

Mean-Field Approximation For The Anyon Gas

Théotime Girardot

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THÈSE

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Approximation de champ moyen pour un gaz d'anyons

MEAN-FIELD APPROXIMATION FOR THE ANYON GAS

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Mean-field approximation for the anyon gas.

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Abstract. Ce manuscrit est dédié à l'étude des gaz d'anyons. Il s'agit de dérivations rigoureuses d'énergies fondamentales de ces systèmes dans la limite de champs moyen, i.e quand le nombre de particule tend vers l'infini.

Au sein des systèmes bi-dimensionnels le choix de la statistique des particules n'est pas limité à la dichotomie boson/fermion. Le paramètre statistique peut prendre n'importe quelle valeur entre 0 et 1 extrapolant entre bosons (paramètre statistique nul) et fermions (paramètre statistique égal à 1). Ces quasi-particules sont décrites comme d'authentiques fermions ou bosons interagissant via un potentiel magnétique longue portée généré par des charges magnétiques de rayon R que les particules transportent. En d'autres termes, nous ajoutons aux particules un tube de flux fictif de type Aharonov-Bohm. Nous avons donc affaire à un problème à N-corps où toutes les particules interagissent entre elles. Nous considérons de plus que les anyons sont piégés dans un potentiel V et soumis à un champ magnétique extérieur. Nous étudions ici deux limites appelées quasi-bosonique et quasi-fermionique où le paramètre statistique tend respectivement vers celui des bosons puis des fermions.

Dans le premier cas le paramètre statistique est vu comme une perturbation du cas bosonique. Nous le prenons égal à $1/N$. Le rayon R tend aussi vers zéro quand N tend vers l'infini. Notre méthode permet d'obtenir la convergence de l'énergie fondamentale de ces gaz ainsi que la convergence des états associés. Nous obtenons une énergie de Hartree. La validité de ce résultat dépend de la vitesse de convergence de R qui ne doit pas décroître trop vite. La discussion autour de la taille des anyons sera centrale dans le manuscrit. L'outil principal dans l'obtention du modèle effectif est la version du théorème de de Finetti de Brandão et Harrow combiné à une localisation dans l'espace de Fock.

Dans le cas quasi-fermionique le paramètre statistique tend vers un. Nous travaillons donc sur une perturbation du cas fermionique. Cette fois-ci nous introduisons la constante de Planck que nous prenons proportionnelle à $1/N^{1/2}$ pour nous trouver dans une limite semi-classique. Nous obtenons à la limite une énergie Thomas-Fermi. La validité de ces résultats est aussi tributaire d'une contrainte sur la taille des charges R. Nous utilisons les états cohérents et les fonctions de Husimi combinés au théorème de Diaconis-Freedman pour obtenir l'énergie effective. Le point central de la preuve est l'obtention d'un principe de Pauli quantitatif semi-classique.

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Part 1

Introduction et résultats

CHAPTER 1

Introduction

C'est une loi de la nature; toutes les particules constitutives de notre monde physique sont soit des bosons, soit des fermions. Les bosons décrivent les interactions entre particules, ils sont les intermédiaires par lesquels les autres particules agissent les unes sur les autres. On compte parmi eux les photons de l'interaction électromagnétique, le gluon de l'interaction forte, échangé par les quarks au cœur des nucléons ou encore le graviton, coursier de l'attraction gravitationnelle. D'un point de vue mathématique les fonctions d'onde utilisées pour décrire les particules sont symétriques pour les bosons et antisymétriques pour les fermions. Cette contrainte de symétrie rend compte d'un grand nombre de propriétés des systèmes macroscopiques. À très basse température nous observons par exemple le phénomène de condensation de Bose-Einstein [**45, 31**]. Dans cet état particulier de la matière presque toutes les particules se trouvent dans le même état quantique. Cette tendance à peupler un seul et même état permet la construction de lasers aux rayons de photons très cohérents. Cela ne se pourrait produire avec des fermions. Ceux-là sont soumis au principe de Pauli et ne peuvent en conséquence jamais occuper le même état. C'est l'argument fondamental dans l'explication de la stabilité de la matière usuelle [**42, 110, 43**]. C'est aussi la raison de la plupart des propriétés chimiques des atomes. À ces deux comportements, bosonique et fermionique, sont respectivement associées les statistiques de Bose-Einstein et de Fermi-Dirac. À l'équilibre thermodynamique la statistique dicte aux particules comment se distribuer parmi les niveaux d'énergie du système. Considérons plus précisément un système avec les niveaux $\{E_i\}_{i=1}^{\infty}$ de dégénérescence g_i à température T et potentiel chimique μ. Le nombre de particules d'un groupe de bosons ou de fermions peuplant en moyenne un état i est donné par les distributions N^{BE} et N^{FD} suivantes

$$N_i^{\mathrm{BE}} = g_i \left(e^{\frac{E_i-\mu}{k_B T}} - 1 \right)^{-1} \qquad \text{pour les bosons} \tag{1.1}$$

et

$$N_i^{\mathrm{FD}} = g_i \left(e^{\frac{E_i-\mu}{k_B T}} + 1 \right)^{-1} \qquad \text{pour les fermions} \tag{1.2}$$

avec k_B la constante de Boltzmann. Quand la température T devient très basse nous observons que

$$N_0^{\mathrm{BE}} \to \infty \text{ alors que } N_i^{\mathrm{FD}} \to g_i \mathbb{1}_{E_i \leq \mu}.$$

Le premier phénomène est appelé condensation. Le second décrit l'apparition de la mer de Fermi, i.e les fermions s'empilent jusqu'à atteindre l'énergie μ.

Cette stricte dichotomie boson/fermion nous amène à nous questionner sur son indubitabilité. Existerait-il une particule fondamentale avec une statistique différente ? la réponse expérimentale est non. Jusqu'à aujourd'hui seules ont été mesurées des particules

bosoniques et fermioniques. La réponse apportée par la théorie est tout aussi claire, non. La découverte des anyons tire son origine d'une question naturellement associée aux dernières réponses: pourquoi ? Là aussi nous trouvons une réponse nette: parce que notre monde est au moins tri-dimensionnel. Mais alors, qu'adviendrait-il si nous vivions dans un plan ? Les espaces $\mathbb{R}^3$ et $\mathbb{R}^2$ présentent d'intrinsèques différences topologiques. Celle qui nous intéresse plus particulièrement est le fait que l'espace $\mathbb{R}^3 \setminus \{0\}$ est simplement connexe alors que $\mathbb{R}^2 \setminus \{0\}$ ne l'est pas. Les groupes fondamentaux de ces espaces sont alors distincts ce qui, dans le cas de la dimension deux, ouvre la voie à de nouvelles possibilités de symétrisation de la fonction d'onde dès lors que la mécanique quantique est formulée d'une manière géométrique [**126**]. Une fonction d'onde décrivant des particules vivant dans un plan peut donc être dotée d'une propriété de symétrie plus exotique que la symétrie ou l'antisymétrie. Mais, comme nous l'avons dit, notre monde physique est constitué de trois dimensions. Nous aurions pu penser que ces particules resteraient d'intrigants objets théoriques pour toujours. Il n'en est rien. Des systèmes quasi-bidimensionnels peuvent habriter d'authentiques quasi-particules deux-dimensionnelles qui elles, n'ont a priori pas à être bosoniques ou fermioniques.

Les pionniers de cette découverte théorique furent Leinas et Myrheim [**78, 28**] en 1977, Wilczek [**165, 163, 164**] et Goldin, Menikoff, Sharp [**62, 63**] dans les années 80. Les dernières décénies ont vu s'accroître l'intérêt porté aux anyons, notamment grâce aux progrès en physique des atomes froids et en informatique quantique. Plusieurs systèmes suspectés d'habriter des anyons ont vu le jour. L'effet Hall quantique fractionnaire (EHQF) est peut-être un des plus connus et étudiés [**61, 67, 66, 77**]. Des anyons ont été mesurés pour la première fois en 2020 dans le régime de l'EHQF [**12, 127**]. Il existe aussi plusieurs espoirs de créer des anyons dans les gaz d'atomes froids [**30, 121, 160, 167, 168, 169**].

CHAPTER 2

Physique à N corps

1. Physique classique

Nous nous intéressons dans cette thèse à un gaz d'anyons. Notre approche combine champ-moyen et limite semi-classique. Nous rappellons ici les bases de la physique à N corps classique. L'espace des configurations d'un gaz de N particules évoluant dans $\mathbb{R}^d$ est $\mathbb{R}^{dN}$. L'espace des phase est le produit cartésien de l'espace des configurations des positions et des vitesses $\mathbb{R}^{dN} \times \mathbb{R}^{dN}$. On appelle état pur une configuration du système où nous connaissons la vitesse et la position de toutes les particules. Il s'agit d'un point de l'espace des phases $Z_N = (X_N, P_N)$. La mesure empirique associée est

$$\mathrm{Emp}_{Z_N} = \frac{1}{N} \sum_{j=1}^{N} \delta_{\mathbf{z}_j} \tag{2.1}$$

où $\mathbf{z}_j = (\mathbf{x}_j, \mathbf{p}_j) \in \mathbb{R}^d \times \mathbb{R}^d$. Les états d'équilibre à température nulle sont des états purs. Dans les autres cas le bruit thermal étale la mesure empirique, on parle alors d'un état mixte décrit par une mesure de probabilité $\mu_N \in \mathcal{P}\left(\mathbb{R}^{dN} \times \mathbb{R}^{dN}\right)$ telle que $\mu_N \geq 0$ avec

$$\int_{\mathbb{R}^{dN} \times \mathbb{R}^{dN}} \mathrm{d}\mu_N = 1. \tag{2.2}$$

Les observables physiques sont calculées à partir d'un état (pur ou mixte) de la manière suivante. Prenons par exemple un Hamiltonien

$$H_N(X;P) = \sum_{j=1}^{N} \frac{|\mathbf{p}_j|^2}{2} + \sum_{j=1}^{N} V(\mathbf{x}_j) \tag{2.3}$$

de N particules qui n'interagissent pas dans un potentiel V. L'observable énergie est alors

$$\langle H_N(Z) \rangle = \int_{\mathbb{R}^{dN} \times \mathbb{R}^{dN}} H_N(X;P)\, \mathrm{d}\mu_N(X;P). \tag{2.4}$$

Afin de prendre en compte l'indicernabilité des particules nous imposons que μ_N satisfasse

$$\mu_N(Z) = \mu_N(Z_\sigma) \tag{2.5}$$

pour toute permutation σ telle que

$$Z_\sigma = \left(\mathbf{x}_{\sigma(1)}, \mathbf{p}_{\sigma(1)}, ..., \mathbf{x}_{\sigma(N)}, \mathbf{p}_{\sigma(N)}\right). \tag{2.6}$$

Cette propriété de symétrie permet d'exprimer un grand nombre d'observables en utilisant seulement les densités réduites

$$\mu^{(k)}(\mathbf{x}_1, \mathbf{p}_1, ..., \mathbf{x}_k, \mathbf{p}_k) = \int_{\mathbb{R}^{d(N-k)} \times \mathbb{R}^{d(N-k)}} \mu_N(\mathbf{x}_1, \mathbf{p}_1, ..., \mathbf{x}_N, \mathbf{p}_N)\, \mathrm{d}\mathbf{x}_{k+1} \mathrm{d}\mathbf{p}_{k+1} ... \mathrm{d}\mathbf{x}_N \mathrm{d}\mathbf{p}_N \tag{2.7}$$

2. Mécanique quantique

En mécanique quantique à N corps un état pur est une fonction complexe de carré sommable sur l'espace des configurations des positions $\mathbb{R}^{dN}$. L'évolution d'un état $\Psi_N \in L^2\left(\mathbb{R}^{dN}\right)$ est pilotée par l'équation de Schrödinger

$$\mathrm{i}\hbar\partial_t \Psi_N\left(\mathbf{x},t\right) = H_N \Psi_N(\mathbf{x},t) \tag{2.8}$$

où $\hbar = h/2\pi$ est la constante de Planck réduite avec $h = 6,62 \times 10^{-34} m^2 kg/s$. H_N est le Hamiltonien du système. La plus petite valeur de son spectre (quand elle existe) est appellée énergie fondamentale. Le carré $|\Psi_N|^2$ est interprété comme la densité de probabilité de trouver les particules dans un volume donné de l'espace des configurations. Nous imposons que le carré de sa transformée de Fourier (définie comme isométrie de L^2), $|\hat{\Psi}_N|^2$ soit la probabilité de trouver les particules dans une certaine configuration de vitesses. Cette interprétation probabiliste demande que

$$\int_{\mathbb{R}^{dN}} |\Psi_N|^2 \,\mathrm{d}\mathbf{x} = 1. \tag{2.9}$$

Les particules doivent aussi être indiscernables. Cela implique que l'étiquetage des particules $X_N = (\mathbf{x}_1, ..., \mathbf{x}_N)$ soit arbitraire et que

$$|U_\sigma \Psi_N|^2 = |\Psi_N|^2 \tag{2.10}$$

pour toute permutation $\sigma \in S_N$ du groupe des permutations de N éléments avec U_σ l'opérateur unitaire permuttant les indices

$$U_\sigma \Psi_N(\mathbf{x}_1, ..., \mathbf{x}_N) = \Psi_N(\mathbf{x}_{\sigma(1)}, ..., \mathbf{x}_{\sigma(N)}). \tag{2.11}$$

L'indiscernabilité doit être effective relativement à toutes les mesures envisageables. La valeur moyenne d'un opérateur A étant donné par

$$\langle \Psi_N, A\Psi_N \rangle \tag{2.12}$$

celle-ci ne doit pas dépendre des indices. La dichotomie boson/fermion suit alors [**132**, Lemma 7.1.1]

Lemme 2.1 (**Bosons et fermions**).
Considerons $\Psi_N \in L^2\left(\mathbb{R}^{dN}\right)$ avec

$$\int_{\mathbb{R}^{dN}} |\Psi_N|^2 \,\mathrm{d}\mathbf{x} = 1 \tag{2.13}$$

satisfaisant, pour n'importe quel opérateur auto-adjoint A sur $L^2\left(\mathbb{R}^{dN}\right)$ et n'importe quel $\sigma \in S_N$

$$\langle \Psi_N, A\Psi_N \rangle = \langle U_\sigma \Psi_N, A U_\sigma \Psi_N \rangle \tag{2.14}$$

alors soit

$$U_\sigma \Psi_N = \Psi_N \ \ \forall \sigma \in S_N \quad \textbf{Bosons} \tag{2.15}$$

ou

$$U_\sigma \Psi_N = (-1)^{\mathrm{sign}(\sigma)} \Psi_N \ \ \forall \sigma \in S_N \quad \textbf{Fermions} \tag{2.16}$$

PROOF. Prenons $A = |\Psi_N\rangle \langle \Psi_N|$. Selon (2.14) nous avons

$$|\langle \Psi_N, U_\sigma \Psi_N \rangle|^2 = 1 \tag{2.17}$$

pour tout $\sigma \in S_N$ et donc

$$U_\sigma \Psi_N = c_\sigma \Psi_N \tag{2.18}$$

avec c_σ un nombre complexe de module un. Nous avons aussi pour des raisons de consistence que

$$c_{\sigma\circ\sigma'} = c_\sigma c_{\sigma'} \tag{2.19}$$

pour tout σ et $\sigma' \in S_N$. L'application $\sigma \mapsto c_\sigma$ est donc une représentation scalaire du groupe des permutations. Il s'ensuit que $c_\sigma = 1$ ou $c_\sigma = (-1)^{\mathrm{sign}(\sigma)}$. □

Nous devons restreindre l'espace des fonctions d'onde $L^2\left(\mathbb{R}^{dN}\right)$ en conséquence.

$$L^2_{\mathrm{sym}}\left(\mathbb{R}^{dN}\right) \simeq \bigotimes_{\mathrm{sym}}^{N} L^2\left(\mathbb{R}^N\right) \quad \text{pour les bosons}$$

et

$$L^2_{\mathrm{asym}}\left(\mathbb{R}^{dN}\right) \simeq \bigotimes_{\mathrm{asym}}^{N} L^2\left(\mathbb{R}^N\right) \quad \text{pour les fermions.}$$

En mécanique quantique à N corps, un état pur Ψ_N s'identifie avec la matrice densité $|\Psi_N\rangle \langle \Psi_N|$. Un état mixte est alors définit comme une superposition d'états purs. Il s'agit d'une combinaison convexe de projecteurs orthogonaux. Eu égard au théorème spectral, l'ensemble des états mixtes correspond à celui des opérateurs auto-adjoints, positifs et de trace un

$$\mathcal{S}^1\left(L^2\left(\mathbb{R}^{dN}\right)\right) = \left\{\Gamma \in \mathfrak{S}^1\left(L^2\left(\mathbb{R}^{dN}\right)\right), \Gamma = \Gamma^*, \Gamma \geq 0, \mathrm{Tr}\,\Gamma = 1\right\} \tag{2.20}$$

où $\mathfrak{S}^1(\mathfrak{H})$ est la classe de Schatten-von Neumann des opérateurs à trace sur l'espace de Hilbert $\mathfrak{H}$. Nous pouvons aussi identifier un état mixte avec son noyau $\Gamma(X;Y)$ de manière que

$$\Gamma\Psi = \int_{\mathbb{R}^{dN}} \Gamma(X;Y)\,\Psi(Y)\,\mathrm{d}Y \tag{2.21}$$

avec

$$\Gamma(X;Y) = \overline{\Psi_N(Y)}\Psi_N(X) \tag{2.22}$$

dans le cas d'un état pur identifié avec Ψ_N. Nous définissons la k-ième matrice densité réduite en prenant la trace partielle sur les dernières $N-k$ particules

$$\Gamma^{(k)} = \mathrm{Tr}_{k+1\to N}\,\Gamma. \tag{2.23}$$

Cela signifie que pour tout opérateur borné A_k agissant sur $L^2\left(\mathbb{R}^{dk}\right)$

$$\mathrm{Tr}\left[A_k\Gamma^{(k)}\right] := \mathrm{Tr}\left[A_k \otimes \mathbb{1}^{\otimes N-k}\Gamma\right] \tag{2.24}$$

avec $\mathbb{1}$ l'identité de $L^2\left(\mathbb{R}^d\right)$. La mécanique quantique habituelle peut alors être exprimée en terme de matrices densité. La valeur moyenne d'une observable A dans l'état Γ est ainsi

$$\langle A\rangle_\Gamma = \mathrm{Tr}\left[A\Gamma\right] \tag{2.25}$$

avec, dans le cas d'un état pur $\Gamma = |\Psi_N\rangle \langle \Psi_N|$,

$$\mathrm{Tr}\left[A\Gamma\right] = \langle \Psi_N, A\Psi_N\rangle. \tag{2.26}$$

Dans le langage des matrices densité les symétries bosoniques et fermioniques sont imposées via

$$U_\sigma \Gamma = \Gamma U_\sigma = \Gamma \text{ pour les bosons} \tag{2.27}$$

$$U_\sigma \Gamma = \Gamma U_\sigma = (-1)^{\mathrm{sign}(\sigma)} \Gamma \text{ pour les fermions.} \tag{2.28}$$

3. Approximation de champ-moyen

Un Hamiltonien comprenant N particules interagissant par l'intermédiaire d'un potentiel d'interaction w_L et piégées dans un potentiel V a la forme

$$H_{N,L} = \sum_{j=1}^{N} \left(-\hbar^2 \Delta_j + V(\mathbf{x}_j)\right) + \lambda \sum_{1 \le k < j \le N} w_L(\mathbf{x}_j - \mathbf{x}_k) \tag{2.29}$$

où λ reprśente la force des interactions. Cet opérateur non borné agit sur $L^2\left(\mathbb{R}^{dN}\right)$. Il est très difficile de calculer l'énergie fondamentale

$$e_N = \inf_{\Psi_N \in L^2\left(\mathbb{R}^{dN}\right)} \langle \Psi_N, H_N \Psi_N \rangle \tag{2.30}$$

pour un Hamiltonien tel que (2.29) lorsque le nombre de particules est grand. Nous introduisons souvent en physique statistique des grandeurs émergentes macroscopiques telles que la température ou la pression. Ces quantités sont définies en moyenne et peuvent être étudiées par des équations effectives comme celle de Navier-Stokes ou d'Euler en mécanique des fluides. Cette méthode est applicable à la mécanique quantique. Nous introduisons des modèles effectifs décrivant le système dans son ensemble dans le but d'obtenir des théories plus maniables. Le modèle le plus simple et peut-être le plus connu est celui obtenu par approximation de champ moyen. Le but est de se ramener à un problème à un corps auto-consistant en partant d'un problème à N corps.

3.1. Approximation de Hartree pour les bosons. Dans le cas du Hamiltonien (2.29) agissant sur des bosons nous allons illustrer trois régimes, chacun menant à un modèle effectif différent. Le modèle limite obtenu dépend directement de la compétition entre la force des interactions λ et leur portée L. Le nombre typique de particules interagissant avec une particule donnée est ρL^d où ρ est la densité de celles-ci. Dans un espace de volume 1 (la taille du système) nous avons $\rho = N/1$. L'énergie d'interaction moyenne par particule est d'ordre $N\lambda L^d$. Cette énergie doit être d'ordre 1 pour que les termes de (2.29) soient comparables lorsque $N \to \infty$. Nous introduisons un paramètre de scaling $\beta \ge 0$ pour décrire la compétition entre L et λ

$$L = N^{-\beta}, \qquad \lambda = \frac{N^{d\beta}}{N-1}. \tag{2.31}$$

Le Hamiltonien (2.29) devient

$$H_N = \sum_{j=1}^{N} \left(-\Delta_j + V(\mathbf{x}_j)\right) + \frac{1}{N-1} \sum_{1 \le k < j \le N} N^{d\beta} w\left(N^\beta \mathbf{x}_j - \mathbf{x}_k\right). \tag{2.32}$$

Nous avons pris $\hbar = 1$, cette constante ne sera utile que dans le cas fermionique. L'énergie fondamentale pour un gaz de bosons est

$$e_N^{\mathrm{sym}} := \inf_{\Psi_N \in L^2_{\mathrm{sym}}\left(\mathbb{R}^{dN}\right)} \langle \Psi_N, H_N \Psi_N \rangle \,. \tag{2.33}$$

L'approximation de champ moyen se base sur l'intuition que dans un gaz avec un grand nombre de particules, chacune d'entre elles interagit avec un potentiel global et non plus avec toute les autres. Elles peuvent ainsi être vues comme indépendantes et identiquement distribuées. La fonction d'onde prend donc la forme

$$\Psi_N^{\mathrm{H}}(\mathbf{x}_1, ..., \mathbf{x}_N) = u(\mathbf{x}_1)u(\mathbf{x}_2)...u(\mathbf{x}_N) \tag{2.34}$$

avec u une densité à une particule de $L^2\left(\mathbb{R}^d\right)$. Cet ansatz couplé avec

$$w_L(\mathbf{x}) = L^{-d} w\left(\frac{\mathbf{x}}{L}\right) \to \left(\int_{\mathbb{R}^d} w\right) \delta_0 \tag{2.35}$$

au sens des distributions quand $L \to 0$ nous amène aux régimes qui suivent.

Régime champ moyen (CM). Quand $\beta = 0$ l'énergie obtenue par insersion de l'ansatz (2.34) est

$$\mathcal{E}_{\mathrm{H}}[u] := \frac{\left\langle \Psi_N^{\mathrm{H}}, H_N \Psi_N^{\mathrm{H}} \right\rangle}{N} = \int_{\mathbb{R}^d} \left(|\nabla u(\mathbf{x})|^2 + V(\mathbf{x}) |u(\mathbf{x})|^2 \right) \mathrm{d}\mathbf{x} \tag{2.36}$$

$$+\lambda \frac{(N-1)}{2} \int_{\mathbb{R}^d} |u(\mathbf{x})|^2 \left(w * |u|^2 \right)(\mathbf{x}) \mathrm{d}\mathbf{x}. \tag{2.37}$$

L'énergie $\mathcal{E}_{\mathrm{H}}$ est une énergie de Hartree associée au problème de minimisation

$$e_{\mathrm{H}} = \inf_{\substack{u \in L^2(\mathbb{R}^d) \\ \|u\|_{L^2}=1}} \mathcal{E}_{\mathrm{H}}[u] \,. \tag{2.38}$$

Nous obtenons alors que

$$\frac{e_N^{\mathrm{sym}}}{N} \to e_{\mathrm{H}} \quad \text{quand} \quad N \to \infty \tag{2.39}$$

comme énoncé dans [**136**, Section 6] ou encore [**52, 87, 40, 47, 54, 55, 81, 83, 85**].

Régime Schrödinger non linéaire (NLS). Lorsque $0 < \beta < 1$ l'ansatz (2.34) reste pertinent. Le potentiel w_L tend vers une masse de Dirac. Si nous le remplaçons simplement dans (2.37) nous obtenons la fonctionnelle NLS

$$\mathcal{E}_{\mathrm{NLS}}[u] := \int_{\mathbb{R}^d} \left(|\nabla u(\mathbf{x})|^2 + V(\mathbf{x}) |u(\mathbf{x})|^2 + \frac{a}{2} |u(\mathbf{x})|^4 \right) \mathrm{d}\mathbf{x}. \tag{2.40}$$

Ici

$$a = \int_{\mathbb{R}^d} w. \tag{2.41}$$

L'énergie fondamentale associée est

$$e_{\mathrm{NLS}} = \inf_{\substack{u \in L^2(\mathbb{R}^d) \\ \|u\|_{L^2}=1}} \mathcal{E}_{\mathrm{NLS}}[u] \,. \tag{2.42}$$

Nous avons transformé un problème linéaire avec une fonction d'onde à N corps en un problème cubique avec une fonction d'onde à un corps. Nous obtenons aussi la convergence

$$\frac{e_N^{\mathrm{sym}}}{N} \to \ e_{\mathrm{NLS}} \ \text{ quand } \ N \to \infty. \tag{2.43}$$

Nous référons à [**136**, Section 7] pour plus de détail. Ce régime contient deux situations physiques très différentes mais aboutisant au même modèle effectif.

- $(\beta < 1/d)$ correspond au cas où les particules interagissent souvent (L est grand) via de faibles interactions $\lambda \leq 1$. Nous sommes dans le régime de champ moyen.
- $(\beta > 1/d)$ correspond au cas où les particules interagissent fortement mais rarement. Nous somme dans un régime dilué.

La transision à lieu quand $\beta = 1/d$ c'est-à-dire β de l'ordre de la distance inter-particules.

Le régime Gross-Pitaevskii (GP). Quand $\beta = 1$ la fonctionnelle limite est toujours (2.40) mais

$$a = 4\pi \times \ \ (\text{longueur de diffusion de} \ \ w)\,. \tag{2.44}$$

La longueur de diffusion est définie [**95**, Appendix C]. L'ansatz (2.34) doit alors être corrigé pour prendre en compte des correlations courte-portées. Un bon candidat serait

$$\Psi_N\left(\mathbf{x}_1, ..., \mathbf{x}_N\right) = \prod_{j=1}^{N} u\left(\mathbf{x}_j\right) \prod_{1 \leq i < j \leq N} f\left(N^{\beta}\left(\mathbf{x}_i - \mathbf{x}_j\right)\right) \tag{2.45}$$

où f vient du problème de diffusion à deux corps défini par w. Nous parlons de régime Gross-Pitaevskii (GP) étudié dans la série de papiers [**107, 106, 145, 97, 96, 98**]. Pour plus de details à propos de la condensation de Bose-Einstein et des théories de champ moyen nous référons à [**137, 87, 136, 81, 95**].

3.2. Approximation de Hartree-Fock pour les fermions. Dans le cas des fermions, l'énergie cinétique augmente plus vite lorsque N grandit en raison du principe de Pauli. Pour équilibrer celle-ci vis-à-vis des autres termes du Hamiltonien (2.29) nous prenons $\hbar = N^{-1/d}$. C'est ce que l'on appelle l'approche semi-classique. Nous n'investigons dans ce qui suit que le régime champ moyen où $\lambda = 1/N$ et $L = 1$. Le Hamiltonien (2.29) devient

$$H_N = \sum_{j=1}^{N} \left(\frac{-\Delta_j}{N^{1/d}} + V\left(\mathbf{x}_j\right) \right) + \frac{1}{N} \sum_{1 \leq k < j \leq N} w\left(\mathbf{x}_j - \mathbf{x}_k\right) \tag{2.46}$$

avec un potentiel d'interactions w fixé. L'état fondamental de (2.29) est

$$e_N^{\mathrm{asym}} := \inf_{\Psi_N \in L^2_{\mathrm{asym}}\left(\mathbb{R}^{dN}\right)} \left\langle \Psi_N, H_N \Psi_N \right\rangle. \tag{2.47}$$

Le principe de Pauli ne permet pas de complètement décorréler les particules au sens de (2.34). L'ansatz de champ moyen pour les fermions consiste à prendre une fonction d'onde sous forme d'un determinant de Slater

$$\Psi_N^{\mathrm{SL}}(\mathbf{x}_1, ..., \mathbf{x}_N) = \det\left(u_j\left(\mathbf{x}_k\right)\right)_{1 \leq j,k \leq N} \tag{2.48}$$

où les fonctions $\{u_j\}_{j=1}^{N}$ des fonctions à 1 corps de $L^2\left(\mathbb{R}^d\right)$ orthogonales entre elles. Nous définissons la première matrice densité associée au determinant de Slater par

$$\gamma = \sum_{j=1}^{N} |u_j\rangle \langle u_j| \quad \text{avec} \quad \rho_\gamma = \sum_{j=1}^{N} |u_j|^2 \tag{2.49}$$

et calculons l'énergie Hartree-Fock

$$\mathcal{E}_{\mathrm{HF}}[\gamma] = \mathrm{Tr}\left[-\Delta\gamma\right] + \int_{\mathbb{R}^d} V(\mathbf{x})\rho_\gamma(\mathbf{x})\mathrm{d}\mathbf{x} \tag{2.50}$$

$$+ \frac{1}{2N}\int_{\mathbb{R}^d\times\mathbb{R}^d} w(\mathbf{x}-\mathbf{y})\left(\rho_\gamma(\mathbf{x})\rho_\gamma(\mathbf{y}) - |\gamma(\mathbf{x},\mathbf{y})|^2\right)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \tag{2.51}$$

dans laquelle nous utilisons le noyau de γ

$$\gamma(x,y) = \sum_{j=1}^{N} u_j(\mathbf{x})\overline{u_j(\mathbf{y})}. \tag{2.52}$$

En notant

$$e_{\mathrm{HF}} := \inf_{\substack{\gamma^2=\gamma=\gamma^* \\ \mathrm{Tr}\,\gamma=N}} \mathcal{E}_{\mathrm{HF}}[\gamma] \tag{2.53}$$

nous pouvons, supposant que V est confinant

$$\lim_{|\mathbf{x}|\to\infty} V(\mathbf{x}) = +\infty \tag{2.54}$$

et que $w \in L^{1+d/2}\left(\mathbb{R}^d\right) + L^\infty_\varepsilon\left(\mathbb{R}^d\right)$, obtenir la convergence

$$\left|\frac{e_N^{\mathrm{asym}}}{N} - e_{\mathrm{HF}}\right| \to 0 \ \text{ quand } \ N\to\infty. \tag{2.55}$$

Dans la plupart des cas nous n'obtenons que le premier ordre de la convergence ci-dessus. Il s'agit d'une énergie Thomas-Fermi, minimum de la fonctionnelle

$$2\pi\int_{\mathbb{R}^2}\rho^2(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2} V(\mathbf{x})\,\rho(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{2.56}$$

Des exemples de résultats similaires peuvent être trouvés dans [**147, 99, 103, 50, 91, 90**].

CHAPTER 3

Modélisation de la statistique anyonique

1. L'espace de configurations

Nous commençerons par discuter des arguments habituels justifiant la dichotomie boson/fermion. L'espace des configurations classique d'un système de N particules en dimension d est $\mathbb{R}^{dN}$. Un vecteur de cet espace sera noté $X_N = (\mathbf{x}_1, ..., \mathbf{x}_N)$ avec chaque $\mathbf{x}_j \in \mathbb{R}^d$. Pour construire une théorie quantique sur cet espace nous introduisons les fonctions d'onde $\Psi_N \in L^2\left(\mathbb{R}^{dN}\right)$ puis symétrisons la théorie obtenue comme en (2.10). De cette manière le Lemme 2.1 fournit une preuve solide de cette dichotomie. Il y a néanmoins certains choix implicites dans la construction présentée ci-dessus [**58, 62, 148, 149**]. Nous avons en effet

(1) quantifié i.e introduit les fonctions d'onde sur l'espace des configurations.
(2) symétrisé i.e imposé une condition de symétrie sur les fonctions d'onde.

Il n'y a cependant aucune nécessité logique imposant d'effectuer (1) puis (2). Étudions ce qui arrive lorsque que nous réalisons (2) puis (1).

1.1. Symétrisation d'un système classique. Considerons N particules impénétrables vivant dans $\mathbb{R}^{dN}$. Un point dans cet espace est

$$X_N := (\mathbf{x}_1, ..., \mathbf{x}_N) \in \mathbb{R}^{dN}. \tag{3.1}$$

Deux particules ne pouvant s'interpénétrer, nous retirons les diagonales de $\mathbb{R}^{dN}$

$$\mathbb{D} := \left\{ X_N \in \mathbb{R}^{dN}, \ \ \mathbf{x}_j = \mathbf{x}_k \ \text{ for some } \ j \neq k \right\}. \tag{3.2}$$

L'espace des configurations de N particules identiques dans $\mathbb{R}^{dN}$ est alors formellement donné par

$$C_N^d = \frac{\mathbb{R}^{dN} \backslash \mathbb{D}}{S_N} \tag{3.3}$$

où S_N est le groupe des permutations de N éléments agissant sur $\mathbb{R}^{dN}$ de la façon suivante

$$\sigma\left(X_N\right) = \left(\mathbf{x}_{\sigma(1)}, ..., \mathbf{x}_{\sigma(N)}\right). \tag{3.4}$$

Dans l'espace C_N^d tout point X_N est identifié avec tous les $\sigma\left(X_N\right)$ pour tout $\sigma \in S_N$. Un échange de particules doit maintenant être appréhendé comme un processus physique au sein de cet espace. En d'autres termes l'échange de deux particules est associé à un chemin dans C_N^d qui se trouve être une boucle puisque le point de départ est identifié avec le point d'arrivé.

Afin de mieux se représenter l'espace C_N^d nous introduisons la coordonée du centre de masse

$$\mathbf{X}\left(X_N\right) = \frac{1}{N} \sum_{j=1}^{N} \mathbf{x}_j \in \mathbb{R}^d \tag{3.5}$$

satisfaisant à

$$\mathbf{X}\left(\sigma\left(X_N\right)\right) = \mathbf{X}\left(X_N\right).$$

Nous pouvons donc factoriser

$$C_N^d \simeq \mathbb{R}^d \times r^d \tag{3.6}$$

avec

$$r^d = \frac{\left\{X_N \in \mathbb{R}^{dN} \backslash \mathbb{D},\ \ \mathbf{X}\left(X_N\right) = 0\right\}}{S_N} \tag{3.7}$$

n'étant rien d'autre que la coordonée relative.

1.2. Quantification d'un système classique. Les fonctions d'onde de N particules identiques sont définies comme des fonctions complexes de carré sommables sur C_N^d munies de conditions de recollemet appropriées

$$\Psi\left(\sigma\left(X_N\right)\right) = e^{\mathrm{i}\pi\alpha}\Psi\left(X_N\right). \tag{3.8}$$

L'amplitude doit être la même mais la phase peut changer à chaque recollement. Nous introduisons le premier groupe d'homotopie de C_N^d

$$\pi^1\left(C_N^d\right) = \Gamma/\sim \tag{3.9}$$

où Γ est l'ensemble des boucles de C_N^d et où $\sim$ désigne la relation d'équivalence entre boucles

$$\gamma_1 \sim \gamma_2 \quad \Leftrightarrow \quad \gamma_1 \ \text{ peut être déformé continûment en } \ \gamma_2. \tag{3.10}$$

Le premier groupe d'homotopie est doté de la loi de concaténation entre boucle. Remarquons que, l'espace C_N^d étant connexe, le groupe d'homotopie reste le même quelque soit le point de départ des boucles. Nous ne notifions donc pas ce point dans nos notations. L'ensemble Γ contient toutes les boucles que les particules peuvent suivre durant un échange. Comme énoncé dans [**28, 78**] la phase acquise lors d'un échange $e^{\mathrm{i}\pi\alpha}$ doit correspondre à une représentation scalaire du groupe $\pi^1\left(C_N^d\right)$. C'est une chose bien connue en topologie [**7**] que

$$\pi^1\left(C_N^d\right) \simeq S_N \ \text{ si } \ d \geq 3. \tag{3.11}$$

Dans le cas $d = 2$ l'espace $\mathbb{R}^d\backslash\{0\}$ n'est plus simplement connexe. Le groupe fondamental s'en trouve fortement affecté puisqu'il s'agit dans ce cas du groupe des tresses [**9**]

$$\pi^1\left(C_N^2\right) \simeq B_N. \tag{3.12}$$

Lorsque nous nous trouvons en dimension supérieure ou égale à trois nous n'avons que deux représentations scalaires possibles pour le groupe fondamental. Il existe en revanche une infinité de telles représentations pour le groupe des tresses. Celles-ci sont indexées par un réel $\alpha \in [0, 2[$.

Afin d'illustrer cette différence topologique entre les espaces des configurations en deux et trois dimensions nous allons considérer le cas de deux particules en dimension 3 et en dimension 2. Les espaces de configurations en question sont C_2^2 and C_2^3. Selon (3.6) $C_2^3 \simeq \mathbb{R}^3 \times r^3$ avec la coordonnée relative

$$\mathbf{r} = \frac{\mathbf{x}_1 - \mathbf{x}_2}{2} \in r^3$$

où $\mathbf{x}_1$ et $\mathbf{x}_2$ peuvent prendre n'importe quelle valeur de $\mathbb{R}^2$ satisfaisant $\mathbf{x}_1 \neq \mathbf{x}_2$ et telles que les points $\mathbf{r}$ et $-\mathbf{r}$ soit identifiés. Notons que

$$|\mathbf{r}| = \sqrt{|\mathbf{x}_1|^2 + |\mathbf{x}_2|^2} \tag{3.13}$$

nous utilisons alors les coordonées sphériques pour écrire $\mathbf{r}$ sous la forme $(|\mathbf{r}|, \theta, \phi)$ et factoriser

$$r^3 \simeq \mathbb{R}_*^+ \times S_{\theta,\phi} \tag{3.14}$$

avec $S_{\theta,\phi}$ une sphère de rayon un dont les points diamétrallement opposés sont identifiés. Le schéma 3.1 représente l'espace $S_{\theta,\phi}$, un hémisphère nord avec les points de l'équateur

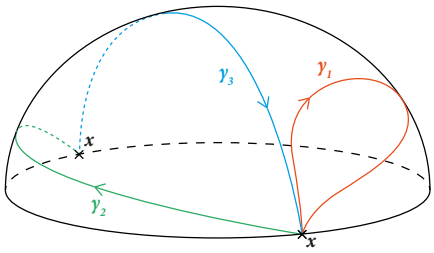


FIGURE 3.1. Espace projectif $S_{\theta,\phi}$

diamètrallement opposés identifiés. En partant d'un point $\mathbf{x}$, il y a deux types de boucles non équivalentes modulo homotopie. La rouge (équivalente à la boucle nulle) et la verte passant d'un coté de l'hémisphère à l'autre. Nous voyons sur le schéma que γ_1 ne peut pas être déformé en γ_2 mais que tout autre boucle basée en $\mathbf{x}$ peut être déformée en l'une des deux. La boucle γ_3 peut être transformée en γ_2 et $\gamma_2 \circ \gamma_3$ en γ_1. Le premier groupe d'homotopie est alors

$$\pi^1\left(C_2^3\right) \simeq S_2 = \{\mathbb{1}, \sigma_{12}\}.$$

Remarquons qu'un double échange est ici toujours équivalent à la boucle nulle. En effet, si nous suivons la courbe rouge (ou l'un de ces équivalents) n fois, la courbe obtenue est toujours équivalente à la boucle nulle. Si nous suivons γ_2 deux fois nous revenons par un équivalent de γ_3, la courbe obtenue est $\gamma_2 \circ \gamma_3 \simeq \gamma_1$. Les échanges exécutés un nombre pair de fois sont donc représentés par la phase e^0 et les impairs par $e^{-\mathrm{i}\pi}$.

Dans le cas de la dimension 2 nous avons $C_2^2 = \mathbb{R}^2 \times r^2$ avec $\mathbf{r} \in r^2$ un point de $\mathbb{R}_*^2$ identifié à $-\mathbf{r}$. Nous pouvons identifier r^2 avec le demi-plan supérieur privé de zéro au sein duquel la demi-droite $\mathbb{R}_*^+$ est identifiée à la demi-droite $\mathbb{R}_*^-$. Nous obtenons donc un cône sans sommet en repliant le plan de sorte que $\mathbb{R}_*^+$ se superpose à $\mathbb{R}_*^-$. Le résultat est schématisé figure 3.2. Ici, γ_1 peut aussi être réduite à un point alors que γ_2 ne le peut pas. Le nouvel aspect topologique du cône sans sommet est l'existence d'une autre sorte de boucle γ_3. Deux boucles, l'une exécutant n tours et l'autre $m \neq n$ tours ne peuvent pas être déformées l'une en l'autre. Ainsi, chaque nouveau tour définit une nouvelle classe de boucles.

Nous obtenons un groupe fondamental constitué d'une infinité de boucles différentes idexées par le nombre de tours qu'elles exécutent autour du cône. Nous voyons donc que

$$\pi^1\left(C_2^2\right) \simeq B_2 \simeq \mathbb{Z}.$$

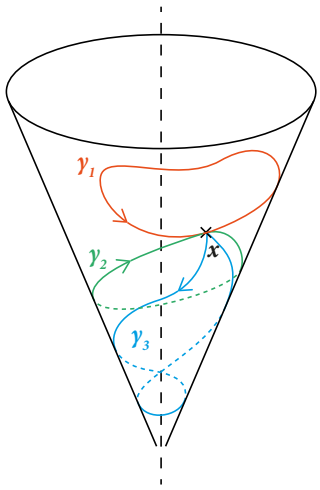


FIGURE 3.2. Cône sans sommet r^2

Une différence notable entre $S_2 \simeq (\{0,1\},+)$ et $B_2 \simeq (\mathbb{Z},+)$ concerne leur générateurs. Les générateur s_i de S_2, et de S_N en général, ont la propriété $s_i^2 = \mathbb{1}$ ce qui n'est plus le cas pour les générateurs de B_N. Dans B_N un double échange de deux particules n'est pas équivalent à l'identité et sa reprśentation peut ne pas être le scalaire e^0. L'échange de deux particules voisines (n'encerclant pas d'autres particules) induit une phase $e^{i\pi\alpha}$. Il est représenté sur la figure 3.3 où chaque brin tient lieu d'une particule. Ce schéma décrit un échange où les particules 1 et 3 encerclent la particules 2

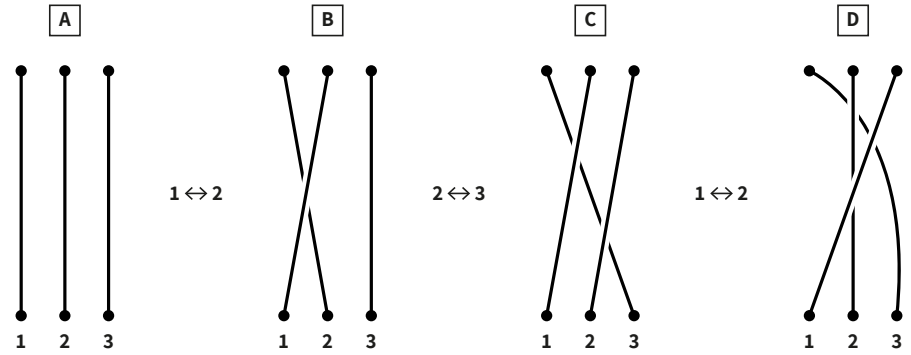


FIGURE 3.3. Diagramme des tresses décrivant l'échange $1-3$ encerclant la particule 2.

Sur ce diagramme le temps évolue du haut vers le bas. Afin d'échanger les brins 1 et 3 trois échanges entre voisins sont nécessaires. Le premier correspond au générateu b_{12} agissant sur A pour donner B en échangeant les brins 1 et 2. Notons que le brin 1 passe en-dessous. C'est pour cela que la répétition de cette action en B ne nous ramène pas en A. Les brins 1 et 2 ont été noués, $b_{12}^2 \neq \mathbb{1}$. En D, trois échanges de voisinnage ont été

exécutés. La phase correspondante est $e^{3i\pi\alpha}$. En généralisant le processus décrit ci-dessus nous voyons que lorsque p particules sont encerclées par la trajectoire des deux particules que nous échangeons la phase acquise est $e^{i(2p+1)\pi\alpha}$. Quand une seule particule fait les tour de p particules la phase est $e^{i(2p)\pi\alpha}$.

2. La formulation par jauge magnétique

2.1. Anyons idéaux. Dans la pratique les anyons sont modélisés comme des bosons ou des fermions munis d'une interaction magnétique de type Aharonov-Bohm entre chaque paire de particules. La fonction d'onde anyonique doit formellement se comporter de la façon suivante lors d'un échange entre deux particules

$$\tilde{\Psi}(\mathbf{x}_1, ..., \mathbf{x}_j, ..., \mathbf{x}_k, ..., \mathbf{x}_N) = e^{i\alpha\pi}\tilde{\Psi}(\mathbf{x}_1, ..., \mathbf{x}_k, ..., \mathbf{x}_j, ..., \mathbf{x}_N)$$

ceci s'obtient en écrivant

$$\tilde{\Psi}(\mathbf{x}_1, ..., \mathbf{x}_N) = \prod_{j<k} e^{i\alpha\phi_{jk}}\Psi(\mathbf{x}_1, ..., \mathbf{x}_N) \text{ où } \phi_{jk} = \text{angle}\frac{\mathbf{x}_j - \mathbf{x}_k}{|\mathbf{x}_j - \mathbf{x}_k|}$$

avec Ψ une fonction bosonique, symétrique par échange de particules. Notons que les cas $\alpha = 1$ et $\alpha = 0$ correspondent respectivement aux fermions et aux bosons. Nous définissons la transformation de jauge singulière U_α

$$\left(U_\alpha\Psi\right)(\mathbf{x}) = \prod_{j<k} e^{i\alpha\phi_{jk}}\Psi(\mathbf{x}) \tag{3.15}$$

Ainsi,

$$U_1 : L^2_{\text{asym/sym}}\left(\mathbb{R}^{2N}\right) \to L^2_{\text{sym/asym}}\left(\mathbb{R}^{2N}\right) \tag{3.16}$$

est un isomorphisme. De plus U_1 est continu sur $\mathbb{R}^{2N}\backslash\mathbb{D}$. Soit l'opérateur $\mathbf{p}_j = -i\hbar\nabla_j$, nous observons que

$$U_\alpha^{-1}\mathbf{p}_j U_\alpha = \mathbf{p}_j + \alpha\sum_{k\neq j}\frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|^2} =: \mathbf{p}_j + \alpha\mathbf{A}(\mathbf{x}_j) := \mathbf{p}_j + \alpha\mathbf{A}_j. \tag{3.17}$$

Nous obtenons l'opérateur magnétique

$$D_j = \mathbf{p}_j + \alpha\mathbf{A}_j \tag{3.18}$$

et l'énergie cinétique associée

$$T_\alpha = \sum_{j=1}^{N}\left(\mathbf{p}_j + \alpha\mathbf{A}_j\right)^2 = \sum_{j=1}^{N} D_j^2 \tag{3.19}$$

agissant sur des fonctions bosoniques. Cette formulation via la **jauge magnétique** permet de décrire des anyons en modifiant l'énergie cinétique de simples bosons. Les anyons peuvent alors être vus comme des bosons interagissant par le biais d'un champ magnétique fictif

$$\text{curl}\mathbf{A}(\mathbf{x}_j) = 2\pi\sum_{k\neq j}\delta(\mathbf{x}_j - \mathbf{x}_k). \tag{3.20}$$

Chaque boson transporte un solénoïde infiniment fin perpendiculaire au plan. La force de ce flux est $2\pi\alpha$. La formulation en jauge magnétique permet aussi de travailler avec des fonctions fermioniques. En définissant $\alpha = 1 + \widetilde{\alpha}$ et usant du changement de jauge U_1 (3.15)

nous pouvons transformer T_α agissant sur L^2_{sym} en $T_{\widetilde{\alpha}}$ agissant sur L^2_{asym}. Nous observons que lorsque $\alpha \to 0$ nous retrouvons l'énergie cinétique des bosons et de la même façon celle des fermions lorsque $\alpha \to 1$. Ces limites seront dénommées

- La limite quasi-bosonique : $\alpha \to 0$ et T_α agit sur $L^2_{\text{sym}}\left(\mathbb{R}^{2N}\right)$.
- La limite quasi-fermionique : $\widetilde{\alpha} \to 0$ et $T_{\widetilde{\alpha}}$ agit sur $L^2_{\text{asym}}\left(\mathbb{R}^{2N}\right)$. Formellement, $\alpha \to 1$ et T_α agissant sur $L^2_{\text{sym}}\left(\mathbb{R}^{2N}\right)$.

Pour tout $\alpha \in [0, 2[$ l'énergie cinétique anyonique agit sur le domaine $\mathcal{D}^N_\alpha$ en tant que forme quadratique

$$q_\alpha : L^2\left(\mathbb{R}^{2N}\right) \to \mathbb{R} \tag{3.21}$$

$$u \mapsto q_\alpha(u) = \sum_{j=1}^{N} \int_{\mathbb{R}^{2N}} \bar{u} D_j^2 u \mathrm{d}\mathbf{x} \tag{3.22}$$

definie sur

$$C_0^\infty\left(\mathbb{R}^{2N} \backslash \mathbb{D}\right) \cap L^2_{\text{sym}}\left(\mathbb{R}^{2N}\right)$$

l'ensemble des fonctions continues de carré intégrable sur $\mathbb{R}^{2N}$ avec support en dehors des diagonales et dotées des propriétés de symétrie adaptées. L'espace des fonctions de N anyons à énergie cinétique finie $\mathcal{D}^N_\alpha$ correspond au domaine de la fermeture de la forme quadratique (3.22). Un autre domaine d'intérêt est $\tilde{\mathcal{D}}^N_\alpha$ définit comme le domaine de l'extension de Friedrichs de l'opérateur auto-adjoint (3.19) associé à la fermeture de la forme (3.22). Il a été montré dans [**118**] que ce dernier correspond au domaine maximal de la form q_α (3.22). Voir [**118**] pour plus de détail. Nous pouvons remarquer que lorsque $\alpha \in 2\mathbb{Z}$ et $\alpha \in 2\mathbb{Z}+1$, $\mathcal{D}^N_\alpha$ est équivalent par changement de jauge à $\mathcal{D}^N_0 = H^1_{\text{sym}}$ et à $D^N_1 = H^1_{\text{asym}}$, l'espace de Sobolev

$$H^1 := W^{1,2} := \left\{ \psi \in C^\infty\left(\mathbb{R}^{2N}\right) : \left(\sum_{|\alpha| \leq 1} \|D^\alpha \psi\|^2_{L^2(\mathbb{R}^{2N})} \right)^{1/2} < \infty \right\}$$

restreint aux fonctions symétriques ou antisymétriques.

3. Les inégalités de Lieb-Thirring

Nous savons que le comportement des anyons est plus proche de celui des fermions que des bosons. Cette proximité est justifiée par le fait que les anyons, comme les fermions, sont sujets à l'inégalité de Lieb-Thirring. Une conséquence du principe d'exclusion de Pauli est l'inégalité suivante énoncant que

$$\sum_{j=1}^{N} \int_{\mathbb{R}^{dN}} \left(|\nabla_j u|^2 + V(\mathbf{x}_j) |u|^2 \right) \mathrm{d}\mathbf{x} \geq - \sum_{k=1}^{N-1} |\lambda_k(h)| \geq -C_d^{\text{LT}} \int_{\mathbb{R}^d} |V_-(\mathbf{x})|^{1+\frac{d}{2}} \mathrm{d}\mathbf{x} \tag{3.23}$$

pour un certain potentiel V donné avec $\lambda_k(h)$ les valeurs propres négatives de l'opérateur de Schrödinger

$$h := -\Delta_{\mathbb{R}^d} + V \tag{3.24}$$

agissant sur $L^2\left(\mathbb{R}^d\right)$ et V_- la partie négative de V, $V_- := \max(-V, 0)$. La première inégalité (3.23) exprime que l'énergie minimale pour N fermions est obtenue en les distribuant dans les N premières orbitales du problème à un corps h. En d'autres termes

$$u = \bigotimes_{k=1 \text{ à } N-1}^{\text{asym}} u_k$$

est un produit antisymétrique de fonctions propres à une particule $u_k(h)$ (appelé determinant de Slater). La seconde inegalité concerne la trace sur la partie négative du spectre de h. De plus l'inégalité (3.23)

$$\sum_{j=1}^{N} \int_{\mathbb{R}^{dN}} |\nabla_j u|^2 \, \mathrm{d}\mathbf{x} \geq C_d^{\mathrm{K}} \int_{\mathbb{R}^d} \rho(\mathbf{x})^{1+\frac{2}{d}} \mathrm{d}\mathbf{x} \tag{3.25}$$

$$\rho(\mathbf{x}) := \sum_{j=1}^{N} \int_{\mathbb{R}^{d(N-1)}} |u(\mathbf{x}_1, ..., \mathbf{x}_j = \mathbf{x}, ..., \mathbf{x}_N)|^2 \prod_{k \neq j} \mathrm{d}\mathbf{x}_k \tag{3.26}$$

est interprétée comme une forme du principe d'exclusion pour les fermions. D'un autre coté le mieux que nous puissions obtenir pour des bosons est

$$\sum_{j=1}^{N} \int_{\mathbb{R}^{dN}} \left(|\nabla_j u|^2 + V(\mathbf{x}_j) \, |u|^2 \right) \mathrm{d}\mathbf{x} \geq -N C_d^{\mathrm{LT}} \int_{\mathbb{R}^d} |V_-(\mathbf{x})|^{1+\frac{d}{2}} \, \mathrm{d}\mathbf{x} \tag{3.27}$$

en mettant toutes les particules dans le même état u_0, $u = u_0 \otimes ... \otimes u_0$ avec u_0 qui satisfait (3.23). Nous obtenons alors directement des inégalités de Sobolev [**92**] le résultat

$$\sum_{j=1}^{N} \int_{\mathbb{R}^{dN}} |\nabla_j u|^2 \, \mathrm{d}\mathbf{x} \geq \frac{C_d}{N^{2/d}} \int_{\mathbb{R}^d} \rho(\mathbf{x})^{1+\frac{2}{d}} \mathrm{d}\mathbf{x} \tag{3.28}$$

devenant trivial quand $N \to \infty$.

L'énergie cinétique anyonique (3.19) étant un authentique opérateur à N corps, rend difficile la transposition des méthodes usuelles pour dériver les inégalités de Lieb-Thirring. En particulier (3.23) n'est plus équivalent à (3.25). Aucun résultat de ce type n'état disponible jusqu'à la série de publications [**115, 118, 113**] donnant accès à plusieurs résultats concernant les énergies fondamentales des gaz d'anyons. Il a été dérivé en particulier l'inégalité suivante

THÉORÈME 3.1 (**Inégalité de Lieb-Thirring pour des anyons idéaux (Lundholm-Seiringer)**).
Il existe une constant $C > 0$ telle que pour tout $0 \leq \alpha \leq 1$, $N \geq 1$ et $\Psi_N \in \mathcal{D}_N^\alpha$

$$\sum_{j=1}^{N} \int_{\mathbb{R}^{dN}} |D_j \Psi_N|^2 \, \mathrm{d}x \geq C\alpha \int_{\mathbb{R}^d} \rho_{\Psi_N}(\mathbf{x})^2 \mathrm{d}\mathbf{x} \tag{3.29}$$

Une conséquence directe de cette dernière inégalité appliquée à un gaz d'anyons dans un potentiel $V(\mathbf{x}_1, .., \mathbf{x}_N) = \sum_{j=1}^{N} V(\mathbf{x}_j)$ avec V un potentiel à un corps est que

$$e_\alpha^V := \inf \mathrm{spec} \left(T_\alpha + \tilde{V} \right) \geq -\frac{1}{4C\alpha} \int_{\mathbb{R}^2} V_-(\mathbf{x})^2 \mathrm{d}\mathbf{x} \tag{3.30}$$

indépendamment de N. En appliquant ceci au cas de l'oscillateur harmonique nous obtenons

$$e_\alpha^V \geq C\frac{4N^{3/2}}{3}\sqrt{\alpha}. \tag{3.31}$$

Les anyons idéaux sont donc soumis au principe de Pauli. Ils sont en ce sens plus proche des fermions que des bosons.

CHAPTER 4

Les anyons dans l'EHQF

Il fut montré par Laughlin [**77, 76**], Halperin [**67**] et Arovas- Schrieffer-Wilczek [**8**] que l'EHQF pouvait être décrit par le truchement d'une fonction d'onde particulière dont les excitations sont des anyons. En 2020 deux expériences [**127**] et [**12**] ont mesuré des particules présentant la caractéristique d'avoir une phase d'échange $e^{\frac{i\pi}{3}}$, signe irréfutable de la statistique anyonique. Les deux expériences utilisent un gaz d'électrons 2D dans le régime de l'EHQF. Nous détaillons dans ce chapitre les caractéristiques principales de l'EHQF permettant de comprendre comment les anyons peuvent y apparaître. Nous terminerons en décrivant à grands traits les expériences [**127**] et [**12**].

1. L'effet Hall quantique

L'effet Hall fut découvert par Edwin Hall en 1879. Lorsqu'un courant électrique I est appliqué à une plaque conductrice bi-dimensionnelle soumise à un champ magnétique perpendiculaire à celle-ci, une tension V_H apparaît perpendiculairement au courant et au champ magnétique. Cette tension est due à l'accumulation d'électrons sur les bords de la plaque déviés par la force de Lorentz. Les découvertes de l'effet Hall quantique entier et fractionnaire, respectivement dénotés EHQE et EHQF, sont plus récentes. L'EHQE fut découvert par von Klitzing en 1980. L'EHQF [**60, 61, 74**] fut mis en évidence par Tsui, Störmer et Gossard [**159**] en 1982. Ces effets doivent leur nom au fait que la conductivité de Hall

$$\sigma_H = \frac{V_H}{I} = \nu \frac{e^2}{h} \tag{4.1}$$

prend des valeurs quantifiées indexées par le nombre ν pouvant être entier (EHQE) ou fractionnaire (EHQF). La précision de la quantification est d'une robustesse impressionnante. Elle est utilisée pour établir le standard de résistivité. Pour mieux comprendre la description quantique des phénomenes évoqués ci-dessus nous nous intéressons au cas d'un gaz d'électrons planaire soumis à un champ magnétique constant perpendiculaire au plan.

2. Les niveaux de Landau

Le Hamiltonien de N électrons évoluant dans le plan (avec charge 1 et masse $m = 1/2$) interagissant avec le champ magnétique $\mathbf{B}$ perpendiculaire au plan s'écrit

$$H_N = \sum_{j=1}^{N} \left(-i\hbar\nabla_j + B\mathbf{x}^{\perp} \right)^2 \tag{4.2}$$

avec $\mathbf{x}^{\perp} = (-y, x)$ lorsque $\mathbf{x} = (x, y)$. Le spectre de (4.2) est bien connu [**74**]. Il présente des niveaux d'énergie

$$E_n = \hbar B \left(n + \frac{1}{2} \right). \tag{4.3}$$

Les états d'énergie E_0 forment le plus bas niveau de Landau (LLL). Dans cet espace les fonctions sont de la forme

$$\Psi^{\mathrm{LLL}}(\mathbf{x}) = f(\mathbf{z}) e^{-\frac{B}{\hbar}\frac{|\mathbf{z}|^2}{4}} \tag{4.4}$$

où $\mathbf{x} \in \mathbb{R}^2$ est identifié avec $\mathbf{z} \in \mathbb{C}$ et où f est une fonction holomorphe. La dégénérescence du LLL est infinie et peut être estimée en unités d'aire de façon heuristique en notant que

$$\mathrm{LLL} = \mathrm{span} \left\{ \phi_n := \mathbf{z}^n e^{-\frac{B}{\hbar}\frac{|\mathbf{z}|^2}{4}} \right\}_{n=0}^{\infty}. \tag{4.5}$$

Les densités radiales $|\phi_n|^2$ sont localisées au niveau du rayon $r_{\max}$ tel que

$$0 = \partial_r |\phi_n|^2 (r_{\max}) \tag{4.6}$$

$$0 = \left(2n r_{\max}^{2n-1} - r_{\max}^{2n+1} \frac{B}{\hbar} \right) e^{-\frac{B}{\hbar}\frac{|\mathbf{z}|^2}{4}} \tag{4.7}$$

$$r_{\max} = \sqrt{\frac{2n\hbar}{B}}. \tag{4.8}$$

Nous pouvons ainsi loger n états dans un disque de centre 0 et de rayon $r_{\max}$. La dégénérescence par unité d'aire est alors

$$D \simeq \frac{n}{\pi r_{\max}^2} \simeq \frac{B}{2\pi\hbar}. \tag{4.9}$$

Le facteur de remplissage est défini en comptant combien de niveaux de Landau sont remplis en présence de N particules

$$\nu = \frac{\rho}{D}. \tag{4.10}$$

3. La fonction d'onde de Laughlin

L'EHQE s'explique par la quantification de l'énergie cinétique des électrons soumis au champ magnétique. L'EHQF est dû aux fortes corrélations entre électrons peuplant un niveau de Landau partiellement rempli. Dans ce dernier régime Laughlin montra [**76**] que les interactions entre électrons devaient être prises en compte et décrit le gaz comme un liquide imcompressible d'électrons fortement corrélés. L'idée principale de Laughlin fut la proposition d'une fonction d'onde prenant en compte ces corrélations

$$\Psi_{\mathrm{Lau}}(\mathbf{z}_1, ..., \mathbf{z}_N) = c_{\mathrm{Lau}} \prod_{1 \leq i < j \leq N} (\mathbf{z}_i - \mathbf{z}_j)^n \, e^{-B|\mathbf{z}_j|^2/4} \tag{4.11}$$

où la position des électrons $\mathbf{y}_i \in \mathbb{R}^2$ est identifiée avec $\mathbf{z}_i \in \mathbb{C}$ et où c_{Lau} est une constante de normalisation. Le paramètre n est un entier (impair pour l'antisymétrie). Le facteur de remplissage ν peut être identifié avec $1/n$ [**134**].
Nous observons que la forme (4.11) assure que l'énergie n'explose pas lorsque deux électrons se rapprochent. On voit aussi qu'à grand B plus de particules sont localisées près du centre $Z_N = (\mathbf{z}_1, ..., \mathbf{z}_N) = 0$. On s'attend donc à ce que le facteur n augmente en conséquence.

3.1. Les quasi-trous. Il a déjà été proposé dans [**169, 30**] (et revisité dans [**121**]) que des particules avec des statistiques fractionnaires pouvaient émerger dans ce contexte. Selon [**121**], sous des hypothèses raisonnables, la fonction d'onde décrivant N particules immobiles $Y_N = (\mathbf{y}_1, ..., \mathbf{y}_N)$ formant un bain, couplé avec M quasi-trous aux emplacements $X_M = (\mathbf{x}_1, ..., \mathbf{x}_M)$ devrait avoir la form:

$$\Psi\left(X_M, Y_N\right) = c_{\mathrm{qh}} \Psi_{\mathrm{qh}}\left(X_M, Y_N\right) \Phi\left(X_M\right) \tag{4.12}$$

où Ψ_{qh} est la fonction de Laughlin

$$\Psi_{\mathrm{qh}}\left(X_M, Y_N\right) = \prod_{j=1}^{M} \prod_{k=1}^{M} \left(\zeta_j - \mathbf{z}_k\right) \prod_{1 \leq i < j \leq N} \left(\mathbf{z}_i - \mathbf{z}_j\right)^n e^{-B|\mathbf{z}_j|^2/4} \tag{4.13}$$

avec $\mathbf{x}_i \in \mathbb{R}^2$ et $\mathbf{y}_i \in \mathbb{R}^2$ respectivement identifiés avec $\zeta_i \in \mathbb{C}$ and $\mathbf{z}_i \in \mathbb{C}$. La constante $c_{\mathrm{qh}} > 0$ est choisie telle que

$$c_{\mathrm{qh}}\left(X_M\right)^2 \int_{\mathbb{R}^{2N}} \left|\Psi_{\mathrm{qh}}\left(X_M, Y_N\right)\right|^2 \mathrm{d}Y_N = 1 \tag{4.14}$$

pour tout X_M. La fonction d'onde totale Ψ est normalisée par

$$\int_{\mathbb{R}^{2M}} \left|\Phi\left(X_M\right)\right|^2 \mathrm{d}X_M = 1. \tag{4.15}$$

Une des caractéristiques de l'ansatz (4.12) est de mettre toute la physique restante dans la fonction Φ. Prenons le champ magnétique $\mathbf{B}$ tel que $\mathrm{curl}\mathbf{A} = \mathbf{B}$ et les quasi-trous $\mathbf{x}_j$, alors l'énergie cinétique du système total

$$\left\langle \Psi, \left(-\mathrm{i}\nabla_j + \mathbf{A}\left(\mathbf{x}_j\right)\right)^2 \Psi \right\rangle \simeq \left\langle \Phi, H^{\mathrm{eff}} \Phi \right\rangle + \frac{BN}{2} \tag{4.16}$$

est gouvernée par le seul Hamiltonien effectif

$$H^{\mathrm{eff}} = \left(-\mathrm{i}\nabla_{\mathbf{x}_j} + \mathbf{A}\left(\mathbf{x}_j\right) + \mathbf{A}_r\left(\mathbf{x}_j\right) + \mathbf{A}_a\left(\mathbf{x}_j\right)\right)^2 \tag{4.17}$$

où deux nouveaux potentiels vecteur ont fait leur apparition. Le premier est

$$\mathbf{A}_r(\mathbf{x}_j) = \frac{B}{2n} \mathbf{x}_j^{\perp} \tag{4.18}$$

$$\mathrm{curl}\mathbf{A}_r = \frac{B}{n} \tag{4.19}$$

générant un champ magnétique constant ayant pour effet de diminuer le champ extérieur. On parle alors de charge fractionnaire puisque le couplage au champ se fait via une charge réduite. Le second potentiel vecteur apparaîssant dans (4.17) est le potentiel anyonique (3.17) étalé sur une échelle R correspondant à une statistique $\alpha = 1/n$

$$\mathbf{A}_a\left(\mathbf{x}_j\right) = -\sum_{\substack{k=1 \\ k \neq j}}^{M} \frac{B}{2\pi n} \int_{\mathbb{R}^2} \frac{\left(\mathbf{x}_j - \mathbf{y}\right)^{\perp}}{\left|\mathbf{x}_j - \mathbf{y}\right|^2} \mathbb{1}_{B(\mathbf{x}_k, R)}(\mathbf{y}) \mathrm{d}\mathbf{y} \tag{4.20}$$

avec une longueur magnétique de $R = \sqrt{2/B}$. Le champ est l'équivalent de tubes de flux de type Aharonov-Bohm transportés par les quasi-trous.

$$\mathrm{curl}\mathbf{A}_a(\mathbf{x}_j) = -\sum_{\substack{k=1\\k\neq j}}^{M} \frac{B}{n} \mathbb{1}_{B(\mathbf{x}_k,R)}(\mathbf{x}_j) = -\frac{2}{n}\sum_{\substack{k=1\\k\neq j}}^{M} \frac{\mathbb{1}_{B(\mathbf{x}_k,R)}}{R^2}(\mathbf{x}_j) \tag{4.21}$$

Nous avons donc retrouvé la description en jauge magnétique avec la contrainte supplémentaire que les disques ne doivent pas s'interpénéter. Nous pouvons voir les quasi-trous comme des anyons libres étendus sur une échelle de longueur R dans un champ magnétique $\mathbf{A} + \mathbf{A}_r$.

4. Résultats expérimentaux

Il existe à ce jour deux mesures expérimentales d'anyons. Ces deux expériences [**127**] et [**12**] se déroulent dans le contexte de l'EHQF. Elles utilisent un gaz d'électrons 2D soumis à un fort champ magnétique de sorte que le facteur de remplissage soit $\nu = 1/3$. Les excitations élémentaires de ce système sont alors des anyons de statistique $\alpha = 1/3$. Nous n'en présenterons que les concepts directeurs.

4.1. Mesure par les interférences. Dans [**127**] la statistique anyonique est mise en évidence à l'aide d'un interféromètre Fabry-Perot. Le schéma 4.1 décrit le montage expérimental. Les régions bleues représentent le gaz, les régions jaunes le matériau. Dans le régime $\nu = 1/3$ de l'effet Hall quantique fractionnaire, le courant circule à l'état de quasi-particules le long des bords dans le sens des flèches rouges. Ces particules ont deux occasions de sauter du coté opposé. Celles qui sautent sont représentées par des pointillés. Elles interférent à la sortie de l'interféromètre où les deux lignes pointillées se rejoingnent. D'autres quasi-particules sont quant à elles localisées par le désordre à l'intérieur de la chambre. Ce sont les tourbillons sur le schéma. De cette manière une des deux particules qui interfèrent a fait le tour de N_{qp} autres particules à la sortie de l'interféromètre. La différence de phase acquise en faisant le tour est la somme de la phase Aharonov-Bohm ϕ_{AB} due au champ magnétique extérieur et de la contribution anyonique $2\pi\alpha N_{qp}$

$$\theta = \phi_{\mathrm{AB}} + 2\pi\alpha N_{qp}. \tag{4.22}$$

Le courant total à la sortie de la chambre dépend de $\cos\theta$. On mesure donc la conductance $G(\theta)$. Tandis que la phase Aharonov-Bohm varie continûment lorsque B s'intensifie, les quasi-trous s'ajoutent discontinûment. Les sauts de N_{qp} se traduisent par des discontinuités dans la mesure de $G(B)$. Nous en déduisons le θ correspondant à l'ajout d'une particule. La valeur mesurée dans l'article est alors de $\alpha = \nu = 0.31 \pm 0,04 \simeq 1/3$ et correspond parfaitement aux prédictions théoriques.

4.2. Mesure via le principe d'exclusion. L'expérience [**12**] est basée sur la fait que, selon leur statistique, les anyons s'excluent plus ou moins au sens du principe de Pauli. Cette expérience a elle aussi lieu dans le régime $\nu = 1/3$ de l'EHQF. Les quasi-particules étudiées sont donc aussi des excitations élémentaires des états de Hall fractionnaires.

Le schéma 4.2 présente le montage expérimental. Le matériau est en jaune et le gaz d'électrons en bleu. Les courants I_1^0 et I_2^0 sont envoyés des points 1 et 2. Ils évoluent ensuite le long des parois du matériau jusqu'aux points de contact quantiques QPC_1 et QPC_2 tenant lieu de générateurs Poissonien aléatoires d'anyons. Les courants d'anyons

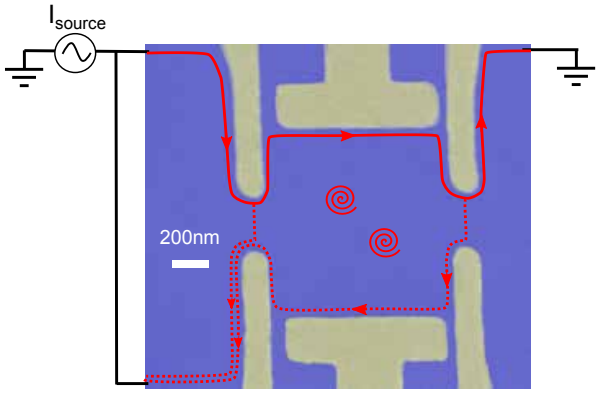


FIGURE 4.1. Représentation schématique de l'interféromètre extraite de [**127**]

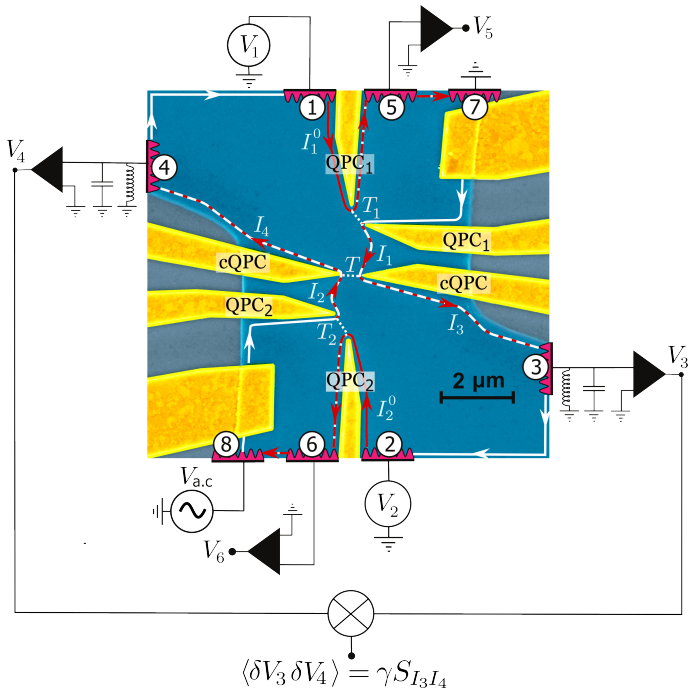


FIGURE 4.2. Représentation schématique du collisionneur extraite de [**12**]

sont donc I_1 et I_2. La collision a lieu au centre de l'image avec la probabilité T à l'aide des points cQPC. Les particules (mélange de celles ayant sauté ou non) poursuivent leur

chemin jusqu'aux détecteurs 3 et 4 qui mesurent les courants I_4 et I_3. La mesure des corrélations $S_{I_3 I_4}$ entre les deux courants I_3 et I_4 dépend alors de la statistique des quasi-particules ayant franchi le gap. Deux fermions, ne pouvant pas se trouver dans le même état quantique engendrent une corrélation croisée $S_{I_3 I_4} = 0$. Dans les autres cas $S_{I_3 I_4}$ doit être proportionnel au courant d'entrée $I_+ = I_1^0 + I_2^0$ avec une pente directement reliée au nombre α. Nous laissons de coté un certain nombre de détails mais il n'est pas faux de dire que les expérimentateurs mesurent directement la pente et en déduisent une valeur $\alpha \simeq 1/3$ excellemment proche de la valeur théorique.

