes_{A\otimes F, v} F_v^s(\!(\mathfrak{j})\!) \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

We deduce that the Hodge-Pink lattice $\mathfrak{q}_{\underline{E}}$ is

$$\begin{aligned}
\mathfrak{q}_{\underline{E}} &= \begin{pmatrix} (\gamma^v_{\underline{M}})^{-1} & -(\gamma^v_{\underline{M}})^{-1}(\xi) \\ 0 & 1 \end{pmatrix} (M \oplus A \otimes F) \otimes_{A\otimes F, v} F_v^s[\![\mathfrak{j}]\!] \\
&= \begin{pmatrix} \mathrm{id} & -(\gamma^v_{\underline{M}})^{-1}(\xi) \\ 0 & 1 \end{pmatrix} \mathfrak{q}_{\underline{M}} \oplus \mathfrak{q}_{\mathbb{1}}.
\end{aligned}$$

It follows that $h = -(\gamma^v_{\underline{M}})^{-1}(\xi)$. $\square$

Special regulator

The general regulator of $\underline{M}$ induces an R-linear morphism

$$\mathscr{R}\!\mathit{eg}^{v}_R(\underline{M}) : \operatorname{Ext}^{1,\mathrm{reg}}_{\mathcal{MM}^{\mathrm{rig}}_F}(\mathbb{1}, \underline{M}) \otimes_A R \longrightarrow \operatorname{Ext}^{1,\mathrm{ha}}_{\mathcal{MHP}^+_R}(\mathbb{1}^+, \mathscr{H}^+_R(\underline{M})).$$

We now assume that $\underline{M}$ is an object of $\mathcal{MM}^{\mathrm{reg}}_F$. We introduce the special regulator.

Definition 5.23. The *special v-regulator of $\underline{M}$* (with coefficients in R) is the R-linear morphism

$$\operatorname{Reg}^v_R(\underline{M}) : \operatorname{Ext}^1_{\mathcal{MM}^{\mathrm{reg}}_F}(\mathbb{1}, \underline{M}) \otimes_A R \longrightarrow \operatorname{Ext}^1_{\mathcal{MH}^+_R}(\mathbb{1}^+, \mathcal{H}^+_R(\underline{M}))$$

which maps the class of $[\underline{E}]$ to the class of $[\mathcal{H}^+_R(\underline{E})]$.

Remark 5.24. By Definition 5.14, $\operatorname{Reg}^v_R(\underline{M})$ is the composition of $\mathscr{R}\!\mathit{eg}^{v}_R(\underline{M})$ and $[\underline{H}] \mapsto [\underline{H}^{\#}]$.

5.2 Extensions with analytic reduction at v

Let $v : F \to \mathbb{C}_\infty$ be a K-algebra morphism and let F_v be the completion of F for $|x|_v = |\sigma(x)|$. Let F_v^s be a separable closure of F_v and let $G_v = \mathrm{Gal}(F_v^s|F_v)$ be the absolute Galois group of F_v. In this section, we introduce the notion of extensions in $\mathcal{MM}_F^{\mathrm{rig}}$ *having analytic reduction at* v. The notion plays a central role in the rest of the text, and answers two main issues:

- In Chapter 6, we showed the kernels of the adic realization maps (Definitions 4.13 and 4.53) are more relevant than $\mathrm{Ext}^1_{\mathcal{MH}_R^+}$ and $\mathrm{Ext}^1_{\mathcal{MHP}_R^+}$ in the study of Beilinson's conjectures (e.g. the beginning of Subsection 4.2.2). Extensions in the category $\mathcal{MM}_F^{\mathrm{rig}}$ *having analytic reduction at* v will provide a natural sub-A-module of $\mathrm{Ext}^1_{\mathcal{MM}_F^{\mathrm{rig}}}(\mathbb{1}, \underline{M})$ whose image through $\mathscr{R}eg_R^v$ lies in the kernel of $d_{\mathscr{H}_R^+(\underline{M})}$ (see Corollary 5.30).
- If $\underline{M}$ is an object in $\mathcal{MM}_F^{\mathrm{reg}}$, say with negative weights, the sub-A-module $\mathrm{Ext}^1_{\mathcal{MM}_F^{\mathrm{reg}},\mathcal{O}_F}(\mathbb{1}, \underline{M})$ of extensions of $\mathbb{1}$ by $\underline{M}$ in the category $\mathcal{MM}_F^{\mathrm{reg}}$ having everywhere good reduction (as in Subsection 2.2.3) is not a finitely generated module unless $\underline{M} = \underline{0}$ (Theorem 6.2). This is in opposition to what is expected in the number field situation. The essential reason for the non-finiteness feature comes from a phenomenon already observed by Taelman in [Tae2] from the side of Drinfeld modules. Contrary to the number field setting where, for a classical mixed motive M over $\mathbb{Q}$ and M_B its Betti realization, $H^1(\mathrm{Gal}(\mathbb{C}|\mathbb{R}), M_B)$ is, conjecturally, a finite abelian group, $H^1(G_v, \Lambda_v(\underline{M}))$ is generally not of finite type over A. In the next chapter, Theorem 6.2, we explain that this prevents $\mathrm{Ext}^1_{\mathcal{MM}_F^{\mathrm{reg}},\mathcal{O}_F}(\mathbb{1}, \underline{M})$ from being finitely generated. We equally show that the natural submodule of extensions having analytic reduction at ∞ is finitely generated over A.

Let $\underline{M}$ be an object in $\mathcal{MM}_F^{\mathrm{rig}}$. Inspired by [Tae2] in the context of Drinfeld modules, we introduce the *v-adic realization map* as follows. Given a short exact sequence of A-motives over F

$$[\underline{E}] : \quad 0 \longrightarrow \underline{M} \longrightarrow \underline{E} \longrightarrow \mathbb{1} \longrightarrow 0, \tag{5.7}$$

the induced sequence of $A[G_v]$-modules

$$0 \longrightarrow \Lambda_v(\underline{M}) \longrightarrow \Lambda_v(\underline{E}) \longrightarrow \Lambda_v(\mathbb{1}) \longrightarrow 0 \tag{5.8}$$

is exact by Corollary 1.61. In the category of A-modules, the above sequence splits, and the choice of a splitting yields an A-linear isomorphism $\Lambda_v(\underline{E}) \cong \Lambda_v(\underline{M}) \oplus \Lambda_v(\mathbb{1})$ on which $\sigma \in G_v$ acts by a matrix of the form $\left(\begin{smallmatrix} \sigma & c \\ 0 & 1 \end{smallmatrix}\right)$ for some application $c : G_v \to \Lambda_v(\underline{M})$. The property that this association is a group morphism translates to the assertion that the mapping $\sigma \mapsto c$ defines a cocycle $c_{\underline{E}} : G_v \to \Lambda_v(\underline{M})$. The latter cocycle is well-known to not depend on the choice of the splitting modulo principal cocycles.

Definition 5.25. We call the *v-adic realization map of $\underline{M}$*, and name it $r_{\underline{M},v}$, the A-linear morphism

$$r_{\underline{M},v} : \mathrm{Ext}^1_{\mathcal{MM}_F^{\mathrm{rig}}}(\mathbb{1}, \underline{M}) \longrightarrow H^1(G_v, \Lambda_v(\underline{M}))$$

induced by the exactness of Λ_v (Corollary 1.61). It maps $[\underline{E}]$ to $c_{\underline{E}}$.

Similarly, the Betti realization functor is exact and, as such, defines an A-module morphism, called the *∞-adic realization map of $\underline{M}$*,

$$r_{\underline{M},\infty} : \mathrm{Ext}^1_{\mathcal{MM}_F^{\mathrm{rig}}}(\mathbb{1}, \underline{M}) \longrightarrow H^1(G_\infty, \Lambda(\underline{M})). \tag{5.9}$$

The morphism $r_{\underline{M},\infty}$ is already covered by Definition 5.25 taking $F = K$ and for $v = i$ the inclusion. Indeed, we have a commutative square of A-modules

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathcal{MM}_F^{\mathrm{rig}}}(\mathbb{1}_F, \underline{M}) & \xrightarrow{=} & \mathrm{Ext}^1_{\mathcal{MM}_K^{\mathrm{rig}}}(\mathbb{1}_K, \mathrm{Res}_{F/K}\, \underline{M}) \\ \Big\downarrow r_{\underline{M},\infty} & & \Big\downarrow r_{\underline{M},i} \\ H^1(G_\infty, \Lambda(\underline{M})) & \xrightarrow{=} & H^1(G_\infty, \Lambda_i(\mathrm{Res}_{F/K}\, \underline{M})) \end{array}$$

Definition 5.26. We say that $[\underline{E}] \in \mathrm{Ext}^1_{\mathcal{MM}_F^{\mathrm{rig}}}(\mathbb{1}, \underline{M})$ has *analytic reduction at v* (resp. *at ∞*) if $[\underline{E}]$ lies in the kernel of $r_{\underline{M},v}$ (resp. $r_{\underline{M},\infty}$). We denote $\mathrm{Ext}^{1,v}_{\mathcal{MM}_F^{\mathrm{rig}}}(\mathbb{1}, \underline{M})$ (resp. $\mathrm{Ext}^{1,\infty}_{\mathcal{MM}_F^{\mathrm{rig}}}(\mathbb{1}, \underline{M})$) the kernel of $r_{\underline{M},v}$ (resp. $r_{\underline{M},\infty}$).

We now assume that $\underline{M}$ has negative weights. We describe explicitly $r_{\underline{M},v}(\iota(m))$, where ι is the isomorphism of Theorem 3.4 and $m \in M[\mathfrak{j}^{-1}]$.

Proposition 5.27. *For $m \in M[\mathfrak{j}^{-1}]$, the morphism $r_{\underline{M},v}$ maps the extension $\iota(m)$ to the cocycle $c_m : \sigma \mapsto \xi_m^\sigma - \xi_m$, where $\xi_m \in M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle A\rangle$ is any solution of the equation $\xi - \tau_M(\tau^*\xi) = m$ (which exists by Corollary 1.68).*

Remark 5.28. Note the similarities between Propositions 5.27 and 5.22. This proximity is at the heart of Theorem 5.31 below.

Proof of Proposition 5.27. Choose $m \in M[\mathfrak{j}^{-1}]$ and let $[\underline{E}] = \iota(m)$. By definition, the v-Betti realization of $\underline{E}$ consists of pairs (ξ, a), $\xi \in M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle A\rangle$ and $a \in F_v\langle A\rangle$, solution of the system

$$\begin{pmatrix} \tau_M & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau^*\xi \\ \tau^* a \end{pmatrix} = \begin{pmatrix} \xi \\ a \end{pmatrix}.$$

It follows that $a \in A$ and $\xi - \tau_M(\tau^*\xi) = am$. A splitting of exact sequence $[\Lambda_v(\underline{E})]$ in the category of A-modules corresponds to the choice of a particular solution ξ_m of the equation $\xi - \tau_M(\tau^*\xi) = m$ in $M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle A\rangle$. To the choice of ξ_m corresponds the splitting

$$\Lambda_v(\underline{M}) \oplus \Lambda_v(\mathbb{1}) \xrightarrow{\sim} \Lambda_v(\underline{E}), \quad (\omega, a) \longmapsto (\omega + a\xi_m, a).$$

An element $\sigma \in G_v$ acts on the right-hand side by

$$(\omega + a\xi_m, a) \mapsto (\omega^\sigma + a\xi_m^\sigma, a) = (\omega^\sigma + a(\xi_m^\sigma - \xi_m) + a\xi_m, a)$$

where $\xi_m^\sigma - \xi_m \in \Lambda_v(\underline{M})$. Hence, σ acts as the matrix $\left(\begin{smallmatrix} \sigma & \xi_m^\sigma - \xi_m \\ 0 & 1 \end{smallmatrix}\right)$ as desired. $\square$

The next definition introduce new appropriate notations.

Definition 5.29. Let $\underline{M}$ be an object of $\mathcal{MM}_F^{\mathrm{rig}}$ with negative weights. We let

$$\mathrm{Ext}^{1,v}_{\mathcal{MHP}_R^+}(\mathbb{1}^+,\mathscr{H}_R^+(\underline{M})) \quad (\text{resp. } \mathrm{Ext}^{1,v}_{\mathcal{MH}_R^+}(\mathbb{1}^+,\mathcal{H}_R^+(\underline{M}))\ )$$

denote the submodule of $\mathrm{Ext}^1_{\mathcal{MHP}_R^+}(\mathbb{1}^+,\mathscr{H}_R^+(\underline{M}))$ (resp. $\mathrm{Ext}^1_{\mathcal{MH}_R^+}(\mathbb{1}^+,\mathscr{H}_R^+(\underline{M}))$) given by the kernel of $d_{\mathscr{H}_R^+(\underline{M})}$ (resp. $d_{\mathcal{H}_R^+(\underline{M})}$) (Definition 4.53, resp. 4.13).

As an immediate consequence of Propositions 5.27 and 5.22, we have the next important result, already announced in the beginning of this section:

Corollary 5.30. *Let $\underline{M}$ be an object of $\mathcal{MM}_F^{rig}$ with negative weights. Then,*

$$\mathscr{R}\!eg^{v}_{R}(\underline{M})\left(\mathrm{Ext}^{1,v}_{\mathcal{MM}_F^{rig}}(\mathbb{1},\underline{M})\right)\subset \mathrm{Ext}^{1,v}_{\mathcal{MHP}_R^+}(\mathbb{1}^+,\mathscr{H}_R^+(\underline{M})).$$

The above corollary is in fact a particular case of the next theorem.

Theorem 5.31. *Let $\underline{M}$ be an object in $\mathcal{MM}_F^{rig}$ with negative weights. We have a commutative diagram of A-modules whose lines are exact:*

$$\begin{tikzcd}
\mathrm{Ext}^{1,v}_{\mathcal{MM}_F^{rig}}(\mathbb{1},\underline{M}) \arrow[r, hook] \arrow[d, "\mathscr{R}\!eg^{v}_{R}(\underline{M})"] & \mathrm{Ext}^{1}_{\mathcal{MM}_F^{rig}}(\mathbb{1},\underline{M}) \arrow[r, "r"] \arrow[d, "\mathscr{R}\!eg^{v}_{R}(\underline{M})"] & H^1(G_v,\Lambda_v(\underline{M})) \arrow[d, "\mathrm{id}\otimes_A 1_R"] \\
\mathrm{Ext}^{1,v}_{\mathcal{MHP}_R^+}(\mathbb{1}^+,\mathscr{H}_R^+(\underline{M})) \arrow[r, hook] & \mathrm{Ext}^{1}_{\mathcal{MHP}_R^+}(\mathbb{1}^+,\mathscr{H}_R^+(\underline{M})) \arrow[r, two heads, "d"] & H^1(G_v,\Lambda_v(\underline{M})\otimes_A R)
\end{tikzcd}$$

where $r=r_{\underline{M},v}\otimes_A \mathrm{id}_R$ and $d=d_{\mathscr{H}_R^+(\underline{M})}$ (see Definitions 5.25 and 4.53).

We begin with a lemma.

Lemma 5.32. *Let $\underline{M}$ be a rigid analytically trivial A-motive. Let $\mathfrak{l}$ be a $F_v^s[\![\mathfrak{j}]\!]$-lattice in $\Lambda(\underline{M})\otimes_A F_v^s(\!(\mathfrak{j})\!)$. Then, $\mathfrak{l}$ is G_v-equivariant and $H^1(G_v,\mathfrak{l})=0$.*

Proof. The $F_v^s[\![\mathfrak{j}]\!]$-lattice $\mathfrak{l}$ is isomorphic to an $F_v^s[\![\mathfrak{j}]\!]$-lattice in $M\otimes_{A\otimes F,v}F_v^s(\!(\mathfrak{j})\!)$ via $\gamma^v_{\underline{M}}$ (5.3). By the elementary divisor Theorem in the discrete valuation ring $F_v^s[\![\mathfrak{j}]\!]$, there exists a G_v-equivariant $F_v^s(\!(\mathfrak{j})\!)$-linear automorphism ψ of the $F_v^s(\!(\mathfrak{j})\!)$-vector space $M\otimes_{A\otimes F,v}F_v^s(\!(\mathfrak{j})\!)$ such that

$$\gamma^v_{\underline{M}}(\mathfrak{l})=\psi(M\otimes_{A\otimes F,v}F_v^s[\![\mathfrak{j}]\!]).$$

This implies that $\mathfrak{l}$ is G_v-equivariant and further that $\mathfrak{l}$ is isomorphic to $M\otimes_{A\otimes F,v}F_v^s[\![\mathfrak{j}]\!]$ as a $F_v^s[\![\mathfrak{j}]\!][G_v]$-module. By the additive Hilbert's 90 Theorem we have $H^1(G_v,F_v^s[\![\mathfrak{j}]\!])=0$ and it follows that $H^1(G_v,\mathfrak{l})=0$. □

Proof of Theorem 5.31. The upper row is exact by construction (Definition 5.26). The exactness of the lower row follows from Lemma 5.32 and Proposition 4.55. Propositions 5.27 and 5.22 imply that the squares are commutative. □

5.3 Rigid analytic description of extension modules

We end this chapter by a description of the extension groups in $\mathcal{M}_F^{\mathrm{rig}}$ in terms of solutions of τ-difference equations (Proposition 5.33). These formulas will be used to describe general and special regulators. In that respect, Theorem 5.34 below show that $\mathrm{Reg}_R^v(\underline{M})$ can be interpreted as an "evaluation at $\mathfrak{j}$" of certain transcendental series given by solutions of these τ-difference equations. This key observation is at the origin of Chapter 7, where we use $\mathrm{Reg}_R^v(\underline{M})$ to study algebraic relations among values of Carlitz's polylogarithms. Also, the content of this section is one of the main ingredients in the proofs of Theorem E and F (Chapter 6).

Let $v : F \to \mathbb{C}_\infty$ be a K-algebra morphism. Recall that F_v is the completion of F with respect to $|x|_v := |v(x)|$. The embedding v extends by continuity to $F_v \to \mathbb{C}_\infty$, so that it makes sense to consider the algebra $F_v\langle A\rangle$ of Section 1.4.

Let $\underline{M}$ be an object of $\mathcal{M}_F^{\mathrm{rig}}$. Let us give a name to various sub-A-modules of $\mathrm{Ext}^1_{\mathcal{M}_F^{\mathrm{rig}}}(\mathbb{1},\underline{M})$. To this end, we identify $\mathrm{Ext}^1_{\mathcal{M}_F^{\mathrm{rig}}}(\mathbb{1},\underline{M})$ as a submodule of $\mathrm{Ext}^1_{\mathcal{M}_F}(\mathbb{1},\underline{M})$.

For $* \in \{\infty, v\}$, we set:

- $\mathrm{Ext}^{1,*}_{\mathcal{M}_F^{\mathrm{rig}},\mathcal{O}_F}(\mathbb{1},\underline{M}) := \mathrm{Ext}^1_{\mathcal{M}_F,\mathcal{O}_F}(\mathbb{1},\underline{M}) \cap \mathrm{Ext}^{1,*}_{\mathcal{M}_F^{\mathrm{rig}}}(\mathbb{1},\underline{M})$,

- $\mathrm{Ext}^{1,\mathrm{reg},*}_{\mathcal{M}_F^{\mathrm{rig}}}(\mathbb{1},\underline{M}) := \mathrm{Ext}^{1,\mathrm{reg}}_{\mathcal{M}_F^{\mathrm{rig}}}(\mathbb{1},\underline{M}) \cap \mathrm{Ext}^{1,*}_{\mathcal{M}_F^{\mathrm{rig}}}(\mathbb{1},\underline{M})$,

- $\mathrm{Ext}^{1,\mathrm{reg},*}_{\mathcal{M}_F^{\mathrm{rig}},\mathcal{O}_F}(\mathbb{1},\underline{M}) := \mathrm{Ext}^{1,*}_{\mathcal{M}_F^{\mathrm{rig}},\mathcal{O}_F}(\mathbb{1},\underline{M}) \cap \mathrm{Ext}^{1,\mathrm{reg}}_{\mathcal{M}_F^{\mathrm{rig}}}(\mathbb{1},\underline{M})$.

The next proposition describes elements of the above modules as solutions of τ-difference equations.

Proposition 5.33. *Let $\underline{M} = (M, \tau_M)$ be a rigid analytically trivial A-motive over F. Let ι be the isomorphism of Theorem 3.4. The A-linear map*

$$\frac{\{\xi \in M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle A\rangle | \xi - \tau_M(\tau^*\xi) \in M[\mathfrak{j}^{-1}]\}}{M + \Lambda_v(\underline{M})} \to \mathrm{Ext}^1_{\mathcal{M}_F^{rig}}(\mathbb{1},\underline{M}), \tag{5.10}$$

mapping the class of ξ to $\iota(\xi - \tau_M(\tau^*\xi))$, is an isomorphism. Let $M^{reg} :=$

$M + \tau_M(\tau^* M)$ *and* $M_{\mathcal{O}}^{reg} := M^{reg} \cap M_{\mathcal{O}}[\mathfrak{j}^{-1}]$. *Then* (5.10) *specializes to:*

$$\frac{\{\xi \in M \otimes_v \mathbb{C}_\infty\langle A\rangle \mid \xi - \tau_M(\tau^*\xi) \in M_{\mathcal{O}}[\mathfrak{j}^{-1}]\}}{M_{\mathcal{O}} + \Lambda_v(\underline{M})} \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{M}_F^{rig},\mathcal{O}_F}(\mathbb{1}, \underline{M}), \tag{5.11}$$

$$\frac{\{\xi \in M \otimes_v \mathbb{C}_\infty\langle A\rangle \mid \xi - \tau_M(\tau^*\xi) \in M^{reg}\}}{M + \Lambda_v(\underline{M})} \xrightarrow{\sim} \mathrm{Ext}^{1,reg}_{\mathcal{M}_F^{rig}}(\mathbb{1}, \underline{M}), \tag{5.12}$$

$$\frac{\{\xi \in M \otimes_v F_v\langle A\rangle \mid \xi - \tau_M(\tau^*\xi) \in M[\mathfrak{j}^{-1}]\}}{M + \Lambda_v(\underline{M})^+} \xrightarrow{\sim} \mathrm{Ext}^{1,v}_{\mathcal{M}_F^{rig}}(\mathbb{1}, \underline{M}), \tag{5.13}$$

$$\frac{\{\xi \in M \otimes_v \mathbb{C}_\infty\langle A\rangle \mid \xi - \tau_M(\tau^*\xi) \in M_{\mathcal{O}}^{reg}\}}{M_{\mathcal{O}} + \Lambda_v(\underline{M})} \xrightarrow{\sim} \mathrm{Ext}^{1,reg}_{\mathcal{M}_F^{rig},\mathcal{O}_F}(\mathbb{1}, \underline{M}), \tag{5.14}$$

$$\frac{\{\xi \in M \otimes_v F_v\langle A\rangle \mid \xi - \tau_M(\tau^*\xi) \in M_{\mathcal{O}}[\mathfrak{j}^{-1}]\}}{M_{\mathcal{O}} + \Lambda_v(\underline{M})^+} \xrightarrow{\sim} \mathrm{Ext}^{1,v}_{\mathcal{M}_F^{rig},\mathcal{O}_F}(\mathbb{1}, \underline{M}), \tag{5.15}$$

$$\frac{\{\xi \in M \otimes_v F_v\langle A\rangle \mid \xi - \tau_M(\tau^*\xi) \in M^{reg}\}}{M + \Lambda_v(\underline{M})^+} \xrightarrow{\sim} \mathrm{Ext}^{1,reg,v}_{\mathcal{M}_F^{rig}}(\mathbb{1}, \underline{M}), \tag{5.16}$$

$$\frac{\{\xi \in M \otimes_v F_v\langle A\rangle \mid \xi - \tau_M(\tau^*\xi) \in M_{\mathcal{O}}^{reg}\}}{M_{\mathcal{O}} + \Lambda_v(\underline{M})^+} \xrightarrow{\sim} \mathrm{Ext}^{1,reg,v}_{\mathcal{M}_F^{rig},\mathcal{O}_F}(\mathbb{1}, \underline{M}). \tag{5.17}$$

Proof. First note that because both $\underline{M}$ and $\mathbb{1}$ are rigid analytically trivial, $\mathrm{Ext}^1_{\mathcal{M}_F^{\mathrm{rig}}}(\mathbb{1}, \underline{M})$ equals $\mathrm{Ext}^1_{\mathcal{M}_F}(\mathbb{1}, \underline{M})$ (Proposition 1.60). Let $N \subset N' \subset M[\mathfrak{j}^{-1}]$ be inclusions of sub-A-modules, and assume that $\tau_M(\tau^* N) \subset N'$. We have the following diagram of A-modules, exact on lines with commutative squares

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\mathrm{id} - \tau_M"] & M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle A\rangle \arrow[r] \arrow[d, "\mathrm{id} - \tau_M"] & \dfrac{M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle A\rangle}{N} \arrow[r] \arrow[d, "\varphi_{N,N'}"] & 0 \\
0 \arrow[r] & N' \arrow[r] & M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle A\rangle \arrow[r] & \dfrac{M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle A\rangle}{N'} \arrow[r] & 0,
\end{tikzcd}$$

where we denoted $\varphi_{N,N'}$ the induced morphism $\mathrm{id} - \tau_M$ on the quotient. By definition, we find

$$\ker \varphi_{N,N'} = \frac{\{\xi \in M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle A\rangle \mid \xi - \tau_M(\tau^*\xi) \in N'\}}{N}.$$

By Theorem 1.56, the middle vertical arrow is surjective. The snake Lemma implies that there is a natural isomorphism

$$\frac{\ker \varphi_{N,N'}}{\Lambda_v(\underline{M})} = \frac{\{\xi \in M \otimes_{A\otimes F,v} \mathbb{C}_\infty\langle A\rangle \mid \xi - \tau_M(\tau^*\xi) \in N'\}}{N + \Lambda_v(\underline{M})} \xrightarrow{\sim} \frac{N'}{(\mathrm{id} - \tau_M)(N)}$$

given explicitly by mapping ξ to $\xi - \tau_M(\tau^*\xi)$.

We obtain the isomorphism (5.10) by taking $N = M$ and $N' = M[\mathfrak{j}^{-1}]$ above. By Theorem 3.18, (5.11) follows from $N = M_{\mathcal{O}}$ and $N' = M_{\mathcal{O}}[\mathfrak{j}^{-1}]$. By Corollary 5.20, (5.12) follows from $N = M$ and $N' = M^{\mathrm{reg}}$. (5.14) follows from $N = M_{\mathcal{O}}$ and $N' = M_{\mathcal{O}}^{\mathrm{reg}}$.

To obtain the remaining isomorphisms, we apply the same argument to the following commutative diagram of A-modules, exact on rows:

$$\begin{tikzcd}
 & \ker(\mathrm{id}-\tau_M|N) \arrow[r] \arrow[d, hook] & \Lambda_v(\underline{M})^+ \arrow[d, hook] & & \\
0 \arrow[r] & N \arrow[r] \arrow[d, "\mathrm{id}-\tau_M"] & M\otimes_{A\otimes F,v} F_v\langle A\rangle \arrow[r] \arrow[d, "\mathrm{id}-\tau_M"] & \dfrac{M\otimes_{A\otimes F,v} F_v\langle A\rangle}{N} \arrow[r] \arrow[d, "\varphi_{N,N'}"] & 0 \\
0 \arrow[r] & N' \arrow[r] \arrow[d, two heads] & M\otimes_{A\otimes F,v} F_v\langle A\rangle \arrow[r] \arrow[d, two heads] & \dfrac{M\otimes_{A\otimes F,v} F_v\langle A\rangle}{N'} \arrow[r] & 0 \\
 & \operatorname{coker}(\mathrm{id}-\tau_M|N) \arrow[r, "r"] & H^1(G_v, \Lambda_v(\underline{M})) & &
\end{tikzcd} \tag{5.18}$$

The middle column is exact by Theorem 1.56.