CHAPTER 5

Résultats

L'objectif de cette section est d'introduire les objets nécessaires à l'énoncé des résultats principaux.
Un gaz d'anyons est un système à N corps. Ne serait-ce qu'un mètre cube d'air ambiant contient approximativement 10^{25} particules. Les expériences utilisant des atomes froids en comptent dans les $10^3 - 10^6$ et l'EHQF autour de 10^{11} par cm^2. Les résultats exacts se comptent sur les doigts de la main. Le fondamental de deux anyons dans un piège harmonique peut être résolu analytiquement [**28**]. Les valeurs propres d'énergie dépendent alors linéairement du paramètre statistique α. Le signe de la pente changeant à chaque valeur bosonique $\alpha = 2$. Nous trouvons aussi dans [**33, 35**] une étude concernant deux anyons soumis à un champ magnétique externe perpendiculaire au plan. Un calcul direct de l'état fondamental de notre système à N particules est hors de portée. Nous entreprenons donc la dérivation de modèles effectifs comme ceux décrits à la section 3. Dans ce qui suit les anyons seront vus comme des bosons ou des fermions interagissant via le potentiel vecteur (3.17) d'intensité $\alpha \to 0$ lorsque $N \to \infty$. Le Hamiltonien du système est

$$H_N = \sum_{j=1}^{N} \left(\mathbf{p}_j^{\mathbf{A}} + \alpha \mathbf{A}_j\right)^2 + V\left(\mathbf{x}_j\right) \tag{5.1}$$

où

$$\mathbf{p}_j^{\mathbf{A}} = -\mathrm{i}\hbar\nabla_j + \mathbf{A}_e\left(\mathbf{x}_j\right) \tag{5.2}$$

avec $\mathbf{A}_e$ tel que

$$\mathrm{curl}\mathbf{A}_e = \mathbf{B}_e$$

et le potentiel confinant V. Nous envisagerons deux scénarios:

Le cas quasi-bosonique: Dans cette situation, $\hbar = 1$ et le Hamiltonien (5.1) agit sur des fonctions d'onde bosoniques. Le paramètre statistique $\alpha \to 0$ quand $N \to \infty$. La phase d'échange des anyons (3.15) tend donc vers 1. Nous retrouvons au premier ordre de cette limite une fonctionnelle de type Hartree. Il s'agit d'une fonctionnelle à un corps avec champ magnétique auto-consistant. L'état fondamental de notre système converge vers le minimum de cette fonctionnelle et ses minimiseurs vers une combinaison convexe d'états purs de type Hartree

$$\int_{\mathcal{M}} \left|u^{\otimes N}\right\rangle \left\langle u^{\otimes N}\right| \mathrm{d}\mu(u)$$

où $\mathcal{M}$ est l'ensemble des minimiseurs de la fonctionnelle limite.

Le cas quasi-fermionique: Le Hamiltonien (5.1) agit ici sur des fonctions fermioniques. Le principe de Pauli nous impose d'aussi considérer la limite $\hbar \to 0$ quand $N \to \infty$ pour contrebalancer la rapide augmentation de l'énergie cinétique. La limite $\alpha \to 0$ nous permet par (3.15) de retrouver à la limite un Hamiltonien fermionique. Nous obtenons

que l'état fondamental converge vers celui d'une énergie Vlasov semi-classique à un corps. Sa minimisation nous donne accès à une mesure de probabilité m sur l'espace des phases $\mathbb{R}^2 \times \mathbb{R}^2$. Nous montrons que les k-ième fonctions de Husimi réduites, (qui sont les versions semi-classiques de la densité de probabilité quantique) construites à partir d'une suite de minimiseurs Ψ_N, tendent vers un produit tensoriel de mesures semi-classiques m.

1. Anyons étendus

Le Hamiltonien (5.1) décrit des anyons idéaux. Cependant, les anyons émergeant au sein de l'EHQF ne sont pas des particules dépourvues d'épaisseur. Dans les expériences, nous nous attendons à ce que les anyons soient étalés sur une échelle de longueur comme discuté section 4. Nous introduisons donc un paramètre R. Le flux attaché aux particules est ainsi étalé sur un cercle centré sur la particule. Dans notre approche, la limite $R \to 0$ nous permettra de retrouver la description des anyons idéaux. Ce modèle d'anyons étendus est discuté dans [**122, 158, 22**]. Le rayon R est assimilable au rayon des particules, i.e. une échelle de longeur au-delà de laquelle la decription en terme de quasi-particules n'est plus valable. Dans l'EHQF, R est identifié dans (4.21) avec la longueur magnétique $l_B \simeq \sqrt{2/B}$.

Nous introduisons le potentiel Coulombien 2D généré par une charge étalée sur le disque de rayon R

$$w_R(\mathbf{x}) = (\log|\,.\,| * \chi_R)(\mathbf{x}), \text{ avec la convention } w_0 = \log|\,.\,| \tag{5.3}$$

et $\chi_R(x)$ un fonction régularisante positive et de masse unité

$$\chi_R = \frac{1}{R^2}\chi\left(\frac{.}{R}\right) \text{ et } \int_{\mathbb{R}^2}\chi = 1, \quad \chi(\mathbf{x}) = \begin{cases} 1/\pi^2 & \text{si } |\mathbf{x}| \leq 1 \\ 0 & \text{si } |\mathbf{x}| \geq 2. \end{cases} \tag{5.4}$$

Nous prenons χ lisse, positive, radiale et décoissante de 1 et 2. Nous avons alors que

$$\nabla^\perp w_0(\mathbf{x}) = \frac{\mathbf{x}^\perp}{|\mathbf{x}|^2}, \text{ avec } B_0(\mathbf{x}) = \nabla^\perp\nabla^\perp w_0 = \Delta w_0 = 2\pi\delta_0.$$

Nous retrouvons le champ magnétique de (3.20) des anyons idéaux (au sens des distributions) à la limite $R \to 0$ en définissant

$$\mathbf{A}^R(\mathbf{x}_j) = \sum_{k\neq j}\nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k). \tag{5.5}$$

L'opérateur d'énergie cinétique est alors modifié en conséquence

$$T_\alpha^R := \sum_{j=1}^N \left(D_j^R\right)^2 := \sum_{j=1}^N \left(-\mathrm{i}\hbar\nabla_j + \alpha\mathbf{A}^R(\mathbf{x}_j)\right)^2. \tag{5.6}$$

Nous travaillons avec le Hamiltonien régularisé décrivant N anyons étendus de rayon R dans un potentiel confinant V et soumis à un champ magnétique extérieur $\mathbf{A}_e$

$$\boxed{H_N^R := \sum_{j=1}^N \left(\left(\mathbf{p}_j^{\mathbf{A}_e} + \alpha\mathbf{A}^R(\mathbf{x}_j)\right)^2 + V(\mathbf{x}_j)\right).} \tag{5.7}$$

L'opérateur $\mathbf{A}^R$ est à $R > 0$ une perturbation bornée du laplacien magnétique. T_α^R est donc essentiellement auto-adjoint sur son domaine $\mathcal{D}_{\alpha,R} = H^2_{\mathrm{a/sym}}(\mathbb{R}^{2N})$ (se référer à [**129**, Theorem X.17] et [**10**]). Son spectre possède une plus petite valeur à $R > 0$, nous la notons

$$E^R(N) = \inf \sigma \left(H_N^R \right). \tag{5.8}$$

Sous nos hypothèses H_N^R aura en fait une résolvante compacte. La valeur ci-dessus est donc une valeur propre. Mais la limite de cet opérateur quand $R \to 0$ avec N fixé n'est pas unique. Le Hamiltonien à $R = 0$ (discuté dans [**119**]) n'est pas essensiellement auto-adjoint (exemples dans [**35, 38, 17**]). Cependant, dans les limites conjointes $R \to 0$ et $N \to \infty$ nous obtenons un unique modèle non-linéaire bien défini.

Il peut être pratique de développer (5.7) pour traiter ses termes un par un

$$\begin{aligned} H_N^R = & \sum_{j=1}^N \left((\mathbf{p}_j^{\mathbf{A}_e})^2 + V(\mathbf{x}_j) \right) \text{ “Terme d'énergie cinétique”} \\ & + \alpha \sum_{j \neq k} \left(\mathbf{p}_j^{\mathbf{A}_e} . \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) + \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) . \mathbf{p}_j^{\mathbf{A}_e} \right) \text{ “Terme à deux corps mixte”} \\ & + \alpha^2 \sum_{j \neq k \neq l} \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) . \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_l) \text{ “Terme à trois corps”} \\ & + \alpha^2 \sum_{j \neq k} \left| \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) \right|^2 \text{ “Terme à deux corps singulier”}. \end{aligned} \tag{5.9}$$

Ce Hamiltonien contient des termes peu communs. Nous voyons dans (5.9) une interaction à trois corps ainsi qu'un terme mélangeant une dépendance en espace et en moment. Notons que le dernier terme de (5.9) est d'un ordre inférieur à tous les autres. Il disparaîtra donc à la limite. Nous sommes maintenant armés pour traiter des limites qui motivent notre intérêt.

2. Limite quasi-bosonique

2.1. Modèle quasi-bosonique. Notre premier résultat s'inscrit dans la continuité des approches de champ-moyen basées sur les théorèmes de de Finetti. Nous avançons plus particulièrement dans les pas de l'article [**112**]. Notre but est d'étendre ses résultats selon deux directions. La première amélioration concerne la situation physique. Nous prenons en compte un champ magnétique extérieur $\mathbf{B}_e$, absent dans [**112**]. Le second concerne l'élargissement des hypothèses de validité de l'approche de champ-moyen. Nous nous basons sur une version des théorèmes de de Finetti tout droit venue de l'informatique théorique. Cette version est due à Brandão et Harrow [**18, 88**], et a été revisitée dans [**135**].

Nous rappelons que le Hamiltonien régularisé (5.7) de N anyons piégés dans un potentiel V sous un champ magnétique $\mathbf{A}_e$ est

$$H_N^R := \sum_{j=1}^N \left(\left(\mathbf{p}_j^{\mathbf{A}_e} + \alpha \mathbf{A}^R(\mathbf{x}_j) \right)^2 + V(\mathbf{x}_j) \right) \tag{5.10}$$

agissant sur $\Psi_N \in L^2_{\text{sym}}\left(\mathbb{R}^{dN}\right)$. Nous fixons le paramètre

$$\beta = \alpha(N-1) \tag{5.11}$$

afin de garder une trace du caractère anyonique. Le paramètre $\beta \in \mathbb{R}$ quantifie la force du champ magnétique porté par le gaz d'anyons. De cette manière, la statistique anyonique laisse une trace au premier ordre se manifestant par l'apparition d'un champ magnétique auto-consitant au sein de la fonctionnelle de champ moyen

$$\mathbf{A}[\rho] = \nabla^{\perp} w_0 * \rho \text{ and } \mathbf{A}^R[\rho] = \nabla^{\perp} w_R * \rho. \tag{5.12}$$

Ainsi $\mathbf{A}$ devient indépendant de la position précise des $\mathbf{x}_j$. Ici ρ est la densité à un corps normalisée dans $L^1(\mathbb{R}^2)$ associée à la fonction d'onde Ψ_N

$$\rho(\mathbf{x}_1) = \int_{\mathbb{R}^{2(N-1)}} |\Psi_N(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N)|^2 \, \mathrm{d}\mathbf{x}_2 ... \mathrm{d}\mathbf{x}_N$$

En remplacant $\mathbf{A}^R(\mathbf{x}_j)$ par $\mathbf{A}^R[\rho](\mathbf{x}_j)$ nous obtenons le Hamiltonien champ-moyen à N corps

$$H_N^{\text{af}} = \sum_{j=1}^{N} \left(\left(\mathbf{p}_j^{\mathbf{A}} + (N-1)\alpha \mathbf{A}^R[\rho] \right)^2 + V(\mathbf{x}_j) \right). \tag{5.13}$$

Le champ magnétique devient

$$\mathbf{B}(\mathbf{x}) = \operatorname{curl} \beta \mathbf{A}[\rho](\mathbf{x}) = 2\pi\beta\rho(\mathbf{x}) \tag{5.14}$$

avec un flux total de $2\pi\beta$. Nous pouvons maintenant insérer l'ansatz

$$\Psi(\mathbf{x}_1, ..., \mathbf{x}_N) = u(\mathbf{x}_1) ... u(\mathbf{x}_N)$$

avec l'équation de consistance

$$|u|^2 = \rho$$

et obtenir la fonctionnelle champ-moyen à une particule

$$\mathcal{E}^{\text{af}}_{R,\beta}[u] = \int_{\mathbb{R}^2} \left(\left| (-\mathrm{i}\nabla + \mathbf{A}_e + \beta \mathbf{A}^R[|u|^2]) u \right|^2 + V|u|^2 \right) \tag{5.15}$$

qui, lorsque $R = 0$, devient

$$\mathcal{E}^{\text{af}}_{\beta}[u] = \int_{\mathbb{R}^2} \left(\left| \left(\mathbf{p}_j^{\mathbf{A}_e} + \beta \mathbf{A}\left[|u|^2\right] \right) u \right|^2 + V|u|^2 \right) \tag{5.16}$$

avec $\mathbf{A}^R[\cdot]$ défini en (5.12). Nous notons son infimum

$$E^{\text{af}}_{R,\beta} := \inf_{\|u\|_{L^2}=1} \mathcal{E}^{\text{af}}_R[u] \text{ and } E^{\text{af}}_{\beta} := E^{\text{af}}_{0,\beta} = \inf_{\|u\|_{L^2}=1} \mathcal{E}^{\text{af}}_{\beta}[u] \tag{5.17}$$

sous la contrainte de masse

$$\int_{\mathbb{R}^2} |u|^2 = 1.$$

Nous pouvons aussi écrire

$$\mathcal{E}^{\text{af}}_R[u] = \int_{\mathbb{R}^2} \left(|\mathbf{p}^{\mathbf{A}_e} u|^2 + V|u|^2 + 2\beta \mathbf{A}^R[|u|^2] \cdot \left(\frac{\mathrm{i}}{2}(u\nabla\overline{u} - \overline{u}\nabla u) + \mathbf{A}_e|u|^2 + \frac{\beta}{2}\mathbf{A}^R[|u|^2]|u|^2 \right) \right) \mathrm{d}\mathbf{x}.$$

Notre but est alors de montrer que

$$\frac{E^R(N)}{N} = \frac{\inf \left\langle \Psi_N, H_R^N \Psi_N \right\rangle}{N} \approx \inf \left\{ \mathcal{E}_R^{\mathrm{af}}[u], \int |u|^2 = 1 \right\} \tag{5.18}$$

quand $N \to \infty$ et $R \to 0$. En d'autre mots, la fonctionnelle de champ-moyen est une bonne approximation pour l'état fondamental de notre système piloté par H_N^R.

2.2. Résultats pour les anyons quasi-bosoniques. Nous énoncons notre premier théorème démontrant la validité de l'approche champ-moyen pour l'état fondamental. Nous supposons que le potentiel de piégeage est confinant

$$V(x) \geq c|\mathbf{x}|^s - C, \quad s > 0, \tag{5.19}$$

pour des raisons techniques ainsi que

Hypothèses 2.1 (**Potentiel magnétique extérieur**).
Le champ magnétique extérieur satisfait aux conditions suivantes

- $\mathbf{A}_e \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$
- $\mathrm{curl}\mathbf{A}_e = \mathbf{B}_e = \mathbf{B}_0 + \tilde{\mathbf{B}}$ avec $\mathbf{B}_0$ constant et $\tilde{\mathbf{B}} \in W^{1,2+\varepsilon}$

Nous demandons aussi que la taille R des anyons ne tende pas trop vite vers zéro i.e $R \sim N^{-\eta}$ pour quelques η.

Théorème 2.1 (**Validité de l'approximation de champ-moyen**).
Supposons $R \sim N^{-\eta}$ avec

$$0 < \eta < \frac{1}{4} \tag{5.20}$$

est prenons le paramètre statistique

$$\alpha = \frac{\beta}{(N-1)}$$

avec $\beta \in \mathbb{R}$ fixé. Alors, dans la limite $N \to \infty$ nous avons

$$\frac{E^R(N)}{N} \to E_\beta^{\mathrm{af}}. \tag{5.21}$$

De plus, si Ψ_N est une suite approchée de minimiseurs H_N^R

$$\left\langle \Psi_N, H_N^R \Psi_N \right\rangle \leq E^R(N) + o(N) \tag{5.22}$$

avec les matrices de densité associées

$$\gamma_N^{(k)} := \mathrm{Tr}_{k+1 \to N} \left[|\Psi_N\rangle\langle\Psi_N| \right] \tag{5.23}$$

alors, le long d'une sous-suite, nous avons

$$\gamma_N^{(k)} \to \int_{\mathcal{M}^{\mathrm{af}}} |u^{\otimes k}\rangle\langle u^{\otimes k}| \, \mathrm{d}\mu(u) \tag{5.24}$$

fortement dans la classe trace quand $N \to \infty$, avec μ une mesure de probabilité de Borel sur l'ensemble des minimiseurs de $\mathcal{E}^{\mathrm{af}}$,

$$\mathcal{M}^{\mathrm{af}} := \{u \in L^2(\mathbb{R}^2) : \|u\|_{L^2} = 1, \, \mathcal{E}_\beta^{\mathrm{af}}[u] = E_\beta^{\mathrm{af}}\}.$$

L'étude de cette limite de champ moyen présente plusieurs aspects nouveaux eu égard à l'approximation de Hartree habituelle décrite dans la section 3.1:

- Nous combinons les deux limites $R \to 0$ et $\alpha \to 0$ quand $N \to \infty$. Cela fait écho aux limites NLS et GP pour les gaz de bosons [**96, 93, 85**]. Les paramètres R et α sont assimilables à L et λ de la Section 3. La nouveauté est l'interaction magnétique impliquant un terme à trois corps et une terme mélangeant opérateurs en espace et en moment (5.9).
- La contrainte $\eta < 1/4$ est une idéalisation du scénario physique attendu. La distance inter-particules d'un gaz dans un volume d'ordre 1 fixé par le potentiel V en deux dimensions est $1/\sqrt{N}$. Puisque $R = N^{-\eta}$ avec $\eta < 1/4$ notre théorème n'est valide que dans le régime de champ moyen où les particules s'interpénetrent largement. Il n'est pas évident de savoir si cette contrainte sur R est une condition irrémissible de la validité de notre approche. La borne supérieure de l'énergie est obtenue via l'ansatz $u^{\otimes N}$ sans aucune restriction sur η [**112**, equation 3.19].
- Il serait souhaitable d'élargir la condition sur η pour atteindre $\eta > 1/2$ et le régime dilué. Dans ce cas, nous aurions de bons candidats pour décrire les quasi-particules de l'EHQF.
- La méthode de réguarisation de [**112**, equation 1.4] utilisait

$$\widetilde{\chi}_R := \frac{\mathbb{1}_{B(0,R)}}{\pi R^2}(\mathbf{x}).$$

  Le χ que nous avons pris dans ce travail en est une version lissée

$$\int_{\mathbb{R}^2} |\widehat{\chi}(\mathbf{p})| \mathrm{d}\mathbf{p} < \infty$$

  avec $\widehat{\chi}$ la transformée de Fourier de χ.

3. Limite quasi-fermionique

Notre second résultat traite d'une limite dans laquelle la statistique tend vers celle des fermions. Nous utilisons une approche de champ-moyen et démontrons que l'état fondamental est atteint au premier ordre par celui d'une énergie Vlasov semi-classique. La statistique anyonique est visible à la limite dans la distribution en moment de cette dernière. Nous sommes donc dans une limite quasi-fermionique $\alpha \to 1$ quand $N \to \infty$. Notre situation est très proche des limites champ-moyen/semi-classiques pour des fermions usuels [**100, 147, 13, 11, 46, 53**].

3.1. Modèle quasi-fermionique. Nous rappelons que le Hamiltonien (5.7) de N anyons piégés dans une potentiel V et soumis à un champ magnénique extérieur $\mathbf{B}_e = \mathrm{curl}\mathbf{A}_e$ est

$$H_N^R := \sum_{j=1}^{N} \left(\left(\mathbf{p}_j^{\mathbf{A}_e} + \alpha \mathbf{A}^R(\mathbf{x}_j) \right)^2 + V(\mathbf{x}_j) \right) \tag{5.25}$$

agissant sur $\Psi_N \in L^2_{\mathrm{asym}}\left(\mathbb{R}^{dN}\right)$. Plutôt que d'étudier l'état fondamental d'un Hamiltonien agissant sur des fonctions d'onde bosoniques nous agissons sur des fonctions fermioniques. La phase d'échange est maintenant $e^{i\pi(\alpha+1)}$ (voir (3.15)) et tend vers la statistique des fermions lorsque $\alpha \to 0$.

Nous nous intéressons à la limite jointe $N \to \infty$, $\alpha \to 0$ dans laquelle le système présentera une dépendance non triviale en α. Comme nous travaillons avec des quasi-fermions, le principe de Pauli implique que l'énergie cinétique croisse plus vite que N. Ce comportement est mis en exergue par l'inégalité de Lieb-Thirring [**94**, Theorem 4.1] qui, pour toute fonction antisymétrique de N particules Ψ_N à support dans un domaine borné Ω^N, nous dit que

$$\sum_{j=1}^{N} \int_{\Omega^N} |\nabla_j \Psi|^2 \geq C|\Omega|^{-\frac{2}{d}} N^{1+\frac{2}{d}}. \tag{5.26}$$

où d est la dimension de l'espace physique. À cause de cette croissance rapide de l'énergie cinétique, la limite de champ-moyen naturelle pour des fermions [**13, 14, 147**] consiste à prendre $\hbar \sim N^{-1/d}$. Ici $d = 2$ et nous obtenons une énergie cinétique proportionnelle à N^2 et donc, à la louche $-\mathrm{i}\nabla_j \sim N^{1/2}$. Nous concentrant sur la dépendance en N de chaque terme de (5.1) nous prévoyons que

$$\langle \Psi_N, H_N \Psi_N \rangle \approx C_1 N \left(\hbar N^{1/2} + \alpha N \right)^2 + C_2 N$$

où les termes entre parenthèses représentent l'énergie cinétique, incluant l'influence de (5.5). Le second terme est l'énergie du potentiel V. Afin d'obtenir une limite non triviale quand $\alpha \to 0$, un choix possible est

$$\hbar = \frac{1}{\sqrt{N}} \text{ et } \alpha = \frac{\beta}{N} \tag{5.27}$$

avec $\beta \in \mathbb{R}$ une constante nous permettant de garder une trace de l'influence de la statistique anyonique. Nous nous intéressons donc à la limite

$$\lim_{N\to\infty} \frac{E^R(N)}{N}$$

avec

$$E^R(N) := \inf \left\{ \left\langle \Psi_N, H_N^R \Psi_N \right\rangle, \Psi_N \in L^2_{\mathrm{asym}}(\mathbb{R}^{2N}), \int_{\mathbb{R}^{2N}} |\Psi_N|^2 = 1 \right\}. \tag{5.28}$$

Cette approche combine deux limites (voir la remarque 5.1 ci-dessous), la limite semi-classique $\hbar \to 0$ et la limite quasi-fermionique $\alpha \to 0$. Nous nous attendons, comme dans d'autres types de limite de champ-moyen e.g. [**137**] et références associées) à ce que, pour N grand, les particules se comportent indépendamment

$$\sum_{j\neq k} \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) \approx \int_{\mathbb{R}^2} \nabla^\perp w_R(\mathbf{x}_j - \mathbf{y}) \rho(\mathbf{y}) \mathrm{d}\mathbf{y} = \nabla^\perp w_R * \rho(\mathbf{x}_j)$$

où ρ est la densité de particule. Étant donnée une densité ρ normalisée dans $L^1(\mathbb{R}^2)$, nous définissons,

$$\mathbf{A}[\rho] = \nabla^\perp w_0 * \rho \text{ and } \mathbf{A}^R[\rho] = \nabla^\perp w_R * \rho$$

avec

$$w_R = \log|.| * \chi_R \text{ and } w_0 = \log|.| \tag{5.29}$$

explicité en (5.3). Les fonctionnelles de champ-moyen naturellement associées à (5.9) sont

$$\mathcal{E}_R^{\mathrm{af}}[\gamma] = \frac{1}{N}\operatorname{Tr}\left[\left(\mathbf{p}_1^{\mathbf{A}} + \alpha \mathbf{A}^R[\rho_\gamma]\right)^2 \gamma\right] + \frac{1}{N}\int_{\mathbb{R}^2} V(\mathbf{x}_1)\rho_\gamma(\mathbf{x}_1)\mathrm{d}\mathbf{x}_1 \tag{5.30}$$

$$\mathcal{E}^{\mathrm{af}}[\gamma] = \frac{1}{N}\operatorname{Tr}\left[\left(\mathbf{p}_1^{\mathbf{A}} + \alpha \mathbf{A}[\rho_\gamma]\right)^2 \gamma\right] + \frac{1}{N}\int_{\mathbb{R}^2} V(\mathbf{x}_1)\rho_\gamma(\mathbf{x}_1)\mathrm{d}\mathbf{x}_1 \tag{5.31}$$

avec γ la matrice densité à un corps de l'état $\Gamma_N = |\Psi_N\rangle\langle\Psi_N|$

$$\gamma := \gamma_N^{(1)} = N \operatorname{Tr}_{2\to N} \Gamma_N, \qquad \rho_\gamma(\mathbf{x}) := \gamma(\mathbf{x},\mathbf{x}).$$

Notons que les conventions de normalisation ont changées. Dans le cas bosonique nous travaillions avec (5.23) où $\operatorname{Tr}\gamma^{(1)} = 1$. Nous avons désormais $\operatorname{Tr}\gamma^{(1)} = N$. L'analogue semi-classique de la fonctionnelle (5.31) est

$$\mathcal{E}_{\mathrm{Vla}}[m] = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} |\mathbf{p} + \mathbf{A}_e(\mathbf{x}) + \beta\mathbf{A}[\rho](\mathbf{x})|^2 m(\mathbf{x},\mathbf{p})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p} + \int_{\mathbb{R}^2} V(\mathbf{x})\rho_m(\mathbf{x})\mathrm{d}\mathbf{x} \tag{5.32}$$

avec $m(\mathbf{x},\mathbf{p})$ une mesure positive sur l'espace des phases $\mathbb{R}^4$ des positions et vitesses

$$\rho_m(\mathbf{x}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} m(\mathbf{x},\mathbf{p})\mathrm{d}\mathbf{p}. \tag{5.33}$$

Notre convention sur la masse de m est

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} m(\mathbf{x},\mathbf{p})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p} = (2\pi)^2. \tag{5.34}$$

et le principe de Pauli prend alors la forme

$$\boxed{0 \le m(\mathbf{x},\mathbf{p}) \le 1 \ \text{ p.p.}} \tag{5.35}$$

Le principe d'exclusion est imposé au niveau classique par le fait qu'un état classique $(\mathbf{x},\mathbf{p})$ ne peux pas être occupé par plus d'un fermion. En utilisant le principe de la baignoire [**92**, Theorem 1.14] nous pouvons minimiser l'énergie Vlasov dans la variable $\mathbf{p}$ explicitement (ce qui rejoint les considérations de [**147**]). Nous trouvons des minimiseurs de la forme

$$m_\rho(\mathbf{x},\mathbf{p}) = \mathbb{1}\Big(|\mathbf{p} + \mathbf{A}_e(\mathbf{x}) + \beta\mathbf{A}[\rho](\mathbf{x})|^2 \le 4\pi\rho(\mathbf{x})\Big)$$

où ρ minimise l'énergie Thomas-Fermi

$$\mathcal{E}_{\mathrm{TF}}[\rho] = \mathcal{E}_{\mathrm{Vla}}[m_\rho] = 2\pi\int_{\mathbb{R}^2}\rho^2(\mathbf{x})\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2} V(\mathbf{x})\rho(\mathbf{x})\mathrm{d}\mathbf{x}. \tag{5.36}$$

Le minimum de l'énergie Thomas-Fermi est

$$e_{\mathrm{TF}} = \inf\Big\{\mathcal{E}_{\mathrm{TF}}[\rho] : 0 \le \rho \in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2), \int_{\mathbb{R}^2}\rho = 1\Big\}. \tag{5.37}$$

Notre première intention est d'obtenir la convergence du fondamental de H_N^R vers cette énergie.

3.2. Résultats pour l'énergie fondamentale quasi-fermionique. Nous travaillons sous les hypothèses suivantes

HYPOTHÈSES 3.1 (**Potentiels extérieurs**).
Les potentiels extérieurs de (5.7) satisfont

- $\mathbf{A}_e \in L^2(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2)$
- $|\mathbf{A}_e|^2 \in W^{1,\infty}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$
- $V(\mathbf{x}) \geq c|\mathbf{x}|^s - C$ pour $s > 1$
- $|\nabla V(\mathbf{x})| \leq c'|\mathbf{x}|^{s-1} + C'$ et $|\Delta V(\mathbf{x})| \leq c''|\mathbf{x}|^{s-2} + C''$

THÉORÈME 3.1 (**Convergence de l'énergie fondamentale quasi-fermionique**).
Nous considérons N anyons étendus de rayon $R = N^{-\eta}$ et paramètre statistique $\alpha = \beta/N$ dans les potentiels V et $\mathbf{A}_e$. Sous les hypothèses (3.1), avec $R = N^{-\eta}$

$$0 < \eta < \frac{1}{4}$$

nous avons, dans le régime (5.27)

$$\boxed{\lim_{N\to\infty} \frac{E^R(N)}{N} = e_{\mathrm{TF}}}$$

où l'énergie $E^R(N)$ et le minimiseur Thomas-Fermi sont respectivement définis en (5.28) et (5.37).

L'étude du régime (5.27) ressemble à une limite Thomas-Fermi d'un systme fermionique à N corps [**100, 147, 155, 90**], avec de nouveaux aspects:

- L'interaction effective comprend un terme à trois corps et un terme mélangeant position et moment (voir l'équation (5.9)).
- Le problème limite (5.32) comprend un champ magnétique effectif auto-consitant $\mathbf{A}[\rho]$.
- Nous devons traiter les limites $R \to 0$, $\alpha \to 0$ et $\hbar \to 0$ en même temps que $N \to \infty$.

Nous faisons, à propos de ce dernier point, la

REMARQUE 5.1 (Scaling des paramètres).
Modulo extraction d'un facteur N dans l'énergie, le régime (5.27) est équivalent à

$$\hbar = 1 \ \text{ et } \ \alpha = \frac{1}{\sqrt{N}} \tag{5.38}$$

pourvu que nous remplacions le potentiel $V(\mathbf{x})$ par $NV(\mathbf{x})$ (et faisions de même pour $\mathbf{A}_e$). Le scaling (5.38) illustre mieux le caractère quasi-fermionique de notre limite. Il fut cependant préférable de travailler avec (5.27) d'un point de vue technique. Le coté semi-classique étant plus apparent.

REMARQUE 5.2 (Cas des anyons authentiques).
Un autre régime est d'un grand intérêt. Pour en discuter, comparons l'énergie cinétique d'une particule dans un domaine Ω tel que $|\Omega| = 1$ à $\hbar = 1$

$$E_c = (-\mathrm{i}\hbar\nabla + \alpha\mathbf{A})^2 \tag{5.39}$$

avec l'énergie de Landau (4.2)

$$E_L = \left(-\mathrm{i}\hbar\nabla + B\mathbf{x}^\perp\right)^2. \tag{5.40}$$

Si nous approximons $\mathbf{A}$ par $\mathbf{A}[\rho]$ dans (5.39) nous obtenons un champ magnétique d'ordre $\alpha\rho$ avec

$$\int_\Omega \mathrm{d}\rho = N. \tag{5.41}$$

Faisons l'hypothèse que ρ est constant et nous obtenons que le cas anyonique est approximé dans la limite $N \to \infty$ par celui de particules soumisent à un champ d'intensité $B = \alpha N$. Rappelons que la dégénéréscence par unité d'aire d'un niveau de Landau est (4.9)

$$D \simeq \frac{B}{2\pi\hbar} = \frac{B}{2\pi} \tag{5.42}$$

nous comparons deux régimes :

(1) $\alpha = 1/\sqrt{N}$, $B = O\left(\sqrt{N}\right)$. C'est le régime de notre étude. La dégénérescence d'un niveau de Landau (4.9) est d'ordre $\sqrt{N} \ll N$. Nous nous attendons qu'à la limite $N \to \infty$, tous les niveaux soient remplis. L'énergie finale étant une énergie Thomas-Fermi pour électron usuelle .
(2) $B = N$ qui est équivalent à fixer α. La dégénérécence du LLL est alors d'ordre N. Un seul niveau de Landau devrait être rempli. L'énergie finale devrait alors être modifiée et devenir une énergie Thomas-Fermi magnétique [**51, 101, 102**].

Remarque 5.3 (Condition de validité).
Concernant le paramètre R, il serait souhaitable d'accéder au régime dilué $\eta > 1/2$ où le rayon des flux magnétiques devient plus petit que la distance inter-particule. Nous pourrions élargir la contrainte $\eta < 1/4$ à $\eta < 1/3$ (partie 5) en supposant une inégalité à la Lieb-Thirring pour les anyons étendus similaire à celle du Théorème (3.1). Elle serait de la forme

$$\left\langle \Psi_N, \sum_{j=1}^N \left(-\mathrm{i}\nabla_j + \alpha\mathbf{A}^R(\mathbf{x}_j)\right)^2 \Psi_N \right\rangle \geq C_\alpha \int_{\mathbb{R}^2} \rho_{\Psi_N}(\mathbf{x})^2 \mathrm{d}\mathbf{x}.$$

et serait valable pour toute fonction d'onde fermionique Ψ_N de densité ρ_{Ψ_N}. La constante C_α serait alors uniformément bornée inférieurement pour tout $\alpha \neq 1$. Une telle inégalité est rendue possible par les travaux [**75**] sur le gaz homogène d'anyons étendus et par l'inégalité de Lieb-Thirring pour les anyons idéaux [**116, 117, 119, 114**]. Nous planifions d'étudier ce problème dans un futur proche. ◇

L'énergie limite e_{TF} ne présente pas de caractéristique anyonique, i.e. ne dépend de α en aucune sorte. Ce fait est trompeur car le caractère anyonique apparaît en fait dans la seconde partie de nos résultats.

3.3. Mesures semi-classiques. États cohérents déformés

Prenons[1]

$$f(\mathbf{x}) := \frac{1}{\sqrt{\pi}} e^{-\frac{|\mathbf{x}|^2}{2}} \tag{5.43}$$

[1]Nous choisissons ici une gaussienne f mais toute autre fonction de L^2 pourrait être utilisée.

Pour tout $(\mathbf{x}, \mathbf{p})$ dans l'espace des phases $\mathbb{R}^2 \times \mathbb{R}^2$ nous définissons l'état cohérent squeesé

$$F_{\mathbf{x},\mathbf{p}}(\mathbf{y}) := \frac{1}{\sqrt{\hbar_x}} f\left(\frac{\mathbf{y}-\mathbf{x}}{\sqrt{\hbar_x}}\right) e^{\mathrm{i}\frac{\mathbf{p}\cdot\mathbf{y}}{\hbar}}. \tag{5.44}$$

Cet état a la propriété d'être étalé sur une échelle $\sqrt{\hbar_x}$ en espace et $\sqrt{\hbar_p}$ en moment, ces deux échelles étant reliées par

$$\hbar = \sqrt{\hbar_x}\sqrt{\hbar_p}. \tag{5.45}$$

En général les états cohérents sont utilisés [**147**, Section 2.1] avec $\sqrt{\hbar_x} = \sqrt{\hbar_p} = \sqrt{\hbar}$. Ces états saturent le principe d'incertitude d'Heisenberg. Ils sont en ce sens, les plus classiques des états quantiques. En effet, prenons les deux observables x_1 et $p_1 = -\mathrm{i}\partial_{x_1}$ et évaluons les dans cet état, nous trouvons

$$\Delta_{x_1}\Delta_{p_1} = \frac{\hbar}{2} \text{ où } \Delta_a = \left\langle F_{\mathbf{x},\mathbf{p}}, a^2 F_{\mathbf{x},\mathbf{p}}\right\rangle - \left\langle F_{\mathbf{x},\mathbf{p}}, aF_{\mathbf{x},\mathbf{p}}\right\rangle^2.$$

Nous prendrons dans notre preuve $\hbar_x \ll \hbar_p \to 0$, choix commode étant donné que notre Hamiltonien est singulier en $\mathbf{x}$ mais pas en $\mathbf{p}$. Nous définissons la transformée de Fourier de paramètre $\hbar$

$$\mathcal{F}_\hbar[\psi](\mathbf{p}) = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2} \psi(\mathbf{x}) e^{-\mathrm{i}\frac{\mathbf{p}\cdot\mathbf{x}}{\hbar}} \mathrm{d}\mathbf{x} \tag{5.46}$$

et notons

$$F_{\hbar_x}(\mathbf{x}) = F_{0,0}(\mathbf{x}) = \frac{1}{\sqrt{\hbar_x}\sqrt{\pi}} e^{-\frac{|\mathbf{x}|^2}{2\hbar_x}}. \tag{5.47}$$

La transformée de Fourier de(5.47)(voir les calculs dans l'appendice 3) est

$$G_{\hbar_p}(\mathbf{p}) = \mathcal{F}_\hbar[F_{\hbar_x}](\mathbf{p}) = \frac{1}{\sqrt{\hbar_p}\sqrt{\pi}} e^{-\frac{|\mathbf{p}|^2}{2\hbar_p}}. \tag{5.48}$$

Elle rend claire la localisation en $\mathbf{p}$ dont nous parlions à l'instant.

Nous associons à $F_{\mathbf{x},\mathbf{p}}$ le projecteur orthogonal $P_{\mathbf{x},\mathbf{p}}$

$$P_{\mathbf{x},\mathbf{p}} := |F_{\mathbf{x},\mathbf{p}}\rangle\langle F_{\mathbf{x},\mathbf{p}}|. \tag{5.49}$$

Nous obtenons ainsi la

LEMME 5.4 (**Résolution de l'identité sur** $L^2\left(\mathbb{R}^2\right)$).
Nous avons

$$\frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} P_{\mathbf{x},\mathbf{p}}\, \mathrm{d}x\mathrm{d}p = \mathbb{1} \tag{5.50}$$

La preuve est rappelée dans l'appendice 3, on peut aussi la trouver dans [**92**, Theorem 12.8].

Convergence des fonctions de Husimi

Les fonctions de Husimi $m^{(k)}_{\Psi_N}(\mathbf{x}_1, \mathbf{p}_1, \ldots, \mathbf{x}_k, \mathbf{p}_k)$[**73, 27, 154**] décrivent comment les particules sont distribuées dans les k boites semi-classique de taille $\sqrt{\hbar_x} \times \sqrt{\hbar_p}$ centrées en $(\mathbf{x}_1, \mathbf{p}_1), \ldots, (\mathbf{x}_k, \mathbf{p}_k)$. Une fonction d'onde Ψ_N étant donnée, nous définissons

$$m^{(k)}_{\Psi_N}(\mathbf{x}_1, \mathbf{p}_1, \ldots, \mathbf{x}_k, \mathbf{p}_k) = \frac{N!}{(N-k)!}\left\langle \Psi_N, \bigotimes_{j=1}^{k} P_{\mathbf{x}_j,\mathbf{p}_j} \otimes \mathbb{1}_{N-k}\Psi_N \right\rangle. \tag{5.51}$$

Ou alternativement,

$$m^{(k)}_{\Psi_N}(\mathbf{x}_1,\mathbf{p}_1,\ldots,\mathbf{x}_k,\mathbf{p}_k)=\frac{N!}{(N-k)!}\int_{\mathbb{R}^{2(N-k)}}\left|\left\langle F_{\mathbf{x}_1,\mathbf{p}_1}\otimes\ldots\otimes F_{\mathbf{x}_k,\mathbf{p}_k},\Psi_N(\cdot,z)\right\rangle_{L^2(\mathbb{R}^{2k})}\right|^2\mathrm{d}z$$

ou encore, en terme d'opérateurs de création et d'annihilation fermionique (voir (5.53) ci-dessous)

$$m^{(k)}_{\Psi_N}(\mathbf{x}_1,\mathbf{p}_1,\ldots,\mathbf{x}_k,\mathbf{p}_k)=\left\langle\Psi_N,a^*\left(F_{\mathbf{x}_1,\mathbf{p}_1}\right)\ldots a^*\left(F_{\mathbf{x}_k,\mathbf{p}_k}\right)a\left(F_{\mathbf{x}_k,\mathbf{p}_k}\right)\ldots a\left(F_{\mathbf{x}_1,\mathbf{p}_1}\right)\Psi_N\right\rangle$$

où a et a^* satisfont les relations canoniques d'anticommutation [**146, 64**]

$$a^*(f)a(g)+a(g)a^*(f)=\langle g,f\rangle \tag{5.52}$$

$$a^*(f)a^*(g)+a^*(g)a^*(f)=0. \tag{5.53}$$

Nous utiliserons aussi les matrices densités réduites à k particules de Ψ_N, i.e. les opérateurs $\gamma^{(k)}_{\Psi_N}$ avec le noyau intégral

$$\gamma^{(k)}_{\Psi_N}(\mathbf{x}_1,.,\mathbf{x}_k;\mathbf{x}'_1,.,\mathbf{x}'_k)=\binom{N}{k}\int_{\mathbb{R}^{2(N-k)}}\Psi_N(\mathbf{x}_1,.,\mathbf{x}_N)\overline{\Psi_N(\mathbf{x}'_1,.,\mathbf{x}'_k,\mathbf{x}_{k+1},.,\mathbf{x}_N)}\mathrm{d}\mathbf{x}_{k+1}\ldots\mathrm{d}\mathbf{x}_N. \tag{5.54}$$

Les fonctions de Husimi s'expriment aussi sous la forme

$$m^{(k)}_{\Psi_N}(\mathbf{x}_1,\mathbf{p}_1,\ldots,\mathbf{x}_k,\mathbf{p}_k)=k!\ \mathrm{Tr}\left[\bigotimes_{j=1}^{k}P_{\mathbf{x}_j,\mathbf{p}_j}\gamma^{(k)}_{\Psi_N}\right]. \tag{5.55}$$

Ces objets définissent à leur tour des marginales de densités en espace et en moment

$$\rho^{(k)}_{\Psi_N}(\mathbf{x}_1,\ldots,\mathbf{x}_k)=\binom{N}{k}\int_{\mathbb{R}^{2(N-k)}}|\Psi_N(\mathbf{x}_1,\ldots,\mathbf{x}_N)|^2\,\mathrm{d}\mathbf{x}_{k+1}\ldots\mathrm{d}\mathbf{x}_N \tag{5.56}$$

$$t^{(k)}_{\Psi_N}(\mathbf{p}_1,\ldots,\mathbf{p}_k)=\binom{N}{k}\int_{\mathbb{R}^{2(N-k)}}|\mathcal{F}_\hbar[\Psi_N](\mathbf{p}_1,\ldots,\mathbf{p}_N)|^2\,\mathrm{d}\mathbf{p}_{k+1}\ldots\mathrm{d}\mathbf{p}_N. \tag{5.57}$$

Nous définissons aussi la fonction de Wigner à k particules [**52, 108**]

$$\begin{aligned}\mathcal{W}^{(k)}_{\Psi_N}(\mathbf{x}_1,\mathbf{p}_1,...,\mathbf{x}_k,\mathbf{p}_k):=&\int_{\mathbb{R}^{2k}}\int_{\mathbb{R}^{2(N-k)}}e^{-\mathrm{i}\sum_{l=1}^k\mathbf{p}_l\cdot\mathbf{y}_l}\\&\times\Psi_N\left(\mathbf{x}_1+\hbar\mathbf{y}_1/2,...,\mathbf{x}_k+\hbar\mathbf{y}_k/2,\mathbf{x}_{k+1},...,\mathbf{x}_N\right)\\&\times\overline{\Psi_N\left(\mathbf{x}_1-\hbar\mathbf{y}_1/2,...,\mathbf{x}_k-\hbar\mathbf{y}_k/2,\mathbf{x}_{k+1},...,\mathbf{x}_N\right)}\\&\times\mathrm{d}\mathbf{y}_1...\mathrm{d}\mathbf{y}_k\mathrm{d}\mathbf{x}_{k+1}...\mathrm{d}\mathbf{x}_N\end{aligned} \tag{5.58}$$

qui n'est pas nécessairement positive mais qui admet la même limite que $m^{(k)}_{f,\Psi_N}$ [**147**, Theorem 2.1]. Nous sommes fin prêts pour énoncer la convergence des états du théorème (3.2) à l'aide des objets ci-dessus. Notons le minimiseur de l'énergie Vlasov(5.32)

$$m_{\rho^{\mathrm{TF}}}(\mathbf{p},\mathbf{x})=\mathbb{1}\left\{|\mathbf{p}+\mathbf{A}_e(\mathbf{x})+\beta\mathbf{A}[\rho^{\mathrm{TF}}](\mathbf{x})|^2\leq 4\pi\rho^{\mathrm{TF}}(\mathbf{x})\right\}. \tag{5.59}$$

Sa marginale en espace étant

$$\rho^{\mathrm{TF}}(\mathbf{x})=\frac{1}{4\pi}\left(\lambda^{\mathrm{TF}}-V(\mathbf{x})\right)_+ \tag{5.60}$$

avec λ^{TF} le multiplicateur de Lagrange assurant la normalisation, et la marginale en moment

$$t^{\mathrm{TF}}(\mathbf{p}) = \int_{\mathbb{R}^2} \mathbb{1}_{\left\{|\mathbf{p}+\mathbf{A}_e(\mathbf{x})+\beta\mathbf{A}[\rho^{\mathrm{TF}}](\mathbf{x})|^2 \leq 4\pi\rho^{\mathrm{TF}}(\mathbf{x})\right\}} \mathrm{d}\mathbf{x}. \tag{5.61}$$

3.4. Convergence des états dans la limite quasi-fermionique.

Théorème 3.2 (**Convergence des états quasi-fermioniques**).
Soit $\{\Psi_N\} \subset L^2_{asym}(\mathbb{R}^{2N}) = \bigwedge^N(\mathbb{R}^2)$ une suite de fonctions normalisées de L^2 telle que

$$\langle \Psi_N, H_N \Psi_N \rangle = E^R(N) + o(N)$$

dans la limite $N \to \infty$ avec (5.27). Pour n'importe quel choix $\hbar_x, \hbar_p \to 0$, tel que (5.45), et sous les hypothèses 3.1, les fonctions de Husimi définies via Ψ_N convergent vers celles de l'énergie Vlasov

$$m^{(k)}_{\Psi_N}(\mathbf{x}_1, \mathbf{p}_1, \ldots, \mathbf{x}_k, \mathbf{p}_k) \to \prod_{j=1}^{k} m_{\rho^{\mathrm{TF}}}(\mathbf{p}_j, \mathbf{x}_j)$$

pour la convergence faible des mesures de probabilité et $k \geq 1$, i.e.

$$\int_{\mathbb{R}^{4k}} m^{(k)}_{\Psi_N} \phi_k \to \int_{\mathbb{R}^{4k}} (m_{\rho^{\mathrm{TF}}})^{\otimes k} \phi_k$$

pour toute fonction continue et bornée $\phi_k \in C_b\left(\mathbb{R}^{4k}\right)$.

Nous obtenons conséquemment la convergence des marginales en position et en moment

$$\binom{N}{k}^{-1} \rho^{(k)}_{\Psi_N}(\mathbf{x}_1, \ldots, \mathbf{x}_k) \to \prod_{j=1}^{k} \rho^{\mathrm{TF}}(\mathbf{x}_j)$$

$$\binom{N}{k}^{-1} t^{(k)}_{\Psi_N}(\mathbf{p}_1, \ldots, \mathbf{p}_k) \to \prod_{j=1}^{k} t^{\mathrm{TF}}(\mathbf{p}_j)$$

pour la convergence faible des mesures de probabilité. La fonction de Wigner $\mathcal{W}^{(k)}_{\Psi_N}$ définie dans (5.58), présente le même comportement

$$\int_{\mathbb{R}^{4k}} \mathcal{W}^{(k)}_{\Psi_N} \phi \to \int_{\mathbb{R}^{4k}} \left(m_{\rho^{\mathrm{TF}}}\right)^{\otimes k} \phi \tag{5.62}$$

pour toute fonction test ϕ telle que

$$\partial^{\alpha_1}_{x_1} \ldots \partial^{\alpha_k}_{x_k} \partial^{\beta_1}_{p_1} \ldots \partial^{\beta_k}_{p_k} \phi \in L^\infty\left(\mathbb{R}^{4k}\right), \tag{5.63}$$

$$\max\left(\alpha_j, \beta_j\right) \leq 1. \tag{5.64}$$

Remarque 5.5.

- La convergence des fonctions de Wigner (5.62) découle directement de celle des fonctions de Husimi par [**147**, Theorem 2.1].
- Nous pouvons aussi montrer que la convergence ne dépend pas de la fonction f choisie pour construire les états cohérents [**147**, Lemma 2.8].
- Dans la preuve du résultat ci-dessus nous travaillerons avec $\hbar_x \ll \hbar_p$ pour mieux contrôler les singularités en $\mathbf{x}$.

- L'énergie Thomas-Fermi ne dépend pas de la statistique anyonique. C'est en revanche le cas des minimiseurs de la fonctionnelle Vlasov. La statistique se retrouve ainsi dans l'objet limite de la distribution en moment (5.61).

4. Illustrations numériques

Nous illustrons les résultats des sections précédentes à l'aide de simulations numériques. Nous commencerons par parler de résultats existants [**32**] pour le cas quasi-bosonique. Nous retrouverons ces résultats à l'aide du code qu'Antoine Levitt m'a aimablement prêté. Dans un second temps, nous utiliserons ce même code (au centre d'un travail en cours d'Antoine Levitt et Nicolas Rougerie) pour montrer la cohérence entre les prédictions théoriques et les calculs numériques. Je tiens à les remercier chaleureusement de m'avoir laissé accéder à leur travail ce qui m'a permis d'obtenir les tracés 5.1, 5.2, 5.3, 5.4 et 5.5 .

4.1. Simulations pour les bosons. L'état fondamental de la fonctionnelle de champ-moyen (5.15) peut être étudié numériquement avec $\mathbf{A}_e = 0$ pour simplifier. Nous rappelons que

$$\mathcal{E}_\beta^{\mathrm{af},B}[u] = \int_{\mathbb{R}^2} \left| \left(-\mathrm{i}\nabla + \beta \mathbf{A}\left[|u|^2 \right] \right) u \right|^2 + V|u|^2 \tag{5.65}$$

est la fonctionnelle limite dans le cas des anyons quasi-bosoniques. Nous notons l'état fondamental associé

$$\inf \left\{ E_\beta^{\mathrm{af},B}[u], \int_{\mathbb{R}^2} |u|^2 = 1 \right\}. \tag{5.66}$$

Le champ magnétique de (5.65) est

$$\beta \mathrm{curl}\mathbf{A}[|u|^2] = 2\pi\beta |u|^2 \tag{5.67}$$

d'un flux total $2\pi\beta$. Cette énergie présente des similarités avec la fonctionnelle Gross-Pitaevskii

$$\mathcal{E}^{\mathrm{GP}}[u] := \int_{\mathbb{R}^2} \left(|-\mathrm{i}\nabla u + \mathbf{A}u|^2 + V|u|^2 + g|u|^4 \right) \tag{5.68}$$

de potentiel vecteur $\mathbf{A}$. La fonctionnelle ci-dessus décrit un gaz de bosons en rotation [**95, 97**]. Le terme quartique d'intensité g provient d'interaction de paires courtes portées. La différence principale entre (5.68) et (5.65) est que nos interactions sont purement magnétiques. Cela implique que la phase de la fonction d'onde se comporte de façon plus compliquée. En écrivant $u = \rho e^{\mathrm{i}\phi}$ nous voyons plus clairement comment nous pouvons minimiser l'énergie

$$\int_{\mathbb{R}^2} |\nabla\sqrt{\rho}|^2 + \rho |\nabla\phi + \beta\mathbf{A}[\rho]|^2. \tag{5.69}$$

Le second terme ci-dessus sera minimal si

$$\nabla\phi \approx -\beta\mathbf{A}[\rho]. \tag{5.70}$$

Lorsqu'on en prend le curl, nous obtenons

$$\mathrm{curl}\nabla\phi \approx -2\pi\beta\rho \tag{5.71}$$

ce qui n'est possible que si ϕ présente des singularités à circulation non triviale i.e des vortex. Notons que la valeur de β tout comme la forme du piège V dans (5.68) peuvent facilement altérer la distribution des vortex. À champ magnétique faible, le condensat GP ne présente aucun vortex. Ceux-ci commencent à appairaître lorsque nous augmentons

le champ magnétique jusqu'à ce que la force centrifuge compense la force de piégeage et que le gaz s'étale complètement. Nous verrons que la fonctionnelle (5.65) présente un comportement similaire.

L'état fondamental de (5.65) a été calculé numériquement dans [**32**] pour différentes valeurs de β et différentes formes de potentiel. Nous présentons ici des simulations similaires obtenues avec le code d'Antoine Levitt[2].

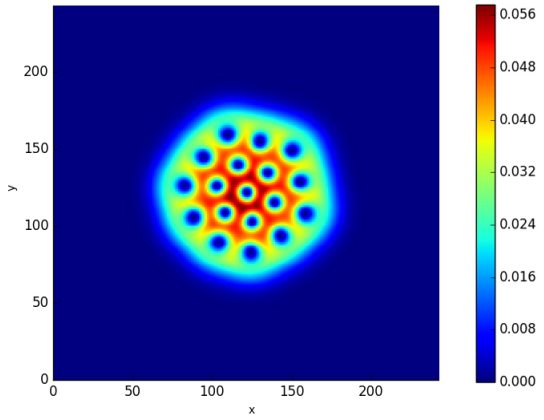


FIGURE 5.1. Densité du minimiseur de (5.65) $\beta = 25$, $V(\mathbf{x}) = |\mathbf{x}|^2$

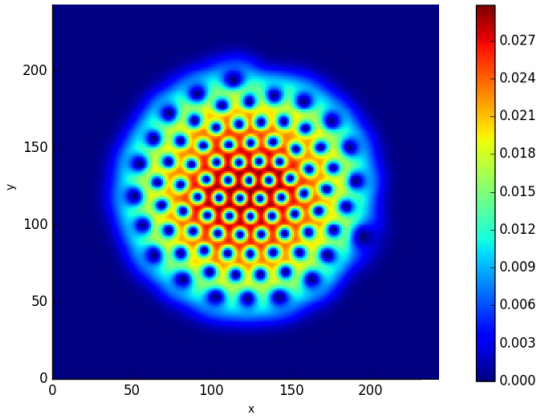


FIGURE 5.2. Densité du minimiseur de (5.65) $\beta = 95$ et $V(\mathbf{x}) = |\mathbf{x}|^2$

Les figures 5.1 et 5.2 présentent la densité du minimiseur de (5.65) avec $V(\mathbf{x}) = |\mathbf{x}|^2$. Nous observons en premier lieu que le gaz s'étale lorsque β augmente. Le condensat dévoloppe de nombreux vortex répartis selon une maille triangulaire similaires à ceux des condensats de Bose-Einstein en rotations [**2, 29, 36, 37, 49**] ou aux superconducteurs de type II [**139, 156, 157**].

4.2. Simulations pour les fermions. Nous nous tournons maintenant vers la fonctionnelle (5.31)

$$\mathcal{E}_\beta^{\mathrm{af},F}[\gamma] = \frac{1}{N}\,\mathrm{Tr}\left[\left(|\mathbf{p}_1 + \beta\mathbf{A}\left[\frac{\rho_\gamma}{N}\right]\right)^2\gamma\right] + \int_{\mathbb{R}^2} V(\mathbf{x}_1)\frac{\rho_\gamma(\mathbf{x}_1)}{N}\mathrm{d}\mathbf{x}_1. \tag{5.72}$$

Le champ magnétique associé est

$$\beta\ \mathrm{curl}\mathbf{A}\left[\frac{\rho_\gamma}{N}\right] = \frac{2\pi\beta}{N}\sum_{j=1}^{N}|u_j|^2 \tag{5.73}$$

d'un flux total $2\pi\beta$. Nous définissons son énergie fondamentale par

$$E_\beta^{\mathrm{af},F} := \inf\left\{\mathcal{E}_\beta^{\mathrm{af},F}[\gamma], \mathrm{Tr}[\gamma] = N,\ \ 0 \le \gamma \le 1, \gamma \text{ de rang } N\right\}. \tag{5.74}$$

[2]Son code est conçu pour des fermions. Nous pouvons cependant retrouver les résultats de [**32**] en ne considérant qu'une seule particule $N = 1$.

Ce problème de minimisation est équivalent à (5.65) quand $N = 1$. Ici γ est un projecteur, nous minimisons donc l'énergie originelle en nous restreignant à un dérterminant de Slater. Le modèle limite est

$$\mathcal{E}_{\mathrm{Vla}}[m] = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} |\mathbf{p} + \beta\mathbf{A}[\rho](\mathbf{x})|^2\, m(\mathbf{x},\mathbf{p})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p} + \int_{\mathbb{R}^2} V(\mathbf{x})\rho_m(\mathbf{x})\mathrm{d}\mathbf{x} \tag{5.75}$$

avec $m(\mathbf{x},\mathbf{p}) \leq 1$ une mesure positive, de masse $4\pi^2$ l'espace des phases $\mathbb{R}^4$ des positions et moments. Les minimiseurs sont de la forme

$$m^{\mathrm{TF}}(\mathbf{x},\mathbf{p}) = \mathbb{1}\Big(\left|\mathbf{p} + \beta\mathbf{A}[\rho^{\mathrm{TF}}](\mathbf{x})\right|^2 \leq 4\pi\rho^{\mathrm{TF}}(\mathbf{x})\Big)$$

avec

$$\rho^{\mathrm{TF}}(\mathbf{x}) = \frac{1}{4\pi}\left(\lambda^{\mathrm{TF}} - V(\mathbf{x})\right)_+ \tag{5.76}$$

où $\lambda^{\mathrm{TF}} = 2\sqrt{2}$ dans le cas $V(\mathbf{x}) = |\mathbf{x}|^2$ assurant la normalisation. La fonction ρ^{TF} minimise l'énergie

$$\mathcal{E}_{\mathrm{TF}}[\rho] = \mathcal{E}_{\mathrm{Vla}}[m^{\mathrm{TF}}] = 2\pi\int_{\mathbb{R}^2}\rho^2(\mathbf{x})\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2} V(\mathbf{x})\rho(\mathbf{x})\mathrm{d}\mathbf{x}. \tag{5.77}$$

Nous obtenons aussi

$$t^{\mathrm{TF}}(\mathbf{x}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} m^{\mathrm{TF}}(\mathbf{x},\mathbf{p})\mathrm{d}\mathbf{x} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\mathbb{1}\left(\left|\mathbf{p} + \beta\mathbf{A}\left[\rho^{\mathrm{TF}}\right](\mathbf{x})\right|^2 \leq 4\pi\rho^{\mathrm{TF}}(\mathbf{x})\right)\mathrm{d}\mathbf{x} \tag{5.78}$$

qui sera calculée numériquement en utilisant la densité explicite (5.76). Nous comparons numériquement les états fondamentaux de (5.72) avec ρ^{TF} et t^{TF}.

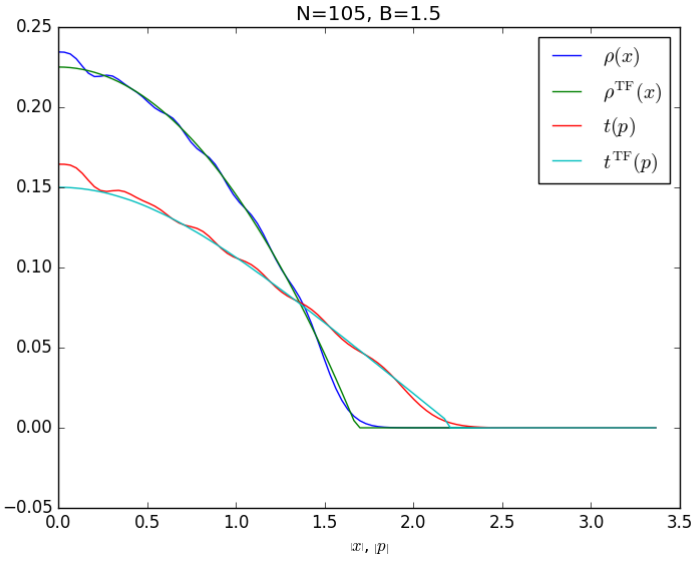


FIGURE 5.3. Densités radiales des minimiseurs de (5.32) comparées à celles de (5.72) $\beta = 1.5$ $N = 105$ et $V(\mathbf{x}) = |\mathbf{x}|^2$.

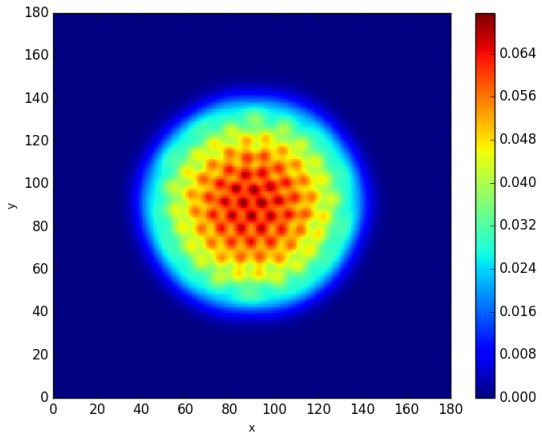


FIGURE 5.4. Densités des minimiseurs de (5.72) $\beta = 25$ $N = 2$ et $V(\mathbf{x}) = |\mathbf{x}|^2$

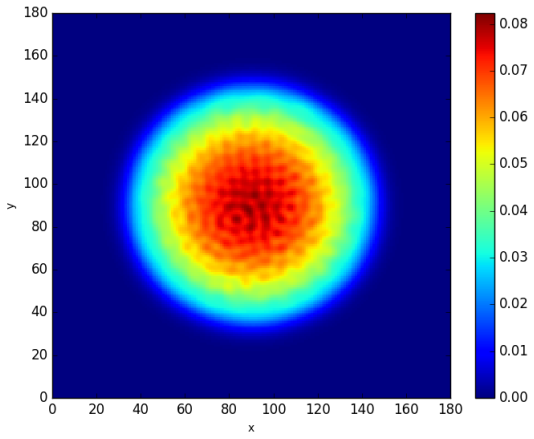


FIGURE 5.5. Densités des minimiseurs de (5.72) $\beta = 25$ $N = 3$ et $V(\mathbf{x}) = |\mathbf{x}|^2$

Les simulations confirment bien les résultats du Théorème 3.2. La figure 5.3 présente un très bon accord entre théorie et calcul avec seulement $N = 105$ particules.

Il suffit de prendre $N > 1$ pour remarquer sur les tracés 5.4 et 5.5 que les vortex disparaîssent. Le champ magnétique créé par la première particule $u \in L^2(\mathbb{R}^2)$ via le potentiel $\beta\mathbf{A}$ est $B = 2\pi\beta|u|^2$. À cet instant, le champ magnétique est fort la où la

première particule se trouve. La seconde particule $v \in L^2\left(\mathbb{R}^2\right)$ va donc essayer d'éviter ces zones ce qui explique le remplissage progressif des trous observés en 5.4 et 5.5.

5. Perspectives: vers une inégalité de Lieb-Thirring pour anyons étendus et quelques autres idées

Nous terminerons en décrivant à larges traits un travail en cours avec Nicolas Rougerie. Nous donnerons ensuite quelques suggestions concernant les études du gaz d'anyons à venir.

Inégalité de Lieb-Thirring pour des anyons étendus

Notre but est l'obtention d'une inégalité similaire à celle du Théorème 3.1 mais pour tout R. Nous avons mentionné à la remarque 5.3 qu'un tel résultat était possible et nous allons expliquer pourquoi. L'inégalité

$$\left\langle \Psi_N, \sum_{j=1}^{N} \left(\mathbf{p}_j + \alpha \mathbf{A}^R(\mathbf{x}_j)\right)^2 \Psi_N \right\rangle \geq C_\alpha \int_{\mathbb{R}^2} \rho_{\Psi_N}(\mathbf{x})^2 \mathrm{d}x \tag{5.79}$$

devrait être valide pour toute fonction fermionique Ψ_N de densité à un corps

$$\rho_{\Psi_N} := \sum_{j=1}^{N} \int_{\mathbb{R}^{2(N-1)}} \left|\Psi_N\left(\mathbf{x}_1, ..., \mathbf{x}_{j-1}, \mathbf{x}, \mathbf{x}_{j+1}, ..., \mathbf{x}_N\right)\right|^2 \prod_{k\neq j} \mathrm{d}\mathbf{x}_k \tag{5.80}$$

avec une constante C_α uniformément bornée inférieurement pour tout

$$\alpha \neq 1 \mod 2.$$

. L'inégalité (5.79) a déjà été démontrée dans le cas particulier $R = 0$ [**114**, Theorem 2.2]. La méthode combine une borne inférieure sur l'énergie de Neumann locale (5.83) avec un procédé de localisation décrit dans [**115**]. Des bornes similaires pour l'énergie de Neumann locale des anyons étendus sont disponibles dans [**75**] mais nous allons voir que la méthode utilisée pour obtenir le cas $R = 0$ ne s'applique pas directement dans le cas $R \neq 0$.

5.1. Le cas $R = 0$. La preuve de l'équation (5.79) à $R = 0$ de [**114**] combine [**114**, Theorem 2.1] et la méthode de [**115**]. Le théorème [**114**, Theorem 2.1] fournit une borne inférieure sur l'énergie fondamentale avec conditions de Neumann de N anyons idéaux dans un domaine $\Omega \subset \mathbb{R}^2$

$$E_N^{\mathcal{N}}(\alpha; \Omega) := \inf \left\{ \sum_{j=1}^{N} \int_{\Omega^N} \left|(-\mathrm{i}\nabla_j + \alpha \mathbf{A}_j)\, \Psi\right|^2 ; \Psi \in \mathcal{D}_N^\alpha, \int_{\Omega^N} |\Psi|^2 = 1 \right\} \tag{5.81}$$

où l'opérateur $D_j = -\mathrm{i}\nabla_j + \alpha \mathbf{A}_j$ est définit en (3.18) et où le domaine $\mathcal{D}_N^\alpha$ est celui de la forme quadratique (3.22), les fonctions d'ondes considérées sont bosoniques. Nous notons

$$E_N^{\mathcal{N}}(\alpha) = E_N^{\mathcal{N}}(\alpha; \Omega_0)$$

dans le cas particulier où $\Omega_0 = [0, 1]^2$. De cette manière

$$E_N^{\mathcal{N}}(\alpha; Q) = |Q|^{-1} E_N^{\mathcal{N}}(\alpha). \tag{5.82}$$

Nous ne citons ici que la partie du [**114**, Theorem 2.1] relative à notre discussion

Théorème 5.1 (**Bornes pour le gaz d'anyons idéaux**).
Il existe une constante C telle que pour tout $0 \leq \alpha \leq 1$

$$\frac{E_N^{\mathcal{N}}(\alpha)}{N^2} \geq C\alpha \left(1 - O\left(N^{-3}\right)\right) \tag{5.83}$$

Cette borne combinée à [**115**] permet d'obtenir

THÉORÈME 5.2 (**Inégalité de Lieb-Thirring pour les anyons idéaux**). Il existe une costante $C > 0$ telle que pour tout $0 \leq \alpha \leq 1$, $N \geq 1$ et $\Psi_N \in \mathcal{D}_N^\alpha$

$$\sum_{j=1}^{N} \int_{\mathbb{R}^{dN}} |D_j \Psi_N|^2 \, \mathrm{d}x \geq C\alpha \int_{\mathbb{R}^d} \rho_{\psi_N}(\mathbf{x})^2 \mathrm{d}\mathbf{x} \tag{5.84}$$

Le théorème (valide pour des fonctions d'ondes bosoniques) est une preuve de (5.79) dans le cas $R = 0$ pourvu que nous remplacions $\Psi_N \to U_1 \Psi_N$ et $\alpha \to 1 - \alpha$ puisque (5.79) est énoncé pour des fonctions d'onde fermioniques.

5.2. Le cas $R > 0$. Dans le cas $R > 0$ il existe une borne inférieure pour la limite thermodynamique du gaz d'anyons [**75**, Theorem 1.5]. Les bornes intermédiaires utilisées afin de l'obtenir sont locales comme en (5.83) mais ici l'énergie $E_N^{\mathcal{N}}$ perd sa propriété d'homogénéité (5.82) qui était importante dans l'approche de [**114**]. Une des bornes locales est [**75**, Lemma 5.3] et est valide pour tout R

$$E_N^{\mathcal{N}}(\alpha; Q) \geq \frac{C\alpha_*^2}{|Q|} \int_Q \rho_{\Psi_N} \tag{5.85}$$

avec

$$\alpha_* = \lim_{N \to \infty} \alpha_N \tag{5.86}$$

$$\alpha_N := \min_{p \subset \{0,1,\ldots,N-2\}} \min_{q \in \mathbb{Z}} |(2p+1)\,\alpha - 2q| \,. \tag{5.87}$$

Le problème est que $\alpha_* = 0$ à moins que α soit une fraction avec un numérateur impair. Dans les autres bornes le problème est que le ratio $\gamma(Q) = R/|Q|$ devient important [**75**, Lemma 5.1,5.2]. Ces bornes sont de la forme

$$E_N^{\mathcal{N}}(\alpha; Q) \geq \frac{e\left(\alpha, \gamma(Q), \int_Q \rho_{\Psi_N}\right)}{|Q|} \tag{5.88}$$

où la dépendence en $\gamma(Q)$ ne permet pas de suivre exactement le même chemin que dans le cas $R = 0$. Nous rappelons que le champ porté par les anyons est

$$\mathrm{curl}\widetilde{\mathbf{A}}^R(\mathbf{x}_j) = \mathrm{curl} \sum_{k \neq j} \log|\,.\,| * \frac{\mathbb{1}_{B(0,R)}}{\pi R^2}(\mathbf{x}_j - \mathbf{x}_k) = 2\pi \sum_{k \neq j} \frac{\mathbb{1}_{B(\mathbf{x}_k,R)}}{\pi R^2}(\mathbf{x}_j) \tag{5.89}$$

et observons que dans le cas où le rayon est plus grand que la diagonale de la boite Q, les particules à l'intérieur de celle-ci induisent un champ constant alors que celles se trouvant à une distance d'au moins $R + \sqrt{2Q}$ ne jouent aucun rôle dans l'énergie. Les particules que nous devons contrôler se trouvent entre ces deux zones. Il semble que ce soit la dernière étape avant l'obtention de l'inégalité désirée.

Autres suggestions: Nous pouvons aussi imaginer poursuivre l'étude des anyons selon trois différents chemins.

- Tenter d'investiguer le cas α fixe et obtenir une énergie Thomas-Fermi magnétique modifiée. Voir remarque 5.2.
- Tenter d'atteindre le régime dilué $\eta > 1/2$ dans le cas bosonique. Voir remarque 3.1.

- Étudier la dynamique des anyons fermioniques ou bosoniques comme dans [**4, 5, 6, 52, 53, 54, 56, 57, 69, 131, 142, 141, 14, 150, 152, 151**] .

CHAPTER 6

Méthodes de preuve

Notre méthode pour prouver 2.1 se base sur le Théorème de de Finetti quantique [**136**]. Le Théorème 3.1 utilise quant à lui les versions de Finetti classiques comme le théorème de Diaconis-Freedman [**41**] ou celui d'Hewitt-Savage [**70**]. Ils consistent tous à transformer une probabilité symétrique à k particules en une combinaison convexe de mesures à un corps factorisées. Nous introduisons ces théorèmes avant de commencer les preuves.

1. Théorèmes de de Finetti classiques pour les cas fermionique

La preuve du Théorème 3.1 utilise une approche semi-classique basée sur les états cohérents. Les mesures mises en jeu ont pour domaine l'espace des phases $\mathbb{R}^{2N} \times \mathbb{R}^{2N}$ des positions et vitesses.

1.1. Le Théorème d'Hewitt-Savage. Ce thérème énonce que toute mesure de probabilité symétrique sur l'espace $\Omega^N = \Omega \times ... \times \Omega$ se rapproche d'une combinaison convexe de mesures produits sur Ω à mesure que N augmente. Nous notons $\mathcal{P}\left(\Omega^N\right)$ l'espace des mesures de probabilité sur Ω^N. Une mesure de probabilité symétrique $\mu_N \in \mathcal{P}\left(\Omega^N\right)$ satisfait à

$$\mu_N\left(A_1 \times ... \times A_N\right) = \mu_N\left(A_{\sigma(1)} \times ... \times A_{\sigma(N)}\right) \tag{6.1}$$

pour tout domaine borélien $A_1, ..., A_N \subset \Omega$ et toute permutation de N indices $\sigma \in S_N$. Cet ensemble est noté $\mathcal{P}_{\text{sym}}\left(\Omega^N\right)$. Nous définissons aussi

$$\mathcal{P}\left(\mathcal{P}\left(\Omega^N\right)\right) \tag{6.2}$$

l'ensemble des mesures de probabilité sur l'ensemble $\mathcal{P}\left(\Omega^N\right)$. Une mesure produit de terme principal $\rho \in \mathcal{P}\left(\Omega\right)$ est de la forme

$$\rho^{\otimes N}\left(A_1 \times ... \times A_N\right) = \rho\left(A_1\right)...\rho\left(A_N\right). \tag{6.3}$$

Une telle mesure est symétrique. Dans la limite $N \to \infty$ nous pouvons considérer des mesures avec un nombre infini de variables $\mu \in \mathcal{P}\left(\Omega^{\mathbb{N}}\right)$ dotées de la notion de symétrie

$$\mu\left(A_1, A_2, ...\right) = \mu\left(A_{\sigma(1)}, A_{\sigma(2)}, ...\right) \tag{6.4}$$

pour toute suite de domaines boréliens $\{A_K\}_{K\in\mathbb{N}} \subset \Omega^{\mathbb{N}}$ et toute permutation avec un nombre infini d'indices $\sigma \in S_\infty$. Notons $\mathcal{P}_{\text{sym}}\left(\Omega^{\mathbb{N}}\right)$ l'ensemble des mesures satisfaisant (6.4). Nous pouvons énoncer le théorème

Théorème 1.1 (**Hewitt-Savage 1955**).
Soit $\mu \in \mathcal{P}_{\text{sym}}\left(\Omega^{\mathbb{N}}\right)$ satisfaisant (6.4). Notons $\mu^{(k)}$ sa k-ième marginale,

$$\mu^{(k)}\left(A_1, ..., A_k\right) = \mu\left(A_1, ..., A_k, \Omega, ..., \Omega, ...\right) \tag{6.5}$$

Il existe une unique mesure de probabilité $P_\mu \in \mathcal{P}(\mathcal{P}(\Omega))$ telle que

$$\mu^{(k)} = \int_{\rho\in\mathcal{P}(\Omega)} \rho^{\otimes k} \mathrm{d}P_\mu(\rho). \tag{6.6}$$

La preuve originelle de ce théorème est géométrique. L'ensemble des mesures de probabilité symétriques sur Ω^N est convexe. Le théorème de Choquet-Krein-Milman énonce que tout point d'un ensemble convexe est une combinaison des points extrémaux. Il suffit alors de montrer que les points extrémaux de $\mathcal{P}_{\mathrm{sym}}\left(\Omega^{\mathbb{N}}\right)$ coïncident avec les mesures produit, correspondant à des suites de marginales de la forme $\left\{\rho^{(k)}\right\}_{k\in\mathbb{N}}$. Cette approche n'est pas constructive, et la preuve du fait que les suites $\left\{\rho^{(k)}\right\}_{k\in\mathbb{N}}$ sont les points extrémaux de $\mathcal{P}_{\mathrm{sym}}\left(\Omega^{\mathbb{N}}\right)$ est par contradiction. Le théorème de Diaconis-Freedman [**41**] est une approche constructive de ce dernier résultat. Lions developpa une approche similaire pour les besoins du champ-moyen en théorie des jeux [**109**]. Voir [**124**] pour une discussion plus détaillée.