When $N = M$ and $N' = M[\mathfrak{j}^{-1}]$, it follows from the explicit description of the v-adic realization map $r_{\underline{M},v}$ in Proposition 5.27 that r coincides with $r_{\underline{M},v}\circ\iota$. Hence (5.13) follows from the snake Lemma. Similarly, (5.15) follows from $N = M_{\mathcal{O}}$, $N' = M_{\mathcal{O}}[\mathfrak{j}^{-1}]$, (5.16) follows from $N = M$, $N' = M^{\mathrm{reg}}$ and (5.17) follows from $N = M_{\mathcal{O}}$, $N' = M_{\mathcal{O}}^{\mathrm{reg}}$. $\square$

We now assume that $\underline{M}$ is an object of $\mathcal{MM}_F^{\mathrm{rig}}$. We regard $\operatorname{Ext}^1_{\mathcal{MM}_F^{\mathrm{rig}}}(\mathbb{1},\underline{M})$ as a submodule of $\operatorname{Ext}^1_{\mathcal{M}_F}(\mathbb{1},\underline{M})$. We assume that the weights of $\underline{M}$ are negative, so that the latter inclusion is an equality (Proposition 3.8). We end this chapter by a conclusive commutative diagram of A-modules which synthetizes results on regulators.

Theorem 5.34. *Let $\underline{M}$ be an object of $\mathcal{MM}_F^{rig}$ with negative weights. We have a commutative diagram of A-modules:*

$$\begin{tikzcd}
\operatorname{Ext}^{1,reg,v}_{\mathcal{MM}_F^{rig}}(\mathbb{1},\underline{M}) \arrow[d, "\mathscr{R}eg^v_R(\underline{M})"'] \arrow[dd, bend right=60, "\operatorname{Reg}^v_R(\underline{M})"'] & \dfrac{\{\xi\in M\otimes_v F_v\langle A\rangle \mid \xi-\tau_M(\tau^*\xi)\in M_{\mathcal{O}}^{reg}\}}{\Lambda_v(\underline{M})^+ + M_{\mathcal{O}}} \arrow[l, "\sim"'] \arrow[d, "\xi\mapsto\xi"] \\
\operatorname{Ext}^{1,ha,v}_{\mathcal{MHP}_R^+}(\mathbb{1}^+,\mathscr{H}_R^+(\underline{M})) \arrow[d, "{[\underline{H}]\mapsto[\underline{H}^\#]}"'] & \dfrac{(M+\tau_M(\tau^*M))\otimes_v F_v[\![\mathfrak{j}]\!]}{\Lambda_v(\underline{M})_R^+ + M\otimes_v F_v[\![\mathfrak{j}]\!]} \arrow[l, "\sim"'] \arrow[d, "\xi\mapsto-(\gamma^v_{\underline{M}})^{-1}(\xi)\ (mod\ \mathfrak{j})"] \\
\operatorname{Ext}^{1,v}_{\mathcal{MH}_R^+}(\mathbb{1}^+,\mathcal{H}_R^+(\underline{M})) & \dfrac{(\Lambda_v(\underline{M})\otimes_A F_v^s)^+}{\Lambda_v(\underline{M})_R^+ + F^0(\Lambda_v(\underline{M})\otimes_A F_v^s)^+} \arrow[l, "\sim"']
\end{tikzcd}$$

Proof. The top isomorphism is Proposition 5.33(5.17). The middle isomorphism results as the composition of

$$\mathrm{Ext}^{1,\mathrm{ha},v}_{\mathcal{MHP}^+_R}(\mathbb{1}^+, \mathscr{H}^+_R(\underline{M})) \xleftarrow{\sim} \frac{\mathfrak{p}^+_{\underline{M}} + \mathfrak{q}^+_{\underline{M}}}{\Lambda(\underline{M})^+_R + \mathfrak{q}^+_{\underline{M}}} \xrightarrow{\sim} \frac{(M + \tau_M(\tau^* M)) \otimes_v F_v[\![\mathfrak{j}]\!]}{\Lambda_v(\underline{M})^+_R + M \otimes_v F_v[\![\mathfrak{j}]\!]}$$

where the first isomorphism follows from Proposition 4.55 and the second is induced by $-(\gamma^v_{\underline{M}})^{-1}$ (Lemma 5.7). That the upper square commutes is due to Proposition 5.22. The lower square commutes by Proposition 4.55. That the loop (with the curved arrow) commutes follows from the definition of Reg^v_R (Definition 5.23). □

Remark 5.35. From Theorem 5.34, $\mathrm{Reg}^v_R(\underline{M})$ can be interpreted as an "evaluation at $\mathfrak{j}$" of certain transcendental series given by solutions of τ-difference equations. This key observation is at the origin of Chapter 7, where we use $\mathrm{Reg}^v_R(\underline{M})$ to study algebraic relations among values of Carlitz's polylogarithms.

Chapter 6

Towards Beilinson's first conjecture for A-motives

This chapter is devoted to the proof of Theorems E, F (6.2 and 6.4 in this chapter) stated in the introduction (Chapter 0), and aims to lay down the statement of Beilinson's first conjecture in the function field setting.

6.1 Statements and methods of proof

6.1.1 Statements

Recall that $(C,\mathcal{O}_C)$ is a geometrically irreducible smooth projective curve over $\mathbb{F}$ and that ∞ is a closed point on C. Let F be a finite field extension of $K=\mathbb{F}(C)$ and let $\mathcal{O}_F$ be the integral closure of $A=H^0(C\setminus\{\infty\},\mathcal{O}_C)$ in F. Let $\underline{M}$ be a rigid analytically trivial A-motive over F and let $\Lambda(\underline{M})$ be its Betti realization equipped with the continuous action of $G_\infty=\mathrm{Gal}(K_\infty^s|K_\infty)$ (Definition 1.48). Recall that we denoted

$$\mathrm{Ext}^{1,\mathrm{reg},\infty}_{\mathcal{M}_F^{\mathrm{rig}},\mathcal{O}_F}(\mathbb{1},\underline{M}) := \ker\left(\mathrm{Ext}^{1,\mathrm{reg}}_{\mathcal{M}_F^{\mathrm{rig}},\mathcal{O}_F}(\mathbb{1},\underline{M}) \stackrel{r_{\underline{M},\infty}}{\longrightarrow} H^1(G_\infty,\Lambda(\underline{M}))\right)$$

where $r_{\underline{M},\infty}$ is the ∞-adic realization map (see Definition 5.25). Let us denote by $U(\underline{M})$ the corresponding cokernel:

$$U(\underline{M}) := \mathrm{coker}\left(\mathrm{Ext}^{1,\mathrm{reg}}_{\mathcal{M}_F^{\mathrm{rig}},\mathcal{O}_F}(\mathbb{1},\underline{M}) \stackrel{r_{\underline{M},\infty}}{\longrightarrow} H^1(G_\infty,\Lambda(\underline{M}))\right). \tag{6.1}$$

Inspired by [Tae2, Rmk. 6.2], we call $U(\underline{M})$ the *class module of* $\underline{M}$.

Remark 6.1. By Theorem 1.56, the computation of the class module $U(\underline{M})$ can be made explicit: we have a natural isomorphism of A-modules:

$$U(\underline{M}) \cong \frac{M\otimes_{A\otimes K} K_\infty\langle A\rangle}{(M+\tau_M(\tau^*M))\cap M_{\mathcal{O}}[\mathfrak{j}^{-1}]+(\mathrm{id}-\tau_M)(M\otimes_{A\otimes K}K_\infty\langle A\rangle)}.$$

The next theorem is the function field analogue of the finiteness conjecture in classical motivic cohomology. Repeated from Theorem E in Chapter 0, we state:

Theorem 6.2. *The A-modules* $\mathrm{Ext}^{1,reg,\infty}_{\mathcal{M}^{rig}_F,\mathcal{O}_F}(\mathbb{1},\underline{M})$ *and* $U(\underline{M})$ *are finitely generated. If all the weights of* $\underline{M}$ *are negative (even if* $\underline{M}$ *is not necessarily mixed), then* $U(\underline{M})$ *is finite.*

Remark 6.3. It follows from Theorem 6.2 that $\mathrm{Ext}^{1,\mathrm{reg}}_{\mathcal{M}^{\mathrm{rig}}_F,\mathcal{O}_F}(\mathbb{1},\underline{M})$ is generally not a finitely generated A-module. To wit, let $\underline{M}$ be a non zero rigid analytically trivial A-motive over K satisfying

$$\Lambda(\underline{M}) = \Lambda(\underline{M})^+$$

so that G_∞ acts trivially on $\Lambda(\underline{M})$. By Theorem 6.2, the non-finite generation of $\mathrm{Ext}^{1,\mathrm{reg}}_{\mathcal{M}^{\mathrm{rig}}_K,A}(\mathbb{1},\underline{M})$ is equivalent to that of $H^1(G_\infty,\Lambda(\underline{M}))$. Any cocycle thereof is described by an additive continuous function $G_\infty \to \Lambda(\underline{M})$. To conclude that $H^1(G_\infty,\Lambda(\underline{M}))$ is not finitely generated, it then suffices to show that G_∞ is not topologically finitely generated. This follows from class field theory: its wild inertia group is isomorphic to the group of one-units in $\mathcal{O}_\infty$, which is isomorphic to a countably infinite product of $\mathbb{Z}_p$.

Inasmuch as $\mathrm{Ext}^{1,\mathrm{reg},\infty}_{\mathcal{M}^{\mathrm{rig}}_F,\mathcal{O}_F}(\mathbb{1},\underline{M})$ is a finitely generated A-module, it is determined, up to isomorphisms, by its rank and its torsion submodule. While the latter seems very hard to determine in practice (see Chapter 7 for explicit computations in the case of Carlitz's tensor powers), the former is accessible from our next theorem which is the highlight of this thesis.

Let $\mathscr{H}^+_{K_\infty} : \mathcal{MM}^{\mathrm{rig}}_K \to \mathcal{MHP}^+_{K_\infty}$ be the Hodge-Pink realization functor with base field K_∞ and coefficients in K_∞ (Definition 5.3). By abuse of notations, we still denote by $\mathscr{H}^+_{K_\infty} : \mathcal{MM}^{\mathrm{rig}}_F \to \mathcal{MHP}^+_{K_\infty}$ the functor given by the composition

$$\mathcal{MM}^{\mathrm{rig}}_F \xrightarrow{\mathrm{Res}_{F/K}} \mathcal{MM}^{\mathrm{rig}}_K \xrightarrow{\mathscr{H}^+_{K_\infty}} \mathcal{MHP}^+_{K_\infty}.$$

Repeated from Theorem F in Chapter 0, we state:

Theorem 6.4. *Assume that* $\underline{M}$ *is mixed and that all the weights of* $\underline{M}$ *are negative. The spaces* $\mathrm{Ext}^{1,reg,\infty}_{\mathcal{MM}^{rig}_F,\mathcal{O}_F}(\mathbb{1},\underline{M})\otimes_A K_\infty$ *and* $\mathrm{Ext}^{1,ha,\infty}_{\mathcal{MHP}^+_{K_\infty}}(\mathbb{1}^+,\mathscr{H}^+_{K_\infty}(\underline{M}))$ *have the same dimension over* K_∞.

The proof of Theorem 6.4 exhibits an isomorphism $\rho(\underline{M})$ of K_∞-vector spaces. Theorem 6.4 motivates the next definition:

Definition 6.5. Assume that $\underline{M}$ is mixed and that all the weights of $\underline{M}$ are negative. We say that *Beilinson's conjecture is true for* $\underline{M}$ whenever $\mathscr{R}eg^\infty_{K_\infty}(\underline{M})$ induces an isomorphism of K_∞-vector spaces:

$$\mathrm{Ext}^{1,\mathrm{reg},\infty}_{\mathcal{MM}^{\mathrm{rig}}_F,\mathcal{O}_F}(\mathbb{1},\underline{M})\otimes_A K_\infty \xrightarrow{\sim} \mathrm{Ext}^{1,\mathrm{ha},\infty}_{\mathcal{MHP}^+_{K_\infty}}(\mathbb{1}^+,\mathscr{H}^+_{K_\infty}(\underline{M})).$$

It will appear from Chapter 7 that Beilinson's conjecture is not true in the case of p tensor power multiple of the Carlitz module, where p is the characteristic of $\mathbb{F}$.

6.1.2 Methods and plan of proof

Let $\underline{M}$ be a mixed rigid analytically trivial A-motive over F. A first observation is that in both Theorems 6.2 and 6.4, one can assume that $F = K$ and that $v = i$ is the inclusion. Indeed, we have (e.g. Proposition 5.33):

$$\mathrm{Ext}^{1,\mathrm{reg},\infty}_{\mathcal{MM}^{\mathrm{rig}}_F,\mathcal{O}_F}(\mathbb{1},\underline{M}) = \mathrm{Ext}^{1,\mathrm{reg},\infty}_{\mathcal{MM}^{\mathrm{rig}}_K,A}(\mathbb{1},\mathrm{Res}_{F/K}\underline{M}).$$

We therefore assume that $\underline{M}$ is over K. A main ingredient in the proof of Theorems 6.2 and 6.4 is what we call *shtuka models of A-motives*. We discuss this notion in Section 6.2. Shtuka models correspond roughly to *compactified versions* of A-motives over K, where the Dedekind scheme $\operatorname{Spec} A \otimes K$ is replaced by $C \times C$ and the ideal $\mathfrak{j}$ is replaced by the diagonal divisor Δ (Definition 6.9). Under the assumption that the weights of $\underline{M}$ are negative, we associate non-canonically a *$C \times C$-shtuka model* to $\underline{M}$ (Theorem 6.11). Our construction shares many similarities with Mornev's *global models* in the context of Drinfeld modules [Mo1, §12], although it is not directly linked. An important feature of Section 6.2 is Subsection 6.2.3 where we show an unexpected link between shtuka models at $\{\infty\} \times \{\infty\}$ and Hodge additive extensions of Hodge-Pink structures (Theorem 6.19). This link is at the heart of the proof of Theorem 6.4.

In Section 6.4, we focus on the proof of the theorems. We begin by cohomological preliminaries in Subsection 6.3.2. The objective there is to develop a simple method to compute the Zariski and formal coherent cohomology on schemes covered by two affine subschemes (Theorems 6.26 and 6.28). In Subsection 6.4.1, the proofs of Theorems 6.2 and 6.4 are achieved as an application of the methods of Subsection 6.3.2 to a $C \times C$-shtuka of $\underline{M}$. Comparison of Zariski and formal cohomology will lead to an isomorphism of K_∞-vector spaces $\rho(\underline{M})$ (see Definition 6.35) which we conjecture to be related to $\mathscr{R}eg^{\infty}_{K_\infty}(\underline{M})$ (Conjecture **??**). We leave this question open for now.

6.2 Shtuka models of A-motives

In this section, we define Shtuka models associated to A-motives $\underline{M}$ over K. Being defined on proper varieties over $\mathbb{F}$, they are better suited for cohomological computations. We consider two types of them: C and $C \times C$-models (introduced in Subsection 6.2.1 and 6.2.2 respectively). While the former exist unconditionally (Proposition 6.7), $C \times C$-models exist if and only if the weights of $\underline{M}$ are non-positive (Lemma 6.12 and Theorem 6.11). The most important part of this section is Subsection 6.2.3 where a link is made between $C \times C$-shtuka models at $\{\infty\} \times \{\infty\}$ and Hodge additive extensions of mixed Hodge-Pink structures (Theorem 6.19).

We denote by $\tau : C \times C \to C \times C$ the morphism of $\mathbb{F}$-schemes which acts as the identity on the left-hand factor C and as the q-Frobenius on the right-hand one. Because C is separated over $\mathbb{F}$, the diagonal morphism $C \to C \times C$ is a closed immersion and its image defines a closed subscheme Δ of $C \times C$ of codimension 1. It defines a divisor Δ on $C \times C$ which we call the *characteristic divisor*. Because $\mathcal{O}(\Delta) \subset \mathcal{O}_{C\times C}$, Δ is an effective divisor. The evaluation of $\mathcal{O}(\Delta)$ at the affine open subscheme $\operatorname{Spec}(A \otimes A)$ of $C \times C$ recovers the ideal $\mathfrak{j} = \mathfrak{j}_{\mathrm{id}_A}$ of $A \otimes A$.

Let us recall some notations introduced earlier in Chapter 1. For R a Noetherian $\mathbb{F}$-algebra, we denoted by $\mathcal{A}_\infty(R)$ the R-algebra

$$\mathcal{A}_\infty(R) = \varprojlim_n (\mathcal{O}_\infty \otimes R)/(\mathfrak{m}_\infty^n \otimes R).$$

This ring was considered to define isocrystals and mixedness in Subsection 1.2. We denoted $\mathcal{B}_\infty(R)$ the R-algebra $K_\infty \otimes_{\mathcal{O}_\infty} \mathcal{A}_\infty(R)$. The formal spectrum $\operatorname{Spf} \mathcal{A}_\infty(R)$ corresponds to the completion of the Noetherian scheme $C \times \operatorname{Spec} R$ at the closed subscheme $\{\infty\} \times R$, that is, $\operatorname{Spf} \mathcal{O}_\infty \hat{\times} \operatorname{Spec} R = \operatorname{Spf} \mathcal{A}_\infty(R)$.

In the context of Betti realizations, Section 1.4, we also considered the algebra $\mathcal{O}_\infty\langle A\rangle$, defined by

$$\mathcal{O}_\infty\langle A\rangle = \varprojlim_n (A \otimes \mathcal{O}_\infty)/(A \otimes \mathfrak{m}_\infty^n)$$

and, given a complete field L in $\mathbb{C}_\infty$ that contains K_∞, we denoted by $L\langle A\rangle$ the algebra $\mathcal{O}_\infty\langle A\rangle \otimes_{\mathcal{O}_\infty} L$. Similarly, $\operatorname{Spf} \mathcal{O}_\infty\langle A\rangle$ is the completion of $\operatorname{Spec}(A \otimes \mathcal{O}_\infty)$ at $\operatorname{Spec} A \times \{\infty\}$.

The closed subscheme $C \times \{\infty\}$ defines an effective divisor on $C \times C$ which we denote ∞_C. Similarly, we let ∞_A be the effective divisor $(\operatorname{Spec} A) \times \{\infty\}$ of $(\operatorname{Spec} A) \times C$.

6.2.1 C-shtuka models

Let $\underline{M}$ be an A-motive over K. Let M_A be the maximal A-model of $\underline{M}$ (Proposition 2.30). We set $N := M + \tau_M(\tau^* M)$ and $N_A := N \cap M_A[\mathfrak{j}^{-1}]$.

Definition 6.6. A *C-shtuka model* for $\underline{M}$ is the datum $(\mathcal{N}, \mathcal{M}, \tau_{\mathcal{M}})$ of

(a) A coherent sheaf $\mathcal{N}$ on $(\operatorname{Spec} A) \times C$ such that $\mathcal{N}(\operatorname{Spec} A \otimes A) = N_A$,

(b) A coherent subsheaf $\mathcal{M}$ of $\mathcal{N}$ such that $\mathcal{M}(\operatorname{Spec} A \otimes A) = M_A$ and for which the cokernel of the inclusion $\iota : \mathcal{M} \to \mathcal{N}$ is supported at Δ,

(c) A morphism $\tau_{\mathcal{M}} : \tau^*\mathcal{M} \to \mathcal{N}(-\infty_A)$ which coincides with $\tau_M : \tau^* M_A \to N_A$ on the affine open subscheme $\operatorname{Spec} A \otimes A$.

In the case of effective A-motives, a reference for the next proposition is [**?**, Prop. 4.5.1].

Proposition 6.7. *A C-shtuka model for $\underline{M}$ exists.*

Proof. Let B be a sub-$\mathbb{F}$-algebra of K such that $(\operatorname{Spec} A)\cup(\operatorname{Spec} B)$ forms an affine open covering of C in the Zariski topology. Let D be the sub-$\mathbb{F}$-algebra of K containing both A and B and such that $\operatorname{Spec} D=(\operatorname{Spec} A)\cap(\operatorname{Spec} B)$[1]. For $S\in\{A,B,D\}$, we let $\mathfrak{j}_S$ be the ideal of $A\otimes S$ given by either $\mathfrak{j}_A:=\mathfrak{j}$, $\mathfrak{j}_D:=\mathfrak{j}(A\otimes D)$ and $\mathfrak{j}_B:=\mathfrak{j}_D\cap(A\otimes B)$. Note that $\mathcal{O}(\Delta)(\operatorname{Spec} A\otimes S)=\mathfrak{j}_S$.

Let M_D be the $A\otimes D$-module $M_A\otimes_A D$, and let M'_B be an $A\otimes B$-lattice in M_D (for instance, if $m_1,...,m_s$ are generators of M_D, consider M'_B to be the $A\otimes B$-submodule spanned by $m_1,...,m_s$).

Since $\tau_M(\tau^*M_A)\subset M_A[\mathfrak{j}^{-1}]$, we have $\tau_M(\tau^*M_D)\subset M_D[\mathfrak{j}_D^{-1}]$. However, it might not be true that $\tau_M(\tau^*M'_B)\subset M'_B[\mathfrak{j}_B^{-1}]$. Yet, there exists $d\in B$ invertible in D such that

$$\tau_M(\tau^*M'_B)\subset d^{-1}M'_B[\mathfrak{j}_B^{-1}].$$

Let $r\in B$ invertible in D which vanishes[2] at ∞ and let $M_B:=(rd)M'_B$. We now have

$$\tau_M(\tau^*M_B)\subset rM_B[\mathfrak{j}_B^{-1}].$$

Since r is invertible in D, the multiplication maps furnish *glueing* isomorphisms

$$M_A\otimes_A D\xrightarrow{=}M_D\xleftarrow{\sim}M_B\otimes_B D. \tag{6.2}$$

For $S\in\{A,B,D\}$, we set $N_S:=(M+\tau_M(\tau^*M))\cap M_S[\mathfrak{j}_S^{-1}]$. N_S is an $A\otimes S$-module of finite type which contains M_S. By flatness of D over A (resp. B), the multiplication maps also are isomorphisms:

$$N_A\otimes_A D\xrightarrow{\sim}N_D\xleftarrow{\sim}N_B\otimes_B D. \tag{6.3}$$

Let $\mathcal{M}$ (resp. $\mathcal{N}$) be the coherent sheaf on $\operatorname{Spec} A\times C$ resulting from the glueing (6.2) (resp. (6.3)). Since $M_A\subset N_A$ and $M_B\subset N_B$, $\mathcal{M}$ is a subsheaf of $\mathcal{N}$. We have further $M_A[\mathfrak{j}^{-1}]=N_A[\mathfrak{j}^{-1}]$ and $M_B[\mathfrak{j}_B^{-1}]=N_B[\mathfrak{j}_B^{-1}]$ which implies that the cokernel of $\mathcal{M}\subset\mathcal{N}$ is supported at Δ.

Because $\tau_M(\tau^*M_S)\subset N_S$ for all $S\in\{A,B,D\}$, one obtains a unique morphism of $\mathcal{O}_{(\operatorname{Spec} A)\times C}$-modules $\tau_{\mathcal{M}}:\tau^*\mathcal{M}\to\mathcal{N}$. Since $\tau_M(\tau^*M_B)\subset rN_B$ and r vanishes at ∞, we also have $\tau_{\mathcal{M}}(\tau^*\mathcal{M})\subset\mathcal{N}(-\infty_A)$. $\square$

The fact that the image of $\tau_{\mathcal{M}}$ lands in $\mathcal{N}(-\infty_A)$ allows to show the following lemma. It will be used later on in Subsection 6.4.1 to simplify some cohomological computations. Let $(\mathcal{N},\mathcal{M},\tau_{\mathcal{M}})$ be a C-shtuka model for $\underline{M}$.

Lemma 6.8. *Let $i:\operatorname{Spec}\mathcal{O}_\infty\langle A\rangle\to\operatorname{Spec} A\otimes\mathcal{O}_\infty\hookrightarrow(\operatorname{Spec} A)\times C$ be the canonical morphism of A-schemes. The inclusion of sheaves $i^*\mathcal{M}\subset i^*\mathcal{N}$ is an equality and the induced morphism*

$$\iota-\tau_{\mathcal{M}}:i^*\mathcal{M}(\operatorname{Spec}\mathcal{O}_\infty\langle A\rangle)\longrightarrow i^*\mathcal{N}(\operatorname{Spec}\mathcal{O}_\infty\langle A\rangle)$$

is an isomorphism of $\mathcal{O}_\infty\langle A\rangle$-modules.

[1]Let x be a closed point on C distinct from ∞. Then $B:=H^0(C\setminus\{x\},\mathcal{O}_C)$ works. In the latter case, we have $D:=H^0(C\setminus\{\infty,x\},\mathcal{O}_C)$.

[2]Such an r always exists: the divisor $D:=\deg(x)\cdot\infty-\deg(\infty)\cdot x$ has degree zero so that nD is principal for n large enough by the fact that $\mathrm{C}^0(K)$ is finite [Ros, Lem. 5.6]. Chosing r such that $(r)=nD$, then $r\in B$ and r is invertible in D.

Proof. By Lemma 1.47, we have $\jmath\mathcal{O}_\infty\langle A\rangle = \mathcal{O}_\infty\langle A\rangle$. In particular, $i^*\Delta$ is the empty divisor of $\operatorname{Spec}\mathcal{O}_\infty\langle A\rangle$. The equality between $i^*\mathcal{M}$ and $i^*\mathcal{N}$ follows.

Let π_∞ be a uniformizer of $\mathcal{O}_\infty$. We denote by Ξ the $\mathcal{O}_\infty\langle A\rangle$-module $i^*\mathcal{M}(\operatorname{Spec}\mathcal{O}_\infty\langle A\rangle)$. Because $\tau_{\mathcal{M}}(\tau^*\mathcal{M}) \subset \mathcal{N}(-\infty_A)$, we have $\tau_{\mathcal{M}}(\tau^*\Xi) \subset \pi_\infty\Xi$. In particular, for all $\xi \in \Xi$, the series

$$\psi := \sum_{n=0}^{\infty} \tau_M^n(\tau^{n*}\xi)$$

converges in Ξ. The assignation $\xi \mapsto \psi$ defines an inverse of $\mathrm{id} - \tau_{\mathcal{M}}$ on Ξ. □

6.2.2 $C \times C$-shtuka models

We want to extend the construction of Proposition 6.7 from $(\operatorname{Spec} A) \times C$ to $C \times C$.

Definition 6.9. A $C \times C$-shtuka model for $\underline{M}$ is the datum $(\mathcal{N}, \mathcal{M}, \tau_{\mathcal{M}})$ of

(a) a coherent sheaf $\mathcal{N}$ on $C \times C$ such that $\mathcal{N}(\operatorname{Spec} A \otimes A) = N_A$,

(b) a coherent subsheaf $\mathcal{M}$ of $\mathcal{N}$ such that $\mathcal{M}(\operatorname{Spec} A \otimes A) = M_A$ and such that the cokernel of the inclusion $\iota : \mathcal{M} \to \mathcal{N}$ is supported at Δ,

(c) a morphism of sheaves $\tau_{\mathcal{M}} : \tau^*\mathcal{M} \to \mathcal{N}(-\infty_C)$ which coincides with $\tau_M : \tau^* M_A \to N_A$ on $\operatorname{Spec} A \otimes A$.

Remark 6.10. The restriction of a $C \times C$-shtuka model for $\underline{M}$ on $(\operatorname{Spec} A) \times C$ is an C-shtuka model for $\underline{M}$.

We prove:

Theorem 6.11. *If the weights of $\underline{M}$ are non-positive, a $C \times C$-shtuka model for $\underline{M}$ exists.*

Proof. We use the notations and definitions of the proof of Proposition 6.7. That is, B is a sub-$\mathbb{F}$-algebra of K such that $(\operatorname{Spec} A) \cup (\operatorname{Spec} B)$ forms an open affine cover of C, D is the sub-$\mathbb{F}$-algebra of K containing A and B such that $\operatorname{Spec} D = (\operatorname{Spec} A) \cap (\operatorname{Spec} B)$.

Recall that, given an $\mathbb{F}$-algebra R, $\mathcal{A}_\infty(R)$ and $\mathcal{B}_\infty(R)$ are defined respectively as the completion of $\mathcal{O}_\infty \otimes R$ at the ideal $\mathfrak{m}_\infty \otimes R$ and as $K_\infty \otimes_{\mathcal{O}_\infty} \mathcal{A}_\infty(R)$.

Because the weights of $\underline{M}$ are non-positive, there exists by Lemma 1.40 an $\mathcal{A}_\infty(K)$-lattice T in $M \otimes_{A\otimes K} \mathcal{B}_\infty(K)$ stable by τ_M. We define two sub-$\mathcal{A}_\infty(B)$-modules of T:

$$T_B := T \cap (M_B \otimes_{A\otimes B} \mathcal{B}_\infty(B)), \quad U_B := T \cap (N_B \otimes_{A\otimes B} \mathcal{B}_\infty(B)),$$

two sub-$\mathcal{A}_\infty(D)$-modules of T:

$$T_D := T \cap (M_D \otimes_{A\otimes D} \mathcal{B}_\infty(D)), \quad U_D := T \cap (N_D \otimes_{A\otimes D} \mathcal{B}_\infty(D)),$$

and two sub-$\mathcal{A}_\infty(A)$-modules of T:

$$T_A := T \cap (M_A \otimes_{A\otimes A} \mathcal{B}_\infty(A)), \quad U_A := T \cap (N_A \otimes_{A\otimes A} \mathcal{B}_\infty(A)).$$

The above two $\mathcal{A}_\infty(A)$-modules are in fact equal. Indeed, as $\mathfrak{j}\mathcal{B}_\infty(A) = \mathcal{B}_\infty(A)$ and since the inclusion $A \otimes A \to \mathcal{B}_\infty(A)$ is flat, we have

$$\begin{aligned} N_A \otimes_{A\otimes A} \mathcal{B}_\infty(A) &= [(M + \tau_M(\tau^* M)) \cap M_A[\mathfrak{j}^{-1}]] \otimes_{A\otimes A} \mathcal{B}_\infty(A) \\ &= [(M + \tau_M(\tau^* M)) \otimes_{A\otimes A} \mathcal{B}_\infty(A)] \cap [M_A[\mathfrak{j}^{-1}] \otimes_{A\otimes A} \mathcal{B}_\infty(A)] \\ &= [M \otimes_{A\otimes K} \mathcal{B}_\infty(K)] \cap [M_A \otimes_{A\otimes A} \mathcal{B}_\infty(A)] \\ &= M_A \otimes_{A\otimes A} \mathcal{B}_\infty(A). \end{aligned}$$

Our aim is to glue together M_A, M_B, T_A and T_B (resp. N_A, N_B, U_A and U_B) to obtain $\mathcal{M}$ (resp. $\mathcal{N}$) along the covering $\operatorname{Spec} A \otimes A$, $\operatorname{Spec} A \otimes B$, $\operatorname{Spec} \mathcal{A}_\infty(A)$ and $\operatorname{Spec} \mathcal{A}_\infty(B)$ of $C \times C$.

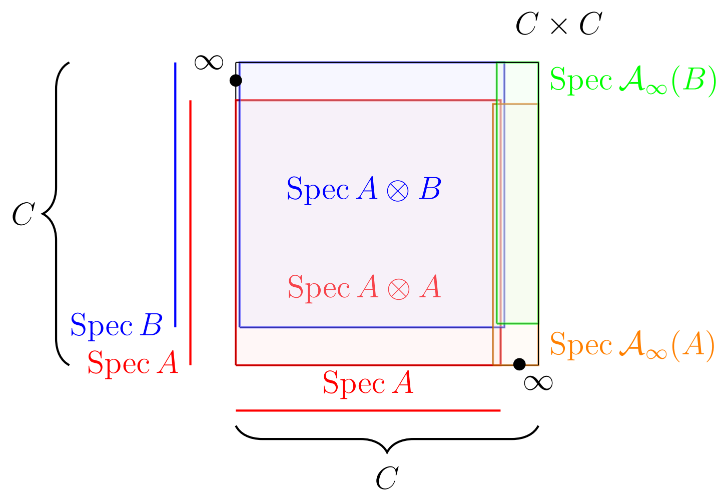

Figure 6.1: The covering $\{\operatorname{Spec} A \otimes A, \operatorname{Spec} A \otimes B, \operatorname{Spec} \mathcal{A}_\infty(A), \operatorname{Spec} \mathcal{A}_\infty(B)\}$ of the $\mathbb{F}$-scheme $C \times C$

This covering is not Zariski, so we will use the Beauville-Lazslo Theorem [BeaLa] to carry out the glueing process. By functoriality, the morphism $\tau_{\mathcal{M}}$ will result as the glueing of

$$\begin{array}{cccc} \tau^* M_A & \tau^* M_B & \tau^* T_A & \tau^* T_B \\ \downarrow & \downarrow & \downarrow & \downarrow \\ N_A & N_B & U_A & U_B \end{array} \tag{6.4}$$

along the corresponding covering. Note that the first two arrows glue together by the proof of Proposition 6.7.

Step 1: the modules T_A, T_B, U_A and U_B are finitely generated. We prove finite generation for T_A (the argument for T_B, U_A and U_B being similar).

The $A \otimes A$-module M_A is finitely generated projective and, as such, is a direct summand in a free $A \otimes A$-module L of finite rank. In particular, the $\mathcal{B}_\infty(K)$-module $L \otimes_{A\otimes A} \mathcal{B}_\infty(K)$ is free of finite rank and contains $M \otimes_{A\otimes K} \mathcal{B}_\infty(K)$ as a submodule. Let $\mathbf{n} = (n_1, ..., n_s)$ be a basis of L. For any element m in $L \otimes_{A\otimes A} \mathcal{B}_\infty(K)$, we denote by $v_\infty(m)$ the minimum of the valuations of the coefficients of m in $\mathbf{n}$. Because T is finitely generated over $\mathcal{A}_\infty(K)$, there exists a positive integer v_T such that $v_\infty(t) \geq -v_T$ for all $t \in T$.

Let $N \subset L \otimes_{A\otimes A} \mathcal{B}_\infty(K)$ be the finite free $\mathcal{A}_\infty(A)$-module generated by $\mathbf{n}$. We have by definition $v_\infty(n) \geq 0$ for any $n \in N$. We have

$$T_A \subset M_A \otimes_{A\otimes A} \mathcal{B}_\infty(A) \subset N \otimes_{\mathcal{A}_\infty(A)} \mathcal{B}_\infty(A) = \bigcup_{n=0}^{\infty} \pi_\infty^{-n} N.$$

For $x \in T_A \setminus \{0\}$, let n be a non-negative integer such that $x = \pi_\infty^{-n} m$ for some $m \in N \setminus \pi_\infty N$. Comparing valuations yields

$$n = v_\infty(m) - v_\infty(x) \leq v_\infty(m) + v_T.$$

The number $v_\infty(m)$ cannot be positive, otherwise we would have $m \in \pi_\infty(N \otimes_{\mathcal{A}_\infty(A)} \mathcal{A}_\infty(K))$ which contradicts $m \notin \pi_\infty N$ because of the equality

$$\pi_\infty N = N \cap \pi_\infty(N \otimes_{\mathcal{A}_\infty(A)} \mathcal{A}_\infty(K)).$$

Therefore, $n \leq v_T$, and we deduce that

$$T_A \subset \bigcup_{n=0}^{v_T} \pi_\infty^{-n} N.$$

Because $\mathcal{A}_\infty(A)$ is Noetherian, it follows that T_A is finitely generated.

Step 2: $T_A \otimes_A D$ and $T_B \otimes_B D$ (resp. $U_A \otimes_A D$ and $U_B \otimes_B D$) are dense in T_D (resp. U_D) for the $\mathfrak{m}_\infty$-adic topology. We only prove the density of $T_A \otimes_A D$ in T_D since the argument for the others follows the same lines.

Let $t \in T_D = T \cap (M_D \otimes_{A\otimes D} \mathcal{B}_\infty(D))$. Let $(m_1, ..., m_s)$ be generators of M_A as an $A \otimes A$-module. t can be written as a sum $\sum_{i=1}^r m_i \otimes b_i$ with coefficients $b_i \in \mathcal{B}_\infty(D)$. For $i \in \{1, ..., r\}$, let $(b_{i,n})_{n\in\mathbb{Z}}$ be a sequence in $\mathcal{B}_\infty(A) \otimes_A D$, such that $b_{i,n} = 0$ for $n \ll 0$, satisfying $b_i - b_{i,n} \in \mathfrak{m}_\infty^n \mathcal{A}_\infty(D)$ for all $n \in \mathbb{Z}$. In particular, $(b_{i,n})_{n\in\mathbb{Z}}$ converges to b_i when n tens to infinity. For $n \in \mathbb{Z}$, we define

$$t_n := \sum_{i=1}^{s} m_i \otimes b_{i,n} \in (M_{\mathcal{O}} \otimes_{A\otimes A} \mathcal{B}_\infty(A)) \otimes_A D.$$

Then $t - t_n$ belongs to $\mathfrak{m}_\infty^n N$ where N is the $\mathcal{A}_\infty(D)$-module generated by $(m_1, ..., m_s)$. For n large enough, $\mathfrak{m}_\infty^n N \subset T$, hence $t - t_n \in T$ and $t_n \in T$. We deduce that $t_n \in T_A \otimes_A D$ for large value of n and that $(t_n)_{n\in\mathbb{Z}}$ converges to t when n goes to infinity. We conclude that $T_A \otimes_A D$ is dense in T_D.

Steps $1\&2 \implies$ **compatibility.** Because T_A and T_B are finitely generated over $\mathcal{A}_\infty(A)$ and $\mathcal{A}_\infty(B)$ respectively, $T_A \otimes_{\mathcal{A}_\infty(A)} \mathcal{A}_\infty(D)$ coincides with the completion of $T_A \otimes_A D$ and $T_B \otimes_{\mathcal{A}_\infty(B)} \mathcal{A}_\infty(D)$ with the completion of $T_B \otimes_B D$ (by [Bou, (AC)§.3 Thm. 3.4.3]). Therefore, the multiplication maps are isomorphisms:

$$T_A \otimes_{\mathcal{A}_\infty(A)} \mathcal{A}_\infty(D) \xrightarrow{\sim} T_D \xleftarrow{\sim} T_B \otimes_{\mathcal{A}_\infty(B)} \mathcal{A}_\infty(D),$$

$$U_A \otimes_{\mathcal{A}_\infty(A)} \mathcal{A}_\infty(D) \xrightarrow{\sim} U_D \xleftarrow{\sim} U_B \otimes_{\mathcal{A}_\infty(B)} \mathcal{A}_\infty(D).$$

Step 3**: the glueing.** We consider the morphisms of formal schemes over $\operatorname{Spf} \mathcal{O}_\infty$

$$\operatorname{Spf} \mathcal{A}_\infty(A) = \operatorname{Spf} \mathcal{O}_\infty \hat{\otimes} A \xrightarrow{\hat{i}} \operatorname{Spf} \mathcal{O}_\infty \hat{\times} C \xleftarrow{\hat{j}} \operatorname{Spf} \mathcal{O}_\infty \hat{\otimes} B = \operatorname{Spf} \mathcal{A}_\infty(B).$$

By the Beauville-Laszlo Theorem [BeaLa], there exists a unique pair of coherent sheaves $(\mathcal{M}, \mathcal{N})$ of $\mathcal{O}_{C\times C}$-modules such that

$$\begin{array}{ll} \mathcal{M}(\operatorname{Spec} A \otimes A) = M_A & \mathcal{N}(\operatorname{Spec} A \otimes A) = N_A \\ \mathcal{M}(\operatorname{Spec} A \otimes B) = M_B & \mathcal{N}(\operatorname{Spec} A \otimes B) = N_B \\ \hat{i}^*\mathcal{M}(\operatorname{Spf} \mathcal{O}_\infty \hat{\otimes} A) = T_A & \hat{i}^*\mathcal{N}(\operatorname{Spf} \mathcal{O}_\infty \hat{\otimes} A) = U_A \\ \hat{j}^*\mathcal{M}(\operatorname{Spf} \mathcal{O}_\infty \hat{\otimes} B) = T_B & \hat{j}^*\mathcal{N}(\operatorname{Spf} \mathcal{O}_\infty \hat{\otimes} B) = U_B \end{array}$$

Since, for each line of the above table, the left-hand side is canonically a submodule of the right-hand side, we have $\mathcal{M} \subset \mathcal{N}$. Because these inclusions become equalities away from Δ, we deduce that the cokernel of the inclusion $\mathcal{M} \subset \mathcal{N}$ is supported at Δ. The glueing of (6.4) then defines a morphism $\tau_{\mathcal{M}} : \tau^*\mathcal{M} \to \mathcal{N}$. Finally, we recall that there exists $r \in B$ invertible in D and vanishing at ∞ such that $\tau_M(\tau^* M_B) \subset rN_B$ and thus $\tau_M(\tau^* T_B) \subset rU_B$. Hence, the image of $\tau_{\mathcal{M}}$ lands in $\mathcal{N}(-\infty_C)$. □

As a matter of fact, the converse of Theorem 6.11 holds (the next lemma is only informative and will not be used later in the text).

Lemma 6.12. *If $\underline{M}$ admits a $C \times C$-shtuka model, then all the weights of $\underline{M}$ are non-positive.*

Proof. Recall that we denoted $\mathcal{A}_\infty(K)$ the completion of $\mathcal{O}_\infty \otimes K$ with respect to the ideal $\mathfrak{m}_\infty \otimes K$. Let $i : \operatorname{Spec} \mathcal{A}_\infty(K) \to C \times C$ be the canonical morphism of schemes. Because $\mathfrak{j}\mathcal{A}_\infty(K) = \mathcal{A}_\infty(K)$, $i^*\Delta$ is the empty divisor of $\operatorname{Spec} \mathcal{A}_\infty(K)$. If $(\mathcal{N}, \mathcal{M}, \tau_{\mathcal{M}})$ is a $C \times C$-shtuka model for $\underline{M}$, then $i^*\mathcal{M} = i^*\mathcal{N}$. The global sections module of $i^*\mathcal{M}$ defines an $\mathcal{A}_\infty(K)$-lattice T in $\mathcal{I}_\infty(M) = i^*\mathcal{M}(\operatorname{Spec} \mathcal{A}_\infty(K)) \otimes_{\mathcal{O}_\infty} K_\infty$ which satisfies $\langle \tau_M T \rangle \subset T$. We conclude by Lemma 1.41 that the weights of $\underline{M}$ are non-positive. □

We assume that the weights of $\underline{M}$ are negative. By Theorem 6.11, let $\underline{\mathcal{M}}$ be a global model for $\underline{M}$. Let $i : \operatorname{Spec} \mathcal{A}_\infty(A) \to C \times C$. We denote:

$$\begin{aligned} L_A &= i^*\mathcal{M}(\operatorname{Spf} \mathcal{A}_\infty(A)) = i^*\mathcal{N}(\operatorname{Spf} \mathcal{A}_\infty(A)) \\ L &= L_A \otimes_{\mathcal{A}_\infty(A)} \mathcal{A}_\infty(K) \end{aligned}$$

(indeed, Δ is not supported at $\operatorname{Spf} \mathcal{A}_\infty(A)$). τ_M induces an $\mathcal{O}_\infty$-linear endormorphism of L (resp. L_A). We record the following counterpart of Lemma 6.8:

Lemma 6.13. *The morphism* $\mathrm{id} - \tau_{\mathcal{M}}$ *induces an* $\mathcal{O}_\infty$*-linear automorphism of* L *and* L_A.

Proof. The statement for L_A implies the one for L. Because the weights of $\underline{M}$ are negative, Lemma 1.40 implies that there exists an $\mathcal{A}_\infty(K)$-lattice T in $M \otimes_{A\otimes K} \mathcal{B}_\infty(K)$ and let h and d be positive integers such that $\langle \tau_M^h L \rangle = \mathfrak{m}_\infty^d L$. There exists a positive integer $k \geq 0$ such that $L_A \subset \mathfrak{m}_\infty^k T$.

To show that $\mathrm{id} - \tau_{\mathcal{M}}$ is injective on L_A, let x be an element of $\ker(\mathrm{id} - \tau_{\mathcal{M}}|L_A)$. Without loss, we assume $x \in T$. For all positive integer n,

$$x = \tau_M^{nh}(\tau^{nh*}x) \in \mathfrak{m}_\infty^{nd} T.$$

Because $d > 0$, $x = 0$.
We turn to surjectivity. Let T' be the $\mathcal{A}_\infty(K)$-lattice generated by T, $\langle \tau_M T \rangle$, ..., $\langle \tau_M^{h-1} T \rangle$. Then T' is stable by τ_M. Let $x \in L_A$ and let $k \geq 0$ be such that $\pi_\infty^k x \in T$. For all $n \geq 0$, we have

$$\tau_M^{nh}(\tau^{nh*}x) \in \mathfrak{m}_\infty^{nd-k} T'$$

and, in particular, for all $q \in \{0, 1, ..., h-1\}$,

$$\tau_M^{nh+q}(\tau^{(nh+q)*}x) \in \mathfrak{m}_\infty^{nd-k} T'.$$

Therefore, the series

$$\sum_{t=0}^{\infty} \tau_M^t(\tau^{t*}x) = \sum_{n=0}^{\infty} \left(\sum_{q=0}^{h-1} \tau_M^{nh+q}(\tau^{(nh+q)*}x) \right)$$

converges to a solution f in L_A of $f - \tau_M(\tau^* f) = x$. □

6.2.3 Shtuka models and extensions of MHPS

Let $\underline{M}$ be a mixed and rigid analytically trivial A-motive over K with negative weights. Let $(\mathcal{N}, \mathcal{M}, \tau_{\mathcal{M}})$ be any $C \times C$-shtuka model for $\underline{M}$, whose existence is ensured by Theorem 6.11. Let $\iota : \mathcal{M} \to \mathcal{N}$ be the inclusion of sheaves. We consider the inclusion of ringed spaces

$$\operatorname{Spf} \mathcal{A}_\infty(\mathcal{O}_\infty) = \operatorname{Spf} \mathcal{O}_\infty \hat{\otimes} \mathcal{O}_\infty \longrightarrow C \times C \tag{6.5}$$

and denote respectively $\hat{\mathcal{N}}$ and $\hat{\mathcal{M}}$ the pullback of $\mathcal{N}$ and $\mathcal{M}$ through (6.5). Finally, denote by $\hat{\mathcal{N}}_\infty$ and $\hat{\mathcal{M}}_\infty$ the finitely generated $\mathcal{A}_\infty(\mathcal{O}_\infty)$-modules:

$$\hat{\mathcal{N}}_\infty := \hat{\mathcal{N}}(\operatorname{Spf} \mathcal{O}_\infty \hat{\otimes} \mathcal{O}_\infty), \quad \hat{\mathcal{M}}_\infty := \hat{\mathcal{M}}(\operatorname{Spf} \mathcal{O}_\infty \hat{\otimes} \mathcal{O}_\infty).$$

The aim of this subsection is to prove that there is an exact sequence of K_∞-vector spaces (Corollary 6.22):

$$\Lambda(\underline{M})^+ \otimes_A K_\infty \hookrightarrow \frac{\hat{\mathcal{N}}_\infty}{(\iota - \tau_{\mathcal{M}})(\hat{\mathcal{M}}_\infty)} \otimes_{\mathcal{O}_\infty} K_\infty \twoheadrightarrow \operatorname{Ext}^{1,\mathrm{ha},\infty}_{\mathcal{MHP}^+_{K_\infty}}(\mathbb{1}^+, \mathscr{H}^+_{K_\infty}(\underline{M})).$$

We start by a proposition.

Proposition 6.14. *There is an isomorphism of K_∞-vector spaces*

$$\frac{\hat{\mathcal{N}}_\infty}{(\iota - \tau_{\mathcal{M}})(\hat{\mathcal{M}}_\infty)} \otimes_{\mathcal{O}_\infty} K_\infty \xrightarrow{\sim} (\hat{\mathcal{N}}_\infty / \hat{\mathcal{M}}_\infty) \otimes_{\mathcal{O}_\infty} K_\infty.$$

We split the proof of Proposition 6.14 into several lemmas.

Lemma 6.15. *There exists an injective $\mathcal{A}_\infty(\mathcal{O}_\infty)$-linear morphism $\iota' : \hat{\mathcal{N}}_\infty \to \hat{\mathcal{M}}_\infty$ and a positive integer e such that $\iota'\iota$ and $\iota\iota'$ coincide with the multiplication by $(\pi_\infty \otimes 1 - 1 \otimes \pi_\infty)^e$ on $\hat{\mathcal{M}}_\infty$ and $\hat{\mathcal{N}}_\infty$ respectively.*

Proof. Let $\mathfrak{d} := \mathcal{O}(\Delta)(\operatorname{Spf} \mathcal{O}_\infty \hat{\otimes} \mathcal{O}_\infty)$ as an ideal of $\mathcal{A}_\infty(\mathcal{O}_\infty)$. The cokernel of the inclusion $\iota : \hat{\mathcal{M}}_\infty \to \hat{\mathcal{N}}_\infty$ is $\mathfrak{d}$-torsion. It is also finitely generated, and since $\pi_\infty \otimes 1 - 1 \otimes \pi_\infty \in \mathfrak{d}$, there exists $e \geq 0$ such that $(\pi_\infty \otimes 1 - 1 \otimes \pi_\infty)^e v \in \hat{\mathcal{M}}_\infty$ for all $v \in \hat{\mathcal{N}}_\infty$. We let $\iota' : \hat{\mathcal{N}}_\infty \to \hat{\mathcal{M}}_\infty$ be the multiplication by $(\pi_\infty \otimes 1 - 1 \otimes \pi_\infty)^e$ and the lemma follows. $\square$

Lemma 6.16. *Let t be a positive integer. Then, $\iota - \tau_{\mathcal{M}}$ and ι respectively induce isomorphisms of K_∞-vector spaces:*

$$\left(\frac{\hat{\mathcal{M}}_\infty}{(1 \otimes \pi_\infty)^t \hat{\mathcal{M}}_\infty}\right) \otimes_{\mathcal{O}_\infty} K_\infty \xrightarrow{\iota - \tau_{\mathcal{M}}} \left(\frac{\hat{\mathcal{N}}_\infty}{(1 \otimes \pi_\infty)^t \hat{\mathcal{N}}_\infty}\right) \otimes_{\mathcal{O}_\infty} K_\infty,$$

$$\left(\frac{\hat{\mathcal{M}}_\infty}{(1 \otimes \pi_\infty)^t \hat{\mathcal{M}}_\infty}\right) \otimes_{\mathcal{O}_\infty} K_\infty \xrightarrow{\iota} \left(\frac{\hat{\mathcal{N}}_\infty}{(1 \otimes \pi_\infty)^t \hat{\mathcal{N}}_\infty}\right) \otimes_{\mathcal{O}_\infty} K_\infty.$$

Proof. Let ι' and $e \geq 0$ be as in Lemma 6.15. The multiplication by

$$\left(\sum_{k=0}^{t-1} \pi_\infty^{-(k+1)} \otimes \pi_\infty^k\right)^e$$

on $(\hat{\mathcal{M}}_\infty / (1 \otimes \pi_\infty)^t \hat{\mathcal{M}}_\infty) \otimes_{\mathcal{O}_\infty} K_\infty$ defines an inverse of $\iota'\iota$. The same argument shows that $\iota\iota'$ is an automorphism of $(\hat{\mathcal{N}}_\infty / (1 \otimes \pi_\infty)^t \hat{\mathcal{N}}_\infty) \otimes_{\mathcal{O}_\infty} K_\infty$.

On the other-hand, we have $(\iota'\tau_{\mathcal{M}})^k(\tau^{*k}\hat{\mathcal{M}}_\infty) \subset (1\otimes\pi_\infty)\hat{\mathcal{M}}_\infty$ for k large enough. Hence, $(\iota'\tau_{\mathcal{M}})$ is nilpotent on $\hat{\mathcal{M}}_\infty/(1\otimes\pi_\infty)^t\hat{\mathcal{M}}_\infty$, and so is $(\iota'\iota)^{-1}(\iota'\tau_{\mathcal{M}})$. In particular,

$$\iota'(\iota-\tau_{\mathcal{M}}) = (\iota'\iota)(\mathrm{id}-(\iota'\iota)^{-1}(\iota'\tau_{\mathcal{M}}))$$

is an isomorphism. It follows that $\iota-\tau_{\mathcal{M}}$ is injective and ι' surjective. Since $\iota\iota'$ is invertible, ι' is injective. We deduce that $\iota-\tau_{\mathcal{M}}$, ι' and thus ι are isomorphisms. □

Lemma 6.17. *Let t be a non-negative integer. Then, the canonical maps*

$$\frac{(1\otimes\pi_\infty)^t\hat{\mathcal{N}}_\infty}{(\iota-\tau_{\mathcal{M}})((1\otimes\pi_\infty)^t\hat{\mathcal{M}}_\infty)}\otimes_{\mathcal{O}_\infty} K_\infty \longrightarrow \frac{\hat{\mathcal{N}}_\infty}{(\iota-\tau_{\mathcal{M}})(\hat{\mathcal{M}}_\infty)}\otimes_{\mathcal{O}_\infty} K_\infty, \tag{6.6}$$

$$\frac{(1\otimes\pi_\infty)^t\hat{\mathcal{N}}_\infty}{\iota((1\otimes\pi_\infty)^t\hat{\mathcal{M}}_\infty)}\otimes_{\mathcal{O}_\infty} K_\infty \longrightarrow \frac{\hat{\mathcal{N}}_\infty}{\iota(\hat{\mathcal{M}}_\infty)}\otimes_{\mathcal{O}_\infty} K_\infty, \tag{6.7}$$

are isomorphisms of K_∞-vector spaces.

Proof. In the category of $\mathcal{O}_\infty$-vector spaces, we have a diagram exact on lines and commutative on squares:

$$\begin{tikzcd} 0 \arrow[r] & (1\otimes\pi_\infty)^t\hat{\mathcal{M}}_\infty \arrow[r]\arrow[d, "\iota-\tau_{\mathcal{M}}"] & \hat{\mathcal{M}}_\infty \arrow[r]\arrow[d, "\iota-\tau_{\mathcal{M}}"] & \hat{\mathcal{M}}_\infty/(1\otimes\pi_\infty)^t\hat{\mathcal{M}}_\infty \arrow[r]\arrow[d, "\iota-\tau_{\mathcal{M}}"] & 0 \\ 0 \arrow[r] & (1\otimes\pi_\infty)^t\hat{\mathcal{N}}_\infty \arrow[r] & \hat{\mathcal{N}}_\infty \arrow[r] & \hat{\mathcal{N}}_\infty/(1\otimes\pi_\infty)^t\hat{\mathcal{N}}_\infty \arrow[r] & 0 \end{tikzcd} \tag{6.8}$$

By Lemma 6.16, the third vertical arrow once tensored with K_∞ over $\mathcal{O}_\infty$ is an isomorphism. The first isomorphism then follows from the Snake Lemma. The second one follows from the very same argument, with ι in place of $\iota-\tau_{\mathcal{M}}$. □

Lemma 6.18. *For t large, we have $(\iota-\tau_{\mathcal{M}})((1\otimes\pi_\infty)^t\hat{\mathcal{M}}_\infty) = \iota((1\otimes\pi_\infty)^t\hat{\mathcal{M}}_\infty)$.*

Proof. Let ι' and $e\geq 0$ be as in Lemma 6.15. We chose t such that $(q-1)t>e$. For $s\geq t$, let $\hat{\mathcal{M}}_s := (1\otimes\pi_\infty)^s\hat{\mathcal{M}}_\infty$. $(\hat{\mathcal{M}}_s)_{s\geq t}$ forms a decreasing family of $\mathcal{A}_\infty(\mathcal{O}_\infty)$-modules for the inclusion. It suffices to show that

$$\iota'(\iota-\tau_{\mathcal{M}})(\hat{\mathcal{M}}_t) = (\iota'\iota)(\hat{\mathcal{M}}_t). \tag{6.9}$$

By our assumption on t, we have $(\iota'\iota)^{-1}\tau_{\mathcal{M}}(\hat{\mathcal{M}}_s)\subset\hat{\mathcal{M}}_{s+1}$ for all $s\geq t$. Hence, the endomorphism $\mathrm{id}-(\iota'\iota)^{-1}\tau_{\mathcal{M}}$ of $\hat{\mathcal{M}}_t$ becomes an automorphism over the completion of $\hat{\mathcal{M}}_t$ with respect to the $(1\otimes\pi_\infty)$-adic topology (equivalently, the topology which makes $(\hat{\mathcal{M}}_s)_{s\geq n}$ a neighbourhood of 0 for all $n\geq t$). To conclude, it suffices to show that $\hat{\mathcal{M}}_t$ is already complete for this topology. Because $\hat{\mathcal{M}}_t$ is Noetherian, we have

$$\widehat{(\hat{\mathcal{M}}_t)}_{(1\otimes\pi_\infty)} \cong \hat{\mathcal{M}}_t\otimes_{\mathcal{A}_\infty(\mathcal{O}_\infty)}\widehat{\mathcal{A}_\infty(\mathcal{O}_S)}_{(1\otimes\pi_\infty)},$$

and it suffices to show that $\mathcal{A}_\infty(\mathcal{O}_\infty)$ is complete for the $(1\otimes\pi_\infty)$-adic topology. We have the identifications

$$\mathcal{A}_\infty(\mathcal{O}_\infty) = (\mathbb{F}_\infty \otimes \mathcal{O}_\infty)[\![\pi_\infty \otimes 1]\!] = (\mathbb{F}_\infty \otimes \mathbb{F}_\infty)[\![1 \otimes \pi_\infty, \pi_\infty \otimes 1]\!]$$

which allows us to conclude that $\mathcal{A}_\infty(\mathcal{O}_\infty)$ is complete for the $(1 \otimes \pi_\infty)$-adic topology. □

Proof of Proposition 6.14. The desired isomorphism results of the composition

$$\begin{tikzcd}
\dfrac{\hat{\mathcal{N}}_\infty}{(\iota-\tau_{\mathcal{M}})(\hat{\mathcal{M}}_\infty)} \otimes_{\mathcal{O}_\infty} K_\infty \arrow[r, "(6.6)", "\sim"'] \arrow[d, dashed] & \dfrac{(1\otimes\pi_\infty)^t\hat{\mathcal{N}}_\infty}{(\iota-\tau_{\mathcal{M}})((1\otimes\pi_\infty)^t\hat{\mathcal{M}}_\infty)} \otimes_{\mathcal{O}_\infty} K_\infty \arrow[d, "\wr"', "\text{Lemma 6.18}"] \\
\dfrac{\hat{\mathcal{N}}_\infty}{\iota(\hat{\mathcal{M}}_\infty)} \otimes_{\mathcal{O}_\infty} K_\infty & \dfrac{(1\otimes\pi_\infty)^t\hat{\mathcal{N}}_\infty}{\iota((1\otimes\pi_\infty)^t\hat{\mathcal{M}}_\infty)} \otimes_{\mathcal{O}_\infty} K_\infty \arrow[l, "(6.7)", "\sim"']
\end{tikzcd}$$

For $v \in \hat{\mathcal{N}}_\infty \otimes_{\mathcal{O}_\infty} K_\infty$, the dashed morphism maps

$$v + (\iota - \tau_{\mathcal{M}})(\hat{\mathcal{M}}_\infty) \otimes_{\mathcal{O}_\infty} K_\infty \longmapsto v' + \iota(\hat{\mathcal{M}}_\infty) \otimes_{\mathcal{O}_\infty} K_\infty,$$

where v' is any element of $\hat{\mathcal{N}}_\infty \otimes_{\mathcal{O}_\infty} K_\infty$ satisfying

$$v' - v \in \iota(\hat{\mathcal{M}}_\infty) \otimes_{\mathcal{O}_\infty} K_\infty + (\iota - \tau_{\mathcal{M}})(\hat{\mathcal{M}}_\infty) \otimes_{\mathcal{O}_\infty} K_\infty.$$

□

Recall that the $A \otimes K$-module $M + \tau_M(\tau^* M)$ was denoted N. We are almost in position to prove the main result of this section.