1.2. Le théorème de Diaconis-Freedman. Le théorème précédent est vrai lorsque $N = +\infty$ mais une version plus quantitative existe. À N fixé l'erreur est quantifiée par la norme en variation totale des mesures de probabilité sur S

$$\|\mu\|_{\mathrm{TV}} = \int_\Omega \mathrm{d}|\mu| = \sup_{\phi\in C_b(\Omega)} \left|\int_\Omega \phi \mathrm{d}\mu\right|. \tag{6.7}$$

avec $C_b(\Omega)$ l'espace des fonctions continues et bornées définies sur Ω. Définissons alors la mesure

$$P_{\mu_N} := \int_{\Omega^N} \delta_{\rho=\rho_Z} \mu_N(Z)\,\mathrm{d}Z \tag{6.8}$$

dans $\mathcal{P}(\mathcal{P}(\Omega))$ dont le sens précis est

$$\int_{\mathcal{P}(\Omega)} \phi(\rho)\,\mathrm{d}P_{\mu_N}(\rho) := \int_{\Omega^N} \phi(\rho_Z)\,\mu_N(Z)\,\mathrm{d}Z$$

pour toute fonction ϕ continue et bornée sur les mesures de probabilité et avec

$$\rho_Z(x) := \frac{1}{N}\sum_{j=1}^N \delta_{z_j=x} \tag{6.9}$$

une mesure empirique définie par

$$\int_\Omega u(x)\,\mathrm{d}\rho_Z := \frac{1}{N}\sum_{j=1}^N u(z_j)$$

pour tout $u \in C_b(\Omega)$. Nous pouvons alors énoncer le

THÉORÈME 1.2 (**Diaconis Freedman**).
Soit $\mu_N \in \mathcal{P}_{\mathrm{sym}}\left(\Omega^N\right)$ une mesure de probabilité symétrique, en posant

$$\widetilde{\mu}_N := \int_{\rho\in\mathcal{P}(\Omega)} \rho^{\otimes N} \mathrm{d}P_{\mu_N} \tag{6.10}$$

nous obtenons que

$$\left\|\widetilde{\mu}_N^{(k)} - \mu_N^{(k)}\right\|_{\mathrm{TV}} \leq 2\frac{k(k-1)}{N}. \tag{6.11}$$

De plus, lorsque la suite $\left(\mu_N^{(1)}\right)_N$ est tendue,

$$\limsup_{R\to\infty}\sup_{N\in\mathbb{N}}\int_{B(0,R)}\mathrm{d}\mu_N^{(1)}=1, \tag{6.12}$$

il existe une sous-suite et une unique mesure $P\in\mathcal{P}\left(\left(\mathcal{P}(\mathbb{R}^4)\right)\right.$ telle que, modulo sous-suite,

$$P_{\mu_N}\to P$$

testée contre toute fonction continue et bornée définie sur $\mathcal{P}(\mathbb{R}^4)$. La mesure P ici obtenue n'est rien d'autre que celle du théorème de Hewitt-Savage [**70**]. Pour tout $n\in\mathbb{N}^*$ fixé

$$\widetilde{\mu}_N^{(n)}\rightharpoonup\int_{\rho\in\mathcal{P}(\Omega)}\rho^{\otimes n}\mathrm{d}P(\rho). \tag{6.13}$$

Observons que

$$\widetilde{\mu}_N\left(X\right):=\int_{\Omega^N}\mu_N\left(Z\right)\sum_{\gamma\in\Gamma_N}N^{-N}\delta_{X=Z_\gamma}\mathrm{d}Z \tag{6.14}$$

où Γ_N est l'ensemble des applications (autorisant les indices répétés) de $\{1,...,N\}$ dans lui-même

$$X_\gamma=\left(\mathbf{x}_{\gamma(1)},...,\mathbf{x}_{\gamma(N)}\right)$$

avec $\gamma\in\Gamma_N$. Le point crucial est que la mesure symétrique μ_N peut être écrite

$$\mu_N\left(X\right):=\int_{\Omega^N}\mu_N\left(Z\right)\sum_{\sigma\in S_N}\left(N!\right)^{-1}\delta_{X=Z_\sigma}\mathrm{d}Z \tag{6.15}$$

et donc

$$\mu_N^{(k)}-\widetilde{\mu}_N^{(k)}=\int_{\Omega^N}\left[\left(\sum_{\sigma\in S_N}\left(N!\right)^{-1}\delta_{X=Z_\sigma}\right)^{(k)}-\left(\sum_{\gamma\in\Gamma_N}N^{-N}\delta_{X=Z_\gamma}\right)^{(k)}\right]\mu_N\left(Z\right)\mathrm{d}Z. \tag{6.16}$$

Ce résultat se comprend en interprétant

$$\left(\sum_{\sigma\in S_N}\left(N!\right)^{-1}\delta_{X=Z_\sigma}\right)^{(k)} \tag{6.17}$$

comme la probabilité de tirer k boules au hasard d'une urne en contenant N sans remplacement, alors que

$$\left(\sum_{\gamma\in\Gamma_N}N^{-N}\delta_{X=Z_\gamma}\right)^{(k)} \tag{6.18}$$

est la loi de probabilité pour un tirage de k boules avec remplacement. Intuitivement, il est clair que lorsque k est petit devant N, le remplacement des boules n'affecte que très peu le résultat final. Nous référons à [**136**][Theorem 2.2] pour une preuve complète. Nous remarquons que (6.14) nous fournit les marginales de $\widetilde{\mu}_N$. C'est une conséquence de l'approch de Dicaonis-Freedman.

LEMME 6.1 (**Premières marginales de la mesure de Diaconis-Freedman**). Soit $\mu_N \in \mathcal{P}_{\mathrm{sym}}(\Omega^N)$ et sa mesure associée par le Théorème de Diaconis-Freedman $\widetilde{\mu}_N$. Nous avons

$$\widetilde{\mu}_N^{(1)}(X_1) = \mu_N^{(1)}(X_1) \tag{6.19}$$

$$\widetilde{\mu}_N^{(2)}(X_1, X_2) = \frac{N-1}{N}\mu_N^{(2)}(X_1, X_2) + \frac{1}{N}\mu_N^{(1)}(X_1)\delta_{X_1=X_2} \tag{6.20}$$

$$\begin{aligned}\widetilde{\mu}_N^{(3)}(X_1, X_2, X_3) &= \frac{N(N-1)(N-2)}{N^3}\mu_N^{(3)}(X_1, X_2, X_3) \\ &+\frac{(N-1)}{N^2}\Big[\mu_N^{(2)}(X_1, X_3)\delta_{X_1=X_2} + \mu_N^{(2)}(X_2, X_1)\delta_{X_2=X_3} + \mu_N^{(2)}(X_3, X_2)\delta_{X_3=X_1}\Big] \\ &+ \frac{1}{N^2}\mu_N^{(1)}(X_1)\delta_{X_1=X_2}\delta_{X_2=X_3}\end{aligned} \tag{6.21}$$

PROOF. Nous pouvons trouver (6.19) et (6.20) dans [**136**, Remark 2.3]. Il suit de [**136**, Equation 2.16] que

$$\widetilde{\mu}_N(X_N) = \int_{\Omega^N} \mu_N(Z) \sum_{\gamma\in\Gamma_N} N^{-N}\delta_{X=Z_\gamma}\mathrm{d}Z \tag{6.22}$$

$$= \int_{\Omega^N} \mu_N(Z) \left(N^{-1}\sum_{j=1}^N \delta_{z_j=(x_j,p_j)}\right)^{\otimes N} \mathrm{d}Z \tag{6.23}$$

où Z_σ est le $4N$-uplet $(x_{\sigma(1)}, p_{\sigma(1)}, ..., x_{\sigma(N)}, p_{\sigma(N)}) = (z_{\sigma(1)}, ...z_{\sigma_N})$, $(z_i = (x_i, p_i))$ et Γ_N l'ensemble des applications de $\{1, ..., N\}$ dans lui-même. Nous exprimons la n-ième marginale

$$\widetilde{\mu}_N^{(n)}(X_N) = \int_{\Omega^N} \mu_N(Z) \left(N^{-1}\sum_{j=1}^N \delta_{z_j=X_j}\right)^{\otimes n} \mathrm{d}Z. \tag{6.24}$$

Nous trouvons en particulier

$$\widetilde{\mu}_N^{(1)}(x_1, p_1) = N^{-1}\int_{\Omega^N} \mu_N(Z) \left(\sum_{j=1}^N \delta_{z_j=X_1}\right)\mathrm{d}Z$$

$$\widetilde{\mu}_N^{(2)}(x_1, p_1, x_2, p_2) = N^{-2}\int_{\Omega^N} \mu_N(Z) \left(\sum_{j=1}^N \delta_{z_j=X_1}\right)\left(\sum_{j=1}^N \delta_{z_j=X_2}\right)\mathrm{d}Z$$

$$\widetilde{\mu}_N^{(3)}(X_1, X_2, X_3) = N^{-3}\int_{\Omega^N} \mu_N(Z) \left(\sum_{j=1}^N \delta_{z_j=X_1}\right)\left(\sum_{j=1}^N \delta_{z_j=X_2}\right)\left(\sum_{j=1}^N \delta_{z_j=X_3}\right)\mathrm{d}Z$$

en éveloppant la n-ième puissance de $\widetilde{\mu}_N^{(n)}$ en termes de μ_N par un calcul explicite. □

2. Théorèmes de de Finetti quantiques pour les bosons

Le théorème 2.1 utilise le Théorèmes de de Finetti quantiques, combiné avec une localisation dans l'espace de Fock. Ces deux points font l'objets de la partie suivante.

2.1. Théorèmes de de Finetti quantiques. En mécanique quantique à N corps un état n'est plus une fonction d'onde mais une matrice densité de la classe trace (2.20) sur l'espace de Hilbert $L^2\left(\mathbb{R}^{dN}\right) \simeq L^2\left(\mathbb{R}^d\right)^{\otimes N}$. Un tel opérateur Γ_N est dit symétrique (ou bosonique) si $\Gamma_N \in \mathcal{S}^1\left(L^2_{\mathrm{sym}}\left(\mathbb{R}^{dN}\right)\right)$

$$\mathcal{S}^1\left(L^2\left(\mathbb{R}^{dN}\right)\right) = \left\{\Gamma \in \mathfrak{S}^1\left(L^2\left(\mathbb{R}^{dN}\right)\right), \Gamma = \Gamma^*, \Gamma \geq 0, \operatorname{Tr}\Gamma = 1\right\}. \tag{6.25}$$

Nous définissons un état bosonique avec une nombre infini de particules comme une suite $\left\{\gamma^{(k)}\right\}_{k\in\mathbb{N}}$ d'opérateur de $\mathcal{S}^1$ satisfaisant

- Chaque état à k particules $\gamma^{(k)} \in \mathcal{S}^1\left(L^2_{\mathrm{sym}}\left(\mathbb{R}^{dN}\right)\right)$ est auto-adjoint, positif et de trace 1.
$$\operatorname{Tr}\left[\gamma^{(k)}\right] = 1. \tag{6.26}$$
- La suite $\left\{\gamma^{(k)}\right\}_{k\in\mathbb{N}}$ est consistante
$$\operatorname{Tr}_{k+1}\left[\gamma^{(k+1)}\right] = \gamma^{(k)} \tag{6.27}$$
où Tr_{k+1} est la trace partielle par rapport à la dernière variable $L^2\left(\mathbb{R}^d\right)^{\otimes(k+1)}$.

la propriété (6.27) assure que la suite décrit un véritable état physique. Un cas particulier de ces états bosoniques sont les états produits. Un état produit avec un nombre infini de particules est une suite d'opérateurs de classe trace $\left\{\gamma^{(k)}\right\}_{k\in\mathbb{N}}$ satisfaisant

- Pour tout $k \geq 0$ $\gamma^{(k)} = \gamma^{\otimes k}$ où γ est un état à un corps.
- Cet état prend alors la forme
$$\gamma^{(k)} = \left(|u\rangle\langle u|\right)^{\otimes k} = \left|u^{\otimes k}\right\rangle\left\langle u^{\otimes k}\right| \tag{6.28}$$
avec $u \in SL^2\left(\mathbb{R}^d\right)$ la sphère unité de l'espace de hilbert à un corps.

Les états produits bosoniques sont tous de la forme (6.28). En effet, si $\gamma \in \mathcal{S}^1\left(L^2_{\mathrm{sym}}\left(\mathbb{R}^{dN}\right)\right)$ n'est pas un projecteur $\gamma^{\otimes 2}$ ne peut pas avoir la symétrie bosonique [**72**]. Le Théorème de de Finetti quantique fort ci-dessous est dû à Hudson et Moody [**72**] et Störmer [**153**].

THÉORÈME 2.1 (**Théorème de de Finetti quantique fort**).
Soit $\mathfrak{H}$ un espace de Hilbert complexe séparable et $\left\{\gamma^{(k)}\right\}_{k\in\mathbb{N}}$ un état bosonique avec un nombre infini de particules sur $\mathfrak{H}$. Il existe une unique mesure de probabilité de Borel $\mu \in \mathcal{P}(S\mathfrak{H})$ sur la sphère $S\mathfrak{H} = \{u \in \mathfrak{H}, \|u\| = 1\}$ de $\mathfrak{H}$, invariante sous l'action de S^1 (multiplication par un complexe de module un $e^{\mathrm{i}\theta}$, $\theta \in \mathbb{R}$) telle que

$$\gamma^{(k)} = \int_{S\mathfrak{H}} \left|u^{\otimes k}\right\rangle\left\langle u^{\otimes k}\right| \mathrm{d}\mu(u) \tag{6.29}$$

pour tout $k \geq 0$.

Ce théorème permet de prouver l'exactitude de l'approximation de champ-moyen au niveau de l'état fondamental (2.39) assez facilement, voir [**136**][Theorem 3.6] ou [**87**]. Des résultats plus forts existent [**136**][Theorem 3.7 et Theorem 3.16] mais nous suivront le

chemin menant à la version en dimension finie. Le premier avantage de travailler en dimension finie est la possibilité d'obtenir des estimations explicites comme celle du théorème de Diaconis-Freedman. Nous définissons la $k$ième matrice densité réduite de $\Gamma_N \in \mathcal{S}^1\left(\mathfrak{H}_{\mathrm{sym}}^N\right)$

$$\gamma_N^{(k)} := \mathrm{Tr}_{k+1\to N}\,\Gamma_N. \tag{6.30}$$

Nous introduisons un espace de Hilbert de dimension finie $\mathfrak{H}$

$$\dim \mathfrak{H} = d \tag{6.31}$$

et énoncons

THÉORÈME 2.2 (**Théorème de de Finetti quantitatif**).
Soit $\Gamma_N \in \mathcal{S}^1\left(\mathfrak{H}_{\mathrm{sym}}^N\right)$ un état bosonique sur $\mathfrak{H}_{\mathrm{sym}}^N$ et $\gamma_N^{(n)}$ ses matrices densité réduites. Il existe une mesure de probilité $\mu_N \in \mathcal{P}\left(S\mathfrak{H}\right)$ telle que, en notant

$$\widetilde{\Gamma}_N = \int_{u\in S\mathfrak{H}} \left|u^{\otimes N}\right\rangle \left\langle u^{\otimes N}\right| \mathrm{d}\mu_N\left(u\right) \tag{6.32}$$

l'état associé et $\widetilde{\gamma}_N^{(k)}$ sa matrice de densité réduite, nous ayons

$$\mathrm{Tr}_{\mathfrak{H}^k}\left|\gamma_N^{(n)} - \widetilde{\gamma}_N^{(n)}\right| \leq \frac{2n\left(d+2n\right)}{N} \tag{6.33}$$

pour tout $n = 1...N$.

Nous devons ce résultat à Christandl, König, Mitchison et Renner [**23**]. Il y a aussi quelques développement dans [**21, 68, 84**] mais nous seront plus intéressé par les variantes de l'informatique théorique [**20, 23, 24**]. Ici l'erreur dans (6.33) ne dépendra plus linéairement de la dimension de l'espace de Hilbert à un corps mais logarithmiquement. La contrepartie est que l'erreur ne sera plus quantifiée en terme de norme de la classe trace et que la mesure construite ne chargera plus seulement des états bosoniques. Cette version du Théorème de de Finetti est due à Brandão et Harrow [**18, 88**] ou encore [**135**][Proposition 3.2].

THÉORÈME 2.3 (**Représentations à la de Finetti des matrices densité projetées**).
Soit $\mathfrak{H}$ un espace de Hilbert complexe séparable avec l'espace bosonique correpondant $\mathfrak{H}_N = \mathfrak{H}^{\otimes_{\mathrm{sym}} N}$. Soit $\gamma_N^{(n)}$ la n-ième matrice densité réduite de l'état à N corps $\Psi_N \in \mathfrak{H}_N$ ou d'un état mixte général:

$$\mathcal{S}^1\left(L^2_{\mathrm{sym}}\left(\mathbb{R}^{2N}\right)\right) = \left\{\Gamma \in \mathfrak{S}^1\left(L^2_{sym}\left(\mathbb{R}^{2N}\right)\right), \Gamma = \Gamma^*, \Gamma \geq 0, \mathrm{Tr}\,\Gamma = 1\right\} \tag{6.34}$$

Soit P un projecteur orthogonal de dimension finie.
Il existe une mesure de Borel positive $\mu_N^{(n)}$ sur l'ensembe des états mixtes à un corps

$$S_P = \mathcal{S}\left(PL^2_{\mathrm{sym}}\left(\mathbb{R}^{2N}\right)\right)$$

telle que

$$\sup_{0\leq A_1,A_2,\ldots A_n\leq 1} \mathrm{Tr}\left|A_1\otimes A_2\otimes\ldots\otimes A_n\left(P^{\otimes n}\gamma_N^{(n)}P^{\otimes n} - \int_{S_P}\gamma^{\otimes n}\mathrm{d}\mu_N^{(n)}(\gamma)\right)\right| \leq C_n\sqrt{\frac{\log(\dim(P))}{N}} \tag{6.35}$$

où le sup est pris parmi les opérateurs de $P\mathfrak{H}$ bornés et auto-adjoints.

Comme dans le Théorème (2.1) l'erreur obtenue dans (6.35) implique de travailler dans un espace de Hilbert de dimension finie est donc de localiser les états de la façon suivante

$$P^{\otimes n}\gamma_N^{(n)}P^{\otimes n}. \tag{6.36}$$

Nous voulons voir $P^{\otimes n}\gamma_N^{(n)}P^{\otimes n}$ comme la densité réduite d'un état localisé Γ_P dont toutes les particules sont dans $P\mathfrak{H}$

$$P^{\otimes n}\gamma_N^{(n)}P^{\otimes n} = (\Gamma_P)^{(n)}. \tag{6.37}$$

L'état Γ_P doit donc vivre dans l'espace de Fock car des particules peuvent être perdues dans le processus de localisation.

2.2. Localisation dans l'espace de Fock. nous définissons l'espace de Fock en tant que somme infinie d'espaces de Hilbert symétriques à k particules

$$\mathcal{F}(\mathfrak{H}) = \mathbb{C} \oplus \mathfrak{H} \oplus \mathfrak{H}^2 ... \tag{6.38}$$

Dans notre cas nous avons

$$\mathcal{F}_{\mathrm{sym}}\left(L^2_{\mathrm{sym}}\left(\mathbb{R}^d\right)\right) = \mathbb{C} \oplus L^2_{\mathrm{sym}}\left(\mathbb{R}^d\right) \oplus L^2_{\mathrm{sym}}\left(\mathbb{R}^{2d}\right) \oplus ... \tag{6.39}$$

Nous définissons l'espace de Fock tronqué

$$\mathcal{F}^{\leq N}_{\mathrm{sym}}\left(L^2_{\mathrm{sym}}\left(\mathbb{R}^d\right)\right) = \mathbb{C} \oplus L^2_{\mathrm{sym}}\left(\mathbb{R}^d\right) \oplus L^2_{\mathrm{sym}}\left(\mathbb{R}^{2d}\right) \oplus ... \oplus L^2_{\mathrm{sym}}\left(\mathbb{R}^{dN}\right) \tag{6.40}$$

Un état bosonique sur l'espace de Fock est un opérateur auto-adjoint, positif de trace 1 sur $\mathcal{F}_{\mathrm{sym}}$. Nous notons l'ensemble de tels états

$$\mathcal{S}\left(\mathcal{F}_{\mathrm{sym}}(\mathfrak{H})\right) = \left\{\Gamma \in \mathcal{S}^1\left(\mathcal{F}_{\mathrm{sym}}(\mathfrak{H})\right), \Gamma = \Gamma^*, \Gamma \geq 0, \mathrm{Tr}_{\mathcal{F}_{\mathrm{sym}}(\mathfrak{H})}[\Gamma] = 1\right\} \tag{6.41}$$

Un état est dit block-diagonal s'il peut être mis sous la forme

$$\Gamma = G_0 \oplus G_1 \oplus ... \oplus G_n \oplus ... \tag{6.42}$$

avec $G_n \in \mathcal{S}^1\left(\mathfrak{H}^n_{\mathrm{sym}}\right)$. Un état block-diagonal sur l'espace tronqué $\mathcal{F}^{\leq N}_{\mathrm{sym}}(\mathfrak{H})$ aura la forme

$$\Gamma = G_{0,N} \oplus G_{1,N} \oplus ... \oplus G_{N,N} \tag{6.43}$$

et sa k-ième matrice de densité réduite $\Gamma_N^{(k)}$ sera un opérateur sur $\mathfrak{H}^k_{\mathrm{sym}}$ défini par

$$\Gamma_N^{(k)} = \binom{N}{k}^{-1} \sum_{j=k}^{N} \binom{j}{k} \mathrm{Tr}_{k+1\to j}\left[G^P_{N,j}\right] \tag{6.44}$$

Pour un état diagonal, les matrices de densité réduites (6.44) caractérisent complètement l'état.

Lemme 6.2 (**Localisation d'un état à N corps**).
Soit $\Gamma_N \in \mathcal{S}\left(\mathfrak{H}^N_{\mathrm{sym}}\right)$ un état bosonique à N corps et P un opérateur auto-adjoint sur $\mathfrak{H}$ avec $0 \leq P^2 \leq 1$. Il existe un unique état diagonal $\Gamma_N^P \in \mathcal{S}\left(\mathcal{F}^{\leq N}_{\mathrm{sym}}(\mathfrak{H})\right)$ tel que

$$\left(\Gamma_N^P\right)^{(k)} = P^{\otimes k}\Gamma_N^{(k)}P^{\otimes k} \tag{6.45}$$

pour tout $0 \leq n \leq N$. De plus, en écrivant Γ_N^P sous la forme

$$\Gamma_N^P = G^P_{0,N} \oplus G^P_{1,N} \oplus ... \oplus G^P_{N,N} \tag{6.46}$$

nous avons la relation fondamentale

$$\mathrm{Tr}_{\mathfrak{H}_{\mathrm{sym}}^{k}}\left[G_{N,k}^{P}\right]=\mathrm{Tr}_{\mathfrak{H}_{\mathrm{sym}}^{N-k}}\left[G_{N,N-k}^{\sqrt{1-P^{2}}}\right]. \tag{6.47}$$

une preuve de ce qui précède se trouve dans [**136**][Lemma 5.2]. La relation (6.47) est centrale dans la méthode. Elle énonce qu'étant donné un état Γ_N, la probabilité d'avoir k particles P-localisées est égale à celle d'en avoir $N-k$ $\sqrt{1-P^2}$-localisées. Ce résultat vient de [**80**] mais d'autre versions existent [**3, 39, 65**]. Dans ce formalisme, un état (6.36) est donné par

$$P^{\otimes n}\gamma_{N}^{(n)}P^{\otimes n}=(G_{N}^{P})^{(n)}=\binom{N}{n}^{-1}\sum_{k=n}^{N}\binom{k}{n}\mathrm{Tr}\left[G_{N,k}^{P}\right] \tag{6.48}$$

pour tout $0\leq n\leq N$.

Part 2

Introduction and results

CHAPTER 7

Introduction

It is a law of nature that all the particles constitutive of our physical world are either bosons or fermions. The first kind of particles are forces carriers, as the photon describing the electromagnetic interactions between charged particles, or the gluon describing the strong interaction between quarks. The fermions constitute matter, as the electron, and the quarks forming nuclei. From a mathematical point of view the wave function describing such particles has to be symmetric for bosons and antisymmetric for fermions. This implies huge differences in the behavior of macroscopic systems. For instance, at very low temperature Bose-Einstein condensation [**45, 31**] can be observed for bosons. In this particular state of matter almost all particles populate the same quantum state. This bunching feature also allows to construct lasers made of coherent beams of photons. This could not happen with fermions. The latter are subjected to the Pauli principle, stating that two of them cannot occupy the same quantum state. This is the fondamental reason for the stability of usual matter [**42, 110, 43**]. This is also the cause of most of the chemical properties of atoms. The bosonic and fermionic behavior are also respectively called the Bose-Einstein statistics and the Fermi-Dirac statistics. In thermal equilibrium each statistics implies a particular distribution of particles among energy levels. More precisely, if we consider a system with energy levels $\{E_i\}_{i=1}^{\infty}$ of degeneracy g_i at temperature T and chemical potential μ then the number of particles of a bunch of bosons or fermions will expectively follow the N^{BE} and N^{FD} distributions

$$N_i^{\mathrm{BE}} = g_i \left(e^{\frac{E_i-\mu}{k_B T}} - 1 \right)^{-1} \qquad \text{for bosons} \tag{7.1}$$

and

$$N_i^{\mathrm{FD}} = g_i \left(e^{\frac{E_i-\mu}{k_B T}} + 1 \right)^{-1} \qquad \text{for fermions} \tag{7.2}$$

where k_B is the Boltzmann constant. When T is very low we observe that

$$N_0^{\mathrm{BE}} \to \infty \text{ whereas } N_i^{\mathrm{FD}} \to g_i \mathbb{1}_{E_i \leq \mu}$$

The first phenomenon is called condensation. The second leads to the Fermi sea, i.e the fermions pile up until they reach the energy μ.

The dichotomy presented above can lead to the question: Is there a possibility for fundamental particles being neither bosons nor fermions ? The experimental answer is clear, all fundamental particles which have been mesured until now were either bosons or fermions. The theoretical answer is also clear, no.

The breakthrough leading to the discovery of anyons comes when we try to understand the last result. Why ? and we also find an unambiguous response: because our physical world is at least three dimensional. Then dawns the next question: would it still be the case if we were living on a plane ? The answer is no. The spaces $\mathbb{R}^3$ and $\mathbb{R}^2$ present an

intrinsic topological difference. The punctured space $\mathbb{R}^3 \setminus \{0\}$ is simply connected whereas the punctured plane $\mathbb{R}^2 \setminus \{0\}$ is not. It implies that the fundamental groups of configuration spaces are different in two and three or more dimensions, and induces new possible symmetries for the wave function when quantum mechanics is stated in a geometrical way see for instance [**126**]. In two dimensions there is no inconsistency in demanding that the wave function to be something else than symmetric or antisymmetric. But our physical word is three dimensional and we could think that these particles will stay curious abstract objects forever. This is not the case. Indeed two-dimentional systems do not exist but almost two dimensional systems do. In these systems guenuine two dimentional quasi-particles can emerge. These need not a priori to be bosons or fermions.

Pioneers in this theoretical discovery were Leinas and Myrheim [**78, 28**] in 1977, Wilczek [**165, 163, 164**] and Goldin, Menikoff, Sharp [**62, 63**] in the 80's. The last decades have seen an upsurge of interest for anyons with the developement of cold atoms physics and quantum computation. Several two dimensional systems presenting quasiparticles looking like anyons have emerged. The fractional quantum Hall effect (FQHE) is maybe the most well-known and the most studied [**61, 67, 66, 77**]. Anyons have actually been mesured for the first time in 2020 in the FQHE regime [**12, 127**]. There are also several hopes to create anyons in cold quantum gases [**30, 121, 160, 167, 168, 169**].

CHAPTER 8

Many-body physics

1. Classical physics

In this thesis we will be interested in anyon gases, in a semi-classical approach and mean-field descriptions thereof. We recall for convenience the basics of many-body classical physics. The configuration space of a gas of N distinguishable particles is $\mathbb{R}^{dN}$ where d is the dimension of the physical space where particles move. The phase space is the position configuration space times the momemtum configuration space $\mathbb{R}^{dN} \times \mathbb{R}^{dN}$. A pure state is a configuration where all positions and velocities of each particle are specified. It corresponds to a unique point $Z_N = (X_N, P_N)$ in the phase space. The associated empirical measure is then

$$\mathrm{Emp}_{Z_N} = \frac{1}{N} \sum_{j=1}^{N} \delta_{\mathbf{z}_j} \tag{8.1}$$

where $\mathbf{z}_j = (\mathbf{x}_j, \mathbf{p}_j) \in \mathbb{R}^d \times \mathbb{R}^d$. Equilibrium states at zero temperature are such states but in other cases the thermal noise usually spreads the measure into a mixed state described by a probability measure $\mu_N \in \mathcal{P}\left(\mathbb{R}^{dN} \times \mathbb{R}^{dN}\right)$ such that $\mu_N \geq 0$ with

$$\int_{\mathbb{R}^{dN} \times \mathbb{R}^{dN}} \mathrm{d}\mu_N = 1. \tag{8.2}$$

When we know this measure, the expected values of physicals observables can be computed. For instance the energy

$$H_N\left(X; P\right) = \sum_{j=1}^{N} \frac{|\mathbf{p}_j|^2}{2} + \sum_{j=1}^{N} V\left(\mathbf{x}_j\right) \tag{8.3}$$

of N non-interacting particles in a potential V can be expressed by

$$\langle H_N\left(Z\right)\rangle = \int_{\mathbb{R}^{dN} \times \mathbb{R}^{dN}} H_N\left(X; P\right) \mathrm{d}\mu_N\left(Z\right). \tag{8.4}$$

To take into account the indistinguishability we need μ_N to satisfy

$$\mu_N\left(Z\right) = \mu_N\left(Z_\sigma\right) \tag{8.5}$$

for any permutation σ such that

$$Z_\sigma = \left(\mathbf{x}_{\sigma(1)}, \mathbf{p}_{\sigma(1)}, ..., \mathbf{x}_{\sigma(N)}, \mathbf{p}_{\sigma(N)}\right) \tag{8.6}$$

Using this symmetry property many expected values can be expressed using only the first marginal of the state

$$\mu^{(k)}\left(\mathbf{x}_1,\mathbf{p}_1,...,\mathbf{x}_k,\mathbf{p}_k\right)=\int_{\mathbb{R}^{d(N-k)}\times\mathbb{R}^{d(N-k)}}\mu_N\left(\mathbf{x}_1,\mathbf{p}_1,...,\mathbf{x}_N,\mathbf{p}_N\right)\mathrm{d}\mathbf{x}_{k+1}\mathrm{d}\mathbf{p}_{k+1}...\mathrm{d}\mathbf{x}_N\mathrm{d}\mathbf{p}_N \tag{8.7}$$

2. Quantum physics

In many-body quantum mechanics pure states are complex-valued wave-functions over the configuration space $\mathbb{R}^{dN}$. The evolution of a given state $\Psi_N\in L^2\left(\mathbb{R}^{dN}\right)$ is governed by the Schrödinger equation

$$\mathrm{i}\hbar\partial_t\Psi_N\left(\mathbf{x},t\right)=H_N\Psi_N(\mathbf{x},t) \tag{8.8}$$

where $\hbar=h/2\pi$ is the reduced Planck constant with $h=6,62\times10^{-34}m^2kg/s$ and H_N is the Hamiltonian of the system. The bottom of its spectrum (when it exists) is called the ground state energy. The square $|\Psi_N|^2$ is interpreted as a probability density for finding some particles in a certain volume of the configuration space. We also impose that the square of the Fourier tranform of Ψ_N (defined as an isometry of L^2), $|\hat{\Psi}_N|^2$ is the probability of finding the particles to have certain velocities. This probability density interpretation enforces that

$$\int_{\mathbb{R}^{dN}}\left|\Psi_N\right|^2\mathrm{d}x=1 \tag{8.9}$$

Particles also need to be considered as indistinguishable. This implies that the labeling $X_N=(\mathbf{x}_1,...,\mathbf{x}_N)$ is arbitrary and thus that

$$\left|U_\sigma\Psi_N\right|^2=\left|\Psi_N\right|^2 \tag{8.10}$$

for any permutation $\sigma\in S_N$ the permutation group of N elements with U_σ the unitary operator which permutes labels

$$U_\sigma\Psi_N(\mathbf{x}_1,...,\mathbf{x}_N)=\Psi_N(\mathbf{x}_{\sigma(1)},...,\mathbf{x}_{\sigma(N)}) \tag{8.11}$$

The indistinguishability has to be implemented for all observables. The expectation value of an observable A being given by

$$\langle\Psi_N,A\Psi_N\rangle \tag{8.12}$$

should not depend on the labeling. With this additional constraint the boson/fermion dichotomy actually follows from [**132**, Lemma 7.1.1]

Lemma 8.1 (**Bosons and fermions**).
Consider $\Psi_N\in L^2\left(\mathbb{R}^{dN}\right)$ with

$$\int_{\mathbb{R}^{dN}}\left|\Psi_N\right|^2\mathrm{d}\mathbf{x}=1 \tag{8.13}$$

satisfying, for any bounded self-adjoint operator A on $L^2\left(\mathbb{R}^{dN}\right)$ and any permutation $\sigma\in S_N$

$$\langle\Psi_N,A\Psi_N\rangle=\langle U_\sigma\Psi_N,AU_\sigma\Psi_N\rangle \tag{8.14}$$

then either

$$U_\sigma\Psi_N=\Psi_N\ \ \forall\sigma\in S_N\quad\textbf{\textit{Bosons}} \tag{8.15}$$

or

$$U_\sigma\Psi_N=(-1)^{\mathrm{sign}(\sigma)}\Psi_N\ \ \forall\sigma\in S_N\quad\textbf{\textit{Fermions}} \tag{8.16}$$

PROOF. Take $A = |\Psi_N\rangle\langle\Psi_N|$. According to (8.14) we must have

$$|\langle\Psi_N, U_\sigma\Psi_N\rangle|^2 = 1 \tag{8.17}$$

for all $\sigma \in S_N$ and hence

$$U_\sigma\Psi_N = c_\sigma\Psi_N \tag{8.18}$$

for some complex number c_σ of modulus one. For consistency we also have that

$$c_{\sigma\circ\sigma'} = c_\sigma c_{\sigma'} \tag{8.19}$$

must hold for all σ and $\sigma' \in S_N$. The map $\sigma \mapsto c_\sigma$ is a thus one-dimensional representation of the permutation group and it follows that either $c_\sigma = 1$ or $c_\sigma = (-1)^{\mathrm{sign}(\sigma)}$. □

We have consequently to restrict the space $L^2\left(\mathbb{R}^{dN}\right)$ to

$$L^2_{\mathrm{sym}}\left(\mathbb{R}^{dN}\right) \simeq \bigotimes_{\mathrm{sym}}^{N} L^2\left(\mathbb{R}^N\right) \quad \text{for bosons}$$

and to

$$L^2_{\mathrm{asym}}\left(\mathbb{R}^{dN}\right) \simeq \bigotimes_{\mathrm{asym}}^{N} L^2\left(\mathbb{R}^N\right) \quad \text{for fermions.}$$

In many-body quantum physics a pure state Ψ_N is then identified with the density matrix $|\Psi_N\rangle\langle\Psi_N|$. We define quantum mixed states by the statistical superpositions of pure states, that is, the convex combinations of orthogonal projectors. Using the spectral theorem, it is clear that the set of mixed states coincides with that of positive self-adjoint operators having trace one

$$\mathcal{S}^1\left(L^2\left(\mathbb{R}^{dN}\right)\right) = \left\{\Gamma \in \mathfrak{S}^1\left(L^2\left(\mathbb{R}^{dN}\right)\right), \Gamma = \Gamma^*, \Gamma \geq 0, \mathrm{Tr}\,\Gamma = 1\right\} \tag{8.20}$$

where $\mathfrak{S}^1(\mathfrak{H})$ is the Schatten-von Neumann class of trace-class operators on a Hilbert space $\mathfrak{H}$. A mixed state can also be identified by its kernel $\Gamma(X;Y)$ such that

$$\Gamma\Psi = \int_{\mathbb{R}^{dN}} \Gamma(X;Y)\,\Psi(Y)\,\mathrm{d}Y \tag{8.21}$$

with

$$\Gamma(X;Y) = \overline{\Psi_N(Y)}\Psi_N(X) \tag{8.22}$$

for a pure state identified with Ψ_N. We also define the k-th reduced density matrix by taking a partial trace over the last $N-k$ particles

$$\Gamma^{(k)} = \mathrm{Tr}_{k+1\to N}\,\Gamma \tag{8.23}$$

which means that for any bounded operator A_k acting on $L^2\left(\mathbb{R}^{dk}\right)$

$$\mathrm{Tr}\left[A_k\Gamma^{(k)}\right] := \mathrm{Tr}\left[A_k \otimes \mathbb{1}^{\otimes N-k}\Gamma\right] \tag{8.24}$$

with $\mathbb{1}$ is the identity on $L^2\left(\mathbb{R}^d\right)$. Usual quantum mechanics can now be formulated in terms of density matrices. The mean value of an observable A in the state Γ being given by

$$\langle A\rangle_\Gamma = \mathrm{Tr}\left[A\Gamma\right] \tag{8.25}$$

which, in the case of a pure state $\Gamma = |\Psi_N\rangle\langle\Psi_N|$, is nothing else than

$$\mathrm{Tr}\left[A\Gamma\right] = \langle\Psi_N, A\Psi_N\rangle \tag{8.26}$$

In the language of density matrices the bosonic and fermionic symmetries are enforced by imposing

$$U_\sigma \Gamma = \Gamma U_\sigma = \Gamma \text{ for bosons} \tag{8.27}$$

$$U_\sigma \Gamma = \Gamma U_\sigma = (-1)^{\mathrm{sign}(\sigma)} \Gamma \text{ for fermions.} \tag{8.28}$$

3. Mean field approximation

A usual Hamiltonian describing N particles interacting via a potential w_L and trapped in a potential V is of the form

$$H_{N,L} = \sum_{j=1}^{N} \left(-\hbar^2 \Delta_j + V(\mathbf{x}_j)\right) + \lambda \sum_{1 \le k < j \le N} w_L(\mathbf{x}_j - \mathbf{x}_k) \tag{8.29}$$

where λ describes the strength of the interactions. This operator acts on $L^2\left(\mathbb{R}^{dN}\right)$ as an unbounded operator. Finding the ground state energy

$$e_N = \inf_{\Psi_N \in L^2\left(\mathbb{R}^{dN}\right)} \langle \Psi_N, H_N \Psi_N \rangle \tag{8.30}$$

for a N-body Hamiltonian like (8.29) is a very difficult task when the particle number N gets large. In classical statistical mechanics we usually introduce macroscopic emergent quantities such as the temperature, or the pressure. Such quantities are well defined in average and can be studied via effective equations as for instance Navier-Stokes' or Euler's. In quantum mechanics we also follow that route by finding effective models describing the whole system in order to obtain more tractable theories from which useful information can be extracted. The simplest and most well-known of these are obtained through mean-field approximation. The goal is to obtain a self-consistent one-body problem starting from the N-body problem.

3.1. Hartree approximation for bosons. In the case of (8.29) acting on bosons we will illustrate three regimes leading to three different effective models. The limit model is directly linked to the competition between the strength of the interaction λ and its range L. The typical number of particles interacting with a single particle is ρL^d with ρ the density of particles. In a space of volume one (the size of the system) we have $\rho = N/1$. The expected interaction energy per particle is of order $N\lambda L^d$. This energy has to be of order one to make the terms of (8.29) of the same order when $N \to \infty$. We introduce a scaling parameter $\beta \ge 0$ to describe the competition between L and λ

$$L = N^{-\beta}, \qquad \lambda = \frac{N^{d\beta}}{N-1}. \tag{8.31}$$

The Hamiltonian (8.29) becomes

$$H_N = \sum_{j=1}^{N} \left(-\Delta_j + V(\mathbf{x}_j)\right) + \frac{1}{N-1} \sum_{1 \le k < j \le N} N^{d\beta} w\left(N^\beta \mathbf{x}_j - \mathbf{x}_k\right) \tag{8.32}$$

where we took $\hbar = 1$. The ground state energy for a gas of bosons is

$$e_N^{\mathrm{sym}} := \inf_{\Psi_N \in L^2_{\mathrm{sym}}\left(\mathbb{R}^{dN}\right)} \langle \Psi_N, H_N \Psi_N \rangle \tag{8.33}$$

The mean-field approximation is based on the intuition that when the number of particle is very large all particles behave as if independent and identically distributed. The wave function is then expected to be of the form

$$\Psi_N^{\mathrm{H}}(\mathbf{x}_1, ..., \mathbf{x}_N) = u(\mathbf{x}_1)u(\mathbf{x}_2)...u(\mathbf{x}_N) \tag{8.34}$$

with u a single particle density belonging to $L^2\left(\mathbb{R}^d\right)$. This ansatz is coupled with the convergence

$$w_L(\mathbf{x}) = L^{-d} w\left(\frac{\mathbf{x}}{L}\right) \to \left(\int_{\mathbb{R}^d} w\right) \delta_0 \tag{8.35}$$

in the sense of distribution when $L \to 0$ and gives rise to three distinct regimes.

Mean-field regime (MP). When $\beta = 0$ computing the energy of the system with such a wave function (8.34) gives

$$\mathcal{E}_{\mathrm{H}}[u] := \frac{\left\langle \Psi_N^{\mathrm{H}}, H_N \Psi_N^{\mathrm{H}} \right\rangle}{N} = \int_{\mathbb{R}^d} \left(|\nabla u(\mathbf{x})|^2 + V(\mathbf{x}) |u(\mathbf{x})|^2 \right) \mathrm{d}\mathbf{x} \tag{8.36}$$

$$+\lambda \frac{(N-1)}{2} \int_{\mathbb{R}^d} |u(\mathbf{x})|^2 \left(w * |u|^2\right)(\mathbf{x}) \mathrm{d}\mathbf{x} \tag{8.37}$$

The energy $\mathcal{E}_{\mathrm{H}}$ is called the Hartree energy associated with the minimization problem

$$e_{\mathrm{H}} = \inf_{\substack{u \in L^2(\mathbb{R}^d) \\ \|u\|_{L^2}=1}} \mathcal{E}_{\mathrm{H}}[u]. \tag{8.38}$$

and we can state

$$\frac{e_N^{\mathrm{sym}}}{N} \to e_{\mathrm{H}} \quad \text{when} \quad N \to \infty \tag{8.39}$$

see [**136**, Section 6] and references therein [**52, 87, 40, 47, 54, 55, 81, 83, 85**].

Non-linear-Schrödinger regime (NLS). When $0 < \beta < 1$ the ansatz (8.34) is still relevant. The potential w_L tends to a Dirac mass. If we simply replace it in (8.37) we obtain the NLS functional

$$\mathcal{E}_{\mathrm{NLS}}[u] := \int_{\mathbb{R}^d} \left(|\nabla u(\mathbf{x})|^2 + V(\mathbf{x}) |u(\mathbf{x})|^2 + \frac{a}{2} |u(\mathbf{x})|^4 \right) \mathrm{d}\mathbf{x} \tag{8.40}$$

Here

$$a = \int_{\mathbb{R}^d} w. \tag{8.41}$$

The associated ground state

$$e_{\mathrm{NLS}} = \inf_{\substack{u \in L^2(\mathbb{R}^d) \\ \|u\|_{L^2}=1}} \mathcal{E}_{\mathrm{NLS}}[u]. \tag{8.42}$$

We have transformed a linear problem for the N-body wave-function into a cubic problem for the one-body wave-function. We also obtain the convergence

$$\frac{e_N^{\mathrm{sym}}}{N} \to \quad e_{\mathrm{NLS}} \quad \text{when} \quad N \to \infty \tag{8.43}$$

See for details [**136**, Section 7]. This regime presents two interesting sub-regimes.

- ($\beta < 1/d$) corresponds to case when particles interact often (L is large) with a weak strength $\lambda \leq 1$. We are still in a mean-field regime.

- $(\beta > 1/d)$ makes particles interact strongly but rarely. It is a dilute regime.

The transition happens at $\beta = 1/d$.

Gross-Pitaevskii regime (GP). When $\beta = 1$ the limit functional is still (8.40) but

$$a = 4\pi \times \ \text{(scattering length of } w\text{)}\,. \tag{8.44}$$

The scattering length is defined in [**95**, Appendix C]. The ansatz (8.34) has to be corrected to include short-range correlations. A candidate would be

$$\Psi_N\left(\mathbf{x}_1, ..., \mathbf{x}_N\right) = \prod_{j=1}^{N} u\left(\mathbf{x}_j\right) \prod_{1 \le i < j \le N} f\left(N^{\beta}\left(\mathbf{x}_i - \mathbf{x}_j\right)\right) \tag{8.45}$$

where f comes from the two-body problem defined by w. This case is called the Gross-Pitaevskii limit (GP) studied in [**107, 106, 145, 97, 96, 98**]. For more details about Bose-Einstein condensation and mean field theories see also [**137, 87, 136, 81, 95**].

3.2. Hartree-Fock approximation for fermions. In the case of fermions, due to the Pauli principle, the kinetic energy increases faster than other terms of the hamiltonian (8.29) when $N \to \infty$. This issue can be circumvented by taking $\hbar = N^{-1/d}$, leading to a semi-classical regime. In the following we only investigate the mean-field regime: $\lambda = 1/N$ and $L = 1$. The Hamiltonian (8.29) becomes

$$H_N = \sum_{j=1}^{N} \left(\frac{-\Delta_j}{N^{1/d}} + V\left(\mathbf{x}_j\right) \right) + \frac{1}{N} \sum_{1 \le k < j \le N} w\left(\mathbf{x}_j - \mathbf{x}_k\right) \tag{8.46}$$

with a fixed interaction potential w. The ground state of (8.29) is

$$e_N^{\text{asym}} := \inf_{\Psi_N \in L^2_{\text{asym}}\left(\mathbb{R}^{dN}\right)} \langle \Psi_N, H_N \Psi_N \rangle \tag{8.47}$$

Because of Pauli's principle fermions can never completely be uncorrelated in the sense of (8.34). The mean-field ansatz for fermions rather consists in taking the Slater determinant

$$\Psi_N^{\text{SL}}(\mathbf{x}_1, ..., \mathbf{x}_N) = \det\left(u_j\left(\mathbf{x}_k\right)\right)_{1 \le j,k \le N} \tag{8.48}$$

where the functions $\{u_j\}_{j=1}^N$ are orthogonal one-body functions of $L^2\left(\mathbb{R}^d\right)$.

$$H_1 = -\Delta_{x_1} + V(\mathbf{x}_1) \tag{8.49}$$

We define the first density matrix associated with the Slater determinant

$$\gamma = \sum_{j=1}^{N} |u_j\rangle \langle u_j| \quad \text{with} \quad \rho_\gamma = \sum_{j=1}^{N} |u_j|^2 \tag{8.50}$$

and compute the Hartree-Fock energy

$$\mathcal{E}_{\text{HF}}\left[\gamma\right] = \text{Tr}\left[-\Delta\gamma\right] + \int_{\mathbb{R}^d} V(\mathbf{x}) \rho_\gamma(\mathbf{x}) \mathrm{d}\mathbf{x} \tag{8.51}$$

$$+ \frac{1}{2N} \int_{\mathbb{R}^d \times \mathbb{R}^d} w(\mathbf{x} - \mathbf{y}) \left(\rho_\gamma(\mathbf{x}) \rho_\gamma(\mathbf{y}) - |\gamma(\mathbf{x}, \mathbf{y})|^2 \right) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \tag{8.52}$$

where we used the kernel of γ

$$\gamma(x,y) = \sum_{j=1}^{N} u_j(\mathbf{x})\overline{u_j(\mathbf{y})} \tag{8.53}$$

Calling

$$e_{\mathrm{HF}} := \inf_{\substack{\gamma^2=\gamma=\gamma^* \\ \mathrm{Tr}\,\gamma=N}} \mathcal{E}_{\mathrm{HF}}[\gamma] \tag{8.54}$$

we can, assuming that V is confining

$$\lim_{|\mathbf{x}|\to\infty} V(\mathbf{x}) = +\infty \tag{8.55}$$

and that $w \in L^{1+d/2}\left(\mathbb{R}^d\right) + L^{\infty}_{\varepsilon}\left(\mathbb{R}^d\right)$ to get the convergence

$$\left|\frac{e_N^{\mathrm{asym}}}{N} - e_{\mathrm{HF}}\right| \to 0 \ \text{ when } \ N \to \infty. \tag{8.56}$$

In most of the cases the above convergence only holds for the first order, a Thomas-Fermi energy, minimum of the functional

$$2\pi \int_{\mathbb{R}^2} \rho^2(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2} V(\mathbf{x})\,\rho(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{8.57}$$

Examples of such results can be found in [**147, 99, 103, 50, 91, 90**].

CHAPTER 9

Models of anyonic statistics

1. The space of configurations

We shall first review the usual theoretical argument supporting the bosons/fermions dichotomy. The classical configuration space of N particles in d dimensions is $\mathbb{R}^{dN}$. A vector of this space will be denoted $X_N = (\mathbf{x}_1, ..., \mathbf{x}_N)$ with each $\mathbf{x}_j \in \mathbb{R}^d$. To build a quantum theory on this space we introduce a wave function $\Psi_N \in L^2\left(\mathbb{R}^{dN}\right)$ and symmetrize it as in (8.10). This way Lemma 8.1 seems to provide a proof for the boson fermion dichotomy but there was an implicit choice made in this argument [**58, 62, 148, 149**]. We have indeed

(1) quantized i.e introduced a wave function on the space of configurations
(2) symmetrized i.e imposed the symmetry condition on the wave function

There is no logical or physical necessity to implement (1) before (2). We shall discuss what happens the other way, when we perform (2) before (1).

1.1. Symmetrization of a classical system. Consider N hard-core particles living in $\mathbb{R}^{dN}$. A vector of this space is

$$X_N := (\mathbf{x}_1, ..., \mathbf{x}_N) \in \mathbb{R}^{dN} \tag{9.1}$$

Two particles cannot be located at the same point, so we need to remove the diagonals of $\mathbb{R}^{dN}$

$$\mathbb{D} := \left\{ X_N \in \mathbb{R}^{dN}, \ \ \mathbf{x}_j = \mathbf{x}_k \ \text{ for some } \ j \neq k \right\}. \tag{9.2}$$

The classical configuration space of N identical particles living in $\mathbb{R}^{dN}$ is then formally given by

$$C_N^d = \frac{\mathbb{R}^{dN} \backslash \mathbb{D}}{S_N} \tag{9.3}$$

where S_N is the permutation group of N elements acting on $\mathbb{R}^{dN}$ the following way

$$\sigma\left(X_N\right) = \left(\mathbf{x}_{\sigma(1)}, ..., \mathbf{x}_{\sigma(N)}\right). \tag{9.4}$$

In C_N^d any point X_N is identified with $\sigma\left(X_N\right)$ for all $\sigma \in S_N$. An exchange of particles must now be seen as a physical process. In other words, to exchange two particles the system has to follow a path in C_N^d which is actually a loop since the starting point is identified with the arrival.

To better understand what is C_N^d we introduce the center of mass coordinate

$$\mathbf{X}\left(X_N\right) = \frac{1}{N} \sum_{j=1}^{N} \mathbf{x}_j \in \mathbb{R}^d \tag{9.5}$$

which satisfies

$$\mathbf{X}\left(\sigma\left(X_N\right)\right) = \mathbf{X}\left(X_N\right)$$

and can be factored out to write

$$C_N^d \simeq \mathbb{R}^d \times r^d \tag{9.6}$$

where

$$r^d = \frac{\left\{X_N \in \mathbb{R}^{dN} \backslash \mathbb{D},\ \ \mathbf{X}\left(X_N\right) = 0\right\}}{S_N} \tag{9.7}$$

is the relative coordinate space.

1.2. Quantization of a classical system. Wave functions of N identical particles are defined as square-integrable complex-valued functions on C_N^d with appropriate gluing conditions

$$\Psi\left(\sigma\left(X_N\right)\right) = e^{\mathrm{i}\pi\alpha}\Psi\left(X_N\right). \tag{9.8}$$

The amplitude has to be the same but the phase may differ under exchange of particles. We indroduce the first homotopy group of C_N^d

$$\pi_X^1\left(C_N^d\right) = \Gamma_X / \sim \tag{9.9}$$

where Γ_X is the set of all loops in C_N^d starting at X and $\sim$ the equivalence relation between loops

$$\gamma_1 \sim \gamma_2 \ \ \Leftrightarrow \ \ \gamma_1 \ \text{ can be continuously deformed into } \ \gamma_2 \tag{9.10}$$

The first homotopy group is endowed with the concatenation law. The space C_N^d being connected the starting point does not affect the homotopy group, we then drop it from the notation. $\Gamma_X = \Gamma$ contains all the different loops the system can follow when it performs an exchange. As claimed in [**28, 78**] the exchange phase $e^{\mathrm{i}\pi\alpha}$ has to be a scalar representation of the group $\pi^1\left(C_N^d\right)$. It is a well known topological fact [**7**] that

$$\pi^1\left(C_N^d\right) \simeq S_N \ \text{ if } \ d \geq 3. \tag{9.11}$$

But in the case $d = 2$ the punctured space $\mathbb{R}^d \backslash \{0\}$ is not simply connected. This affects drastically the first homotopy group which becomes the braid group

$$\pi^1\left(C_N^2\right) \simeq B_N \tag{9.12}$$

see [**9**]. In three or higher dimension there are only two unitary scalar representations of the fundamental group, corresponding to the usual boson/fermion dichotomy. In two dimensional systems there exists a continuum of scalar representations of B_N, indexed by an integer $\alpha \in [0, 2[$.

To illustrate the topological difference between the configuration spaces in three an two dimensions we will consider only two partices and describe C_2^2 and C_2^3 in more detail. According to (9.6) $C_2^3 \simeq \mathbb{R}^3 \times r^3$ with the relative coordinate

$$\mathbf{r} = \frac{\mathbf{x}_1 - \mathbf{x}_2}{2} \in r^3$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ can take any value in $\mathbb{R}^2$ satisfying $\mathbf{x}_1 \neq \mathbf{x}_2$ and the points $\mathbf{r}$ and $-\mathbf{r}$ being identified. Notice that

$$|\mathbf{r}| = \sqrt{|\mathbf{x}_1|^2 + |\mathbf{x}_2|^2} \tag{9.13}$$

we then use spherical coordinates to write $\mathbf{r}$ under the form $(|\mathbf{r}|, \theta, \phi)$ and factorize

$$r^3 \simeq \mathbb{R}_*^+ \times S_{\theta,\phi} \tag{9.14}$$

with $S_{\theta,\phi}$ a sphere of radius one where two points are identified by the central symmetry. Figure 9.1 depicts the space $S_{\theta,\phi}$, a northern hemisphere with opposite points on the equator

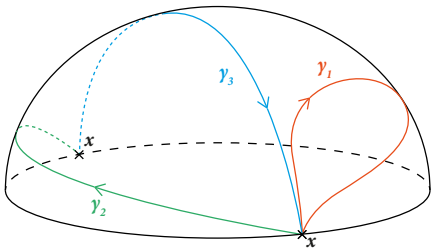


FIGURE 9.1. Projective space $S_{\theta,\phi}$

identified. There are two kinds of non homotopicaly equivalent loops starting at $\mathbf{x}$, the red one, equivalent to the empty loop, and the green one. This picture makes clear that γ_1 cannot be continuously deformed into γ_2. Any other loop is then either equivalent to γ_2, this is case of γ_3, or equivalent to γ_1, this the case of $\gamma_2 \circ \gamma_3$. We notice that a double exchange is equivalent to the identity. Indeed, $\gamma_1 \circ \gamma_1 \simeq \gamma_1$ and $\gamma_2 \circ \gamma_3 \simeq \gamma_1$. The first homotopy group is then

$$\pi^1\left(C_2^3\right) \simeq S_2 = \{\mathbb{1}, \sigma_{12}\}\,.$$

On the other hand we have $C_2^2 = \mathbb{R}^2 \times r^2$ with $\mathbf{r} \in r^2$ being a point in $\mathbb{R}^2_*$ identified with $-\mathbf{r}$. Then r^2 is equivalent to the half-plane without zero, with the half line $\mathbb{R}^+_*$ identified with $\mathbb{R}^-_*$ and can be seen as a cone without tip as represented in Figure 9.2. Here, γ_1 can be also shrunk to a point whereas γ_2 cannot. The new topological aspect of the cone is the fact that, without the tip, the blue loop cannot be deformed into γ_1 or γ_2. A loop can turn around the cone any times, which defines each time a new non equivalent loop.

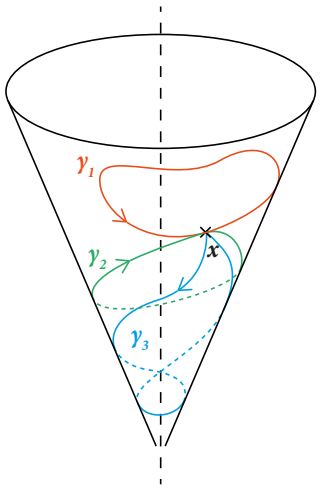


FIGURE 9.2. Cone without tip r^2

We then obtain an infinite set of different loops up to homotopy. Each of them is identified by counting the number of time a loop encircle the cone. We then have that

$$\pi^1\left(C_2^2\right) \simeq B_2 \simeq \mathbb{Z}.$$

The exchange of particles (9.8) is here described by the unitary scalar representation of B_2. The main diffence of interest between $S_2 \simeq (\{0,1\},+)$ and $B_2 \simeq (\mathbb{Z},+)$ is that any generator s_i of S_2, and S_N in general, has the property that $s_i^2 = \mathbb{1}$ which is not the case for the generators of B_N. In B_N a double exchange of two particles is not equivalent to the identity. For instance, $\gamma_2 \circ \gamma_2 \simeq \gamma_3 \neq \mathbb{1}$. Its representation has consequently not to be the scalar e^0. The exchange of two neighbouring (not encircling other particles) strands is represented by $e^{\mathrm{i}\pi\alpha}$. We illustrated in Figure 9.3 the exchange of particles 1 and 3 encircling particle 2.

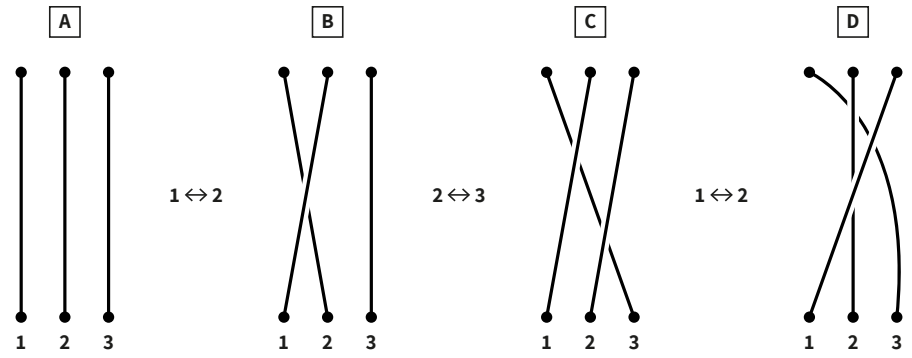


FIGURE 9.3. Braid diagrams corresponding to the exchange $1-3$ encircling particle 2.

On these braid diagrams the evolution in time is downwards. To exchange braids 1 and 3 three neighbouring exchanges are needed. The first generator b_{12} leads from A to B by exchanging braids 1 and 2. Notice that the braid 1 goes underneath. Consequently the repetition of exchange $1-2$ at B does not brink back the diagram A. Strands 1 and 2 have been knotted, $b_{12}^2 \neq \mathbb{1}$. In D, three neighbouring exchanges have been achieved. The corresponding phase is $e^{3\mathrm{i}\pi\alpha}$. As a generalization of the above process, when p particles are encircled by the trajectory of the two exchanged particles the phase is $e^{\mathrm{i}(2p+1)\pi\alpha}$. When a single particle makes a loop around p others the phase is $e^{\mathrm{i}(2p)\pi\alpha}$.

2. Magnetic gauge picture

2.1. Ideal anyons. For a convenient and rigorous treatment of anyons, one can model them as identical bosons or fermions in the plane, but with a magnetic interaction of Aharonov-Bohm type between each pair of particles, giving rise to the correct statistics phase as the particles encircle each other. In other words we want the wave function to formally behave as

$$\tilde{\Psi}(\mathbf{x}_1,...,\mathbf{x}_j,...,\mathbf{x}_k,...,\mathbf{x}_N) = e^{\mathrm{i}\alpha\pi}\tilde{\Psi}(\mathbf{x}_1,...,\mathbf{x}_k,...,\mathbf{x}_j,...,\mathbf{x}_N)$$

and we can obtain this by writting

$$\tilde{\Psi}(\mathbf{x}_1,...,\mathbf{x}_N)=\prod_{j<k}e^{\mathrm{i}\alpha\phi_{jk}}\Psi(\mathbf{x}_1,...,\mathbf{x}_N)\ \text{ where }\ \phi_{jk}=\mathrm{angle}\frac{\mathbf{x}_j-\mathbf{x}_k}{|\mathbf{x}_j-\mathbf{x}_k|}$$

with Ψ a bosonic wave function, symmetric under particle exchange. Note that cases $\alpha=1$ and $\alpha=0$ are respectively the fermionic and the bosonic one. We define the singular gauge transformation U_α

$$\left(U_\alpha\Psi\right)(\mathbf{x})=\prod_{j<k}e^{\mathrm{i}\alpha\phi_{jk}}\Psi(\mathbf{x}) \tag{9.15}$$

Note that

$$U_1:L^2_{\mathrm{asym/sym}}\left(\mathbb{R}^{2N}\right)\to L^2_{\mathrm{sym/asym}}\left(\mathbb{R}^{2N}\right) \tag{9.16}$$

is an isomorphism. Furthermore U_1 is smooth on $\mathbb{R}^{2N}\backslash\mathbb{D}$. Define the operator $p_j=-\mathrm{i}\hbar\nabla_j$ we observe that, formally

$$U_\alpha^{-1}\mathbf{p}_jU_\alpha=\mathbf{p}_j+\alpha\sum_{k\neq j}\frac{(\mathbf{x}_j-\mathbf{x}_k)^\perp}{|\mathbf{x}_j-\mathbf{x}_k|^2}=:\mathbf{p}_j+\alpha\mathbf{A}(\mathbf{x}_j):=\mathbf{p}_j+\alpha\mathbf{A}_j. \tag{9.17}$$

We have added a new vector potential

$$\mathbf{A}(\mathbf{x}_j):=\sum_{k\neq j}\frac{(\mathbf{x}_j-\mathbf{x}_k)^\perp}{|\mathbf{x}_j-\mathbf{x}_k|^2} \tag{9.18}$$

We then get the new magnetic operator

$$D_j=\mathbf{p}_j+\alpha\mathbf{A}_j \tag{9.19}$$

and the associated kinetic energy of N anyons

$$T_\alpha=\sum_{j=1}^N\left(\mathbf{p}_j+\alpha\mathbf{A}_j\right)^2=\sum_{j=1}^N D_j^2 \tag{9.20}$$

acting on bosonic wave functions. The above describes anyons via the **magnetic gauge picture**. Anyons are seen as usual bosons or fermions with a vector potential corresponding to the magnetic field

$$\mathrm{curl}\mathbf{A}(\mathbf{x}_j)=2\pi\sum_{k\neq j}\delta(\mathbf{x}_j-\mathbf{x}_k) \tag{9.21}$$

Each boson or fermion carries an infinitely thin solenoid perpendicular to the plane. In other words we have added fictitious Aharonov-Bohm-like flux tubes whose vector potential influences the other particles. The strengh of the magnetic flux is $2\pi\alpha$. The magnetic gauge picture also allows to work with fermionic wave functions. Defining $\alpha=1+\widetilde{\alpha}$ and using the change of gauge U_1 (9.15) we transform T_α acting on L^2_{sym} into $T_{\widetilde{\alpha}}$ acting on L^2_{asym}. We observe that when $\alpha\to0$ we recover the kinetic energy for free bosons and when $\alpha\to1$ the kinetic energy of free fermions. These limits will be respectively called in the following:

- The almost bosonic limit : $\alpha\to0$ and T_α acts on $L^2_{\mathrm{sym}}\left(\mathbb{R}^{2N}\right)$.
- The almost fermionic limit : $\widetilde{\alpha}\to0$ and $T_{\widetilde{\alpha}}$ acts on $L^2_{\mathrm{asym}}\left(\mathbb{R}^{2N}\right)$. Formally, $\alpha\to1$ and T_α acts on $L^2_{\mathrm{sym}}\left(\mathbb{R}^{2N}\right)$.

For $\alpha \in [0,2[$ the anyionic kinetic energy acts on the domain $\mathcal{D}_\alpha^N$ as a quadratic form

$$q_\alpha : L^2\left(\mathbb{R}^{2N}\right) \to \mathbb{R} \tag{9.22}$$

$$u \mapsto q_\alpha(u) = \sum_{j=1}^{N} \int_{\mathbb{R}^{2N}} \bar{u} D_j^2 u \mathrm{d}\mathbf{x} \tag{9.23}$$

defined on

$$C_0^\infty\left(\mathbb{R}^{2N} \backslash \mathbb{D}\right) \cap L^2_{\mathrm{sym}}\left(\mathbb{R}^{2N}\right)$$

the smooth symmetric or antisymmetric square-integrable functions on $\mathbb{R}^{2N}$ with support away from diagonals. The space of finite kinetic energy N-anyonic wave functions $\mathcal{D}_\alpha^N$ is the domain of the closure of this quadratic form. Another space of interest is $\tilde{\mathcal{D}}_\alpha^N$ defined as the domain of the Friedrichs extension of the self-adjoint operator (9.20) associated to the closure of the quadratic form (9.23). It was shown in [**118**] that it corresponds to the maximal domain of the quadratic form q_α (9.23). We refer to [**118**] for more detail. We can notice that in the cases $\alpha \in 2\mathbb{Z}$ and respectively $\alpha \in 2\mathbb{Z}+1$, $\mathcal{D}_\alpha^N$ is gauge-equivalent to the Sobolev space $\mathcal{D}_0^N = H^1_{\mathrm{sym}}$ and respectively to $D_1^N = H^1_{\mathrm{asym}}$.

3. Lieb-Thirring inequality

We know that anyons are closer to fermions that bosons in the sens that they obey a Lieb-Thirring type inequality. A consequence of the Pauli principle for fermions is the Lieb-Thirring inequality which, given a potential V on $\mathbb{R}^d$, states that

$$\sum_{j=1}^{N} \int_{\mathbb{R}^{dN}} \left(|\nabla_j u|^2 + V(\mathbf{x}_j)\,|u|^2 \right) \mathrm{d}\mathbf{x} \geq -\sum_{k=1}^{N-1} |\lambda_k(h)| \geq -C_d^{\mathrm{LT}} \int_{\mathbb{R}^d} |V_-(\mathbf{x})|^{1+\frac{d}{2}}\,\mathrm{d}\mathbf{x} \tag{9.24}$$

where $\lambda_k(h)$ are the negative eigenvalues of the one particle Schrödinger operator

$$h := -\Delta_{\mathbb{R}^d} + V \tag{9.25}$$

acting on $L^2\left(\mathbb{R}^d\right)$ and V_- the negative part of V, $V_- := \max(-V, 0)$. The first inequality of (9.24) shows that the lowest possible energy for N fermions is obtained by distributing them in the N first orbitals of the one body operator h. In other words

$$u = \bigotimes_{k=1 \text{ to } N-1}^{\text{asym}} u_k$$

is an antisymmetrized product of the one-particles eigenfunctons $u_k(h)$ (also called a Slater determinant). The second inequality concerns the trace over the negative spectrum of the one body operator h. Furthermore the inequality (9.24) is equivalent to the kinetic energy inequality

$$\sum_{j=1}^{N} \int_{\mathbb{R}^{dN}} |\nabla_j u|^2\,\mathrm{d}\mathbf{x} \geq C_d^{\mathrm{K}} \int_{\mathbb{R}^d} \rho(\mathbf{x})^{1+\frac{2}{d}}\,\mathrm{d}\mathbf{x} \tag{9.26}$$

$$\rho(\mathbf{x}) := \sum_{j=1}^{N} \int_{\mathbb{R}^{d(N-1)}} |u(\mathbf{x}_1, ..., \mathbf{x}_j = \mathbf{x}, ..., \mathbf{x}_N)|^2 \prod_{k \neq j} \mathrm{d}\mathbf{x}_k \tag{9.27}$$

which is often interpreted as a form of the Pauli principle for fermions. On the other hand, the best we can achieve with bosons is

$$\sum_{j=1}^{N}\int_{\mathbb{R}^{dN}}\left(|\nabla_j u|^2+V(\mathbf{x}_j)\,|u|^2\right)\mathrm{d}\mathbf{x}\geq -NC_d^{\mathrm{LT}}\int_{\mathbb{R}^d}|V_-(\mathbf{x})|^{1+\frac{d}{2}}\,\mathrm{d}\mathbf{x} \tag{9.28}$$

by putting all particles in the same state $u=u_0\otimes...\otimes u_0$ with u_0 satisfying (9.24) and then get a kinetic inequality

$$\sum_{j=1}^{N}\int_{\mathbb{R}^{dN}}|\nabla_j u|^2\,\mathrm{d}\mathbf{x}\geq\frac{C_d}{N^{2/d}}\int_{\mathbb{R}^d}\rho(\mathbf{x})^{1+\frac{2}{d}}\mathrm{d}\mathbf{x} \tag{9.29}$$

which becomes trivial as $N\to\infty$. The inequality (9.29) is actually the Sobolev inequality.

The anyonic kinetic energy (9.20) being a genuine many-body operator, a bound as (9.24) cannot be found because there are no one-body operator eigenvalues to sum. The usual approach for obtaining Lieb-Thirring inequalities (to use (9.24) to get (9.26)) fails in the case of anyons. No such results were available for anyons until the series of paper [**115, 118, 113**] which gave birth to several results concerning anyonic ground states. In particular the following Lieb-Thirring inequality for ideal anyons

THEOREM 9.1 (**Lieb-Thirring inequality for ideal anyons (Lundholm-Seiringer)**).

There exists a constant $C>0$ such that for any $0\leq\alpha\leq 1$, $N\geq 1$ and $\Psi_N\in\mathcal{D}_N^\alpha$

$$\sum_{j=1}^{N}\int_{\mathbb{R}^{dN}}|D_j\Psi_N|^2\,\mathrm{d}x\geq C\alpha\int_{\mathbb{R}^d}\rho_{\Psi_N}(\mathbf{x})^2\mathrm{d}\mathbf{x} \tag{9.30}$$

A simple consequence of the above Theorem concerns the ground-state energy of a trapped gas of anyons in a potential $V(\mathbf{x}_1,..,\mathbf{x}_N)=\sum_{j=1}^N V(\mathbf{x}_j)$ with V a single particle potential which is

$$e_\alpha^V:=\inf\mathrm{spec}\left(T_\alpha+\tilde{V}\right)\geq-\frac{1}{4C\alpha}\int_{\mathbb{R}^2}V_-(\mathbf{x})^2\mathrm{d}\mathbf{x} \tag{9.31}$$

independent of N. Applying this to an oscillator potential we get

$$e_\alpha^V\geq C\frac{4N^{3/2}}{3}\sqrt{\alpha} \tag{9.32}$$

Ideal anyons are thus submited to the Pauli principle. In this sense they are closer from fermions than bosons.

CHAPTER 10

Anyons arising as quasi-particles in the FQHE

It was shown by Laughlin [**77, 76**], Halperin [**67**] and Arovas- Schrieffer-Wilczek [**8**] that the FQHE may be described in terms of a particular highly correlated wave function, whose excitations are anyons. In 2020 two experiments [**127**] and [**12**] measured particles presenting an anyonic behavior. They both used a 2D electron gas device in the FQHE regime. In this section we will explain some features of the FQHE in order to understand how anyons can theoretically emerge. We will conclude by quickly describing the experiments [**127**] and [**12**].

1. Quantum Hall effect

The Hall effect was discovered by Edwin Hall in 1879. If an electric current I is imposed on a 2D sample in the presence of a perpendicular magnetic field, a voltage drop V_H perpendicular to both the current and the magnetic field develops, due to the Lorentz force acting on charge carriers. The discoveries of the integer and the fractional quantum Hall effect (IQHE) and (FQHE) are more recent. The IQHE was dicovered by von Klitzing in 1980. The Fractional quantum Hall effect [**60, 61, 74**] was discovered by Tsui, Störmer and Gossard [**159**] in 1982. They are named from the fact that the Hall conductivity takes quantized values

$$\sigma_H = \frac{V_H}{I} = \nu \frac{e^2}{h} \tag{10.1}$$

depending on ν which can take integer (IQHE) or fractional (FQHE) values. The precision of the quantization is impressively independent of the sample's details. It is nowadays used to set the resistivity standard. In order to understand the quantum description of the above phenomenon we will begin with the case of a planar free electron gas submitted to a constant perpendicular magnetic field.