Theorem 6.19. *Let $(\mathcal{M}, \mathcal{N}, \tau_{\mathcal{M}})$ be a $C \times C$-shtuka model for $\underline{M}$. Then, there is an isomorphism of K_∞-vector spaces*

$$\frac{\hat{\mathcal{N}}(\operatorname{Spf} \mathcal{O}_\infty \hat{\otimes} \mathcal{O}_\infty)}{(\iota - \tau_{\mathcal{M}})\hat{\mathcal{M}}(\operatorname{Spf} \mathcal{O}_\infty \hat{\otimes} \mathcal{O}_\infty)} \otimes_{\mathcal{O}_\infty} K_\infty \xrightarrow{\sim} \frac{N \otimes_{A\otimes K} K_\infty[\![\mathfrak{j}]\!]}{M \otimes_{A\otimes K} K_\infty[\![\mathfrak{j}]\!]}$$

where the K_∞-vector space structure on the right-hand side is given through $\nu : K_\infty \to K_\infty[\![\mathfrak{j}]\!]$ of Lemma 4.16.

We begin by two preliminary lemmas concerning the ring $\mathcal{B}_\infty(\mathcal{O}_\infty)$.

Lemma 6.20. *Let $\mathfrak{d} \subset \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ be the ideal generated by elements of the form $a \otimes 1 - 1 \otimes a$ for $a \in \mathcal{O}_\infty$. The canonical morphism*

$$\frac{K_\infty \otimes \mathcal{O}_\infty}{\mathfrak{d}^m K_\infty \otimes \mathcal{O}_\infty} \longrightarrow \frac{\mathcal{B}_\infty(\mathcal{O}_\infty)}{\mathfrak{d}^m \mathcal{B}_\infty(\mathcal{O}_\infty)}$$

is an isomorphism for all $m \geq 1$.

Proof. The sequence of $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$-modules $0 \to \mathfrak{d} \to \mathcal{O}_\infty \otimes \mathcal{O}_\infty \to \mathcal{O}_\infty \to 0$ is exact, and extending the coefficients from $\mathcal{O}_\infty$ to K_∞ reads

$$0 \longrightarrow (K_\infty \otimes \mathcal{O}_\infty) \otimes_{\mathcal{O}_\infty \otimes \mathcal{O}_\infty} \mathfrak{d} \longrightarrow K_\infty \otimes \mathcal{O}_\infty \longrightarrow K_\infty \longrightarrow 0. \tag{6.10}$$

The morphisms appearing in (6.10) are continuous with respect to the $\pi_\infty \otimes 1$-adic topology on $K_\infty \otimes \mathcal{O}_\infty$ and the topology on K_∞. Taking the completions yields

$$0 \longrightarrow \mathfrak{d}\mathcal{B}_\infty(\mathcal{O}_\infty) \longrightarrow \mathcal{B}_\infty(\mathcal{O}_\infty) \longrightarrow K_\infty \longrightarrow 0$$

and the case $m = 1$ follows. Before treating the general m-case, note that $\mathfrak{d}/\mathfrak{d}^2$ is the $\mathcal{O}_\infty$-module $\Omega^1_{\mathcal{O}_\infty/\mathbb{F}}$ of Kähler differentials. In particular, $\mathfrak{d}/\mathfrak{d}^2$ is a free $\mathcal{O}_\infty$-module of rank 1. We deduce that for any $r \in \mathfrak{d} \setminus \mathfrak{d}^2$, the multiplication by r induces an isomorphism of K_∞-vector spaces of dimension 1

$$K_\infty \otimes_{\mathcal{O}_\infty} (\mathcal{O}_\infty \otimes \mathcal{O}_\infty/\mathfrak{d}) \xrightarrow{\sim} K_\infty \otimes_{\mathcal{O}_\infty} (\mathfrak{d}/\mathfrak{d}^2).$$

It follows that $\mathfrak{d}(K_\infty \otimes \mathcal{O}_\infty) = \mathfrak{d}^2(K_\infty \otimes \mathcal{O}_\infty) + r(K_\infty \otimes \mathcal{O}_\infty)$ and hence $\mathfrak{d}^{m-1}(K_\infty \otimes \mathcal{O}_\infty) = \mathfrak{d}^m(K_\infty \otimes \mathcal{O}_\infty) + r\mathfrak{d}^{m-1}(K_\infty \otimes \mathcal{O}_\infty)$ for all $m \geq 1$. From Nakayama's Lemma, $\mathfrak{d}^{m-1} \neq \mathfrak{d}^m$ and we deduce from the sequence of isomorphisms

$$K_\infty \otimes_{\mathcal{O}_\infty} (\mathfrak{d}/\mathfrak{d}^2) \overset{\times r}{\to} K_\infty \otimes_{\mathcal{O}_\infty} (\mathfrak{d}^2/\mathfrak{d}^3) \overset{\times r}{\to} \cdots \overset{\times r}{\to} K_\infty \otimes_{\mathcal{O}_\infty} (\mathfrak{d}^{m-1}/\mathfrak{d}^m)$$

that $K_\infty \otimes_{\mathcal{O}_\infty} (\mathfrak{d}^{m-1}/\mathfrak{d}^m)$ has dimension 1 over K_∞. It follows that there is an exact sequence

$$0 \longrightarrow \mathfrak{d}^m K_\infty \otimes \mathcal{O}_\infty \longrightarrow \mathfrak{d}^{m-1} K_\infty \otimes \mathcal{O}_\infty \longrightarrow K_\infty \longrightarrow 0.$$

Similarly, taking completions yields

$$0 \longrightarrow \mathfrak{d}^m \mathcal{B}_\infty(\mathcal{O}_\infty) \longrightarrow \mathfrak{d}^{m-1}\mathcal{B}_\infty(\mathcal{O}_\infty) \longrightarrow K_\infty \longrightarrow 0.$$

Hence, for all $m \geq 1$, the canonical map

$$\mathfrak{d}^{m-1} K_\infty \otimes \mathcal{O}_\infty/\mathfrak{d}^m K_\infty \otimes \mathcal{O}_\infty \xrightarrow{\sim} \mathfrak{d}^{m-1}\mathcal{B}_\infty(\mathcal{O}_\infty)/\mathfrak{d}^m \mathcal{B}_\infty(\mathcal{O}_\infty). \tag{6.11}$$

is an isomorphism.

Back to the proof of the lemma, where we so far only proved the case $m = 1$. The general m-case follows by induction using the Snake Lemma on the diagram

$$\begin{tikzcd}
\mathfrak{d}^{m-1} K_\infty \otimes \mathcal{O}_\infty/\mathfrak{d}^m \arrow[r, hook] \arrow[d, "(6.11)"', "\wr"] & K_\infty \otimes \mathcal{O}_\infty/\mathfrak{d}^m \arrow[r, two heads] \arrow[d] & K_\infty \otimes \mathcal{O}_\infty/\mathfrak{d}^{m-1} \arrow[d, "\wr\ \text{hypothesis}"] \\
\mathfrak{d}^{m-1}\mathcal{B}_\infty(\mathcal{O}_\infty)/\mathfrak{d}^m\mathcal{B}_\infty(\mathcal{O}_\infty) \arrow[r, hook] & \mathcal{B}_\infty(\mathcal{O}_\infty)/\mathfrak{d}^m\mathcal{B}_\infty(\mathcal{O}_\infty) \arrow[r, two heads] & \mathcal{B}_\infty(\mathcal{O}_\infty)/\mathfrak{d}^{m-1}\mathcal{B}_\infty(\mathcal{O}_\infty)
\end{tikzcd}$$

where our induction hypothesis implies that the middle vertical map is an isomorphism. $\square$

Lemma 6.21. *Let P (resp. Q) be a finitely generated module over $K_\infty \otimes \mathcal{O}_\infty$ (resp. over $A \otimes K$) which is $\mathfrak{d}$-power torsion (resp. $\mathfrak{j}$-power torsion), that is, for all $x \in P$ (resp. $\delta \in Q$) there exists $m \geq 0$ such that $\mathfrak{j}^m x = 0$ (resp. $\delta^m x = 0$). Assume further that we are given a $K_\infty \otimes \mathcal{O}_\infty$-linear isomorphism*

$$Q \otimes_{A\otimes K} (K_\infty \otimes K_\infty) \xrightarrow{\sim} P \otimes_{K_\infty \otimes \mathcal{O}_\infty} (K_\infty \otimes K_\infty). \tag{6.12}$$

Then, there is a $K_\infty \otimes \mathcal{O}_\infty$-linear morphism extending (6.12)

$$Q \otimes_{A\otimes K} K_\infty[\![\mathfrak{j}]\!] \xrightarrow{\sim} P \otimes_{K_\infty\otimes\mathcal{O}_\infty} \mathcal{B}_\infty(\mathcal{O}_\infty).$$

Proof. By Lemma 4.16, the inclusion $A \otimes K_\infty \to K_\infty[\![\mathfrak{j}]\!]$ extends canonically to $K_\infty \otimes K_\infty \to K_\infty[\![\mathfrak{j}]\!]$. By Lemma 4.17, we have an exact sequence $0 \to \mathfrak{v} \to K_\infty \otimes K_\infty \to K_\infty[\![\mathfrak{j}]\!]$ where $\mathfrak{v}$ is the ideal of $K_\infty \otimes K_\infty$ generated by $\{f \otimes 1 - 1 \otimes f \mid f \in \mathbb{F}_\infty\}$. Thus, we have an isomorphism

$$\frac{K_\infty \otimes K_\infty}{\mathfrak{v} + \mathfrak{j}^m K_\infty \otimes K_\infty} \xrightarrow{\sim} \frac{K_\infty[\![\mathfrak{j}]\!]}{\mathfrak{j}^m}.$$

Because Q is $\mathfrak{j}$-power torsion and finitely generated, for m large enough we have

$$Q \otimes_{A\otimes K} (K_\infty \otimes K_\infty) = Q \otimes_{A\otimes K} \frac{K_\infty \otimes K_\infty}{\mathfrak{j}^m K_\infty \otimes K_\infty}.$$

From (6.12), there exists $n \geq 0$ such that for all $x \in Q \otimes_{A\otimes K} (K_\infty \otimes K_\infty)$, $\mathfrak{d}^n \cdot x = 0$. Because $\mathfrak{v} \subset \mathfrak{d}$, we thus have $\mathfrak{v} \cdot x = \mathfrak{v}^{q^{nd_\infty}} \cdot x = 0$. Hence, we can refine the above to:

$$Q \otimes_{A\otimes K} (K_\infty \otimes K_\infty) = Q \otimes_{A\otimes K} \frac{K_\infty \otimes K_\infty}{\mathfrak{v} + \mathfrak{j}^m K_\infty \otimes K_\infty} \cong Q \otimes_{A\otimes K} K_\infty[\![\mathfrak{j}]\!].$$

On the other-hand, that the map

$$(K_\infty \otimes \mathcal{O}_\infty)/\mathfrak{d}^m \longrightarrow \mathcal{B}_\infty(\mathcal{O}_\infty)/\mathfrak{d}^m \mathcal{B}_\infty(\mathcal{O}_\infty)$$

is an isomorphism for all $m \geq 1$ by Lemma 6.20. Because P is $\mathfrak{d}$-power torsion and finitely generated, we deduce that the canonical morphism

$$P \longrightarrow P \otimes_{K_\infty\otimes\mathcal{O}_\infty} \mathcal{B}_\infty(\mathcal{O}_\infty)$$

is an isomorphism. On the other-hand, because the ideals of the form $(1 \otimes a) \subset K_\infty \otimes \mathcal{O}_\infty$ for $a \in \mathcal{O}_\infty$ are coprime to $\mathfrak{d}$, the map

$$P \longrightarrow P \otimes_{K_\infty\otimes\mathcal{O}_\infty} (K_\infty \otimes K_\infty)$$

is also an isomorphism. Hence P has a natural structure of $K_\infty \otimes K_\infty$-module extending the one over $\mathcal{O}_\infty \otimes K_\infty$. Combining both, we get the claimed isomorphism of $K_\infty \otimes K_\infty$-modules

$$Q \otimes_{A\otimes F} K_\infty[\![\mathfrak{j}]\!] \xrightarrow{\sim} P \otimes_{K_\infty\otimes\mathcal{O}_\infty} \mathcal{B}_\infty(\mathcal{O}_\infty)$$

extending (6.12). $\square$

Proof of Theorem 6.19. We apply Lemma 6.21 with the $A\otimes K$-module N/M for Q and the $K_\infty\otimes\mathcal{O}_\infty$-module $\hat{\mathcal{N}}_\infty/\hat{\mathcal{M}}_\infty$ for P. The isomorphism (6.12) follows from the sheaf property of $\mathcal{N}$ and $\mathcal{M}$. From the flatness of $A\otimes K\to K_\infty[\![\mathfrak{j}]\!]$ and $K_\infty\otimes\mathcal{O}_\infty\to\mathcal{B}_\infty(\mathcal{O}_\infty)$, we obtain the desired isomorphism

$$\frac{\hat{\mathcal{N}}_\infty}{\hat{\mathcal{M}}_\infty}\otimes_{\mathcal{O}_\infty}K_\infty\xrightarrow{\sim}\frac{N\otimes_{A\otimes K}K_\infty[\![\mathfrak{j}]\!]}{M\otimes_{A\otimes K}K_\infty[\![\mathfrak{j}]\!]}.$$

Pre-composition with the isomorphism of Proposition 6.14 gives the desired isomorphism. □

As announced, we have:

Corollary 6.22. *There is an exact sequence of K_∞-vector spaces:*

$$\Lambda(\underline{M})^+\otimes_A K_\infty\hookrightarrow\frac{\hat{\mathcal{N}}_\infty}{(\iota-\tau_{\mathcal{M}})(\hat{\mathcal{M}}_\infty)}\otimes_{\mathcal{O}_\infty}K_\infty\twoheadrightarrow\operatorname{Ext}^{1,ha,\infty}_{\mathcal{MHP}^+_{K_\infty}}(\mathbb{1}^+,\mathscr{H}^+_{K_\infty}(\underline{M})).$$

Proof. By Theorem 5.34, we have an exact sequence

$$\Lambda(\underline{M})^+\otimes_A K_\infty\hookrightarrow\frac{\mathfrak{p}^+_{\underline{M}}+\mathfrak{q}^+_{\underline{M}}}{\mathfrak{q}^+_{\underline{M}}}\twoheadrightarrow\operatorname{Ext}^{1,\mathrm{ha},\infty}_{\mathcal{MHP}^+_{K_\infty}}(\mathbb{1}^+,\mathscr{H}^+_{K_\infty}(\underline{M})).$$

Let $\gamma^\infty_{\underline{M}}:\Lambda(\underline{M})\otimes_{A,\nu}K_\infty[\![\mathfrak{j}]\!]\to M\otimes_{A\otimes K}K_\infty[\![\mathfrak{j}]\!]$ be the isomorphism (5.3) with respect to the choice of $F=K$ and $v=i:K\to\mathbb{C}_\infty$ the inclusion. We have:

$$\frac{\mathfrak{p}^+_{\underline{M}}+\mathfrak{q}^+_{\underline{M}}}{\mathfrak{q}^+_{\underline{M}}}\xrightarrow{\sim}\frac{N\otimes_{A\otimes K}K_\infty[\![\mathfrak{j}]\!]}{M\otimes_{A\otimes K}K_\infty[\![\mathfrak{j}]\!]}\overset{\text{Thm.6.19}}{\cong}\frac{\hat{\mathcal{N}}_\infty}{(\iota-\tau_{\mathcal{M}})(\hat{\mathcal{M}}_\infty)}\otimes_{\mathcal{O}_\infty}K_\infty$$

where the first isomorphism is induced by $\gamma^\infty_{\underline{M}}$. The corollary follows. □

6.3 Cohomological computations

In this section, we establish general preliminary observations related to sheaf cohomology. We refer to [Wei] for the definitions of homological algebra (*cones*, *distinguished triangles*, *derived categories*, etc.)

6.3.1 Change of coefficients

To fix the setting, we consider the following commutative square in the category of schemes over C:

$$\begin{array}{ccc}\operatorname{Spec}K_\infty\times C & \xrightarrow{j} & \operatorname{Spec}\mathcal{O}_\infty\times C\\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle q}\\ \operatorname{Spec}K\times C & \xrightarrow{p} & C\times C\end{array}\tag{6.13}$$

Our first result is the following.

Proposition 6.23. *Let $\mathcal{F}$ be a sheaf of modules on $C \times C$. In the derived category of K_∞-modules, there is a quasi-isomorphism*

$$R\Gamma(\operatorname{Spec} A \times C, \mathcal{F}) \otimes_A K_\infty \cong R\Gamma(\operatorname{Spec} \mathcal{O}_\infty \times C, q^*\mathcal{F}) \otimes_{\mathcal{O}_\infty} K_\infty$$

which is functorial in $\mathcal{F}$.

Proof. For $\mathcal{G}$ a sheaf of modules on $\operatorname{Spec} A \times C$, we first claim that

$$R\Gamma(\operatorname{Spec} A \times C, \mathcal{G}) \otimes_A K_\infty \cong R\Gamma(\operatorname{Spec} K \times C, \mathcal{G}) \otimes_K K_\infty. \tag{6.14}$$

This follows from the composition Theorem on derived functors (015M) applied to the commutative square of categories

$$\begin{tikzcd} \mathsf{Mod}(\operatorname{Spec} A \times C) \arrow[r, "\Gamma(\operatorname{Spec} A\times C)"] \arrow[d, "\Gamma(\operatorname{Spec} K\times C)"'] & \mathsf{Mod}(A) \arrow[d, "\otimes_A K_\infty"] \\ \mathsf{Mod}(K) \arrow[r, "\otimes_K K_\infty"] & \mathsf{Mod}(K_\infty) \end{tikzcd}$$

On the other-hand, for $\mathcal{H}$ a sheaf of modules on $\operatorname{Spec} K \times C$, we have

$$R\Gamma(\operatorname{Spec} K \times C, \mathcal{H}) \otimes_K K_\infty \cong R\Gamma(\operatorname{Spec} K_\infty \times C, i^*\mathcal{H}) \tag{6.15}$$

which again follows from the composition Theorem on derived functors applied to the commutative square of categories

$$\begin{tikzcd} \mathsf{Mod}(\operatorname{Spec} K \times C) \arrow[r, "i^*"] \arrow[d, "\Gamma(\operatorname{Spec} K\times C)"'] & \mathsf{Mod}(\operatorname{Spec} K_\infty \times C) \arrow[d, "\Gamma(\operatorname{Spec} K_\infty\times C)"] \\ \mathsf{Mod}(K) \arrow[r, "\otimes_K K_\infty"] & \mathsf{Mod}(K_\infty) \end{tikzcd}$$

(we used that i is flat, and hence that i^* is an exact functor). Finally, for a sheaf of modules $\mathcal{J}$ on $\operatorname{Spec} \mathcal{O}_\infty \times C$, we have

$$R\Gamma(\operatorname{Spec} K_\infty \times C, j^*\mathcal{J}) \cong R\Gamma(\operatorname{Spec} \mathcal{O}_\infty \times C, \mathcal{J}) \otimes_{\mathcal{O}_\infty} K_\infty \tag{6.16}$$

using the commutative square

$$\begin{tikzcd} \mathsf{Mod}(\operatorname{Spec} \mathcal{O}_\infty \times C) \arrow[r, "j^*"] \arrow[d, "\Gamma(\operatorname{Spec} \mathcal{O}_\infty\times C)"'] & \mathsf{Mod}(\operatorname{Spec} K_\infty \times C) \arrow[d, "\Gamma(\operatorname{Spec} K_\infty\times C)"] \\ \mathsf{Mod}(\mathcal{O}_\infty) \arrow[r, "\otimes_{\mathcal{O}_\infty} K_\infty"] & \mathsf{Mod}(K_\infty) \end{tikzcd}$$

together with the flatness of j. The composition:

$$\begin{aligned} R\Gamma(\operatorname{Spec} A \times C, \mathcal{F}) \otimes_A K_\infty &= R\Gamma(\operatorname{Spec} A \times C, p^*\mathcal{F}) \otimes_A K_\infty \\ &\cong R\Gamma(\operatorname{Spec} K \times C, p^*\mathcal{F}) \otimes_K K_\infty \quad (\text{by } (6.14)) \\ &\cong R\Gamma(\operatorname{Spec} K_\infty \times C, i^*p^*\mathcal{F}) \quad (\text{by } (6.15)) \\ &= R\Gamma(\operatorname{Spec} K_\infty \times C, j^*q^*\mathcal{F}) \quad (\text{by } (6.13)) \\ &\cong R\Gamma(\operatorname{Spec} \mathcal{O}_\infty \times C, q^*\mathcal{F}) \otimes_{\mathcal{O}_\infty} K_\infty \quad (\text{by } (6.16)). \end{aligned}$$

is the claimed isomorphism of the proposition. $\square$

6.3.2 Cech cohomology of schemes covered by two affines

Let S be a scheme and let T be a separated scheme over S. Let U, V and W be affine schemes over S which insert in a commutative diagram of S-schemes

$$\begin{tikzcd} U \arrow[r, "i"] & T \\ W \arrow[u] \arrow[r] \arrow[ur, dashed, "k"] & V \arrow[u, "j"'] \end{tikzcd}$$

such that $\{U \to T, V \to T\}$ forms a covering of T.

For $\mathcal{F}$ a sheaf of $\mathcal{O}_T$-modules, we denote by $S(\mathcal{F})$ the sequence of $\mathcal{O}_T$-modules:

$$0 \longrightarrow \mathcal{F} \longrightarrow i_*i^*\mathcal{F} \oplus j_*j^*\mathcal{F} \longrightarrow k_*k^*\mathcal{F} \longrightarrow 0$$

where the morphisms are given by the adjunction unit (note that the data of $S(\mathcal{F})$ is functorial in $\mathcal{F}$). The next lemma is of fundamental importance for our cohomological computations:

Lemma 6.24. *Assume that $S(\mathcal{O}_T)$ is exact. Then, for any finite locally free sheaf $\mathcal{F}$ of $\mathcal{O}_T$-modules, $S(\mathcal{F})$ is exact. In particular, the natural map*

$$R\Gamma(T, \mathcal{F}) \longrightarrow [\mathcal{F}(U) \oplus \mathcal{F}(V) \longrightarrow \mathcal{F}(W)], \tag{6.17}$$

where the right-hand side is a complex concentrated in degrees 0 and 1, is a quasi-isomorphism.

Proof. We show that $S(\mathcal{F})$ is an exact sequence (the second assertion follows, since applying $R\Gamma(T,-)$ to $S(\mathcal{F})$ yields the distinguished triangle computing (6.17)). To prove exactness of $S(\mathcal{F})$, first note that i, j and k are affine morphisms because T is separated (01SG). Thus, the pushforward functors appearing in $S(\mathcal{F})$ are naturally isomorphic to their right-derived functor (0G9R). Thereby, $S(\mathcal{F})$ in $\mathrm{D}_{qc}(T)$, the derived category of quasi-coherent sheaves over T, is naturally isomorphic to the triangle

$$\mathcal{F} \longrightarrow Ri_*i^*\mathcal{F} \oplus Rj_*j^*\mathcal{F} \longrightarrow Rk_*k^*\mathcal{F} \longrightarrow [1] \tag{6.18}$$

and it is sufficient to show that the latter is distinguished. Yet, because $\mathcal{F}$ is finite locally-free, the projection formula (01E8) implies that (6.18) is naturally isomorphic to

$$\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_T} \mathcal{O}_T \longrightarrow \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_T} (Ri_*\mathcal{O}_U \oplus Rj_*\mathcal{O}_V) \longrightarrow \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_T} Rk_*\mathcal{O}_W \longrightarrow [1]$$

Because $\mathcal{F}$ is locally-free, the functor $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_T} -$ is exact on $\mathrm{D}_{qc}(T)$ and it suffices to show the distinguishness of

$$\mathcal{O}_T \longrightarrow Ri_*\mathcal{O}_U \oplus Rj_*\mathcal{O}_V \longrightarrow Rk_*\mathcal{O}_W \longrightarrow [1].$$

But because $\mathcal{O}_U = i^*\mathcal{O}_T$, $\mathcal{O}_V = j^*\mathcal{O}_T$ and $\mathcal{O}_W = k^*\mathcal{O}_T$, this follows from our assumption that $S(\mathcal{O}_T)$ is exact. We conclude that (6.18) is distinguished. $\square$

Assuming that T is a smooth variety[3] over a field allows us to relax the "locally free" assumption in Lemma 6.24 to "coherent".

Proposition 6.25. *Let k be a field and assume that $S = \operatorname{Spec} k$. Assume further that T is a smooth variety over k, and that i, j and k are flat. Let $\mathcal{F}$ be a coherent sheaf on X. Then, $S(\mathcal{F})$ is exact. In particular, the natural map*

$$R\Gamma(T,\mathcal{F}) \longrightarrow [\mathcal{F}(U) \oplus \mathcal{F}(V) \longrightarrow \mathcal{F}(W)]$$

is a quasi-isomorphism.

Proof. Choose a resolution of $\mathcal{F}$ by finite locally free sheaves $0 \to \mathcal{F}_n \to \cdots \to \mathcal{F}_0 \to \mathcal{F} \to 0$. Because i (resp. j, k) is flat, i^* (resp. j^*, k^*) is an exact functor on quasi-coherent sheaves. Because it is affine, i_* (resp. j_*, k_*) is an exact functor on quasi-coherent sheaves. Thereby, for all $s \in \{0, ..., n\}$, the sequence $S(\mathcal{F}_s)$ is exact by Lemma 6.24. Using the $n \times n$-Lemma in the abelian category of quasi-coherent sheaves of $\mathcal{O}_T$-modules, we deduce that $S(\mathcal{F})$ is exact. $\square$

The main result of this subsection is:

Theorem 6.26. *Assume the setting of Proposition 6.25. Let $\mathcal{F}'$ be a coherent sheaf of $\mathcal{O}_T$-module and let $f : \mathcal{F} \to \mathcal{F}'$ be a morphism of sheaves of abelian groups. Then, the rows and the lines of the following diagram*

$$\begin{tikzcd}
R\Gamma(T,\mathcal{F}) \arrow[r] \arrow[d, "f_T"] & \mathcal{F}(U) \oplus \mathcal{F}(V) \arrow[r] \arrow[d, "f_U \oplus f_V"] & \mathcal{F}(W) \arrow[r] \arrow[d, "f_W"] & {[1]} \\
R\Gamma(T,\mathcal{F}') \arrow[r] \arrow[d] & \mathcal{F}'(U) \oplus \mathcal{F}'(V) \arrow[r] \arrow[d] & \mathcal{F}'(W) \arrow[r] \arrow[d] & {[1]} \\
\operatorname{cone}(f_T) \arrow[r] \arrow[d] & \operatorname{cone}(f_U) \oplus \operatorname{cone}(f_V) \arrow[r] \arrow[d] & \operatorname{cone}(f_W) \arrow[r] \arrow[d] & {[1]} \\
{[1]} & {[1]} & {[1]} &
\end{tikzcd} \tag{6.19}$$

form distinguished triangles in the derived category of abelian groups, where $f_Y := R\Gamma(Y, f)$ (for $Y \in \{T, U, V, W\}$).

Proof. We lift the first two lines in the category of chain complexes: by Lemma 6.24, the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, "f"] & i_*i^*\mathcal{F} \oplus j_*j^*\mathcal{F} \arrow[r] \arrow[d, "i_*i^*f \oplus j_*j^*f"] & k_*k^*\mathcal{F} \arrow[r] \arrow[d, "k_*k^*f"] & 0 \\
0 \arrow[r] & \mathcal{F}' \arrow[r] & i_*i^*\mathcal{F}' \oplus j_*j^*\mathcal{F}' \arrow[r] & k_*k^*\mathcal{F}' \arrow[r] & 0
\end{tikzcd} \tag{6.20}$$

[3]By variety over k, we mean that T is integral and that $T \to \operatorname{Spec} k$ is separated and of finite type.

is exact on lines and commutative on squares in the category of quasi-coherent sheaves of $\mathcal{O}_T$-modules. From (013T) we can find injective resolutions $\mathcal{F} \to I_1^\bullet$, $i_*i^*\mathcal{F} \oplus j_*j^*\mathcal{F} \to I_2^\bullet$ and $k_*k^*\mathcal{F} \to I_3^\bullet$ (respectively $\mathcal{F}' \to J_1^\bullet$, $i_*i^*\mathcal{F}' \oplus j_*j^*\mathcal{F}' \to J_2^\bullet$ and $k_*k^*\mathcal{F}' \to J_3^\bullet$) such that

$$\begin{tikzcd} 0 \arrow[r] & I_1^\bullet \arrow[r] \arrow[d, "i_1"] & I_2^\bullet \arrow[r] \arrow[d, "i_2"] & I_3^\bullet \arrow[r] \arrow[d, "i_3"] & 0 \\ 0 \arrow[r] & J_1^\bullet \arrow[r] & J_2^\bullet \arrow[r] & J_3^\bullet \arrow[r] & 0 \end{tikzcd}$$

is an injective resolution of the whole diagram (6.20). Completing the vertical maps into distinguished triangles gives:

$$\begin{tikzcd} 0 \arrow[r] & I_1^\bullet \arrow[r] \arrow[d, "i_1"] & I_2^\bullet \arrow[r] \arrow[d, "i_2"] & I_3^\bullet \arrow[r] \arrow[d, "i_3"] & 0 \\ 0 \arrow[r] & J_1^\bullet \arrow[r] \arrow[d] & J_2^\bullet \arrow[r] \arrow[d] & J_3^\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{cone}(i_1) \arrow[r] \arrow[d] & \mathrm{cone}(i_2) \arrow[r] \arrow[d] & \mathrm{cone}(i_3) \arrow[r] \arrow[d] & 0 \\ & {[1]} & {[1]} & {[1]} & \end{tikzcd} \tag{6.21}$$

where the rows are distinguished triangles. The third line is a direct sum of exact sequences and therefore is exact. The horizontal exact sequences transform to distinguished triangles in the derived category of abelian module. This concludes. □

Under Noetherianity assumptions, Theorem 6.26 can be extended to the case of formal schemes. Our main reference is [KaF, §I]. From now on, we assume that T, U, V and W are Noetherian schemes over S. Let $T' \subset T$, $U' \subset U$, $V' \subset V$ and $W' \subset W$ be closed subschemes such that $i^{-1}(T') = U'$, $j^{-1}(T') = V'$, and $p^{-1}(U') = W' = q^{-1}(V')$. It follows that $k^{-1}(T') = W'$. Let $\hat{T}$, $\hat{U}$, $\hat{V}$ and $\hat{W}$ be the formal completions along the corresponding closed subschemes [KaF, §I.1.4]. We obtain a commutative diagram of formal schemes

$$\begin{tikzcd} \hat{U} \arrow[r, "\hat{i}"] & \hat{T} \\ \hat{W} \arrow[u] \arrow[r] \arrow[ur, dashed, "\hat{k}"] & \hat{V} \arrow[u, "\hat{j}"'] \end{tikzcd}$$

Given an adically quasi-coherent sheaf[4] $\mathcal{F}$ of $\mathcal{O}_{\hat{T}}$-modules [KaF, §I,Def.3.1.3], we consider the sequence

$$\hat{S}(\mathcal{F}): \quad 0 \to \mathcal{F} \to \hat{i}_*\hat{i}^*\mathcal{F} \oplus \hat{j}_*\hat{j}^*\mathcal{F} \to \hat{k}_*\hat{k}^*\mathcal{F} \to 0.$$

[4]e.g. the formal completion of a quasi-coherent sheaf with respect to a closed subscheme of finite presentation is adically quasi-coherent by [KaF, §I, Prop.3.1.5]

Lemma 6.27. *Let $\mathcal{F}$ be a quasi-coherent sheaf on T. Then $\widehat{S(\mathcal{F})} \cong \hat{S}(\hat{\mathcal{F}})$, where $\mathcal{G} \mapsto \hat{\mathcal{G}}$ denotes the formal completion functor along T'. In particular, if $S(\mathcal{F})$ is exact, then $\hat{S}(\hat{\mathcal{F}})$ is exact.*

Proof. This almost follows from the flat-base change Theorem (02KH). Indeed, the diagram

$$\begin{tikzcd} \hat{U} \arrow[r, "f_U"] \arrow[d, "\hat{\imath}"] & U \arrow[d, "i"] \\ \hat{T} \arrow[r, "f_T"] & T \end{tikzcd}$$

where f_U and f_T are the canonical maps, is Cartesian. Because i is affine, i is quasi-compact and quasi-separated (01S7). On the other-hand, f_T is flat and the flat-base change Theorem applies. It states that for any quasi-coherent sheaf $\mathcal{G}$ of $\mathcal{O}_U$-modules, the natural map

$$f_T^* Ri_* \mathcal{G} \longrightarrow R\hat{\imath}_*(f_U^* \mathcal{G})$$

is a quasi-isomorphism in the derived category of $\mathcal{O}_U$-modules. Because i is affine, the functors $R\hat{\imath}_*$ and i_* are isomorphic on the category of coherent sheaves (0G9R). Similarly, but in the setting of formal geometry, $\hat{\imath}$ is also affine [KaF, §I,Def.4.1.1], and the formal analogue of the previous argument [KaF, §I,Thm.7.1.1] reads that the functors $R\hat{\imath}_*$ and $\hat{\imath}_*$ are isomorphic on the category of adically quasi-coherent sheaves. Therefore, in the derived category of $\mathcal{O}_U$-modules, we have an isomorphism

$$f_T^* i_* \mathcal{G} \xrightarrow{\sim} \hat{\imath}_* f_U^* \mathcal{G}.$$

Applied to $\mathcal{G} = i^*\mathcal{F}$ for a quasi-coherent $\mathcal{F}$ on T, we obtain $f_T^* i_* i^* \mathcal{F} \cong \hat{\imath}_* \hat{\imath}^* f_T^* \mathcal{F}$ functorially in $\mathcal{F}$. In other words,

$$\widehat{i_* i^* \mathcal{F}} \cong \hat{\imath}_* \hat{\imath}^* \hat{\mathcal{F}}.$$

The very same argument for j and k in place of i yields respectively $\widehat{j_* j^* \mathcal{F}} \cong \hat{\jmath}_* \hat{\jmath}^* \hat{\mathcal{F}}$ and $\widehat{k_* k^* \mathcal{F}} \cong \hat{k}_* \hat{k}^* \hat{\mathcal{F}}$. It follows that $\widehat{S(\mathcal{F})} \cong \hat{S}(\hat{\mathcal{F}})$. Since the formal completion functor is exact, $\hat{S}(\hat{\mathcal{F}})$ is exact if $S(\mathcal{F})$ is. $\square$

Thanks to Lemma 6.27, the proof of Theorem 6.26 blithely applies to the formal situation:

Theorem 6.28. *Assume the setting of Theorem 6.26. Then, each rows and* *each lines of the following diagram*

$$\begin{tikzcd}
R\Gamma(\hat{T},\hat{\mathcal{F}}) \arrow[r] \arrow[d, "\hat{f}_{\hat{T}}"] & \hat{\mathcal{F}}(\hat{U}) \oplus \hat{\mathcal{F}}(\hat{V}) \arrow[r] \arrow[d, "\hat{f}_{\hat{U}} \oplus \hat{f}_{\hat{V}}"] & \hat{\mathcal{F}}(\hat{W}) \arrow[r] \arrow[d, "\hat{f}_{\hat{W}}"] & {[1]} \\
R\Gamma(\hat{T},\hat{\mathcal{F}}') \arrow[r] \arrow[d] & \hat{\mathcal{F}}'(\hat{U}) \oplus \hat{\mathcal{F}}'(\hat{V}) \arrow[r] \arrow[d] & \hat{\mathcal{F}}'(\hat{W}) \arrow[r] \arrow[d] & {[1]} \\
\mathrm{cone}(\hat{f}_{\hat{T}}) \arrow[r] \arrow[d] & \mathrm{cone}(\hat{f}_{\hat{U}}) \oplus \mathrm{cone}(\hat{f}_{\hat{V}}) \arrow[r] \arrow[d] & \mathrm{cone}(\hat{f}_{\hat{W}}) \arrow[r] \arrow[d] & {[1]} \\
{[1]} & {[1]} & {[1]} &
\end{tikzcd}$$

form distinguished triangles in the derived category of abelian groups, where $\hat{f}_{\hat{Y}} := R\Gamma(\hat{Y},\hat{f})$ *(for* $Y \in \{T,U,V,W\}$*).*

6.4 Proof of Theorems E and F

6.4.1 Proof of the first part of Theorem 6.2

Let $\underline{M}$ be a rigid analytically trivial A-motive over K. We let $U(\underline{M})$ be the class module of $\underline{M}$ (6.1). In this subsection, we prove the first part of Theorem 6.2, that is

Theorem 6.29 (First part of 6.2)**.** *The* A*-modules* $\mathrm{Ext}^{1,reg,\infty}_{\mathcal{M}_K,A}(\mathbb{1},\underline{M})$ *and* $U(\underline{M})$ *are finitely generated.*

The proof of Theorem 6.29 is organized as follows. We first introduce the $\mathcal{G}$-*complex* $\mathcal{G}_{\underline{M}}$ *of* $\underline{M}$, a complex of A-modules in direct link with $\mathrm{Ext}^{1,\mathrm{reg},\infty}_{\mathcal{M}_K,A}(\mathbb{1},\underline{M})$ and $U(\underline{M})$ as we show in Proposition 6.30. From it, Theorem 6.29 becomes equivalent to the fact that the A-modules $H^0(\mathcal{G}_{\underline{M}})$ and $H^1(\mathcal{G}_{\underline{M}})$ are finitely generated. In Proposition 6.33, we show that $\mathcal{G}_{\underline{M}}$ is quasi-isomorphic to a complex of A-modules involving the global sections modules of a C-sthuka model for $\underline{M}$. Theorem 6.29 will ultimately follow from the finiteness Theorem for proper varieties.

The $\mathcal{G}$-complex of an A-motive

Let M_A be the maximal A-model of $\underline{M}$ (whose existence is prescribed by Proposition 2.30). Let $N_A := M_A[\mathfrak{j}^{-1}] \cap (M + \tau_M(\tau^*M))$. We introduce the complex of A-modules

$$\mathcal{G}_{\underline{M}} := \left[\frac{M \otimes_{A\otimes K} K_\infty\langle A\rangle}{M_A} \xrightarrow{\mathrm{id}-\tau_M} \frac{M \otimes_{A\otimes K} K_\infty\langle A\rangle}{N_A}\right] \tag{6.22}$$

placed in degrees 0 and 1. We call $\mathcal{G}_{\underline{M}}$ the $\mathcal{G}$*-complex of* $\underline{M}$.

Proposition 6.30. *There is a long exact sequence of A-modules, given functorially in $\underline{M}$:*

$$\begin{aligned} 0 \longrightarrow \operatorname{Ext}^0_{\mathcal{M}_K^{rig}}(\mathbb{1}, \underline{M}) \longrightarrow \Lambda(\underline{M})^+ \longrightarrow H^0(\mathcal{G}_{\underline{M}}) \\ \longrightarrow \operatorname{Ext}^{1,reg}_{\mathcal{M}_K^{rig},A}(\mathbb{1}, \underline{M}) \stackrel{r_{\underline{M},\infty}}{\longrightarrow} H^1(G_\infty, \Lambda(\underline{M})) \longrightarrow H^1(\mathcal{G}_{\underline{M}}) \longrightarrow 0. \end{aligned}$$

Proof. By Proposition 5.33 applied in the case $F = K$ and $v = i : K \to \mathbb{C}_\infty$, the A-module morphism

$$\frac{\{\xi \in M \otimes_{A\otimes K} K_\infty\langle A\rangle \mid \xi - \tau_M(\tau^*\xi) \in N_A\}}{M_A + \Lambda(\underline{M})^+} \longrightarrow \operatorname{Ext}^{1,\mathrm{reg},\infty}_{\mathcal{M}_K^{\mathrm{rig}},A}(\mathbb{1}, \underline{M}),$$

which maps ξ to $\iota(\xi - \tau_M(\tau^*\xi))$, is an isomorphism. The proposition then follows from the snake Lemma applied to the diagram of A-modules:

$$\begin{tikzcd} 0 \arrow[r] & M_A \arrow[r] \arrow[d, "\mathrm{id}-\tau_M"] & M \otimes_{A\otimes K} K_\infty\langle A\rangle \arrow[r] \arrow[d, "\mathrm{id}-\tau_M"] & \dfrac{M \otimes_{A\otimes K} K_\infty\langle A\rangle}{M_A} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & N_A \arrow[r] & M \otimes_{A\otimes K} K_\infty\langle A\rangle \arrow[r] & \dfrac{M \otimes_{A\otimes K} K_\infty\langle A\rangle}{N_A} \arrow[r] & 0. \end{tikzcd}$$

together with Proposition 5.27. □

We are going to prove that the A-modules $H^0(\mathcal{G}_{\underline{M}})$ and $H^1(\mathcal{G}_{\underline{M}})$ are finitely generated. By Proposition 6.30, this will imply Theorem 6.29.

The $\mathcal{G}$-complex is perfect

The main ingredient are the cohomological preliminaries of Section 6.3. We consider the particular setting of $S = \operatorname{Spec}\mathbb{F}$ and of the commutative diagram of S-schemes

$$\begin{tikzcd} \operatorname{Spec}\mathcal{O}_\infty\langle A\rangle \arrow[r, "i"] & (\operatorname{Spec} A) \times C \\ \operatorname{Spec} K_\infty\langle A\rangle \arrow[u, "p"] \arrow[r, "q"] \arrow[ur, dashed, "k"] & \operatorname{Spec} A \otimes A \arrow[u, "j"'] \end{tikzcd}$$

Because A is geometrically irreducible over $\mathbb{F}$, $(\operatorname{Spec} A)\times C$ is a smooth variety over $\mathbb{F}$. To use the results of Section 6.3, one requires the next two lemmas.

Lemma 6.31. *The morphisms i, j, k are flat.*

Proof. We consider the affine open cover $(\operatorname{Spec} A \otimes A) \cup (\operatorname{Spec} A \otimes B)$ of $(\operatorname{Spec} A) \times C$. We first show that i is flat. We have $i^{-1}(\operatorname{Spec} A \otimes A) = \operatorname{Spec} K_\infty\langle A\rangle$ and $i^{-1}(\operatorname{Spec} A \otimes B) = \operatorname{Spec}\mathcal{O}_\infty\langle A\rangle$. The morphism $A \otimes B \to \mathcal{O}_\infty\langle A\rangle$ is flat (because it is the completion of the Noetherian ring $A \otimes B$ and the ideal $\mathfrak{m}_\infty \subset B$) and thus, so is $A \otimes A \to K_\infty\langle A\rangle$. By (01U5), i is flat.

We have $j^{-1}(\operatorname{Spec} A\otimes B) = \operatorname{Spec} A\otimes D$, where $D \subset K$ is the sub-$\mathbb{F}$-algebra such that $\operatorname{Spec} D = \operatorname{Spec} A \cap \operatorname{Spec} B$. The inclusion $B \to D$ is a localization, and hence $A \otimes B \to A \otimes D$ is flat. Thereby, j is flat.

Because $K_\infty\langle A\rangle \cong K_\infty \otimes_{\mathcal{O}_\infty} \mathcal{O}_\infty\langle A\rangle$, p is flat. Since compositions of flat morphisms are flat, $k = i \circ p$ is flat. $\square$

Lemma 6.32. *For $T = (\operatorname{Spec} A) \times C$, the sequence $0 \to \mathcal{O}_T \to i_*i^*\mathcal{O}_T \oplus j_*j^*\mathcal{O}_T \to k_*k^*\mathcal{O}_T \to 0$ is exact.*

Proof. We need to show that the complex $Z := [\mathcal{O}_\infty\langle A\rangle \oplus (A\otimes A) \to K_\infty\langle A\rangle]$, where the morphism is the difference of the canonical inclusions, represents the sheaf cohomology in the Zariski topology of $\mathcal{O}_{\operatorname{Spec} A\times C}$, the latter being quasi-isomorphic to

$$R\Gamma(\operatorname{Spec} A \times C, \mathcal{O}_{\operatorname{Spec} A\times C}) = [(A \otimes B) \oplus (A \otimes A) \to A \otimes D].$$

Let $(t_i)_{i\geq 0}$ be a (countable) basis of A over $\mathbb{F}$. Any element f in $K_\infty\langle A\rangle$ can be represented uniquely by a converging series

$$f = \sum_{i=0}^{\infty} t_i \otimes f_i, \quad t_i \in K_\infty,\ t_i \to 0\ (i \to \infty).$$

Elements of $\mathcal{O}_\infty\langle A\rangle$ are the ones for which $f_i \in \mathcal{O}_\infty$ ($\forall i \geq 0$) and elements of $A \otimes A$ are the ones for which $f_i \in A$ ($\forall i \geq 0$) and $f_i = 0$ for i large enough. Therefore, it is clear that $\mathcal{O}_\infty\langle A\rangle \cap (A \otimes A)$ is $A \otimes (\mathcal{O}_\infty \cap A)$. Yet, $\mathcal{O}_\infty \cap A$ is the constant field of C, showing that $H^0(Z) = H^0(\operatorname{Spec} A \times C, \mathcal{O}_{\operatorname{Spec} A\times C})$.

Because $K_\infty = \mathcal{O}_\infty + A + D$, the canonical map

$$\frac{A\otimes D}{A\otimes B + A\otimes A} \to \frac{K_\infty\langle A\rangle}{\mathcal{O}_\infty\langle A\rangle + A\otimes A}$$

is surjective. Because $(A\otimes D)\cap \mathcal{O}_\infty\langle A\rangle \subset A\otimes B + A\otimes A$, it is also injective. It follows that $H^1(Z) = H^1(\operatorname{Spec} A \times C, \mathcal{O}_{\operatorname{Spec} A\times C})$. $\square$

Let $\underline{\mathcal{M}} = (\mathcal{N}, \mathcal{M}, \tau_M)$ be a C-shtuka model for $\underline{M}$ (Definition 6.6). We have

$$\begin{aligned} j^*\mathcal{M}(\operatorname{Spec} A \otimes A) &= \mathcal{M}(\operatorname{Spec} A \otimes A) = M_A, \\ j^*\mathcal{N}(\operatorname{Spec} A \otimes A) &= \mathcal{N}(\operatorname{Spec} A \otimes A) = N_A, \\ k^*\mathcal{M}(\operatorname{Spec} K_\infty\langle A\rangle) &= k^*\mathcal{N}(\operatorname{Spec} K_\infty\langle A\rangle) = M \otimes_{A\otimes K} K_\infty\langle A\rangle. \end{aligned}$$

Theorem 6.26 yields a morphism of distinguished triangles

$$\begin{tikzcd} R\Gamma(\operatorname{Spec} A\times C,\mathcal{M}) \arrow[r] \arrow[d, "\iota-\tau_{\mathcal{M}}"] & i^*\mathcal{M}(\operatorname{Spec}\mathcal{O}_\infty\langle A\rangle) \arrow[r] \arrow[d, "\mathrm{id}-\tau_{\mathcal{M}}"] & \dfrac{M\otimes_{A\otimes K} K_\infty\langle A\rangle}{M_A} \arrow[r] \arrow[d, "\mathrm{id}-\tau_M"] & {[1]} \\ R\Gamma(\operatorname{Spec} A\times C,\mathcal{N}) \arrow[r] \arrow[d] & i^*\mathcal{N}(\operatorname{Spec}\mathcal{O}_\infty\langle A\rangle) \arrow[r] \arrow[d] & \dfrac{M\otimes_{A\otimes K} K_\infty\langle A\rangle}{N_A} \arrow[r] \arrow[d] & {[1]} \\ \operatorname{cone}(\iota-\tau_{\mathcal{M}}|\operatorname{Spec} A\times C) \arrow[r] & 0 \arrow[r] & \mathcal{G}_{\underline{M}}[1] \arrow[r] & {[1]} \end{tikzcd}$$

where the cone of the middle upper vertical morphism is zero by Lemma 6.8. The third row is a distinguished triangle, hence we have proved:

Proposition 6.33. *Let $(\mathcal{N},\mathcal{M},\tau_M)$ be a C-shtuka model for $\underline{M}$. Let ι denotes the inclusion of $\mathcal{M}$ in $\mathcal{N}$. There is a quasi-isomorphism of A-module complexes*

$$\mathcal{G}_{\underline{M}} \xrightarrow{\sim} \operatorname{cone}\left(R\Gamma(\operatorname{Spec} A\times C,\mathcal{M}) \xrightarrow{\iota-\tau_{\mathcal{M}}} R\Gamma(\operatorname{Spec} A\times C,\mathcal{N})\right).$$

We have all the tools in hand to prove Theorem 6.29:

Proof of Theorem 6.29. As $\operatorname{Spec} A\times C$ is proper over $\operatorname{Spec} A$ and both $\mathcal{M}$ and $\mathcal{N}$ are coherent sheaves of $\mathcal{O}_{\operatorname{Spec} A\times C}$-modules, both $R\Gamma(\operatorname{Spec} A\times C,\mathcal{M})$ and $R\Gamma(\operatorname{Spec} A\times C,\mathcal{N})$ are perfect complexes. Hence

$$\operatorname{cone}\left(R\Gamma(\operatorname{Spec} A\times C,\mathcal{M}) \xrightarrow{\iota-\tau_{\mathcal{M}}} R\Gamma(\operatorname{Spec} A\times C,\mathcal{N})\right)$$

is a perfect complex, and so is $\mathcal{G}_{\underline{M}}$ by Proposition 6.33. The theorem follows from Proposition 6.30. $\square$

6.4.2 Proof of Theorems 6.2 and 6.4

Theorems 6.2 (second part) and 6.4 will follow from the study of the cohomology of a $C\times C$-shtuka model of $\underline{\mathcal{M}}$ at $\operatorname{Spf}\mathcal{O}_\infty\hat{\times}C$. The latter corresponds to the completion of the Noetherian scheme $C\times C$ at the closed subscheme $\{\infty\}\times C$. The argument given here is a refinement of the one given in the previous subsection where we use $C\times C$-shtuka models instead of C-shtuka models.

We apply the results of Section 6.3 under a different setting. We consider the commutative square of schemes over $\operatorname{Spec}\mathcal{O}_\infty$:

$$\begin{tikzcd} \operatorname{Spec}\mathcal{O}_\infty\otimes A \arrow[r, "i"] & (\operatorname{Spec}\mathcal{O}_\infty)\times C \\ \operatorname{Spec}\mathcal{O}_\infty\otimes K_\infty \arrow[r] \arrow[u] \arrow[ur, dashed, "k"] & \operatorname{Spec}\mathcal{O}_\infty\otimes\mathcal{O}_\infty \arrow[u, "j"'] \end{tikzcd}$$

Similarly to Lemma 6.31, one shows that i, j and k are flat morphisms. For the sake of compatibility of notations with subsection 6.3.2, we let $T=\operatorname{Spec}\mathcal{O}_\infty\times C$, $U=\operatorname{Spec}(\mathcal{O}_\infty\otimes A)$, $V=\operatorname{Spec}(\mathcal{O}_\infty\otimes\mathcal{O}_\infty)$ and $W=\operatorname{Spec}(\mathcal{O}_\infty\otimes K_\infty)$. Consider the respective closed subschemes $T'=\{\infty\}\times C$, $U'=\{\infty\}\times\operatorname{Spec} A$, $V=\{\infty\}\times\operatorname{Spec}\mathcal{O}_\infty$ and $W=\{\infty\}\times\operatorname{Spec} K_\infty$ and the formal completions $\hat{T}=\operatorname{Spf}\mathcal{O}_\infty\hat{\times}C$, $\hat{U}=\operatorname{Spf}\mathcal{O}_\infty\hat{\times}\operatorname{Spec} A$, $\hat{V}=\operatorname{Spf}\mathcal{O}_\infty\hat{\times}\operatorname{Spec}\mathcal{O}_\infty$ and $\hat{W}=\operatorname{Spf}\mathcal{O}_\infty\hat{\times}\operatorname{Spec} K_\infty$. We obtain the commutative square of formal schemes over $\operatorname{Spf}\mathcal{O}_\infty$:

$$\begin{tikzcd} \operatorname{Spf}\mathcal{O}_\infty\hat{\otimes}A \arrow[r, "\hat{i}"] & (\operatorname{Spf}\mathcal{O}_\infty)\hat{\times}C \\ \operatorname{Spf}\mathcal{O}_\infty\hat{\otimes}K_\infty \arrow[r] \arrow[u] \arrow[ur, dashed, "\hat{k}"] & \operatorname{Spf}\mathcal{O}_\infty\hat{\otimes}\mathcal{O}_\infty \arrow[u, "\hat{j}"'] \end{tikzcd}$$

We let $q : \operatorname{Spec} \mathcal{O}_\infty \times C \to C \times C$ be the inclusion of schemes. To the morphism of sheaves $\tau_{\mathcal{M}} : \tau^*(q^*\mathcal{M}) \to (q^*\mathcal{N})$ on $(\operatorname{Spec} \mathcal{O}_\infty) \times C$, one associates functorially the morphism of the formal coherent sheaves $\tau^* : \hat{\mathcal{M}} \to \hat{\mathcal{N}}$ on the formal spectrum $(\operatorname{Spf} \mathcal{O}_\infty)\hat{\times} C$. Because both $q^*\mathcal{M}$ and $q^*\mathcal{N}$ are coherent sheaves, their formal completion corresponds to their pullback along the completion morphism

$$(\operatorname{Spf} \mathcal{O}_\infty)\hat{\times} C \longrightarrow (\operatorname{Spec} \mathcal{O}_\infty) \times C.$$

Recall that $\hat{\mathcal{N}}_\infty$ and $\hat{\mathcal{M}}_\infty$ were the respective $\mathcal{A}_\infty(\mathcal{O}_\infty)$-modules $\hat{\mathcal{N}}(\operatorname{Spf} \mathcal{O}_\infty \hat{\otimes} \mathcal{O}_\infty)$ and $\hat{\mathcal{M}}(\operatorname{Spf} \mathcal{O}_\infty \hat{\otimes} \mathcal{O}_\infty)$. Let also L and L_A be given respectively by

$$\begin{aligned} L &:= \hat{\mathcal{M}}(\operatorname{Spf} \mathcal{O}_\infty \hat{\otimes} K) = \hat{\mathcal{N}}(\operatorname{Spf} \mathcal{O}_\infty \hat{\otimes} K) \\ L_A &:= \hat{\mathcal{M}}(\operatorname{Spf} \mathcal{O}_\infty \hat{\otimes} A) = \hat{\mathcal{N}}(\operatorname{Spf} \mathcal{O}_\infty \hat{\otimes} A). \end{aligned}$$

Note that L defines an $\mathcal{A}_\infty(K)$-lattice stable by τ_M for the isocrystal $\mathcal{I}_\infty(\underline{M})$ (Definition 1.28).