2. Landau Levels

The Hamiltonian of N 2D free electrons of charge one and mass $m = 1/2$ interacting with a constant magnetic field $\mathbf{B}$ perpendicular to the plane is

$$H_N = \sum_{j=1}^{N} \left(-\mathrm{i}\hbar\nabla_j + B\mathbf{x}^{\perp} \right)^2 \tag{10.2}$$

with $\mathbf{x}^{\perp} = (-y, x)$ if $\mathbf{x} = (x, y)$. The spectrum of (10.2) is well known [**74**] and presents energy levels of the form

$$E_n = \hbar B \left(n + \frac{1}{2} \right) \tag{10.3}$$

The states of energy E_0 form the lowest Landau level (LLL). LLL wave functions take the form

$$\Psi^{\mathrm{LLL}}(\mathbf{x}) = f(\mathbf{z}) e^{-\frac{B}{\hbar}\frac{|\mathbf{z}|^2}{4}} \tag{10.4}$$

where $\mathbf{x} \in \mathbb{R}^2$ is identified with $\mathbf{z} \in \mathbb{C}$ and where f is an holomorphic function. The LLL is infinitely degenerated. We can estimate heuristically its degeneracy per unit area by selecting a basis

$$\mathrm{LLL} = \mathrm{span}\left\{ \phi_n := \mathbf{z}^n e^{-\frac{B}{\hbar}\frac{|\mathbf{z}|^2}{4}} \right\}_{n=0}^{\infty}. \tag{10.5}$$

Each radial density $|\phi_n|^2$ is localized around $r_{\max}$ such that

$$0 = \partial_r |\phi_n|^2 (r_{\max}) \tag{10.6}$$

$$0 = \left(2n r_{\max}^{2n-1} - r_{\max}^{2n+1} \frac{B}{\hbar} \right) e^{-\frac{B}{\hbar}\frac{|\mathbf{z}|^2}{4}} \tag{10.7}$$

$$r_{\max} = \sqrt{\frac{2n\hbar}{B}}. \tag{10.8}$$

Intuitively we can accomodate n states in a disc of center 0 and radius $r_{\max}$. The degeneracy per unit area is then

$$D \simeq \frac{n}{\pi r_{\max}^2} \simeq \frac{B}{2\pi\hbar}. \tag{10.9}$$

The filling factor counting how many Landau levels are filled is approximatively

$$\nu = \frac{\rho}{D} = \frac{2\pi N\hbar}{B}. \tag{10.10}$$

where we assumed the density ρ to be constant.

3. Laughlin's wave function

Whereas the IQHE is understood as the result of the kinetic energy quantisation of independent electrons in a magnetic field, the FQHE is due to strong electronic correlations when a Landau level is only partially filled. In this regime Laughlin showed [**76**] that Coulomb interactions between the electrons become relevant and described the gas as a correlated incompressible electron liquid. The main idea of Laughlin was to suggest that the ground state energy of a 2D electron gas could be described by the Laughlin wave function

$$\Psi_{\mathrm{Lau}}(\mathbf{z}_1, ..., \mathbf{z}_N) = c_{\mathrm{Lau}} \prod_{1 \le i < j \le N} (\mathbf{z}_i - \mathbf{z}_j)^n e^{-B|\mathbf{z}_j|^2/4} \tag{10.11}$$

where the electron positions $\mathbf{y}_i \in \mathbb{R}^2$ are identified with $\mathbf{z}_i \in \mathbb{C}$ and c_{Lau} is a constant ensuring the normalization. Here n is an odd integer for symmetry reasons. The filling factor ν can be identified with $1/n$ [**134**].

The ansatz (10.11) is very good at reducing repulsive interaction energy. We observe from the expression (10.11) that the larger B is, the more particles will be located near $Z_N = (z_1, ..., z_N) = 0$. In this case the Coulombian interactions would increase drastically. The polynomial factor in front of the gaussian in (10.11) allows to avoid this scenario.

3.1. Quasi-holes. It has been proposed [**169, 30**] (and revisited in [**121**]) that the fractional statistics of anyons could emerge in the context of test particles immersed in an atomic gas forming a quantum Hall droplet. According to [**121**], under reasonable assumptions, the wave function describing N bath particles $Y_N = (y_1, ..., y_N)$, coupled to M quasi-holes at the locations $X_M = (x_1, ..., x_M)$ should have the form:

$$\Psi\left(X_M, Y_N\right) = c_{\rm qh} \Psi_{\rm qh}\left(X_M, Y_N\right) \Phi\left(X_M\right) \tag{10.12}$$

where $\Psi_{\rm qh}$ is the Laughlin state with quasi-holes

$$\Psi_{\rm qh}\left(X_M, Y_N\right) = \prod_{j=1}^{M} \prod_{k=1}^{M} \left(\zeta_j - \mathbf{z}_k\right) \prod_{1 \leq i < j \leq N} \left(\mathbf{z}_i - \mathbf{z}_j\right)^n e^{-B|\mathbf{z}_j|^2/4} \tag{10.13}$$

with $\mathbf{x}_i \in \mathbb{R}^2$ and $\mathbf{y}_i \in \mathbb{R}^2$ respectively identified with $\zeta_i \in \mathbb{C}$ and $\mathbf{z}_i \in \mathbb{C}$. In the Laughlin approach quasi-holes are first seen as classical parameters. In Equation (10.12) we see them as real quantum quasi-holes interacting repulsively with the particles. We choose $c_{\rm qh} > 0$ to enforce

$$c_{\rm qh}\left(X_M\right)^2 \int_{\mathbb{R}^{2N}} \left|\Psi_{\rm qh}\left(X_M, Y_N\right)\right|^2 {\rm d}Y_N = 1 \tag{10.14}$$

for any X_M and normalize the full wave function Ψ by imposing that

$$\int_{\mathbb{R}^{2M}} \left|\Phi\left(X_M\right)\right|^2 {\rm d}X_M = 1. \tag{10.15}$$

One of the virtues of the ansatz (10.12) is to relegate the physics in the quasi-holes wave function Φ. Denote $\mathbf{B}$ the perpendicular magnetic field and $\mathbf{A}$ the associated vector potential ${\rm curl}\mathbf{A} = \mathbf{B}$, take a quasi-hole $\mathbf{x}_j$, its kinetic energy

$$\left\langle \Psi, \left(-{\rm i}\nabla_j + \mathbf{A}\left(\mathbf{x}_j\right)\right)^2 \Psi \right\rangle \simeq \left\langle \Phi, H^{\rm eff} \Phi \right\rangle + \frac{BN}{2} \tag{10.16}$$

is only governed by an effective Hamiltonian taking the form

$$H^{\rm eff} = \left(-{\rm i}\nabla_{\mathbf{x}_j} + \mathbf{A}\left(\mathbf{x}_j\right) + \mathbf{A}_r\left(\mathbf{x}_j\right) + \mathbf{A}_a\left(\mathbf{x}_j\right)\right)^2 \tag{10.17}$$

where two vector potentials have appeared. The first

$$\mathbf{A}_r(\mathbf{x}_j) \simeq \frac{B}{2n} \mathbf{x}_j^{\perp} \tag{10.18}$$

$$ {\rm curl}\mathbf{A}_r \simeq \frac{B}{n} \tag{10.19}$$

generates a new constant magnetic field with the effect of reducing the external one. This can be seen as a change of the charge into an effective fractionnal one. The second vector potential to appear in (10.17) is the anyonic vector potential (9.18) spread over a disc of radius R with a statistics $\alpha = 1/n$

$$\mathbf{A}_a\left(\mathbf{x}_j\right) \simeq -\sum_{\substack{k=1 \\ k \neq j}}^{M} \frac{B}{2\pi n} \int_{\mathbb{R}^2} \frac{\left(\mathbf{x}_j - \mathbf{y}\right)^{\perp}}{\left|\mathbf{x}_j - \mathbf{y}\right|^2} \mathbb{1}_{B(\mathbf{x}_k, R)}(\mathbf{y}) {\rm d}\mathbf{y} \tag{10.20}$$

with the magnetic lenght $R = \sqrt{2/B}$. This field corresponds to Aharonov-Bohm-like units of flux carried by the tracer particles

$$\mathrm{curl}\mathbf{A}_a(\mathbf{x}_j) \simeq -\sum_{\substack{k=1\\k\neq j}}^{M} \frac{B}{n} \mathbb{1}_{B(\mathbf{x}_k,R)}(\mathbf{x}_j) = -\frac{2}{n}\sum_{\substack{k=1\\k\neq j}}^{M} \frac{\mathbb{1}_{B(\mathbf{x}_k,R)}}{R^2}(\mathbf{x}_j) \tag{10.21}$$

We retrieve the magnetic gauge picture of anyons with the requirement that the discs don't overlap to be able to recover the ideal anyonic potential

$$\mathbf{A}_j = \sum_{k<j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^{\perp}}{|\mathbf{x}_j - \mathbf{x}_k|^2} \tag{10.22}$$

via Newton's Theorem [**92**, Theorem 9.7]. The above allows us to see the quasi-holes as extended (spread over a lengh R) interacting free anyons in a magnetic field $\mathbf{A} + \mathbf{A}_r$.

4. Experimental results

We can find at least two experimental measurements [**127**] and [**12**] of particles in the FQHE regime presenting an anyonic behavior. Both of them use a 2D electron gas device at filling factor $\nu = 1/3$ and study its elementary excitations, expected to be anyons with statistics $\alpha = 1/3$. We briefly present the basic ideas behind these experiments.

4.1. Measure of braiding statistics. In [**127**] anyonic braiding statistics are observed using a Fabry-Perot interferometer. Figure 10.1 describes the experimental setup. Blue regions indicate the gas and the bulk is highlighted in yellow. When operated at the $\nu = 1/3$ fractional quantum Hall state, the current is carried by quasiparticles traveling in chiral edge states (red arrows), and dotted arrows indicate the backscattered quasiparticle paths which may interfere. Quasiparticles may be localized inside the chamber of the interferometer by disorder, as represented by the red vortices, and the backscattered paths enclose a loop around these quasiparticles, making the interferometer sensitive to the phase of anyons. By means of quantum point contacts one uses the edge current to make quasiparticles tunneling currents (dashed lines on Figure 10.1) turn around the N_{qp} particles located inside the interferometer symbolized by vortices on the picture. This way we expect the phase difference θ of the total system to be the sum of the Aharonov-Bohm phase ϕ_{AB} due to the external magnetic field and the anyonic contribution $2\pi\alpha N_{qp}$

$$\theta = \phi_{\mathrm{AB}} + 2\pi\alpha N_{qp}. \tag{10.23}$$

The total current backscattered by the interferometer will depend on $\cos\theta$. One then measures the conductance G of the interferometer, directly depending on θ. Whereas the Aharonov-Bohm phase continuously varies when B increases, quasi-holes are created discontinuously. The jumps of N_{qp} correspond to leaps in the measured conductance $G(B)$. We can deduce from these jumps the change of θ when a particle is added or subtracted, leading to the experimental value $\alpha = \nu = 0.31 \pm 0,04 \simeq 1/3$ in good agreement with the predicted one.

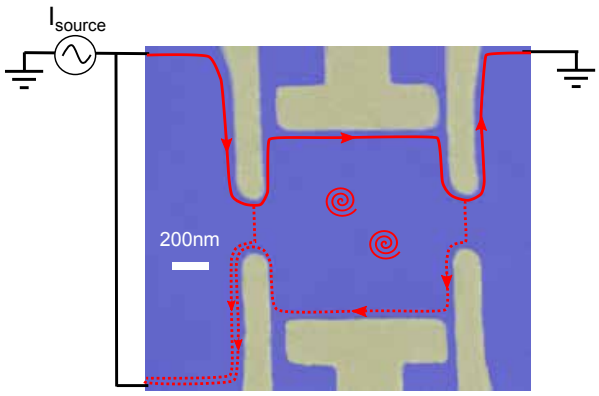


FIGURE 10.1. Schematic representation of the interferometer extracted from [**127**]

4.2. Measure via the exclusion principle. The experiment [**12**] is rather based on the fact that, depending on their statistics parameter α anyons are more or less submitted to the Pauli principle. In this experiment one also works with elementary excitations of the fractional quantum Hall effect at $\nu = 1/3$.

Figure 10.2 describes the experimental setup. The material is in yellow and the electron gas in blue. Two currents I_1^0 and I_2^0 are sent from 1 and 2, are carried by the edges, split by the quantum point contacts QPC_1 and QPC_2 acting as random Poissonian sources of anyons I_1 and I_2. These two currents finally collide via the beam-splitter of transmission T made of two cQPC in the center of the figure and continue their path to the detectors 3 and 4 measuring the currents I_4 and I_3. One measures the correlations $S_{I_3I_4}$ between two quasiparticles currents I_3 and I_4 generated by the collision of the two currents I_1 and I_2. The point is that two genuine fermions cannot be found in the same state. This case corresponds to $S_{I_3I_4} = 0$. Otherwise, in a specific regime we expect $S_{I_3I_4}$ to be proportionnal to the input current $I_+ = I_1^0 + I_2^0$ with a slope directly related to the statistics parameter α. One finds the statistic parameter $\alpha = 1/3$, which corresponds to the theoretical expectations. See [**111, 19**] for more detail about the exchange and exclusion properties of anyons involved by the two above experiments.

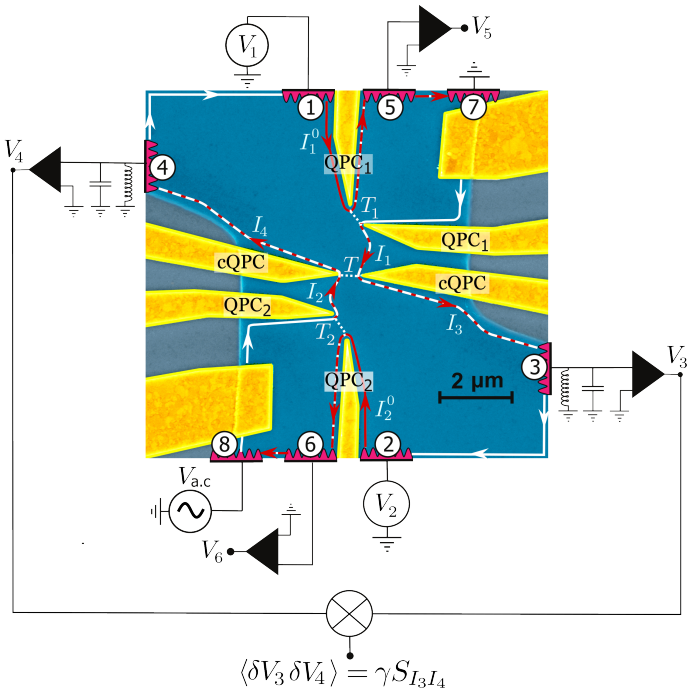


FIGURE 10.2. Schematic representation of the collisioner extracted from [**12**]

CHAPTER 11

Results

This section is dedicated to the main statements of the thesis. We first introduce the complete model we want to study and state the results.

A gas of anyons is a many-body system. One cubic meter of usual air contains approximatively 10^{25} particles. The experiments in cold atoms deal with $\sim 10^3 - 10^6$ particles and the FQHE with 10^{11} electrons per cm^2. Exact results are rare. The two-anyons ground state energy problem in a harmonic trap can be solved analytically [**28**]. The energy eigenvalues then depend linearly on the statistics parameter α with a fixed slope. The sign of the slope changes at every bosonic value $\alpha = 2$. On can find in [**33, 35**] the study of a Hamiltonian describing two anyons moving inside a plane in presence of an external magnetic field. We cannot expect to directly calculate the ground state of a general anyon system. We then aim at deriving approximate models. Here anyons are seen as bosons or fermions in the magnetix gauge picture (9.20) with the statistical parameter $\alpha \to 0$ and $N \to \infty$. The Hamiltonian of the system is

$$H_N = \sum_{j=1}^{N} \left(\mathbf{p}_j^{\mathbf{A}_e} + \alpha \mathbf{A}_j \right)^2 + V(\mathbf{x}_j) \tag{11.1}$$

where

$$\mathbf{p}_j^{\mathbf{A}_e} = -\mathrm{i}\hbar\nabla_j + \mathbf{A}_e(\mathbf{x}_j) \tag{11.2}$$

with V a trapping potential and $\mathbf{A}_e$ chosen such that

$$\mathrm{curl}\mathbf{A}_e = \mathbf{B}_e.$$

We consider two scenarios in the sequel:

Almost-bosonic case: $\hbar = 1$ and the Hamiltonian (11.1) acts on bosonic wavefunctions. The statistical parameter $\alpha \to 0$ when $N \to \infty$ so the exchange phase (9.15) tends to 1 making our system more and more bosonic. In the limit we recover at leading order a Hartree-type functional. More precisely, the ground state of the system is well approximated by that of an effective one-body self-consistent magnetic functional. The associated minimizers are shown to converge in a certain sense to a convex combination of Hartree minimizers of the form

$$\int_{\mathcal{M}} \left| u^{\otimes N} \right\rangle \left\langle u^{\otimes N} \right| \mathrm{d}\mu(u)$$

where $\mathcal{M}$ is the set of average-field energy functional minimizers.

Almost-fermionic case: the Hamiltonian (11.1) acts on fermionic wave functions. The Pauli principle requires to take $\hbar \to 0$ at the same time as $N \to \infty$ to balance the kinetic energy increasing faster than any other term. The limit $\alpha \to 0$ allows by (9.15) to

recover guenuine fermions in the limit. We obtain that the ground state converges to that of a semi-classical one-body Vlasov energy. The minimization of this functional involves probability measures m on the phase space $\mathbb{R}^2 \times \mathbb{R}^2$. The marginals of the k-reduced Husimi functions (semi-classical version of the quantum probability density) constructed from a minimizing sequence Ψ_N are shown to converge to a tensor product of those of the semi-classical minimizer m.

1. Extended anyons

The Hamiltonian (11.1) describes ideal anyons. However, anyons arising as quasi-particles in condensed-matter systems are not point-like objects. In physics experiments we expect extended anyons to arise, see Section 10. Consequently we introduce a length R over which the magnetic flux attached to our particles is smeared. In our approach, the limit $R \to 0$ will make us recover the point-like anyons point of view. This "extended anyons" model is discussed in [**122, 158, 22**]. The radius R corresponds to the size of the quasi-particle, i.e. a length-scale below which the quasi-particle description breaks down. In the FQHE context R is identified in (10.21) to the magnetic length $l_B \simeq \sqrt{2/B}$.

We introduce the 2D Coulomb potential generated by a unit charge smeared over the disc of radius R

$$w_R(\mathbf{x}) = (\log|\,.\,| * \chi_R)(\mathbf{x}), \text{ with the convention } w_0 = \log|\,.\,| \tag{11.3}$$

and $\chi_R(x)$ a positive, regularizing function of unit mass

$$\chi_R = \frac{1}{R^2}\chi\left(\frac{.}{R}\right) \text{ and } \int_{\mathbb{R}^2}\chi = 1, \quad \chi(\mathbf{x}) = \begin{cases} 1/\pi^2 \text{ for } |\mathbf{x}| \leq 1 \\ 0 \text{ for } |\mathbf{x}| \geq 2. \end{cases} \tag{11.4}$$

We take χ to be smooth, positive, radial and decreasing between 1 and 2. Remark that

$$\nabla^\perp w_0(\mathbf{x}) = \frac{\mathbf{x}^\perp}{|\mathbf{x}|^2}, \text{ with } B_0(\mathbf{x}) = \nabla^\perp\nabla^\perp w_0 = \Delta w_0 = 2\pi\delta_0$$

and that we will recover the same magnetic field as in (9.21) for the ideal anyon case (in a distributional sense) in the limit $R \to 0$ by defining

$$\mathbf{A}^R(\mathbf{x}_j) = \sum_{k\neq j} \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k). \tag{11.5}$$

We also define the extended kinetic energy operator

$$T_\alpha^R := \sum_{j=1}^N \left(D_j^R\right)^2 := \sum_{j=1}^N \left(-\mathrm{i}\hbar\nabla_j + \alpha\mathbf{A}^R(\mathbf{x}_j)\right)^2. \tag{11.6}$$

We work with the regularized Hamiltonian of N extended anyons of radius R in a trapping potential V feeling an exterior magnetic field $\mathbf{A}_e$

$$\boxed{H_N^R := \sum_{j=1}^N \left(\left(\mathbf{p}_j^{\mathbf{A}_e} + \alpha\mathbf{A}^R(\mathbf{x}_j)\right)^2 + V(\mathbf{x}_j)\right).} \tag{11.7}$$

Since $\mathbf{A}^R$ is for $R > 0$ a bounded perturbation of a usual magnetic laplacian the kinetic energy T_α^R is essentially self-adjoint on its domain $\mathcal{D}_{\alpha,R} = H^2_{\mathrm{a/sym}}(\mathbb{R}^{2N})$ (see [**129**, Theorem X.17] and [**10**]). The bottom of its spectrum exists for any fixed $R > 0$, we denote it

$$E^R(N) = \inf \sigma \left(H_N^R \right). \tag{11.8}$$

Under our assumptions H_N^R will actually have compact resolvent, so the above is an eigenvalue. However, this operator does not have a unique limit as $R \to 0$ with N fixed and the Hamiltonian at $R = 0$ (discussed e.g. in [**119**]) is *not* essentially self-adjoint (see for instance [**35, 38, 17**]). Nevertheless, in the joint limit $R \to 0$ and $N \to \infty$ we recover a unique well-defined (non-linear) model. It is convenient to expand (11.7) and treat summands separately

$$\begin{aligned} H_N^R = &\sum_{j=1}^N \left((\mathbf{p}_j^{\mathbf{A}_e})^2 + V(\mathbf{x}_j) \right) \text{ “Kinetic and potential terms”} \\ &+ \alpha \sum_{j \neq k} \left(\mathbf{p}_j^{\mathbf{A}_e} . \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) + \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) . \mathbf{p}_j^{\mathbf{A}_e} \right) \text{ “Mixed two-body term”} \\ &+ \alpha^2 \sum_{j \neq k \neq l} \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) . \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_l) \text{ “Three-body term”} \\ &+ \alpha^2 \sum_{j \neq k} \left| \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) \right|^2 \text{ “Singular two-body term”}. \end{aligned} \tag{11.9}$$

This Hamiltonian contains unusual terms. Indeed, we see in (11.9) a three-body term as well as a mixed term containing both space and momentum dependency. We are now ready to discuss the two limits we shall consider.

2. Almost-bosonic limit

2.1. The almost-bosonic model. Our first result is a continuation of the quantum de Finetti-based approach to the mean-field limit of anyonic ground states of [**112**]. The motivation is to extend this result in two different ways. The first extension concerns the physical situation. We take into account an external magnetic field $\mathbf{B}_e$, absent in [**112**]. The second concerns the improvement of the range of valididy of the average field approximation. It is based on an information-theoretic variant of the quantum de Finetti Theorem due to Brandão and Harrow [**18, 88**], and revisited in [**135**].

We recall the regularized Hamiltonian (11.7) of N trapped anyons under an external magnetic field $\mathbf{A}_e$

$$H_N^R := \sum_{j=1}^N \left(\left(\mathbf{p}_j^{\mathbf{A}_e} + \alpha \mathbf{A}^R(\mathbf{x}_j) \right)^2 + V(\mathbf{x}_j) \right) \tag{11.10}$$

acting on $\Psi_N \in L^2_{\mathrm{sym}} \left(\mathbb{R}^{dN} \right)$. We fix the parameter

$$\beta = \alpha(N-1) \tag{11.11}$$

to enforce the almost-bosonic limit. The fixed parameter $\beta \in \mathbb{R}$ quantifies the strenght of the magnetic field carried by the total anyons gas. In this way the anyon statistics has a

leading order effect manifest through the average field approximation we now introduce. We set

$$\mathbf{A}[\rho] = \nabla^{\perp} w_0 * \rho \text{ and } \mathbf{A}^R[\rho] = \nabla^{\perp} w_R * \rho \tag{11.12}$$

which makes $\mathbf{A}$ independent of the precise positions x_j. Here ρ is the one-body density normalized in $L^1(\mathbb{R}^2)$ of the bosonic wave function Ψ_N

$$\rho(\mathbf{x}_1) = \int_{\mathbb{R}^{2(N-1)}} |\Psi_N(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N)|^2 \, \mathrm{d}\mathbf{x}_2 ... \mathrm{d}\mathbf{x}_N$$

We obtain by replacing $\mathbf{A}^R(\mathbf{x}_j)$ by $\mathbf{A}^R[\rho](\mathbf{x}_j)$ the average field N-body Hamiltonian,

$$H_N^{\mathrm{af}} = \sum_{j=1}^{N} \left(\left(\mathbf{p}_j^{\mathbf{A}} + (N-1)\alpha \mathbf{A}^R[\rho] \right)^2 + V(\mathbf{x}_j) \right). \tag{11.13}$$

the magnetic field becomes

$$\mathbf{B}(\mathbf{x}) = \mathrm{curl}\beta \mathbf{A}[\rho](\mathbf{x}) = 2\pi\beta\rho(\mathbf{x}) \tag{11.14}$$

with a total flux of $2\pi\beta$. We can now consider the usual mean field ansatz, taking

$$\Psi(\mathbf{x}_1, ..., \mathbf{x}_N) = u(\mathbf{x}_1) ... u(\mathbf{x}_N)$$

with the consistency equation

$$|u|^2 = \rho$$

and obtain the one particle average-field energy functional

$$\mathcal{E}_{R,\beta}^{\mathrm{af}}[u] = \int_{\mathbb{R}^2} \left(\left| (-\mathrm{i}\nabla + \mathbf{A}_e + \beta \mathbf{A}^R[|u|^2]) u \right|^2 + V|u|^2 \right) \tag{11.15}$$

and when $R = 0$

$$\mathcal{E}_{\beta}^{\mathrm{af}}[u] = \int_{\mathbb{R}^2} \left(\left| \left(\mathbf{p}_j^{\mathbf{A}} + \beta \mathbf{A}\left[|u|^2\right] \right) u \right|^2 + V|u|^2 \right) \tag{11.16}$$

with $\mathbf{A}^R[\cdot]$ defined in (11.12). We denote the corresponding infima

$$E_{R,\beta}^{\mathrm{af}} := \inf_{\|u\|_{L^2}=1} \mathcal{E}_R^{\mathrm{af}}[u] \text{ and } E_{\beta}^{\mathrm{af}} := E_{0,\beta}^{\mathrm{af}} = \inf_{\|u\|_{L^2}=1} \mathcal{E}_{\beta}^{\mathrm{af}}[u] \tag{11.17}$$

under the unit mass constraint

$$\int_{\mathbb{R}^2} |u|^2 = 1.$$

Alternatively, we can write

$$\mathcal{E}_R^{\mathrm{af}}[u] = \int_{\mathbb{R}^2} \left(|\mathbf{p}^{\mathbf{A}_e} u|^2 + V|u|^2 + 2\beta \mathbf{A}^R[|u|^2] \cdot \left(\frac{\mathrm{i}}{2}(u\nabla\overline{u} - \overline{u}\nabla u) + \mathbf{A}_e|u|^2 + \frac{\beta}{2}\mathbf{A}^R[|u|^2]|u|^2 \right) \right) \mathrm{d}\mathbf{x}$$

Our aim is now well defined and consists in proving

$$\frac{E^R(N)}{N} = \frac{\inf \left\langle \Psi_N, H_R^N \Psi_N \right\rangle}{N} \approx \inf \left\{ \mathcal{E}_R^{\mathrm{af}}[u], \int |u|^2 = 1 \right\} \tag{11.18}$$

when $N \to \infty$ and $R \to 0$. In other words, the average mean field energy is a good approximation for the ground state energy of H_N^R.

2.2. Results for the almost-bosonic case. We state our first main theorem, justifying the average field approximation in the almost-bosonic limit at the level of the ground state. For technical reasons we assume that the one-body potential is confining

$$V(x) \geq c|\mathbf{x}|^s - C, \quad s > 0, \tag{11.19}$$

and

Assumption 11.1 (**External Magnetic Potentials**).
The external magnetic field satisfies

- $\mathbf{A}_e \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$
- $\mathrm{curl}\mathbf{A}_e = \mathbf{B}_e = \mathbf{B}_0 + \tilde{\mathbf{B}}$ *with* $\mathbf{B}_0$ *constant and* $\tilde{\mathbf{B}} \in W^{1,2+\varepsilon}$

We also need that the size R of the extended anyons does not go to zero too fast in the limit $N \to \infty$ i.e $R \sim N^{-\eta}$ for some η.

Theorem 11.1 (**Validity of the average field approximation**).
Assume $R \sim N^{-\eta}$ *for some*

$$0 < \eta < \frac{1}{4} \tag{11.20}$$

and set the statistics parameter

$$\alpha = \frac{\beta}{(N-1)}$$

for some fixed $\beta \in \mathbb{R}$. *Then, in the limit* $N \to \infty$ *we have for the ground-state energy*

$$\frac{E^R(N)}{N} \to E^{\mathrm{af}}_\beta. \tag{11.21}$$

Moreover, if Ψ_N *is a sequence of approximate minimizers for* H^R_N

$$\left\langle \Psi_N, H^R_N \Psi_N \right\rangle \leq E^R(N) + o\left(N\right) \tag{11.22}$$

with associated reduced density matrices

$$\gamma^{(k)}_N := \mathrm{Tr}_{k+1\to N}\left[|\Psi_N\rangle\langle\Psi_N|\right] \tag{11.23}$$

then along a subsequence we have

$$\gamma^{(k)}_N \to \int_{\mathcal{M}^{\mathrm{af}}} |u^{\otimes k}\rangle\langle u^{\otimes k}| \, \mathrm{d}\mu(u) \tag{11.24}$$

strongly in the trace-class when $N \to \infty$, *where* μ *is a Borel probability measure supported on the set of minimizers of* $\mathcal{E}^{\mathrm{af}}$,

$$\mathcal{M}^{\mathrm{af}} := \{u \in L^2(\mathbb{R}^2) : \|u\|_{L^2} = 1,\, \mathcal{E}^{\mathrm{af}}_\beta[u] = E^{\mathrm{af}}_\beta\}.$$

The study of this average field limit presents several new aspects compared to a usual Hatree approximation as sketched in Section 3.1:

- We need to deal with the limit $R \to 0$ at the same time as $N \to \infty$ which is reminiscent of the NLS and GP limits for trapped Bose gases [**96, 93, 85**]. The new feature is the magnetic interaction involving a mixed two-body term and a three-body term (11.9).

- The constraint $\eta < 1/4$ is an idealization of the expected physical scenario. The inter-particle space in a volume of order one fixed by the potential V in two dimensions is $1/\sqrt{N}$. Since $R = N^{-\eta}$ we only reach in the above Theorem the mean-field regime. It is in fact not clear if some lower bound on R is an unavoidable condition for the average field description to be correct. The energy upper bound can be obtained by computing a trial state $u^{\otimes N}$ without any restriction on η [**112**, equation 3.19].
- It would be desirable to relax the condition to reach $\eta > 1/2$ and reach the dilute regime. In this case anyons would be good candidates to describe the quasi-particles of the FQHE.
- The regularization method in [**112**, equation 1.4] where

$$\widetilde{\chi}_R := \frac{\mathbb{1}_{B(0,R)}}{\pi R^2}(\mathbf{x})$$

was used. The χ we use in this work is a smooth version of the former, namely,

$$\int_{\mathbb{R}^2} |\widehat{\chi}(\mathbf{p})| \mathrm{d}\mathbf{p} < \infty$$

with $\widehat{\chi}$ the Fourier tranform of χ.

3. Almost-fermionic limit

Our second result concerns a limit situation where the statistics/magnetic interaction is seen as a "perturbation from the fermionic end". We vindicate a mean-field approximation, proving that the ground state of a gas of anyons is described to leading order by a semi-classical, Vlasov-like, energy functional. The ground state of the latter displays anyonic behavior in its momentum distribution. We study an "almost fermionic limit" $\alpha \to 1$ in dependence with the particle number. The relevant scaling in dependence of N leads to a problem resembling the usual mean-field/semi-classical limit for interacting fermions [**100, 147, 13, 11, 46, 53**].

3.1. The almost-fermionic model. We recall the regularized Hamiltonian (11.7) for N trapped anyons under an external magnetic field $\mathbf{B}_e = \mathrm{curl}\mathbf{A}_e$

$$H_N^R := \sum_{j=1}^N \left(\left(\mathbf{p}_j^{\mathbf{A}_e} + \alpha \mathbf{A}^R(\mathbf{x}_j) \right)^2 + V(\mathbf{x}_j) \right) \tag{11.25}$$

acting on $\Psi_N \in L^2_{\mathrm{asym}}\left(\mathbb{R}^{dN}\right)$. Instead of looking at the ground state of an Hamiltonian acting on bosonic wave-functions we now consider fermionic wave functions. The exchange phase is now $e^{i\pi(\alpha+1)}$ due to the fermionic nature of Ψ in (9.15).

We are interested in a joint limit $N \to \infty$, $\alpha \to 0$ in which the problem would simplify but still display signatures of anyonic behavior, i.e. a non-trivial dependence on α. Since we perturb around fermions, the Pauli principle implies that the kinetic energy grows faster than N. This behavior is encoded in the Lieb-Thirring inequality [**94**, Theorem 4.1] which, for any N-particles normalized antisymmetric function Ψ_N with support in a bounded

domain Ω^N, implies

$$\sum_{j=1}^{N} \int_{\Omega^N} |\nabla_j \Psi|^2 \geq C|\Omega|^{-\frac{2}{d}} N^{1+\frac{2}{d}}. \tag{11.26}$$

where d is the dimension of the physical space. Because of this rapid growth of the kinetic energy, the natural mean-field limit [**13, 14, 147**] for fermions involves a scaling $\hbar \sim N^{-1/d}$. Here with $d = 2$ we get a kinetic energy proportional to N^2 and thus, roughly speaking $-\mathrm{i}\nabla_j \sim N^{1/2}$. Investigating the N-dependence of each term of (11.1) we thus expect

$$\langle \Psi_N, H_N \Psi_N \rangle \approx C_1 N \left(\hbar N^{1/2} + \alpha N \right)^2 + C_2 N$$

where the terms in parenthesis account for kinetic energy, including the influence of (11.5), and the second term is the energy in the external potential V. To obtain a non trivial limit when $\alpha \to 0$, a natural choice is to take

$$\hbar = \frac{1}{\sqrt{N}} \text{ and } \alpha = \frac{\beta}{N} \tag{11.27}$$

with $\beta \in \mathbb{R}$ a constant allowing to keep track of the anyonic behavior. We consider the limit

$$\lim_{N\to\infty} \frac{E^R(N)}{N}$$

with

$$E^R(N) := \inf \left\{ \left\langle \Psi_N, H_N^R \Psi_N \right\rangle, \Psi_N \in L^2_{\mathrm{asym}}(\mathbb{R}^{2N}), \int_{\mathbb{R}^{2N}} |\Psi_N|^2 = 1 \right\}. \tag{11.28}$$

Such an approach combines two limits (see also Remark 11.3 below), the semi-classical $\hbar \to 0$ and the almost-fermionic $\alpha \to 0$. We expect (as in other types of mean-field limits, see e.g. [**137**] and references therein) that, for large N, particles behave independently, and thus

$$\sum_{j\neq k} \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) \approx \int_{\mathbb{R}^2} \nabla^\perp w_R(\mathbf{x}_j - \mathbf{y}) \rho(\mathbf{y}) \mathrm{d}\mathbf{y} = \nabla^\perp w_R * \rho(\mathbf{x}_j)$$

where ρ is the density of particles. Therefore, it is convenient to define, given a one-body density ρ normalized in $L^1(\mathbb{R}^2)$

$$\mathbf{A}[\rho] = \nabla^\perp w_0 * \rho \text{ and } \mathbf{A}^R[\rho] = \nabla^\perp w_R * \rho$$

with

$$w_R = \log|.| * \chi_R \text{ and } w_0 = \log|.| \tag{11.29}$$

defined in (11.3). Natural mean-field functionals associated with (11.9) are

$$\mathcal{E}_R^{\mathrm{af}}[\gamma] = \frac{1}{N} \operatorname{Tr}\left[\left(\mathbf{p}_1^{\mathbf{A}} + \alpha \mathbf{A}^R[\rho_\gamma] \right)^2 \gamma \right] + \frac{1}{N} \int_{\mathbb{R}^2} V(\mathbf{x}_1) \rho_\gamma(\mathbf{x}_1) \mathrm{d}\mathbf{x}_1 \tag{11.30}$$

$$\mathcal{E}^{\mathrm{af}}[\gamma] = \frac{1}{N} \operatorname{Tr}\left[\left(\mathbf{p}_1^{\mathbf{A}} + \alpha \mathbf{A}[\rho_\gamma] \right)^2 \gamma \right] + \frac{1}{N} \int_{\mathbb{R}^2} V(\mathbf{x}_1) \rho_\gamma(\mathbf{x}_1) \mathrm{d}\mathbf{x}_1 \tag{11.31}$$

with γ the 1-body reduced density matrix of a state $\Gamma_N = |\Psi_N\rangle \langle \Psi_N|$

$$\gamma := \gamma_N^{(1)} = N \operatorname{Tr}_{2\to N} \Gamma_N \text{ and } \rho_\gamma(\mathbf{x}) := \gamma(\mathbf{x}, \mathbf{x}).$$

Note that conventions have changed. In the bosonic case we worked with (11.23) where $\operatorname{Tr} \gamma^{(1)} = 1$. Now we have $\operatorname{Tr} \gamma^{(1)} = N$. Similar Hartree-like functionals were obtained in

the almost-bosonic limit above. Here we simplify things one step further by considering a semi-classical analogue

$$\mathcal{E}_{\mathrm{Vla}}[m] = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} |\mathbf{p}+\mathbf{A}_e(\mathbf{x})+\beta\mathbf{A}[\rho](\mathbf{x})|^2\, m(\mathbf{x},\mathbf{p})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p} + \int_{\mathbb{R}^2} V(\mathbf{x})\rho_m(\mathbf{x})\mathrm{d}\mathbf{x} \tag{11.32}$$

with $m(\mathbf{x},\mathbf{p})$ a positive measure on the phase-space $\mathbb{R}^4$ of positions/momenta and

$$\rho_m(\mathbf{x}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} m(\mathbf{x},\mathbf{p})\mathrm{d}\mathbf{p}. \tag{11.33}$$

Our convention is that m satisfies the mass constraint

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} m(\mathbf{x},\mathbf{p})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p} = (2\pi)^2\,. \tag{11.34}$$

and the semi-classical Pauli principle

$$\boxed{0 \le m(\mathbf{x},\mathbf{p}) \le 1 \ \text{ a.e.}} \tag{11.35}$$

The latter is imposed at the classical level by the requirement that a classical state $(\mathbf{x},\mathbf{p})$ cannot be occupied by more than one fermion. By the Bathtub principle [**92**, Theorem 1.14] we can perform the minimization in $\mathbf{p}$ explicitly (this parallels the considerations in [**147**]). We find minimizers of the form

$$m_\rho(\mathbf{x},\mathbf{p}) = \mathbb{1}\left\{|\mathbf{p}+\mathbf{A}_e(\mathbf{x})+\beta\mathbf{A}[\rho](\mathbf{x})|^2 \le 4\pi\rho(\mathbf{x})\right\}$$

where ρ minimizes the Thomas-Fermi energy

$$\mathcal{E}_{\mathrm{TF}}[\rho] = \mathcal{E}_{\mathrm{Vla}}[m_\rho] = 2\pi\int_{\mathbb{R}^2}\rho^2(\mathbf{x})\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2} V(\mathbf{x})\rho(\mathbf{x})\mathrm{d}\mathbf{x}. \tag{11.36}$$

We define the minimum of the Thomas-Fermi functional

$$e_{\mathrm{TF}} = \inf\left\{\mathcal{E}_{\mathrm{TF}}[\rho] : 0 \le \rho \in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2), \int_{\mathbb{R}^2}\rho = 1\right\}. \tag{11.37}$$

Our first purpose is to obtain the above as the limit of the true many-body energy.

3.2. Results for the almost-fermionic ground state. We shall work under the following assumptions

Assumption 11.1 (**External Potentials**).
The external potentials entering (11.7) *satisfy*

- $\mathbf{A}_e \in L^2(\mathbb{R}^2)\cap W^{2,\infty}(\mathbb{R}^2)$
- $|\mathbf{A}_e|^2 \in W^{1,\infty}(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$
- $V(\mathbf{x}) \ge c|\mathbf{x}|^s - C$ *for some* $s > 1$
- $|\nabla V(\mathbf{x})| \le c'|\mathbf{x}|^{s-1} + C'$ *and* $|\Delta V(\mathbf{x})| \le c''|\mathbf{x}|^{s-2} + C''$

Theorem 11.2 (**Convergence of the almost-fermionic ground state energy**).
We consider N extended anyons of radius $R = N^{-\eta}$ and statistical parameter $\alpha = \beta/N$ in an external potential V. Under Assumption (11.1) *and setting*

$$0 < \eta < \frac{1}{4}$$

we have, in the parameter regime (11.27)

$$\boxed{\lim_{N\to\infty} \frac{E^R(N)}{N} = e_{\mathrm{TF}}}$$

where the many-anyon ground state energy $E^R(N)$ *and the Thomas-Fermi energy are defined in* (11.28) *and* (11.37)*, respectively.*

The study of the regime (11.27) resembles the usual Thomas-Fermi limit for a large fermionic system [**100, 147, 155, 90**], but with important new aspects:

- The effective interaction comprises a three-body term, and a two-body term (see Equation (11.9)) which mixes position and momentum variables.
- The limit problem (11.32) comprises an effective self-consistent magnetic field $\mathbf{A}[\rho]$.
- One should deal with the limit $R \to 0$, $\alpha \to 0$ and $\hbar \to 0$ at the same time as $N \to \infty$.

Regarding the last point, we make the

Remark 11.3 (Scaling of parameters).
Extracting a multiplicative factor N from the energy, the parameter regime (11.27) we consider is equivalent to

$$\hbar = 1 \text{ and } \alpha = \frac{1}{\sqrt{N}} \tag{11.38}$$

provided we replace the external potential $V(\mathbf{x})$ by $NV(\mathbf{x})$ and a similar replacement for the external vector potential $\mathbf{A}_e$. The scaling (11.38) better highlights the quasi-fermionic character of the limit we take. We however find it preferable technically to work with (11.27), where the semi-classical aspect is more apparent.

Remark 11.4 (Guenuine anyonic case).
Another regime could be of interest. Let us we compare the kinetic energy of one particle in a domain Ω such that $|\Omega| = 1$ at $\hbar = 1$

$$E_c = (-\mathrm{i}\hbar\nabla + \alpha\mathbf{A})^2 \tag{11.39}$$

with the Landau energy (10.2)

$$E_L = \left(-\mathrm{i}\hbar\nabla + B\mathbf{x}^\perp\right)^2. \tag{11.40}$$

Approximating $\mathbf{A}$ by $\mathbf{A}[\rho]$ in (11.39) we find a magnetic field of order $\alpha\rho$ with

$$\int_\Omega \mathrm{d}\rho = N \tag{11.41}$$

Assuming ρ to be a constant we find in the limit $N \to \infty$ that the anyonic case can be seen as usual particles with a constant magnetic field of strengh $B = \alpha N$. Recall that the degeneracy per unit area of a Landau level (10.9) is

$$D \simeq \frac{B}{2\pi\hbar} = \frac{B}{2\pi} \tag{11.42}$$

We compare two regimes :

(1) $\alpha = 1/\sqrt{N}$, $B = O\left(\sqrt{N}\right)$. The regime of our study. The degeneracy of a Landau level (10.9) is of order $\sqrt{N} \ll N$. In the limit $N \to \infty$ we expect all the Landau levels to be filled, leading to a usual Thomas-fermi energy for electrons.
(2) $B \sim N$ equivalent to fixing α. The degerancy of the LLL would be of order N. In this case the final energy should be modified to become a magnetic Thomas-Fermi energy [**51, 101, 102**].

REMARK 11.5 (Range of validity).
Regarding the scaling of R, it would be highly desirable (and challenging) to be able to deal with a dilute regime $\eta > 1/2$ where the radius of the magnetic flux is much smaller than the typical inter-particle distance. We can relax our constraint $\eta < 1/4$ to $\eta < 1/3$ (see Chapter 5) if we assume a Lieb-Thirring inequality for extended anyons similar to Theorem (9.1) of the form

$$\left\langle \Psi_N, \sum_{j=1}^{N} \left(-\mathrm{i}\nabla_j + \alpha \mathbf{A}^R(\mathbf{x}_j)\right)^2 \Psi_N \right\rangle \geq C_\alpha \int_{\mathbb{R}^2} \rho_{\Psi_N}(\mathbf{x})^2 \mathrm{d}\mathbf{x}.$$

The inequality would hold for any fermionic wave-function Ψ_N with one-particle density ρ_{Ψ_N}. In the above we would need a C_α uniformly bounded from below for any $\alpha \neq 1$. Such an inequality is made very plausible by the results of [**75**] on the homogeneous extended anyon gas and the known Lieb-Thirring inequalities for point-like anyons [**116, 117, 119, 114**]. We plan to return to this matter in the future. ◇

The limit ground state energy e_{TF} does not reveal any anyonic behavior, i.e. it does not depend on α. This is however deceptive, for the behavior of the system's ground state *does* depend on α and shows the influence of the non-trivial statistics. This will become apparent when we state the second part of our result, the convergence of ground states. It is expressed in terms of Husimi functions, constructed using coherent states. We discuss this next.

3.3. Semi-classical measures. Squeezed coherent states

Let[1]

$$f(\mathbf{x}) := \frac{1}{\sqrt{\pi}} e^{-\frac{|\mathbf{x}|^2}{2}} \tag{11.43}$$

For every $(\mathbf{x}, \mathbf{p})$ in the phase space $\mathbb{R}^2 \times \mathbb{R}^2$ we define the squeezed coherent state

$$F_{\mathbf{x},\mathbf{p}}(\mathbf{y}) := \frac{1}{\sqrt{\hbar_x}} f\left(\frac{\mathbf{y} - \mathbf{x}}{\sqrt{\hbar_x}}\right) e^{\mathrm{i}\frac{\mathbf{p}\cdot\mathbf{y}}{\hbar}}. \tag{11.44}$$

It has the property of being localized on a scale $\sqrt{\hbar_x}$ in space and $\sqrt{\hbar_p}$ in momentum, these two scales being related by

$$\hbar = \sqrt{\hbar_x}\sqrt{\hbar_p}. \tag{11.45}$$

The usual coherent state [**147**, Section 2.1] is the particular case $\sqrt{\hbar_x} = \sqrt{\hbar_p} = \sqrt{\hbar}$. Any such state saturates Heisenberg's uncertainty principle, and is thus as classical as a quantum

[1]We make the standard choice that f is a gaussian but any radial L^2 function could be used instead.

state can be. Indeed, for the two observables x_1 and $p_1 = -\mathrm{i}\partial_{x_1}$ evaluated in this state we get

$$\Delta_{x_1}\Delta_{p_1} = \frac{\hbar}{2} \text{ where } \Delta_a = \left\langle F_{\mathbf{x},\mathbf{p}}, a^2 F_{\mathbf{x},\mathbf{p}} \right\rangle - \left\langle F_{\mathbf{x},\mathbf{p}}, a F_{\mathbf{x},\mathbf{p}} \right\rangle^2 .$$

In our proofs we will take $\hbar_x \ll \hbar_p \to 0$, which is convenient because our Hamiltonian is singular in x but not in p.

We define the $\hbar$-Fourier tranform

$$\mathcal{F}_\hbar[\psi](\mathbf{p}) = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2} \psi(\mathbf{x}) e^{-\mathrm{i}\frac{\mathbf{p}.\mathbf{x}}{\hbar}} \mathrm{d}\mathbf{x} \tag{11.46}$$

and denote

$$F_{\hbar_x}(\mathbf{x}) = F_{0,0}(\mathbf{x}) = \frac{1}{\sqrt{\hbar_x}\sqrt{\pi}} e^{-\frac{|\mathbf{x}|^2}{2\hbar_x}}. \tag{11.47}$$

The Fourier transform of (11.47)(see the calculation in Appendix 3) is

$$G_{\hbar_p}(\mathbf{p}) = \mathcal{F}_\hbar[F_{\hbar_x}](\mathbf{p}) = \frac{1}{\sqrt{\hbar_p}\sqrt{\pi}} e^{-\frac{|\mathbf{p}|^2}{2\hbar_p}} \tag{11.48}$$

makes apparent the localization in momentum space we just claimed.

To $F_{\mathbf{x},\mathbf{p}}$ we associate the orthogonal projector $P_{\mathbf{x},\mathbf{p}}$

$$P_{\mathbf{x},\mathbf{p}} := |F_{\mathbf{x},\mathbf{p}}\rangle\langle F_{\mathbf{x},\mathbf{p}}|. \tag{11.49}$$

Then we have the well-known

LEMMA 11.6 (**Resolution of the identity on $L^2\left(\mathbb{R}^2\right)$**).
We have

$$\frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} P_{\mathbf{x},\mathbf{p}}\, \mathrm{d}x\mathrm{d}p = \mathbb{1} \tag{11.50}$$

The proof is recalled in Appendix 3, see also [**92**, Theorem 12.8].

3.3.1. *Convergence of Husimi functions.* The Husimi function $m^{(k)}_{\Psi_N}(\mathbf{x}_1,\mathbf{p}_1,\ldots,\mathbf{x}_k,\mathbf{p}_k)$[**73, 27, 154**] describes how many particles are distributed in the k semi-classical boxes of size $\sqrt{\hbar_x}\times\sqrt{\hbar_p}$ centered at its arguments $(\mathbf{x}_1,\mathbf{p}_1),\ldots,(\mathbf{x}_k,\mathbf{p}_k)$. A wave function Ψ_N being given we define

$$m^{(k)}_{\Psi_N}(\mathbf{x}_1,\mathbf{p}_1,\ldots,\mathbf{x}_k,\mathbf{p}_k) = \frac{N!}{(N-k)!}\left\langle \Psi_N, \bigotimes_{j=1}^{k} P_{\mathbf{x}_j,\mathbf{p}_j}\otimes \mathbb{1}_{N-k}\Psi_N \right\rangle \tag{11.51}$$

Alternatively,

$$m^{(k)}_{\Psi_N}(\mathbf{x}_1,\mathbf{p}_1,\ldots,\mathbf{x}_k,\mathbf{p}_k) = \frac{N!}{(N-k)!}\int_{\mathbb{R}^{2(N-k)}} \left|\left\langle F_{\mathbf{x}_1,\mathbf{p}_1}\otimes\ldots\otimes F_{\mathbf{x}_k,\mathbf{p}_k}, \Psi_N(\cdot,z)\right\rangle_{L^2(\mathbb{R}^{2k})}\right|^2 \mathrm{d}z$$

or, in terms of fermionic annihilation and creation operators (see (11.53) below)

$$m^{(k)}_{\Psi_N}(\mathbf{x}_1,\mathbf{p}_1,\ldots,\mathbf{x}_k,\mathbf{p}_k) = \left\langle \Psi_N, a^*\left(F_{\mathbf{x}_1,\mathbf{p}_1}\right)\ldots a^*\left(F_{\mathbf{x}_k,\mathbf{p}_k}\right) a\left(F_{\mathbf{x}_k,\mathbf{p}_k}\right)\ldots a\left(F_{\mathbf{x}_1,\mathbf{p}_1}\right)\Psi_N\right\rangle$$

where a and a^* satisfy the Canonical Anticommutation Relations [**146, 64**]

$$a^*(f)a(g) + a(g)a^*(f) = \langle g, f\rangle \tag{11.52}$$

$$a^*(f)a^*(g) + a^*(g)a^*(f) = 0. \tag{11.53}$$

for $f, g \in L^2$. We will also use the k-particles density matrices of Ψ_N, i.e. the operator $\gamma^{(k)}_{\Psi_N}$ with integral kernel

$$\gamma^{(k)}_{\Psi_N}(\mathbf{x}_1, ., \mathbf{x}_k; \mathbf{x}'_1, ., \mathbf{x}'_k) = \binom{N}{k} \int_{\mathbb{R}^{2(N-k)}} \Psi_N(\mathbf{x}_1, ., \mathbf{x}_N)\overline{\Psi_N(\mathbf{x}'_1, ., \mathbf{x}'_k, \mathbf{x}_{k+1}, ., \mathbf{x}_N)}\mathrm{d}\mathbf{x}_{k+1} \dots \mathrm{d}\mathbf{x}_N. \tag{11.54}$$

We can also express the Husimi function in terms of it

$$m^{(k)}_{\Psi_N}(\mathbf{x}_1, \mathbf{p}_1, \dots, \mathbf{x}_k, \mathbf{p}_k) = k!\ \mathrm{Tr}\left[\bigotimes_{j=1}^{k} P_{\mathbf{x}_j, \mathbf{p}_j} \gamma^{(k)}_{\Psi_N}\right]. \tag{11.55}$$

These objects are further related to the density marginals in space and in momentum, respectively defined as

$$\rho^{(k)}_{\Psi_N}(\mathbf{x}_1, \dots, \mathbf{x}_k) = \binom{N}{k} \int_{\mathbb{R}^{2(N-k)}} |\Psi_N(\mathbf{x}_1, \dots, \mathbf{x}_N)|^2 \,\mathrm{d}\mathbf{x}_{k+1} \dots \mathrm{d}\mathbf{x}_N \tag{11.56}$$

$$t^{(k)}_{\Psi_N}(\mathbf{p}_1, \dots, \mathbf{p}_k) = \binom{N}{k} \int_{\mathbb{R}^{2(N-k)}} |\mathcal{F}_\hbar[\Psi_N](\mathbf{p}_1, \dots, \mathbf{p}_N)|^2 \,\mathrm{d}\mathbf{p}_{k+1} \dots \mathrm{d}\mathbf{p}_N \tag{11.57}$$

We also define the k-particle Wigner function [**52, 108**]

$$\begin{aligned} \mathcal{W}^{(k)}_{\Psi_N}(\mathbf{x}_1, \mathbf{p}_1, ..., \mathbf{x}_k, \mathbf{p}_k) := \int_{\mathbb{R}^{2k}} \int_{\mathbb{R}^{2(N-k)}} & e^{-\mathrm{i}\sum_{l=1}^{k} \mathbf{p}_l \cdot \mathbf{y}_l} \\ & \times \Psi_N\left(\mathbf{x}_1 + \hbar\mathbf{y}_1/2, ..., \mathbf{x}_k + \hbar\mathbf{y}_k/2, \mathbf{x}_{k+1}, ..., \mathbf{x}_N\right) \\ & \times \overline{\Psi_N\left(\mathbf{x}_1 - \hbar\mathbf{y}_1/2, ..., \mathbf{x}_k - \hbar\mathbf{y}_k/2, \mathbf{x}_{k+1}, ..., \mathbf{x}_N\right)} \\ & \times \mathrm{d}\mathbf{y}_1 ... \mathrm{d}\mathbf{y}_k \mathrm{d}\mathbf{x}_{k+1} ... \mathrm{d}\mathbf{x}_N \end{aligned} \tag{11.58}$$

which is not necessarily positive but has the same limit as $m^{(k)}_{f,\Psi_N}$ see [**147**, Theorem 2.1]. We can now express the convergence of states corresponding to Theorem (11.2) using the above objects. Let the minimizer of the Vlasov energy (11.32) be

$$m_{\rho^{\mathrm{TF}}}(\mathbf{p}, \mathbf{x}) = \mathbb{1}\left\{|\mathbf{p} + \mathbf{A}_e(\mathbf{x}) + \beta\mathbf{A}[\rho^{\mathrm{TF}}](\mathbf{x})|^2 \leq 4\pi\rho^{\mathrm{TF}}(\mathbf{x})\right\}. \tag{11.59}$$

Its marginals in space and momentum are respectively given by

$$\rho^{\mathrm{TF}}(\mathbf{x}) = \frac{1}{4\pi}\left(\lambda^{\mathrm{TF}} - V(\mathbf{x})\right)_+ \tag{11.60}$$

where λ^{TF} is a Lagrange multiplier ensuring normalization, and

$$t^{\mathrm{TF}}(\mathbf{p}) = \int_{\mathbb{R}^2} \mathbb{1}_{\left\{|\mathbf{p}+\mathbf{A}_e(\mathbf{x})+\beta\mathbf{A}[\rho^{\mathrm{TF}}](\mathbf{x})|^2 \leq 4\pi\rho^{\mathrm{TF}}(\mathbf{x})\right\}} \mathrm{d}\mathbf{x}. \tag{11.61}$$

3.4. Convergence of states in almost-fermionic limit.

THEOREM 11.7 (**Convergence of states**).
Let $\{\Psi_N\} \subset L^2_{asym}(\mathbb{R}^{2N}) = \bigwedge^N(\mathbb{R}^2)$ be any L^2-normalized sequence such that

$$\langle \Psi_N, H_N \Psi_N \rangle = E^R(N) + o(N)$$

in the limit $N \to \infty$ *with* (11.27). *For any choice of* $\hbar_x, \hbar_p \to 0$, *related by* (11.45), *and under Assumption 11.1, the Husimi functions of* Ψ_N *converge to those of the Vlasov minimizer*

$$m^{(k)}_{\Psi_N}(\mathbf{x}_1, \mathbf{p}_1, \dots, \mathbf{x}_k, \mathbf{p}_k) \to \prod_{j=1}^{k} m_{\rho^{\mathrm{TF}}}(\mathbf{p}_j, \mathbf{x}_j)$$

weakly as measures for all $k \geq 1$, *i.e.*

$$\int_{\mathbb{R}^{4k}} m^{(k)}_{\Psi_N} \phi_k \to \int_{\mathbb{R}^{4k}} (m_{\rho^{\mathrm{TF}}})^{\otimes k} \phi_k$$

for any bounded continous function $\phi_k \in C_b\left(\mathbb{R}^{4k}\right)$.

Consequently we also have the convergence of the k*-particles marginals in position and momentum*

$$\binom{N}{k}^{-1} \rho^{(k)}_{\Psi_N}(\mathbf{x}_1, \dots, \mathbf{x}_k) \to \prod_{j=1}^{k} \rho^{\mathrm{TF}}(\mathbf{x}_j)$$

$$\binom{N}{k}^{-1} t^{(k)}_{\Psi_N}(\mathbf{p}_1, \dots, \mathbf{p}_k) \to \prod_{j=1}^{k} t^{\mathrm{TF}}(\mathbf{p}_j)$$

weakly as measures. For the Wigner function $\mathcal{W}^{(k)}_{\Psi_N}$ *defined in* (11.58), *we have the same limit*

$$\int_{\mathbb{R}^{4k}} \mathcal{W}^{(k)}_{\Psi_N} \phi \to \int_{\mathbb{R}^{4k}} \left(m_{\rho^{\mathrm{TF}}}\right)^{\otimes k} \phi \tag{11.62}$$

for every test function ϕ *such that*

$$\partial^{\alpha_1}_{x_1} \dots \partial^{\alpha_k}_{x_k} \partial^{\beta_1}_{p_1} \dots \partial^{\beta_k}_{p_k} \phi \in L^\infty\left(\mathbb{R}^{4k}\right), \tag{11.63}$$

$$\max\left(\alpha_j, \beta_j\right) \leq 1. \tag{11.64}$$

Remark 11.8.

- The converge of the Wigner functions (11.62) directly follows from the convergence of the Husimi functions via [**147**, Theorem 2.1].
- We could also have proved that the result does not depend on the choice of function f used to construct the coherent states [**147**, Lemma 2.8].
- In our proof of energy convergence it will however be convenient to make a particular choice, and in particular use squeezed coherent states, $\hbar_x \ll \hbar_p$.

Recall that the semi-classical Thomas-Fermi energy does not depend on α, but the Vlasov minimizer does. Anyonic features are retained e.g. in the limiting object's momentum distribution (11.61).

4. Numerical illustrations

We illustrate our results with numerics. First we describe some numerical results of [**32**] for the almost-bosonic case. In a second time we use a code from a joint work in progress of Antoine Levitt and Nicolas Rougerie to illustrate the almost-fermionic results. I thank them for allowing me to play with the code and obtain Figures 11.1 and 11.3.

4.1. Numerics for bosons. The ground state of the averaged field functional (11.15) can been studied numerically with $\mathbf{A}_e = 0$ for simplicity. We recall for convenience

$$\mathcal{E}_\beta^{\mathrm{af},B}[u] = \int_{\mathbb{R}^2} \left| \left(-\mathrm{i}\nabla + \beta \mathbf{A}\left[|u|^2 \right] \right) u \right|^2 + V|u|^2 \tag{11.65}$$

the energy for almost-bosonic anyons. We define the associated ground state

$$\inf \left\{ E_\beta^{\mathrm{af},B}[u], \int_{\mathbb{R}^2} |u|^2 = 1 \right\} \tag{11.66}$$

The associated magnetic field is

$$\beta \mathrm{curl}\mathbf{A}[|u|^2] = 2\pi\beta |u|^2 \tag{11.67}$$

of total flux $2\pi\beta$. These two energy present similarities with the Gross-Pitaevskii functional

$$\mathcal{E}^{\mathrm{GP}}[u] := \int_{\mathbb{R}^2} \left(|-\mathrm{i}\nabla u + \mathbf{A}u|^2 + V|u|^2 + g|u|^4 \right) \tag{11.68}$$

with a constant vector potential $\mathbf{A}$. It describes a rotating gas of interacting bosons [**95, 97**]. The new quartic term of strength g originates from short-range pair interactions. The main difference between (11.68) and average field functionals for anyons like (11.65) is that in this last case interactions are purely magnetic. This involves the phase of the wave function in a more complicated way. Writing $u = \rho e^{\mathrm{i}\phi}$ makes clearer how the kinetic energy can be minimized

$$\int_{\mathbb{R}^2} |\nabla\sqrt{\rho}|^2 + \rho |\nabla\phi + \beta\mathbf{A}[\rho]|^2 \tag{11.69}$$

by separating density ρ and phase ϕ. The minimization of the second term would demand that

$$\nabla\phi \approx -\beta\mathbf{A}[\rho]. \tag{11.70}$$

Taking the curl

$$\mathrm{curl}\nabla\phi \approx -2\pi\beta\rho \tag{11.71}$$

which is possible only if ϕ has singularities with non-trivial circulation i.e vortices. Note that the value of β and the shape of the trap V in (11.68) can alter a lot the distribution of the vortices. For small β a GP condensate does not present any vortices. By increasing it more and more vortices appear and when the centrifugal force compensates that of the trap the gas complety spreads out. We will see in the following that such behaviors can be encoutered with anyonic functionals.

The ground state of (11.65) has been numericaly calculated in [**32**] with different trapping potentials. We will present similar plots obtained with Antoine Levitt's code[2].

Figure 11.1 and 11.2 show the density of the minimizer of (11.65) with $V(\mathbf{x}) = |\mathbf{x}|^2$. We first observe that the spreading of the gaz increases with β as expected. The condensate develops triangular vortex lattices [**34**] similar to those found in rotating trapped Bose-Einstein condensates [**2, 29, 36, 37, 49**] or type II superconductors [**139, 156, 157**].

[2]This code is for fermions. It simulates (11.72) below. However, it allows to recover the bosonic results of [**32**] by setting $N = 1$ in (11.72) below.

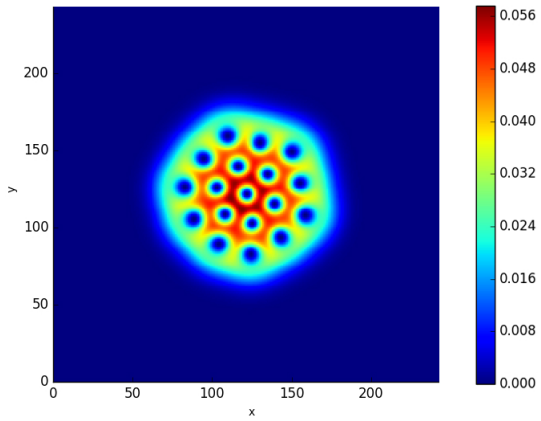


FIGURE 11.1. Density of the minimizer of (11.65) $\beta = 25$, $V(\mathbf{x}) = |\mathbf{x}|^2$

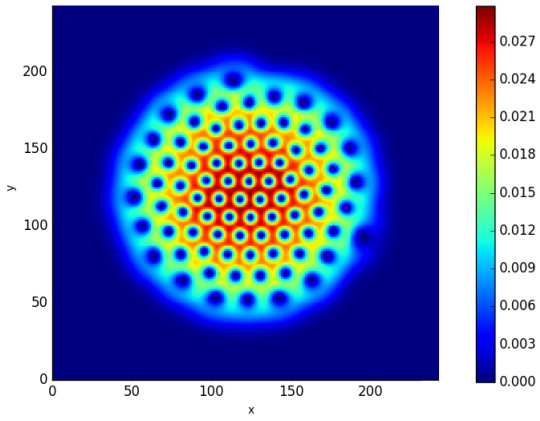


FIGURE 11.2. Density of the minimizer of (11.65) $\beta = 95$ and $V(\mathbf{x}) = |\mathbf{x}|^2$

4.2. Numerics for fermions. We now turn to the average field functional (11.31)

$$\mathcal{E}_\beta^{\mathrm{af},F}[\gamma] = \frac{1}{N}\,\mathrm{Tr}\left[\left(|\mathbf{p}_1 + \beta\mathbf{A}\left[\frac{\rho_\gamma}{N}\right]\right)^2 \gamma\right] + \int_{\mathbb{R}^2} V(\mathbf{x}_1)\frac{\rho_\gamma(\mathbf{x}_1)}{N}\mathrm{d}\mathbf{x}_1 \tag{11.72}$$

The associated magnetic field is

$$\beta\,\mathrm{curl}\mathbf{A}\left[\frac{\rho_\gamma}{N}\right] = \frac{2\pi\beta}{N}\sum_{j=1}^{N}|u_j|^2 \tag{11.73}$$

with total flux $2\pi\beta$. We define the ground state energy

$$E_\beta^{\mathrm{af},F} := \inf\left\{\mathcal{E}_\beta^{\mathrm{af},F}[\gamma], \mathrm{Tr}[\gamma] = N,\ \ 0 \le \gamma \le 1, \gamma \text{ of rank } N\right\}. \tag{11.74}$$

Note that this energy is equal to (11.65) for $N = 1$. Here γ is a projector, thus we minimize the original energy over Slater states. We recall the limit model

$$\mathcal{E}_{\mathrm{Vla}}[m] = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2\times\mathbb{R}^2}|\mathbf{p} + \beta\mathbf{A}[\rho](\mathbf{x})|^2\, m(\mathbf{x},\mathbf{p})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p} + \int_{\mathbb{R}^2} V(\mathbf{x})\rho_m(\mathbf{x})\mathrm{d}\mathbf{x} \tag{11.75}$$

with $m(\mathbf{x},\mathbf{p}) \le 1$ a positive measure of mass $4\pi^2$ on the phase-space $\mathbb{R}^4$ of positions/momentas. We find minimizers of the form

$$m^{\mathrm{TF}}(\mathbf{x},\mathbf{p}) = \mathbb{1}\Big(\left|\mathbf{p} + \beta\mathbf{A}[\rho^{\mathrm{TF}}](\mathbf{x})\right|^2 \le 4\pi\rho^{\mathrm{TF}}(\mathbf{x})\Big)$$

with

$$\rho^{\mathrm{TF}}(\mathbf{x}) = \frac{1}{4\pi}\left(\lambda^{\mathrm{TF}} - V(\mathbf{x})\right)_+ \tag{11.76}$$

where λ^{TF} is set by the normalisation of ρ^{TF} ($2\sqrt{2}$ in the case $V(\mathbf{x}) = |\mathbf{x}|^2$). The function ρ^{TF} minimizes

$$\mathcal{E}_{\mathrm{TF}}[\rho] = \mathcal{E}_{\mathrm{Vla}}[m^{\mathrm{TF}}] = 2\pi \int_{\mathbb{R}^2} \rho^2(\mathbf{x})\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^2} V(\mathbf{x})\rho(\mathbf{x})\mathrm{d}\mathbf{x}. \tag{11.77}$$

We also obtain

$$t^{\mathrm{TF}}(\mathbf{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} m^{\mathrm{TF}}(\mathbf{x}, \mathbf{p})\mathrm{d}\mathbf{x} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathbb{1}\left(\left|\mathbf{p} + \beta \mathbf{A}\left[\rho^{\mathrm{TF}}\right](\mathbf{x})\right|^2 \leq 4\pi\rho^{\mathrm{TF}}(\mathbf{x})\right) \mathrm{d}\mathbf{x}. \tag{11.78}$$

numerically calculated using the explicit density (11.76). We can numerically compare the ground state of (11.72) with ρ^{TF} and t^{TF}.

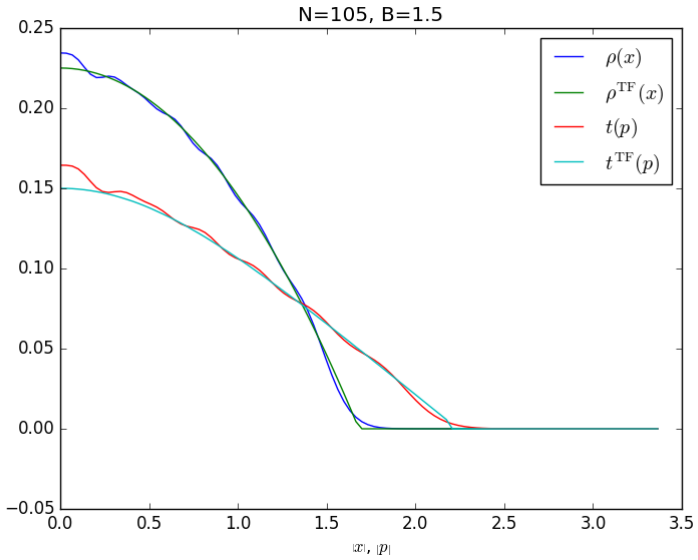


FIGURE 11.3. Density of the minimizers of the LDA (11.32) compared to those of the energy (11.72) $\beta = 1.5$ $N = 105$ and $V(\mathbf{x}) = |\mathbf{x}|^2$.

Numerics confirm the convergence of measures of Theorem 11.7. We can observe on Figure 11.3 that the agreement is rather good with only $N = 105$ particles.

It suffices to take $N > 1$ to notice on Figures 11.4 and 11.5 that vortices disappear. Indeed, the magnetic field created by the first particle with the wave function $u \in L^2(\mathbb{R}^2)$ is $B = 2\pi\beta|u|^2$. The magnetic field is strong where the first particle density is. The second particle $v \in L^2(\mathbb{R}^2)$ will try to avoid these locations and preferably chose the centers of the

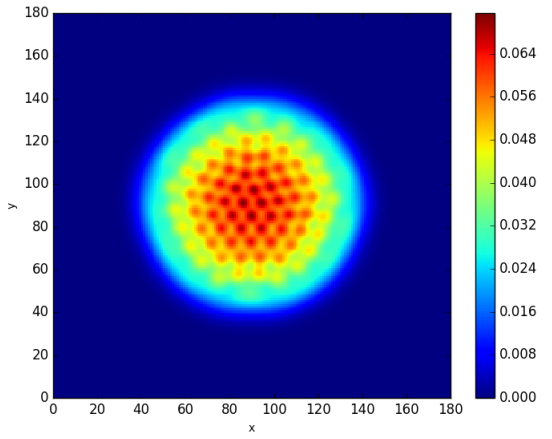


FIGURE 11.4. Density of the minimizer of (11.72) $\beta = 25$ $N = 2$ and $V(\mathbf{x}) = |\mathbf{x}|^2$

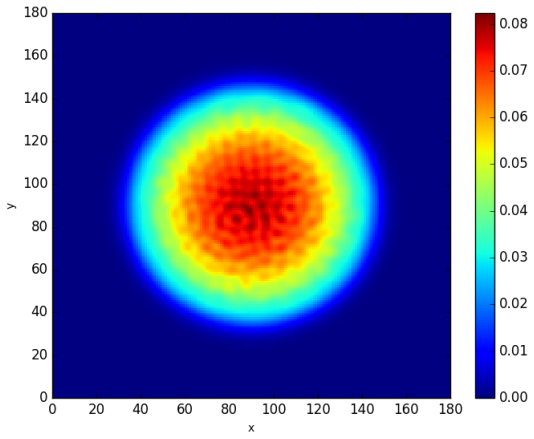


FIGURE 11.5. Density of the minimizer of (11.72) $\beta = 25$ $N = 3$ and $V(\mathbf{x}) = |\mathbf{x}|^2$

vortices created by u. Increasing N more and more will then lead to a complete averaging of the vortices. This behavior is well illustrated on Figure 11.5.

5. Outlook: Towards a Lieb-Thirring inequality for extended anyons and some suggestions

We close this introduction by describing a joint work in progress with Nicolas Rougerie. We also quickly propose a few future research directions.

We claimed in Remark 16.3 that a Lieb-Thirring inequality for extended anyons would be very plausible. We here explain why, simplifying the discussion by taking $\mathbf{A}_e = 0$. The inequality

$$\left\langle \Psi_N, \sum_{j=1}^{N} \left(\mathbf{p}_j + \alpha \mathbf{A}^R(\mathbf{x}_j)\right)^2 \Psi_N \right\rangle \geq C_\alpha \int_{\mathbb{R}^2} \rho_{\Psi_N}(\mathbf{x})^2 \mathrm{d}x \tag{11.79}$$

should be valid for any fermionic wave-function Ψ_N with the one-particle density

$$\rho_{\Psi_N} := \sum_{j=1}^{N} \int_{\mathbb{R}^{2(N-1)}} \left|\Psi_N\left(\mathbf{x}_1, ..., \mathbf{x}_{j-1}, \mathbf{x}, \mathbf{x}_{j+1}, ..., \mathbf{x}_N\right)\right|^2 \prod_{k \neq j} \mathrm{d}\mathbf{x}_k \tag{11.80}$$

and C_α uniformly bounded from below for any

$$\alpha \neq 1 \mod 2$$

Equation (11.79) has already been proved for the particular case $R = 0$ [**114**, Theorem 2.2]. The method uses a lower bound on the local Neumann energy, see (11.83) below. Such bounds are available for the extended case in [**75**] but we will below see that the method for $R = 0$ does not directly provide the bound for $R \neq 0$.

5.1. The case $R = 0$. Actually, Equation (11.79) has been proved in the particular case $R = 0$ in [**114**, Theorem 2.2]. This article provids lower and upper bounds on the Neumann ground-state energy for N ideal anyons in a domain $\Omega \subset \mathbb{R}^2$

$$E_N^{\mathcal{N}}(\alpha; \Omega) := \inf \left\{ \sum_{j=1}^{N} \int_{\Omega^N} \left|(-\mathrm{i}\nabla_j + \alpha \mathbf{A}_j)\, \Psi\right|^2 ; \Psi \in \mathcal{D}_N^\alpha, \int_{\Omega^N} |\Psi|^2 = 1 \right\} \tag{11.81}$$

where the operator $D_j = -\mathrm{i}\nabla_j + \alpha \mathbf{A}_j$ is defined in (9.19). Recall that the domain $\mathcal{D}_N^\alpha$ is that of the quadratic form (9.23) and contains symmetric wave functions. We denote

$$E_N^{\mathcal{N}}(\alpha) = E_N^{\mathcal{N}}(\alpha; \Omega_0)$$

in the special case $\Omega_0 = [0,1]^2$. This way

$$E_N^{\mathcal{N}}(\alpha; Q) = |Q|^{-1} E_N^{\mathcal{N}}(\alpha). \tag{11.82}$$

We only state the part of [**114**, Theorem 2.1] related to our concerns

THEOREM 11.9 (**Bounds for the ideal anyon gas**).
There exists a constant C such that for any $0 \leq \alpha \leq 1$

$$\frac{E_N^{\mathcal{N}}(\alpha)}{N^2} \geq C\alpha \left(1 - O\left(N^{-3}\right)\right) \tag{11.83}$$

This last bound, combined with the method of [**115**] provides

THEOREM 11.10 (**Lieb-Thirring inequality for ideal anyons (Lundholm-Seiringer)**).

There exists a constant $C > 0$ such that for any $0 \leq \alpha \leq 1$, $N \geq 1$ and $\Psi_N \in \mathcal{D}_N^\alpha$

$$\sum_{j=1}^{N} \int_{\mathbb{R}^{dN}} |D_j \Psi_N|^2 \, \mathrm{d}x \geq C\alpha \int_{\mathbb{R}^d} \rho_{\psi_N}(\mathbf{x})^2 \mathrm{d}\mathbf{x} \tag{11.84}$$

The above Theorem (valid for bosonic wave-functions) provides a proof of (11.79) in the case $R = 0$ by remplacing $\Psi_N \to U_1 \Psi_N$ to obtain $\alpha \to 1 - \alpha$ since (11.79) is stated for fermionic wave-functions.

5.2. The case $R > 0$. In the case $R > 0$ we can find a lower bound on the thermodynamic energy of the system in [**75**, Theorem 1.5]. The intermediate bounds used to prove it are local as (11.83) but without the energy $E_N^{\mathcal{N}}$ homogeneity property (11.82). We have indeed a first bound [**75**, Lemma 5.3]

$$E_N^{\mathcal{N}}(\alpha; Q) \geq \frac{C\alpha_*^2}{|Q|} \int_Q \rho_{\Psi_N} \tag{11.85}$$

with

$$\alpha_* = \lim_{N \to \infty} \alpha_N \tag{11.86}$$

$$\alpha_N := \min_{p \subset \{0,1,\ldots,N-2\}} \min_{q \in \mathbb{Z}} |(2p+1)\alpha - 2q| \, . \tag{11.87}$$

A problem is that $\alpha_* = 0$ unless α is an odd-numerator fraction. In the other bounds[**75**, Lemma 5.1,5.2] the range of the ratio $\gamma(Q) = R/|Q|$ becomes relevant. They are of the form

$$E_N^{\mathcal{N}}(\alpha; Q) \geq \frac{e\left(\alpha, \gamma(Q), \int_Q \rho_{\Psi_N}\right)}{|Q|} \tag{11.88}$$

where e is a function providing a bound similar to (11.83) except in the case $\gamma \geq \sqrt{2}$. Recalling the magnetic field carried by anyons

$$\mathrm{curl}\widetilde{\mathbf{A}}^R(\mathbf{x}_j) = \mathrm{curl} \sum_{k \neq j} \log|\,.\,| * \frac{\mathbb{1}_{B(0,R)}}{\pi R^2}(\mathbf{x}_j - \mathbf{x}_k) = 2\pi \sum_{k \neq j} \frac{\mathbb{1}_{B(\mathbf{x}_k,R)}}{\pi R^2}(\mathbf{x}_j) \tag{11.89}$$

we see that in the case $\gamma \geq 2$ the radius of anyons is larger than the diagonal of the square Q. The particles inside Q create a constant magnetic field. The particles apart from Q by a distance larger than $R + \sqrt{2Q}$ do not play a role in the energy and can be gauged away. The particles we really need to control are between these two areas. It seems to be the last step before obtaining Lieb-Thirring inequalities for extended anyons.

Other research directions:

- Try to investigate the α fixed case in the fermionic picture to obtain an magnetic Thomas-Fermi energy. See Remark 11.4.
- Try to reach the dilute regime $\eta > 1/2$ in the bosonic case. See Section 3.1.
- Investigate the dynamic of fermionic or bosonic anyons as in [**4, 5, 6, 52, 53, 54, 56, 57, 69, 131, 142, 141, 14, 150, 152, 151**] .

CHAPTER 12

Methods of proof

The method used to prove Theorem 13.1 is based on the Quantum de Finetti theorem [**136**]. Theorem 11.2 utilizes classical de Finetti theorems instead, the Diaconis-Freedman [**41**] and Hewitt-Savage [**70**] theorems. They all aim at transform a k-body symmetric probability measure into a convex combination of 1-body facrorized measures. We will introduce these theorems before starting the proofs.

1. Classical de Finetti theorems for the fermionic case

The proof of Theorem 11.2 follows a semi-classical approach, using coherent states. The measures we deal with are classical probability measures on the phase space $\mathbb{R}^{2N} \times \mathbb{R}^{2N}$ of positions and momemta. We consequently need the classical versions of the de Finetti type theorems.

1.1. The Hewitt-Savage theorem. Roughly speaking this theorem states that every symmetric probability measure over a space $\Omega^N = \Omega \times ... \times \Omega$ approaches a convex combination of product probability measures when N becomes large. We denote $\mathcal{P}\left(\Omega^N\right)$ the set of probability measures over Ω^N. A symmetric probability measure $\mu_N \in \mathcal{P}\left(\Omega^N\right)$ satisfies

$$\mu_N\left(A_1 \times ... \times A_N\right) = \mu_N\left(A_{\sigma(1)} \times ... \times A_{\sigma(N)}\right) \tag{12.1}$$

for every borelian domains $A_1, ..., A_N \subset \Omega$ and every permutation of N indices $\sigma \in S_N$. The set of these symmetric measure is denoted $\mathcal{P}_{\text{sym}}\left(\Omega^N\right)$. We also define

$$\mathcal{P}\left(\mathcal{P}\left(\Omega^N\right)\right) \tag{12.2}$$

the set of probability measure over probability measures $\mathcal{P}\left(\Omega^N\right)$. We call a product measure built on $\rho \in \mathcal{P}\left(\Omega\right)$ a measure of the form

$$\rho^{\otimes N}\left(A_1 \times ... \times A_N\right) = \rho\left(A_1\right)...\rho\left(A_N\right) \tag{12.3}$$

such a measure is symmetric. In the limit $N \to \infty$ we will consider a probability with infinitely many variables $\mu \in \mathcal{P}\left(\Omega^{\mathbb{N}}\right)$ with the inherited notion of symmetry

$$\mu\left(A_1, A_2, ...\right) = \mu\left(A_{\sigma(1)}, A_{\sigma(2)}, ...\right) \tag{12.4}$$

for every sequence of borelian domains $\left\{A_K\right\}_{K\in\mathbb{N}} \subset \Omega^{\mathbb{N}}$ and every permutation of infinitely many indices $\sigma \in S_\infty$. Denoting $\mathcal{P}_{\text{sym}}\left(\Omega^{\mathbb{N}}\right)$ the set of measure satisfying (12.4) we can state the theorem

THEOREM 12.1 (**Hewitt-Savage 1955**).
Let $\mu \in \mathcal{P}_{\text{sym}}\left(\Omega^{\mathbb{N}}\right)$ satisfy (12.4). Denote by $\mu^{(k)}$ its k-th marginal,

$$\mu^{(k)}\left(A_1, ..., A_k\right) = \mu\left(A_1, ..., A_k, \Omega, ..., \Omega, ...\right) \tag{12.5}$$

There exists a unique probability measure $P_\mu \in \mathcal{P}(\mathcal{P}(\Omega))$ *such that*

$$\mu^{(k)} = \int_{\rho\in\mathcal{P}(\Omega)} \rho^{\otimes k} \mathrm{d}P_\mu(\rho) \tag{12.6}$$

The original proof of the above theorem is geometrical. The set of symmetric probability measures over Ω^N is convex. The Choquet-Krein-Milman theorem says that any point of a convex set is a convex combination of the extremal points. It thus suffices to show that the extremal points of $\mathcal{P}_{\mathrm{sym}}\left(\Omega^{\mathbb{N}}\right)$ coincide with product measures, corresponding to sequence of marginals of the form $\left\{\rho^{(k)}\right\}_{k\in\mathbb{N}}$. This approach is not constructive, and the proof that the sequences $\left\{\rho^{(k)}\right\}_{k\in\mathbb{N}}$ are the extremal points of $\mathcal{P}_{\mathrm{sym}}\left(\Omega^{\mathbb{N}}\right)$ is by contradiction. The Diaconis-Freedman theorem [**41**] is a constructive approach of the same result. Lions also developed an approach for the needs of mean-field game theory [**109**] which is close to the method of Diaconis and Freedman. See [**124**] for a more detailed discussion.

1.2. The Diaconis-Freedman theorem. The above theorem is true when $N = +\infty$ but a more quantitative approach exists. At fixed N the error will be quantified in the natural norm for probability measures over a set S, the total variation norm

$$\|\mu\|_{\mathrm{TV}} = \int_\Omega \mathrm{d}|\mu| = \sup_{\phi\in C_b(\Omega)} \left|\int_\Omega \phi \mathrm{d}\mu\right| \tag{12.7}$$

Define the measure

$$P_{\mu_N} := \int_{\Omega^N} \delta_{\rho=\rho_Z} \mu_N(Z)\,\mathrm{d}Z \tag{12.8}$$

in $\mathcal{P}(\mathcal{P}(\Omega))$ whose the precise meaning is

$$\int_{\mathcal{P}(\Omega)} \phi(\rho)\,\mathrm{d}P_{\mu_N}(\rho) := \int_{\Omega^N} \phi(\rho_Z)\,\mu_N(Z)\,\mathrm{d}Z$$

for all bounded continuous function ϕ on $\mathcal{P}(\Omega)$ and with

$$\rho_Z(x) := \frac{1}{N}\sum_{j=1}^N \delta_{z_j=x} \tag{12.9}$$

an empirical measure defined by

$$\int_\Omega u(x)\,\mathrm{d}\rho_Z := \frac{1}{N}\sum_{j=1}^N u(z_j)$$

for any $u \in C_b(\Omega)$. We can state the theorem

THEOREM 12.2 (**Diaconis Freedman**).
Let $\mu_N \in \mathcal{P}_{\mathrm{sym}}\left(\Omega^N\right)$ *be a symmetric probability measure. Setting*

$$\widetilde{\mu}_N := \int_{\rho\in\mathcal{P}(\Omega)} \rho^{\otimes N} \mathrm{d}P_{\mu_N} \tag{12.10}$$

we have

$$\left\|\widetilde{\mu}_N^{(k)} - \mu_N^{(k)}\right\|_{\mathrm{TV}} \leq 2\frac{k(k-1)}{N} \tag{12.11}$$

Moreover, if the sequence $\left(\mu_N^{(1)}\right)_N$ *is tight,*

$$\limsup_{R\to\infty}\sup_{N\in\mathbb{N}}\int_{B(0,R)}\mathrm{d}\mu_N^{(1)}=1, \tag{12.12}$$

there exists a subsequence and a unique probability measure $P\in\mathcal{P}\left(\left(\mathcal{P}(\mathbb{R}^4)\right)\right)$ *such that (up to extraction)*

$$P_{\mu_N}\to P$$

tested against any $\phi\in C_b\left(\mathcal{P}\left(\Omega\right)\right)$*. The measure* P *obtained in the limit is that appearing in the Hewitt-Savage theorem* [**70**]*. For any fixed* $n\in\mathbb{N}^*$

$$\widetilde{\mu}_N^{(n)}\rightharpoonup\int_{\rho\in\mathcal{P}(\Omega)}\rho^{\otimes n}\mathrm{d}P(\rho). \tag{12.13}$$

Observe that

$$\widetilde{\mu}_N\left(X\right):=\int_{\Omega^N}\mu_N\left(Z\right)\sum_{\gamma\in\Gamma_N}N^{-N}\delta_{X=Z_\gamma}\mathrm{d}Z \tag{12.14}$$

where Γ_N is the set of all map (allowing repeated indices) from $\{1,...,N\}$ to itself and

$$X_\gamma=\left(\mathbf{x}_{\gamma(1)},...,\mathbf{x}_{\gamma(N)}\right)$$

with $\gamma\in\Gamma_N$. The point is that the symmetric function μ_N can be written

$$\mu_N\left(X\right):=\int_{\Omega^N}\mu_N\left(Z\right)\sum_{\sigma\in S_N}\left(N!\right)^{-1}\delta_{X=Z_\sigma}\mathrm{d}Z \tag{12.15}$$

and thus

$$\mu_N^{(k)}-\widetilde{\mu}_N^{(k)}=\int_{\Omega^N}\left[\left(\sum_{\sigma\in S_N}\left(N!\right)^{-1}\delta_{X=Z_\sigma}\right)^{(k)}-\left(\sum_{\gamma\in\Gamma_N}N^{-N}\delta_{X=Z_\gamma}\right)^{(k)}\right]\mu_N\left(Z\right)\mathrm{d}Z \tag{12.16}$$

We can understand the result by interpreting

$$\left(\sum_{\sigma\in S_N}\left(N!\right)^{-1}\delta_{X=Z_\sigma}\right)^{(k)} \tag{12.17}$$

as the probability law for drowing k balls at random from an urn containing N balls without replacement, whereas

$$\left(\sum_{\gamma\in\Gamma_N}N^{-N}\delta_{X=Z_\gamma}\right)^{(k)} \tag{12.18}$$

is the probability law for drawing k balls at random from an urn containing N , with replacement. Intuitively it is clear that when k is small compared to N, replacing the balls or not after each drawing does not significantly influence the result. See [**136**, Theorem 2.2]for the full proof. We remark that (12.14) easily gives the first marginals of $\widetilde{\mu}_N$ as a consequence of the Diaconis-Freedman construction.

LEMMA 12.3 (**First marginals of the Diaconis-Freedman measure**).
Let $\mu_N \in \mathcal{P}_{\mathrm{sym}}(\Omega^N)$ and the associated Diaconis-Freedman approximation $\widetilde{\mu}_N$ be given by the previous theorem. We have

$$\widetilde{\mu}_N^{(1)}(X_1) = \mu_N^{(1)}(X_1) \tag{12.19}$$

$$\widetilde{\mu}_N^{(2)}(X_1, X_2) = \frac{N-1}{N}\mu_N^{(2)}(X_1, X_2) + \frac{1}{N}\mu_N^{(1)}(X_1)\delta_{X_1=X_2} \tag{12.20}$$

$$\begin{aligned}\widetilde{\mu}_N^{(3)}(X_1, X_2, X_3) &= \frac{N(N-1)(N-2)}{N^3}\mu_N^{(3)}(X_1, X_2, X_3) \\ &+\frac{(N-1)}{N^2}\Big[\mu_N^{(2)}(X_1, X_3)\delta_{X_1=X_2} + \mu_N^{(2)}(X_2, X_1)\delta_{X_2=X_3} + \mu_N^{(2)}(X_3, X_2)\delta_{X_3=X_1}\Big] \\ &+ \frac{1}{N^2}\mu_N^{(1)}(X_1)\delta_{X_1=X_2}\delta_{X_2=X_3}\end{aligned} \tag{12.21}$$

PROOF. One can find (12.19) and (12.20) in [**136**, Remark 2.3]. It follows from [**136**, Equation 2.16] that

$$\widetilde{\mu}_N(X_N) = \int_{\Omega^N} \mu_N(Z) \sum_{\gamma\in\Gamma_N} N^{-N}\delta_{X=Z_\gamma}\mathrm{d}Z \tag{12.22}$$

$$= \int_{\Omega^N} \mu_N(Z) \left(N^{-1}\sum_{j=1}^N \delta_{z_j=(x_j,p_j)}\right)^{\otimes N} \mathrm{d}Z \tag{12.23}$$

where Z_σ is the $4N$-uple $(x_{\sigma(1)}, p_{\sigma(1)}, ..., x_{\sigma(N)}, p_{\sigma(N)}) = (z_{\sigma(1)}, ...z_{\sigma_N})$, $(z_i = (x_i, p_i))$ and Γ_N the set of applications from $\{1, ..., N\}$ to itself (which allows repeated indices). We look at the n-th marginal

$$\widetilde{\mu}_N^{(n)}(X_N) = \int_{\Omega^N} \mu_N(Z) \left(N^{-1}\sum_{j=1}^N \delta_{z_j=X_j}\right)^{\otimes n} \mathrm{d}Z. \tag{12.24}$$

In particular

$$\widetilde{\mu}_N^{(1)}(x_1, p_1) = N^{-1}\int_{\Omega^N} \mu_N(Z) \left(\sum_{j=1}^N \delta_{z_j=X_1}\right)\mathrm{d}Z$$

$$\widetilde{\mu}_N^{(2)}(x_1, p_1, x_2, p_2) = N^{-2}\int_{\Omega^N} \mu_N(Z) \left(\sum_{j=1}^N \delta_{z_j=X_1}\right)\left(\sum_{j=1}^N \delta_{z_j=X_2}\right)\mathrm{d}Z$$

$$\widetilde{\mu}_N^{(3)}(X_1, X_2, X_3) = N^{-3}\int_{\Omega^N} \mu_N(Z) \left(\sum_{j=1}^N \delta_{z_j=X_1}\right)\left(\sum_{j=1}^N \delta_{z_j=X_2}\right)\left(\sum_{j=1}^N \delta_{z_j=X_3}\right)\mathrm{d}Z$$

Expanding the n-th tensor power yields $\widetilde{\mu}_N^{(n)}$ in terms of the marginals of μ_N by an explicit calculation. □

2. Quantum de Finetti theorems for the bosonic case

Theorem 13.1 requires the use of a quantum de Finetti theorem, combined with a Fock-space localisation. We explain these two points in the following.

2.1. Quantum de Finetti theorems. In quantum mechanics a state is no longer a density but a trace-class operator (8.20) over the Hilbert space $L^2\left(\mathbb{R}^{dN}\right) \simeq L^2\left(\mathbb{R}^d\right)^{\otimes N}$. Such an operator Γ_N is said symmetric or bosonic if $\Gamma_N \in \mathcal{S}^1\left(L^2_{\mathrm{sym}}\left(\mathbb{R}^{dN}\right)\right)$

$$\mathcal{S}^1\left(L^2\left(\mathbb{R}^{dN}\right)\right) = \left\{\Gamma \in \mathfrak{S}^1\left(L^2\left(\mathbb{R}^{dN}\right)\right), \Gamma = \Gamma^*, \Gamma \geq 0, \mathrm{Tr}\,\Gamma = 1\right\}. \tag{12.25}$$

We define a bosonic state with infinitely many particles as a sequence $\left\{\gamma^{(k)}\right\}_{k\in\mathbb{N}}$ of trace-class operator satisfying

- Each bosonic k-particle state $\gamma^{(k)} \in \mathcal{S}^1\left(L^2_{\mathrm{sym}}\left(\mathbb{R}^{dN}\right)\right)$ is self-adjoint, positive and trace one
$$\mathrm{Tr}\left[\gamma^{(k)}\right] = 1 \tag{12.26}$$
- The sequence $\left\{\gamma^{(k)}\right\}_{k\in\mathbb{N}}$ is consistent
$$\mathrm{Tr}_{k+1}\left[\gamma^{(k+1)}\right] = \gamma^{(k)} \tag{12.27}$$
where Tr_{k+1} is the partial trace with respect to the last variable in $L^2\left(\mathbb{R}^d\right)^{\otimes(k+1)}$

Property (12.27) ensures that the sequence describes a real physical state. A particular case of these bosonic states are product states. A product state with infinitely many particles is a sequence of trace-class operators $\left\{\gamma^{(k)}\right\}_{k\in\mathbb{N}}$ satisfying

- For all $k \geq 0$ $\gamma^{(k)} = \gamma^{\otimes k}$ where γ is a one-particle state.
- This state is of the form
$$\gamma^{(k)} = \left(|u\rangle\langle u|\right)^{\otimes k} = \left|u^{\otimes k}\right\rangle\left\langle u^{\otimes k}\right| \tag{12.28}$$
with $u \in SL^2\left(\mathbb{R}^d\right)$ the unit sphere of the one-body Hilbert space.

Bosonic product states are all of the form (12.28). Indeed, if $\gamma \in \mathcal{S}^1\left(L^2_{\mathrm{sym}}\left(\mathbb{R}^{dN}\right)\right)$ is not a projector then $\gamma^{\otimes 2}$ cannot have bosonic symmetry [**72**]. The strong quantum de Finetti theorem below is due to Hudson and Moody [**72**] and Störmer [**153**].

THEOREM 12.4 (**Strong quantum de Finetti**).
Let $\mathfrak{H}$ be a complex separable Hilbert space and $\left\{\gamma^{(k)}\right\}_{k\in\mathbb{N}}$ a bosonic state with infinitely many particles on $\mathfrak{H}$. There exists a unique Borel probability measure $\mu \in \mathcal{P}\left(S\mathfrak{H}\right)$ on the sphere $S\mathfrak{H} = \{u \in \mathfrak{H}, \|u\| = 1\}$ of $\mathfrak{H}$, invariant under the action of S^1 (multiplication by a constant phase $e^{\mathrm{i}\theta}$, $\theta \in \mathbb{R}$) such that

$$\gamma^{(k)} = \int_{S\mathfrak{H}} \left|u^{\otimes k}\right\rangle\left\langle u^{\otimes k}\right| \mathrm{d}\mu\left(u\right) \tag{12.29}$$

for all $k \geq 0$.

This theorem allows to prove the exactness of the mean-field approximation (8.39) at the level of the ground state relatively easily see [**136**][Theorem 3.6] or [**87**]. Stronger results exist [**136**][Theorem 3.7 and Theorem 3.16] but we will follow the route of the finite dimensional one. The main advantage of working in finite dimensions is the possibility to

obtain explicit estimates, in the spirit of the Diaconis-Freedman theorem. We define the k-reduced density matrices of $\Gamma_N \in \mathcal{S}^1\left(\mathfrak{H}_{\mathrm{sym}}^N\right)$

$$\gamma_N^{(k)} := \mathrm{Tr}_{k+1\to N}\,\Gamma_N. \tag{12.30}$$

We introduce a finite dimensional complex Hilbert space $\mathfrak{H}$

$$\dim \mathfrak{H} = d \tag{12.31}$$

and state the

THEOREM 12.5 (**Quantitative quantum de Finetti**).
Let $\Gamma_N \in \mathcal{S}^1\left(\mathfrak{H}_{\mathrm{sym}}^N\right)$ be a bosonic state over $\mathfrak{H}_{\mathrm{sym}}^N$ and $\gamma_N^{(n)}$ its reduced density matrices. There exists a probability measure $\mu_N \in \mathcal{P}\left(S\mathfrak{H}\right)$ such that, denoting

$$\widetilde{\Gamma}_N = \int_{u\in S\mathfrak{H}} \left|u^{\otimes N}\right\rangle \left\langle u^{\otimes N}\right| \mathrm{d}\mu_N\left(u\right) \tag{12.32}$$

the associated state and $\widetilde{\gamma}_N^{(k)}$ its reduced density matrices, we have

$$\mathrm{Tr}_{\mathfrak{H}^k}\left|\gamma_N^{(n)} - \widetilde{\gamma}_N^{(n)}\right| \leq \frac{2n\left(d+2n\right)}{N} \tag{12.33}$$

for all $n = 1...N$.

This result is due to Christandl, König, Mitchison and Renner [**23**]. There also exists developments in [**21, 68, 84**] but we will be more interested in the quantum information variants [**20, 23, 24**]. There, the error in (12.33) which depends on the dimension linearly will only depends logarithmically on the dimension of the one-body Hilbert space. The trade-off is that the error is not quantified in the usual trace-class norm, and that the measure constructed there does not charge only bosonic states. This version of the de Finetti theorem is due to Brandão and Harrow [**18, 88**] see also [**135**][Proposition 3.2].

THEOREM 12.6 (**de Finetti representation of projected density matrices**).
Let $\mathfrak{H}$ be a complex separable Hilbert space with the corresponding bosonic space $\mathfrak{H}_N = \mathfrak{H}^{\otimes_{\mathrm{sym}} N}$. Let $\gamma_N^{(n)}$ be the n-body reduced density matrix of a N-body state vector $\Psi_N \in \mathfrak{H}_N$ or a general mixed state in:

$$\mathcal{S}^1\left(L^2_{\mathrm{sym}}\left(\mathbb{R}^{2N}\right)\right) = \left\{\Gamma \in \mathfrak{S}^1\left(L^2_{sym}\left(\mathbb{R}^{2N}\right)\right), \Gamma = \Gamma^*, \Gamma \geq 0, \mathrm{Tr}\,\Gamma = 1\right\} \tag{12.34}$$

Let P be a finite dimensional orthogonal projector.
There exists a positive Borel measure $\mu_N^{(n)}$ on the set of one-body mixed states

$$S_P = \mathcal{S}\left(PL^2_{\mathrm{sym}}\left(\mathbb{R}^{2N}\right)\right)$$

such that

$$\sup_{0\leq A_1,A_2,...A_n\leq 1} \mathrm{Tr}\left|A_1\otimes A_2\otimes ...\otimes A_n\left(P^{\otimes n}\gamma_N^{(n)}P^{\otimes n} - \int_{S_P}\gamma^{\otimes n}\mathrm{d}\mu_N^{(n)}(\gamma)\right)\right| \leq C_n\sqrt{\frac{\log(\dim(P))}{N}} \tag{12.35}$$

where the sup is over bounded hermitian operators on $P\mathfrak{H}$.

As in Theorem (12.4) the right hand side of (12.35) requires to work with a finite Hilbert space and then with localized states

$$P^{\otimes n}\gamma_N^{(n)}P^{\otimes n}. \tag{12.36}$$

We want to see $P^{\otimes n}\gamma_N^{(n)}P^{\otimes n}$ as the reduced density of a localized state Γ_P with all particles living on $P\mathfrak{H}$, i.e

$$P^{\otimes n}\gamma_N^{(n)}P^{\otimes n} = (\Gamma_P)^{(n)}. \tag{12.37}$$

The state Γ_P has then to live on the Fock space because particles can be lost in the localisation procedure.

2.2. Fock-space localisation. We define the symmetric Fock space as an infinite sum of k-particles symmetric Hilbert spaces

$$\mathcal{F}_{\mathrm{sym}}(\mathfrak{H}) = \mathbb{C}\oplus\mathfrak{H}\oplus\mathfrak{H}^2... \tag{12.38}$$

In our case we will work with

$$\mathcal{F}_{\mathrm{sym}}\left(L^2\left(\mathbb{R}^d\right)\right) = \mathbb{C}\oplus L^2\left(\mathbb{R}^d\right)\oplus L^2\left(\mathbb{R}^{2d}\right)\oplus ... \tag{12.39}$$

We then define the truncated Fock spaces

$$\mathcal{F}_{\mathrm{sym}}^{\leq N}\left(L^2\left(\mathbb{R}^d\right)\right) = \mathbb{C}\oplus L^2\left(\mathbb{R}^d\right)\oplus L^2\left(\mathbb{R}^{2d}\right)\oplus ...\oplus L^2\left(\mathbb{R}^{dN}\right) \tag{12.40}$$

We define a bosonic state on the Fock space to be a positive self-adjoint operator with trace one on $\mathcal{F}_{\mathrm{sym}}$.We denote the set of such states

$$\mathcal{S}\left(\mathcal{F}_{\mathrm{sym}}(\mathfrak{H})\right) = \left\{\Gamma\in\mathcal{S}^1\left(\mathcal{F}_{\mathrm{sym}}(\mathfrak{H})\right), \Gamma=\Gamma^*, \Gamma\geq 0, \mathrm{Tr}_{\mathcal{F}_{\mathrm{sym}}(\mathfrak{H})}[\Gamma]=1\right\} \tag{12.41}$$

A state is called block-diagonal if it can be written in the form

$$\Gamma = G_0\oplus G_1\oplus ...\oplus G_n\oplus ... \tag{12.42}$$

with $G_n\in\mathcal{S}^1\left(\mathfrak{H}_{\mathrm{sym}}^n\right)$. A block-diagonal state on the truncated space $\mathcal{F}_{\mathrm{sym}}^{\leq N}(\mathfrak{H})$ will be of the form

$$\Gamma = G_{0,N}\oplus G_{1,N}\oplus ...\oplus G_{N,N} \tag{12.43}$$

and its k-th reduced density matrix $\Gamma_N^{(k)}$ an operator on $\mathfrak{H}_{\mathrm{sym}}^k$ defined by

$$\Gamma_N^{(k)} = \binom{N}{k}^{-1}\sum_{j=k}^{N}\binom{j}{k}\mathrm{Tr}_{k+1\to j}\left[G_{N,j}^P\right] \tag{12.44}$$

For a diagonal state, the reduced density matrices (12.44) characterize the state completely.

LEMMA 12.7 (**Localisation of a N-body state**).
Let $\Gamma_N\in\mathcal{S}\left(\mathfrak{H}_{\mathrm{sym}}^N\right)$ be a bosonic N-body state and P a self-adjoint operator on $\mathfrak{H}$ with $0\leq P^2\leq 1$. There exists a unique diagonal state $\Gamma_N^P\in\mathcal{S}\left(\mathcal{F}_{\mathrm{sym}}^{\leq N}(\mathfrak{H})\right)$ such that

$$\left(\Gamma_N^P\right)^{(k)} = P^{\otimes k}\Gamma_N^{(k)}P^{\otimes k} \tag{12.45}$$

for all $0\leq n\leq N$. Moreover, writing Γ_N^P in the form

$$\Gamma_N^P = G_{0,N}^P\oplus G_{1,N}^P\oplus ...\oplus G_{N,N}^P \tag{12.46}$$

we have the fundamental relation

$$\mathrm{Tr}_{\mathfrak{H}^k_{\mathrm{sym}}}\left[G^P_{N,k}\right] = \mathrm{Tr}_{\mathfrak{H}^{N-k}_{\mathrm{sym}}}\left[G^{\sqrt{1-P^2}}_{N,N-k}\right] \tag{12.47}$$

A proof of the above can be found in [**136**][Lemma 5.2]. The relation (12.47) is central to the method. It says that in the state Γ_N the probability of having k particles P-localized is equal to the probability of having $N-k$ particles $\sqrt{1-P^2}$ localized. This result comes from [**80**] but other versions exist [**3, 39, 65**]. In this formalism the state (12.36) is given by

$$P^{\otimes n}\gamma_N^{(n)}P^{\otimes n} = (G^P_N)^{(n)} = \binom{N}{n}^{-1}\sum_{k=n}^{N}\binom{k}{n}\mathrm{Tr}\left[G^P_{N,k}\right] \tag{12.48}$$

for any $0 \leq n \leq N$.

Part 3

Average field approximation for almost-bosonic anyons in a magnetic field

Abstract. We study the ground state of a large number N of 2D extended anyons in an external magnetic field. We consider a scaling limit where the statistics parameter α is proportional to N^{-1} when $N \to \infty$ which allows the statistics to be seen as a"perturbation around the bosonic end". Our model is that of bosons in a magnetic field and interacting through long-range magnetic potential generated by magnetic charges carried by each particle, smeared over discs of radius R. Our method allows to take $R \to 0$ not too fast at the same time as $N \to \infty$: $R = N^{-1/4+\varepsilon}$. We use the information theoretic version of the de Finetti theorem of Brandão and Harrow to justify the so-called "average field approximation": the particles behave like independent, identically distributed bosons interacting via a self-consistent magnetic field.

CHAPTER 13

Introduction

This first study is a continuation of the quantum de Finetti-based approach to the mean-field limit of anyonic grounds states of [**112**] with the motivation to extend its results in two different ways. The first extension concerns the physical situation. Indeed we take into account an external magnetic field $\mathbf{B}_e$ absent in [**112**]. The second concerns the improvement of the range of valididy of the average field approximation thanks to an information-theoretic variant of the quantum de Finetti theorem due to Brandão and Harrow [**18, 88**], and revisited in [**135**].

In lower dimensions there are possibilities for quantum statistics other than the bosonic one and the fermionic one, so called intermediate or fractional statistics. Such particles, termed anyons (as in *any*thing in between bosons and fermions), could arise as effective quasiparticles in many-body quantum systems confined to lower dimensions. For an example of these intermediate exchange statistic properties see [**117, 119**] or [**116**]. These quasiparticles have been conjectured [**8**] to be relevant for the fractional quantum Hall effect (see [**61, 77, 120**] for review) and that is why we study the behavior of their ground state in a magnectic field.

1. The model

To describe anyons, the "wave function" has to (formally) behave as

$$\tilde{\Psi}(\mathbf{x}_1, ..., \mathbf{x}_j, ..., \mathbf{x}_k, ..., \mathbf{x}_N) = e^{\mathrm{i}\alpha\pi}\tilde{\Psi}(\mathbf{x}_1, ..., \mathbf{x}_k, ..., \mathbf{x}_j, ..., \mathbf{x}_N)$$

with $\alpha \in [0, 1]$ and we can obtain this by writing:

$$\tilde{\Psi}(\mathbf{x}_1, ..., \mathbf{x}_N) = \prod_{j<k} e^{\mathrm{i}\alpha\phi_{jk}}\Psi(\mathbf{x}_1, ..., \mathbf{x}_N) \text{ where } \phi_{jk} = \arg\frac{\mathbf{x}_j - \mathbf{x}_k}{|\mathbf{x}_j - \mathbf{x}_k|}$$

with Ψ a bosonic wave function, symmetric under particle exchange. Note that cases $\alpha = 1$ and $\alpha = 0$ are respectively the fermionic and the bosonic one. Let us now consider an usual Hamiltonian with a magnetic field $\mathbf{B}_e = \operatorname{curl} \mathbf{A}_e$ and a trapping potential V:

$$\mathcal{H}_N = \sum_{j=1}^{N}(-\mathrm{i}\nabla_j + \mathbf{A}_e(\mathbf{x}_j))^2 + V(\mathbf{x}_j)$$

with:

$$V(\mathbf{x}) \geq c|\mathbf{x}|^s - C, \quad s > 0.$$

One can show that acting with it on $\tilde{\Psi}$ is equivalent to acting on Ψ with the Hamiltonian:

$$H_N = \sum_{j=1}^{N} \left((\mathbf{p}_j + \mathbf{A}_e(\mathbf{x}_j) + \alpha \mathbf{A}(\mathbf{x}_j))^2 + V(\mathbf{x}_j) \right) \tag{13.1}$$

where, denoting $\mathbf{x}^\perp = (x, y)^\perp = (-y, x)$,

$$\mathbf{p}_j = -\mathrm{i}\nabla_j \text{ and } \mathbf{A}(\mathbf{x}_j) = \sum_{j \neq k} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|^2}. \tag{13.2}$$

The operator $\mathbf{A}(\mathbf{x}_j)$ is the statistical gauge vector potential felt by the particle j due to the influence of all the other particles. The statistics parameter is denoted by α, corresponding to a statistical phase $e^{i\alpha\pi}$ under a continuous simple interchange of two particles. In this so-called "magnetic gauge picture", 2D anyons are thus described as bosons, each of them carrying an Aharonov-Bohm magnetic flux of strength α.

We henceforth work with the Hamiltonian H_N acting on symmetric wave functions

$$\Psi \in \mathfrak{H} = L^2_{\mathrm{sym}}(\mathbb{R}^{2N}).$$

In this work we will assume that $\alpha \to 0$ when $N \to \infty$, a limit that we call "almost bosonic". More precisely we define a fixed constant β such that

$$\alpha = \frac{\beta}{N-1} \xrightarrow[N\to\infty]{} 0. \tag{13.3}$$

In this way, the anyon statistics has a leading order effect manifest through *the average field approximation.*

2. Extended anyons

The Hamiltonian (13.1) is too singular to be considered as acting on a pure tensor product $u^{\otimes N} \in \bigotimes^N_{\mathrm{sym}} \mathfrak{H}$, and, consequently, to consider a mean field limit. In order to circumvent this problem we will introduce a length R over which the equivalent magnetic charge is smeared. In our approach, $R \to 0$ will make us recover the point-like anyons point of view.

Let us consider the 2D Coulomb potential generated by a unit charge smeared over the disc of radius R:

$$w_R(\mathbf{x}) = (\log|\,.\,| * \chi_R)(\mathbf{x}), \text{ with the convention } w_0 = \log|\,.\,| \tag{13.4}$$

and $\chi_R(\mathbf{x})$ a positive regularizing function with a unit mass and built as follows

$$\chi_R = \frac{1}{R^2}\chi\left(\frac{\cdot}{R}\right) \text{ and } \int_{\mathbb{R}^2} \chi = 1$$

such that

$$\chi(\mathbf{x}) = \begin{cases} 1/\pi^2 & |\mathbf{x}| \leq 1 \\ 0 & |\mathbf{x}| \geq 2 \end{cases} \tag{13.5}$$

being smooth, positive and decreasing between 1 and 2. Since the magnetic field associated to $\mathbf{A}(\mathbf{x}_j)$ is:

$$\mathrm{curl}\mathbf{A}(\mathbf{x}_j) = 2\pi \sum_{k \neq j} \delta(\mathbf{x}_j - \mathbf{x}_k)$$

we will recover the same magnetic field in a distributional sense in the limit $R \to 0$ defining the magnetic potential vector:

$$\mathbf{A}^R(\mathbf{x}_j) := \sum_{k \neq j} \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) \tag{13.6}$$

leading to the regularized Hamiltonian

$$H_N^R = \sum_{j=1}^N \left(\left(\mathbf{p}_j + \mathbf{A}_e(\mathbf{x}_j) + \alpha \mathbf{A}^R(\mathbf{x}_j) \right)^2 + V(\mathbf{x}_j) \right). \tag{13.7}$$

We denote

$$E^R(N) = \inf \sigma \left(H_N^R \right) \tag{13.8}$$

with H_N^R acting on $L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$.

For $R > 0$ this operator is essentially self-adjoint on $L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$ (see [**129**, Theorem X.17] and [**10**]). It does however not have a unique limit as $R \to 0$ and the Hamiltonian at $R = 0$ (discussed e.g. in [**119**]) is *not* essentially self-adjoint (see for instance [**35, 38, 17**]). Nevertheless, in the joint limit $R \to 0$ and $N \to \infty$ we recover a unique well-defined (non-linear) model. This regularization method is a generalization of the one used in [**112**, equation 1.4] or in [**75**] where many-body magnetic Hardy inequalities and local exclusion principles are obtained. There,

$$\chi = \frac{\mathbb{1}_{B(0,R)}}{\pi R^2}(x)$$

was used but we will need a smoother χ, namely,

$$\int_{\mathbb{R}^2} |\widehat{\chi}(\mathbf{p})| \mathrm{d}\mathbf{p} < \infty$$

with $\widehat{\chi}$ the Fourier tranform of χ.
We denote :

$$\mathbf{p}_j^{\mathbf{A}} = \mathbf{p}_j + \mathbf{A}_e(\mathbf{x}_j).$$

We expand (13.7) in four terms that we will treat one by one:

$$\begin{aligned} H_R^N = & \sum_{j=1}^N \left((\mathbf{p}_j^{\mathbf{A}})^2 + V(\mathbf{x}_j) \right) \text{ “Kinetic and potential terms”} \\ & + \alpha \sum_{j \neq k} \left(\mathbf{p}_j^{\mathbf{A}} . \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) + \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) . \mathbf{p}_j^{\mathbf{A}} \right) \text{ “Mixed two-body term”} \\ & + \alpha^2 \sum_{j \neq k \neq l} \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) . \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_l) \text{ “Three-body term”} \\ & + \alpha^2 \sum_{j \neq k} \left| \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) \right|^2 \text{ “Singular two-body term”}. \end{aligned} \tag{13.9}$$

3. Average field approximation

The model we have built until now is still hard to study. That is why we will set:

$$\mathbf{A}[\rho] = \nabla^{\perp} w_0 * \rho \text{ and } \mathbf{A}^R[\rho] = \nabla^{\perp} w_R * \rho \tag{13.10}$$

which makes $\mathbf{A}$ independent of the precise positions $\mathbf{x}_j$. Here ρ is a fixed one-body density normalized in $L^1(\mathbb{R}^2)$. We obtain from this the average field N-body Hamiltonian:

$$H_N^{\mathrm{af}} = \sum_{j=1}^{N} \left((\mathbf{p}_j^{\mathbf{A}} + (N-1)\alpha \mathbf{A}^R[\rho])^2 + V(\mathbf{x}_j) \right). \tag{13.11}$$

This average field approximation Hamiltonian is the usual way to describe almost-bosonic anyons. It has also been studied in [**34, 32**] with, in these cases, β as a parameter. In this way, the magnetic field becomes:

$$\mathbf{B}(\mathbf{x}) = \mathrm{curl}\beta\mathbf{A}[\rho](\mathbf{x}) = 2\pi\beta\rho(\mathbf{x}).$$

We can now consider the usual mean field ansatz taking $\Psi(\mathbf{x}_1, ..., \mathbf{x}_N) = u^{\otimes N}$ with the consistency equation:

$$\rho(\mathbf{x}_1) = \int_{\mathbb{R}^{2(N-1)}} |\Psi(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N)|^2 \, \mathrm{d}\mathbf{x}_2 ... \mathrm{d}\mathbf{x}_N$$

which means $|u|^2 = \rho$ and get the one particle average-field energy functional:

$$\mathcal{E}_R^{\mathrm{af}}[u] = \int_{\mathbb{R}^2} \left(\left| (-\mathrm{i}\nabla + \mathbf{A}_e + \beta \mathbf{A}^R[|u|^2]) u \right|^2 + V|u|^2 \right) \tag{13.12}$$

with

$$E_R^{\mathrm{af}} = \inf \mathcal{E}_R^{\mathrm{af}}[u] \tag{13.13}$$

under the unit mass constraint:

$$\int_{\mathbb{R}^2} |u|^2 = 1.$$

Alternatively, we can write

$$\mathcal{E}_R^{\mathrm{af}}[u] = \int_{\mathbb{R}^2} \left(|\mathbf{p}^{\mathbf{A}} u|^2 + V|u|^2 + 2\beta \mathbf{A}^R[|u|^2] \cdot \left(\frac{\mathrm{i}}{2}(u\nabla\overline{u} - \overline{u}\nabla u) + \mathbf{A}_e|u|^2 + \frac{\beta}{2}\mathbf{A}^R[|u|^2]|u|^2 \right) \right) \mathrm{d}\mathbf{x}$$

Our aim is now well defined and consists in proving:

$$\frac{\inf \left\langle \Psi_N, H_N^R \Psi_N \right\rangle}{N} \approx \inf \left\{ \mathcal{E}_R^{\mathrm{af}}[u], \int |u|^2 = 1 \right\} \tag{13.14}$$

when $N \to \infty$. In other words, the average mean field energy is a good approximation for the ground state energy of H_N^R (see Proposition (15.7) in the appendix for a discussion about the existence of such minimizers).

4. Main results

We state our main theorem, justifying the average field approximation in the almost-bosonic limit at the level of the ground state. For technical reasons we assume that the one-body potential is confining

$$V(\mathbf{x}) \geq c|\mathbf{x}|^s - C, \quad s > 0, \tag{13.15}$$

and

$$\mathbf{A}_e \in L^2_{loc}(\mathbb{R}^2), \ \ \mathrm{curl}\mathbf{A}_e = \mathbf{B}_e = \mathbf{B}_0 + \tilde{\mathbf{B}}, \ \ \mathbf{B}_0 = \mathrm{Cst}, \ \ \tilde{\mathbf{B}} \in W^{1,2+\varepsilon}. \tag{13.16}$$

We also need that the size R of the extended anyons does not go to zero too fast in the limit $N \to \infty$ i.e $R = N^{-\eta}$ for η having to be determined.

Theorem 13.1 (**Validity of the average field approximation**).
Assume that we have N extended anyons of radius $R = N^{-\eta}$ for some

$$0 < \eta < \frac{1}{4} \tag{13.17}$$

and with the statistics parameter

$$\alpha = \frac{\beta}{(N-1)}$$

for fixed $\beta \in \mathbb{R}$. Then, in the limit $N \to \infty$ we have for the ground-state energy

$$\frac{E_R(N)}{N} \to E_0^{\mathrm{af}} = E^{\mathrm{af}}. \tag{13.18}$$

Moreover, if Ψ_N is a sequence of ground states for H_N^R, with associated reduced density matrices

$$\gamma_N^{(k)} := \mathrm{Tr}_{k+1\to N}\left[|\Psi_N\rangle\langle\Psi_N|\right]$$

then along a subsequence we have

$$\gamma_N^{(k)} \to \int_{\mathcal{M}^{\mathrm{af}}} |u^{\otimes k}\rangle\langle u^{\otimes k}| \, \mathrm{d}\mu(u) \tag{13.19}$$

strongly in trace class norm when $N \to \infty$, where μ is a Borel probability measure supported on the set of minimizers of $\mathcal{E}^{\mathrm{af}}$,

$$\mathcal{M}^{\mathrm{af}} := \{u \in L^2(\mathbb{R}^2) : \|u\|_{L^2} = 1, \, \mathcal{E}^{\mathrm{af}}[u] = E^{\mathrm{af}}\}.$$

There are two main improvements in this theorem compared with [**112**, Theorem 1.1]. The first is the removal of the constraint

$$\eta < \eta_0 = \frac{1}{4}\left(1 + \frac{1}{s}\right)^{-1}$$

providing us a less restritive result holding for every $s > 0$. We achieve this using the method of [**135**] which revisits the quantum de Finetti theorem due to Brandão and Harrow [**18**]. This implies to write the operators in (13.7) in a tensorized form (see (15.1)). This forces us to use a smooth χ complicating the estimate of the 3-body term (third line in (13.9)).

The second improvement is the addition of an external magnetic term $\mathbf{A}_e$ in H_N. Transforming $\mathbf{p}_j^0$ in $\mathbf{p}_j^{\mathbf{A}}$ the estimates of the cross product term becomes more difficult. The non commutativity

$$\left[\left(p_j^{\mathbf{A}}\right)_1, \left(p_j^{\mathbf{A}}\right)_2\right] \neq 0$$

causes this difficulty.

The needed new bounds will be the aim of the second section. Indeed, we will obtain a control of the Hamiltonian terms as a function of the kinetic energy in the limit $R \to \infty$. In the third section we combine these results with an a priori bound on the kinetic energy also derived in Section 2 and the de Finetti theorem to get a lower bound on $E^R(N)/N$ as E^{af} plus an error depending on R, N and a kinetic energy cut-off Λ that we finally optimize in the limit $N \to \infty$. We also prove the convergence of states in this section.
We finally derive some auxiliary results such as the convergence $E_R^{\mathrm{af}} \to E^{\mathrm{af}}$ or the existence of minimizers for $\mathcal{E}^{\mathrm{af}}$ in the appendix.

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CHAPTER 14

The extended anyon Hamiltonian

In this section we establish some bounds allowing to control the Hamiltonian (13.7). We can expand it as in (13.9) (and (15.41) for the energy expression) in order to estimate it term by term.

Because of the boundness of the interaction at fixed R, H_N^R is defined uniquely as a self-adjoint operator on $L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$ with the same form domain as the non-interacting bosonic Hamiltonian, (see [**92**, Proposition 7.20] for a definition of $H^1_{\mathbf{A}_e}$):

$$\sum_{j=1}^{N}\left(\left(\mathbf{p}_j^{\mathbf{A}}\right)^2+V(\mathbf{x}_j)\right),\ \ \mathbf{A}_e\in L^2_{\mathrm{loc}}.$$

As we will finally take the limit $R\to 0$ we need to deduce precise bounds depending on R assuming $R\ll R_0$, $R_0>0$ a fixed reference length scale. In the following, the generic constant C may implicitly depends on R_0. Bounds are here similar to [**112**] but with $\mathbf{p}_j^{\mathbf{A}}$ instead of $\mathbf{p}_j$ which complicates the proof of lemma 14.3 concerning the mixed two-body term. As for the three-body term (lemma 14.4), the new regularizing function χ makes its proof based on the Hardy inequality harder.

1. Operator bounds for the interaction terms

Exploiting the regularizing effect of taking $R>0$ we will estimate the different terms in (15.41) .

Lemma 14.1 (**The smeared Coulomb potential**).
Let w_R be defined as in (13.4). *There is a constant $C>0$ depending only on R_0 such that*

$$\sup_{\mathbb{R}^2}|\nabla w_R|\leq\frac{C}{R}\ \ \textit{and}\ \ ||\nabla w_R||_{L^p(\mathbb{R}^2)}\leq C_pR^{2/p-1}\tag{14.1}$$

for any $2<p<\infty$.

Proof. We apply Newton's theorem [**92**, Theorem 9.7] to calculate w_R in each domain:

$$|\nabla w_R(\mathbf{x})|=\begin{cases}1/|\mathbf{x}|,\ \ |\mathbf{x}|\in[2R,+\infty[\\ 1/|\mathbf{x}|\int_{B(0,|\mathbf{x}|)}\chi_R(u)\mathrm{d}u,\ \ |\mathbf{x}|\in]R,2R[\\ |\mathbf{x}|/\pi R^2,\ \ |\mathbf{x}|\in[0,R]\end{cases}\tag{14.2}$$

For $|\mathbf{x}|$ between 0 and $2R$ the given formula comes from:

$$w_R=\log|\mathbf{x}|\int_0^{2\pi}\int_0^{|\mathbf{x}|}\chi_R(y)y\mathrm{d}y\mathrm{d}\theta+\int_0^{2\pi}\int_{|\mathbf{x}|}^{2R}\log(y)\chi_R(y)y\mathrm{d}y\mathrm{d}\theta$$

which gives the first conclusion of the lemma.

To obtain the second we need to compute

$$||\nabla w_R||^p_{L^p(\mathbb{R}^2)} \leq 2\pi \int_0^R \frac{r^p}{\pi^p R^{2p}} r\mathrm{d}r + 2\pi \int_R^{+\infty} r^{-p} r\mathrm{d}r \leq C_p^p R^{2-p} \tag{14.3}$$

with $C_p > 0$ only depending on $p > 2$. □

LEMMA 14.2 (**Singular two-body term**).
For any $\varepsilon > 0$ we have as multiplication operators on $L^2(\mathbb{R}^4)$:

$$|\nabla w_R(\mathbf{x}_1 - \mathbf{x}_2)|^2 \leq C_\varepsilon R^{-\varepsilon} \left((\mathbf{p}_1^{\mathbf{A}})^2 + 1\right) \tag{14.4}$$

$$|\nabla w_R(\mathbf{x}_1 - \mathbf{x}_2)|^2 \leq C_\varepsilon R^{-\varepsilon} \left(\mathbf{p}_1^2 + 1\right) \tag{14.5}$$

PROOF. Taking any w: $\mathbb{R}^2 \mapsto \mathbb{R}$ and $f \in C_c^\infty(\mathbb{R}^4)$ with $u = |f|$, we compute

$$\begin{aligned}
\langle f, w(\mathbf{x}_1 - \mathbf{x}_2) f\rangle &= \langle u, w(\mathbf{x}_1 - \mathbf{x}_2) u\rangle \\
&= \iint_{\mathbb{R}^2\times\mathbb{R}^2} w(\mathbf{x}_1 - \mathbf{x}_2)|u(\mathbf{x}_1, \mathbf{x}_2)|^2 \mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2 \text{ because } w \text{ does not see the phase} \\
&\leq \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |w(\mathbf{x}_1 - \mathbf{x}_2)|^p \mathrm{d}\mathbf{x}_1\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^2} |u(\mathbf{x}_1, \mathbf{x}_2)|^{2q} \mathrm{d}\mathbf{x}_1\right)^{\frac{1}{q}} \mathrm{d}\mathbf{x}_2 b
\end{aligned}$$

by Hölder's inequality. Then, using Sobolev's inequality [**92**, Theorem 8.5 ii]

$$\begin{aligned}
\langle f, w(\mathbf{x}_1 - \mathbf{x}_2) f\rangle &\leq ||w||_{L^p} \int_{\mathbb{R}^2} ||u(., \mathbf{x}_2)||^2_{L^{2q}} \mathrm{d}\mathbf{x}_2 \\
&\leq C||w||_{L^p} \int_{\mathbb{R}^2} \left(||\nabla u(., \mathbf{x}_2)||^2_{L^2} + ||u(., \mathbf{x}_2)||^2_{L^2}\right) \mathrm{d}\mathbf{x}_2 \\
&= C||w||_{L^p} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(|\nabla u(\mathbf{x}_1, \mathbf{x}_2)|^2 + |u(\mathbf{x}_1, \mathbf{x}_2)|^2\right) \mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2 \\
&\leq C||w||_{L^p} \left\langle f, \left((\mathbf{p}_x^{\mathbf{A}})^2 + 1\right) f\right\rangle \text{ using the diamagnetic inequality (15.37)}
\end{aligned}$$

with $p > 1$ and $q = \frac{p}{p-1}$. We now use (14.1) with $w = |\nabla w_R|^2$ and $p = 1 + \varepsilon'$ to get

$$||w||_{L^p} = ||\nabla w_R||^2_{L^{2p}} \leq C_{2p}^2 R^{2/p-2} \leq C_\varepsilon R^{-\varepsilon}$$

with a constant $C_\varepsilon > 0$ for given $\varepsilon > 0$. The proof of the second inequality is the same without the diamagnetic inequality step. □

We next deal with the term mixing the position and the momentum. The first inequality we will get will have a bad R-dependence but a better behavior for large momenta than the other.

LEMMA 14.3 (**Mixed two-body term**).
Consider a magnetic field

$$\mathbf{B}_e = \mathrm{curl}\mathbf{A}_e = \mathbf{B}_0 + \tilde{\mathbf{B}}$$

with $\mathbf{B}_0 = \mathrm{Cst}$ and $\tilde{\mathbf{B}} \in W^{1,2+\varepsilon}$.

For $R < R_0$ and $\varepsilon > 0$ we have, as operators on $L^2_{sym}(\mathbb{R}^4)$:

$$\left|\mathbf{p}_1^{\mathbf{A}}.\nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2) + \nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2).\mathbf{p}_1^{\mathbf{A}}\right| \leq CR^{-1}|\mathbf{p}_1^{\mathbf{A}}| \tag{14.6}$$

$$\left|\mathbf{p}_1^{\mathbf{A}}.\nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2) + \nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2).\mathbf{p}_1^{\mathbf{A}}\right| \leq C_\varepsilon R^{-\varepsilon} \left(\left(\mathbf{p}_1^{\mathbf{A}}\right)^2 + 1\right) \tag{14.7}$$

$$\left|\mathbf{p}_1.\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2)+\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2).\mathbf{p}_1\right| \leq C_\varepsilon R^{-\varepsilon}(\mathbf{p}_1^2+1) \tag{14.8}$$

Proof. Proof of (14.6):
We first note that $\nabla_{\mathbf{x}_1}.\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2)=0$, consequently

$$\mathbf{p}_1^{\mathbf{A}}.\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2)=\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2).\mathbf{p}_1^{\mathbf{A}}$$

and we square the expression we want to estimate

$$\left(\mathbf{p}_1^{\mathbf{A}}.\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2)+\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2).\mathbf{p}_1^{\mathbf{A}}\right)^2 = 4\mathbf{p}_1^{\mathbf{A}}.\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2)\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2).\mathbf{p}_1^{\mathbf{A}}.$$

Then for any $f(\mathbf{x}_1,\mathbf{x}_2)\in C_c^\infty(\mathbb{R}^4)$

$$\begin{aligned}&\left|\left\langle f,\left(\mathbf{p}_1^{\mathbf{A}}.\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2)+\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2).\mathbf{p}_1^{\mathbf{A}}\right)^2 f\right\rangle\right|\\ &\leq 4\iint_{\mathbb{R}^2\times\mathbb{R}^2}\left|\mathbf{p}_1^{\mathbf{A}}f(\mathbf{x}_1,\mathbf{x}_2)\right|^2\left|\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2)\right|^2\mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2\end{aligned}$$

using (14.4) we get

$$\left|\left\langle f,\left(\mathbf{p}_1^{\mathbf{A}}.\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2)+\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2).\mathbf{p}_1^{\mathbf{A}}\right)^2 f\right\rangle\right| \leq \frac{C}{R^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\left|\mathbf{p}_1^{\mathbf{A}}f(\mathbf{x}_1,\mathbf{x}_2)\right|^2\mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2$$

$$\left(\mathbf{p}_1^{\mathbf{A}}.\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2)+\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2).\mathbf{p}_1^{\mathbf{A}}\right)^2 \leq \frac{C}{R^2}\left(\mathbf{p}_1^{\mathbf{A}}\right)^2$$

and we deduce (14.6) because the square root is operator monotone, see [**15**, Chapter 5].

For the proof of (14.7), writing $\mathbf{x}=(x,y)$ the coordinates of the particle located in $\mathbf{x}$, we start from

$$\begin{aligned}T=&\left|\left\langle f,\left(\mathbf{p}_1^{\mathbf{A}}.\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2)+\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2).\mathbf{p}_1^{\mathbf{A}}\right)^2 f\right\rangle\right|\\ &\leq 4\iint_{\mathbb{R}^2\times\mathbb{R}^2}\left|\mathbf{p}_1^{\mathbf{A}}f(\mathbf{x}_1,\mathbf{x}_2)\right|^2\left|\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2)\right|^2\mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2\\ &= 4\iint_{\mathbb{R}^2\times\mathbb{R}^2}\left(\left|p_x^{\mathbf{A}}f(\mathbf{x}_1,\mathbf{x}_2)\right|^2+\left|p_y^{\mathbf{A}}f(\mathbf{x}_1,\mathbf{x}_2)\right|^2\right)\left|\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2)\right|^2\mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2\\ &= 4\left(\left\langle p_x^{\mathbf{A}}f,\left|\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2)\right|^2 p_x^{\mathbf{A}}f\right\rangle+\left\langle p_y^{\mathbf{A}}f,\left|\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2)\right|^2 p_y^{\mathbf{A}}f\right\rangle\right)\\ &\leq \frac{C_\varepsilon}{R^\varepsilon}\left(\langle p_x^{\mathbf{A}}f,((\mathbf{p}_1^{\mathbf{A}})^2+1)p_x^{\mathbf{A}}f\rangle+\langle p_y^{\mathbf{A}}f,((\mathbf{p}_1^{\mathbf{A}})^2+1)p_y^{\mathbf{A}}f\rangle\right)\end{aligned}$$

Using the bound (14.4).
In order to recover $(\mathbf{p}_{x,y}^{\mathbf{A}})^2$ we have to calculate commutators involving $p_x^{\mathbf{A}}$ and $\left(\mathbf{p}_1^{\mathbf{A}}\right)^2$.

$$\begin{aligned}T&\leq \frac{C_\varepsilon}{R^\varepsilon}\left\langle f,((\mathbf{p}_1^{\mathbf{A}})^2+1)\left(\mathbf{p}_1^{\mathbf{A}}\right)^2 f\right\rangle+\frac{C_\varepsilon}{R^\varepsilon}\left\langle f,\left([p_x^{\mathbf{A}},\left(\mathbf{p}_y^{\mathbf{A}}\right)^2]p_x^{\mathbf{A}}+[p_y^{\mathbf{A}},\left(\mathbf{p}_x^{\mathbf{A}}\right)^2]p_y^{\mathbf{A}}\right)f\right\rangle\\ &\leq \frac{C_\varepsilon}{R^\varepsilon}\left\langle f,\left((\mathbf{p}_1^{\mathbf{A}})^2+1\right)^2 f\right\rangle+\frac{C_\varepsilon}{R^\varepsilon}\langle f,Ff\rangle\end{aligned}$$

To control F we calculate:

$$[p_x^{\mathbf{A}},p_y^{\mathbf{A}}]=-\mathrm{i}\left(\nabla\wedge\mathbf{A}_e\right)=-\mathrm{i}\mathbf{B}_e$$

Note that we are in two dimensions and that is why $\mathbf{B}_e$ is a scalar (physically carried by the axis perpendicular to the plane). We compute:

$$[p_x^{\mathbf{A}}, -\mathrm{i}\mathbf{B}_e] = -\partial_x \mathbf{B}_e \text{ and } [p_y^{\mathbf{A}}, -\mathrm{i}\mathbf{B}_e] = \partial_y \mathbf{B}_e.$$

Thus we obtain:

$$\begin{aligned} F &= p_y^{\mathbf{A}}(-\mathrm{i}\mathbf{B}_e)p_x^{\mathbf{A}} - \mathrm{i}\mathbf{B}_e p_y^{\mathbf{A}} p_x^{\mathbf{A}} - p_x^{\mathbf{A}}(-\mathrm{i}\mathbf{B}_e)p_y^{\mathbf{A}} + \mathrm{i}\mathbf{B}_e p_x^{\mathbf{A}} p_y^{\mathbf{A}} \\ &= (\partial_y \mathbf{B}_e - \mathrm{i}\mathbf{B}_e p_y^{\mathbf{A}})p_x^{\mathbf{A}} - \mathrm{i}\mathbf{B}_e p_y^{\mathbf{A}} p_x^{\mathbf{A}} - (\partial_x \mathbf{B}_e - \mathrm{i}\mathbf{B}_e p_x^{\mathbf{A}})p_y^{\mathbf{A}} + \mathrm{i}\mathbf{B}_e p_x^{\mathbf{A}} p_y^{\mathbf{A}} \\ &= \partial_y \mathbf{B}_e p_x^{\mathbf{A}} + \partial_x \mathbf{B}_e p_y^{\mathbf{A}} - 2\mathrm{i}\mathbf{B}_e(p_y^{\mathbf{A}} p_x^{\mathbf{A}} - p_x^{\mathbf{A}} p_y^{\mathbf{A}}) \\ &= 2\mathbf{B}_e^2 + (\partial_y \mathbf{B}_e)p_x^{\mathbf{A}} + (\partial_x \mathbf{B}_e)p_y^{\mathbf{A}} \\ &\leq 2\mathbf{B}_e^2 + \frac{1}{2}\left((\partial_x \mathbf{B}_e)^2 + (\partial_y \mathbf{B}_e)^2 + (p_x^{\mathbf{A}})^2 + (p_y^{\mathbf{A}})^2\right) \\ &= 2\mathbf{B}_e^2 + \frac{1}{2}|\nabla \mathbf{B}_e|^2 + \frac{1}{2}|\mathbf{p}^{\mathbf{A}}|^2 \end{aligned}$$

Here $\nabla \mathbf{B}_e$ is the vector $(\partial_x \mathbf{B}_e, \partial_y \mathbf{B}_e)$. Splitting the magnetic field as $\mathbf{B}_e = \mathbf{B}_0 + \tilde{\mathbf{B}}$ with $\mathbf{B}_0 = \mathrm{Cst}$ we get for the $\mathbf{B}_e^2 = \mathbf{B}_0^2 + 2\mathbf{B}_0.\tilde{\mathbf{B}} + \tilde{\mathbf{B}}^2$ and the $(\nabla \mathbf{B}_e)^2 = (\nabla \tilde{\mathbf{B}})^2$ terms:

$$\left\langle f, \mathbf{B}_e^2 f \right\rangle = \mathbf{B}_0^2 + 2\mathbf{B}_0 \left\langle f, \tilde{\mathbf{B}} f \right\rangle + \left\langle f, \tilde{\mathbf{B}}^2 f \right\rangle$$

$$\leq \mathbf{B}_0^2 + \left(2\mathbf{B}_0 ||\tilde{\mathbf{B}}||_{L^p(\mathbb{R}^2)} + ||\tilde{\mathbf{B}}||^2_{L^{2p}(\mathbb{R}^2)}\right) \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |f(x,y)|^{2q} \, \mathrm{d}x \right)^{1/q} \mathrm{d}y \leq C \left\langle f, \left(\left(\mathbf{p}_1^{\mathbf{A}}\right)^2 + 1 \right) f \right\rangle$$

$$\left\langle f, (\nabla \tilde{\mathbf{B}})^2 f \right\rangle \leq ||(\nabla \tilde{\mathbf{B}})^2||_{L^p} \int_{\mathbb{R}^2} ||f||^2_{L^{2q}(\mathbb{R}^2)} \mathrm{d}y \leq C ||(\nabla \tilde{\mathbf{B}})^2||_{L^p} \left\langle f, \left(\left(\mathbf{p}_1^{\mathbf{A}}\right)^2 + 1 \right) f \right\rangle$$

using Hölder's inequality with $1 = 1/p + 1/q$, that $\mathbf{B}$ and $(\nabla \mathbf{B})^2$ are multiplicator operators and the Sobolev' inequality [**92**, Theorem 8.5 ii]:

$$||g||^2_{L^{2q}(\mathbb{R}^2)} \leq C_q \left(||\nabla g||^2_{L^2(\mathbb{R}^2)} + ||g||^2_{L^2(\mathbb{R}^2)} \right)$$

for $1 \leq q < \infty$.
So because we have $\tilde{\mathbf{B}} \in W^{1,2+\varepsilon}$:

$$F \leq C\left((\mathbf{p}_1^{\mathbf{A}})^2 + 1\right)$$

and we get

$$T \leq \frac{C_\varepsilon}{R^\varepsilon} \left\langle f, \left((\mathbf{p}_1^{\mathbf{A}})^2 + 1\right)^2 f \right\rangle$$

$$\left(\mathbf{p}_1^{\mathbf{A}}.\nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2) + \nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2).\mathbf{p}_1^{\mathbf{A}}\right)^2 \leq \frac{C_\varepsilon}{R^\varepsilon}\left((\mathbf{p}_1^{\mathbf{A}})^2 + 1\right)^2$$

and the result follows by taking the square root. The proof of the third inequality (14.8) is a simpler version of what we just did. □

LEMMA 14.4 (**Three-body term**).
We have that, as multiplication operators on $L^2_{sym}(\mathbb{R}^6)$*:*

$$\left|\nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2).\nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_3)\right| \leq C\left(\left(\mathbf{p}_1^{\mathbf{A}}\right)^2 + 1\right) \tag{14.9}$$

$$\left|\nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2).\nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_3)\right| \leq C\left(\mathbf{p}_1^2 + 1\right) \tag{14.10}$$

To prove this lemma we will use the three-particle Hardy inequality [**71**, Lemma 3.6] as done in [**112**, Lemma 2.4] but with our new $\chi \neq \mathbb{1}_{B(0,R)}$ which implies to take into account the sign of each considered term. The second difference is that we do not use Clifford algebra.

LEMMA 14.5 (**Three-body Hardy inequality**,[**71**].).
Let $d = 2$ and $u : \mathbb{R}^6 \to \mathbb{C}$. Let $\mathcal{R}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ be the circumradius of the triangle with vertices $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$, and $\rho(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \sqrt{|\mathbf{x}-\mathbf{y}|^2 + |\mathbf{y}-\mathbf{z}|^2 + |\mathbf{z}-\mathbf{x}|^2}$. Then $\mathcal{R}^{-2} \leq 9\rho^{-2}$ pointwise, and, for all $u \in H^1$

$$3\int_{\mathbb{R}^6} \frac{|u(\mathbf{x},\mathbf{y},\mathbf{z})|^2}{\rho(\mathbf{x},\mathbf{y},\mathbf{z})^2} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{z} \leq \int_{\mathbb{R}^6} \left(|\nabla_{\mathbf{x}} u|^2 + |\nabla_{\mathbf{y}} u|^2 + |\nabla_{\mathbf{z}} u|^2 \right) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{z} \tag{14.11}$$

PROOF OF LEMMA 14.4. Since we consider the multiplication operator as acting on symmetric wave functions it is equivalent to estimate the sum:

$$S = \sum_{\mathbf{x},\mathbf{y},\mathbf{z}} \nabla w_R(\mathbf{x}-\mathbf{y}).\nabla w_R(\mathbf{x}-\mathbf{z}) = \sum_{\mathbf{x},\mathbf{y},\mathbf{z}} \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}.\frac{\mathbf{x}-\mathbf{z}}{|\mathbf{x}-\mathbf{z}|} v(\mathbf{x}-\mathbf{y}) v(\mathbf{x}-\mathbf{z}) \tag{14.12}$$

where $\mathbf{x}, \mathbf{y}, \mathbf{z}$ under the sum means cyclic in $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and $v(u) = \partial_u w_R(u)$ is a radial function whose expression has been calculated in (14.2) such that

$$v\left(|\mathbf{x}|\right) = |\nabla w_R(\mathbf{x})| = \begin{cases} 1/|\mathbf{x}|, & |\mathbf{x}| \in [2R, +\infty[\\ 1/|\mathbf{x}| \int_{B(0,|\mathbf{x}|)} \chi_R(u)\mathrm{d}u, & |\mathbf{x}| \in]R, 2R[\\ |\mathbf{x}|/\pi R^2, & |\mathbf{x}| \in [0, R] \end{cases} \tag{14.13}$$

In general, let each $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$ denote the vertices of a triangle. Then we claim the following geometric fact:

$$|S| = \left| \sum_{\mathbf{x},\mathbf{y},\mathbf{z}} \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}.\frac{\mathbf{x}-\mathbf{z}}{|\mathbf{x}-\mathbf{z}|} v(\mathbf{x}-\mathbf{y}) v(\mathbf{x}-\mathbf{z}) \right| \leq \frac{C}{\rho(\mathbf{x},\mathbf{y},\mathbf{z})^2}. \tag{14.14}$$

where C is a positive contant independent of R. Observe that for $R \leq |\mathbf{x}| \leq 2R$, $v(\mathbf{x}) \geq 0$ and that

$$\frac{C_1}{R} \leq v(\mathbf{x}) \leq \frac{C_2}{R}$$

where $C_1 \leq C_2$ are two non-negative constants independent of R. We will proceed in several steps respectively considering the cases:

(1) All lengths of the triangle are larger than $2R$

(2) All lengths are smaller than $2R$

(3) Two of the lengths are smaller than $2R$ and one greater.

(4) Two of the lengths are larger than $2R$ and one smaller.

and establish for each the inequality $S \leq C\rho^{-2}$. The second, $S \geq -C\rho^{-2}$ being easy or similar, depending on the case, will not be treated.

Step 1. All lengths of the triangle are greater than $2R$ i.e $|\mathbf{x}| \geq 2R$. Then

$$S = \sum_{\mathbf{x},\mathbf{y},\mathbf{z}} \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2}.\frac{\mathbf{x}-\mathbf{z}}{|\mathbf{x}-\mathbf{z}|^2} \tag{14.15}$$

We proceed as in [**71**, Lemma 3.2]: $a = \mathbf{x}-\mathbf{y}$, $b = \mathbf{x}-\mathbf{z}$ and ϕ the angle between a and b. A simple computation of S leads to:

$$\begin{aligned} S &= \frac{a.b}{|a|^2|b|^2} - \frac{a}{|a|^2}.\frac{(b-a)}{|b-a|^2} - \frac{b}{|b|^2}.\frac{(a-b)}{|b-a|^2} \\ &= \frac{2(|a|^2|b|^2-(a.b)^2)}{|a|^2|b|^2|b-a|^2} = \frac{2\sin(\phi)^2}{|\mathbf{y}-\mathbf{z}|^2} = \frac{1}{2}R^{-2}(\mathbf{x},\mathbf{y},\mathbf{z}) \leq \frac{9}{2}\rho^{-2} \end{aligned}$$

by Lemma 14.5.

Step 2. Before initiating the proof we need to enlight some general facts about this geometric configuration. Indeed we will need to know which term of the sum is positive or negative in each case we will study. First, as the sum of the triangle's angles is π we know that if one of them is larger than $\pi/2$ the others have to be smaller. That implies that only one of the scalar products of the sum can be negative in each configuration and we are able to identify it. For example if $\widehat{\mathbf{xyz}} \geq \pi/2$ then the first term of the sum has to be negative and the two others positive. If all of them are under $\pi/2$, then the sum only consists of positive terms. This case being easier and involving the same reasonning than the others we will systematically ignore it.