By Theorem 6.28 we have a morphism of distinguished triangles:

$$\begin{tikzcd} R\Gamma(\operatorname{Spf} \mathcal{O}_\infty \hat{\times} C, \hat{\mathcal{M}}) \arrow[r] \arrow[d, "\iota-\tau_{\mathcal{M}}"] & \hat{\mathcal{M}}_\infty \arrow[r] \arrow[d, "\iota-\tau_{\mathcal{M}}"] & \dfrac{L \otimes_{\mathcal{A}_\infty(K)} \mathcal{A}_\infty(K_\infty)}{L_A} \arrow[r] \arrow[d, "\mathrm{id}-\tau_M"] & {[1]} \\ R\Gamma(\operatorname{Spf} \mathcal{O}_\infty \hat{\times} C, \hat{\mathcal{N}}) \arrow[r] & \hat{\mathcal{N}}_\infty \arrow[r] & \dfrac{L \otimes_{\mathcal{A}_\infty(K)} \mathcal{A}_\infty(K_\infty)}{L_A} \arrow[r] & {[1]} \end{tikzcd} \tag{6.23}$$

The third vertical arrow is an isomorphism by the next lemma:

Lemma 6.34. *The morphism $\iota - \tau_{\mathcal{M}} : \hat{\mathcal{N}}_\infty \otimes_{\mathcal{O}_\infty} K_\infty \to \hat{\mathcal{M}}_\infty \otimes_{\mathcal{O}_\infty} K_\infty$ is injective.*

Proof. For t a positive integer, $a \geq 0$ and $x \in \hat{\mathcal{M}}_\infty$, we have

$$(\iota - \tau_{\mathcal{M}})((1 \otimes \pi_\infty)^{t+a} x) \equiv (1 \otimes \pi_\infty)^{t+a} \iota(x) \pmod{(1 \otimes \pi_\infty)^{t+a+1} \hat{\mathcal{N}}_\infty}.$$

In particular, the first vertical arrow in diagram 6.8 is injective. The lemma then follows from Lemma 6.16 together with the snake Lemma. $\square$

Theorem 6.19 together with Lemma 6.34 implies the existence of a quasi-isomorphism

$$\operatorname{cone}([\hat{\mathcal{M}}_\infty \xrightarrow{\iota-\tau_{\mathcal{M}}} \hat{\mathcal{N}}_\infty]) \otimes_{\mathcal{O}_\infty} K_\infty \cong \frac{N \otimes_{A \otimes K} K_\infty[\![\mathfrak{j}]\!]}{M \otimes_{A \otimes K} K_\infty[\![\mathfrak{j}]\!]}.$$

Applying the inverse of the isomorphism $\gamma_{\underline{M}}^\infty$ (defined in (5.3)), we even obtain by Lemma 5.7 a quasi-isomorphism:

$$\operatorname{cone}([\hat{\mathcal{M}}_\infty \xrightarrow{\iota-\tau_{\mathcal{M}}} \hat{\mathcal{N}}_\infty]) \otimes_{\mathcal{O}_\infty} K_\infty \cong \frac{\mathfrak{p}_{\underline{M}}^+ + \mathfrak{q}_{\underline{M}}^+}{\mathfrak{q}_{\underline{M}}^+}.$$

Theorem 6.28 then implies that

$$\operatorname{cone}\left(R\Gamma(\operatorname{Spf}\mathcal{O}_\infty\hat{\times}C,\hat{\mathcal{M}})\xrightarrow{\iota-\tau_{\mathcal{M}}}R\Gamma(\operatorname{Spf}\mathcal{O}_\infty\hat{\times}C,\hat{\mathcal{N}})\right)\otimes_{\mathcal{O}_\infty}K_\infty\cong\frac{\mathfrak{p}^+_{\underline{M}}+\mathfrak{q}^+_{\underline{M}}}{\mathfrak{q}^+_{\underline{M}}}. \tag{6.24}$$

Because $(\operatorname{Spec} A)\times C\to\operatorname{Spec} A$ is proper, Grothendieck's comparison Theorem [EGA, Thm. 4.1.5] provides natural quasi-isomorphisms

$$R\Gamma(\operatorname{Spec}\mathcal{O}_\infty\times C,\mathcal{F})\cong R\Gamma(\operatorname{Spf}\mathcal{O}_\infty\hat{\times}C,\hat{\mathcal{F}})$$

for $\mathcal{F}$ being either $q^*\mathcal{M}$ or $q^*\mathcal{N}$. This allows us to rewrite (6.24) as

$$\operatorname{cone}\left(R\Gamma(\operatorname{Spec}\mathcal{O}_\infty\times C,q^*\mathcal{M})\xrightarrow{\iota-\tau_{\mathcal{M}}}R\Gamma(\operatorname{Spec}\mathcal{O}_\infty\times C,q^*\mathcal{N})\right)\otimes_{\mathcal{O}_\infty}K_\infty\cong\frac{\mathfrak{p}^+_{\underline{M}}+\mathfrak{q}^+_{\underline{M}}}{\mathfrak{q}^+_{\underline{M}}},$$

and we use Proposition 6.23 to obtain

$$\operatorname{cone}\left(R\Gamma(\operatorname{Spec}A\times C,\mathcal{M})\xrightarrow{\iota-\tau_{\mathcal{M}}}R\Gamma(\operatorname{Spec}A\times C,\mathcal{N})\right)\otimes_A K_\infty\cong\frac{\mathfrak{p}^+_{\underline{M}}+\mathfrak{q}^+_{\underline{M}}}{\mathfrak{q}^+_{\underline{M}}}.$$

From Proposition 6.33, we deduce

$$\operatorname{cone}(\mathcal{G}_{\underline{M}})\otimes_A K_\infty\cong\frac{\mathfrak{p}^+_{\underline{M}}+\mathfrak{q}^+_{\underline{M}}}{\mathfrak{q}^+_{\underline{M}}}. \tag{6.25}$$

Proof of Theorem 6.2. The first part is Theorem 6.29, it remains to prove that $U(\underline{M})$ is torsion. But $U(\underline{M})$ is indentified with $H^1(\mathcal{G}_{\underline{M}})$ by Proposition 6.30. The latter is a torsion A-module, because the cohomology of $\mathcal{G}_{\underline{M}}\otimes_A K_\infty$ is concentrated in degree 0 by (6.25). □

It remains to prove Theorem 6.4 (see the next page). Let us first give a notation to an essential ingredient of the proof.

Definition 6.35. We let $\rho(\underline{M})$ be the isomorphism of K_∞-vector spaces

$$\rho(\underline{M}) : \frac{\{\xi \in M \otimes_{A\otimes K} K_\infty\langle A\rangle \mid \xi - \tau_M(\tau^*\xi) \in N_A\}}{M_A} \otimes_A K_\infty \xrightarrow{\sim} \frac{\mathfrak{p}^+_{\underline{M}} + \mathfrak{q}^+_{\underline{M}}}{\mathfrak{q}^+_{\underline{M}}}$$

obtained by the vertical composition of the quasi-isomorphisms of complexes of K_∞-vector spaces:

$$\begin{array}{c}
\dfrac{\{\xi \in M \otimes_{A\otimes K} K_\infty\langle A\rangle \mid \xi - \tau_M(\tau^*\xi) \in N_A\}}{M_A} \otimes_A K_\infty \\
\Big\downarrow \wr\ {\scriptstyle U(\underline{M})\otimes_A K_\infty = 0} \\
\mathcal{G}_{\underline{M}} \otimes_A K_\infty \\
\Big\downarrow \wr\ {\scriptstyle \text{Proposition 6.33}} \\
\operatorname{cone}\left[R\Gamma(\operatorname{Spec} A \times C, \mathcal{M}) \xrightarrow{\iota - \tau_{\mathcal{M}}} R\Gamma(\operatorname{Spec} A \times C, \mathcal{N})\right] \otimes_A K_\infty \\
\Big\downarrow \wr\ {\scriptstyle \text{Proposition 6.23}} \\
\operatorname{cone}\left[R\Gamma(\operatorname{Spec} \mathcal{O}_\infty \times C, q^*\mathcal{M}) \xrightarrow{\iota - \tau_{\mathcal{M}}} R\Gamma(\operatorname{Spec} \mathcal{O}_\infty \times C, q^*\mathcal{N})\right] \otimes_{\mathcal{O}_\infty} K_\infty \\
\Big\downarrow \wr\ {\scriptstyle \text{Grothendieck's comparison Theorem}} \\
\operatorname{cone}\left[R\Gamma(\operatorname{Spf} \mathcal{O}_\infty \hat{\times} C, \hat{\mathcal{M}}) \xrightarrow{\iota - \tau_{\mathcal{M}}} R\Gamma(\operatorname{Spf} \mathcal{O}_\infty \hat{\times} C, \hat{\mathcal{N}})\right] \otimes_{\mathcal{O}_\infty} K_\infty \\
\Big\downarrow \wr\ {\scriptstyle \text{(6.23) and Lemma 6.13}} \\
\operatorname{cone}\left[\hat{\mathcal{M}}_\infty \xrightarrow{\mathrm{id} - \tau_M} \hat{\mathcal{N}}_\infty\right] \otimes_{\mathcal{O}_\infty} K_\infty \\
\Big\downarrow \wr\ {\scriptstyle \text{Lemma 6.34}} \\
\dfrac{\hat{\mathcal{N}}_\infty}{(\iota - \tau_{\mathcal{M}})(\hat{\mathcal{M}}_\infty)} \otimes_{\mathcal{O}_\infty} K_\infty \\
\Big\downarrow \wr\ {\scriptstyle \text{Theorem 6.19}} \\
\dfrac{N \otimes_{A\otimes K} K_\infty[\![\mathfrak{j}]\!]}{M \otimes_{A\otimes K} K_\infty[\![\mathfrak{j}]\!]} \\
\Big\downarrow \wr\ {\scriptstyle (\gamma^\infty_{\underline{M}})^{-1}} \\
\dfrac{\mathfrak{p}^+_{\underline{M}} + \mathfrak{q}^+_{\underline{M}}}{\mathfrak{q}^+_{\underline{M}}}
\end{array}$$

Proof of Theorem 6.4. By Proposition 5.33, we have an exact sequence of K_∞-vector spaces:

$$0 \longrightarrow \Lambda(\underline{M})^+ \otimes_A K_\infty \to \frac{\{\xi \in M \otimes_{A\otimes K} K_\infty\langle A\rangle \mid \xi - \tau_M(\tau^*\xi) \in N_A\}}{M_A} \otimes_A K_\infty \longrightarrow \operatorname{Ext}^{1,\mathrm{reg},\infty}_{\mathcal{MM}^{\mathrm{rig}}_{K},A}(\mathbb{1}, \underline{M}) \longrightarrow 0.$$

On the other-hand, by Theorem 5.34, we have an exact sequence of K_∞-vector spaces:

$$0 \longrightarrow \Lambda(\underline{M})^+ \otimes_A K_\infty \longrightarrow \frac{\mathfrak{p}^+_{\underline{M}} + \mathfrak{q}^+_{\underline{M}}}{\mathfrak{q}^+_{\underline{M}}} \longrightarrow \operatorname{Ext}^{1,\mathrm{ha},\infty}_{\mathcal{MHP}^+_{K_\infty}}(\mathbb{1}^+, \mathcal{H}^+(\underline{M})) \longrightarrow 0.$$

Theorem 6.4 follows from the fact that $\rho(\underline{M})$ is an isomorphism. $\square$

Chapter 7

Application to Carlitz tensor powers

This chapter is devoted to the application of the results of Chapter 1 to 6 in the simplest case of $\underline{M}$ being the tensor powers of the Carlitz's motive $\underline{C}^n$ (defined in example 1.5) over the function field of $C = \mathbb{P}^1_{\mathbb{F}}$. We describe explicitly the A-motivic cohomology of $\underline{C}^n$ and obtain, as a consequence, new results on the algebraic relations among values of Carlitz's polylogarithms at K-rational points (Corollary 7.30).

The t-setting

In this chapter, C is the projective line $\mathbb{P}^1_{\mathbb{F}}$ over $\mathbb{F}$ and ∞ is the point of coordinates $[0:1]$. The ring $A = H^0(C \setminus \{\infty\}, \mathcal{O}_C)$ is identified with $\mathbb{F}[t]$, where t^{-1} is a uniformizer in K of $\mathcal{O}_\infty$. Thus, K is identified with $\mathbb{F}(t)$, K_∞ with $\mathbb{F}(\!(t^{-1})\!)$ and $\mathcal{O}_\infty$ with $\mathbb{F}[\![t^{-1}]\!]$. The valuation v_∞ at ∞ corresponds to the opposite of the degree in t. We recall that $\mathbb{C}_\infty$ is the completion of an algebraic closure of K_∞, and we denote $|\cdot|$ a norm on $\mathbb{C}_\infty$ associated to v_∞.

Let L be an A-algebra. To make notations not to heavy and agree with the existing literature, we identify $\mathbb{F}[t] \otimes L$ with $L[t]$, and denote by t the element $t \otimes 1$ and by θ the element $1 \otimes t$. Under these notations, $\mathbb{C}_\infty\langle A\rangle$ is identified with the Tate algebra over $\mathbb{C}_\infty$

$$\mathbb{C}_\infty\langle t\rangle = \left\{ f = \sum_{n=0}^{\infty} a_n t^n \;\middle|\; a_n \in \mathbb{C}_\infty,\ \lim_{n\to\infty} a_n = 0 \right\}.$$

The *Gauss norm* of $f = \sum_{n\geq 0} a_n t^n \in \mathbb{C}_\infty\langle t\rangle$ is given by $\|f\| := \max_{n\geq 0}\{|a_n|\}$ (see [GazMa, §2] for a general construction of the Gauss norm). We let $f \mapsto f^{(1)}$ be the map on $\mathbb{C}_\infty\langle t\rangle$ which raises the coefficients to the q-th power:

$$f^{(1)} = \sum_{n=0}^{\infty} a_n^q t^n.$$

(it corresponds to τ over $\mathbb{C}_\infty\langle A\rangle$), and let $f \mapsto f^{(i)}$ denote its i-th iterates. We denote by $\mathbb{C}_\infty\langle\langle t\rangle\rangle$ the sub-$\mathbb{C}_\infty$-algebra of $\mathbb{C}_\infty\langle t\rangle$ which have infinite radius of convergence:

$$\mathbb{C}_\infty\langle\langle t\rangle\rangle = \left\{ f = \sum_{n=0}^{\infty} a_n t^n \;\middle|\; a_n \in \mathbb{C}_\infty,\ \forall \rho > 0,\ \lim_{n\to\infty} a_n\rho^n = 0 \right\}.$$

It corresponds to the algebra $\mathbb{C}_\infty\langle\langle A\rangle\rangle$ introduced in subsection 1.4.2 (see Example 1.62). In this setting, $\mathbb{C}_\infty\langle\langle A\rangle\rangle_{\mathrm{j}}$ (Definition 1.63) corresponds to the algebra of elements $g \in \operatorname{Quot}\mathbb{C}_\infty\langle\langle t\rangle\rangle$ such that g is regular outside $\{\theta, \theta^q, ...\}$ in $\mathbb{C}_\infty$ and for which there exists $n > 0$ in such a way that g has a pole of order at most n at the elements $\{\theta, \theta^q, ...\}$.

For n an integer, let $\underline{C}^n = (K[t], \tau_{C^n})$ be the nth tensor power of the Carlitz $\mathbb{F}[t]$-motive over K (Example 1.5), where τ_{C^n} maps $\tau^*p(t)$ to $(t-\theta)^n p(t)^{(1)}$. It defines an A-motive over K. Let also $\underline{A}(n) := \underline{C}^{-n} = (\underline{C}^n)^\vee$. This is the function fields analogue of the nth Tate twist $\mathbb{Z}(n)$.

Results

After showing in Section 7.1 that $\underline{C}^n$ (respectively $\underline{A}(n)$) is an object of $\mathcal{MM}_K^{\mathrm{reg}}$, we compute in Section 7.2 the modules $\operatorname{Ext}^{1,\mathrm{reg},\infty}_{\mathcal{MM}_K,A}(\mathbb{1}, \underline{A}(n))$ for various values of n. Our results can be summarized as follows:

Theorem 7.1. *There are isomorphisms of $\mathbb{F}[t]$-modules*

$$\operatorname{Ext}^{1,\infty}_{\mathcal{MM}_K^{reg},A}(\mathbb{1}, \underline{A}(n)) \cong \begin{cases} 0 & \text{if } n \le 0, \\ \mathbb{F}[t]^n & \text{if } n > 0 \text{ and } q-1 \not| n, \\ \mathbb{F}[t]^{n-1} \oplus \mathbb{F}[t]/(d_n(t)) & \text{if } n > 0 \text{ and } q-1 | n, \end{cases}$$

where $d_n(t)$ is a certain monic polynomial in $\mathbb{F}[t]$.

We refer respectively to Proposition 7.12, Lemma 7.15 and Proposition 7.19 for Theorem 7.1.

We prove that $d_n(t) = t^{q^{k+1}} - t^{q^k}$ whenever $n = q^k(q-1)$ for some $k \ge 0$ (Corollary 7.25). Yet, an explicit expression of $d_n(t)$ is rather difficult to obtain in practice. For general values of n multiple of $q-1$, we show that $d_n(t)$ is described by the linear relations among the *Carlitz period* and values of *Carlitz polylogarithms* at polynomials (Theorem 7.23).

For $\alpha \in K$ whose degree is $< nq/q-1$, we let $\operatorname{Li}_n(\alpha)$ be its *n-th Cartliz polylogarithm*, defined by the converging series in K_∞:

$$\operatorname{Li}_n(\alpha) := \alpha + \sum_{k=1}^{\infty} \frac{\alpha^{q^k}}{(\theta-\theta^q)^n(\theta-\theta^{q^2})^n\cdots(\theta-\theta^{q^k})^n}.$$

Fix η a $(q-1)$st root of $(-\theta)$ in K_∞^s. Let $\tilde{\pi}$ be the *Carlitz period*[1], given by the converging product in the separable closure K_∞^s of K_∞ in $\mathbb{C}_\infty$:

$$\tilde{\pi} = (-\theta\eta)^{-1} \prod_{i=1}^{\infty} \left(1 - \frac{\theta}{\theta^{q^i}}\right)^{-1}. \tag{7.1}$$

An unexpected consequence of our results is the intimate relation between the module $\operatorname{Ext}^{1,\mathrm{reg},\infty}_{\mathcal{MM}_K,A}(\mathbb{1}, \underline{A}(n))$ and the module of $\mathbb{F}[\theta]$-linear relations between $\tilde{\pi}^n$ and values of polylogarithms at polynomials (see Proposition 7.24). Using the ABP criterion [ABP, Thm. 3.1.1] combined with methods of Chang-Yu [ChaYu], we prove that the series $\mathrm{Li}_n(1)$, ..., $\mathrm{Li}_n(\theta^{n-1})$ are algebraically independent over $\bar{K}$ (Corollary 7.30). If $q-1 \nmid n$, we can further deduce that $(\tilde{\pi}, \mathrm{Li}_n(1), ..., \mathrm{Li}_n(\theta^{n-1}))$ forms an algebraically independent family over $\bar{K}$ (Corollary 7.28). More generally, we provide a criterion to spot algebraic independence among the series $\mathrm{Li}_n(\alpha)$ for $\alpha \in K$. The case of finding algebraic relations among $\mathrm{Li}_n(\alpha)$ for $\alpha \in F$ for a finite field extension F of K in $\mathbb{C}_\infty$ seems to require more investigations. We leave it open.

We end this text by Section 7.4, where we give an equivalent formulation of Beilinson's conjecture (6.5) in the case of $\underline{M} = \underline{A}(n)$ $(n > 0)$ by mean of *higher* polylogarithms. It will follow that Beilinson's conjecture is true for $\underline{A}(1)$, but false for $\underline{A}(n)$ whenever n is a multiple of the characteristic p of $\mathbb{F}$.

7.1 Carlitz's tensor powers' tool box

7.1.1 Properties of the Carlitz's tensor powers

We begin by some general and well-known facts on the Carlitz's tensor powers (mixedness, Betti realization, maximal A-model, mixed Hodge-Pink structure). Our aim is to prove that $\underline{A}(n)$ is an object of the category $\mathcal{MM}_K^{\mathrm{reg}}$, so that the module of extensions

$$\operatorname{Ext}^1_{\mathcal{MM}_K^{\mathrm{reg}}}(\mathbb{1}, \underline{A}(n)),$$

to be studied in the incoming sections, is well-defined.

Mixedness

We have the identifications $\mathcal{A}_\infty(K) = K[\![t^{-1}]\!]$ and $\mathcal{B}_\infty(K) = K(\!(t^{-1})\!)$. The isocrystal $\mathcal{I}_\infty(\underline{A}(n))$ (see Definition 1.28) admits $\mathcal{B}_\infty(K) = K(\!(t^{-1})\!)$ as underlying $K(\!(t^{-1})\!)$-module, and the τ-linear isomorphism acts by multiplication by $(t-\theta)^n$.

Proposition 7.2. *The isocrystal $\mathcal{I}_\infty(\underline{A}(n))$ is pure of slope n (Definition 1.12). In particular, $\underline{A}(n)$ is pure of weight $-n$ (Definition 1.30) and hence is mixed.*

[1] $\tilde{\pi}$ depends on η as $2\pi i$ depends on the choice of i, a square root of -1.

Proof. We have $(t-\theta)^n = t^{-n}(1-\theta/t)^{-n}$. Because $(1-\theta/t)$ is a unit in $\mathcal{A}_\infty(K) = K[\![t^{-1}]\!]$ and t^{-1} a uniformizer, we have

$$\tau_{A(n)}(\tau^*\mathcal{A}_\infty(K)) = (t^{-1})^n\mathcal{A}_\infty(K).$$

Since $\mathcal{A}_\infty(K)$ is an $\mathcal{A}_\infty(K)$-lattice in $\mathcal{B}_\infty(K)$, $\mathcal{I}_\infty(\underline{A}(n))$ is pure of slope n. The fact that $\underline{A}(n)$ is pure of weight $-n$ follows by definition. □

Rigid analytic triviality

Let $\omega(t)$ be the Anderson-Thakur's generating function:

$$\omega(t) = \eta\prod_{i=0}^{\infty}\left(1-\frac{t}{\theta^{q^i}}\right)^{-1} \in K_\infty(\eta)\langle t\rangle,$$

(see [AndT, Proof of Lemma 2.5.4]). The following proposition is well-known:

Proposition 7.3. *We have $\Lambda(\underline{A}(n)) = \mathbb{F}[t]\omega(t)^n$. In particular, $\underline{A}(n)$ is rigid analytically trivial.*

Proof. Note that, by Definition 1.48, we have

$$\Lambda(\underline{A}(n)) = \{f \in \mathbb{C}_\infty\langle t\rangle \mid f^{(1)} = (t-\theta)^n f\}.$$

The series ω satisfies $\omega(t)^{(1)} = (t-\theta)\omega(t)$ and it follows that $\mathbb{F}[t]\omega(t)^n \subset \Lambda(\underline{A}(n))$. Conversely, ω is invertible in $\mathbb{C}_\infty\langle t\rangle$ so that, if $f \in \Lambda(\underline{A}(n))$, then the element $g := f/\omega^n$ satisfies $g^{(1)} = g \in \mathbb{C}_\infty\langle t\rangle$. Hence, $g \in \mathbb{F}[t]$ and $f \in \mathbb{F}[t]\omega(t)^n$. We conclude by Proposition 1.52 that $\underline{A}(n)$ is rigid analytically trivial. □

We recall that $\Lambda(\underline{A}(n))$ is equipped with a continuous action of $G_\infty = \operatorname{Gal}(K_\infty^s|K_\infty)$. It is given as follows: from Proposition 1.54, we know that any $f \in \Lambda(\underline{A}(n))$ belongs to $K_\infty^s\langle t\rangle$ (this also follows from Proposition 7.3). Given $\sigma \in G_\infty$, it acts on f via

$$f = \sum_{n=0}^{\infty} a_n t^n, \quad f^\sigma = \sum_{n=0}^{\infty} a_n^\sigma t^n.$$

We let $\Lambda(\underline{A}(n))^+$ be the submodule of $\Lambda(\underline{A}(n))$ fixed by G_∞. Let $\mathbf{1}_{q-1|n}$ be 1 if $q-1|n$ and 0 otherwise. From Proposition 7.3, we obtain:

Corollary 7.4. *We have $\Lambda(\underline{A}(n))^+ = \mathbf{1}_{q-1|n}\mathbb{F}[t]\omega(t)^n$ (here, $\mathbf{1}_{q-1|n}$ equals 1 if $q-1 \mid n$ and is zero otherwise).*

Maximal A-model

Let M_A denote the maximal A-model of $\underline{A}(n)$, which we know to exist by Proposition 2.30.

Proposition 7.5. *We have $M_A = \mathbb{F}[\theta, t]$.*

Proof. If $L = \mathbb{F}[\theta,t] \subset K[t]$, then clearly $\tau_{A(n)}(\tau^*L)[\mathfrak{j}^{-1}] = L[\mathfrak{j}^{-1}]$ as $\mathfrak{j}$ corresponds to the principal ideal $(t-\theta)$ of $\mathbb{F}[\theta,t]$. It follows that L is the maximal A-model of $\underline{A}(n)$ by Proposition 2.36. □

Mixed Hodge-Pink structure

To end this section, we compute the mixed Hodge-Pink structure associated to $\underline{A}(n)$ (Section 5.1). Note that we have an identification $K_\infty^s[\![\mathfrak{j}]\!] = K_\infty^s[\![t-\theta]\!]$. The isomorphism $\gamma_{\underline{A}(n)}^\infty$ of (5.3) takes the form

$$\gamma_{\underline{A}(n)}^\infty : K_\infty^s(\!(t-\theta)\!) \cdot \omega(t)^n \xrightarrow{\sim} K_\infty^s(\!(t-\theta)\!), \quad f \cdot \omega(t)^n \longmapsto f\omega(t)^n.$$

It follows that $\mathfrak{p}_{\underline{A}(n)} = K_\infty^s[\![t-\theta]\!] \cdot \omega(t)^n$ and $\mathfrak{q}_{\underline{A}(n)} = (t-\theta)^n K_\infty^s[\![t-\theta]\!] \cdot \omega(t)^n$ (see Definition 5.3). According to Definition 5.3 and the above computations, we obtain:

Proposition 7.6. *Let R be a Noetherian subring of K_∞ containing $\mathbb{F}[t]$. The mixed Hodge-Pink structure $\mathscr{H}_R^+(\underline{A}(n))$ has underlying module $R \cdot \omega(t)^n$, is pure of weight $-n$, has Hodge-Pink lattice $\mathfrak{q}_{\underline{A}(n)} = (t-\theta)^n K_\infty^s[\![t-\theta]\!] \cdot \omega(t)^n$, and its infinite Frobenius maps $\sigma \in G_\infty$ to $(\omega \mapsto \omega^\sigma)$.*

As an application of Corollary 4.46, it follows that $\underline{A}(n)$ is regulated (Definition 5.10). As announced, we obtain:

Corollary 7.7. *$\underline{A}(n)$ is an object of the category $\mathcal{MM}_K^{reg}$.*

7.1.2 Regulators

In view of what has been established in the subsection 7.1.1, the module of extension:

$$\operatorname{Ext}^1_{\mathcal{MM}_K^{\mathrm{reg}}}(\mathbb{1}, \underline{A}(n))$$

is well-defined. We assume that $n > 0$ so that $\underline{A}(n)$ has negative weights. In this subsection, we compute the general and special regulators of $\underline{A}(n)$ at *Poylog-classes* (Definition 7.8) and show how they are related to Carlitz polylogarithms (Proposition 7.10). We will deduce algebraic dependence among values of polylogarithms thanks to this description (see Section 7.3).