Here all lengths are smaller than $2R$ and the negative term is the second. This term can be bounded in three different ways. First:

$$\frac{(\mathbf{y}-\mathbf{z}).(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{z}||\mathbf{y}-\mathbf{x}|}v(\mathbf{y}-\mathbf{z})v(\mathbf{y}-\mathbf{x}) \leq \frac{(\mathbf{y}-\mathbf{z}).(\mathbf{y}-\mathbf{x})}{\pi^2 R^4}$$

if $|\mathbf{y}-\mathbf{z}|$ and $|\mathbf{y}-\mathbf{x}|$ are smaller than R (here the inequality is actually an equality). Or it can be bounded as

$$\frac{(\mathbf{y}-\mathbf{z}).(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{z}||\mathbf{y}-\mathbf{x}|}v(\mathbf{y}-\mathbf{z})v(\mathbf{y}-\mathbf{x}) \leq C_1\frac{(\mathbf{y}-\mathbf{z}).(\mathbf{y}-\mathbf{x})}{2R^4}$$

if only $|\mathbf{y}-\mathbf{z}|$ or $|\mathbf{y}-\mathbf{x}|$ is smaller than R and by

$$\frac{(\mathbf{y}-\mathbf{z}).(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{z}||\mathbf{y}-\mathbf{x}|}v(\mathbf{y}-\mathbf{z})v(\mathbf{y}-\mathbf{x}) \leq \frac{C_1^2}{4}\frac{(\mathbf{y}-\mathbf{z}).(\mathbf{y}-\mathbf{x})}{R^4}$$

if $|\mathbf{y}-\mathbf{z}|$ and $|\mathbf{y}-\mathbf{x}|$ are between R and $2R$. Arguing similarly for the positive terms we get

$$\begin{aligned} R^4 S \leq{}& \max\left(1, C_2, C_2^2\right)(\mathbf{x}-\mathbf{y}).(\mathbf{x}-\mathbf{z}) + \min\left(1, C_1/2, C_1^2/4\right)(\mathbf{y}-\mathbf{z}).(\mathbf{y}-\mathbf{x}) \\ &+ \max\left(1, C_2, C_2^2\right)(\mathbf{z}-\mathbf{x}).(\mathbf{z}-\mathbf{y}) \\ ={}& \max\left(1, C_2, C_2^2\right)((\mathbf{x}-\mathbf{y}).(\mathbf{x}-\mathbf{z}) + (\mathbf{y}-\mathbf{z}).(\mathbf{y}-\mathbf{x}) + (\mathbf{z}-\mathbf{x}).(\mathbf{z}-\mathbf{y})) \\ &+ \left(\min\left(1, C_1/2, C_1^2/4\right) - \max\left(1, C_2, C_2^2\right)\right)(\mathbf{y}-\mathbf{z}).(\mathbf{y}-\mathbf{x}) \end{aligned}$$

and we obtain two positive constants C_3 and C_4 such that

$$|S| \leq \frac{C_3}{R^4}\rho^{-2} \leq C_4\rho^{-2}. \tag{14.16}$$

Step 3. We consider the case where two of the edges are short and one long. Let's take $|\mathbf{x}-\mathbf{y}|, |\mathbf{y}-\mathbf{z}| \leq 2R$ and $|\mathbf{x}-\mathbf{z}| \geq 2R$. In this configuration the negative term is the second. As we did in Step 2 we start by writing

$$\begin{aligned} S_3 \leq & \max(1, C_2)\frac{(\mathbf{x}-\mathbf{y}).(\mathbf{x}-\mathbf{z})}{R^2|\mathbf{x}-\mathbf{z}|^2} + \min\left(1, C_1/2, C_1^2\right)\frac{(\mathbf{y}-\mathbf{z}).(\mathbf{y}-\mathbf{x})}{R^4} \\ & + \max(1, C_2)\frac{(\mathbf{y}-\mathbf{z}).(\mathbf{y}-\mathbf{x})}{R^2|\mathbf{x}-\mathbf{z}|^2} \\ = & \max(1, C_2)\left[\frac{(\mathbf{x}-\mathbf{y}).(\mathbf{x}-\mathbf{z})}{R^2|\mathbf{x}-\mathbf{z}|^2} + \frac{(\mathbf{y}-\mathbf{z}).(\mathbf{y}-\mathbf{x})}{R^4} + \frac{(\mathbf{y}-\mathbf{z}).(\mathbf{y}-\mathbf{x})}{R^2|\mathbf{x}-\mathbf{z}|^2}\right] \\ & + \frac{(\mathbf{y}-\mathbf{z}).(\mathbf{y}-\mathbf{x})}{R^4}\left[\min\left(1, C_1/2, C_1^2\right) - \max(1, C_2)\right] \end{aligned}$$

From there, we proceed as in [**112**, Lemma 2.4]. For the first term of the above

$$\begin{aligned} & (\mathbf{x}-\mathbf{y}).(\mathbf{x}-\mathbf{z})R^2 + (\mathbf{y}-\mathbf{z}).(\mathbf{y}-\mathbf{x})|\mathbf{x}-\mathbf{z}|^2 + R^2(\mathbf{z}-\mathbf{x}).(\mathbf{z}-\mathbf{y}) \\ & = |\mathbf{x}-\mathbf{z}|^2(\mathbf{y}-\mathbf{z}).(\mathbf{y}-\mathbf{x}) + R^2(\mathbf{x}-\mathbf{z})(\mathbf{x}-\mathbf{y}-\mathbf{z}+\mathbf{y}) \\ & \leq 5R^2|\mathbf{x}-\mathbf{z}|^2 \end{aligned}$$

and for the second

$$\left|\frac{(\mathbf{y}-\mathbf{z}).(\mathbf{y}-\mathbf{x})}{R^4}\right||\mathbf{x}-\mathbf{z}|^2 \leq \frac{4}{R^2}|\mathbf{x}-\mathbf{z}|^2.$$

Comparing to

$$\begin{aligned} |\mathbf{x}-\mathbf{z}|^2R^4\rho^{-2}\frac{|\mathbf{x}-\mathbf{z}|^2R^4}{|\mathbf{x}-\mathbf{z}|^2+|\mathbf{x}-\mathbf{y}|^2+|\mathbf{y}-\mathbf{z}|^2} & \geq \frac{R^4|\mathbf{x}-\mathbf{z}|^2}{4R^2+|\mathbf{x}-\mathbf{z}|^2} \\ & \geq \frac{R^2}{20}|\mathbf{x}-\mathbf{z}|^2 \end{aligned}$$

because $|\mathbf{x}-\mathbf{z}| \leq |\mathbf{x}-\mathbf{y}| + |\mathbf{y}-\mathbf{z}| \leq 4R$, we conclude that

$$|S| \leq C_5\rho^{-2}$$

Step 4. We now study the last case when only one edge is short and the two others long. Let's take $|\mathbf{x}-\mathbf{y}| \leq 2R$ and $|\mathbf{y}-\mathbf{z}|, |\mathbf{z}-\mathbf{x}| \geq 2R$. We first notice that the two longer edges have to be of the same order:

$$\begin{aligned} |\mathbf{y}-\mathbf{z}| \leq |\mathbf{y}-\mathbf{x}| + |\mathbf{x}-\mathbf{z}| \leq 2|\mathbf{x}-\mathbf{z}| \\ |\mathbf{x}-\mathbf{z}| \leq |\mathbf{x}-\mathbf{y}| + |\mathbf{y}-\mathbf{z}| \leq 2|\mathbf{y}-\mathbf{z}| \\ \frac{1}{2}|\mathbf{x}-\mathbf{z}| \leq |\mathbf{y}-\mathbf{z}| \leq 2|\mathbf{x}-\mathbf{z}|. \end{aligned} \tag{14.17}$$

We write

$$|\mathbf{x}-\mathbf{y}|^2|\mathbf{x}-\mathbf{z}|^2|\mathbf{y}-\mathbf{z}|^2 S = \sum_{\mathbf{x},\mathbf{y},\mathbf{z}} (\mathbf{x}-\mathbf{y}).(\mathbf{x}-\mathbf{z})|\mathbf{y}-\mathbf{z}|^2|\mathbf{x}-\mathbf{y}||\mathbf{x}-\mathbf{z}|v(\mathbf{x}-\mathbf{y})v(\mathbf{x}-\mathbf{z})$$

where $|\mathbf{x}-\mathbf{z}|v(\mathbf{x}-\mathbf{z}) = 1$. We can now evaluate the right side of the inequality (14.14) multiplied by $|\mathbf{x}-\mathbf{y}|^2|\mathbf{x}-\mathbf{z}|^2|\mathbf{y}-\mathbf{z}|^2$.

$$\rho^{-2}|\mathbf{x}-\mathbf{y}|^2|\mathbf{x}-\mathbf{z}|^2|\mathbf{z}-\mathbf{y}|^2 \geq C_6 R^2 \frac{|\mathbf{x}-\mathbf{z}|^4}{|\mathbf{x}-\mathbf{z}|^2+|\mathbf{z}-\mathbf{y}|^2} \geq C_6 R^2 |\mathbf{x}-\mathbf{z}|^2$$

because $|\mathbf{x}-\mathbf{y}|^2 \leq |\mathbf{x}-\mathbf{z}|^2$. All that is left is to show that

$$\begin{aligned}
&(\mathbf{x}-\mathbf{y}).(\mathbf{x}-\mathbf{z})|\mathbf{z}-\mathbf{y}|^2 + (\mathbf{y}-\mathbf{z}).(\mathbf{y}-\mathbf{x})|\mathbf{x}-\mathbf{z}|^2 \\
=&|\mathbf{z}-\mathbf{y}|^2|\mathbf{x}-\mathbf{y}|^2 + |\mathbf{z}-\mathbf{y}|^2(\mathbf{x}-\mathbf{y}).(\mathbf{y}-\mathbf{z}) + |\mathbf{x}-\mathbf{z}|^2|\mathbf{x}-\mathbf{y}|^2 + |\mathbf{x}-\mathbf{z}|^2(\mathbf{x}-\mathbf{y}).(\mathbf{z}-\mathbf{x}) \\
=&|\mathbf{x}-\mathbf{z}|^2|\mathbf{x}-\mathbf{y}|^2 + (\mathbf{x}-\mathbf{y}).(\mathbf{z}-\mathbf{x})\left[|\mathbf{x}-\mathbf{z}|^2 - |\mathbf{z}-\mathbf{y}|^2\right] \\
=&|\mathbf{x}-\mathbf{z}|^2|\mathbf{x}-\mathbf{y}|^2 + (\mathbf{x}-\mathbf{y}).(\mathbf{z}-\mathbf{x})\left[(|\mathbf{z}-\mathbf{x}| + |\mathbf{z}-\mathbf{y}|)\,(|\mathbf{z}-\mathbf{x}| - |\mathbf{z}-\mathbf{y}|)\right] \\
\leq&C_7 R^2 |\mathbf{x}-\mathbf{z}|^2
\end{aligned}$$

Using the triangular inequality

$$||\mathbf{z}-\mathbf{x}| - |\mathbf{z}-\mathbf{y}|| \leq |\mathbf{z}-\mathbf{x}-(\mathbf{z}-\mathbf{y})|$$

so we deduce

$$\begin{aligned}
|\mathbf{x}-\mathbf{y}|^2|\mathbf{x}-\mathbf{z}|^2|\mathbf{y}-\mathbf{z}|^2 S = |\mathbf{x}-\mathbf{y}|v(\mathbf{x}-\mathbf{y})\left[(\mathbf{x}-\mathbf{y}).(\mathbf{x}-\mathbf{z})|\mathbf{z}-\mathbf{y}|^2 + (\mathbf{y}-\mathbf{z}).(\mathbf{y}-\mathbf{x})|\mathbf{x}-\mathbf{z}|^2\right] \\
+ (\mathbf{z}-\mathbf{x}).(\mathbf{z}-\mathbf{y})|\mathbf{x}-\mathbf{y}|^2 \leq C_8 R^2 |\mathbf{x}-\mathbf{z}|^2
\end{aligned}$$

and conclude that

$$|S| \leq C\rho^{-2}.$$

Now we have

$$\left|\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2).\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_3)\right| \leq C\left((\mathbf{p}_1)^2+1\right)$$

using Lemma 14.5. Then, as a multiplication operator $\nabla^\perp w_R$ does not see the phase so

$$\begin{aligned}
\left\langle u \left|\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2).\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_3)\right| u\right\rangle &= \left\langle |u| \left|\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2).\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_3)\right| |u|\right\rangle \\
&\leq C\left(|\nabla|u||^2 + |u|^2\right) \\
&\leq C\left\langle u, \left(\left(\mathbf{p}_1^{\mathbf{A}}\right)^2+1\right) u\right\rangle
\end{aligned}$$

which gives the expected result using the diamagnetic inequality.

□

2. A priori bound for the ground state

For the estimates of the previous subsection to apply efficiently, we need an a priori bound on ground states (or approximate ground states) of the N-body Hamiltonian (13.7), provided in the following:

PROPOSITION 14.6 (**A priori bound for many-body ground states**).
Let $\Psi_N \in L^2_{sym}(\mathbb{R}^{2N})$ be a sequence of approximate ground state for H_N^R that is:

$$\left\langle \Psi_N, H_N^R \Psi_N \right\rangle \leq E^R(N)(1+o(1)) \ \ when \ \ N \to \infty$$

Denote by $\gamma_N^{(1)}$ the associated sequence of one-body density matrices. In the regime $\alpha = \beta/(N-1) \to 0$, assuming a bound $R \geq N^{-\eta}$ for some $\eta > 0$ independent of N, we have:

$$\mathrm{Tr}\left[\left(\left(\mathbf{p}^{\mathbf{A}}\right)^2 + V\right)\gamma_N^{(1)}\right] \leq C(1+\beta)E_R^{\mathrm{af}} \tag{14.18}$$

PROOF. We proceed in two steps:
Step 1. Using a trial state $u^{\otimes N}$ such that $\mathcal{E}_R^{\mathrm{af}}[u] = E_R^{\mathrm{af}}$ we can obtain from (15.41), (15.44),(15.45) and from the bounds (14.5), (14.8) and (14.10) plus the diamagnetic inequality (15.38):

$$\begin{aligned}
\frac{E^R(N)}{N} &\leq E_R^{\mathrm{af}} - \frac{\beta^2}{N-1}\mathrm{Tr}\left[\left(\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2).\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_3)\right)\gamma_N^{\mathrm{af}(3)}\right] \\
&\quad + \frac{\beta^2}{N-1}\mathrm{Tr}\left[\left|\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2)\right|^2 \gamma_N^{\mathrm{af}(2)}\right] \\
&\leq E_R^{\mathrm{af}} + \left(E_R^{\mathrm{af}}+1\right)\left(\frac{C\beta^2}{N-1} + \frac{C_\varepsilon \beta^2 R^{-\varepsilon}}{N-1}\right) \\
&= E_R^{\mathrm{af}} + o_N(1) \to E^{\mathrm{af}}
\end{aligned} \tag{14.19}$$

since $R = N^{-\eta}$ with $N \to \infty$, and we used Proposition (15.9).

We now use the diamagnetic inequality in each variable to obtain:

$$\begin{aligned}
\left\langle \Psi_N, H_N^R \Psi_N \right\rangle &= \sum_{j=1}^N \int_{\mathbb{R}^{2N}} \left(\left|(-\mathrm{i}\nabla_j + \mathbf{A}_e + \alpha \mathbf{A}_j^R)\Psi_N\right|^2 + V(\mathbf{x}_j)|\Psi_N|^2\right) \mathrm{d}\mathbf{x}_1 ... \mathrm{d}\mathbf{x}_N \\
&\geq \sum_{j=1}^N \int_{\mathbb{R}^{2N}} \left(\left|\nabla_j |\Psi_N|\right|^2 + V(\mathbf{x}_j)|\Psi_N|^2\right) \mathrm{d}\mathbf{x}_1 ... \mathrm{d}\mathbf{x}_N
\end{aligned}$$

We deduce

$$\mathrm{Tr}\left[\left(\mathbf{p}^2 + V\right)\gamma_{N,+}^{(1)}\right] \leq C E_R^{\mathrm{af}} \tag{14.20}$$

where we denote the reduced k-body density matrix of $|\Psi_N|$

$$\gamma_{N,+}^{(k)} = \mathrm{Tr}_{k+1\to N}\left[|\left|\Psi_N\right|\rangle\langle|\Psi_N|\right|]$$

Step 2. We expand the Hamiltonian and use the Cauchy-Schwarz inequality for operators to get:

$$\begin{aligned} H_N^R &= \sum_{j=1}^N \left(\left(\mathbf{p}_j^{\mathbf{A}}\right)^2 + \alpha \mathbf{p}_j^{\mathbf{A}}.\mathbf{A}_j^R + \alpha \mathbf{A}_j^R.\mathbf{p}_j^{\mathbf{A}} + \alpha^2 |\mathbf{A}_j^R|^2 + V(\mathbf{x}_j) \right) \\ &\geq \sum_{j=1}^N \left((1-2\delta^{-1}) \left(\mathbf{p}_j^{\mathbf{A}}\right)^2 + (1-2\delta)\alpha^2 |\mathbf{A}_j^R|^2 + V(\mathbf{x}_j) \right) \\ &= \sum_{j=1}^N \left(\frac{1}{2}(\left(\mathbf{p}_j^{\mathbf{A}}\right)^2 + V(\mathbf{x}_j)) - 7\frac{\beta^2}{(N-1)^2} |\mathbf{A}_j^R|^2 \right) \end{aligned}$$

choosing $\delta = 4$. Thus using (14.19) we have:

$$\mathrm{Tr}\left[\left(\left(\mathbf{p}^{\mathbf{A}}\right)^2 + V \right) \gamma_N^{(1)} \right] \leq C E_R^{\mathrm{af}} + \frac{C\beta}{N(N-1)^2} \left\langle \Psi_N, \sum_{j=1}^N |\mathbf{A}_j^R|^2 \Psi_N \right\rangle \tag{14.21}$$

and since the second term is a purely potential term which does not see the phase:

$$\left\langle \Psi_N, \sum_{j=1}^N |\mathbf{A}_j^R|^2 \Psi_N \right\rangle = \left\langle |\Psi_N|, \sum_{j=1}^N |\mathbf{A}_j^R|^2 |\Psi_N| \right\rangle.$$

We can now expand the square and use (14.5) and (14.8) to obtain:

$$\begin{aligned} &\frac{1}{N(N-1)^2} \left\langle |\Psi_N|, \sum_{j=1}^N |\mathbf{A}_j^R|^2 |\Psi_N| \right\rangle \leq C\mathrm{Tr}\left[\left(\nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2).\nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_3) \right) \gamma_{N,+}^{(3)} \right] \\ &+ CN^{-1}\mathrm{Tr}\left[\left| \nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2) \right|^2 \gamma_{N,+}^{(2)} \right] \\ &\leq C\mathrm{Tr}\left[(\mathbf{p}_1^2 + 1) \otimes \mathbb{1} \otimes \mathbb{1} \gamma_{N,+}^{(3)} \right] + C_\varepsilon R^{-\varepsilon} N^{-1} \mathrm{Tr}\left[(\mathbf{p}_1^2 + 1) \otimes \mathbb{1} \gamma_{N,+}^{(2)} \right] \\ &\leq C(1 + C_\varepsilon R^{-\varepsilon} N^{-1}) \mathrm{Tr}\left[(\mathbf{p}_1^2 + 1) \gamma_{N,+}^{(1)} \right]. \end{aligned}$$

Recalling that $V + C \geq 0$, we conclude by inserting the estimate (14.20) and coming back to (14.21).

□

CHAPTER 15

Mean-field limit

We start the study of the mean-field limit. We use the same strategy as in [**112**] but another version of the quantum de Finetti theorem. This version comes from quantum information theory (see [**88, 18**]) and provides us with less restrictive constraint for the domain of validity of our main theorem but forces us to put all operators under a tensorized form. We mean: each N-particles operator Θ will be written under the form:

$$\Theta = \int \Theta_1(\mathbf{p}) \otimes \Theta_2(\mathbf{p}) ... \otimes \Theta_N(\mathbf{p}) \mathrm{d}P(\mathbf{p}) \tag{15.1}$$

where each Θ_i is a 1-particle operator.
The rest is similar to what is done in [**112**] combining with the methods of [**135**, Results and discussion].

1. Energy cut-off

We introduce the spectral projector P of the operator $h = \left(\mathbf{p}^{\mathbf{A}}\right)^2 + V$ below an energy cut-off Λ to be eventually optimized later.

$$P = \mathbb{1}_{h \leq \Lambda}, \quad Q = \mathbb{1} - P \tag{15.2}$$

We denote

$$N_\Lambda = \dim\left(PL^2(\mathbb{R}^2)\right)$$

the number of energy levels obtained in this way and recall the following Cwikel-Lieb-Rozenblum type inequality proved in [**83**, Lemma 3.3](see [**26, 143**] for more details).

LEMMA 15.1 (**Number of energy levels below the cut-off**).
For Λ large enough we have

$$N_\Lambda \leq C\Lambda^{1+2/s} \tag{15.3}$$

2. Localisation in Fock space

Let γ_N be an arbitrary N-body mixed state. The Fock space

$$\mathcal{F}(\mathfrak{H}) = \mathbb{C} \oplus \mathfrak{H} \oplus \mathfrak{H}^2 \oplus ...$$

$$\mathcal{F}_{\mathrm{sym}}(L^2) = \mathbb{C} \oplus L^2\left(\mathbb{R}^2\right) \oplus L^2_{\mathrm{sym}}\left(\mathbb{R}^4\right) \oplus ...$$

A (diagonal) state on the truncated Fock space $\mathcal{F}^{\leq N}_{\mathrm{sym}}(\mathfrak{H})$ can be written in the form

$$\Gamma_N = G_{0,N} \oplus G_{1,N} \oplus ... \oplus G_{N,N} \tag{15.4}$$

which is a positive self-adjoint trace-class one operator on $\mathcal{F}_{\mathrm{sym}}^{\leq N}(\mathfrak{H})$. Associated with the projector P, there are localisated states $G_{N,k}^P$

$$G_{N,k}^P := \binom{N}{k} \mathrm{Tr}_{k+1\to N}\left[P^{\otimes k}\otimes Q^{\otimes(N-k)}\Gamma_N P^{\otimes k}\otimes Q^{\otimes(N-k)}\right] \tag{15.5}$$

The property that the density matrices satisfy is

$$P^{\otimes n}\gamma_N^{(n)}P^{\otimes n} = (G_N^P)^{(n)} = \binom{N}{n}^{-1}\sum_{k=n}^{N}\binom{k}{n}\mathrm{Tr}\left[G_{N,k}^P\right] \tag{15.6}$$

for any $0 \leq n \leq N$.
We use the convention that

$$\gamma_N^{(n)} = \mathrm{Tr}_{n+1\to N}\left[\gamma_N\right].$$

The localized state G_N^Q corresponding to the projector Q is defined similarly. The relations (15.6) determine the localized states G_N^P and G_N^Q uniquely ensuring they are mixed states on the projected Fock spaces $\mathcal{F}(P\mathfrak{H})$ and $\mathcal{F}(Q\mathfrak{H})$, respectively:

$$\sum_{k=0}^{N}\mathrm{Tr}\left[G_{N,k}^{P/Q}\right] = 1 \tag{15.7}$$

The precise construction can be found in [**133**, Chapter 5]. We will now apply the de Finetti representation of projected density matrices of [**135**, Proposition 3.2] in order to approximate its three-body density matrix. This inequality combine Fock-space localisation and the information-theoretic quantum de Finetti theorem ([**135**, Appendix A]) as we can find in the proof of [**135**, Proposition 3.2].

3. Quantum de Finetti theorem

We call $\mathfrak{S}^1$ the set of trace-class operators.

THEOREM 15.2 (**de Finetti representation of projected density matrices**). *Let $\mathfrak{H}$ be a complex separable Hilbert space with the corresponding bosonic space $\mathfrak{H}_N = \mathfrak{H}^{\otimes_{\mathrm{sym}}N}$. Let $\gamma_N^{(3)}$ be the 3-body reduced density matrix of a N-body state vector $\Psi_N \in \mathfrak{H}_N$ or a general mixed state in:*

$$\mathcal{S}^1\left(L^2_{\mathrm{sym}}\left(\mathbb{R}^{2N}\right)\right) = \left\{\Gamma \in \mathfrak{S}^1\left(L^2_{\mathrm{sym}}\left(\mathbb{R}^{2N}\right)\right), \Gamma = \Gamma^*, \Gamma \geq 0, \mathrm{Tr}\,\Gamma = 1\right\}$$

Let P be a finite dimensional orthogonal projector on $L^2\left(\mathbb{R}^2\right)$.
There exists a positive Borel measure $\mu_N^{(3)}$ on the set of one-body mixed states

$$S_P = \mathcal{S}^1\left(PL^2\left(\mathbb{R}^2\right)\right)$$

such that for all A, B and C hermitian operators:

$$\mathrm{Tr}\left|A\otimes B\otimes C\left(P^{\otimes 3}\gamma_N^{(3)}P^{\otimes 3} - \int_{S_P}\gamma^{\otimes 3}\mathrm{d}\mu_N^{(3)}(\gamma)\right)\right| \leq C\sqrt{\frac{\log(\dim(P))}{N}}||A||.||B||.||C|| \tag{15.8}$$

where the norm $||A||_{\mathrm{op}}$ is the operator norm defined as usual by: $\sup_{||u||=1}||Au||_{L^2(\mathbb{R}^2)}$.

The theorem gives us a better dependency in $\dim(P)$. This comes at the cost of the measure charging general mixed states $\gamma^{\otimes 3}$ instead of $(|u\rangle\langle u|)^{\otimes 3}$. Comparing to [**135**, Proposition 3.2] we extend the theorem to all hermitian operators.

PROOF. To begin, we call

$$\Delta\gamma^{(3)} = \left(P^{\otimes 3}\gamma_N^{(3)}P^{\otimes 3} - \int \gamma^{\otimes 3}\mathrm{d}\mu_N^{(3)}(\gamma)\right)$$

where the measure μ_N is defined as in [**135**] and $A = A^+ + A^- = \Pi_{A\geq 0}A + \Pi_{A\leq 0}A$ with Π an orthogonal projector acting on $P\mathfrak{H}$ defined by the projection on the positive and negative part of A (recall that this operator describes an observables and is thus hermitian). The first term is then

$$\begin{aligned}
&\mathrm{Tr}\left|A\otimes B\otimes C\left(\Delta\gamma^{(3)}\right)\right| \\
=&\mathrm{Tr}\left|\sum_{(i,j,k)\in\{+,-\}}||A^i||.||B^j||.||C^k||\frac{A^i\otimes B^j\otimes C^k\left(\Delta\gamma^{(3)}\right)}{||A^i||.||B^j||.||C^k||}\right| \\
\leq& C\sqrt{\frac{\log(\dim(P))}{N}}\sum_{(i,j,k)\in\{+,-\}}||A^i||.||B^j||.||C^k||
\end{aligned}$$

using the fact that $\frac{A^i}{||A^i||}$ is a bounded by 1 operator which allows us to use [**135**, Theorem A.3] with the method of [**135**, Proposition 3.2] but considering three operators instead and using that $||A^+|| \leq ||A||$, $||A^-|| \leq ||A||$. □

3.1. Truncated Hamiltonian. Our aim is now to find an energy lower bound to H_N^R. We introduce the effective three-body Hamiltonian $\tilde{H}_3^R$ such that for $||u|| = 1$:

$$\left\langle u^{\otimes 3}, \tilde{H}_3^R u^{\otimes 3}\right\rangle = \mathcal{E}_R^{\mathrm{af}}[u] \geq E_R^{\mathrm{af}}$$

and denote

$$\mathrm{Tr}\left[\tilde{H}_3^R\gamma^{\otimes 3}\right] = \mathcal{E}_R^{\mathrm{af}}[\gamma].$$

Posing

$$h = \left(\mathbf{p}^{\mathbf{A}}\right)^2 + V \tag{15.9}$$

we note:

$$\begin{aligned}
\tilde{H}_3^R =& \frac{1}{3}(h_1+h_2+h_3) \\
&+\frac{\beta}{6}\sum_{1\leq j\neq k\leq 3}\left(\mathbf{p}_j^{\mathbf{A}}.\nabla^\perp w_R(\mathbf{x}_j-\mathbf{x}_k) + \nabla^\perp w_R(\mathbf{x}_j-\mathbf{x}_k).\mathbf{p}_j^{\mathbf{A}}\right) \\
&+\beta^2\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2).\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_3)
\end{aligned} \tag{15.10}$$

We denote for shortness

$$W_2 = \mathbf{p}_1.\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2) + \nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2).\mathbf{p}_1 \tag{15.11}$$

the two-body part and

$$W_3 = \nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_2).\nabla^\perp w_R(\mathbf{x}_1-\mathbf{x}_3) \tag{15.12}$$

the three-body part. Thus

$$\tilde{H}_3^R = \frac{1}{3}(h_1 + h_2 + h_3) + \frac{\beta}{6} \sum_{1 \leq j \neq k \leq 3} W_2(j,k) + \beta^2 W_3$$

We now bound the full energy from below in terms of a projected version of $\tilde{H}_3^R$.

LEMMA 15.3 (**Truncated three-body Hamiltonian**).
Let Ψ_N be a sequence of approximate ground states for H_N^R with associated reduced matrices $\gamma_N^{(k)}$. Then for $\varepsilon > 0$ and R small enough:

$$N^{-1} \left\langle \Psi_N, H_N^R \Psi_N \right\rangle \geq \mathrm{Tr}\left[\tilde{H}_3^R P^{\otimes 3} \gamma_N^{(3)} P^{\otimes 3}\right] + C\mathrm{Tr}\left[hQ\gamma_N^{(1)}Q\right] - C_\beta \left(\frac{C_\varepsilon R^{-\varepsilon}}{\sqrt{\Lambda} R} + \frac{C}{\Lambda R^2}\right) - C_\beta N^{-1} \tag{15.13}$$

PROOF. The proof is quite the same as in [**112**, Proposition 3.3]. We just have to use our new bounds (14.4),(14.7), (14.9) and Proposition 14.6 instead of those of the original paper. □

4. Application of the de Finetti theorem

We need to apply the de Finetti Theorem 15.2 to the first term of (15.13) in order to get our energy

$$\int_{S_P} \mathrm{Tr}[\tilde{H}_3^R \gamma^{\otimes 3}] \mathrm{d}\mu_N^{(3)}(\gamma)$$

with a controled error.

LEMMA 15.4 (**Error for the truncated Hamiltonian**).

$$\mathrm{Tr}\left[\tilde{H}_3^R P^{\otimes 3} \gamma_N^{(3)} P^{\otimes 3}\right] \geq \int_{S_P} \mathcal{E}_R^{\mathrm{af}}[\gamma] d\mu_N^{(3)}(\gamma) - \frac{C_\beta}{R^2}\left(R^2\Lambda + R\sqrt{\Lambda} + 1\right)\sqrt{\frac{\log N_\Lambda}{N}} \tag{15.14}$$

PROOF. The only thing we have to do to apply Theorem 15.2 is to put our operators under a tensorized form (15.1).

Step 1. Our first operator is h_i defined in (15.9). As it is a one particle observable its tensorized form is simply

$$h_1 \otimes \mathbb{1} \otimes \mathbb{1}$$

if $i = 1$.

Step 2. To put W_2 defined in (15.11) in the form (15.1) we will use the Fourier transform (FT[.]) of $\nabla^\perp w_R(x)$. We can then write

$$W_2 = -2\pi \mathrm{i} \int \widehat{\chi}_R \frac{\mathbf{e}_{p^\perp}}{|\mathbf{p}|} e^{-\mathrm{i}\mathbf{p}.\mathbf{x}_2} \mathbf{p}_1^{\mathbf{A}} e^{\mathrm{i}\mathbf{p}.\mathbf{x}_1} \mathrm{d}\mathbf{p} - 2\pi \mathrm{i} \int \widehat{\chi}_R \frac{\mathbf{e}_{p^\perp}}{|\mathbf{p}|} e^{-\mathrm{i}\mathbf{p}.\mathbf{x}_2} e^{\mathrm{i}\mathbf{p}.\mathbf{x}_1} \mathbf{p}_1^{\mathbf{A}} \mathrm{d}\mathbf{p}$$

where $\widehat{\chi}_R = \mathrm{FT}[\chi_R]$ and with $\mathbf{e}_{p^\perp}$ being the direction of the vector $\mathbf{p}^\perp$ such that $||\mathbf{e}_{p^\perp}|| = 1$. In order to bring together the two parts of W_2 we note that

$$\mathbf{p}_1^{\mathbf{A}} e^{\mathrm{i}\mathbf{p}.\mathbf{x}_1} \Psi = (\mathbf{p}_1 + \mathbf{A}_e) e^{\mathrm{i}\mathbf{p}.\mathbf{x}_1} \Psi = \left(\mathrm{i}\mathbf{p} e^{\mathrm{i}\mathbf{p}.\mathbf{x}_1} + e^{\mathrm{i}\mathbf{p}.\mathbf{x}_1} \mathbf{p}_1^{\mathbf{A}}\right) \Psi$$

and obtain

$$W_2 = -2\mathrm{i}\pi \int \widehat{\chi}_R(\mathbf{p}) \frac{\mathbf{e}_{p^\perp}}{|\mathbf{p}|} e^{-\mathrm{i}\mathbf{p}.\mathbf{x}_2} e^{\mathrm{i}\mathbf{p}.\mathbf{x}_1}. \left[\mathrm{i}\mathbf{p} + 2\mathbf{p}_1^{\mathbf{A}}\right] \mathrm{d}\mathbf{p} \tag{15.15}$$

$$= -4\mathrm{i}\pi \int \frac{\widehat{\chi}_R(\mathbf{p})}{|\mathbf{p}|} e^{-\mathrm{i}\mathbf{p}.\mathbf{x}_2} \otimes e^{\mathrm{i}\mathbf{p}.\mathbf{x}_1} \mathbf{e}_{p^\perp}.\mathbf{p}_1^{\mathbf{A}} \otimes \mathbb{1} \mathrm{d}\mathbf{p} \tag{15.16}$$

because $\mathbf{e}_{p^\perp}.\mathbf{p} = 0$. The last problem is that $e^{\mathrm{i}px}$ is a complex number and we need hermitian operators. We rewrite

$$\mathrm{i}e^{-\mathrm{i}\mathbf{p}.\mathbf{x}_2} e^{\mathrm{i}\mathbf{p}.\mathbf{x}_1} = \mathrm{i}\left[\cos(\mathbf{p}.\mathbf{x}_1) + \mathrm{i}\sin(\mathbf{p}.\mathbf{x}_1)\right]\left[\cos(\mathbf{p}.\mathbf{x}_2)) - \mathrm{i}\sin(\mathbf{p}.\mathbf{x}_2)\right] \tag{15.17}$$

$$= \mathrm{i}\left[\cos(\mathbf{p}.\mathbf{x}_1)\cos(\mathbf{p}.\mathbf{x}_2) + \sin(\mathbf{p}.\mathbf{x}_1)\sin(\mathbf{p}.\mathbf{x}_2)\right] \tag{15.18}$$

$$+ \left[\sin(\mathbf{p}.\mathbf{x}_2)\cos(\mathbf{p}.\mathbf{x}_1) - \sin(\mathbf{p}.\mathbf{x}_1)\cos(\mathbf{p}.\mathbf{x}_2)\right] \tag{15.19}$$

Here the imaginary part is an even function whereas $\widehat{\chi}_R(\mathbf{p})\frac{\mathbf{e}_{p^\perp}}{|\mathbf{p}|}$ is odd. Consequently this part of the integral gives zero and we get:

$$W_2 = 4\pi \int \frac{\widehat{\chi}_R}{|\mathbf{p}|} \left(\cos(\mathbf{p}.\mathbf{x}_2) \otimes \sin(\mathbf{p}.\mathbf{x}_1) - \sin(\mathbf{p}.\mathbf{x}_2) \otimes \cos(\mathbf{p}.\mathbf{x}_1)\right) \mathbf{e}_{p^\perp}.\mathbf{p}_1^{\mathbf{A}} \otimes \mathbb{1} \mathrm{d}\mathbf{p}. \tag{15.20}$$

Step 3. We now consider the three-body operator W_3 defined in (15.12).

$$W_3 = \int \widehat{\chi}_R(\mathbf{p}) \frac{\mathbf{e}_{p^\perp}}{|\mathbf{p}|} \mathrm{i}e^{\mathrm{i}\mathbf{p}.(\mathbf{x}_1 - \mathbf{x}_2)} \mathrm{d}\mathbf{p}. \int \widehat{\chi}_R(\mathbf{p}') \frac{\mathbf{e}_{p'^\perp}}{|\mathbf{p}'|} \mathrm{i}e^{\mathrm{i}\mathbf{p}'.(\mathbf{x}_1 - \mathbf{x}_3)} \mathrm{d}\mathbf{p}'$$

Then, rewriting the exponential term as in (15.19) we obtain the tensorized form:

$$\begin{aligned} W_3 = \int \widehat{\chi}_R(\mathbf{p})\widehat{\chi}_R(\mathbf{p}') \frac{\mathbf{e}_{p^\perp}}{|\mathbf{p}|}.\frac{\mathbf{e}_{p'^\perp}}{|\mathbf{p}'|} \\ \Big[\cos(\mathbf{p}.\mathbf{x}_1)\cos(\mathbf{p}'.\mathbf{x}_1) \otimes \sin(\mathbf{p}.\mathbf{x}_2) \otimes \sin(\mathbf{p}'.\mathbf{x}_3) \\ + \sin(\mathbf{p}.\mathbf{x}_1)\sin(\mathbf{p}'.\mathbf{x}_1) \otimes \cos(\mathbf{p}.\mathbf{x}_2) \otimes \cos(\mathbf{p}'.\mathbf{x}_3) \\ - \cos(\mathbf{p}.\mathbf{x}_1)\sin(\mathbf{p}'.\mathbf{x}_1) \otimes \sin(\mathbf{p}.\mathbf{x}_2) \otimes \cos(\mathbf{p}'.\mathbf{x}_3) \\ - \sin(\mathbf{p}.\mathbf{x}_1)\sin(\mathbf{p}'.\mathbf{x}_1) \otimes \cos(\mathbf{p}.\mathbf{x}_2) \otimes \cos(\mathbf{p}'.\mathbf{x}_3)\Big] \mathrm{d}\mathbf{p}\mathrm{d}\mathbf{p}' \end{aligned}$$

Step 4. All that is left is to compute the entire Hamiltonian and to apply (p being fixed) Theorem 15.2. Thus we get

$$\left|\mathrm{Tr}\left[\frac{1}{3}\sum_{i=1}^{3} P^{\otimes 3}\mathbb{1} \otimes h_i \otimes \mathbb{1}P^{\otimes 3}\left(\Delta\gamma^{(3)}\right)\right]\right| \leq C\sqrt{\frac{\log N_\Lambda}{N}}||PhP|| \leq C\Lambda\sqrt{\frac{\log N_\Lambda}{N}} \tag{15.21}$$

Recalling that

$$\Delta\gamma^{(3)} = \left(P^{\otimes 3}\gamma_N^{(3)}P^{\otimes 3} - \int \gamma^{\otimes 3}\mathrm{d}\mu_N^{(3)}(\gamma)\right)$$

We do the same for W_2 and W_3 using that $||\cos(x.p)||_{\rm op} \leq 1$ as for the sine, and the triangle inequality

$$\left|\mathrm{Tr}\left[\frac{\beta}{6}\sum_{1\leq j\neq k\leq 3} P^{\otimes 3}W_2(i,j)\otimes \mathbb{1}P^{\otimes 3}\left(\Delta\gamma^{(3)}\right)\right]\right| \leq C_\beta\sqrt{\frac{\log N_\Lambda}{N}}||P\mathbf{e}_{p^\perp}.\mathbf{p}^{\mathbf{A}}P||\int\left|\frac{\widehat{\chi}_R(\mathbf{p})}{\mathbf{p}}\right|\mathrm{d}\mathbf{p}$$
$$\leq \frac{C_\beta\sqrt{\Lambda}}{R}\sqrt{\frac{\log N_\Lambda}{N}} \tag{15.22}$$

where we used

$$\widehat{\chi}_R(\mathbf{p}) = \widehat{\chi}(R\mathbf{p})$$

and for the third term

$$\left|\mathrm{Tr}\left[\beta^2 P^{\otimes 3}W_3P^{\otimes 3}\left(\Delta\gamma^{(3)}\right)\right]\right| \leq C_\beta\sqrt{\frac{\log N_\Lambda}{N}}\int\left|\frac{\widehat{\chi}_R(\mathbf{p})}{\mathbf{p}}\frac{\widehat{\chi}_R(\mathbf{p}')}{\mathbf{p}'}\right|\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{p}' \leq \frac{C_\beta}{R^2}\sqrt{\frac{\log N_\Lambda}{N}} \tag{15.23}$$

We now combine (15.21), (15.22) and (15.23) to get (15.14) by the triangle inequality. □

5. Energy bound

As we did in (14.19) we test (15.41) against a factorized trial state $\Psi_N = \left(u_R^{\rm af}\right)^{\otimes N}$ such as $\mathcal{E}_R^{\rm af}[\Psi_N] = E_R^{\rm af}$ to get an energy upper bound.

$$\frac{E^R(N)}{N} \leq \mathcal{E}_R^{\rm af}[u_R^{\rm af}] + o(1) \to E^{\rm af} \tag{15.24}$$

as $R = N^{-\eta}$ with $N \to \infty$ we also used (15.40) to get $E_R^{\rm af} \to E^{\rm af}$ when $R \to 0$.

We derive the lower bound by using the quantitative quantum de Finetti for localized states Theorem 15.2 on the first term of the estimate (15.13) and we get for any sequence of ground states Ψ_N that

$$N^{-1}\left\langle\Psi_N, H_N^R\Psi_N\right\rangle \geq \int_{S_P}\mathcal{E}_R^{\rm af}[\gamma]\mathrm{d}\mu_N^{(3)}(\gamma) + C\Lambda\mathrm{Tr}\left[Q\gamma_N^1\right]$$
$$-\frac{C_\beta}{R^2}\left(R^2\Lambda + R\sqrt{\Lambda}+1\right)\sqrt{\frac{\log N_\Lambda}{N}}$$
$$-C_\beta\left(\frac{1}{N}+\frac{C_\varepsilon R^{-\varepsilon}}{\sqrt{\Lambda}R}+\frac{1}{\Lambda R^2}\right) \tag{15.25}$$

To get this result we used (15.14) and will use (15.3) to bound N_Λ.

We now choose Λ to minimize the error in (15.13) with R behaving at worst as

$$R = N^{-\eta}$$

Changing a little bit η to $\eta-\varepsilon$ we can ignore the R^ε factor. Our choice of Λ has to minimize

$$\left(\frac{C_\varepsilon}{\sqrt{\Lambda}R}+\frac{1}{\Lambda R^2}\right)+\frac{1}{\sqrt{N}}\left(\Lambda+\frac{\sqrt{\Lambda}}{R}+\frac{1}{R^2}\right)$$

We see that the limiting terms are $\frac{\Lambda}{\sqrt{N}}$ and $\frac{1}{\Lambda R^2}$. We pick $\Lambda = N^{2\eta}$ to minimize their sum, with

$$\eta < \frac{1}{4}$$

Under this assumption we can get rid of the $C\mathrm{Tr}\left[Q\gamma_N^1\right]$ when $\Lambda \to \infty$. Indeed, we claim that:

$$\mathcal{E}_R^{\mathrm{af}}[\gamma] \geq 0 \tag{15.26}$$

Then it follows from (15.25) that for a constant C

$$C \geq \Lambda \mathrm{Tr}\left[Q\gamma_N^{(1)}\right] - C_\eta o_N(1)$$

and the only way the last inequality remains true when $\Lambda \to \infty$ is

$$\mathrm{Tr}\left[Q\gamma_N^{(1)}\right] \xrightarrow[N\to\infty]{} 0. \tag{15.27}$$

To see that (15.26) holds and because γ can be a mixed state it is convenient to write its kernel

$$\gamma(\mathbf{x}_1;\mathbf{x}_2) = \sum_j \lambda_j u_j(\mathbf{x}_1)\overline{u_j(\mathbf{x}_2)}, \quad \lambda_i \geq 0$$

Then

$$\begin{aligned}
\mathrm{Tr}\left[\tilde{H}_3^R \gamma^{\otimes 3}\right] &= \sum_i \lambda_i \int_{\mathbb{R}^2} h|u_i(\mathbf{x}_1)|^2 \mathrm{d}\mathbf{x} + \beta \sum_{i,j} \lambda_i\lambda_j \iint_{\mathbb{R}^2\times\mathbb{R}^2} W_2 |u_i(\mathbf{x}_1)|^2 |u_j(\mathbf{x}_2)|^2 \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2 \\
&+ \beta^2 \sum_{i,j,k} \lambda_i\lambda_j\lambda_k \iiint_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} W_3 |u_i(\mathbf{x}_1)|^2 |u_j(\mathbf{x}_2)|^2 |u_k(\mathbf{x}_3)|^2 \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2 \mathrm{d}\mathbf{x}_3 \\
&= \sum_i \lambda_i \int_{\mathbb{R}^2} \left(\left| \left(\nabla + i\mathbf{A}_e + i\beta \mathbf{A}^R \left[\sum_i \lambda_j |u_j|^2 \right] \right) u_i \right|^2 + V|u_i|^2 \right) \\
&\geq \sum_i \lambda_i \int |\nabla |u_i||^2 \geq 0
\end{aligned}$$

by (15.38). We conclude that:

$$N^{-1} \left\langle \Psi_N, H_N^R \Psi_N \right\rangle \geq \int_{S_P} \mathcal{E}_R^{\mathrm{af}}[\gamma] \mathrm{d}\mu_N^{(3)}(\gamma) + o(1).$$

6. Convergence of states

We first prove that the de Finetti measures converge to a measure living on S_P. We next show that it only charges pure states.

Convergence of the de Finetti measure.
Given the previous constructions and energy estimates, the proof of (13.19) follows almost exactly the one used in [**135**].
From Proposition 14.6 we know that

$$\left(\left(\mathbf{p}^{\mathbf{A}}\right)^2 + V \right) \gamma_N^{(1)}$$

is uniformly bounded in trace-class. Under our assumptions,

$$\left(\left(\mathbf{p}^{\mathbf{A}}\right)^2 + V\right)^{-1}$$

is compact (see [**10**, Theorem 2.7]) and we may thus, modulo a further extraction, assume that

$$\gamma_N^{(1)} \underset{N\to\infty}{\longrightarrow} \gamma^{(1)}$$

strongly in trace-class norm. Modulo extraction of subsequences we have

$$\gamma_N^{(k)} \rightharpoonup_* \gamma^{(k)}$$

weakly-$*$ in the trace-class as $N \to \infty$. So by [**87**, Corollary 2.4] we have

$$\gamma_N^{(k)} \to \gamma^{(k)}$$

strongly in class-trace norm for $k \geq 0$.
Returning to the proof of [**135**, Proposition 3.2] we have that

$$\begin{aligned}\int_{\mathcal{S}_P} \mathrm{d}\mu_N^{(3)}(\gamma) = \mathrm{Tr}\left(P^{\otimes 3}\gamma_N^{(3)}P^{\otimes 3}\right) &= \sum_{\ell=3}^{N} \mathrm{Tr}\left[G_{N,\ell}^P\right] \frac{\ell(\ell-1)(\ell-2)}{N(N-1)(N-2)} \\ &\geq \sum_{\ell=0}^{N} \mathrm{Tr}\left[G_{N,\ell}^P\right] \frac{\ell^3}{N^3} - \frac{C}{N} \geq \left(\sum_{\ell=0}^{N} \mathrm{Tr}\left[G_{N,\ell}^P\right] \frac{\ell}{N}\right)^3 - \frac{C}{N} \\ &= \left(\mathrm{Tr}\left(P\gamma_N^{(1)}P\right)\right)^3 - \frac{C}{N} \end{aligned} \tag{15.28}$$

where we used Jensen's inequality.
But (15.27) implies

$$\mathrm{Tr}\left(P\gamma_N^{(1)}P\right) \underset{N\to\infty}{\longrightarrow} 1$$

and then

$$\mu_N^{(3)}(\mathcal{S}_P) \underset{N\to\infty}{\longrightarrow} 1. \tag{15.29}$$

Recall that the de Finetti theorem 15.2 provides

$$\mathrm{Tr}\left|P^{\otimes 3}\gamma_N^{(3)}P^{\otimes 3} - \int_{S_P} \gamma^{\otimes 3}\mathrm{d}\mu_N^{(3)}(\gamma)\right| \to 0 \tag{15.30}$$

for the above choice of $\Lambda = N^{2\eta}$, $\eta < 1/4$. This last equation combined with (15.29) gives

$$\mathrm{Tr}\left|\gamma_N^{(3)} - \int_{S_P} \gamma^{\otimes 3}\mathrm{d}\mu_N^{(3)}(\gamma)\right| \to 0 \tag{15.31}$$

by using the triangle and Cauchy-Schwarz inequalities. Testing this with a sequence of finite rank orthogonal projectors

$$P_K \underset{K\to\infty}{\longrightarrow} \mathbb{1}$$

and using the strong convergence of $\gamma_N^{(3)}$, we obtain that

$$\lim_{K\to\infty} \lim_{N\to\infty} \mu_N^{(3)}\left(P_K S_P\right) = 1. \tag{15.32}$$

This tightness condition allows to use Prokhorov's theorem and [**144**, Lemma 1] to ensure that, up to extraction, $\mu_N^{(3)}$ converges weakly to a measure $\mu^{(3)}$ on the unit ball

$$B := \left\{\gamma \text{ positive trace class operator on } L^2\left(\mathbb{R}^2\right); \operatorname{Tr}\gamma \leq 1\right\}.$$

But $\gamma_N^{(3)} \to \gamma^{(3)}$ strongly in trace-class so $\mu^{(3)}$ only charges

$$S := \left\{\gamma \text{ positive trace class operator on } L^2\left(\mathbb{R}^2\right); \operatorname{Tr}\gamma = 1\right\}.$$

de Finetti measure charges pure states.
Applying the weak quantum de Finetti theorem [**86**, Theorem 2.2] we deduce that there exists a measure ν on the unit ball of $L^2(\mathbb{R}^2)$ such as

$$\gamma^{(k)} = \int |u^{\otimes k}\rangle\langle u^{\otimes k}| \mathrm{d}\nu(u).$$

But since $\gamma^{(1)}$ must have trace 1, the measure ν must actually live on

$$SL^2(\mathbb{R}^d) = \left\{u \in L^2(\mathbb{R}^2),\ \int_{\mathbb{R}^2} |u|^2 = 1\right\},$$

the unit sphere of $L^2(\mathbb{R}^2)$.

Next we claim that the two measures $\mu = \mu^{(3)}$ of above and ν just found are related by

$$\int |u^{\otimes 3}\rangle\langle u^{\otimes 3}| \mathrm{d}\nu(u) = \int |u^{\otimes 3}\rangle\langle u^{\otimes 3}| \mathrm{d}\mu(u). \tag{15.33}$$

Indeed, let

$$\tilde{P} = \mathbb{1}_{h \leq \tilde{\Lambda}}$$

where $\tilde{\Lambda}$ is a fixed cut-off (different from Λ above). Testing (15.8) with A_1, A_2, A_3 finite rank operators whose ranges lie within that of $\tilde{P}$ we get

$$\operatorname{Tr}\left(A_1 \otimes A_2 \otimes A_3 \gamma_N^{(3)}\right) \underset{N\to\infty}{\longrightarrow} \operatorname{Tr}\left(A_1 \otimes A_2 \otimes A_3 \int_{\mathcal{S}_{\tilde{P}}} \gamma^{\otimes 3} \mathrm{d}\mu(\gamma)\right)$$

using the convergence of $\mu_N^{(3)}$ to μ. On the other hand, by the convergence of $\gamma_N^{(3)}$ to $\gamma^{(3)}$ we also have

$$\begin{aligned}\operatorname{Tr}\left(A_1 \otimes A_2 \otimes A_3 \gamma_N^{(3)}\right) &\underset{N\to\infty}{\longrightarrow} \operatorname{Tr}\left(A_1 \otimes A_2 \otimes A_3 \gamma^{(3)}\right) \\ &= \operatorname{Tr}\left(A_1 \otimes A_2 \otimes A_3 \int_{SL^2(\mathbb{R}^2)} |u^{\otimes 3}\rangle\langle u^{\otimes 3}| \mathrm{d}\nu(u)\right)\end{aligned}$$

Thus

$$\operatorname{Tr}\left(A_1 \otimes A_2 \otimes A_3 \int_{\mathcal{S}_{\tilde{P}}} \gamma^{\otimes 3} \mathrm{d}\mu(\gamma)\right) = \operatorname{Tr}\left(A_1 \otimes A_2 \otimes A_3 \int_{SL^2(\mathbb{R}^2)} |u^{\otimes 3}\rangle\langle u^{\otimes 3}| \mathrm{d}\nu(u)\right) \tag{15.34}$$

for any A_1, A_2, A_3 with range within that of $\tilde{P}$. Letting finally $\tilde{\Lambda} \to \infty$ yields $\tilde{P} \to \mathbb{1}$ and thus (15.34) holds for any compact operators A_1, A_2, A_3. This implies (15.33). In particular, since the left-hand side of (15.33) is $\gamma^{(3)}$, a bosonic operator, μ must be supported on pure states $\gamma = |u\rangle\langle u|$, see [**72**].

Final passage to the liminf. Let us return to (15.25). We split the integral over one-body states γ between low and high kinetic energy states:

$$\mathcal{S}^- = \{\gamma \in \mathcal{S},\, \mathrm{Tr}\,(h\gamma) \leq \mathrm{L}\}\,, \quad \mathcal{S}^+ = \mathcal{S} \setminus \mathcal{S}^-.$$

Using Lemma 14.6 we obtain

$$\begin{aligned}\int_{\mathcal{S}_P} \mathcal{E}_R^{\mathrm{af}}[\gamma]\mathrm{d}\mu_N^{(3)}(\gamma) &\geq \mathrm{L}\int_{\mathcal{S}^+} \mathrm{d}\mu_N^{(3)}(\gamma) + \int_{\mathcal{S}^-} \mathcal{E}_R^{\mathrm{af}}[\gamma]\mathrm{d}\mu_N^{(3)}(\gamma) - o_N(1)\\ &\geq \int_{\mathcal{S}_P} \min\left(\mathrm{L}, \mathcal{E}_R^{\mathrm{af}}[\gamma]\right)\mathrm{d}\mu_N^{(3)}(\gamma) - o_N(1)\end{aligned}$$

Inserting in (15.25) and passing to the liminf in $N \to \infty$ this implies

$$E^{\mathrm{af}} \geq \liminf_{N\to\infty} \frac{E(N)}{N} \geq \int_{\mathcal{S}} \min\left(\mathrm{L}, \mathcal{E}^{\mathrm{af}}[\gamma]\right)\mathrm{d}\mu(\gamma).$$

Finally, we pass to the limit $\mathrm{L} \to \infty$ to deduce

$$E^{\mathrm{af}} \geq \liminf_{N\to\infty} \frac{E(N)}{N} \geq \int_{\mathcal{S}} \mathcal{E}^{\mathrm{af}}[\gamma]\mathrm{d}\mu(\gamma). \tag{15.35}$$

But as we saw above μ must be supported on pure states $\gamma = |u\rangle\langle u|$ which yields both the energy lower bound concluding the proof of (13.18) and the fact that μ must be supported on $\mathcal{M}^{\mathrm{af}}$. Because $\mathcal{E}^{\mathrm{af}}[\gamma]$ is a linear function of $\gamma^{\otimes 3}$ we can also combine (15.35) with (15.33) to deduce that also ν must be supported on $\mathcal{M}^{\mathrm{af}}$, which proves (13.19) and

$$E^{\mathrm{af}} \geq \liminf_{N\to\infty} \frac{E(N)}{N} \geq E^{\mathrm{af}}.$$

7. Appendix for the almost-bosonic limit

In this appendix we etablish some of the fundamental properties of the functional $\mathcal{E}_R^{\mathrm{af}}$ and its limit $R \to 0$. We call $\nabla_{\mathbf{A}_e} = \nabla + \mathrm{i}\mathbf{A}_e$ the covariant derivative. For $\beta \in \mathbb{R}$, $\mathbf{A}_e \in L^2_{loc}\left(\mathbb{R}^2\right)$ and $V : \mathbb{R}^2 \to \mathbb{R}^+$ we define the average-field energy functional

$$\mathcal{E}^{\mathrm{af}}[u] := \int_{\mathbb{R}^2} \left(\left|\left(\nabla_{\mathbf{A}_e} + \mathrm{i}\beta\mathbf{A}[|u|^2]\right)u\right|^2 + V|u|^2\right), \tag{15.36}$$

with the self-generated magnetic potential

$$\mathbf{A}[\rho] := \nabla^\perp w_0 * \rho = \int_{\mathbb{R}^2} \frac{(\mathbf{x}_1 - \mathbf{x}_2)^\perp}{|\mathbf{x}_1 - \mathbf{x}_2|^2}\rho(\mathbf{x}_2)\,\mathrm{d}\mathbf{x}_2, \qquad \mathrm{curl}\,\mathbf{A}[\rho] = 2\pi\rho.$$

Lemma 15.5 (**Bound on the magnetic term**).
We have for any $u \in L^2(\mathbb{R}^2)$ that

$$\int_{\mathbb{R}^2} \left|\mathbf{A}[|u|^2]\right|^2 |u|^2 \leq \frac{3}{2}\|u\|^4_{L^2(\mathbb{R}^2)} \int_{\mathbb{R}^2} |\nabla|u||^2 \leq \frac{3}{2}\|u\|^4_{L^2(\mathbb{R}^2)} \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}_e} u|^2$$

Proof. See [**112**, Lemma A.1] combined with (15.37). □

Using the same reasonning than in [**112**, Equation A2] we can, using (15.5) define the domain of $\mathcal{E}^{\mathrm{af}}$ to be (see [**92**][Proposition 7.20]) and otherwise let $\mathcal{E}^{\mathrm{af}}[u] := +\infty$

$$\mathscr{D}^{\mathrm{af}} := \left\{u \in H^1_{\mathbf{A}_e}(\mathbb{R}^2) : \int_{\mathbb{R}^2} V|u|^2 < \infty\right\},$$

We find using Cauchy-Schwarz and Lemma 15.5, that for $u \in \mathscr{D}^{\mathrm{af}}$

$$0 \leq \mathcal{E}^{\mathrm{af}}[u] \leq 2\|\nabla_{\mathbf{A}_e} u\|^2 + 2\| \mathbf{A}_e u\|^2 + 3\beta^2 \|\mathbf{A}[|u|^2]u\|^2 + \int V|u|^2$$
$$\leq (2 + 3\beta^2\|u\|^4)\|\nabla_{\mathbf{A}_e} u\|^2 + C\int V|u|^2 < \infty.$$

The ground-state energy of the average-field functional is then given by

$$E^{\mathrm{af}} := \inf\left\{\mathcal{E}^{\mathrm{af}}[u] : u \in \mathscr{D}^{\mathrm{af}}, \int_{\mathbb{R}^2} |u|^2 = 1\right\}.$$

For convenience we also make the assumption on V that $V(x) \to +\infty$ as $|x| \to \infty$. Note that $C_c^\infty(\mathbb{R}^2) \subseteq \mathscr{D}^{\mathrm{af}}$ is dense in $H^1_{\mathbf{A}_e}(\mathbb{R}^2)$.

Lemma 15.6 (**Basic magnetic inequalities**).
We have for $u \in \mathscr{D}^{\mathrm{af}}$ that (diamagnetic inequality)

$$\int_{\mathbb{R}^2} |(\nabla + \mathrm{i}\mathbf{A}_e)u|^2 \geq \int_{\mathbb{R}^2} |\nabla|u||^2 \tag{15.37}$$

$$\int_{\mathbb{R}^2} \left|(\nabla + \mathrm{i}\mathbf{A}_e + \mathrm{i}\beta\mathbf{A}[|u|^2])u\right|^2 \geq \int_{\mathbb{R}^2} |\nabla|u||^2, \tag{15.38}$$

and

$$\int_{\mathbb{R}^2} \left|(\nabla + \mathrm{i}\mathbf{A}_e + \mathrm{i}\beta\mathbf{A}[|u|^2])u\right|^2 \geq \int_{\mathbb{R}^2} \mathbf{B}|u|^2 + 2\pi|\beta| \int_{\mathbb{R}^2} |u|^4. \tag{15.39}$$

Proof. This is an application of [Lemma 3.2][**34**].

□

Proposition 15.7 (**Existence of minimizers**).
For any value of $\beta \in \mathbb{R}$ there exists $u^{\mathrm{af}} \in \mathscr{D}^{\mathrm{af}}$ with $\int_{\mathbb{R}^2} |u^{\mathrm{af}}|^2 = 1$ and $\mathcal{E}^{\mathrm{af}}[u^{\mathrm{af}}] = E^{\mathrm{af}}$.

Proof. First note that for $u \in \mathscr{D}^{\mathrm{af}}$, by Lemma 15.5 and Lemma 15.6,

$$\|\nabla_{\mathbf{A}_e} u\|_2 = \left\|\nabla_{\mathbf{A}_e} u + \mathrm{i}\beta\mathbf{A}[|u|^2]u - \mathrm{i}\beta\mathbf{A}[|u|^2]u\right\|_2 \leq \mathcal{E}^{\mathrm{af}}[u]^{1/2} + |\beta| \left\|\mathbf{A}[|u|^2]u\right\|_2$$
$$\leq \mathcal{E}^{\mathrm{af}}[u]^{1/2} + |\beta|\sqrt{\frac{3}{2}}\|u\|_2^2 \||\nabla|u|\|_2 \leq \left(1 + |\beta|\sqrt{\frac{3}{2}}\|u\|_2^2\right)\mathcal{E}^{\mathrm{af}}[u]^{1/2}.$$

Now take a minimizing sequence

$$(u_n)_{n\to\infty} \subset \mathscr{D}^{\mathrm{af}},\ \|u_n\|_2 = 1,\ \lim_{n\to\infty} \mathcal{E}^{\mathrm{af}}[u_n] = E^{\mathrm{af}}.$$

Then clearly (u_n) is uniformly bounded in both $L^2(\mathbb{R}^2)$, L^2_V, and $H^1_{\mathbf{A}_e}(\mathbb{R}^2)$ and therefore there exists $u^{\mathrm{af}} \in \mathscr{D}^{\mathrm{af}}$ and a weakly convergent subsequence (still denoted u_n) such that

$$u_n \rightharpoonup u^{\mathrm{af}} \text{ in } L^2(\mathbb{R}^2) \cap L^2_V \cap H^1_{\mathbf{A}_e}(\mathbb{R}^2), \quad \nabla_{\mathbf{A}_e} u_n \rightharpoonup \nabla_{\mathbf{A}_e} u^{\mathrm{af}} \text{ in } L^2(\mathbb{R}^2).$$

Moreover, since $((\mathbf{p}^{\mathbf{A}})^2 + V + 1)^{-1/2}$ is compact we have that

$$u_n = \left((\mathbf{p}^{\mathbf{A}})^2 + V + 1\right)^{-1/2}\left((\mathbf{p}^{\mathbf{A}})^2 + V + 1\right)^{1/2} u_n$$

is actually strongly convergent (again, extracting a subsequence), hence

$$u_n \to u^{\mathrm{af}} \text{ in } L^2(\mathbb{R}^2).$$

It then follows that

$$\big\|(\nabla_{\mathbf{A}_e} + \mathrm{i}\beta\mathbf{A}[|u|^2])u\big\|_2^2 = \big\langle(\nabla_{\mathbf{A}_e} + \mathrm{i}\beta\mathbf{A}[|u|^2])u, (\nabla_{\mathbf{A}_e} + \mathrm{i}\beta\mathbf{A}[|u|^2])u\big\rangle$$

with

$$\begin{aligned}
\liminf_{n\to\infty} \langle \nabla_{\mathbf{A}_e} u_n, \nabla_{\mathbf{A}_e} u_n\rangle &\geq \int |\nabla_{\mathbf{A}_e} u|^2 \\
\lim_{n\to\infty} \big\langle \nabla_{\mathbf{A}_e} u_n, \mathbf{A}[|u_n|^2]u_n\big\rangle &= \big\langle \nabla_{\mathbf{A}_e} u, \mathbf{A}[|u|^2]u\big\rangle \\
\lim_{n\to\infty} \big\langle \mathbf{A}[|u_n|^2]u_n, \mathbf{A}[|u_n|^2]u_n\big\rangle &= \int |\mathbf{A}[|u^2|]u|^2
\end{aligned}$$

by the weak lower semicontinuity of the norm, the weak-strong convergence and the fact that the functions $\mathbf{A}[|u_n|^2]u_n$ are strongly converging to $\mathbf{A}[|u|^2]u$ in $L^2(\mathbb{R}^2)$, see [**112**, Proposition A.3]. And since $\|\cdot\|_{L^2_V}$ is also weakly lower semicontinuous (see, e.g., [**128**, Supplement to IV.5]), we have $\liminf_{n\to\infty} \mathcal{E}^{\mathrm{af}}[u_n] \geq \mathcal{E}^{\mathrm{af}}[u^{\mathrm{af}}]$. Thus, with $\|u^{\mathrm{af}}\| = \lim_{n\to\infty}\|u_n\| = 1$, we also have $\mathcal{E}^{\mathrm{af}}[u^{\mathrm{af}}] = E^{\mathrm{af}}$. □

Proposition 15.8 (**Regularized functional**).
Let us now consider the corresponding situation for the regularized functional (extended anyons)

$$\mathcal{E}^{\mathrm{af}}_R[u] := \int_{\mathbb{R}^2} \Big(\big| \big(\nabla + \mathrm{i}\mathbf{A}_e + \mathrm{i}\beta\mathbf{A}^R[|u|^2]\big)\, u\big|^2 + V|u|^2\Big), \quad \mathbf{A}^R[\rho] := \nabla^\perp w_R * \rho, \quad R > 0.$$

Since $\nabla w_R \in L^\infty(\mathbb{R}^2)$ we have $\mathbf{A}^R[|u|^2] \in L^\infty(\mathbb{R}^2)$ with

$$\big\|\mathbf{A}^R[|u|^2]\big\|_\infty \leq \frac{C}{R}\|u\|_2^2$$

and instead of Lemma 15.5 we have

$$\big\|\mathbf{A}^R[|u|^2]u\big\|_2 \leq C\|u\|_2^2 \||u|\|_{H^1_{\mathbf{A}_e}}$$

using Lemma 14.4.

Hence the natural domain is again $\mathscr{D}^{\mathrm{af}}$ and all properties established above for $\mathcal{E}^{\mathrm{af}}$ are also found to be valid for $\mathcal{E}^{\mathrm{af}}_R$ (except (15.39) which now has regularized versions). Denoting

$$E^{\mathrm{af}}_R := \min\{\mathcal{E}^{\mathrm{af}}_R[u] : u \in \mathscr{D}^{\mathrm{af}}, \|u\|_2 = 1\},$$

we furthermore have the following relationship:

Proposition 15.9 (**Convergence to point-like anyons**).
The functional $\mathcal{E}^{\mathrm{af}}_R$ converges pointwise to $\mathcal{E}^{\mathrm{af}}$ as $R \to 0$. More precisely, for any $u \in \mathscr{D}^{\mathrm{af}}$

$$\Big|\mathcal{E}^{\mathrm{af}}_R[u] - \mathcal{E}^{\mathrm{af}}[u]\Big| \leq C_u\beta^2\mathcal{E}^{\mathrm{af}}[u]^{1/2}(1 + \mathcal{E}^{\mathrm{af}}[u])^{1/2}R, \tag{15.40}$$

where C_u depends only on $\|u\|_2$. Hence,

$$E^{\mathrm{af}}_R \underset{R\to 0}{\to} E^{\mathrm{af}},$$

and if $(u_R)_{R\to 0} \subset \mathscr{D}^{\mathrm{af}}$ denotes a sequence of minimizers of $\mathcal{E}^{\mathrm{af}}_R$, then there exists a subsequence $(u_{R'})_{R'\to 0}$ s.t. $u_{R'} \to u^{\mathrm{af}}$ as $R' \to 0$, where u^{af} is some minimizer of $\mathcal{E}^{\mathrm{af}}$.

PROOF. This is an easy variant of the proof of [**112**, Proposition A.5]. □

PROPOSITION 15.10 (**Link with the regularized functional**).
The Hamiltonian (13.7) *can be expanded (with the notation* $\mathbf{p}_j + \mathbf{A}_e = \mathbf{p}_j^{\mathbf{A}}$*) and corresponds to the energy:*

$$\begin{aligned} N^{-1}\left\langle \Psi_N, H_R^N \Psi_N \right\rangle &= \mathrm{Tr}\left[\left((\mathbf{p}_j^{\mathbf{A}})^2 + V(\mathbf{x}_j) \right) \gamma_N^{(1)} \right] \\ &+ \beta \mathrm{Tr}\left[\left(\mathbf{p}_1^{\mathbf{A}}.\nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2) + \nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2).\mathbf{p}_1^{\mathbf{A}} \right) \gamma_N^{(2)} \right] \\ &+ \beta^2 \frac{N-2}{N-1} \mathrm{Tr}\left[\left(\nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2).\nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_3) \right) \gamma_N^{(3)} \right] \\ &+ \beta^2 \frac{1}{N-1} \mathrm{Tr}\left[\left| \nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2) \right|^2 \gamma_N^{(2)} \right] \end{aligned} \tag{15.41}$$

In particular, taking a trial state $\psi_N = u^{\otimes N}$ *we observe that:*

$$\mathrm{Tr}\left[\left(\mathbf{p}_1^{\mathbf{A}}.\nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2) + \nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2).\mathbf{p}_1^{\mathbf{A}} \right) \left| u^{\otimes 2} \right\rangle \left\langle u^{\otimes 2} \right| \right]$$

$$= \mathrm{i} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla \overline{u}(\mathbf{x}_1) \overline{u}(\mathbf{x}_2).\nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2) u(\mathbf{x}_1) u(\mathbf{x}_2) \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2 \tag{15.42}$$

$$+ \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \mathbf{A}_e(\mathbf{x}_1).\nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2) |u(\mathbf{x}_1)|^2 |u(\mathbf{x}_2)|^2 \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2$$

$$- \mathrm{i} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \overline{u}(\mathbf{x}_1) \overline{u}(\mathbf{x}_2).\nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2) \nabla u(\mathbf{x}_1) u(\mathbf{x}_2) \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2 \tag{15.43}$$

$$+ \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \mathbf{A}_e(\mathbf{x}_1).\nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2) |u(\mathbf{x}_1)|^2 |u(\mathbf{x}_2)|^2 \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2$$

$$= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |u(\mathbf{x}_2)|^2 \nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2). \left(\mathrm{i}\left[u(\mathbf{x}_1) \nabla \overline{u}(\mathbf{x}_1) - \overline{u}(\mathbf{x}_1) \nabla u(\mathbf{x}_1) \right] + 2\mathbf{A}_e(\mathbf{x}_1) |u(\mathbf{x}_1)|^2 \right) \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2$$

$$= 2 \int_{\mathbb{R}^2} \mathbf{A}^R[|u|^2]. \left(\mathbf{J}[u(\mathbf{x}_1)] + \mathbf{A}_e(\mathbf{x}_1) |u(\mathbf{x}_1)|^2 \right) \mathrm{d}\mathbf{x}_1 \tag{15.44}$$

where

$$\mathbf{J}[u] = \frac{\mathrm{i}}{2}(u \nabla \overline{u} - \overline{u} \nabla u)$$

$$\mathbf{J}^{\mathrm{tot}}[u] = \mathbf{J}[u] + \mathbf{A}_e |u|^2$$

with

$$|\mathbf{A}^R[|u|^2] = \int_{\mathbb{R}^2} |u(\mathbf{x}_2)|^2 \nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2) \mathrm{d}\mathbf{x}_2$$

for the second term and:

$$\mathrm{Tr}\left[\left(\nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2).\nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_3) \right) \gamma_N^{(3)} \right]$$

$$= \iiint_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} |u(\mathbf{x}_1)|^2 |u(\mathbf{x}_2)|^2 |u(\mathbf{x}_3)|^2 .\nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2).\nabla^\perp \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2 \mathrm{d}\mathbf{x}_3$$

$$= \int_{\mathbb{R}^2} |u|^2 \left| \mathbf{A}^R[|u|^2] \right|^2 \mathrm{d}\mathbf{x}_1 \tag{15.45}$$

for the third.

Part 4

Semi-classical limit for almost-fermionic anyons

Abstract. In two-dimensional space there are possibilities for quantum statistics continuously interpolating between the bosonic and the fermionic one. Quasi-particles obeying such statistics can be described as ordinary bosons and fermions with magnetic interactions. We study a limit situation where the statistics/magnetic interaction is seen as a "perturbation from the fermionic end". We vindicate a mean-field approximation, proving that the ground state of a gas of anyons is described to leading order by a semi-classical, Vlasov-like, energy functional. The ground state of the latter displays anyonic behavior in its momentum distribution. Our proof is based on coherent states, Husimi functions, the Diaconis-Freedman theorem and a quantitative version of a semi-classical Pauli pinciple.

CHAPTER 16

Model and main results

1. Introduction

In two dimensional systems there are possibilities for quantum statistics other than the bosonic one and the fermionic one, so called intermediate or fractional statistics. Particles following such statistics, termed anyons (as in *any*thing in between bosons and fermions), can arise as effective quasi-particles in correlated many-body quantum systems confined to two dimensions. They have been conjectured [**8, 121, 132**] to be relevant for the fractional quantum Hall effect (see [**61, 77, 74**] for review and [**12, 127**] for recent experimental developments). They can also be simulated in certain cold atoms systems with synthetic gauge fields [**44, 25, 161, 166, 167, 32, 12, 127**].

The statistics of anyonic particles is labeled by a single parameter $\alpha \in [0, 2[$ where the cases $\alpha = 0, 1$ correspond to ordinary bosons and fermions, respectively. Since the many-anyon problem is not exactly soluble (even in the absence of interactions other than the statistical ones) except for the latter special values, it makes sense to study limiting parameter regimes where reduced models are obtained as effective descriptions.

The "almost bosonic" limit $\alpha \to 0$ is studied in [**59, 112**], in the joint limit of the particle number $N \to \infty, \alpha \sim N^{-1}$, which turns out to be the relevant one for the non-trivial statistics to have an effect at leading order [**34, 32**]. Here we study an "almost fermionic limit" $\alpha \to 1$ in dependence with the particle number. The relevant scaling in dependence of N leads to a problem resembling the usual mean-field/semi-classical limit for interacting fermions [**100, 147, 13, 11, 46, 53**].

Our starting point is a gas of anyonic particles evolving in $\mathbb{R}^2$ subjected to a magnetic field

$$\mathbf{B}_e = \mathrm{curl}\mathbf{A}_e$$

and a trapping external potential

$$V(x) \underset{|x|\to\infty}{\to} \infty$$

To model anyonic behavior we introduce the statistical gauge vector potential felt by particle j due to the influence of all the other particles

$$\mathbf{A}(x_j) = \sum_{j\neq k} \frac{(x_j - x_k)^\perp}{|x_j - x_k|^2}. \tag{16.1}$$

The associated magnetic field is

$$\mathrm{curl}\mathbf{A}(x_j) = 2\pi \sum_{k\neq j} \delta(x_j - x_k). \tag{16.2}$$

The statistics parameter is denoted by α, corresponding to a statistical phase $e^{i\pi(1+\alpha)}$ under a continuous simple interchange of two particles. In this so-called "magnetic gauge picture", 2D anyons are thus described as fermions, each of them carrying an Aharonov-Bohm magnetic flux of strength α. Instead of looking at the ground state of an non-interacting Hamiltonian acting on anyonic wave-functions we thus study the following Hamiltonian

$$H_N = \sum_{j=1}^{N} \left((-\mathrm{i}\hbar\nabla_j + \mathbf{A}_e(x_j) + \alpha\mathbf{A}(x_j))^2 + V(x_j) \right) \text{ acting on } L^2_{\mathrm{asym}}(\mathbb{R}^{2N}). \tag{16.3}$$

Note that in the above we view the anyons statistics parameter α couting from the fermionic representation of anyons. The limit $\alpha \to 0$ for (16.3) is then formally equivalent to $\alpha \to 1$ in the bosonic representation alluded to above [**59, 112**].

We are interested in a joint limit $N \to \infty$, $\alpha \to 0$ in which the problem would simplify but still display signatures of anyonic behavior, i.e. a non-trivial dependence on α. Since we perturb around fermions, the Pauli principle implies that the kinetic energy grows faster than N. This behavior is encoded in the Lieb-Thirring inequality [**105, 104**] which, for any N-particle normalized antisymmetric wave function Ψ_N with support in a bounded domain Ω^N, gives

$$\sum_{j=1}^{N} \int_{\Omega^N} |\nabla_j \Psi|^2 \geq C|\Omega|^{-\frac{2}{d}} N^{1+\frac{2}{d}} \tag{16.4}$$

where d is the dimension of the physical space. Because of this rapid growth of the kinetic energy, the natural mean-field limit [**13, 14, 147**] for fermions involves a scaling $\hbar \sim N^{-1/d}$. Here with $d = 2$ we get a kinetic energy proportional to N^2 and thus, in expectation value $-\mathrm{i}\nabla_j \sim N^{1/2}$. Investigating the N-dependence of each term of (16.3) we thus expect for any $\Psi_N \in L^2\left(\mathbb{R}^{2N}\right)$

$$\langle \Psi_N, H_N \Psi_N \rangle \approx C_1 N \left(\hbar N^{1/2} + \alpha N \right)^2 + C_2 N$$

where the terms in parenthesis account for kinetic energy, including the influence of (16.1), and the second term is the energy in the external potential V. To obtain a non trivial limit when $\alpha \to 0$, a natural choice is to take

$$\hbar = \frac{1}{\sqrt{N}} \text{ and } \alpha = \frac{\beta_0}{N} \tag{16.5}$$

$\beta_0 \in \mathbb{R}$ a constant allowing us to keep a track of the anyonic behavior and consider the limit

$$\lim_{N\to\infty} \frac{E(N)}{N}$$

with

$$E(N) := \inf \left\{ \langle \Psi_N, H_N \Psi_N \rangle, \Psi_N \in L^2_{\mathrm{asym}}(\mathbb{R}^{2N}), \int_{\mathbb{R}^{2N}} |\Psi_N|^2 = 1 \right\}. \tag{16.6}$$

Such an approach combines two limits (see also Remark 16.2 below), the semi-classical $\hbar \to 0$ and the quasi-fermionic $\alpha \to 0$.

2. Extended anyons: energy convergence

The Hamiltonian (16.3) is actually too singular. Consequently we introduce a length R over which the magnetic flux attached to our particles is smeared. In our approach, $R \to 0$ will make us recover the point-like anyons point of view. This "extended anyons" model is discussed in [**122, 158, 22**]. Anyons arising as quasi-particles in condensed-matter systems are not point-like objects. The radius R then corresponds to their size of the quasi-particle, i.e. a length-scale below which the quasi-particle description breaks down.

We introduce the 2D Coulomb potential generated by a unit charge smeared over the disc of radius R

$$w_R(x) = (\log|\,.\,| * \chi_R)(x), \text{ with the convention } w_0 = \log|\,.\,| \tag{16.7}$$

and $\chi_R(x)$ a positive smooth function of unit mass

$$\chi_R = \frac{1}{R^2}\chi\left(\frac{\cdot}{R}\right) \text{ and } \int_{\mathbb{R}^2}\chi = 1, \quad \chi(x) = \begin{cases} 1/\pi^2 & |x| \le 1 \\ 0 & |x| \ge 2. \end{cases} \tag{16.8}$$

We take χ to be smooth, positive, radial and decreasing between 1 and 2. The choice of this particular χ in (16.8) is made for convenience. Others regularizing functions could be used. We will recover the magnetic field (16.2) (in a distributional sense) in the limit $R \to 0$ by defining

$$\mathbf{A}^R(x_j) = \sum_{k\neq j} \nabla^\perp w_R(x_j - x_k). \tag{16.9}$$

We henceforth work with the regularized Hamiltonian of N anyons of radius R confined in a trapping potential $V(x) = |x|^s$, $s > 1$

$$\boxed{H_N^R := \sum_{j=1}^N \left(\left(p_j^{\mathbf{A}} + \alpha \mathbf{A}^R(x_j)\right)^2 + V(x_j)\right)} \tag{16.10}$$

denoting

$$p_j^{\mathbf{A}} = -\mathrm{i}\hbar\nabla_j + \mathbf{A}_e(x_j). \tag{16.11}$$

It is essentially self-adjoint on $L^2_{\mathrm{asym}}(\mathbb{R}^{2N})$ (see [**129**, Theorem X.29] and [**10**]). The bottom of its spectrum then exists for any fixed $R > 0$, we denote it

$$E^R(N) = \inf\sigma\left(H_N^R\right). \tag{16.12}$$

Under our assumptions H_N^R will actually have compact resolvent, so the above is an eigenvalue. However, this operator does not have a unique limit as $R \to 0$ and the Hamiltonian at $R = 0$ is *not* essentially self-adjoint, see for instance [**119, 35, 38, 1, 17**]. Nevertheless, in the joint limit $R \to 0$ and $N \to \infty$ we recover a unique well-defined (non-linear) model.

It is convenient to expand (16.10) and treat summands separately

$$\begin{aligned} H_N^R = & \sum_{j=1}^N \left((p_j^{\mathbf{A}})^2 + V(x_j)\right) \text{ "Kinetic and potential terms"} \\ & + \alpha \sum_{j\neq k} \left(p_j^{\mathbf{A}}.\nabla^\perp w_R(x_j - x_k) + \nabla^\perp w_R(x_j - x_k).p_j^{\mathbf{A}}\right) \text{ "Mixed two-body term"} \\ & + \alpha^2 \sum_{j\neq k\neq l} \nabla^\perp w_R(x_j - x_k).\nabla^\perp w_R(x_j - x_l) \text{ "Three-body term"} \\ & + \alpha^2 \sum_{j\neq k} \left|\nabla^\perp w_R(x_j - x_k)\right|^2 \text{ "Singular two-body term".} \end{aligned} \tag{16.13}$$

The fourth term of the above, being N times smaller than the others in the regime (16.5), will easily be estimated. Further heuristics will only involve the mixed two-body term and the three-body term. These being of the same order, we expect (as in other types of mean-field limits, see e.g. [**138**] and references therein) that, for large N, particles behave independently, and thus

$$\sum_{j\neq k} \nabla^\perp w_R(x_j - x_k) \approx \int_{\mathbb{R}^2} \nabla^\perp w_R(x_j - y)\rho(y)\mathrm{d}y = \nabla^\perp w_R * \rho(x_j)$$

where ρ is the density of particles. Therefore, it is convenient to define, given a one-body density ρ normalized in $L^1(\mathbb{R}^2)$

$$\mathbf{A}[\rho] = \nabla^\perp w_0 * \rho \text{ and } \mathbf{A}^R[\rho] = \nabla^\perp w_R * \rho.$$

Natural mean-field functionals associated with (16.13) are then

$$\mathcal{E}_R^{\mathrm{af}}[\gamma] = \frac{1}{N} \operatorname{Tr}\left[\left(p_1^{\mathbf{A}} + \alpha \mathbf{A}^R[\rho_\gamma]\right)^2 \gamma\right] + \frac{1}{N}\int_{\mathbb{R}^2} V(x_1)\rho_\gamma(x_1)\mathrm{d}x_1 \tag{16.14}$$

$$\mathcal{E}^{\mathrm{af}}[\gamma] = \frac{1}{N} \operatorname{Tr}\left[\left(p_1^{\mathbf{A}} + \alpha \mathbf{A}[\rho_\gamma]\right)^2 \gamma\right] + \frac{1}{N}\int_{\mathbb{R}^2} V(x_1)\rho_\gamma(x_1)\mathrm{d}x_1 \tag{16.15}$$

with

$$\rho_\gamma(x) = \gamma(x,x)$$

and where the argument is a trace-class operator

$$\gamma \in \mathfrak{S}^1\left(L^2\left(\mathbb{R}^2\right)\right), 0 \leq \gamma \leq 1, \operatorname{Tr}\gamma = 1. \tag{16.16}$$

Similar Hartree-like functionals were obtained in the almost-bosonic limit [**59, 112**]. Here we simplify things one step further by considering a semi-classical analogue

$$\mathcal{E}_{\mathrm{Vla}}[m] = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} |p + \mathbf{A}_e(x) + \beta_0 \mathbf{A}[\rho](x)|^2 m(x,p)\mathrm{d}x\mathrm{d}p + \int_{\mathbb{R}^2} V(x)\rho_m(x)\mathrm{d}x \tag{16.17}$$

with $m(x,p)$ a positive measure on the phase-space $\mathbb{R}^4$ of positions/momenta and

$$\rho_m(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} m(x,p)\mathrm{d}p. \tag{16.18}$$

Our convention is that m satisfies the mass constraint

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} m(x,p)\mathrm{d}x\mathrm{d}p = (2\pi)^2 \tag{16.19}$$

and the semi-classical Pauli principle

$$\boxed{0 \leq m(x,p) \leq 1 \text{ a.e.}} \tag{16.20}$$

The latter is imposed at the classical level by the requirement that a classical state (x,p) cannot be occupied by more than one fermion. By the Bathtub principle [**92**, Theorem 1.14] we can perform the minimization in p explicitly (this parallels the considerations in [**147**]). We find minimizers of the form

$$m_\rho(x,p) = \mathbb{1}\Big(|p + \mathbf{A}_e(x) + \beta_0 \mathbf{A}[\rho](x)|^2 \leq 4\pi\rho(x)\Big)$$

where ρ minimizes the Thomas-Fermi energy

$$\mathcal{E}_{\mathrm{TF}}[\rho] = \mathcal{E}_{\mathrm{Vla}}[m_\rho] = 2\pi\int_{\mathbb{R}^2}\rho^2(x)\mathrm{d}x + \int_{\mathbb{R}^2} V(x)\rho(x)\mathrm{d}x. \tag{16.21}$$

We define the minimum of the Thomas-Fermi functional

$$e_{\mathrm{TF}} = \inf\Big\{\mathcal{E}_{\mathrm{TF}}[\rho] : 0 \leq \rho \in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2), \int_{\mathbb{R}^2}\rho = 1\Big\}. \tag{16.22}$$

Our first purpose is to obtain the above as the limit of the true many-body energy. We shall prove this under the following assumptions

Assumption 16.1 (**External Potentials**).
The external potentials entering (16.10) *satisfy*

- $\mathbf{A}_e \in L^2(\mathbb{R}^2)\cap W^{2,\infty}(\mathbb{R}^2)$
- $|\mathbf{A}_e|^2 \in W^{1,\infty}(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$

and, for some $s > 1$ *and* $c, C, c', C', c'', C'' > 0$

- $V(x) \geq c|x|^s - C$
- $|\nabla V(x)| \leq c'|x|^{s-1} + C'$
- $|\Delta V(x)| \leq c''|x|^{s-2} + C''$

Theorem 16.1 (**Convergence of the ground state energy**).
We consider N extended anyons of radius $R = N^{-\eta}$ in an external potential V. Under Assumption (16.1) *and setting*

$$0 < \eta < \frac{1}{4}$$

we have, in the parameter regime (16.5)

$$\boxed{\lim_{N\to\infty}\frac{E^R(N)}{N} = e_{\mathrm{TF}}}$$

where the many-anyon ground state energy $E^R(N)$ and the Thomas-Fermi energy are defined in (16.12) *and* (16.22)*, respectively.*

The study of the regime (16.5) resembles the usual Thomas-Fermi limit for a large fermionic system [**100, 147, 155, 90**], but with important new aspects:

- The effective interaction comprises a three-body term, and a two-body term (see Equation (16.13)) which mixes position and momentum variables.
- The limit problem (16.17) comprises an effective self-consistent magnetic field $\mathbf{A}[\rho]$.
- One should deal with the limit $R \to 0$, $\alpha \to 0$ and $\hbar \to 0$ at the same time as $N \to \infty$.

Regarding the last point, we make the

REMARK 16.2 (Scaling of parameters).
Extracting a multiplicative factor N from the energy, the parameter regime (16.5) we consider is equivalent to

$$\hbar = 1 \text{ and } \alpha = \frac{\beta_0}{\sqrt{N}} \tag{16.23}$$

provided we replace the external potential $V(x)$ by $NV(x)$ and a similar replacement for the external vector potential $\mathbf{A}_e$. The scaling (16.23) better highlights the quasi-fermionic character of the limit we take. We however find it preferable technically to work with (16.5), where the semi-classical aspect is more apparent.

REMARK 16.3 (Improvement on R).
Regarding the scaling of R, it would be highly desirable (and challenging) to be able to deal with a dilute regime $\eta > 1/2$ where the radius of the magnetic flux is much smaller than the typical inter-particle distance. As we will show in Chapter 5, we can relax our constraint $\eta < 1/4$ to $\eta < 1/3$ if we assume a Lieb-Thirring inequality for extended anyons of the form

$$\left\langle \Psi_N, \sum_{j=1}^{N} \left(p_j^{\mathbf{A}} + \alpha \mathbf{A}^R(x_j)\right)^2 \Psi_N \right\rangle \geq C_{\alpha,R} \int_{\mathbb{R}^2} \rho_{\Psi_N}(x)^2 \mathrm{d}x$$

for any fermionic wave-function Ψ_N with one-particle density ρ_{Ψ_N}. In the above we would need a $C_{\alpha,R}$ uniformly bounded from below when $N \to \infty, \alpha \to 0$ and $R \gg N^{-1/2}$. Such an inequality is made very plausible by the results of [**75**] on the homogeneous extended anyon gas and the known Lieb-Thirring inequalities for point-like anyons [**116, 117, 119, 114**]. We plan to return to this matter in the future. ◇

The limit ground state energy e_{TF} does not reveal any anyonic behavior, i.e. it does not depend on α. This is however deceptive, for the behavior of the system's ground state *does* depend on α and shows the influence of the non-trivial statistics. This will become apparent when we state the second part of our result, the convergence of ground states. It is expressed in terms of Husimi functions, constructed using coherent states. We discuss this next.

3. Squeezed coherent states

Let[1]

$$f(x) := \frac{1}{\sqrt{\pi}} e^{-\frac{|x|^2}{2}}. \tag{16.24}$$

For every (x, p) in the phase space $\mathbb{R}^2 \times \mathbb{R}^2$ we define the squeezed coherent state

$$F_{x,p}(y) := \frac{1}{\sqrt{\hbar_x}} f\left(\frac{y-x}{\sqrt{\hbar_x}}\right) e^{\mathrm{i}\frac{p\cdot y}{\hbar}}. \tag{16.25}$$

[1]We make the standard choice that f is a gaussian but any radial L^2 function could be used instead.

It has the property of being localized on a scale $\sqrt{\hbar_x}$ in space and $\sqrt{\hbar_p}$ in momentum, these two scales being related by

$$\hbar = \sqrt{\hbar_x}\sqrt{\hbar_p}. \tag{16.26}$$

The usual coherent state [**147**, Section 2.1] is the particular case $\sqrt{\hbar_x} = \sqrt{\hbar_p} = \sqrt{\hbar}$. Any such state saturates Heisenberg's uncertainty principle, and is thus as classical as a quantum state can be. Indeed, for the two observables x_1 and $p_1 = -\mathrm{i}\partial_{x_1}$ evaluated in this state we get

$$\Delta_{x_1}\Delta_{p_1} = \frac{\hbar}{2} \text{ where } \Delta_a = \left\langle F_{x,p}, a^2 F_{x,p}\right\rangle - \left\langle F_{x,p}, aF_{x,p}\right\rangle^2.$$

In our proofs we will take $\hbar_x \ll \hbar_p \to 0$, which is convenient because our Hamiltonian is singular in x but not in p.

We define the $\hbar$-Fourier tranform

$$\mathcal{F}_\hbar[\psi](p) = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2} \psi(x) e^{-\mathrm{i}\frac{p.x}{\hbar}}\mathrm{d}x \tag{16.27}$$

and denote

$$F_{\hbar_x}(x) = F_{0,0}(x) = \frac{1}{\sqrt{\hbar_x}\sqrt{\pi}} e^{-\frac{|x|^2}{2\hbar_x}}. \tag{16.28}$$

Its Fourier transform (see the calculation in Appendix 3)

$$G_{\hbar_p}(p) = \mathcal{F}_\hbar[F_{\hbar_x}](p) = \frac{1}{\sqrt{\hbar_p}\sqrt{\pi}} e^{-\frac{|p|^2}{2\hbar_p}} \tag{16.29}$$

makes apparent the localization in momentum space we just claimed.

To $F_{x,p}$ we associate the orthogonal projector $P_{x,p}$

$$P_{x,p} := |F_{x,p}\rangle\langle F_{x,p}|. \tag{16.30}$$

Then we have the well-known

LEMMA 16.4 (**Resolution of the identity on $L^2\left(\mathbb{R}^2\right)$**).
We have

$$\frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} P_{x,p}\,\mathrm{d}x\mathrm{d}p = \mathbb{1}_{L^2(\mathbb{R}^2)} \tag{16.31}$$

The proof is recalled in Appendix 3, see also [**92**, Theorem 12.8].

4. Convergence of Husimi functions

The Husimi function $m^{(k)}_{\Psi_N}(x_1, p_1, \ldots, x_k, p_k)$[**73, 27, 154**] describes how many particles are distributed in the k semi-classical boxes of size $\sqrt{\hbar_x}\times\sqrt{\hbar_p}$ centered at its arguments $(x_1, p_1), \ldots, (x_k, p_k)$. A wave function Ψ_N being given we define

$$m^{(k)}_{\Psi_N}(x_1, p_1, \ldots, x_k, p_k) = \frac{N!}{(N-k)!}\left\langle \Psi_N, \bigotimes_{j=1}^{k} P_{x_j,p_j}\otimes \mathbb{1}_{N-k}\Psi_N\right\rangle \tag{16.32}$$

Alternatively,

$$m^{(k)}_{\Psi_N}(x_1, p_1, \ldots, x_k, p_k) = \frac{N!}{(N-k)!}\int_{\mathbb{R}^{2(N-k)}}\left|\left\langle F_{x_1,p_1}\otimes\ldots\otimes F_{x_k,p_k}, \Psi_N(\cdot, z)\right\rangle_{L^2(\mathbb{R}^{2k})}\right|^2 \mathrm{d}z$$

or, in terms of fermionic annihilation and creation operators (see (17.2) below)

$$m^{(k)}_{\Psi_N}(x_1,p_1,\dots,x_k,p_k) = \langle \Psi_N, a^*\left(F_{x_1,p_1}\right)\dots a^*\left(F_{x_k,p_k}\right) a\left(F_{x_k,p_k}\right)\dots a\left(F_{x_1,p_1}\right)\Psi_N\rangle$$

where a and a^* satisfy the canonical anticommutation relations [**146, 64**]

$$\begin{aligned} a^*(f)a(g) + a(g)a^*(f) &= \langle g, f\rangle \\ a^*(f)a^*(g) + a^*(g)a^*(f) &= 0. \end{aligned}$$

We will also use the k-particle density matrices of Ψ_N, i.e. the operator $\gamma^{(k)}_{\Psi_N}$ with integral kernel

$$\gamma^{(k)}_{\Psi_N}(x_1,.,x_k;x_1',.,x_k') = \binom{N}{k}\int_{\mathbb{R}^{2(N-k)}} \Psi_N(x_1,.,x_N)\overline{\Psi_N(x_1',.,x_k',x_{k+1},.,x_N)}\mathrm{d}x_{k+1}\dots\mathrm{d}x_N. \tag{16.33}$$

We can also express the Husimi function in terms of the k-particle density matrices of above

$$m^{(k)}_{\Psi_N}(x_1,p_1,\dots,x_k,p_k) = k!\ \mathrm{Tr}\left[\bigotimes_{j=1}^{k} P_{x_j,p_j}\gamma^{(k)}_{\Psi_N}\right]. \tag{16.34}$$

These objects are furthermore related to the density marginals in space and in momentum, respectively defined as

$$\rho^{(k)}_{\Psi_N}(x_1,\dots,x_k) = \binom{N}{k}\int_{\mathbb{R}^{2(N-k)}} |\Psi_N(x_1,\dots,x_N)|^2\,\mathrm{d}x_{k+1}\dots\mathrm{d}x_N \tag{16.35}$$

$$t^{(k)}_{\Psi_N}(p_1,\dots,p_k) = \binom{N}{k}\int_{\mathbb{R}^{2(N-k)}} |\mathcal{F}_\hbar[\Psi_N](p_1,\dots,p_N)|^2\,\mathrm{d}p_{k+1}\dots\mathrm{d}p_N. \tag{16.36}$$

We can now express the convergence of states corresponding to Theorem (16.1) using the above objects. The minimizer of the Vlasov energy (16.17) is

$$m_{\rho^{\mathrm{TF}}}(p,x) = \mathbb{1}\left(|p + \mathbf{A}_e(x) + \beta_0\mathbf{A}[\rho^{\mathrm{TF}}](x)|^2 \le 4\pi\rho^{\mathrm{TF}}(x)\right). \tag{16.37}$$

Its marginals in space and momentum are respectively given by

$$\rho^{\mathrm{TF}}(x) = \frac{1}{4\pi}\left(\lambda^{\mathrm{TF}} - V(x)\right)_+ \tag{16.38}$$

where λ^{TF} is a Lagrange multiplier ensuring normalization, and

$$t^{\mathrm{TF}}(p) = \int_{\mathbb{R}^2} \mathbb{1}_{\left\{|p+\mathbf{A}_e(x)+\beta_0\mathbf{A}[\rho^{\mathrm{TF}}](x)|^2 \le 4\pi\rho^{\mathrm{TF}}(x)\right\}}\mathrm{d}x. \tag{16.39}$$

We will prove the

Theorem 16.5 (**Convergence of states**).
Let $\{\Psi_N\} \subset L^2_{asym}(\mathbb{R}^{2N}) = \bigotimes^N_{\mathrm{asym}} L^2(\mathbb{R}^2)$ be any L^2-normalized sequence such that

$$\left\langle \Psi_N, H_N^R\Psi_N\right\rangle = E^R(N) + o(N)$$

in the limit $\hbar = 1/\sqrt{N}$, $\alpha = \beta_0/N$, $N\to\infty$. Let $R = N^{-\eta}$ for some $0<\eta<1/4$. For any choice of $\hbar_x,\hbar_p\to 0$, related by (16.26)*, and under Assumption 16.1, the Husimi functions*

of Ψ_N converge to those of the Vlasov minimizer

$$m^{(k)}_{\Psi_N}(x_1,p_1,\dots,x_k,p_k) \to \prod_{j=1}^{k} m_{\rho^{\mathrm{TF}}}(p_j,x_j)$$

weakly as measure for all $k \geq 1$, i.e.

$$\int_{\mathbb{R}^{4k}} m^{(k)}_{\Psi_N}\phi_k \to \int_{\mathbb{R}^{4k}} (m_{\rho^{\mathrm{TF}}})^{\otimes k}\phi_k$$

for any bounded continous function $\phi_k \in C_b\left(\mathbb{R}^{4k}\right)$.

Consequently we also have the convergence of the k-particles marginals in position and momentum

$$\binom{N}{k}^{-1} \rho^{(k)}_{\Psi_N}(x_1,\dots,x_k) \to \prod_{j=1}^{k} \rho^{\mathrm{TF}}(x_j)$$

$$\binom{N}{k}^{-1} t^{(k)}_{\Psi_N}(p_1,\dots,p_k) \to \prod_{j=1}^{k} t^{\mathrm{TF}}(x_j)$$

weakly as measures.

The convergence of Wigner functions follows [**147**, Theorem 2.1]. The result above does not depend on the choice of function f used to construct the coherent states [**147**, Lemma 2.8]. Likewise, the result is the same with any choice of scaling of $\hbar_x, \hbar_p \to 0$ as long as (16.26) is satisfied. In our proof of energy convergence it will however be convenient to make a particular choice, and in particular use squeezed coherent states, $\hbar_x \ll \hbar_p$.

Recall that the semi-classical Thomas-Fermi energy does not depend on α, but the Vlasov minimizer does. Anyonic features are retained e.g. in the limiting object's momentum distribution (16.39).

5. Outline and Sketch of the proof

Our general strategy is inspired by works on mean-field limits for interacting fermions [**100, 102, 103, 50, 82, 155**], in particular by the method of [**147**]. Several improvements are required to handle the singularity of the anyonic Hamiltonian that emerges in the limit $R \to 0$. In particular

- we replace the use of the Hewitt-Savage theorem by that of its' quantitative version, the Diaconis-Freedman theorem.
- we implement the Pauli principle quantitatively at the semi-classical level.

To obtain an energy upper bound (Section 17) we use a Slater determinant as trial state, leading by Wick's theorem to a Hartree-Fock-like energy. We then discard exchange terms to get to the simpler Hartree energy (16.14). Finally, a suitable choice of the trial state's one-body density matrix γ allows to take the semi-classical limit and establish an upper bound in terms of the Vlasov energy(16.17).

For the correponding lower bound we first express (Section 18) the energy semi-classically, in terms of the Husimi functions. The remainder terms are shown to be negligible when $N \to \infty$. The use of squeezed coherent states with $\hbar_x \ll \hbar_p$ takes into account the singularities of the Hamiltonian, more severe in position than in momentum space.

Section 19 contains the main novelties compared to [**147**]. There we tackle the mean-field limit of the semi-classical functional expressed in terms of Husimi functions. We use the Diaconis-Freedman theorem [**41**] to express the latter as statistical superpositions of factorized measures, with a quantitatively controled error. We then estimate quantitatively the probability for the Diaconis-Freedman measure to violate the Pauli-principle (16.20). This leads to the sought-after energy lower bound and completes the proof of Theorem 16.1. The convergence of states Theorem 16.5 follows as a corollary.

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CHAPTER 17

Energy upper bound

In this section we derive an upper bound to the ground state energy (16.12). We first introduce a Hartree-Fock trial state and apply Wick's theorem to calculate its energy. We next discard lower contributions (exchange terms) to obtain a regularized ($R > 0$) Hartree energy in Lemma 17.4 and use Lemma 17.5 to send the regularization to 0. We conclude by using a particular sequence of semi-classical states, which will make the Hartree energy converge to the Thomas-Fermi one. All in all this will prove the

PROPOSITION 17.1 (**Energy upper bound**).
Under Assumption 16.1, setting $R = N^{-\eta}$ with $\eta < 1/2$, we have

$$\limsup_{N\to\infty} \frac{E^R(N)}{N} \leq e_{\mathrm{TF}}. \tag{17.1}$$

1. Hartree-Fock trial state

We recall the definition of creation and annihlation operators of a one-body state $f \in L^2(\mathbb{R}^2)$ (for more details about second quantization see [**146, 64**])

$$\begin{aligned} a(f) \sum_{\sigma\in\Sigma_N} (-1)^{\mathrm{sgn}(\sigma)} f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(N)} &= \sqrt{N} \sum_{\sigma\in\Sigma_N} (-1)^{\mathrm{sgn}(\sigma)} \left\langle f, f_{\sigma(1)} \right\rangle f_{\sigma(2)} \otimes \ldots \otimes f_{\sigma(N)} \\ a^\dagger(f) \sum_{\sigma\in\Sigma_N} (-1)^{\mathrm{sgn}(\sigma)} f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(N)} &= \frac{1}{\sqrt{N+1}} \sum_{\sigma\in\Sigma_{N+1}} (-1)^{\mathrm{sgn}(\sigma)} f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(N+1)}. \end{aligned} \tag{17.2}$$

Sums are over the permutation group and $\mathrm{sgn}(\sigma)$ is the signature of a permutation. Let now $\{\psi_j\}_{j=1\ldots N} \in L^2(\mathbb{R}^2)$ be an orthonormal family. We use it to construct a Slater determinant Ψ_N^{SL} (also called Hartree-Fock state)

$$\Psi_N^{\mathrm{SL}}(x_1, ..., x_N) = \frac{1}{\sqrt{N!}} \det\left(\psi_i(x_j)\right) = \sum_{\sigma\in\Sigma_N} (-1)^{\mathrm{sgn}(\sigma)} \prod_{j=1}^{N} \psi_{\sigma(j)}(x_j). \tag{17.3}$$

Denoting $a_j^\dagger = a^\dagger(\psi_j)$ and $a_j = a(\psi_j)$ we have:

$$\left\langle a_j^\dagger a_k \right\rangle_{\mathrm{SL}} = \left\langle \Psi_N^{\mathrm{SL}}, a_j^\dagger a_k \Psi_N^{\mathrm{SL}} \right\rangle = \delta_{jk}.$$

Hence the associated 1-particle reduced density matrix (16.33) and density are

$$\gamma^{(1)}_{\Psi_N^{\mathrm{SL}}} = N \,\mathrm{Tr}_{2\to N}\left[\left|\Psi_N^{\mathrm{SL}}\right\rangle\left\langle\Psi_N^{\mathrm{SL}}\right|\right] = \sum_j \left|\psi_j\right\rangle\left\langle\psi_j\right|, \quad \rho_\gamma = \sum_j \left|\psi_j\right|^2. \tag{17.4}$$

with $\mathrm{Tr}_{2\to N}$ the partial trace. Replacing the last formula in (16.34) gives us the 1-particle reduced Husimi measure

$$m^{(1)}_{\Psi_N^{\mathrm{SL}}}(x,p) = \sum_{j=1}^{N} \Big|\big\langle \psi_j, F_{x,p}\big\rangle\Big|^2.$$

We will use Wick's theorem, see [**125**, Corollary IV.6] or [**146**].

THEOREM 17.2 (**Corollary of Wick's theorem**).
Let $a_1^\sharp, \dots, a_{2n}^\sharp$ be creation or annihilation operators. We have that

$$\Big\langle a_1^\sharp \dots a_{2n}^\sharp \Big\rangle_{\mathrm{SL}} = \sum_\sigma (-1)^{\mathrm{sgn}(\sigma)} \prod_{j=1}^{n} \Big\langle a_{\sigma(2j-1)}^\sharp a_{\sigma(2j)}^\sharp \Big\rangle_{\mathrm{SL}}$$

where the sum is over all pairings, i.e. permutations of the $2n$ indices such that $\sigma(2j-1) < \min\{\sigma(2j), \sigma(2j+1)\}$ for all j.

We evaluate

$$\big\langle \Psi_N^{\mathrm{SL}}, H_N^R \Psi_N^{\mathrm{SL}}\big\rangle = \big\langle H_N^R\big\rangle_{\mathrm{SL}}$$

term by term as in (16.13). Until the end of this section $\langle\cdot\rangle$ means $\langle\cdot\rangle_{\mathrm{SL}}$. We will use the following notation (recall (16.11))

$$W_j = \big(p_j^{\mathbf{A}}\big)^2 + V(x_j) \tag{17.5}$$

$$W_{jk} = p_j^{\mathbf{A}} \cdot \nabla^\perp w_R(x_j - x_k) + \nabla^\perp w_R(x_j - x_k) \cdot p_j^{\mathbf{A}} \tag{17.6}$$

$$W_{jk\ell} = \nabla^\perp w_R(x_j - x_k) \cdot \nabla^\perp w_R(x_j - x_\ell). \tag{17.7}$$

Recalling the definition of the Hartree energy functional

$$\mathcal{E}_R^{\mathrm{af}}[\gamma] = \frac{1}{N}\,\mathrm{Tr}\left[\big(p_1^{\mathbf{A}} + \alpha \mathbf{A}^R[\rho_\gamma]\big)^2 \gamma\right] + \frac{1}{N}\int_{\mathbb{R}^2} V(x)\rho_\gamma(x)\mathrm{d}x$$

we have the following

LEMMA 17.3 (**Hartree-Fock energy**).
With Ψ^{SL} as above and γ the associated one-particle density matrix (17.4) *we have*

$$\begin{aligned}
\big\langle H_N^R\big\rangle_{\mathrm{SL}} &= N\mathcal{E}_R^{\mathrm{af}}[\gamma] - \alpha\,\mathrm{Tr}\,(W_{12}U_{12}\gamma\otimes\gamma) \\
&\quad + \alpha^2 \int_{\mathbb{R}^4} \Big|\nabla^\perp w_R(x_1 - x_2)\Big|^2 \Big(\rho_\gamma(x_1)\rho_\gamma(x_2) - |\gamma(x_1,x_2)|^2\Big)\mathrm{d}x_1\mathrm{d}x_2 \\
&\quad - \alpha^2 \int_{\mathbb{R}^6} W_{123}\Big[\rho_\gamma(x_1)|\gamma(x_2,x_3)|^2 + \rho_\gamma(x_2)|\gamma(x_1,x_3)|^2 + \rho_\gamma(x_3)|\gamma(x_1,x_2)|^2\Big]\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\
&\quad + \alpha^2 \int_{\mathbb{R}^6} W_{123}\Big[2\Re\Big(\gamma(x_1,x_2)\gamma(x_2,x_3)\gamma(x_3,x_1)\Big)\Big]\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3.
\end{aligned} \tag{17.8}$$

In the first line we have denoted by U_{12} the exchange operator

$$\big(U_{12}\psi\big)(x_1,x_2) = \psi(x_2,x_1). \tag{17.9}$$

PROOF. By definition

$$\left\langle \sum_{j=1}^{N} (p_j^{\mathbf{A}})^2 + V(x_j) \right\rangle = \mathrm{Tr}\left[\big((p^{\mathbf{A}})^2 + V\big)\,\gamma\right].$$

For the two- and three-body terms we express higher density matrices of Ψ^{SL} using Wick's theorem. To this end, let $(\psi_j)_j \in L^2(\mathbb{R}^2)$ be an orthonormal basis, and $a_j, a_j^\dagger$ the associated annihilation and creation operators.

Two-body terms. We have

$$\left\langle a_\alpha^\dagger a_\beta^\dagger a_\varepsilon a_\theta \right\rangle = \delta_{\alpha\theta}\delta_{\beta\varepsilon} - \delta_{\alpha\varepsilon}\delta_{\beta\theta}$$

where δ_{ij} is the Kronecker delta. Recall that $\gamma := \gamma^{(1)}_{\Psi^{\mathrm{SL}}}$. It follows that

$$\gamma^{(2)}_{\Psi^{\mathrm{SL}}} = \gamma \otimes \gamma - U_{12}\gamma \otimes \gamma \tag{17.10}$$

with the exchange operator defined as in (17.9). Using [**146**, Lemma 7.12] we have

$$\sum_{j\neq k} W_{jk} = \sum_{\alpha,\beta,\theta,\varepsilon} \langle \psi_\alpha \otimes \psi_\beta, W_{12}\ \psi_\theta \otimes \psi_\varepsilon \rangle\ a_\alpha^\dagger a_\beta^\dagger a_\varepsilon a_\theta$$

on $\bigotimes^N_{\mathrm{asym}} L^2\left(\mathbb{R}^2\right)$ and thus

$$\left\langle \sum_{j\neq k} W_{j,k} \right\rangle = \mathrm{Tr}\left[\left(p^{\mathbf{A}}.\mathbf{A}^R[\rho_\gamma] + \mathbf{A}^R[\rho_\gamma].p^{\mathbf{A}}\right)\gamma\right] - \mathrm{Tr}\left(W_{12}U_{12}\gamma\otimes\gamma\right). \tag{17.11}$$

Similarly we obtain

$$\left\langle \sum_{j\neq k} \left|\nabla^\perp w_R(x_j - x_k)\right|^2 \right\rangle = \int_{\mathbb{R}^4} \left|\nabla^\perp w_R(x_1-x_2)\right|^2 \Big(\rho_\gamma(x_1)\rho_\gamma(x_2) - |\gamma(x_1,x_2)|^2\Big)\mathrm{d}x_1\mathrm{d}x_2 \tag{17.12}$$

Three-body term. Using [**146**, Lemma 7.12] again we write

$$\sum_{j\neq k\neq l} W_{jkl} = \sum_{\alpha,\beta,\theta,\delta,\varepsilon,\zeta} \langle \psi_\alpha \otimes \psi_\beta \otimes \psi_\theta, W_{123}\ \psi_\varepsilon \otimes \psi_\zeta \otimes \psi_\eta \rangle\ a_\alpha^\dagger a_\beta^\dagger a_\theta^\dagger a_\eta a_\zeta a_\varepsilon.$$

Applying Wick's theorem (17.2) we obtain

$$\left\langle a_\alpha^\dagger a_\beta^\dagger a_\theta^\dagger a_\eta a_\zeta a_\varepsilon \right\rangle = \delta_{\alpha\varepsilon}\delta_{\beta\zeta}\delta_{\theta\eta} + \delta_{\alpha\zeta}\delta_{\beta\eta}\delta_{\theta\varepsilon} + \delta_{\alpha\eta}\delta_{\beta\varepsilon}\delta_{\theta\zeta} - \delta_{\alpha\varepsilon}\delta_{\beta\eta}\delta_{\theta\zeta} - \delta_{\alpha\zeta}\delta_{\beta\varepsilon}\delta_{\theta\eta} - \delta_{\alpha\eta}\delta_{\beta\zeta}\delta_{\theta\varepsilon}$$

and we deduce the expression

$$\gamma^{(3)}_{\Psi^{\mathrm{SL}}} = (\mathbb{1} + U_{12}U_{23} + U_{23}U_{12} - U_{12} - U_{13} - U_{23})\,\gamma\otimes\gamma\otimes\gamma \tag{17.13}$$

for the 3-body density matrix of the Slater determinant, where the exchange operators U_{ij} are natural extensions of (17.9) to the three-particles space. Gathering the above expressions yields

$$\begin{aligned}\left\langle \sum_{j\neq k\neq l} W_{jkl} \right\rangle &= \int_{\mathbb{R}^6} W_{123}\Big[\rho_\gamma(x_1)\rho_\gamma(x_2)\rho_\gamma(x_3)\Big]\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\ &+ \int_{\mathbb{R}^6} W_{123}\Big[2\Re\Big(\gamma(x_1,x_2)\gamma(x_2,x_3)\gamma(x_3,x_1)\Big)\Big]\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \qquad (17.14)\\ &- \int_{\mathbb{R}^6} W_{123}\Big[\rho_\gamma(x_1)|\gamma(x_2,x_3)|^2 + \rho_\gamma(x_2)|\gamma(x_1,x_3)|^2 + \rho_\gamma(x_3)|\gamma(x_1,x_2)|^2\Big]\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3. \qquad (17.15)\end{aligned}$$

□

We now want to discard the exchange terms and the singular two-body terms to reduce the above expression to the Hartree functional.

LEMMA 17.4 (**From Hartree-Fock to Hartree**).
Recalling the notation (16.11)*, assume that*

$$\operatorname{Tr}\left(|p^{\mathbf{A}}|^2\gamma\right) \leq CN.$$

Then

$$E^R(N) \leq N\mathcal{E}^{\mathrm{af}}_R[\gamma] + \frac{C}{R^2}. \tag{17.16}$$

PROOF. We recall the bound

$$\left\|\nabla^\perp w_R\right\|_{L^\infty} \leq \frac{C}{R} \tag{17.17}$$

from [**59**, Lemma 2.1] and the identities $\gamma = \gamma^2 = \gamma^*$ with

$$\operatorname{Tr}\gamma = \int_{\mathbb{R}^2} \rho_\gamma(x)dx = \int_{\mathbb{R}^2} \gamma(x,x)\mathrm{d}x = N$$

for the density matrix (17.4) of a Slater determinant. It follows that

$$\left|\int_{\mathbb{R}^4} \left|\nabla^\perp w_R(x_1-x_2)\right|^2 \Big(\rho_\gamma(x_1)\rho_\gamma(x_2) - |\gamma(x_1,x_2)|^2\Big)\mathrm{d}x_1\mathrm{d}x_2\right| \leq C\frac{N^2}{R^2}.$$

Next we use Cauchy-Schwarz to obtain

$$|W_{12}+W_{21}| \leq \varepsilon\left(|p_1^{\mathbf{A}}|^2 + |p_2^{\mathbf{A}}|^2\right) + \frac{2}{\varepsilon}|\nabla^\perp w_R(w_1-x_2)|^2$$

and hence, since both sides commute with U_{12},

$$|\alpha\operatorname{Tr}\left(W_{12}U_{12}\gamma\otimes\gamma\right)| \leq \frac{C\varepsilon}{N}\operatorname{Tr}\left(|p^{\mathbf{A}}|^2\gamma\right) + \frac{C}{N\varepsilon}\int_{\mathbb{R}^4}\left|\nabla^\perp w_R(x_1-x_2)\right|^2|\gamma(x_1,x_2)|^2\mathrm{d}x_1\mathrm{d}x_2$$

where we used that

$$\operatorname{Tr}\left(|p_1^{\mathbf{A}}|^2 U_{12}\gamma\otimes\gamma\right) = \operatorname{Tr}\left(|p^{\mathbf{A}}|^2\gamma\right).$$

Optimizing over ε gives

$$|\alpha\operatorname{Tr}\left(W_{12}U_{12}\gamma\otimes\gamma\right)| \leq \frac{C}{R}.$$

Next, using (17.17) again,

$$\left|\int_{\mathbb{R}^6} W_{123}\rho_\gamma(x_1)|\gamma(x_2,x_3)|^2\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\right| \leq \frac{C}{R^2}\left(\int_{\mathbb{R}^2}\rho_\gamma(x)dx\right)\left(\int_{\mathbb{R}^4}|\gamma(x,y)|^2\,dxdy\right) \leq C\frac{N^2}{R^2}$$

and we obtain a similar bound on the last term of (17.8) noticing that

$$\begin{aligned}
|\gamma(x_1,x_2)\gamma(x_2,x_3)\gamma(x_3,x_1)| &\leq \sum_j |\gamma(x_1,x_2)\psi_j(x_3)|\,|\gamma(x_2,x_3)\psi_j(x_1)| \\
&\leq \frac{1}{2}\sum_j |\gamma(x_1,x_2)|^2\,|\psi_j(x_3)|^2 + \frac{1}{2}\sum_j |\gamma(x_2,x_3)|^2\,|\psi_j(x_1)|^2 \\
&= \frac{1}{2}|\gamma(x_1,x_2)|^2\,\rho_\gamma(x_3) + \frac{1}{2}|\gamma(x_2,x_3)|^2\,\rho_\gamma(x_1).
\end{aligned}$$

□

We now want to take $R \to 0$ in (17.16) and show that $\mathcal{E}_R^{\rm af}$ can be replaced by $\mathcal{E}^{\rm af}$.

LEMMA 17.5 (**Convergence of the regularized energy**).
The functional $\mathcal{E}_R^{\rm af}$ (16.14) converges pointwise to $\mathcal{E}^{\rm af}$ (16.15) as $R \to 0$. More precisely, for any fermionic one-particle density matrix γ_N with

$$\operatorname{Tr} \gamma_N = N$$

and associated density ρ_N we have

$$\left| \mathcal{E}_R^{\rm af}[\gamma_N] - \mathcal{E}^{\rm af}[\gamma_N] \right| \leq CR\mathcal{E}^{\rm af}\left[\gamma_N\right]^{3/2}. \tag{17.18}$$

PROOF. In terms of

$$\widetilde{\gamma} = N^{-1}\gamma_N \text{ and } \widetilde{\rho} = N^{-1}\rho_{\gamma_N}$$

we have the expression

$$\mathcal{E}_R^{\rm af}[\gamma_N] = \operatorname{Tr}\left(\left(p^{\mathbf{A}} + \beta_0 \mathbf{A}_R[\widetilde{\rho}]\right)^2 \widetilde{\gamma} \right) + \int_{\mathbb{R}^2} V\widetilde{\rho} \tag{17.19}$$

and similarly for $\mathcal{E}^{\rm af}$, which is the case $R = 0$. The Lieb-Thirring inequality [**94**, Theorem 4.3] yields

$$\operatorname{Tr}\left(\left(p^{\mathbf{A}} + \beta_0 \mathbf{A}_R[\widetilde{\rho}]\right)^2 \widetilde{\gamma} \right) \geq C \int_{\mathbb{R}^2} \widetilde{\rho}^2$$

and

$$\operatorname{Tr}\left(\left(p^{\mathbf{A}}\right)^2 \widetilde{\gamma} \right) \geq C \int_{\mathbb{R}^2} \widetilde{\rho}^2$$

while the weak Young inequality (see e.g. similar estimates in [**59**, Appendix A], [**112**, Appendix A] or Appendix 1 below) gives

$$\operatorname{Tr}\left(\left|\mathbf{A}_R[\widetilde{\rho}]\right|^2 \widetilde{\gamma} \right) = \int_{\mathbb{R}^2} \left|\mathbf{A}_R[\widetilde{\rho}]\right|^2 \widetilde{\rho} \leq C \int_{\mathbb{R}^2} \widetilde{\rho}^2.$$

Hence a use of the Cauchy-Schwarz inequality implies

$$\mathcal{E}_R^{\rm af}[\gamma_N] \geq C \operatorname{Tr}\left(\left(p^{\mathbf{A}}\right)^2 \widetilde{\gamma} \right).$$

We now expand the two energies of (17.19) and begin by estimating the squared terms using the Hölder and Yound inequalities

$$\begin{aligned} \left\| \left(\left|\mathbf{A}^R[\widetilde{\rho}]\right|^2 - \left|\mathbf{A}[\widetilde{\rho}]\right|^2 \right) \widetilde{\rho} \right\|_{L^1} &\leq \left\| \left|\mathbf{A}^R[\widetilde{\rho}]\right|^2 - \left|\mathbf{A}[\widetilde{\rho}]\right|^2 \right\|_{L^2}^2 \|\widetilde{\rho}\|_{L^2} \\ &\leq \|(\nabla w_R - \nabla w_0) * \widetilde{\rho}\|_{L^4} \|(\nabla w_R + \nabla w_0) * \widetilde{\rho}\|_{L^4} \|\widetilde{\rho}\|_{L^2} \\ \leq \|\nabla w_R - \nabla w_0\|_{L^{4/3}} \|\nabla w_R + \nabla w_0\|_{L^{4/3}} \|\widetilde{\rho}\|_{L^2}^3 &\leq CR\mathcal{E}^{\rm af}\left[\gamma_N\right]^{3/2} \end{aligned}$$

as per the above estimates, and bounds on ∇w_R following from [**59**, Lemma 2.1]. As for the mixed term

$$\begin{aligned}\mathrm{Tr}\left[\left(\mathbf{A}^R[\widetilde{\rho}]-\mathbf{A}[\widetilde{\rho}]\right).p^{\mathbf{A}}\widetilde{\gamma}\right] &\leq \left\|p^{\mathbf{A}}\sqrt{\widetilde{\gamma}}\right\|_{\mathfrak{S}^2}\left\|\left(\mathbf{A}^R[\widetilde{\rho}]-\mathbf{A}[\widetilde{\rho}]\right)\sqrt{\widetilde{\gamma}}\right\|_{\mathfrak{S}^2}\\ &\leq C\mathcal{E}^{\mathrm{af}}\left[\widetilde{\gamma}\right]^{1/2}\left\|\left|\mathbf{A}^R[\widetilde{\rho}]-\mathbf{A}[\widetilde{\rho}]\right|\right\|_{L^4}\|\widetilde{\rho}\|_{L^2}^{1/2}\\ &\leq C\mathcal{E}^{\mathrm{af}}\left[\widetilde{\gamma}\right]^{1/2}\|(\nabla w_R-\nabla w_0)*\widetilde{\rho}\|_{L^4}\|\widetilde{\rho}\|_{L^2}^{1/2}\\ &\leq C\mathcal{E}^{\mathrm{af}}\left[\widetilde{\gamma}\right]^{1/2}\|\nabla w_R-\nabla w_0\|_{L^{4/3}}\|\widetilde{\rho}\|_{L^2}^{3/2}\\ &\leq CR\mathcal{E}^{\mathrm{af}}\left[\widetilde{\gamma}\right]^{3/2}\end{aligned}$$

where $||W||_{\mathfrak{S}^p}=\mathrm{Tr}\left[|W|^p\right]^{\frac{1}{p}}$ is the Schatten norm [**140**, Chapter 2] and where we used Young's inequality. □

Inserting (17.18) in (17.4) we get

$$\frac{E^R(N)}{N}\leq \mathcal{E}^{\mathrm{af}}[\gamma_N]+CR+\frac{C}{NR^2} \tag{17.20}$$

for a sequence γ_N with uniformly bounded Hartree energy. The last step consists in constructing such a sequence γ_N, whose energy will behave semi-classicaly.

2. Semi-classical upper bound for the Hartree energy

We use [**147**, Lemma 3.2], whose statement we reproduce for the convenience of the reader.

LEMMA 17.6 (**Semi-classical limit of the Hartree energy**).
Let $\rho\geq 0$ be a fixed function in $C_c^\infty(\mathbb{R}^2)$ with support in the square $C_r=(-r/2,r/2)^2$ such that $\rho\geq 0$ and $\int_{C_r}\rho=1$. Let $\mathbf{A}\in L^4(C_r)$ be a magnetic vector potential. If we define

$$\gamma_N=\mathbb{1}\Big((-\mathrm{i}\hbar\nabla+\mathbf{A})^2_{C_r}-4\pi\rho(x)\leq 0\Big) \tag{17.21}$$

where $((-\mathrm{i}\hbar\nabla+\mathbf{A})^2_{C_r}$ is the magnetic Dirichlet Laplacian in the cube and $\hbar=1/\sqrt{N}$ we have

$$\lim_{N\to\infty}N^{-1}\,\mathrm{Tr}\left(\left(-\mathrm{i}\hbar\nabla+\mathbf{A}\right)^2\gamma_N\right)=2\pi\int_{\mathbb{R}^2}\rho(x)^2\mathrm{d}x \tag{17.22}$$

and

$$\lim_{N\to\infty}N^{-1}\,\mathrm{Tr}\,\gamma_N=\int_{\mathbb{R}^2}\rho(x)\mathrm{d}x\ \ \textit{with}\ \ \frac{\rho_{\gamma_N}}{N}\to\rho \tag{17.23}$$

and weakly- in $L^\infty(\mathbb{R}^2)$, strongly in $L^1(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$. Furthermore, the same properties stay true if we replace γ_N by $\widetilde{\gamma}_N$ the projection onto the N lowest eigenvectors of $(-\mathrm{i}\hbar\nabla+\mathbf{A})^2_{C_r}-4\pi\rho(x)$.*

The convergences we claim for $N^{-1}\rho_{\gamma_N}$ are stronger than in the statement of [**147**, Lemma 3.2]. They easily follow from the proof therein.

We are now able to complete the proof of Proposition 17.1. We take

$$\gamma_N=\mathbb{1}\Big(\left(-\mathrm{i}\hbar\nabla+\mathbf{A}_e+\beta_0\mathbf{A}[\rho^{\mathrm{TF}}]\right)^2_{C_r}-4\pi\rho^{\mathrm{TF}}(x)\leq 0\Big) \tag{17.24}$$

and $\widetilde{\gamma}_N$ the associated rank-N projector, as in the above lemma. Therefore we know by the above theorem that

$$\frac{\rho_{\widetilde{\gamma}_N}}{N} \to \rho^{\mathrm{TF}}$$

weakly-* in $L^\infty(\mathbb{R}^2)$, strongly in $L^1(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$. We have that (the definition of the weak L^2 space is recalled in Appendix 1 below)

$$\nabla^\perp w_0 = \nabla^\perp \log|x| \in L^1_{\mathrm{loc}}(C_r) \cap L^{2,w}(\mathbb{R}^2).$$

We deduce that

$$\alpha \mathbf{A}[\rho_{\widetilde{\gamma}_N}] \to \beta_0 \mathbf{A}[\rho^{\mathrm{TF}}]$$

strongly in $L^\infty(\mathbb{R}^2)$, by the weak Young inequality [**92**, Chapter 4]. We thus have

$$\mathcal{E}^{\mathrm{af}}\left[\widetilde{\gamma}_N\right] = \frac{(1+o_N(1))}{N} \operatorname{Tr}\left[\left(p^{\mathbf{A}} + \beta_0 \mathbf{A}[\rho^{\mathrm{TF}}]\right)^2 \widetilde{\gamma}_N\right] + \frac{1}{N}\int_{\mathbb{R}^2} V(x)\rho_{\widetilde{\gamma}_N}(x)\mathrm{d}x + o_N(1) \quad (17.25)$$

We now use (17.22) to take the limit of the trace and (17.23) to take the limit of the potential term ($V \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ by assumption).

Combining with the bounds from Lemmas 17.5 and 17.4 we finally obtain

$$\begin{aligned}
\frac{E^R(N)}{N} &\leq \mathcal{E}^{\mathrm{af}}\left[\widetilde{\gamma}_N\right] + CR + \frac{C}{R^2 N} \\
&\leq 2\pi \int_{\mathbb{R}^2} \rho_{\mathrm{TF}}^2(x)\mathrm{d}x + \int_{\mathbb{R}^2} V(x)\rho_{\mathrm{TF}}(x)\mathrm{d}x + o_N(1) \\
&= e_{\mathrm{TF}} + o_N(1)
\end{aligned}$$

provided $R = N^{-\eta}$ with $0 < \eta < \frac{1}{2}$, completing the proof of Proposition 17.1.

CHAPTER 18

Energy lower bound: semi-classical limit

In this section we first derive some useful a priori bounds and properties of the Husimi functions. Next we express our energy in terms of the latter and obtain a classical energy approximation on the phase space, plus some errors terms that we show to be negligible. We will use this expression in Section 19 to construct our lower bound.

1. A priori estimates.

LEMMA 18.1 (**Kinetic energy bound**).
Let $\Psi_N \in L_{\mathrm{asym}}\left(\mathbb{R}^{2N}\right)$ be a fermionic wave-function such that

$$\left\langle \Psi_N, H_N^R \Psi_N \right\rangle \leq CN.$$

Then

$$\frac{1}{N^2}\left\langle \Psi_N, \left(\sum_{j=1}^N -\Delta_j\right)\Psi_N \right\rangle \leq \frac{C}{R^2} \tag{18.1}$$

and

$$\int_{\mathbb{R}^2}\left(\frac{\rho_{\Psi_N}^{(1)}}{N}\right)^2 \leq \frac{C}{R^2}. \tag{18.2}$$

Moreover

$$\int_{\mathbb{R}^2} V(x)\rho_{\Psi_N}^{(1)}(x)\mathrm{d}x \leq CN. \tag{18.3}$$

PROOF. We expand the Hamiltonian and use the Cauchy-Schwarz inequality for operators to get

$$\begin{aligned} H_N^R &= \sum_{j=1}^N \left(\left(p_j^{\mathbf{A}}\right)^2 + \alpha p_j^{\mathbf{A}} \cdot \mathbf{A}_j^R + \alpha \mathbf{A}_j^R . p_j^{\mathbf{A}} + \alpha^2 |\mathbf{A}_j^R|^2 + V(x_j)\right) \\ &\geq \sum_{j=1}^N \left((1-2\delta^{-1})\left(p_j^{\mathbf{A}}\right)^2 + (1-2\delta)\alpha^2|\mathbf{A}_j^R|^2 + V(x_j)\right) \\ &= \sum_{j=1}^N \left(\frac{1}{2}(\left(p_j^{\mathbf{A}}\right)^2 + V(x_j)) - 7\alpha^2|\mathbf{A}_j^R|^2\right) \end{aligned} \tag{18.4}$$

choosing $\delta = 4$. Thus we have

$$\mathrm{Tr}\left[\left(\left(p^{\mathbf{A}}\right)^2 + V\right)\gamma_N^{(1)}\right] \leq CN + 7\alpha^2\left\langle \Psi_N, \sum_{j=1}^N |\mathbf{A}_j^R|^2 \Psi_N \right\rangle$$

where $\gamma_N^{(1)}$ is the 1-particle reduced density matrix

$$N\mathrm{Tr}_{2\to N}\left[|\Psi_N\rangle\langle\Psi_N|\right].$$

We estimate the second term of the above using [**59**, Lemma 2.1]:

$$\left\langle \Psi_N, \sum_{j=1}^N |\mathbf{A}_j^R|^2 \Psi_N \right\rangle \leq \frac{CN^3}{R^2}.$$

Recalling that $\alpha = N^{-1}$ we get

$$\mathrm{Tr}\left[\left(\left(p^{\mathbf{A}}\right)^2 + V\right)\gamma_N^{(1)}\right] \leq \frac{CN}{R^2}$$

Now we use that

$$|-\mathrm{i}\hbar\nabla + \mathbf{A}_e|^2 \geq -\frac{\hbar^2}{2}\Delta - |\mathbf{A}_e|^2$$

and that $\hbar^2 = N^{-1}$ to obtain

$$C + \frac{CN}{R^2} \geq \frac{C}{N}\left\langle \Psi_N, \left(\sum_{j=1}^N -\Delta_j\right)\Psi_N \right\rangle - \mathrm{Tr}\left[|\mathbf{A}_e|^2\gamma_N^{(1)}\right]$$

recalling that $V + C \geq 0$. But $|\mathbf{A}_e|^2 \in L^\infty(\mathbb{R}^2)$, we thus deduce (18.1). We obtain (18.2) by using the Lieb-Thirring inequality [**104, 105**]

$$\mathrm{Tr}\left[-\Delta\gamma_N^{(1)}\right] \geq C\int_{\mathbb{R}^2}\left(\rho_{\psi_N}^{(1)}(x)\right)^2 \mathrm{d}x$$

combined with (18.1).

Finally (18.3) is a straightforward consequence of the positivity of the full kinetic energy (first term in (16.3)). □

2. Properties of the phase space measures

The following connects the space and momentum densities (16.35) and (16.36) with the Husimi function and is extracted from [**147**, Lemma 2.4]. The generalization to squeezed states we use does not change the properties of the phase space measures.

LEMMA 18.2 (**Densities and fermionic semi-classical measures**).
Let $F_{\hbar_x}$, $G_{\hbar_p}$ be defined as in (16.28) *and* (16.29) *(recall that $\hbar = \sqrt{\hbar_x}\sqrt{\hbar_p}$). Let $\Psi_N \in L^2_{\mathrm{asym}}(\mathbb{R}^{2N})$ be any normalized fermionic wave function. We have*

$$\frac{1}{(2\pi)^{2k}}\int_{\mathbb{R}^{2k}} m_{\Psi_N}^{(k)}(x_1,p_1,...,x_k,p_k)\mathrm{d}p_1...\mathrm{d}p_k = k!\hbar^{2k}\rho_{\Psi_N}^{(k)} * \left(|F_{\hbar_x}|^2\right)^{\otimes k} \tag{18.5}$$

and

$$\frac{1}{(2\pi)^{2k}}\int_{\mathbb{R}^{2k}} m_{\Psi_N}^{(k)}(x_1,p_1,...,x_k,p_k)\mathrm{d}x_1...\mathrm{d}x_k = k!\hbar^{2k}t_{\Psi_N}^{(k)} * \left(|G_{\hbar_p}|^2\right)^{\otimes k} \tag{18.6}$$

The proof is similar to considerations from [**147**]. We recall it in Appendix 3 for completeness.

LEMMA 18.3 (**Properties of the phase space measures**).
Let $\Psi_N \in L^2_{\rm asym}(\mathbb{R}^{2N})$ be a normalized fermionic wave function. For every $1 \leq k \leq N$ the function $m^{(k)}_{\Psi_N}$ defined in (16.32) *is symmetric and satisfies*

$$0 \leq m^{(k)}_{\Psi_N} \leq 1 \text{ a.e. on } \mathbb{R}^{4k}. \tag{18.7}$$

In addition

$$\frac{1}{(2\pi)^{2k}} \int_{\mathbb{R}^{4k}} m^{(k)}_{\Psi_N}(x_1, p_1, ..., x_k, p_k) \mathrm{d}x_1 ... \mathrm{d}p_k = N(N-1)...(N-k+1)\hbar^{2k} \to 1 \tag{18.8}$$

when N tends to infinity and

$$\frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} m^{(k)}_{\Psi_N}(x_1, p_1, ..., x_k, p_k) \mathrm{d}x_k \mathrm{d}p_k = \hbar^2 (N-k+1) m^{(k-1)}_{\Psi_N}(x_1, p_1, ..., x_{k-1}, p_{k-1})$$

We refer to [**147**, Lemma 2.2] for the proof.

3. Semi-classical energy

We now define a semi-classical analogue of the original energy functional, given in terms of Husimi functions. We denote $\mathrm{d}x_{ij} = \mathrm{d}x_i \mathrm{d}x_j$:

$$\begin{aligned}
\mathcal{E}^R_{\rm C}\left[m^{(3)}_{\Psi_N}\right] = &\frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \left(|p + \mathbf{A}_e(x)|^2 + V(x)\right) m^{(1)}_{\Psi_N}(x,p) \mathrm{d}x \mathrm{d}p \\
&+ \frac{2\beta_0}{(2\pi)^4} \int_{\mathbb{R}^8} (p_1 + \mathbf{A}_e(x_1)) \cdot \nabla^\perp w_R(x_1 - x_2) m^{(2)}_{\Psi_N}(x_{12}, p_{12}) \mathrm{d}x_{12} \mathrm{d}p_{12} \\
&+ \frac{\beta_0^2}{(2\pi)^6} \int_{\mathbb{R}^{12}} \nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3) m^{(3)}_{\Psi_N}(x_{123}, p_{123}) \mathrm{d}x_{123} \mathrm{d}p_{123}.
\end{aligned} \tag{18.9}$$

Our aim is to show that it correctly captures the leading order of the full quantum energy.

PROPOSITION 18.4 (**Semi-classical energy functional**).
Pick some $\varepsilon > 0$ and set

$$\sqrt{\hbar_x} = N^{-1/2+\varepsilon}, \quad \sqrt{\hbar_p} = N^{-\varepsilon}. \tag{18.10}$$

Pick a sequence Ψ_N of fermionic wave-functions such that

$$\left\langle \Psi_N, H^R_N \Psi_N \right\rangle \leq CN. \tag{18.11}$$

Under Assumption 16.1 we have

$$\frac{\left\langle \Psi_N, H^R_N \Psi_N \right\rangle}{N} \geq (1 - o_N(1)) \, \mathcal{E}^R_{\rm C}\left[m^{(3)}_{\Psi_N}\right] - CN^{4\eta - 1 + 2\varepsilon} - CN^{2\eta - 1} - CN^{-2\varepsilon} \tag{18.12}$$

We first derive an exact expression, whose extra unwanted terms will be estimated below.

LEMMA 18.5 (**Semi-classical energy with errors**).
In order to lighten the notation, we denote in this lemma and its proof

$$\langle A \rangle = \langle \Psi_N, A\Psi_N \rangle .$$

Let Ψ_N be a fermionic wave-function satisfying the bound (18.11)*, we have the semi-classical bound*

$$\left\langle \sum_{j=1}^{N} \left(p_j^{\mathbf{A}} + \alpha \mathbf{A}^R(x_j)\right)^2 + V(x_j) \right\rangle \geq N\mathcal{E}_{\mathrm{C}}^R \left[m_{\Psi_N}^{(3)}\right]$$

$$+ 2\alpha\Re \left\langle \sum_{j=1}^{N}\sum_{k\neq j} \left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2\right)(x_j - x_k)\cdot(-\mathrm{i}\hbar\nabla_j) \right\rangle \tag{18.13}$$

$$+ 2\alpha \left\langle \sum_{j=1}^{N}\sum_{k\neq j} \mathbf{A}_e(x_j)\cdot\nabla^\perp w_R(x_j - x_k) \right.$$

$$\left. - \int_{\mathbb{R}^2} \mathbf{A}_e(x_j - u)\cdot\left(\nabla^\perp w_R * |F_{\hbar_x}|^2\right)(x_j - x_k - u)|F(u)|^2 \mathrm{d}u \right\rangle \tag{18.14}$$

$$+ \alpha^2 \sum_{j=1\neq k\neq l}^{N} \left\langle \nabla^\perp w_R(x_j - x_k)\cdot\nabla^\perp w_R(x_j - x_l) - \int_{\mathbb{R}^2} \left(\nabla^\perp w_R * |F_{\hbar_x}|^2\right)(x_j - x_k - u) \right.$$

$$\left. \cdot\left(\nabla^\perp w_R * |F_{\hbar_x}|^2\right)(x_j - x_l - u)|F_{\hbar_x}(u)|^2 \mathrm{d}u \right\rangle \tag{18.15}$$

$$+ \mathrm{Err}_1 + \mathrm{Err}_2 \tag{18.16}$$

with

$$\mathrm{Err}_1 = 2\Re \left\langle \sum_{j=1}^{N} (\mathbf{A}_e - \mathbf{A}_e * |F_{\hbar_x}|^2)(x_j)\cdot(-\mathrm{i}\hbar\nabla_j) \right\rangle \tag{18.17}$$

$$+ \left\langle \sum_{j=1}^{N} (|\mathbf{A}_e|^2 - |\mathbf{A}_e|^2 * |F_{\hbar_x}|^2)(x_j) + V - V * |F_{\hbar_x}|^2)(x_j) \right\rangle \tag{18.18}$$

and

$$\mathrm{Err}_2 = \alpha^2 \left\langle \sum_{j=1}^{N}\sum_{k\neq j} |\nabla^\perp w_R(x_j - x_k)|^2 \right\rangle - N\hbar_p \int_{\mathbb{R}^2} |\nabla f(y)|^2 \mathrm{d}y. \tag{18.19}$$

PROOF. We expand the whole energy as in (16.10) and express each term using Husimi's function (16.32). We begin with calculations involving the p variable. Using (16.36)

and (18.6) we can write, for the purely kinetic term,

$$\begin{aligned}
&\left\langle \Psi_N, \left(\sum_{j=1}^N -\hbar^2\Delta_j\right)\Psi_N\right\rangle = \int_{\mathbb{R}^2}|p|^2 t^{(1)}_{\Psi_N}(p)\mathrm{d}p\\
=&\frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} m^{(1)}_{\Psi_N}(x,p)|p|^2\mathrm{d}x\mathrm{d}p + \int_{\mathbb{R}^2\times\mathbb{R}^2} t^{(1)}_{\Psi_N}(p)|G_{\hbar_p}(q-p)|^2(|p|^2-|q|^2)\mathrm{d}p\mathrm{d}q\\
=&\frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} m^{(1)}_{f,\Psi_N}(x,p)|p|^2\mathrm{d}x\mathrm{d}p - \int_{\mathbb{R}^2\times\mathbb{R}^2} t^{(1)}_{\Psi_N}(p)|G_{\hbar_p}(q-p)|^2(|q-p|^2)\mathrm{d}p\mathrm{d}q\\
&-2\left(\int_{\mathbb{R}^2} p t^{(1)}_{\Psi_N}(p)\mathrm{d}p\right)\cdot\left(\int_{\mathbb{R}^2} p|G_{\hbar_p}(p)|^2\mathrm{d}p\right).
\end{aligned}$$

We now use that $G_{\hbar_p}(-p) = \overline{G_{\hbar_p}(p)}$ which makes $|G_{\hbar_p}(p)|^2$ an even function (recall that $G_{\hbar_p}$ is the Fourier tranform of $F_{\hbar_x}$ (16.28)) to discard the last term of the above. On the other hand, using (16.36) we find

$$\begin{aligned}
&\int_{\mathbb{R}^2\times\mathbb{R}^2} t^{(1)}_{\Psi_N}(p)|G_{\hbar_p}(q-p)|^2(|q-p|^2)\mathrm{d}p\mathrm{d}q = N\int_{\mathbb{R}^2}|G_{\hbar_p}(p)|^2|p|^2\mathrm{d}p\\
=&\frac{N}{(2\pi\hbar)^2}\int_{\mathbb{R}^6}\frac{|p|^2}{\hbar_x} f\left(\frac{x}{\sqrt{\hbar_x}}\right) f\left(\frac{x'}{\sqrt{\hbar_x}}\right) e^{\frac{\mathrm{i}p\cdot x}{\hbar}} e^{-\frac{\mathrm{i}p\cdot x'}{\hbar}}\mathrm{d}p\mathrm{d}x\mathrm{d}x'\\
=&\frac{N\hbar^2}{(2\pi\hbar)^2}\int_{\mathbb{R}^6}\nabla f(u)\nabla f(v) e^{\frac{\mathrm{i}p\cdot(u-v)}{\sqrt{\hbar_p}}}\mathrm{d}p\mathrm{d}u\mathrm{d}v = -N\hbar_p\int_{\mathbb{R}^2}|\nabla f(y)|^2\mathrm{d}y,
\end{aligned}$$

thus concluding that

$$\left\langle \Psi_N, \left(\sum_{j=1}^N -\hbar^2\Delta_j\right)\Psi_N\right\rangle = \frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} m^{(1)}_{f,\Psi_N}(x,p)|p|^2\mathrm{d}x\mathrm{d}p - N\hbar_p\int_{\mathbb{R}^2}|\nabla f(y)|^2\mathrm{d}y. \tag{18.20}$$

For the magnetic cross term (18.17) we use (20.24) to obtain, for any $u\in L^2(\mathbb{R}^2)$,

$$\begin{aligned}
&\frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} p\cdot\mathbf{A}_e(x)|\left\langle u, F_{x,p}\right\rangle|^2\mathrm{d}x\mathrm{d}p = \int_{\mathbb{R}^2\times\mathbb{R}^2} p\cdot\mathbf{A}_e(x)|\mathcal{F}_\hbar[F_{x,0}u](p)|^2\mathrm{d}p\mathrm{d}x\\
=&\int_{\mathbb{R}^2}\mathbf{A}_e(x)\cdot\int_{\mathbb{R}^2} p|\mathcal{F}_\hbar[F_{x,0}u](p)|^2\mathrm{d}p\mathrm{d}x = \hbar\int_{\mathbb{R}^2}\mathbf{A}_e(x)\cdot\Im\int_{\mathbb{R}^2} F_{x,0}(y)\overline{u}(y)\nabla[F_{x,0}u](y)\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

Then since $F_{x,0}$ and $|u|^2$ are real we obtain

$$\begin{aligned}
\frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} p\cdot\mathbf{A}_e(x)|\left\langle u, F_{x,p}\right\rangle|^2\mathrm{d}x\mathrm{d}p &= \hbar\int_{\mathbb{R}^2}\mathbf{A}_e(x)\cdot\int_{\mathbb{R}^2}|F_{\hbar_x}(y-x)|^2\Im\left[\overline{u}(y)\nabla u(y)\right]\mathrm{d}y\mathrm{d}x\\
&= \hbar\int_{\mathbb{R}^2}(\mathbf{A}_e(x)*|F_{\hbar_x}|^2)(y)\cdot\Im\left[\overline{u}(y)\nabla u(y)\right]\mathrm{d}y.
\end{aligned}$$

Combining the spectral decomposition of the one-particle reduced density matrix with the above we get

$$\hbar\sum_{j=1}^N\left\langle p_j\cdot\mathbf{A}_e(x_j)\right\rangle = \frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^4} p\cdot\mathbf{A}_e\mathrm{d}m^{(1)}_{\Psi_N} + \Re\left\langle\sum_{j=1}^N(\mathbf{A}_e - \mathbf{A}_e * |F_{\hbar_x}|^2)(x_j)\cdot(-\mathrm{i}\hbar\nabla_j)\right\rangle.$$

Now for the mixed two-body term (18.13):

$$\begin{aligned}M :=& \frac{1}{N(N-1)}\frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^4\times\mathbb{R}^4} m^{(2)}_{\Psi_N}(x_1,x_2,p_1,p_2)\Big(p_1\cdot\nabla^\perp w_R(x_1-x_2)\Big)\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}p_1\mathrm{d}p_2\\ =& -\int_{\mathbb{R}^{2(N-2)}}\int_{\mathbb{R}^4\times\mathbb{R}^4} p_1\cdot\nabla^\perp w_R(x_1-x_2)\left|\langle F_{x_1,p_1}\otimes F_{x_2,p_2}(\cdot)\Psi_N(\cdot,y)\rangle\right|^2_{L^2(\mathbb{R}^4)}\mathrm{d}x_{12}\mathrm{d}p_{12}\mathrm{d}y\\ =& -\int_{\mathbb{R}^{2(N-2)}}\frac{\mathrm{i}\hbar}{2}\int_{\mathbb{R}^4}\nabla^\perp w_R(x_1-x_2)\cdot\int_{\mathbb{R}^8}[\nabla_{y_1}-\nabla_{z_1}]\Big(F_{x_1,0}(y_1)F_{x_1,0}(z_1)\overline{\Psi}(y_1,y_2,y)\\ &\qquad\Psi_N(z_1,z_2,y)F_{x_2,0}(y_2)F_{x_2,0}(z_2)\Big)\delta(y_2-z_2)\delta(y_1-z_1)\mathrm{d}x_{12}\mathrm{d}y_{12}\mathrm{d}z_{12}\mathrm{d}y\end{aligned}$$

and since $-\frac{\mathrm{i}}{2}(u-\overline{u})=\Im[u]$, we get

$$\begin{aligned}M =& -\hbar\int_{\mathbb{R}^{2(N-2)}}\int_{\mathbb{R}^4\times\mathbb{R}^4}\nabla^\perp w_R(x_1-x_2)|F_{\hbar_x}(y_1-x_1)|^2|F_{\hbar_x}(y_2-x_2)|^2\cdot\Im\Big[\overline{\Psi}_N\nabla_{y_1}\Psi_N\Big]\mathrm{d}x_{12}\mathrm{d}y_{12}\mathrm{d}y\\ =& \int_{\mathbb{R}^{2(N-2)}}\int_{\mathbb{R}^4}\Big(\Big(\nabla^\perp w_R * |F_{\hbar_x}|^2\Big) * |F_{\hbar_x}|^2\Big)(x_1-x_2)\Re\Big[-\mathrm{i}\hbar\nabla_{x_1}\Psi_N\overline{\Psi}_N\Big]\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y\end{aligned}$$

and conclude

$$\begin{aligned}\hbar\sum_{j=1}^N\sum_{k\neq j}\Big\langle p_j\cdot\nabla^\perp w_R(x_j-x_k)\Big\rangle =& \frac{1}{(2\pi\hbar)^4}\int_{\mathbb{R}^4\times\mathbb{R}^4}\Big(p_1\cdot\nabla^\perp w_R(x_1-x_2)\Big)\,\mathrm{d}m^{(2)}_{\Psi_N}\\ +\Re\Big\langle\sum_{j=1}^N\sum_{k\neq j}\Big(\nabla^\perp w_R-\nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2\Big)(x_j-x_k)\cdot(-\mathrm{i}\hbar\nabla_j)\Big\rangle&.\end{aligned}$$

For the terms which do not involve the variable p we only discuss the three-body term (18.15), the others being treated in the same way. We apply Lemma (18.2)

$$\begin{aligned}&\frac{1}{(2\pi\hbar)^3}\int_{\mathbb{R}^6\times\mathbb{R}^6} W_{123}(x_1,x_2,x_3)m^{(3)}_{\Psi_N}(x_1,p_1,..,x_3,p_3)\mathrm{d}x_{123}\mathrm{d}p_{123}\\ &=\int_{\mathbb{R}^6\times\mathbb{R}^6} W_{123}(x_1,x_2,x_3)\rho^{(3)}_{\Psi_N} * \Big(|F_{\hbar_x}|^2\Big)^{\otimes 3}\,\mathrm{d}x_{123}\\ &=\int_{\mathbb{R}^{12}}\nabla^\perp w_R(u-v+y_1-y_2)\cdot\nabla^\perp w_R(u-w+y_1-y_3)\\ &\qquad\qquad\rho^{(3)}_{\Psi_N}(u,v,w)\,|F_{\hbar_x}|^2(y_1)\,|F_{\hbar_x}|^2(y_2)\,|F_{\hbar_x}|^2(y_3)\mathrm{d}y_{123}\mathrm{d}u\mathrm{d}v\mathrm{d}w\\ =&\int_{\mathbb{R}^6}\Big(\nabla^\perp w_R * |F_{\hbar_x}|^2\Big)(v-u-y_1)\cdot\Big(\nabla^\perp w_R * |F_{\hbar_x}|^2\Big)(w-u-y_1)\,|F_{\hbar_x}(y_1)|^2\,\rho^{(3)}_{\Psi_N}\mathrm{d}u\mathrm{d}v\mathrm{d}w\mathrm{d}y_1\end{aligned}$$

and then

$$\sum_{j=1}^{N}\sum_{k\neq j}\sum_{l\neq k}\left\langle \nabla^{\perp}w_R(x_j-x_k)\nabla^{\perp}w_R(x_j-x_l)\right\rangle = \frac{1}{(2\pi\hbar)^6}\int_{\mathbb{R}^{12}}\left(\nabla^{\perp}w_R\cdot\nabla^{\perp}w_R\right)\mathrm{d}m^{(3)}_{\Psi_N}$$
$$+\sum_{j=1\neq k\neq l}^{N}\Big\langle \nabla^{\perp}w_R(x_j-x_k)\cdot\nabla^{\perp}w_R(x_j-x_l)-\int_{\mathbb{R}^2}\left(\nabla^{\perp}w_R*|F_{\hbar_x}|^2\right)(x_j-x_k-u)$$
$$\cdot\left(\nabla^{\perp}w_R*|F_{\hbar_x}|^2\right)(x_j-x_l-u)|F_{\hbar_x}(u)|^2\mathrm{d}u\Big\rangle.$$

□

We now show that all errors in (18.16) are smaller than the classical energy $\mathcal{E}^{\mathrm{C}}$. This conclude the proof of Proposition 18.5.