Let ι be the isomorphism of Theorem 3.4 in the case $\underline{M} = \underline{A}(n)$. By Corollary 5.20, ι induces an isomorphism of $\mathbb{F}[t]$-modules:

$$\iota : \frac{(t-\theta)^{-n} K[t]}{\{p(t) - (t-\theta)^{-n} p(t)^{(1)} \mid p(t) \in K[t]\}} \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{MM}_K^{\mathrm{reg}}}(\mathbb{1}, \underline{A}(n)).$$

Definition 7.8. For $\alpha \in K$, we define the *nth Polylog-class of* α, and denote it by $[\underline{L}_n(\alpha)]$, the image of the class of $\dfrac{\alpha}{(t-\theta)^n}$ through ι.

Remark 7.9. The terminology will be justified by Proposition 7.10 below.

Recall that in Section 5.3 we have defined a submodule

$$\operatorname{Ext}^{1,\infty}_{\mathcal{MM}_K^{\mathrm{reg}}}(\mathbb{1}, \underline{A}(n)) \subset \operatorname{Ext}^1_{\mathcal{MM}_K^{\mathrm{reg}}}(\mathbb{1}, \underline{A}(n))$$

consisting in extensions in $\mathcal{MM}^{\mathrm{reg}}_K$ having analytic reduction at ∞ (Definition 5.26). For R a Noetherian subring of K_∞ which contains A, let $\mathscr{Reg}^{\infty}_R(\underline{A}(n))$ and $\mathrm{Reg}^{\infty}_R(\underline{A}(n))$ be respectively the general and special ∞-regulators of $\underline{A}(n)$ (Definitions 5.21 and 5.23). We rewrite the commutative diagram of Theorem 5.34 in the t-setting with $\underline{M} = \underline{A}(n)$:

$$\begin{tikzcd}
\mathrm{Ext}^{1,\infty}_{\mathcal{MM}^{\mathrm{reg}}_K,A}(\mathbb{1},\underline{A}(n)) \arrow[d, "\mathscr{Reg}^{\infty}_R(\underline{A}(n))"'] \arrow[dd, bend right=60, "\mathrm{Reg}^{\infty}_R(\underline{A}(n))"'] & \dfrac{\left\{\xi \in K_\infty\langle t\rangle \mid (t-\theta)^n\xi - \xi^{(1)} \in \mathbb{F}[\theta,t]\right\}}{\mathbf{1}_{q-1|n}\mathbb{F}[t]\omega(t)^n + \mathbb{F}[\theta,t]} \arrow[l, "(A)"', "\sim"] \arrow[d, "\xi \mapsto \xi"] \\
\mathrm{Ext}^{1,\infty}_{\mathcal{MHP}^{\mathrm{ha},+}_R}(\mathbb{1}^+,\mathscr{H}^+_R(\underline{A}(n))) \arrow[d, "{[\underline{H}] \mapsto [\underline{H}^{\#}]}"'] & \dfrac{(t-\theta)^{-n}K_\infty[\![t-\theta]\!]}{\mathbf{1}_{q-1|n}R\omega(t)^n + K_\infty[\![t-\theta]\!]} \arrow[l, "(B)"', "\sim"] \arrow[d, "\xi \mapsto -\omega(t)^n\otimes(\xi\omega^{-n})|_{t=\theta}"] \\
\mathrm{Ext}^{1,\infty}_{\mathcal{MH}^+_R}(\mathbb{1}^+,\mathcal{H}^+_R(\underline{A}(n))) & \dfrac{(\mathbb{F}[t]\omega(t)^n \otimes_{\mathbb{F}[t]} K^s_\infty)^+}{\mathbf{1}_{q-1|n}R\omega(t)^n} \arrow[l, "(C)"', "\sim"]
\end{tikzcd}$$

We labeled the horizontal isomorphisms: (A), (B), (C). We recall that (A) is given explicitly by mapping the class of ξ to $\iota(\xi - (t-\theta)^{-n}\xi^{(1)})$ (see Proposition 5.33).

For $\alpha \in K$ such that $\deg(\alpha) < nq/(q-1)$ as a polynomial, we consider the series

$$\xi_\alpha(t) := \frac{\alpha}{(t-\theta)^n} + \sum_{k=1}^{\infty} \frac{\alpha^{q^k}}{(t-\theta)^n(t-\theta^q)^n\cdots(t-\theta^{q^k})^n} \tag{7.2}$$

converging both in $K_\infty\langle t\rangle$ and in $(t-\theta)^{-n}K_\infty[\![t-\theta]\!]$. We next show how the general regulator $\mathscr{Reg}^{\infty}_R(\underline{A}(n))$ is related to ξ_α and how $\mathrm{Reg}^{\infty}_R(\underline{A}(n))$ is related to the Carlitz polylogarithm $\mathrm{Li}_n(\alpha)$.

Proposition 7.10. *Let $\alpha \in K$ be such that $\deg\alpha < nq/(q-1)$ and let $[\underline{L}_n(\alpha)]$ be its nth Polylog-class. Then $[\underline{L}_n(\alpha)]$ belongs to $\mathrm{Ext}^{1,\infty}_{\mathcal{MM}^{reg}_K}(\mathbb{1},\underline{A}(n))$. In addition,*

1. *$\mathscr{Reg}^{\infty}_R(\underline{A}(n))$ maps the extension $[\underline{L}_n(\alpha)]$ to the image of the class of $\xi_\alpha(t) \in (t-\theta)^{-n}K_\infty[\![t-\theta]\!]$ through (B).*
2. *$\mathrm{Reg}^{\infty}_R(\underline{A}(n))$ maps the extension $[\underline{L}_n(\alpha)]$ to the image of the class of $\omega(t)^n \otimes \frac{\mathrm{Li}_n(\alpha)}{\tilde{\pi}^n} \in (\mathbb{F}[t]\omega(t)^n \otimes_{\mathbb{F}[t]} K^s_\infty)^+$ through (C).*

Proof. As a converging series in $K_\infty\langle t\rangle$, ξ_α satisfies the functional equation:

$$\xi_\alpha(t) - \frac{\xi_\alpha(t)^{(1)}}{(t-\theta)^n} = \frac{\alpha}{(t-\theta)^n}. \tag{7.3}$$

Because $[\underline{L}_n(\alpha)] = \iota((t-\theta)^{-n}\alpha)$, it follows from Proposition 5.27 and the fact that ξ_α is invariant through the action of G_∞ on $K_\infty^s\langle t\rangle$, that $[\underline{L}_n(\alpha)]$ has analytic reduction at ∞. Hence

$$[\underline{L}_n(\alpha)] \in \operatorname{Ext}^{1,\infty}_{\mathcal{MM}_K^{\mathrm{reg}}}(\mathbb{1}, \underline{A}(n)).$$

The inverse image of $[\underline{L}_n(\alpha)]$ via (A) is then well-defined and equals the class of $\xi_\alpha(t)$ by (7.3). By commutativity of the above diagram (Theorem 5.34), the first assertion follows. Note that, from (7.2), $\xi(t)\omega(t)^{-n}$ is regular at $t=\theta$ and $(\xi(t)\omega(t)^{-n})|_{t=\theta} = -\operatorname{Li}_n(\alpha)\tilde{\pi}^{-n}$. The second assertion follows. $\square$

Remark 7.11. Given $\alpha \in A$, one easily sees that $[\underline{L}_n(\alpha)]$ has everywhere good reduction.

7.2 Motivic cohomology of Carlitz's tensor powers

Let n be an integer. The aim of this section is to describe the submodule of extensions

$$\operatorname{Ext}^{1,\infty}_{\mathcal{MM}_K^{\mathrm{reg}},A}(\mathbb{1}, \underline{A}(n))$$

having everywhere good reduction (in addition to having analytic reduction at ∞). Subsection 7.2.1 deals with the $n \leq 0$ case and Subsection 7.2.2 the case $n > 0$.

7.2.1 Non positive tensor powers

We begin by a description of the extension modules of $\mathbb{1}$ by $\underline{A}(n)$ in various categories of A-motives, for general values of n.

In the category $\mathcal{M}_K$

Let $\mathcal{M}_K$ be the category of A-motives over K. By Theorem 3.4, there is an isomorphism of A-modules

$$\iota : \frac{K[t][(t-\theta)^{-1}]}{\{p(t) - (t-\theta)^{-n}p(t)^{(1)} \mid p(t) \in K[t]\}} \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{M}_K}(\mathbb{1}, \underline{A}(n))$$

given explicitly by mapping the class of m in $K[t][(t-\theta)^{-1}]$ to the extension of $\mathbb{1}$ by $\underline{A}(n)$ whose underlying module is $K[t]^{\oplus 2}$ and where the τ-action takes the form $\left(\begin{smallmatrix} \tau_{C^n} & m \\ 0 & 1 \end{smallmatrix}\right)$. By Theorem 3.18 combined with the description of the maximal A-model of $\underline{A}(n)$ in Proposition 7.5, we deduce that ι induces an isomorphism

$$\iota : \frac{\mathbb{F}[\theta,t][(t-\theta)^{-1}]}{\{p(t) - (t-\theta)^{-n}p(t)^{(1)} \mid p(t) \in \mathbb{F}[\theta,t]\}} \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{M}_K,A}(\mathbb{1}, \underline{A}(n))$$

where $\operatorname{Ext}^1_{\mathcal{M}_K,A}$ denotes the submodule of extensions having everywhere good reduction.

In the category $\mathcal{MM}_K$

Let $\mathcal{MM}_K$ be the category of mixed A-motives over K. If $n \geq 0$, we obtain by Propositions 3.8 and Proposition 3.11 the equality

$$\mathrm{Ext}^1_{\mathcal{MM}_K}(\mathbb{1}, \underline{A}(n)) = \mathrm{Ext}^1_{\mathcal{M}_K}(\mathbb{1}, \underline{A}(n)).$$

On the contrary, if $n < 0$, then by Proposition 3.9, we have an equality:

$$\mathrm{Ext}^1_{\mathcal{M}_K}(\mathbb{1}, \underline{A}(n))^{\mathrm{tors}} = \mathrm{Ext}^1_{\mathcal{MM}_K}(\mathbb{1}, \underline{A}(n))$$

and $\mathrm{Ext}^1_{\mathcal{MM}_K}(\mathbb{1}, \underline{A}(n))$, for $n < 0$, is torsion. However we do not know whether it is zero.

In the category $\mathcal{MM}_K^{\mathbf{reg}}$

Let $\mathcal{MM}_K^{\mathrm{reg}}$ be the category of mixed, rigid analytically trivial and regulated A-motives over K (Definition 5.10). Let $\mathrm{Ext}^1_{\mathcal{MM}_K^{\mathrm{reg}},A}(\mathbb{1}, \underline{A}(n))$ be the submodule of extensions having everywhere good reduction in the category $\mathcal{MM}_K^{\mathrm{reg}}$ (Definition 3.17). We claim:

Proposition 7.12. *Let $n < 0$. We have* $\mathrm{Ext}^1_{\mathcal{MM}_K^{reg},A}(\mathbb{1}, \underline{A}(n)) = 0$.

Proof. Let $m = -n > 0$. In vertue of Corollary 5.20, it suffices to show that the $\mathbb{F}[t]$-module

$$\frac{\mathbb{F}[\theta, t]}{\{p(t) - (t-\theta)^m p(t)^{(1)} \mid p(t) \in \mathbb{F}[\theta, t]\}}$$

is torsion-free. Equivalently, that if there exists $a(t) \in \mathbb{F}[t]$ and $p(t) \in \mathbb{F}[\theta, t]$ such that $a(t)p(t) \in (\mathrm{id} - (t-\theta)^m \tau)(\mathbb{F}[\theta, t])$, then $p(t) \in (\mathrm{id} - (t-\theta)^m \tau)(\mathbb{F}[\theta, t])$. Let us assume without loss of generality that $a(t) \neq 0$. Let $q(t) \in \mathbb{F}[\theta, t]$ be such that $a(t)p(t) = q(t) - (t-\theta)^m q(t)^{(1)}$. We want to show that $a(t)$ divides $q(t)$.
For all $N \geq 1$, setting $b_i = (t-\theta)\cdots(t-\theta^{q^{i-1}})$, we have

$$q(t) = a(t) \sum_{i=0}^{N-1} b_i^m p(t)^{(i)} + b_N^m q(t)^{(N)}.$$

Hence, for all $j \geq 0$, there exists a polynomial $H_j(t) \in \mathbb{F}[\theta, t]$ such that

$$q(\theta^{q^j}) = a(\theta^{q^j}) H_j(\theta^{q^j}).$$

By Euclidean division we have, on the other-hand

$$q(t) = a(t)H(t) + r(t), \quad \text{where } H(t),\ r(t) \in \mathbb{F}[\theta, t]$$

and $\rho_1 := \deg_t r < \deg_t a =: \alpha$ (the euclidean division works over $\mathbb{F}[\theta]$ since the leading coefficient of $a(t)$ divides the leading coefficient of $q(t)$). For all j we get

$$r(\theta^{q^j}) = a(\theta^{q^j})(H_j(\theta^{q^j}) - H(\theta^{q^j})).$$

If $r \neq 0$, then for j large enough $v_\infty(r(\theta^{q^j}))$ equals $\rho_0 - \rho_1 q^j$ where ρ_0 is the valuation of the leading coefficient of r. Comparing with the valuation on the right-hand side, we obtain:

$$\delta_j = v_\infty(H_j(\theta^{q^j}) - H(\theta^{q^j})) = \rho_0 - \rho_1 q^j + \alpha q^j.$$

Hence, δ_j tends to infinity as j grows. But this contradicts $H_j(\theta^{q^j}) - H(\theta^{q^j}) \in \mathbb{F}[\theta]$. Thus, $r = 0$. □

In the case $n = 0$, we prove:

Proposition 7.13. *There are isomorphisms of A-modules*

$$\frac{\mathbb{F}[\theta][t]}{(1-\tau)(\mathbb{F}[\theta])[t]} \cong \operatorname{Ext}^1_{\mathcal{MM}_K^{reg},A}(\mathbb{1},\mathbb{1}) \cong \operatorname{Hom}^c_{grp}(G_\infty, A),$$

where $\operatorname{Hom}^c_{grp}(G_\infty, A)$ is the A-module of continuous group morphisms from G_∞ (equipped with profinite topology) to $(A,+)$ (equipped with the discrete topology).

Let $U(\mathbb{1})$ be the class module of $\mathbb{1}$ (Definition 6.1). We begin by two lemmas that might interest the reader beyond the proof of the proposition.

Lemma 7.14. *The class module $U(\mathbb{1})$ is zero.*

Proof. By Remark 6.1, we have

$$U(\mathbb{1}) \cong K_\infty\langle t\rangle / (\mathbb{F}[\theta,t] + \{g - g^{(1)} | g \in K_\infty\langle t\rangle\}).$$

We have $K_\infty = \mathfrak{m}_\infty + \mathbb{F}[\theta]$, where $\mathfrak{m}_\infty = \theta^{-1}\mathbb{F}[\![\theta^{-1}]\!]$ is the maximal ideal of the local ring $\mathcal{O}_\infty = \mathbb{F}[\![\theta^{-1}]\!]$ in the local field $K_\infty = \mathbb{F}(\!(\theta^{-1})\!)$. This implies that, for any

$$f = \sum_{n=0}^{\infty} f_n t^n \in K_\infty\langle t\rangle,$$

up to an element of $\mathbb{F}[\theta,t]$, we can assume that $f_n \in \mathfrak{m}_\infty$ for all $n \geq 0$. In particular, the series

$$g_n := f_n + f_n^q + f_n^{q^2} + \ldots$$

converges in K_∞ and satisfies $v_\infty(g_n) = v_\infty(f_n)$. Hence, $g := \sum_n g_n t^n$ is an element of $K_\infty\langle t\rangle$ such that $g - g^{(1)} = f$. We conclude that $U(\mathbb{1}) = 0$. □

Lemma 7.15. *We have* $\operatorname{Ext}^{1,\infty}_{\mathcal{MM}_K^{reg},A}(\mathbb{1},\mathbb{1}) = 0$.

Proof. Let $\xi = \sum_{i\geq 0} c_i t^i \in K_\infty\langle t\rangle$ be such that $\xi - \xi^{(1)} \in \mathbb{F}[\theta,t]$. For i large enough $c_i \in \mathbb{F}$, and then $c_i = 0$. We obtain $\xi \in K_\infty[t]$. For all $i \geq 0$, we have

$$c_i - c_i^q = x_i$$

for a certain $x_i \in \mathbb{F}[\theta]$. By Lemma 3.21, $K(c_i)$ is unramified at closed points of $\mathbb{P}^1_{\mathbb{F}}$ distinct from ∞. But since $c_i \in K_\infty$, $K(c_i)$ is not ramified at ∞ neither.

Hence $K(c_i)$ is nowhere ramified and we deduce $K = K(c_i)$ (that is $c_i \in K$). This amounts to $\xi \in K[t]$ and then $\xi \in \mathbb{F}[\theta, t]$ by considering valuations. It follows that

$$\frac{\{\xi \in K_\infty\langle t\rangle \mid \xi - \xi^{(1)} \in \mathbb{F}[\theta, t]\}}{\mathbb{F}[\theta, t]} = 0.$$

We conclude by Proposition 5.33. □

Proof of Proposition 7.13. The first isomorphism is induced by ι (Corollary 5.20). For the second one, note that

$$r_{\mathbb{1},\infty} : \mathrm{Ext}^1_{\mathcal{MM}_K^{\mathrm{reg}},A}(\mathbb{1}, \mathbb{1}) \longrightarrow H^1(G_\infty, \Lambda(\mathbb{1}))$$

has kernel $\mathrm{Ext}^{1,\mathrm{reg},\infty}_{\mathcal{MM}_K,A}(\mathbb{1}, \mathbb{1})$ and cokernel $U(\mathbb{1})$. Both are zero by Lemmas 7.14 and 7.15. It follows that $r_{\mathbb{1},\infty}$ is an isomorphism. As $\Lambda(\mathbb{1})$ is isomorphic to A with trivial G_∞-action, we have $H^1(G_\infty, \Lambda(\mathbb{1})) \cong \mathrm{Hom}^c_{\mathrm{grp}}(G_\infty, A)$ as desired. □

Remark 7.16. The fact that $\mathrm{Ext}^1_{\mathcal{MM}_K^{\mathrm{reg}}}(\mathbb{1}, \mathbb{1}) \neq 0$ is in opposition to what is expected in the number fields setting [Del, 1.3 (c)].

7.2.2 Positive tensor powers

In view of Proposition 7.12 and Lemma 7.15, we have proved that

$$\mathrm{Ext}^{1,\infty}_{\mathcal{MM}_K^{\mathrm{reg}},A}(\mathbb{1}, \underline{A}(n)) = 0$$

for $n \leq 0$. Our next objective is to consider the case $n > 0$. Because $\underline{C}^n$ has negative weight, we know by Theorem 6.2 that $\mathrm{Ext}^{1,\infty}_{\mathcal{MM}_K^{\mathrm{reg}},A}(\mathbb{1}, \underline{A}(n))$ is a finitely generated $\mathbb{F}[t]$-module. Up to isomorphism, it is then determined by its rank and its torsion submodule.

We begin with a key lemma.

Lemma 7.17. *Let $n > 0$ and let $(\alpha_1, ..., \alpha_s)$ be a generating family of $\{\alpha \in \mathbb{F}[\theta] | \deg \alpha < n\}$ over $\mathbb{F}$. The $\mathbb{F}[t]$-module*

$$\frac{\left\{\xi \in K_\infty\langle t\rangle \mid (t-\theta)^n\xi - \xi^{(1)} \in \mathbb{F}[\theta, t]\right\}}{\Lambda(\underline{A}(n))^+ + \mathbb{F}[\theta, t]}$$

admits $(\xi_{\alpha_1}(t), ..., \xi_{\alpha_s}(t))$ as generators.

Proof. Let $\xi \in K_\infty\langle t\rangle$ be such that $(t-\theta)^n\xi - \xi^{(1)} \in \mathbb{F}[\theta, t]$. Because $K_\infty = \mathbb{F}[\theta] + \mathfrak{m}_\infty$, up to an element of $\mathbb{F}[\theta, t]$, one can assume that the coefficients of ξ are in $\mathfrak{m}_\infty$, that is $\|\xi\| < 1$. If $m \in \mathbb{F}[\theta, t]$ is such that

$$(t-\theta)^n\xi - \xi^{(1)} = m,$$

we have $\|m\| < q^n$. Therefore, we can write m as a sum

$$m = \sum_{i=1}^{s} a_i(t)\alpha_i, \quad (\forall\ 0 \leq i \leq s\ :\ a_i(t) \in \mathbb{F}[t])$$

and we find $\xi - \sum_{i=1}^{s} a_i(t)\xi_{\alpha_i} \in \Lambda(\underline{A}(n))^+$. This concludes. $\square$

We immediately deduce:

Proposition 7.18. *Let $n > 0$ and let $(\alpha_1, ..., \alpha_s)$ be a generating family of the $\mathbb{F}$-vector space $\{\alpha \in \mathbb{F}[\theta] | \deg \alpha < n\}$. Then $([\underline{L}_n(\alpha_i)] \mid 1 \leq i \leq s)$ forms a generating family of the $\mathbb{F}[t]$-module $\operatorname{Ext}^{1,\infty}_{\mathcal{MM}^{reg}_K,A}(\mathbb{1}, \underline{A}(n))$.*

Proof. By Proposition 5.33, the map $\xi \to \iota(\xi - \tau_{C^n}(\tau^*\xi))$ is an isomorphism of $\mathbb{F}[t]$-modules:

$$\frac{\left\{\xi \in K_\infty\langle t\rangle \mid (t-\theta)^n\xi - \xi^{(1)} \in \mathbb{F}[\theta, t]\right\}}{\mathbb{F}[\theta, t] + \Lambda(\underline{A}(n))^+} \xrightarrow{\sim} \operatorname{Ext}^{1,\infty}_{\mathcal{MM}^{\text{reg}}_K,A}(\mathbb{1}, \underline{A}(n)) \tag{7.4}$$

which maps the class of $\xi_\alpha(t)$ to the class $[\underline{L}_n(\alpha)]$, for $\alpha \in K$ of degree $< nq/(q-1)$. We conclude by Lemma 7.17. $\square$

We can already answers the question of the rank of $\operatorname{Ext}^{1,\infty}_{\mathcal{MM}^{\text{reg}}_K,A}(\mathbb{1}, \underline{A}(n))$ for $n > 0$, and discuss its torsion submodule in the case $q-1 \nmid n$.

Proposition 7.19. *For $n > 0$ and $q-1 \nmid n$, the $\mathbb{F}[t]$-module $\operatorname{Ext}^{1,\infty}_{\mathcal{MM}^{reg}_K,A}(\mathbb{1}, \underline{A}(n))$ is free of rank n. If $q-1|n$, then $\operatorname{Ext}^{1,\infty}_{\mathcal{MM}^{reg}_K,A}(\mathbb{1}, \underline{A}(n))$ has rank $n-1$.*

Proof. We have an exact sequence of K_∞-vector spaces:

$$\Lambda(\underline{A}(n))^+_{K_\infty} \hookrightarrow \frac{(t-\theta)^{-n}K_\infty[\![t-\theta]\!]}{K_\infty[\![t-\theta]\!]} \twoheadrightarrow \operatorname{Ext}^{1,\text{ha},\infty}_{\mathcal{MHP}^+_{K_\infty}}(\mathbb{1}^+, \mathscr{H}^+_{K_\infty}(\underline{A}(n))). \tag{7.5}$$

Using the additivity of the dimension in short exact sequences, we find that $\operatorname{Ext}^{1,\text{ha},\infty}_{\mathcal{MHP}^+_{K_\infty}}(\mathbb{1}^+, \mathscr{H}^+_{K_\infty}(\underline{A}(n)))$ has dimension n if $q-1 \nmid n$ and dimension $n-1$ if $q-1|n$. By Theorem 6.4, the same is true for the rank of $\operatorname{Ext}^{1,\infty}_{\mathcal{MM}^{\text{reg}}_K}(\mathbb{1}, \underline{A}(n))$. If $q-1 \nmid n$, then the latter module has rank n and admits a family of generators with exactly n elements (Proposition 7.18). We deduce that it is free. $\square$

We now describe the torsion submodule of $\operatorname{Ext}^{1,\infty}_{\mathcal{MM}^{\text{reg}}_K}(\mathbb{1}, \underline{A}(n))$ for n a positive multiple of $q-1$. Let $d_n(t)$ be the monic polynomial in $\mathbb{F}[t]$ such that

$$\operatorname{Ext}^{1,\infty}_{\mathcal{MM}^{\text{reg}}_K,A}(\mathbb{1}, \underline{A}(n)) \cong \mathbb{F}[t]^{\oplus n-1} \oplus \mathbb{F}[t]/(d_n(t)).$$

To give a formula for $d_n(t)$, we require the following lemma.

Lemma 7.20. *Let n be a positive multiple of $q-1$. There exists a family $(a_0(\theta), a_1(\theta), ..., a_n(\theta))$ in $\mathbb{F}[\theta]$ such that $a_0 \neq 0$ and the following identity holds:*

$$a_0(\theta)\tilde{\pi}^n = a_1(\theta)\operatorname{Li}_n(1) + a_2(\theta)\operatorname{Li}_n(\theta) + ... + a_n(\theta)\operatorname{Li}_n(\theta^{n-1}).$$

Proof. First, the Euler-Carlitz formula [Carli] implies:

$$\zeta_C(n) \in \mathbb{F}(\theta)^\times \tilde{\pi}^n.$$

On the other-hand, Anderson-Thakur's identity [AndT, Thm 3.8.3] yields:

$$\zeta_C(n) \in \operatorname{Span}_{\mathbb{F}(\theta)}\left\{\operatorname{Li}_n(\theta^i) \mid 0 \leq i < nq/(q-1)\right\}.$$

It remains to show that

$$\operatorname{Span}_{\mathbb{F}(\theta)}\left\{\operatorname{Li}_n(\theta^i) \mid 0 \leq i < \frac{nq}{q-1}\right\} = \operatorname{Span}_{\mathbb{F}(\theta)}\left\{\operatorname{Li}_n(\theta^i) \mid 0 \leq i < n\right\}. \quad (7.6)$$

To prove the above equality, note the elementary relation given for $0 \leq a < n/(q-1)$:

$$\operatorname{Li}_n(\theta^{aq}) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \theta^{n-m} \operatorname{Li}_n(\theta^{a+m}).$$

It implies that $\operatorname{Li}_n(\theta^{n+a}) \in \operatorname{Span}_{\mathbb{F}(\theta)}\{\operatorname{Li}_n(\theta^i) \mid 0 \leq i < n+a\}$, and (7.6) follows by induction. □

Remark 7.21. If we impose coprimality of $(a_0(\theta), a_1(\theta), ..., a_n(\theta))$, a relation as in Lemma 7.20 is unique up to an element of $\mathbb{F}^\times$ (see Corollary 7.26 below).

Remark 7.22. Let $\alpha \in \mathbb{F}[\theta]$ with $\deg \alpha < nq/(q-1)$. We deduce from the proof of Lemma 7.20 that $\operatorname{Li}_n(\alpha)$ belongs to $\operatorname{Span}_{\mathbb{F}(\theta)}\{\operatorname{Li}_n(\theta^i) \mid 0 \leq i < n\}$.

The next theorem computes $d_n(t)$ in terms of the linear relations among $\tilde{\pi}^n$ and the values of polylogarithms.

Theorem 7.23. *Let $(a_0(\theta), a_1(\theta), ..., a_n(\theta))$ be a family of relatively prime elements of $\mathbb{F}[\theta]$ such that $a_0 \neq 0$ for which one has the identity*

$$a_0(\theta)\tilde{\pi}^n = a_1(\theta)\operatorname{Li}_n(1) + a_2(\theta)\operatorname{Li}_n(\theta) + ... + a_n(\theta)\operatorname{Li}_n(\theta^{n-1}). \quad (7.7)$$

Then, $d_n(t)$ equals the monic gcd of $(a_1(t), ..., a_n(t))$.

We begin with a central proposition.

Proposition 7.24. *Let α_1, ..., α_m be elements in K whose degrees are $< nq/q-1$. If there exists $a_0(\theta)$, ..., $a_m(\theta)$ in $\mathbb{F}[\theta]$ such that*

$$a_0(\theta)\tilde{\pi}^n + a_1(\theta)\operatorname{Li}_n(\alpha_1) + ... + a_m(\theta)\operatorname{Li}_n(\alpha_m) = 0$$

then $(-1)^n a_0(t)\omega(t)^n + a_1(t)\xi_{\alpha_1}(t) + ... + a_m(t)\xi_{\alpha_m}(t) \in K[t]$. In particular

$$a_1(t) \cdot [\underline{L}_n(\alpha_1)] + ... + a_m(t) \cdot [\underline{L}_n(\alpha_m)] = 0$$

in $\operatorname{Ext}^{1,\infty}_{\mathcal{MM}_K^{reg}}(\mathbb{1}, \underline{A}(n))$.

Before proving it, we introduce some notations from [ChaYu]. Let

$$L_{\alpha,n}(t) := \alpha + \sum_{k=1}^{\infty} \frac{\alpha^{q^k}}{(t-\theta^q)^n(t-\theta^{q^2})^n\cdots(t-\theta^{q^k})^n},$$

The series $L_{\alpha,n}(t)$ converges to an element of $K_\infty\langle\langle t\rangle\rangle_{\mathrm{j}}$, and according to the definition of $\xi_\alpha(t)$ in (7.2), we have $\xi_\alpha(t) := (t-\theta)^{-n}L_{\alpha,n}(t)$. We also let

$$\Omega(t) := \eta^{-q}\prod_{i=1}^{\infty}\left(1-\frac{t}{\theta^{q^i}}\right) = (t-\theta)^{-1}\omega(t)^{-1}$$

which defines an element in $\mathbb{C}_\infty\langle\langle t\rangle\rangle$. Note that we have $L_{\alpha,n}(\theta) = \mathrm{Li}_n(\alpha)$ and $\Omega(\theta) = -1/\tilde{\pi}$. Let us also mention their functional equations:

$$\Omega^{(-1)} = (t-\theta)\Omega, \quad (\Omega^n L_{\alpha,n})^{(-1)} = \alpha^{1/q}(t-\theta)^n\Omega^n + \Omega^n L_{\alpha,n}. \tag{7.8}$$

The above proposition is proved using the Anderson-Brownawell-Papanikolas (ABP) criterion [ABP, Thm. 3.11], and we use the dual t-motive introduced by Chang and Yu in [ChaYu, (3.5)] to apply the ABP-criterion.

Proof of Proposition 7.24. We write $K = \mathbb{F}(\theta)$ and we fix $\bar{K}$ an algebraic closure of K in $\mathbb{C}_\infty$. We assume that we have a non trivial $\mathbb{F}[\theta]$-linear relation

$$a_0(\theta)\tilde{\pi}^n + a_1(\theta)\,\mathrm{Li}_n(\alpha_1) + \ldots + a_m(\theta)\,\mathrm{Li}_n(\alpha_m) = 0$$

and further that m is chosen minimal with this property. By [ChaYu, Thm 3.1] this implies that any strict subset of $\{\tilde{\pi}, \mathrm{Li}_n(\alpha_1), \ldots, \mathrm{Li}_n(\alpha_m)\}$ forms a set of algebraically independent series over $\bar{K}$.

We use the notation of [ChaYu], in particular let $\mathbb{E}$ be the union in $\mathbb{C}_\infty\langle\langle t\rangle\rangle$ of $L\langle\langle t\rangle\rangle$ ranging over finite field extensions L of K inside $\mathbb{C}_\infty$. We consider a *thickening* of the system of Chang and Yu [ChaYu, (3.5)]:

$$\Phi = \Phi(\alpha_1,\ldots,\alpha_m) := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & (t-\theta)^n & 0 & \cdots & 0 \\ 0 & \alpha_1^{1/q}(t-\theta)^n & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha_m^{1/q}(t-\theta)^n & 0 & \cdots & 1 \end{pmatrix},$$

$$\Psi = \Psi(\alpha_1,\ldots,\alpha_m) = \begin{pmatrix} 1 \\ \Omega(t)^n \\ \Omega(t)^n L_{\alpha_1,n}(t) \\ \vdots \\ \Omega(t)^n L_{\alpha_m,n}(t) \end{pmatrix}.$$

Note that the entries of Ψ are in $\mathbb{E}$ and that Φ is invertible over $\bar{K}(t)$ with determinant $(t-\theta)^n$. The functional equations (7.8) imply that $\Psi^{(-1)} = \Phi\Psi$.

Therefore, we are in position to apply the criterion of Anderson-Brownawell-Papanikolas [ABP, Thm 3.1.1]. The latter implies that the $\bar{K}[t]$-module

$$L=\{q_\omega,q_0,...,q_m\in\bar{K}[t]\mid q_\omega(t)\Omega^n+q_0(t)+q_1(t)\Omega^nL_{\alpha_1,n}+...+q_m(t)\Omega^nL_{\alpha_m,n}=0\}$$

is nonzero. It is also torsion-free. To compute its rank, let $q=(q_\omega,q_0,...,q_m)$ and $q'=(q'_\omega,q'_0,...,q'_m)$ be two elements of L. Then set $r:=q'_mq-q_mq'\in L$ and write r_i ($i\in\{\omega,0,...,m\}$) for its coordinate, knowing that $r_m=0$. We want to show that $r=0$. If one r_i was not divisible by $(t-\theta)$, then evaluating at $t=\theta$ would give a non trivial $\bar{K}$-linear relation among elements of the set $\{1,\tilde{\pi}^n,\mathrm{Li}_n(\alpha_1),...,\mathrm{Li}_n(\alpha_{m-1})\}$. Because such a relation does not exist by minimality (and using [ChaYu, Thm. 3.1]), $(t-\theta)$ divides r_i for all $i\in\{\omega,0,...,m\}$. If $r\neq 0$, there would exist a maximal integer k such that $(t-\theta)^k$ divides r_i for all $i\in\{\omega,0,...,m\}$. Dividing r by $(t-\theta)^k$ and evaluating at $t=\theta$ would give a non trivial $\bar{K}$-linear relation between $1,\tilde{\pi}^n,\mathrm{Li}_n(\alpha_1),...,\mathrm{Li}_n(\alpha_{m-1})$, which does not exist by assumption.

We deduce that L is a free $\bar{K}[t]$-module of rank 1. Let $(p_\omega(t),p_0(t),...,p_m(t))$ be a generator of L. By definition, we have

$$p_\omega(t)\Omega^n+p_0(t)+p_1(t)\Omega^nL_{\alpha_1,n}+...+p_m(t)\Omega^nL_{\alpha_m,n}=0 \tag{7.9}$$

Applying map $f\mapsto f^{(-1)}$ to this equation and using the functional equation yields a new relation in L

$$\left(p_\omega^{(-1)}(t)-\sum_{j=1}^m\alpha_j^{1/q}p_j^{(-1)}(t)\right)(t-\theta)^n\Omega^n \\ +p_0(t)^{(-1)}+p_1^{(-1)}(t)\Omega^nL_{\alpha_1,n}+...+p_m^{(-1)}(t)\Omega^nL_{\alpha_m,n}=0$$

which has to be the same as (7.9) up to a factor in $\bar{K}[t]$. We let $h(t)\in\bar{K}[t]$ be the proportionality factor (note that h has degree 0). For all $j\in\{0,...,m\}$, $p_j(t)$ is a solution of the linear equation

$$(E):\ p(t)^{(-1)}=h(t)p(t),\quad p(t)\in\bar{K}[t].$$

Because at least one of the $p_j(t)$ is nonzero, (E) admits a non-zero solution. The solutions of (E) forms a free $\mathbb{F}[t]$-module of rank 1, and we let $\Delta(t)\in\bar{K}[t]$ be a generator. Therefore, for all $j\in\{0,...,m\}$, there exists $b_j(t)\in\mathbb{F}[t]$ such that $p_j(t)=b_j(t)\Delta(t)$.

Let us consider

$$\Gamma(t):=b_0(t)\Omega(t)^{-n}+\sum_{j=1}^m b_j(t)L_{\alpha_j,n}(t).$$

We have $\Gamma(t)=-p_\omega(t)/\Delta(t)$ and therefore is an element of $\bar{K}(t)$. $\Gamma(t)$ further satisfies the functional equation

$$\Gamma(t)^{(-1)}-\frac{\Gamma(t)}{(t-\theta)^n}=\sum_{j=1}^m b_j(t)\alpha_j^{1/q}. \tag{7.10}$$

By (7.10), if Γ has a pole at x in $\bar{K}$ then Γ has a pole at x^q. Thus, if $\Gamma(t) \neq 0$, all its poles must be in $\mathbb{F}$. In particular, there exists $d(t) \in \mathbb{F}[t]$ nonzero such that $d(t)\Gamma(t) \in \bar{K}[t]$. It also follows from (7.10) that $d(t)\Gamma(t)$ has a zero of order n at $t = \theta$. Since the power series expansion of $L_{\alpha,n}(t)$ in t belongs to $K[\![t]\!]$, we deduce that $d(t)\Gamma(t) \in (t-\theta)^n K[t]$. We obtain the relation

$$d(t)b_0(t)\Omega(t)^{-n} + \sum_{j=1}^{m} d(t)b_j(t)L_{\alpha_j,n}(t) \in (t-\theta)^n K[t] \tag{7.11}$$

where the $d(t)b_j(t) \in \mathbb{F}[t]$ ($j \in \{0,...,m\}$) are not all zero. If the tuples $((-1)^n a_0,...,a_m)$ and $(db_0,...,db_m)$ were linearly independent in $\mathbb{F}[t]^{m+1}$, evaluating (7.11) at $t=\theta$ would have give another independent relation between $\tilde{\pi}^n, \mathrm{Li}_n(\alpha_1),...,\mathrm{Li}_n(\alpha_m)$ which does not exists by minimality. Hence $(a_0,...,a_m)$ depends linearly on $(db_0,...,db_m)$ and

$$(-1)^n a_0(t)\Omega(t)^{-n} + \sum_{j=1}^{m} a_j(t)L_{\alpha_j,n}(t) \in (t-\theta)^n K[t].$$

Dividing out by $(t-\theta)^n$ yields the desired relation. $\square$

Proof of Theorem 7.23. Let f be the $\mathbb{F}[t]$-linear morphism

$$\mathbb{F}[t]\cdot 1 \oplus \mathbb{F}[t]\cdot\theta \oplus \cdots \oplus \mathbb{F}[t]\cdot\theta^{n-1} \longrightarrow \mathrm{Ext}^{1,\infty}_{\mathcal{MM}^{\mathrm{reg}}_{K},A}(\mathbb{1},\underline{A}(n))$$

which maps $q(t,\theta)$ to $\iota((t-\theta)^{-n}q(t,\theta))$. It is surjective, and its kernel is a free $\mathbb{F}[t]$-module of rank 1. Because of (7.7), Proposition 7.24 implies that

$$a_1(t)\cdot[\underline{L}_n(1)] + ... + a_n(t)\cdot[\underline{L}_n(\theta^{n-1})] = 0$$

in $\mathrm{Ext}^{1,\infty}_{\mathcal{MM}^{\mathrm{reg}}_{K},A}(\mathbb{1},\underline{A}(n))$. This yields $\mathbb{F}[t](a_1(t)\cdot 1 + ... + a_n(t)\cdot\theta^{n-1}) \subset \ker f$. Let $d(t)$ be the gcd of $(a_1(t),...,a_n(t))$. Then, $\ker f$ is of the form

$$\ker f = \mathbb{F}[t]\cdot g(t)\cdot\left(\frac{a_1(t)}{d(t)}\cdot 1 + ... + \frac{a_n(t)}{d(t)}\cdot\theta^{n-1}\right)$$

for some monic polynomial $g(t)$ with $g(t)|d(t)$. The theorem follows once we proved that $g(t) = d(t)$. By definition of $\ker f$, we have

$$g(t)\cdot\left(\frac{a_1(t)}{d(t)}\cdot[\underline{L}_n(1)] + \cdots + \frac{a_n(t)}{d(t)}\cdot[\underline{L}_n(\theta^{n-1})]\right) = 0$$

in $\mathrm{Ext}^{1,\infty}_{\mathcal{MM}^{\mathrm{reg}}_{K},A}(\mathbb{1},\underline{A}(n))$. By applying $\mathrm{Reg}^{\infty}_{A}(\underline{A}(n))$ to the above element and using Proposition 7.10, we obtain that the series:

$$\frac{a_0(\theta)g(\theta)}{d(\theta)}\tilde{\pi}^n = \frac{g(\theta)}{d(\theta)}(a_1(\theta)\,\mathrm{Li}_n(1) + a_2(\theta)\,\mathrm{Li}_n(\theta) + ... + a_n(\theta)\,\mathrm{Li}_n(\theta^{n-1}))$$

belongs to $\mathbb{F}[\theta]\tilde{\pi}^n$. Because $d(\theta)$ is prime to $a_0(\theta) \neq 0$, we have $d(\theta)|g(\theta)$ which concludes. $\square$

Corollary 7.25. *Assume that $n = q^k(q-1)$ for some $k \geq 0$. Then, as $\mathbb{F}[t]$-modules:*

$$\mathrm{Ext}^{1,reg,\infty}_{\mathcal{MM}_K,A}(\mathbb{1}, \underline{A}(n)) \cong \mathbb{F}[t]^{n-1} \oplus \mathbb{F}[t]/(t^{q^{k+1}} - t^{q^k}).$$

Proof. The case $k = 0$ follows from the well-known identity (e.g [ChaYu, Thm. 4.1+4.2]):

$$\mathrm{Li}_{q-1}(1) = \frac{\tilde{\pi}^{q-1}}{\theta^q - \theta}$$

which implies $d_{q-1}(t) = t^q - t$ by Theorem 7.23. Raising the latter identity to the power q^k, for $k \geq 0$, yields $(\theta^{q^{k+1}} - \theta^{q^k})\,\mathrm{Li}_n(1) = \tilde{\pi}^n$. We obtain the general case by Theorem 7.23. □

Corollary 7.26. *If n is a positive multiple of $q-1$, a relation as in Lemma 7.20 such that $(a_0(\theta), ...a_n(\theta))$ are relatively prime is unique, up to a multiplicative factor in $\mathbb{F}^\times$.*

Proof. If it was not true, then the $\mathbb{F}[t]$-module

$$\frac{\{\xi \in K_\infty\langle t\rangle \mid (t-\theta)^n\xi - \xi^{(1)} \in \mathbb{F}[\theta, t]\}}{\mathbb{F}[\theta, t] + \Lambda(\underline{A}(n))}$$

would have have rank $\leq n-2$ by Proposition 7.24. This is absurd by Theorem 7.1. □

7.3 Algebraic independence of Carlitz's polylogarithms

Let $n > 0$ and let α_1,..., α_m be elements of $K = \mathbb{F}(\theta)$ whose degrees are $< nq/(q-1)$. The following theorem states that the knowledge of the module structure of $\mathrm{Ext}^1_{\mathcal{MM}^{\mathrm{reg}}_K}(\mathbb{1}, \underline{A}(n))$ provides a criterion to spot the algebraic independence of the series $\tilde{\pi}$, $\mathrm{Li}_n(\alpha_1)$, ..., $\mathrm{Li}_n(\alpha_m)$.

Theorem 7.27. *The following are equivalent:*

(*i*) *The series $\tilde{\pi}$, $\mathrm{Li}_n(\alpha_1)$, ..., $\mathrm{Li}_n(\alpha_m)$ are algebraically independent over $\bar{K}$,*

(*ii*) *The extensions $[\underline{L}_n(\alpha_1)]$, ..., $[\underline{L}_n(\alpha_m)]$ are linearly independent in the $\mathbb{F}[t]$-module $\mathrm{Ext}^1_{\mathcal{MM}^{reg}_K}(\mathbb{1}, \underline{A}(n))$.*

Proof. Assume (i). Then, in particular, $\tilde{\pi}^n$, $\mathrm{Li}_n(\alpha_1)$, ..., $\mathrm{Li}_n(\alpha_m)$ are linearly independent over K. This implies that the series $\xi_{\alpha_1}(t)$, ..., $\xi_{\alpha_m}(t)$ are $\mathbb{F}[t]$-linearly independent in

$$\frac{\left\{\xi \in \mathbb{C}_\infty\langle t\rangle \mid (t-\theta)^n\xi - \xi^{(1)} \in \mathbb{F}(\theta)[t]\right\}}{\mathbf{1}_{q-1|n}\mathbb{F}[t]\omega(t)^n + \mathbb{F}(\theta)[t]} \tag{7.12}$$

(indeed, lifting such a linear relation in $\mathbb{C}_\infty(\!(t-\theta)\!)$ would have led to a K-linear relation among $\tilde{\pi}^n$, $\mathrm{Li}_n(\alpha_1)$, ..., $\mathrm{Li}_n(\alpha_m)$ by observing the first nonzero coefficient in the $(t-\theta)$-expansion). By Proposition 5.33, the $\mathbb{F}[t]$-module (7.12) is isomorphic to $\mathrm{Ext}^1_{\mathcal{MM}_K^{\mathrm{reg}}}(\mathbb{1},\underline{A}(n))$ and to the class of $\xi_\alpha(t)$ corresponds $[\underline{L}_n(\alpha)]$. *(ii)* follows.

Conversely, suppose the series $\tilde{\pi}$, $\mathrm{Li}_n(\alpha_1)$, ..., $\mathrm{Li}_n(\alpha_m)$ are algebraically dependent over $\bar{K}$. By [ChaYu, Cor. 3.2] (and its proof), $\tilde{\pi}^n$, $\mathrm{Li}_n(\alpha_1)$, ..., $\mathrm{Li}_n(\alpha_m)$ are linearly dependent over K. By Proposition 7.24, we deduce that the series $\xi_{\alpha_1}(t)$, ..., $\xi_{\alpha_m}(t)$ are $\mathbb{F}[t]$-linearly dependent in (7.12). It yields that $[\underline{L}_n(\alpha_1)]$, ..., $[\underline{L}_n(\alpha_m)]$ are linearly dependent in the $\mathbb{F}[t]$-module $\mathrm{Ext}^1_{\mathcal{MM}_K^{\mathrm{reg}}}(\mathbb{1},\underline{A}(n))$ as desired. $\square$

Let $n>0$ and let $(\alpha_1,...,\alpha_n)$ be a basis of $\{\alpha\in\mathbb{F}[\theta]|\deg\alpha<n\}$ over $\mathbb{F}$. Repeated from Theorem G in Chapter 0, we prove the following two consequences of the previous sections:

Corollary 7.28. *If $q-1\nmid n$, the series $\tilde{\pi}$, $\mathrm{Li}_n(\alpha_1)$, $\mathrm{Li}_n(\alpha_2)$, ..., $\mathrm{Li}_n(\alpha_n)$ are algebraically independent over $\bar{K}$.*

Proof. If $q-1\nmid n$, the family of elements $[\underline{L}_n(\alpha_i)]$ $(i\in\{1,...,n\})$ forms a basis of $\mathrm{Ext}^{1,\infty}_{\mathcal{MM}_K^{\mathrm{reg}},A}(\mathbb{1},\underline{A}(n))$ and therefore are linearly independent in $\mathrm{Ext}^1_{\mathcal{MM}_K^{\mathrm{reg}}}(\mathbb{1},\underline{A}(n))$. We conclude by Theorem 7.27. $\square$

Remark 7.29. If $q-1|n$, a linear relation among $\tilde{\pi}^n$, $\mathrm{Li}_n(\alpha_1)$, $\mathrm{Li}_n(\alpha_2)$, ..., $\mathrm{Li}_n(\alpha_n)$ was shown to exist in Lemma 7.20.

Corollary 7.30. *For $n>0$, the family $(\mathrm{Li}_n(\alpha_1),...,\mathrm{Li}_n(\alpha_n))$ is algebraically independent over $\bar{K}$.*

Proof. The case where $q-1$ does not divide n is contained in Corollary 7.30. We prove the case where $q-1|n$. By Lemma 7.20, there exists a relation

$$a_0(\theta)\tilde{\pi}^n = a_1(\theta)\,\mathrm{Li}_n(\alpha_1)+...+a_n(\theta)\,\mathrm{Li}_n(\alpha_n) \tag{7.13}$$

where $(a_0(\theta),...,a_n(\theta))$ is a coprime family in $\mathbb{F}[\theta]$ with $a_0\neq 0$. By Proposition 7.24 together with the fact that $\mathrm{Ext}^{1,\infty}_{\mathcal{MM}_K^{\mathrm{reg}}}(\mathbb{1},\underline{A}(n))$ has rank $n-1$, any relation of the form $b_1(\theta)\,\mathrm{Li}_n(\alpha_1)+...+b_n(\theta)\,\mathrm{Li}_n(\alpha_n)=0$ $(b_i(\theta)\in\mathbb{F}[\theta])$ is proportional to the relation (7.13). Because $a_0\neq 0$, the proportionality factor must be zero. Hence, the family $(\mathrm{Li}_n(\alpha_1),...,\mathrm{Li}_n(\alpha_n))$ is linearly independent over K. It is algebraically independent over K by [ChaYu, Cor. 3.2] as desired. $\square$

7.4 Beilinson's first conjecture for Carlitz tensor powers

We end this text by giving an equivalent formulation of Beilinson's conjecture (Definition 6.5) in the case $\underline{M}=\underline{A}(n)$ (with $n>0$).

Let $\alpha \in K$ be such that $\deg \alpha < nq/(q-1)$. For $0 \leq k < n$, let $\mathrm{Li}_n^{[k]}(\alpha)$ be the element of K_∞ defined by the formula:

$$\xi_\alpha(t) = \frac{\mathrm{Li}_n^{[0]}(\alpha)}{(t-\theta)^n} + \frac{\mathrm{Li}_n^{[1]}(\alpha)}{(t-\theta)^{n-1}} + \cdots + \frac{\mathrm{Li}_n^{[n-1]}(\alpha)}{(t-\theta)} + \mathscr{O}(K_\infty[\![t-\theta]\!])$$

where $\xi_\alpha(t)$ is defined by (7.2). By construction, $\mathrm{Li}_n^{[0]}(\alpha) = \mathrm{Li}_n(\alpha)$. We record:

Proposition 7.31. *For $n > 0$, the following are equivalent:*

(*i*) *Beilinson's conjecture is true for $\underline{M} = \underline{A}(n)$,*

(*ii*) *Given a basis $(\alpha_1, ..., \alpha_n)$ of $\{\alpha \in \mathbb{F}[\theta] | \deg(\alpha) < n\}$ over $\mathbb{F}$, the matrix*

$$\begin{pmatrix} \mathrm{Li}_n^{[0]}(\alpha_1) & \mathrm{Li}_n^{[0]}(\alpha_2) & \cdots & \mathrm{Li}_n^{[0]}(\alpha_n) \\ \mathrm{Li}_n^{[1]}(\alpha_1) & \mathrm{Li}_n^{[1]}(\alpha_2) & \cdots & \mathrm{Li}_n^{[1]}(\alpha_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Li}_n^{[n-1]}(\alpha_1) & \mathrm{Li}_n^{[n-1]}(\alpha_2) & \cdots & \mathrm{Li}_n^{[n-1]}(\alpha_n) \end{pmatrix} \tag{7.14}$$

is invertible in $\mathcal{M}_n(K_\infty)$.

Remark 7.32. Assertion (*ii*) does not depend on the choice of the basis.

Proof of Proposition 7.31. We have the following commutative diagram with exact rows (Theorem 5.34):

$$\begin{tikzcd} \Lambda(\underline{A}(n))^+_{K_\infty} \arrow[r, hook] \arrow[d, "\mathrm{id}"] & \dfrac{\{\xi \in K_\infty\langle t\rangle | (t-\theta)^n\xi - \xi^{(1)} \in \mathbb{F}[\theta,t]\}}{\mathbb{F}[\theta,t]} \underset{A}{\otimes} K_\infty \arrow[r, two heads] \arrow[d, "\xi\mapsto\xi"] & \mathrm{Ext}^{1,\mathrm{reg},\infty}_{\mathcal{MM}^{\mathrm{rig}}_K, A}(\mathbb{1}, \underline{A}(n)) \otimes_A K_\infty \arrow[d, "\mathscr{R}eg^\infty_{K_\infty}(\underline{A}(n))"] \\ \Lambda(\underline{A}(n))^+_{K_\infty} \arrow[r, hook] & \dfrac{(t-\theta)^{-n}K_\infty[\![t-\theta]\!]}{K_\infty[\![t-\theta]\!]} \arrow[r, two heads] & \mathrm{Ext}^{1,\mathrm{ha},\infty}_{\mathcal{MHP}^+_{K_\infty}}(\mathbb{1}^+, \mathscr{H}^+_{K_\infty}(\underline{A}(n))) \end{tikzcd}$$

By Theorem 6.4, the K_∞-vector space

$$\frac{\{\xi \in K_\infty\langle t\rangle | (t-\theta)^n\xi - \xi^{(1)} \in \mathbb{F}[\theta,t]\}}{\mathbb{F}[\theta,t]} \tag{7.15}$$

has dimension n. Because the class of the family $(\xi_{\alpha_1}, ..., \xi_{\alpha_m})$ in (7.15) is generating (Lemma 7.17), it is a basis. The matrix $(\mathrm{Li}_n^{[j]}(\alpha_j))_{i,j}$ represents the middle vertical K_∞-linear morphism in the bases $(\xi_{\alpha_i}(t) \mid 1 \leq i \leq n)$ of the source space and $((t-\theta)^{n-j} \mid 0 \leq j < n)$ of the target space. Therefore, $(\mathrm{Li}_n^{[j]}(\alpha_j))_{i,j}$ is invertible if and only if the middle vertical morphism is an isomorphism. By the Snake Lemma, the latter assertion is equivalent to Beilinson's conjecture for $\underline{A}(n)$ (Definition 6.5). The proposition follows. □

Corollary 7.33. *Beilinson's conjecture is true for $\underline{M} = \underline{A}(1)$.*

Corollary 7.34. *Beilinson's conjecture is false for $\underline{M} = \underline{A}(n)$ whenever n is a multiple of the characteristic p of $\mathbb{F}$.*

Proof. If $p|n$, then, from (7.2), $\mathrm{Li}_n^{[i]}(\alpha) = 0$ whenever $p \nmid i$ and for every α. In particular, the matrix in (*ii*) is not invertible. □

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Cohomologie motivique en arithmétique des corps de fonctions

Résumé : Les invariants arithmétiques les plus profonds attachés à une variété algébrique définie sur un corps de nombres sont conjecturalement capturés par sa dénommée *cohomologie motivique*. Les valeurs de fonctions L et les K-groupes de variétés en sont quelques exemples. Cette thèse dépeint le portrait analogue pour les corps globaux de caractéristique positive. L'objectif principal est de décrire les groupes d'extensions dans certaines catégories de A-modules d'Anderson et de montrer un théorème de finitude. Nous concluons par une discussion sur la première conjecture de Beilinson en arithmétique des corps de fonctions. Pour terminer, nous expliquons comment nos résultats s'appliquent pour étudier les relations algébriques entre les valeurs des polylogarithmes de Carlitz.

Mots clés : Cohomologie motivique ; A-modules d'Anderson ; Conjectures de Beilinson ; Polylogarithme de Carlitz.

Motivic cohomology in the arithmetic of function fields

Abstract : The deepest arithmetic invariants attached to an algebraic variety defined over a number field are conjecturally captured by its so-called *motivic cohomology*. Values of L-functions and K-groups of varieties are some examples. This thesis describes the analogous picture for global fields in equal characteristic. The main objective is to compute the extension modules in various categories of Anderson A-motives and to prove a finiteness theorem. We conclude with a discussion on Beilinson's first conjecture in function fields arithmetic. Finally, we explain how our results apply to investigate algebraic relations among values of Carlitz polylogarithms.

Keywords : Motivic cohomology ; Anderson A-modules ; Beilinson's conjectures ; Carlitz polylogarithm.

Image en couverture : *Un ermite.* Crédit image : croquis personnel.