Lemma 18.6 (**Convolution terms**).
For any function W on $L^{\infty}\left(\mathbb{R}^2\right)$, consider the n-fold convoluted product of W with $|F_{\hbar_x}|^2$.

$$W*_n|F_{\hbar_x}|^2 = W*|F_{\hbar_x}|^2*|F_{\hbar_x}|^2*...*|F_{\hbar_x}|^2$$

We have the estimate

$$\left\|W*_n|F_{\hbar_x}|^2-W\right\|_{L^{\infty}}\leq C_n\hbar_x\left\|\Delta W\right\|_{L^{\infty}} \tag{18.21}$$

Proof. For any $x,y\in\mathbb{R}^2$ there exists some $z\in\mathbb{R}^2$ such that

$$W\left(x-\sqrt{\hbar_x}y\right)=W(x)-\sqrt{\hbar_x}\nabla W(x)\cdot y+\frac{\hbar_x^2}{2}\langle y,\mathrm{Hess}W(z)y\rangle$$

and hence

$$\begin{aligned}\left\|W-W*|F_{\hbar_x}|^2(x)\right\|_{L^{\infty}} &= \left|W(x)-\frac{1}{\pi\hbar_x}\int_{\mathbb{R}^2}W(x-y)e^{-\frac{|y|^2}{\hbar_x}}\mathrm{d}y\right|\\ &= \left|W(x)-\frac{1}{\pi}\int_{\mathbb{R}^2}W\left(x-\sqrt{\hbar_x}y\right)e^{-|y|^2}\mathrm{d}y\right|\\ &\leq C\hbar_x\left\|\Delta W\right\|_{L^{\infty}}\end{aligned}$$

using the radial symmetry of $y\mapsto e^{-|y|^2}$ to discard the term of order $\sqrt{\hbar_x}$. It follows that for any $n\geq 2$

$$\left\|W*_n|F_{\hbar_x}|^2-W\right\|_{L^{\infty}}\leq C_n\hbar_x\left\|\Delta W\right\|_{L^{\infty}}$$

because

$$\left\|\Delta W*_{n-1}|F_{\hbar_x}|^2\right\|_{L^{\infty}}=\left\|(\Delta W)*_{n-1}|F_{\hbar_x}|^2\right\|_{L^{\infty}}\leq C\left\|\Delta W\right\|_{L^{\infty}}.$$

□

Proof of Proposition (18.4). Let Ψ_N be a sequence of fermionic wave functions such that

$$\left\langle\Psi_N,H_N^R\Psi_N\right\rangle=O(N).$$

We will systematically use the bounds (18.2) and (18.1)

$$\left\|\rho_{\gamma_N}\right\|^2_{L^2(\mathbb{R}^2)}\leq\frac{CN^2}{R^2}\quad\text{and}\quad\left\langle\Psi_N,\left(\sum_{j=1}^{N}-\Delta_j\right)\Psi_N\right\rangle\leq\frac{CN^2}{R^2}$$

to deal with our error terms. Recall that the main term in the energy (18.16) is of order N.

Estimate of (18.17)**.** We have

$$\begin{aligned}\left|\mathrm{Tr}\left(\left(\mathbf{A}_e-\mathbf{A}_e*|F_{\hbar_x}|^2\right)\cdot\left(-\mathrm{i}\hbar\nabla\right)\gamma_N\right)\right| &\leq \|(-\mathrm{i}\hbar\nabla)\sqrt{\gamma_N}\|_{\mathfrak{S}^2}\left\|(\mathbf{A}_e-\mathbf{A}_e*|F_{\hbar_x}|^2)\sqrt{\gamma_N}\right\|_{\mathfrak{S}^2}\\ &\leq \hbar\sqrt{\mathrm{Tr}(-\Delta)\gamma_N}\left\|(\mathbf{A}_e-\mathbf{A}_e*|F_{\hbar_x}|^2)\sqrt{\rho_{\gamma_N}}\right\|_{L^2(\mathbb{R}^2)}\\ &\leq C\hbar\frac{N}{R}\left\|\mathbf{A}_e-\mathbf{A}_e*|F_{\hbar_x}|^2\right\|_{L^\infty(\mathbb{R}^2)}\|\rho_{\gamma_N}\|_{L^1(\mathbb{R}^2)}^{1/2}\\ &\leq \frac{CN}{R}||\mathbf{A}_e-\mathbf{A}_e*|F_{\hbar_x}|^2||_{L^\infty(\mathbb{R}^2)}\end{aligned}$$

where $||W||_{\mathfrak{S}^p}=\mathrm{Tr}\left[|W|^p\right]^{\frac{1}{p}}$ is the Schatten p-norm. We used (18.1) to bound the kinetic term and we now use Lemma 18.6 to deduce

$$\left|\mathrm{Tr}\left(\left(\mathbf{A}_e-\mathbf{A}_e*|F_{\hbar_x}|^2\cdot\left(-\mathrm{i}\hbar\nabla\right)\gamma_N\right)\right|\leq\frac{CN\hbar_x}{R}\|\Delta\mathbf{A}_e\|_{L^\infty}\leq CN^{\eta+2\varepsilon}\tag{18.22}$$

Estimate of (18.18)**.** We have

$$\begin{aligned}\left\|(|\mathbf{A}_e|^2-|\mathbf{A}_e|^2*|F_{\hbar_x}|^2)\rho_{\gamma_N}\right\|_{L^1(\mathbb{R}^2)} &\leq \left\||\mathbf{A}_e|^2-|\mathbf{A}_e|^2*|F_{\hbar_x}|^2\right\|_{L^\infty(\mathbb{R}^2)}\|\rho_{\gamma_N}\|_{L^1(\mathbb{R}^2)}\\ &\leq CN\left\||\mathbf{A}_e|^2-|\mathbf{A}_e|^2*|F_{\hbar_x}|^2\right\|_{L^\infty(\mathbb{R}^2)}\\ &\leq CN\hbar_x\left\|\Delta|\mathbf{A}_e|^2\right\|_{L^\infty(\mathbb{R}^2)}\leq CN^{2\varepsilon}\end{aligned}\tag{18.23}$$

where we used Cauchy-Schwarz's inequality and the estimate of Lemma (18.6). For the second term of (18.18) we use Assumption 16.1 and Lemma 18.6 to write

$$\begin{aligned}\left\|\left(V-V*|F_{\hbar_x}|^2\right)\rho(x)\right\|_{L^1} &\leq C\hbar_x\int_{\mathbb{R}^2}|\Delta V(x)|\,\rho(x)\mathrm{d}x\\ &\leq C\hbar_x\int_{\mathbb{R}^2}\left(|x|^{s-2}+1\right)\rho(x)\mathrm{d}x\\ &\leq C\hbar_x\int_{\mathbb{R}^2}\left(|x|^s+1\right)\rho(x)\leq CN^{2\varepsilon}\end{aligned}\tag{18.24}$$

by using Young's inequality

$$r^{s-2}\leq\frac{s-2}{s}r^s+\frac{2}{s}$$

the choice (18.10) and the a priori bound (18.3).

Estimate of (18.19)**.** We have

$$\alpha^2\left\langle\sum_{j=1}^N\sum_{k\neq j}|\nabla^\perp w_R(x_j-x_k)|^2\right\rangle\leq\frac{C}{R^2}\leq CN^{2\eta}$$

and

$$\hbar_p\int_{\mathbb{R}^2}|\nabla f(y)|^2\mathrm{d}y\leq CN^{-2\varepsilon}$$

recalling that f is a fixed function and the choice (18.10).

Estimate of (18.13):

$$\begin{aligned}
&4\alpha \left| \mathrm{Tr}\left[\left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 \right) \cdot (-\mathrm{i}\hbar\nabla_1)\gamma_N^{(2)} \right] \right| \\
&\quad \leq C\alpha \, \|(-\mathrm{i}\hbar\nabla)\sqrt{\gamma_N}\|_{\mathfrak{S}^2} \left\| \left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 \right) \sqrt{\gamma_N^{(2)}} \right\|_{\mathfrak{S}^2} \\
&\quad \leq C\alpha\hbar \frac{N}{R} \left\| \rho_{\gamma_N}^{(2)} \right\|_{L^1}^{1/2} \left\| \left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 \right) \right\|_{L^\infty} \\
&\quad \leq C\alpha\hbar \frac{N^2}{R} \left\| \left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 \right) \right\|_{L^\infty} \qquad (18.25)
\end{aligned}$$

where we used (18.2). We then use Lemma 18.6 and the estimate (20.12) to get

$$4\alpha \left| \mathrm{Tr}\left[\left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 \right) \cdot (-\mathrm{i}\hbar\nabla_1)\gamma_N^{(2)} \right] \right| \leq CN \frac{\hbar\hbar_x}{R^4} \leq CN^{4\eta+2\varepsilon-1/2}$$

Estimate of (18.14). we denote

$$\mathrm{I} = \mathbf{A}_e(x_j)\cdot\nabla^\perp w_R(x_j - x_k) - \int_{\mathbb{R}^2} \mathbf{A}_e(x_j - u)\cdot \left(\nabla^\perp w_R * |F_{\hbar_x}|^2 \right)(x_j - x_k - u)|F_{\hbar_x}(u)|^2 \mathrm{d}u. \quad (18.26)$$

Then

$$\begin{aligned}
4\alpha \, \mathrm{Tr}\left[\mathrm{I}\gamma_N^{(2)} \right] &\leq C\alpha \left\| \rho_{\gamma_N}^{(2)} \right\|_{L^1} \|\mathbf{A}_e\|_{L^\infty} \left\| \nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}| \right\|_{L^\infty} \\
&\leq C\alpha N^2 \hbar_x \left\| \Delta\nabla^\perp w_R \right\|_{L^\infty} \\
&\leq \frac{CN\hbar_x}{R^3} \leq CN^{3\eta+2\varepsilon} \qquad (18.27)
\end{aligned}$$

where we used (18.2) and (18.6) combined with the estimate (20.12).

Estimate of (18.15). We denote

$$\begin{aligned}
&\mathrm{II} = W_{123}(x_1, x_2, x_3) \\
&- \int_{\mathbb{R}^2} \left(\nabla^\perp w_R * |F_{\hbar_x}|^2 \right)(x_1 - x_2 - u) \cdot \left(\nabla^\perp w_R * |F_{\hbar_x}|^2 \right)(x_1 - x_3 - u)|F_{\hbar_x}(u)|^2 \mathrm{d}u. \quad (18.28)
\end{aligned}$$

Then

$$\begin{aligned}
6\alpha^2 \, \mathrm{Tr}\left[\mathrm{II}\gamma_N^{(3)} \right] &\leq \frac{CN\alpha^2}{R} \left\| \left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 \right) \rho_{\gamma_N}^{(2)} \right\|_{L^1} \\
&\leq \frac{CN\alpha^2}{R} \left\| \rho_{\gamma_N}^{(2)} \right\|_{L^1} \left\| \nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 \right\|_{L^\infty} \\
&\leq C\alpha^2 \frac{N^3\hbar_x}{R} \left\| \Delta\nabla^\perp w_R \right\|_{L^\infty} \\
&\leq \frac{CN\hbar_x}{R^4} \leq CN^{4\eta+2\varepsilon} \qquad (18.29)
\end{aligned}$$

where we used [**59**, Lemma 2.1] and (18.21) combined with (20.12) to bound the last norm. Collecting all the previous estimates leads to the result. □

CHAPTER 19

Energy lower bound: mean-field limit

This section is dedicated to a lower bound on (16.6). We have reduced the problem to a classical one in the previous section. We now perform a mean-field approximation of the classical energy (18.9), showing that its infimum is (asymptotically) attained by a factorized Husimi function. The crucial point is to keep track of the semi-classical Pauli principle (16.20).

We first recall the Diaconis-Freedman theorem. We use it to rewrite many-particle probability measures as statistical superpositions of factorized ones. We then estimate quantitatively the probability for the measures in the superposition to violate the Pauli principle. This is the main novelty compared to the approach of [**147**]. We finally prove the convergence of the energy and states, Theorems 16.1 and 16.5.

1. The Diaconis-Freedman measure

For $Z_N \in \mathbb{R}^{4N}$ a set of points in the phase space, we define the associated empirical measure

$$\mathrm{Emp}_{Z_N} = \frac{1}{N}\sum_{i=1}^{N} \delta_{z_i}. \tag{19.1}$$

The following originates in [**41**].

THEOREM 19.1 (**Diaconis-Freedman**).
Let $\mu_N \in \mathcal{P}_{\mathrm{sym}}(\mathbb{R}^{4N})$ the set of symmetric probability measures over $\mathbb{R}^{4N}$. Define the probability measure P_{μ_N} over $\mathcal{P}(\mathbb{R}^4)$

$$P_{\mu_N}(\sigma) := \int_{\mathbb{R}^{4N}} \delta_{\sigma=\mathrm{Emp}_{Z_N}} \mathrm{d}\mu_N(z_1,\dots,z_N) \tag{19.2}$$

which means

$$\int_{\mathcal{P}(\Omega)} \phi\left(\rho\right) \mathrm{d}P_{\mu_N}\left(\rho\right) := \int_{\Omega^N} \phi\left(\rho_Z\right) \mu_N\left(Z\right) \mathrm{d}Z$$

for all $\phi \in C_b\left(\mathcal{P}(\mathbb{R}^4)\right)$, and set

$$\tilde{\mu}_N := \int_{\mathcal{P}(\mathbb{R}^4)} \rho^{\otimes N} \mathrm{d}P_{\mu_N}(\rho)$$

with associated marginals

$$\widetilde{\mu}_N^{(n)} = \int_{\rho\in\mathcal{P}(\mathbb{R}^4)} \rho^{\otimes n} \mathrm{d}P_{\mu_N}(\rho). \tag{19.3}$$

Then

$$\left\| \mu_N^{(n)} - \widetilde{\mu}_N^{(n)} \right\|_{\mathrm{TV}} \leq \frac{2n(n-1)}{N} \tag{19.4}$$

in total variation norm

$$\|\mu\|_{\mathrm{TV}} = \sup_{\phi\in C_b(\mathbb{R}^4),\|\phi\|_{L^\infty}\leq 1} \left|\int_{\mathbb{R}^4} \phi \mathrm{d}\mu\right|$$

with $C_b(\Omega)$ the set of continuous bounded functions on Ω.

Moreover, if the sequence $\left(\mu_N^{(1)}\right)_N$ is tight, there exists a subsequence and a unique probability measure $P \in \mathcal{P}\left(\left(\mathcal{P}(\mathbb{R}^4)\right)\right)$ such that (up to extraction)

$$P_{\mu_N} \rightharpoonup P$$

and, for any fixed $n \in \mathbb{N}^*$

$$\widetilde{\mu}_N^{(n)} \rightharpoonup \int_{\rho\in\mathcal{P}(\mathbb{R}^4)} \rho^{\otimes n} \mathrm{d}P(\rho). \tag{19.5}$$

The measure P obtained in the limit is that appearing in the Hewitt-Savage theorem [**70**]. See [**133, 136**, Theorems 2.1 and 2.2] for proofs and references. We will use the following consequence of the Diaconis-Freedman construction.

LEMMA 19.2 (**First marginals of the Diaconis-Freedman measure**).
Let $\mu_N \in \mathcal{P}_{\mathrm{sym}}(\mathbb{R}^{4N})$ and the associated Diaconis-Freedman approximation $\widetilde{\mu}_N$ be given by the previous theorem. Using the notation $X_i = (x_i, p_i)$ we have

$$\widetilde{\mu}_N^{(1)}(x_1,p_1) = \mu_N^{(1)}(x_1,p_1) \tag{19.6}$$

$$\widetilde{\mu}_N^{(2)}(x_1,p_1,x_2,p_2) = \frac{N-1}{N}\mu_N^{(2)}(X_1,X_2) + \frac{1}{N}\mu_N^{(1)}(X_1)\delta_{(x_1,p_1)=(x_2,p_2)} \tag{19.7}$$

$$\begin{aligned}\widetilde{\mu}_N^{(3)}(X_1,X_2,X_3) &= \frac{N(N-1)(N-2)}{N^3}\mu_N^{(3)}(X_1,X_2,X_3)\\ &+ \frac{(N-1)}{N^2}\left[\mu_N^{(2)}(X_1,X_3)\delta_{X_1=X_2} + \mu_N^{(2)}(X_2,X_1)\delta_{X_2=X_3} + \mu_N^{(2)}(X_3,X_2)\delta_{X_3=X_1}\right]\\ &+ \frac{1}{N^2}\mu_N^{(1)}(X_1)\delta_{X_1=X_2}\delta_{X_2=X_3}\end{aligned} \tag{19.8}$$

PROOF. One can find (19.6) and (19.7) in [**136**, Remark 2.3]. It follows from [**133, 136**, Equation 2.16] that

$$\widetilde{\mu}_N(X_N) = \int_{\mathbb{R}^{4N}} \mu_N(Z) \sum_{\gamma\in\Gamma_N} N^{-N}\delta_{X=Z_\gamma}\mathrm{d}Z = \int_{\mathbb{R}^{4N}} \mu_N(Z)\left(N^{-1}\sum_{j=1}^N \delta_{z_j=(x_j,p_j)}\right)^{\otimes N}\mathrm{d}Z \tag{19.9}$$

where Z_σ is the $4N$-uple $(x_{\sigma(1)},p_{\sigma(1)},...,x_{\sigma(N)},p_{\sigma(N)}) = (z_{\sigma(1)},...z_{\sigma_N})$, $(z_i = (x_i,p_i))$ and Γ_N the set of maps from $\{1,...,N\}$ to itself (which allows repeated indices). We look at the n-th marginal

$$\widetilde{\mu}_N^{(n)}(X_N) = \int_{\mathbb{R}^{4N}} \mu_N(Z)\left(N^{-1}\sum_{j=1}^N \delta_{z_j=X_j}\right)^{\otimes n}\mathrm{d}Z. \tag{19.10}$$

In particular

$$\widetilde{\mu}_N^{(1)}(x_1,p_1) = N^{-1}\int_{\mathbb{R}^{4N}} \mu_N(Z)\left(\sum_{j=1}^N \delta_{z_j=X_1}\right)\mathrm{d}Z$$

$$\widetilde{\mu}_N^{(2)}(x_1,p_1,x_2,p_2) = N^{-2}\int_{\mathbb{R}^{4N}} \mu_N(Z)\left(\sum_{j=1}^N \delta_{z_j=X_1}\right)\left(\sum_{j=1}^N \delta_{z_j=X_2}\right)\mathrm{d}Z$$

$$\widetilde{\mu}_N^{(3)}(X_1,X_2,X_3) = N^{-3}\int_{\mathbb{R}^{4N}} \mu_N(Z)\left(\sum_{j=1}^N \delta_{z_j=X_1}\right)\left(\sum_{j=1}^N \delta_{z_j=X_2}\right)\left(\sum_{j=1}^N \delta_{z_j=X_3}\right)\mathrm{d}Z$$

Expanding the n-th tensor power yields $\widetilde{\mu}_N^{(n)}$ in terms of the marginals of μ_N by an explicit calculation. □

We now apply the above constructions to the probability measure

$$m_N := \frac{m_{\Psi_N}^{(N)}(x_1,p_1,...,x_N,p_N)}{N!(2\pi\hbar)^{2N}} \tag{19.11}$$

to obtain an approximation of the marginals (cf Lemma 18.3)

$$m_N^{(k)} = \frac{m_{\Psi_N}^{(k)}(x_1,p_1,...,x_k,p_k)}{N\,(N-1)\,...\,(N-k+1)\,(2\pi\hbar)^{2k}} \tag{19.12}$$

We use the notation W_{12} and W_{123} respectively defined in (17.6) and (17.7) to denote the two and the three body operators. In the former case, this is with a slight abuse of notation: we replace $p^{\mathbf{A}}$ by the classical vector $p_1 + \mathbf{A}_e(x)$ in the phase space.

Lemma 19.3 (**Energy in terms of the Diaconis-Freedman measure**).
Let $m_N \in \mathcal{P}_{\mathrm{sym}}(\mathbb{R}^{4N})$ be as in (19.11)*. Let $P_N^{\mathrm{DF}} \in \mathcal{P}(\mathcal{P}(\mathbb{R}^4))$ be the associated Diaconis-Freedman measure, as in Theorem 19.1. Evaluating each term of* (18.12) *gives*

$$\frac{1}{(2\pi)^2}\int_{\mathbb{R}^4}\left(|p^{\mathbf{A}}|^2 + V(x)\right)m_{f,\Psi_N}^{(1)}(x,p)\mathrm{d}x\mathrm{d}p = \int_{\mathbb{R}^4}\int_{\mu\in\mathcal{P}(\mathbb{R}^4)}\left(|p^{\mathbf{A}}|^2+V(x)\right)\mathrm{d}\mu(x,p)\mathrm{d}P_N^{\mathrm{DF}}(\mu) \tag{19.13}$$

$$\frac{1}{(2\pi)^4}\int_{\mathbb{R}^8} W_{12}\, m_{f,\Psi_N}^{(2)}(x_{12},p_{12})\mathrm{d}x_{12}\mathrm{d}p_{12} = \int_{\mathbb{R}^8}\int_{\mu\in\mathcal{P}(\mathbb{R}^4)} W_{12}\,\mathrm{d}\mu^{\otimes 2}\mathrm{d}P_N^{\mathrm{DF}}(\mu) - \frac{C}{NR^2} \tag{19.14}$$

$$\frac{1}{(2\pi)^6}\int_{\mathbb{R}^{12}} W_{123}\,\mathrm{d}m_{f,\Psi_N}^{(3)} = \int_{\mathbb{R}^{12}}\int_{\mu\in\mathcal{P}(\mathbb{R}^4)} W_{123}\,\mathrm{d}\mu^{\otimes 3}\mathrm{d}P_N^{\mathrm{DF}}(\mu) - \frac{C}{NR^2} \tag{19.15}$$

Proof. To apply the Diaconis-Freedman theorem (19.1) to the measures $m_{f,\Psi_N}^{(i)}$, $i = 1,2,3$ they have to be normalized as in (19.12). We then use the measure $m_N^{(i)}$ instead of $m_{f,\Psi_N}^{(i)}$ in our expressions, keeping track of the adequate normalization factors. Denoting $Z_i = (x_i,p_i)$ and recalling

$$(2\pi\hbar)^{2k}N(N-1)...(N-k+1)m_N^{(k)} = m_{\Psi_N}^{(k)}.$$

Using Lemma 19.2, we calculate

$$\frac{1}{(2\pi)^2}\int_{\mathbb{R}^4}|p_1^{\mathbf{A}}|^2 m^{(1)}_{f,\Psi_N}(x_1,p_1)\mathrm{d}x_1\mathrm{d}p_1 = \int_{\mathbb{R}^4}\int_{\mu\in\mathcal{P}(\mathbb{R}^4)}|p_1^{\mathbf{A}}|^2\mu(x_1,p_1)\mathrm{d}x_1\mathrm{d}p_1\mathrm{d}P_N^{\mathrm{DF}}(\mu)$$

and

$$\begin{aligned}
&\frac{1}{(2\pi)^4}\int_{\mathbb{R}^8}(p_1^{\mathbf{A}})\cdot\nabla^\perp w_R(x_1-x_2)m^{(2)}_{f,\Psi_N}(Z_1,Z_2)\mathrm{d}Z_1\mathrm{d}Z_2\\
&\qquad = N(N-1)\hbar^4\int_{\mathbb{R}^8}(p_1^{\mathbf{A}})\cdot\nabla^\perp w_R(x_1-x_2)m^{(2)}_N(Z_1,Z_2)\mathrm{d}Z_1\mathrm{d}Z_2\\
&\qquad = N^2\hbar^4\int_{\mathbb{R}^8}\int_{\mu\in\mathcal{P}(\mathbb{R}^4)}(p_1^{\mathbf{A}})\cdot\nabla^\perp w_R(x_1-x_2)\mathrm{d}\mu^{\otimes 2}\mathrm{d}P_N^{\mathrm{DF}}(\mu)\\
&\qquad - N\hbar^4\int_{\mathbb{R}^8}(p_1^{\mathbf{A}})\cdot\nabla^\perp w_R(x_1-x_2)m^{(1)}_N(X_1)\delta_{Z_1=Z_2}\mathrm{d}Z_1\mathrm{d}Z_2. \qquad (19.16)
\end{aligned}$$

Moreover

$$\begin{aligned}
&\int_{\mathbb{R}^{12}}\nabla^\perp w_R(x_1-x_2)\cdot\nabla^\perp w_R(x_1-x_3)m^{(3)}_N(Z_1,Z_2,Z_3)\mathrm{d}Z_1\mathrm{d}Z_2\mathrm{d}Z_3\\
&= \frac{N^2}{(N-1)(N-2)}\int_{\mathbb{R}^{12}}\int_{\mu\in\mathcal{P}(\mathbb{R}^4)}\nabla^\perp w_R(x_1-x_2)\cdot\nabla^\perp w_R(x_1-x_3)\mathrm{d}\mu^{\otimes 3}\mathrm{d}P_N^{\mathrm{DF}}(\mu)\\
&- \frac{1}{(N-2)}\int_{\mathbb{R}^{12}}W_{123}\left(m^{(2)}_N(Z_1,Z_3)\delta_{Z_1=Z_2}+m^{(2)}_N(Z_2,Z_1)\delta_{Z_2=Z_3}+m^{(2)}_N(Z_3,Z_2)\delta_{Z_3=Z_1}\right) \qquad (19.17)\\
&- \frac{1}{(N-1)(N-2)}\int_{\mathbb{R}^{12}}W_{123}m^{(1)}_N(Z_1)\delta_{Z_1=Z_2}\delta_{Z_2=Z_3}. \qquad (19.18)
\end{aligned}$$

We will discard the error terms in (19.16) (19.17) and (19.18) using [**59**, Lemma 2.1]

$$\left\|\nabla^\perp w_R\right\|_{L^\infty}\leq R^{-1}$$

and the kinetic energy bound (18.1). The latter, combined with equation (18.20) gives

$$\iint_{\mathbb{R}^4}|p|^2\mathrm{d}m^{(1)}_{f,\Psi_N}(x,p)\leq\frac{C}{R^2} \qquad (19.19)$$

Applying the Cauchy-Schwarz inequality to the two particles term we obtain

$$\begin{aligned}
&\left|\frac{1}{N}\int_{\mathbb{R}^8}(p_1^{\mathbf{A}})\cdot\nabla^\perp w_R(x_1-x_2)m^{(1)}_N(Z_1)\delta_{Z_1=Z_2}\mathrm{d}Z_1\mathrm{d}Z_2\right|\\
&\leq\frac{C||\mathbf{A}_e||_{L^\infty}}{NR}+\frac{1}{N}\left(\int_{\mathbb{R}^8}\left|\nabla^\perp w_R(x_1-x_2)\right|^2 m^{(1)}_N\delta_{Z_1=Z_2}\mathrm{d}Z_1\mathrm{d}Z_2\right)^{1/2}\left(\int_{\mathbb{R}^4}|p|^2\mathrm{d}m^{(1)}_N\right)^{1/2}\\
&\leq\frac{C}{NR^2}
\end{aligned}$$

As for the three-particles term we have

$$\frac{1}{N}\left|\int_{\mathbb{R}^{12}} W_{123}\left(m_N^{(2)}(Z_1,Z_3)\delta_{Z_1=Z_2}\right)\right| \leq \frac{C}{NR^2}$$
$$\frac{1}{N^2}\left|\int_{\mathbb{R}^{12}} W_{123}\left(m_N^{(1)}(Z_1)\delta_{Z_1=Z_2}\delta_{Z_2=Z_3}\right)\right| \leq \frac{C}{N^2R^2}.$$

□

2. Quantitative semi-classical Pauli principle

Lemma (19.3) allows to write the energy as an integral of $\mathcal{E}_V^R$ over $\mathcal{P}(\mathbb{R}^4)$ plus some negligible error terms. Assuming $R = N^{-\eta}$ for some $\eta < 1/4$, we get from Proposition 18.4 and the above lemma:

$$\frac{\left\langle \Psi_N, H_N^R \Psi_N \right\rangle}{N} \geq \int_{\mathcal{P}(\mathbb{R}^4)} \mathcal{E}_V^R[\mu] \mathrm{d}P_{m_N}^{\mathrm{DF}}(\mu) - o_N(1) \tag{19.20}$$

where

$$\mathcal{E}_V^R[\mu] = \int_{\mathbb{R}^4} \left|p + \mathbf{A}_e(x) + \beta_0 \mathbf{A}^R[\rho](x)\right|^2 \mu(x,p)\mathrm{d}x\mathrm{d}p + \int_{\mathbb{R}^2} V(x)\rho(x)\mathrm{d}x \tag{19.21}$$

with

$$\rho(x) = \int_{\mathbb{R}^2} \mu(x,p)\mathrm{d}p. \tag{19.22}$$

Note that

$$\mathcal{E}_V^R\left[\frac{\mu}{(2\pi)^2}\right] = \mathcal{E}_{\mathrm{Vla}}^R[\mu]$$

defined in (16.17). However, P_N^{DF} only charges empirical measures, which do not satisfy the Pauli principle (18.7). To circumvent this issue we divide the phase-space

$$\cup_{m\in\mathbb{N}} \Omega_m = \mathbb{R}^4 \tag{19.23}$$

in hyperrectangles labeled $(\Omega_m)_{m\in\mathbb{N}}$. We take them to have side-length l_x in the space coordinates and l_p in the momentum coordinates, ensuring that

$$|\Omega_m| = l_x^2 l_p^2 = N^{-\beta}. \tag{19.24}$$

The parameter $\beta > 0$ and the lengths l_x, l_p will be chosen in the sequel.

If each Ω_m contains less than $(1+\varepsilon)(2\pi)^{-2}|\Omega_m|$ points (for some small ε) our measure will approximately satisfy the Pauli principle. We thus have to estimate the probability for a box to have the right density of points. This is the purpose of the following lemmas. We denote for a given measure μ:

$$\mathbb{P}_\mu(\Omega) = \int_\Omega \mathrm{d}\mu. \tag{19.25}$$

THEOREM 19.4 (**Probability of violating the Pauli principle in a phase-space box**). *Let m_N be as in* (19.11) *and P_N^{DF} the associated Diaconis-Freedman measure. Recall the tiling* (19.23) *and assume* (19.24) *with $\beta < 1$. For any*

$$0 < \delta < \frac{1-\beta}{2}$$

we have, for constant $C_\delta > 0, c_\delta > 0$, that

$$\mathbb{P}_{P_N^{\mathrm{DF}}}\left(\left\{\mathrm{Emp}_{Z_N}, \int_{\Omega_m} \mathrm{Emp}_{Z_N} \geq \frac{(1+\varepsilon)}{(2\pi)^2}|\Omega_m|\right\}\right) \leq C_\delta e^{-c_\delta N^\delta \ln(1+\varepsilon)}. \tag{19.26}$$

Note that the condition $\beta < 1$ means $l_x^2 l_p^2 \geq N^{-1}$. Since the average interparticle distance in phase-space is $N^{-1/4}$, our hyperrectangles typically contain a large number of particles.

We start the proof of the above theorem with the

LEMMA 19.5 (**Expansion of the Diaconis-Freedman measure**).
Let

$$\widetilde{m}_N = \int_{\mathcal{P}(\mathbb{R}^4)} \mu^{\otimes N} \mathrm{d}P_N^{\mathrm{DF}}(\mu)$$

be associated to (19.11) *as in Theorem 19.1, and $\widetilde{m}_N^{(n)}$ the associated marginals. For any $\Omega \subset \mathbb{R}^4$ we have*

$$\int_{\Omega^n} \widetilde{m}_N^{(n)} \leq N^{-n} \sum_{k=1}^{n} \frac{|\Omega|^k S(n,k)}{(2\pi\hbar)^{2k}} \tag{19.27}$$

with $S(n,k)$ the Stirling number of the second kind, the number of ways to partition a set of n objects into k non-empty subsets.

We refer to [**16**] for combinatorial definitions and estimates.

PROOF OF LEMMA 19.5. The measure $\widetilde{\mu}_N$ constructed from a $\mu_N \in \mathcal{P}_{\mathrm{sym}}(\mathbb{R}^{4N})$ in Theorem 19.1 satisfies (19.9). Hence

$$\widetilde{\mu}_N^{(n)} = N^{-n} \sum_{k=1}^{n} \sum_{\gamma \in \Gamma_{n,k}} \int_{\mathbb{R}^{4(N-n)}} \mu_N(Z) \delta_{X_N = Z_\gamma} \mathrm{d}Z \tag{19.28}$$

with $\Gamma_{n,k}$ the set of ordered samples of n indices amongst N (allowing repetitions) with exactly k distinct indices. Integrating on each side of (19.28) and using the symmetry of (recall Lemma 18.3)

$$\mu_N^{(k)} = m_N^{(k)} \leq \frac{1}{(2\pi\hbar)^{2k} N(N-1)...(N-k+1)}$$

we obtain

$$\int_{\Omega^n} \widetilde{m}_N^{(n)} \leq N^{-n} \sum_{k=1}^{n} \frac{|\Gamma_{n,k}||\Omega|^k}{(2\pi\hbar)^{2k} N(N-1)...(N-k+1)}.$$

Now we calculate the cardinal $|\Gamma_{n,k}|$. For a given k, choose k indices among N (without order) to, next, be distributed in n boxes. The first step gives us

$$\frac{N!}{(N-k)!k!}$$

choices. For each of these choices we have to distribute n balls in k boxes (boxes can be empty) which corresponds to the number of surjections from $\{1,2,...,n\}$ to $\{1,2,...,k\}$, $M(n,k)$ (with $k \leq n$). Then

$$|\Gamma_{n,k}| = \frac{N!}{(N-k)!k!} M(n,k)$$

with [**16**, Theorem 1.17]

$$M(n,k) = k!S(n,k)$$

and $S(n,k)$ is the Stirling number of the second kind [**16**, Lemma 1.16]. Thus

$$|\Gamma_{n,k}| = \frac{N!}{(N-k)!}S(n,k).$$

□

We now need a rough control of the sum appearing in (19.27) when the set Ω is not too small.

LEMMA 19.6 (**Control of the sum**).
Take $|\Omega_m| = N^{-\beta}$ with $\beta < 1$. Choose some

$$0 < \delta < \frac{1-\beta}{2}$$

and define

$$n = \left\lfloor N^{\delta} \right\rfloor$$

the integer part of N^{δ}. We have that

$$(2\pi)^{2n} N^{-n} \sum_{k=1}^{n} \frac{|\Omega_m|^{(k-n)} S(n,k)}{(2\pi\hbar)^{2k}} \leq C N^{\delta'}$$

for some exponent $\delta' > 0$ depending only on δ.

PROOF. We recall Stirling's formula

$$C_1\sqrt{N}\left(\frac{N}{e}\right)^N \leq N! \leq C_2\sqrt{N}\left(\frac{N}{e}\right)^N \tag{19.29}$$

and a bound for the Stirling number of the second kind from [**130**](see also [**123, 89**])

$$S(n,k) \leq C\binom{n}{k} k^{(n-k)}. \tag{19.30}$$

Inserting these bounds we have that

$$(2\pi)^{2n} N^{-n} \frac{|\Omega_m|^{(k-n)} S(n,k)}{(2\pi\hbar)^{2k}} \leq C\sqrt{\frac{n}{k(n-k)}} \exp(f_\beta(k))$$

with

$$\begin{aligned}
f_\beta(k) &= (n-k)\Big(\log k + (\beta-1)\log N + 2\log(2\pi)\Big) + n\log n - k\log k - (n-k)\log(n-k) \\
&= (n-k)\Big(2\log k + \log n + (\beta-1)\log N + 2\log(2\pi) - \log(n-k)\Big) + k\log n - n\log k \\
&= (n-k)\Big(\log k + \log n + (\beta-1)\log N + 2\log(2\pi) - \log(n-k)\Big) + k\log\left(1 + \frac{n-k}{k}\right) \\
&\leq (n-k)\Big(\log k + \log n + (\beta-1)\log N + 2\log(2\pi) + 1 - \log(n-k)\Big),
\end{aligned}$$

using $\log(1+x) \leq x$. Since $k \leq n \sim N^{\delta}$ we find that the leading order of the above is

$$f_\beta(k) \lesssim (n-k)\,(2\delta+\beta-1)\log N \leq 0$$

under our assumptions. Hence we find

$$(2\pi)^{2n}\,N^{-n}\sum_{k=1}^{n}\frac{|\Omega_m|^{(k-n)}S(n,k)}{(2\pi\hbar)^{2k}} \leq C\sum_{k=1}^{n}\sqrt{\frac{n}{k(n-k)}},$$

which yields the result. □

Now we can complete the

Proof of Theorem 19.4. We call

$$\Gamma = \left\{\mathrm{Emp}_{Z_N}, \int_{\Omega_m}\mathrm{Emp}_{Z_N} \geq \frac{(1+\varepsilon)}{(2\pi)^2}|\Omega_m|\right\} \tag{19.31}$$

the set of all empirical measure violating the Pauli principle in Ω_m by a finite amount. We have, using (19.3) that, for any $n \leq N$,

$$\frac{(1+\varepsilon)^n}{(2\pi)^{2n}}|\Omega_m|^n\mathbb{P}_{P_N^{\mathrm{DF}}}(\Gamma) \leq \int_{\rho\in\Gamma}\left(\int_{\Omega_m}\rho^{\otimes n}\right)\mathrm{d}P_N^{\mathrm{DF}}(\rho) = \int_{(\Omega_m)^n}\widetilde{\mu}_N^{(n)}.$$

Next, using Lemma 19.5 we obtain

$$\mathbb{P}_{P_N^{\mathrm{DF}}}(\Gamma) \leq (1+\varepsilon)^{-n}\,(2\pi)^{2n}\,N^{-n}\sum_{k=1}^{n}\frac{|\Omega_m|^{(k-n)}S(n,k)}{(2\pi\hbar)^{2k}}$$

Choosing n as in Lemma 19.6 and inserting the latter result we deduce

$$\mathbb{P}_{P_N^{\mathrm{DF}}}(\Gamma) \leq C\frac{N^{\delta'}}{(1+\varepsilon)^n} \leq CN^{\delta'}e^{-N^{\delta}\ln(1+\varepsilon)}$$

and the result follows. □

3. Averaging the Diaconis-Freedman measure

We divide $\mathbb{R}^4$ into hyperrectangles

$$\Omega_m = \Omega_{m_x}\times\Omega_{m_p}$$

as in (19.23). We set

$$|\Omega_m| = |\Omega_{m_x}||\Omega_{m_p}| = l_x^2 l_p^2 = N^{-\beta} \tag{19.32}$$

for some $0<\beta<1$ to be fixed later. We know from Theorem 19.4 that the probability for one such hyperrectangle to violate the Pauli principle is exponentially small. This means that the contribution to (19.20) of empirical measures

$$\mu = \frac{1}{N}\sum_{j=1}^{N}\delta_{z_j}$$

with $Z_N = (z_1,\ldots,z_j)\in\Gamma$ (see (19.31)) will be negligible (by the union bound), provided the number of hyperrectangles is not too large. The next steps of our proof are thus

- to reduce estimates to the contribution of a finite set of phase-space.
- in the latter set, to average empirical measures to replace μ in (19.20) by true Pauli-principle abiding measures (in the sense of (16.20)).

As for the first step, let $L > 0$ be a number of the form

$$L = nN^{-\frac{\beta}{4}} \tag{19.33}$$

for some $n \in \mathbb{N}$ and define the hypercube

$$S_L = [-L, L]^4$$

of side $2L$ centered at the origin. This way the number of Ω_m-hyperrectangles contained within S_L is $2^4 n^4$. We first discard the energetic contribution of S_L^c.

LEMMA 19.7 (**Reduction to the energy in the phase-space hypercube** S_L). *Let $\sigma > 0$ and μ a probability measure satisfying*

$$\int_{\mathbb{R}^4} \left(|p|^2 + V(x)\right) \mathrm{d}\mu(x,p) \leq D. \tag{19.34}$$

Its energy can be bounded from below as

$$\mathcal{E}_V^R[\mu] \geq (1-\sigma) \int_{\mathbb{R}^4} \left(\left| p^{\mathbf{A}} + \beta_0 \mathbf{A}^R[\mathbb{1}_{S_L}\mu] \right|^2 + V \right) \mathbb{1}_{S_L} \mathrm{d}\mu(x,p) - C \frac{D^2}{\sigma R^2 \inf(L^4, L^{2s})} \tag{19.35}$$

where

$$\mathbf{A}^R[\mathbb{1}_{S_L}\mu] = \int_{\mathbb{R}^4} \nabla^\perp w_R(x-y) \mathbb{1}_{S_L}(y,p) \mathrm{d}\mu(y,p) = \mathbf{A}^R\left[\rho_{\mathbb{1}_{S_L}\mu}\right].$$

and

$$p^{\mathbf{A}} = p + \mathbf{A}_e(x).$$

The parameter σ will tend to 0 at the very end of the proof (after we take $N \to \infty, L \to \infty, R \to 0$ while ensuring $R^6 \inf(L^4, L^{2s}) \to \infty$).

PROOF. First, the positivity of the integrand gives

$$\mathcal{E}_V^R[\mu] \geq \int_{S_L} \left(\left| p^{\mathbf{A}} + \mathbf{A}^R[\mu] \right|^2 + V \right) \mathrm{d}\mu(x,p)$$

Then we split $\mathbf{A}^R[\mu]$ into a contribution from S_L and one from its complement,

$$\mathbf{A}^R[\mu] = \mathbf{A}^R[\mathbb{1}_{S_L}\mu] + \mathbf{A}^R\left[\mathbb{1}_{S_L^c}\mu\right],$$

obtaining

$$\begin{aligned}\left(p^{\mathbf{A}} + \mathbf{A}^R[\mathbb{1}_{S_L}\mu] + \mathbf{A}^R\left[\mathbb{1}_{S_L^c}\mu\right]\right)^2 &= \left(p^{\mathbf{A}} + \mathbf{A}^R[\mathbb{1}_{S_L}\mu]\right)^2 + \left|\mathbf{A}^R\left[\mathbb{1}_{S_L^c}\mu\right]\right|^2 \\ &\quad + 2\left(p^{\mathbf{A}} + \mathbf{A}^R[\mathbb{1}_{S_L}\mu]\right) \cdot \mathbf{A}^R\left[\mathbb{1}_{S_L^c}\mu\right].\end{aligned}$$

Next we use that $ab \leq \frac{a^2}{2\sigma} + \frac{\sigma b^2}{2}$ to deduce

$$2\left(p^{\mathbf{A}} + \mathbf{A}^R[\mathbb{1}_{S_L}\mu]\right) \cdot \mathbf{A}^R\left[\mathbb{1}_{S_L^c}\mu\right] \geq -\sigma\left(p^{\mathbf{A}} + \mathbf{A}^R[\mathbb{1}_{S_L}\mu]\right)^2 - \frac{1}{\sigma}\left|\mathbf{A}^R\left[\mathbb{1}_{S_L^c}\mu\right]\right|^2.$$

But

$$\begin{aligned}\int_{S_L} \left|\mathbf{A}^R\left[\mathbb{1}_{S_L^c}\mu\right]\right|^2(x)\,\mathrm{d}\mu(x,p) &\leq \left\|\mathbf{A}^R\left[\mathbb{1}_{S_L^c}\mu\right]\right\|_{L^\infty}^2 \\ &\leq \left\|\nabla^\perp w_R * \mathbb{1}_{S_L^c}\mu\right\|_{L^\infty}^2 \\ &\leq \left\|\mathbb{1}_{S_L^c}\mu\right\|_{L^1}^2 \left\|\nabla^\perp w_R\right\|_{L^\infty}^2 \\ &\leq \frac{C}{R^2}\left\|\mathbb{1}_{S_L^c}\mu\right\|_{L^1}^2\end{aligned}$$

where we used Young's inequality combined with [**59**, Lemma 2.1]. The measure μ satisfies (19.34) so, using Assumption 16.1,

$$\int_{\mathbb{R}^4} \mathbb{1}_{S_L^c}\mu(x,p)\mathrm{d}x\mathrm{d}p \leq \frac{D}{\inf(L^2, L^s)}$$

which concludes the proof. □

We now turn to the averaging procedure turning empirical measures based on configurations from Γ^c (recall (19.31)) into bounded measures satisfying the Pauli principle (16.20):

Definition (**Averaging map**).
Let μ be a positive measure on the phase-space hypercube $S_L = [-L, L]^4$. We associate to it its local average with respect to the grid (19.23)*:*

$$\mathrm{Ave}\,[\mu] = \sum_{m=1}^{16n^4} \mathbb{1}_{\Omega_m} \int_{\Omega_m} \frac{\mu}{|\Omega_m|} \tag{19.36}$$

with the associated position-space density (recall that $\Omega_m = \Omega_{m_x} \times \Omega_{m_p}$)

$$\rho_{\mathrm{Ave}[\mu]} := \int \mathrm{Ave}\,[\mu]\,\mathrm{d}p = \sum_{m=1}^{16n^4} \mathbb{1}_{\Omega_{m_x}} \int_{\Omega_m} \frac{\mu}{|\Omega_{m_x}|}. \tag{19.37}$$

Note that $\mathrm{Ave}\,[\mu]$ lives on S_L by definition. We next quantify the energetic cost of the above procedure:

Theorem 19.8 (**Averaging the empirical measure**).
Assume $\beta < 1$ in (19.24) *as well as*

$$l_x \ll R, \quad l_p \ll 1 \text{ in the limit } N \to \infty. \tag{19.38}$$

Let $Z_N = (x_1, p_1, ..., x_N, p_N)$ with $\mathrm{Emp}_{Z_N} \in \Gamma_\varepsilon^c$,

$$\Gamma_\varepsilon = \left\{\mathrm{Emp}_{Z_N}, \exists \Omega_m \subset S_L \text{ such that } \int_{\Omega_m} \mathrm{Emp}_{Z_N} \geq \frac{(1+\varepsilon)}{(2\pi)^2}|\Omega_m|\right\}. \tag{19.39}$$

Assume that (19.34) *holds for $\mu = \mathrm{Emp}_{Z_N}$. Then, for any $\sigma > 0$,*

$$\begin{aligned}\mathcal{E}_V^R[\mathrm{Emp}_{Z_N}] \geq (1-\sigma)\,(1-o_N(1))\,\mathcal{E}_V^R\left[\mathrm{Ave}\left[\mathrm{Emp}_{Z_N}\right]\right] \\ - o_N(1) - C\frac{D^2}{\sigma R^2 \inf(L^4, L^{2s})}.\end{aligned} \tag{19.40}$$

PROOF. In this proof we denote

$$\mu = \mathrm{Emp}_{Z_N}$$

for brevity. We first use Lemma 19.7 to write

$$\mathcal{E}_V^R[\mu] \geq (1-\sigma)\mathcal{E}_V^R[\mathbb{1}_{S_L}\mu] - C\frac{D^2}{\sigma R^2 \inf(L^4, L^{2s})}$$

and now estimate $\left|\mathcal{E}_V^R[\mathbb{1}_{S_L}\mu] - \mathcal{E}_V^R[\mathrm{Ave}[\mu]]\right|$. We expand $\mathcal{E}_V^R$ (19.21) and proceed term by term. Recall from (19.33) that

$$S_L = \cup_{m=1}^{2^4 n^4} \Omega_m.$$

We first calculate

$$\int_{S_L} \left((p^{\mathbf{A}})^2 + V(x)\right) \mathrm{d}(\mu - \mathrm{Ave}[\mu]) = \int_{S_L} \left(\left(p^{\mathbf{A}}\right)^2 + V(x)\right) \mathrm{Emp}_{Z_N} \mathrm{d}x\mathrm{d}p$$
$$- \frac{1}{N}\sum_{m=1}^{16n^4} \int_{\Omega_m} \frac{\left(p^{\mathbf{A}}\right)^2 + V(x)}{|\Omega_m|} \mathrm{d}x\mathrm{d}p \int_{\Omega_m} \sum_{i=1}^{N} \delta_{z_i=(x,p)} \mathrm{d}x\mathrm{d}p$$

We denote

$$G_N = \{i, z_i \in S_L\}$$
$$G_N^c = \{i, z_i \in S_L^c\}$$

and let $m(i)$ be the label of the box Ω_m particle z_i is in. This way

$$\sum_{i=1}^{N} \int_{\Omega_m} \delta_{z_i=(x,p)} \mathrm{d}x\mathrm{d}p = \sum_{i \in G_N} \delta_{m=m(i)}$$

and

$$\int_{S_L} \left((p^{\mathbf{A}})^2 + V(x)\right) \mathrm{d}(\mu - \mathrm{Ave}\left[\mathrm{Emp}_{Z_N}\right]) = \frac{1}{N}\sum_{i \in G_N} \left(\left(p_i^{\mathbf{A}}\right)^2 + V(x_i)\right)$$
$$- \int_{\Omega_{m(i)}} \frac{\left(p^{\mathbf{A}}\right)^2 + V(x)}{|\Omega_{m(i)}|} \mathrm{d}x\mathrm{d}p$$

By the mean value theorem applied in each Ω_m there exists a point $\widetilde{z}_i = (\widetilde{x}_i, \widetilde{p}_i) \in \Omega_{m(i)}$ such that

$$\int_{\Omega_{m(i)}} \frac{\left(p^{\mathbf{A}}\right)^2 + V(x)}{|\Omega_{m(i)}|} \mathrm{d}x\mathrm{d}p = \left(\widetilde{p}_i^{\mathbf{A}}\right)^2 + V(\widetilde{x}_i)$$

and we have, using Assumption 16.1 to control the variations of $\mathbf{A}_e$,

$$\left|\int_{S_L} \left((p^{\mathbf{A}})^2 + V(x)\right) \mathrm{d}(\mathrm{Ave}\left[\mathrm{Emp}_{Z_N}\right] - \mu)\right| \leq \frac{1}{N}\sum_{i \in G_N} \left|\left((p_i^{\mathbf{A}})^2 - (\tilde{p}_i^{\mathbf{A}})^2 + V(x_i) - V(\tilde{x}_i)\right)\right|$$
$$\leq \frac{C}{N}\sum_{i \in G_N} \left(\sup_{z=(x,p)\in\Omega_{m(i)}} l_p|p| + l_x \sup_{z=(x,p)\in\Omega_{m(i)}} |\nabla V(x)|\right) + Cl_x$$
$$= \frac{C}{N}\sum_{\Omega_m \subset S_L} \left(N_m l_p \sup_{z=(x,p)\in\Omega_m} |p| + N_m l_x \sup_{z=(x,p)\in\Omega_m} |\nabla V(x)|\right) + Cl_x$$

where we denote

$$N_m := \sharp\{z_i \in Z_N \cap \Omega_m\} \leq C|\Omega_m|. \tag{19.41}$$

Since our assumptions imply that for any $\Omega_m \subset S_L$

$$N_m \leq C|\Omega_m| \tag{19.42}$$

we may recognize Riemann sums and use Assumption 16.1 again to deduce

$$\begin{aligned}\left|\int_{S_L}\left((p^{\mathbf{A}})^2 + V(x)\right)\mathrm{d}(\mathrm{Ave}\left[\mathrm{Emp}_{Z_N}\right] - \mu)\right| \\ \leq Cl_p\int_{\mathbb{R}^4}|p|\mathrm{dAve}\left[\mu\right] + Cl_x\int_{\mathbb{R}^4}|x|^{s-1}\mathrm{dAve}\left[\mu\right] + Cl_p \\ \leq Cl_p\int_{\mathbb{R}^4}|p|^2\mathrm{dAve}\left[\mu\right] + Cl_x\int_{\mathbb{R}^4}V\mathrm{dAve}\left[\mu\right] + Cl_x + Cl_p\end{aligned}$$

using Young's inequality

$$r^{s-1} \leq \frac{s-1}{s}r^s + \frac{1}{s}.$$

We next treat the two body operator W_{12} defined in (17.6) (understanding now p as a momentum variable rather than an operator) in the same way. We set

$$\mathrm{d}x = \mathrm{d}x_1\mathrm{d}x_2, \quad \mathrm{d}p = \mathrm{d}p_1\mathrm{d}p_2.$$

By the mean-value theorem, for each $(x_i, p_i) \in \Omega_{m(i)}$ there exists some $\widetilde{z}_i = (\widetilde{x}_i, \widetilde{p}_i) \in \Omega_{m(i)}$ such that

$$\begin{aligned}\int_{\mathbb{R}^8} W_{12}\,\mathrm{d}\left(\mathrm{Ave}\left[\mathrm{Emp}_{Z_N}\right]^{\otimes 2} - (\mathbb{1}_{S_L}\mu)^{\otimes 2}\right) = \int_{S_L\times S_L} W_{12}\left[\mu^{\otimes 2} - \mathrm{Ave}\left[\mathrm{Emp}_{Z_N}\right]^{\otimes 2}\right]\mathrm{d}x\mathrm{d}p \\ = \frac{1}{N^2}\sum_{i\in G_N}\sum_{j\in G_N}\left[\left(p_i + \mathbf{A}_e(x_i)\right).\nabla^\perp w_R(x_i,x_j) - \left(\widetilde{p}_i + \mathbf{A}_e(\widetilde{x}_i)\right).\nabla^\perp w_R(\widetilde{x}_i,\widetilde{x}_j)\right]\end{aligned} \tag{19.43}$$

Expanding the functions around the points $\widetilde{z}_i$ of the sum there exists a $c_i \in \Omega_{m(i)}$ such that

$$\begin{aligned}\left|\nabla^\perp w_R(x_i) - \nabla^\perp w_R(\widetilde{x}_i)\right| \leq J_{\nabla w_R}(c_i)\cdot(x_i - \widetilde{x}_i) \\ |p_i - \widetilde{p}_i| \leq l_p\end{aligned}$$

with $J_{\nabla w_R}$ the Jacobian matrix of ∇w_R. We denote

$$W = \partial_u^2 w_R \tag{19.44}$$

where ∂_u is the radial derivative explicitly calculated in (20.18). We then use (20.11) and estimate

$$\begin{aligned}\left|\int_{\mathbb{R}^8} W_{12}\,\mathrm{d}\left(\mathrm{Ave}\left[\mathrm{Emp}_{Z_N}\right]^{\otimes 2} - (\mathbb{1}_{S_L}\mu)^{\otimes 2}\right)\right| \\ \leq \frac{C}{N^2}\sum_{\Omega_m\subset S_L}\sum_{\Omega_q\subset S_L} l_x N_m N_q \sup_{z_m\in\Omega_m, z_q\in\Omega_q}|p_m|\,|W(x_m - x_q)| \\ + \frac{C}{N^2}\sum_{\Omega_m\subset S_L}\sum_{\Omega_q\subset S_L} N_m N_q l_p \sup_{z_m\in\Omega_m, z_q\in\Omega_q}\left|\nabla^\perp w_R\left(x_m - x_q\right)\right|\end{aligned} \tag{19.45}$$

where we used that $\mathbf{A}_e$ and its derivatives are in L^∞ by assumption and use the convention $z_i = (x_i, p_i)$ systematically. We denote

$$\begin{aligned}\overline{W_1}(x_1 - x_2) &= \sup\{|W(y_1 - y_2)|, y_1 \in \Omega_{m_x}(x_1), y_2 \in \Omega_{m_x}(x_2)\}\\ \overline{W_2}(x_1 - x_2) &= \sup\left\{|\nabla^\perp w_R(y_1 - y_2)|, y_1 \in \Omega_{m_x}(x_1), y_2 \in \Omega_{m_x}(x_2)\right\}\\ P(p_1) &= \sup\left\{|p|, p \in \Omega_{m_p}(p)\right\}\end{aligned}$$

with $\Omega_{m_x}(x) \subset \mathbb{R}^2$ and $\Omega_{m_p}(p) \subset \mathbb{R}^2$ the square of our grid (19.32) that x belongs to (respectively $\Omega_{m_p}(p)$ the square p belongs to). Hence

$$\begin{aligned}&\left|\int_{\mathbb{R}^8} W_{12}\,\mathrm{d}\left(\mathrm{Ave}\left[\mathrm{Emp}_{Z_N}\right]^{\otimes 2} - (\mathbb{1}_{S_L}\mu)^{\otimes 2}\right)\right|\\ &\qquad \le Cl_x \iint_{\mathbb{R}^8} P(p_1)\overline{W_1}(x_1 - x_2)\mathrm{Ave}[\mu](x_1, p_1)\mathrm{Ave}[\mu](x_2, p_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}p_1\mathrm{d}p_2\\ &\qquad\quad + Cl_p \iint_{\mathbb{R}^8} \overline{W_2}(x_1 - x_2)\mathrm{Ave}[\mu](x_1, p_1)\mathrm{Ave}[\mu](x_2, p_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}p_1\mathrm{d}p_2 =: \mathrm{I} + \mathrm{II}.\end{aligned}$$

Recognizing a Riemann similarly as above yields, using the properties of W (see (20.18)),

$$\begin{aligned}\mathrm{I} &\le Cl_x \int_{\mathbb{R}^4} |p_1|\mathrm{Ave}[\mu](x_1, p_1) \int_{\mathbb{R}^2} \left(\frac{1}{R}|\nabla w_0(x_1 - x_2)| + \frac{1}{R^2}\mathbb{1}_{|x_1 - x_2| \le R}\right) \rho_{\mathrm{Ave}[\mu]}(x_2)\mathrm{d}x_2\mathrm{d}x_1\mathrm{d}p_1\\ &\le C\frac{l_x}{R}\left\|\rho_{\mathrm{Ave}[\mu]}\right\|_{L^2} \int |p|\mathrm{Ave}[\mu](x, p)\mathrm{d}x\mathrm{d}p\\ &\le C\frac{l_x}{R}\left\|\rho_{\mathrm{Ave}[\mu]}\right\|_{L^2} \left(\int |p|^2\mathrm{Ave}[\mu](x, p)\mathrm{d}x\mathrm{d}p\right)^{1/2}\end{aligned}$$

where we also used the weak Young and the Cauchy-Schwarz inequality. Similarly, since $\nabla w_R \in L^{2,w}$,

$$\mathrm{II} \le Cl_p \iint_{\mathbb{R}^4} |\nabla w_R(x_1 - x_2)|\, \rho_{\mathrm{Ave}[\mu](x_1)}\rho_{\mathrm{Ave}[\mu](x_2)}\mathrm{d}x_1\mathrm{d}x_2 \le Cl_p \left\|\rho_{\mathrm{Ave}[\mu]}\right\|_{L^2}.$$

We next turn to the three-body term (17.7)

$$\mathrm{III} := \left|\int_{\mathbb{R}^{12}} W_{123}\mathrm{d}\left((\mathbb{1}_{S_L}\mu)^{\otimes 3} - \mathrm{Ave}\left[\mathrm{Emp}_{Z_N}\right]^{\otimes 3}\right)\right|$$

with

$$W_{123} = \nabla^\perp w_R\left(x_1 - x_2\right)\nabla^\perp w_R\left(x_1 - x_3\right).$$

We denote (with W as in (19.44))

$$\begin{aligned}G(x_1, x_2, x_3) &= |W(x_1 - x_2)|\left|\nabla^\perp w_R(x_1 - x_3)\right| + \left|\nabla^\perp w_R(x_1 - x_2)\right||W(x_1 - x_3)|\\ \overline{G}(x_1, x_2, x_3) &= \sup\{G(y_1, y_2, y_3), y_1 \in \Omega_{m_x}(x_1), y_2 \in \Omega_{m_x}(x_2), y_3 \in \Omega_{m_x}(x_3)\}\end{aligned}$$

and observe that G gives a pointwise upper bound to gradients of W_{123} in any variable. Hence, arguing as above

$$\begin{aligned}\mathrm{III} &\leq Cl_x \int_{R^6} \overline{G}(x_1,x_2,x_3)\rho_{\mathrm{Ave}[\mu]}(x_1)\rho_{\mathrm{Ave}[\mu]}(x_2)\rho_{\mathrm{Ave}[\mu]}(x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\ &\leq C\frac{l_x}{R}\left\|\rho_{\mathrm{Ave}[\mu]}\right\|_{L^2}^2\end{aligned}$$

where we used the weak Young inequality twice. Collecting the previous inequalities we conclude

$$\begin{aligned}\mathcal{E}_V^R[\mu] &\geq (1-\sigma)\mathcal{E}_V^R[\mathrm{Ave}[\mu]] - C(l_x+l_p) \\ &\quad - C\frac{l_x}{R}\left\|\rho_{\mathrm{Ave}[\mu]}\right\|_{L^2}\left(\left\|\rho_{\mathrm{Ave}[\mu]}\right\|_{L^2} + \left(\int |p|^2 \mathrm{dAve}[\mu]\right)^{1/2}\right) \\ &\quad - Cl_p\left\|\rho_{\mathrm{Ave}[\mu]}\right\|_{L^2} - C\frac{D^2}{\sigma R^2 \inf(L^4, L^{2s})} \\ &\geq (1-\sigma)\mathcal{E}_V^R[\mathrm{Ave}[\mu]] - o_N(1) - o_N(1)\int |p|^2\mathrm{dAve}[\mu] - C\frac{D^2}{\sigma R^2\inf(L^4,L^{2s})},\end{aligned}$$

inserting our assumptions on the various parameters involved $l_x^2 = N^{-\beta+\varepsilon} \ll R^2$ and $l_p^2 = N^{-\varepsilon}$. We have also used the fact that, since

$$\mathrm{Ave}[\mu] \leq \frac{1+\varepsilon}{2\pi^2}, \tag{19.46}$$

the bathtub principle [**92**, Theorem 1.14] implies that

$$\int |p|^2 \mathrm{dAve}[\mu] \geq C_\varepsilon \left\|\rho_{\mathrm{Ave}[\mu]}\right\|^2.$$

Finally, arguing as in Appendix 1 below (using (19.46) again),

$$\mathcal{E}_V^R[\mathrm{Ave}[\mu]] \geq \frac{1}{2}\int |p|^2\mathrm{dAve}[\mu] - C\left\|\rho_{\mathrm{Ave}[\mu]}\right\|^2$$

and we can absorb an error term in the main term to obtain

$$\mathcal{E}_V^R[\mu] \geq (1-\sigma)(1-o_N(1))\mathcal{E}_V^R[\mathrm{Ave}[\mu]] - o_N(1) - o_N(1)\left\|\rho_{\mathrm{Ave}[\mu]}\right\|^2 - C\frac{D^2}{\sigma R^2\inf(L^4,L^{2s})}.$$

A last use of the bathtub principle gives

$$\mathcal{E}_V^R[\mathrm{Ave}[\mu]] \geq C\left\|\rho_{\mathrm{Ave}[\mu]}\right\|^2.$$

Thus the proof is concluded by absorbing one last error term in the main expression. □

4. Convergence of the energy

Going back to our lower bound (19.20), assuming $R = N^{-\eta}, 0 < \eta < 1/4$, we have

$$\begin{aligned}\frac{E^R(N)}{N} &= \frac{\langle \Psi_N, H_N^R \Psi_N \rangle}{N} \\ &\geq \int_{\mathcal{P}(\mathbb{R}^4)} \mathcal{E}_V^R[\mu] \mathrm{d}P_{m_N}^{\mathrm{DF}}(\mu) - o_N(1) \\ &\geq \int_{\Gamma_\varepsilon^c \cap \Xi_\tau} \mathcal{E}_V^R[\mu] \mathrm{d}P_{m_N}^{\mathrm{DF}}(\mu) - o_N(1) \end{aligned} \tag{19.47}$$

where

$$\mathcal{E}_V^R[\mu] = \int_{\mathbb{R}^4} \left(\left| p^{\mathbf{A}} + \beta_0 \mathbf{A}^R[\mu](x) \right|^2 + V(x) \right) \mathrm{d}\mu, \tag{19.48}$$

Γ^ε is as in (19.31),

$$\Xi_\tau = \left\{ \mu \in \mathcal{P}(\mathbb{R}^4) : \int_{\mathbb{R}^4} \left(|p|^2 + V(x) \right) \mathrm{d}\mu(x,p) \leq \frac{\tau}{R^2} \right\},$$

and we have used the positivity of the integrand to go to the second line of (19.47). We next pick l_x, l_p to enforce

$$l_x \ll R, \quad l_p \ll 1, \quad l_x^2 l_p^2 = N^{-\beta}$$

for some $\beta < 1$. This is certainly compatible with our other requirements, since $R \gg N^{-1/4}$ by assumption. We can then use Theorem 19.8 to deduce

$$\frac{E^R(N)}{N} \geq (1-\sigma)(1-o_N(1)) \int_{\Gamma_\varepsilon^c \cap \Xi_\tau} \mathcal{E}_V^R\left[\mathrm{Ave}[\mu]\right] \mathrm{d}P_{m_N}^{\mathrm{DF}}(\mu) - o_N(1) - C\frac{\tau^2}{\sigma R^6 \inf(L^4, L^{2s})}. \tag{19.49}$$

Next, denote, for some $\gamma > 0$

$$\Theta_\gamma = \left\{ \mu \in \mathcal{P}(\mathbb{R}^4) \text{ such that } \int_{S_L^c} \mu < \gamma \right\} \tag{19.50}$$

the set of measures whose mass lie mostly in S_L and note that, as per Lemmas 20.4 and 20.5,

$$\mathcal{E}_V^R\left[\mathrm{Ave}[\mu]\right] \geq (1 - CR)(1-2\varepsilon)(1-\gamma) e_{\mathrm{TF}}$$

for any $\mu \in \Gamma_\varepsilon^c \cap \Theta_\gamma$. Clearly, if we set

$$\gamma = c\frac{\tau}{R^2 \inf(L^2, L^{2s})} \tag{19.51}$$

with a suitable $c > 0$ we have that

$$\Theta_\gamma \subset \Xi_\tau.$$

Hence, with this choice,

$$\begin{aligned}\frac{E^R(N)}{N} &\geq (1-CR)(1-2\varepsilon)(1-\gamma)(1-\sigma)(1-o_N(1))e_{\mathrm{TF}}\int_{\Gamma_\varepsilon^c\cap\Xi_\tau}\mathrm{d}P_{m_N}^{\mathrm{DF}}(\mu)-o_N(1)\\ &\quad-\frac{C\tau^2}{\sigma R^6\inf(L^2,L^{2s})}\\ &\geq (1-2\varepsilon)(1-\gamma)(1-\sigma)(1-o_N(1))e_{\mathrm{TF}}\left(1-\mathbb{P}_{m_N}^{\mathrm{DF}}(\Gamma_\varepsilon)-\mathbb{P}_{m_N}^{\mathrm{DF}}(\Xi_\tau^c)\right)\\ &\quad-\frac{C\tau^2}{\sigma R^6\inf(L^2,L^{2s})},\end{aligned}\tag{19.52}$$

with a self-explanatory notation for $\mathbb{P}_{m_N}^{\mathrm{DF}}$. There remains to estimate the probability of the "bad events" in the last inequality. First, there is a $\delta>0$ such that

$$\begin{aligned}\mathbb{P}_{m_N}^{\mathrm{DF}}(\Gamma_\varepsilon)&=\mathbb{P}_{m_N}^{\mathrm{DF}}\left(\cup_{S_K\in S_L}\left\{\mathrm{Emp}_{Z_N},\int_{\Omega_k}\mathrm{Emp}_{Z_N}\geq\frac{(1+\varepsilon)}{(2\pi)^2}|\Omega_k|\right\}\right)\\ &\leq\sum_{k=1}^{16n^4}\mathbb{P}_{m_N}^{\mathrm{DF}}\left(\left\{\mathrm{Emp}_{Z_N},\int_{\Omega_k}\mathrm{Emp}_{Z_N}\geq\frac{(1+\varepsilon)}{(2\pi)^2}|\Omega_k|\right\}\right)\\ &\leq C_\delta L^4N^\beta e^{-c_\delta N^\delta\ln(1+\varepsilon)}.\end{aligned}\tag{19.53}$$

We used the union bound, the bound on the probability of violating the Pauli principle in a single box from Theorem 19.4 and the relation (19.33).

On the other hand, using Markov's inequality, (18.1), Lemma 18.2 and (19.6) we find

$$\begin{aligned}\mathbb{P}_{m_N}^{\mathrm{DF}}(\Xi_\tau^c)&\leq R^2\tau^{-1}\int_{\mathcal{P}(\mathbb{R}^4)}\left(\int_{\mathbb{R}^4}\left(|p|^2+V(x)\right)\mathrm{d}\mu(x,p)\right)\mathrm{d}P_{m_N}^{\mathrm{DF}}(\mu)\\ &=CR^2\tau^{-1}\int_{\mathbb{R}^4}\left(|p|^2+V(x)\right)\mathrm{d}m_N^{(1)}(x,p)\\ &\leq C\tau^{-1}.\end{aligned}\tag{19.54}$$

There remains to insert (19.53) and (19.54) in (19.52). We can now set $L=N^\kappa$ with $\kappa>0$ a suitably large, fixed number and $\tau=N^{\kappa'}$ with κ' a suitably small, fixed number. This makes all errors depending on N negligible because, crucially, (19.53) contains an exponentially small term in N. In particular, recalling (19.51), $\gamma\to0$ with such a choice of a parameters. Hence we can pass to the limit in (19.52), first letting $N,R^{-1},L\to\infty$ with the previously specified scalings, and finally let $\varepsilon,\sigma\to0$, obtaining the energy lower bound claimed in 16.1, matching the upper bound from Section 17.

5. Convergence of states

We turn to the proof of Theorem 16.5. Coming back to (19.49) and inserting the previous choice of parameters we have that

$$e^{\mathrm{TF}}\geq(1-\sigma)(1-o_N(1))\int_{\mathcal{P}(\mathbb{R}^4)}\mathbb{1}_{\Xi_\tau}(\mu)\mathbb{1}_{\Gamma_\varepsilon^c}(\mu)\mathcal{E}_V\left[\mathrm{Ave}\left[\mu\right]\right]\mathrm{d}P_{m_N}^{\mathrm{DF}}(\mu)-o_N(1)(1+\sigma^{-1}).\tag{19.55}$$

It follows from (19.55) and results of Appendix 1 that $m_N^{(1)}$ is tight. Applying the last part of Theorem 19.1 we thus obtain that the Diaconis-Freedman measure $P_{m_N}^{\mathrm{DF}}$ weakly converges

to a probability measure $P \in \mathcal{P}(\mathcal{P}(\mathbb{R}^4))$ (this is the Hewitt-Savage measure). It follows from Theorem 19.4 (or [**147**, Theorem 2.6]) that P is concentrated on probability measures satisfying $\mu \leq (2\pi)^{-2}$. Combining with Lemma 18.3 and (19.12) we have that, for any fixed $k \geq 0$

$$\frac{1}{(2\pi)^{2k}} m^{(k)}_{\Psi_N} \underset{N\to\infty}{\to} \int_{\mathcal{P}(\mathbb{R}^4)} \mu^{\otimes k} \mathrm{d}P(\mu)$$

weakly as measures. We now identify the limit measure P to conclude the proof of Theorem 16.5.

To this aim it is convenient to write the integral above as

$$\int_{\mathcal{P}(\mathbb{R}^4)} \mathbb{1}_{\Xi_\tau}(\mu)\mathbb{1}_{\Gamma_\varepsilon^c}(\mu)\mathcal{E}_V\left[\mathrm{Ave}\left[\mu\right]\right]\mathrm{d}P_N^{\mathrm{DF}}(\mu) = \int_{\mathcal{P}(\mathbb{R}^4)} \mathcal{E}_V\left[\mathrm{Ave}\left[\mu\right]\right]\mathrm{d}Q_N(\mu) \tag{19.56}$$

with

$$\mathrm{d}Q_N[\mu] := \mathbb{1}_{\Xi_\tau}(\mu)\mathbb{1}_{\Gamma_\varepsilon^c}(\mu)\mathrm{d}P_N^{\mathrm{DF}}(\mu). \tag{19.57}$$

We will see $\mathcal{E}_V\left[\mathrm{Ave}\left[\mu\right]\right]$ as a lower semi-continuous function on the space $\left(\mathcal{P}\left(\mathbb{R}^4\right), d_1^W\right)$ where

$$d_1^W(\mu,\nu) = \sup_{\|\phi\|_{\mathrm{Lip}}\leq 1} \left| \int_{\mathbb{R}^4} \phi \mathrm{d}\mu - \int_{\mathbb{R}^4} \phi \mathrm{d}\nu \right|$$

is the Monge-Kantorovitch-Wasserstein distance, known to metrize the weak convergence of measures [**162**, Theorem 6.9] (and we use Kantorovitch-Rubinstein duality [**162**, Remark 6.5]). We have

Lemma 19.9 (**Limit of** Ave).
Let $\mu \in \mathcal{P}(\mathbb{R}^4)$. In the d_1^W metric,

$$\mathrm{Ave}\left[\mu\right] = \sum_m^{16n^4} \mathbb{1}_{\Omega_m} \int_{\Omega_m} \frac{\mathrm{d}\mu}{|\Omega_m|} \to \mu, \tag{19.58}$$

when $N \to \infty$ and $L \to \infty$.

Proof. We pick a function ϕ, $\|\nabla\phi\|_{L^\infty} \leq 1$ and denote

$$\mathrm{a}\left[\phi\right]_m = \int_{\Omega_m} \frac{\phi(x,p)}{|\Omega_m|} \mathrm{d}x\mathrm{d}p$$

its average over a box Ω_m. We have that, for any $x \in \Omega_m$,

$$|\phi(x) - \mathrm{a}\left[\phi\right]_m| \leq Cl \|\nabla\phi\|_{L^\infty} \tag{19.59}$$

where $l = \max(l_x, l_p)$ is the largest dimension of the box Ω_m. We thus have

$$\int_{\mathbb{R}^4} \phi(x)\mathrm{dAve}\left[\mu\right] = \sum_m^{16n^4} \int_{\Omega_m} \mathrm{a}\left[\phi\right]_m \mathrm{d}\mu = \sum_m \int_{\Omega_m} \phi \mathrm{d}\mu + o_N(1)$$

where the $o_N(1)$ is uniform in $\|\nabla\phi\|_{L^\infty}$ by (19.59). □

It follows that, if we take a sequence $\mu_n \to \mu$ when $N \to \infty$,

$$d_1^W\left(\mathrm{Ave}\left[\mu_n\right], \mu\right) \leq d_1^W\left(\mathrm{Ave}\left[\mu_n\right], \mu'\right) + d_1^W\left(\mu_n, \mu\right) \leq o_N(1)$$

Hence, combining with Lemma (20.1), we deduce

$$\liminf_{N\to\infty,\,\mu'\to\mu} \mathcal{E}_V\left[\mathrm{Ave}\left[\mu'\right]\right] \geq \mathcal{E}_V\left[\mu\right]. \tag{19.60}$$

We turn to the convergence of Q_N defined in (19.57). For any continous bounded function $\Phi \in C_b(\mathcal{P}(\mathbb{R}^4))$ over probability measures,

$$\left|\int_{\mathcal{P}(\mathbb{R}^4)} \Phi(\mu)\left(\mathrm{d}Q_N\left(\mu\right) - \mathrm{d}P^{\mathrm{DF}}_{m_N}\left(\mu\right)\right)\right| \underset{N\to\infty}{\to} 0$$

by (19.53) and (19.54). Hence, applying Theorem 19.1 we have that

$$Q_N \to P$$

as measures over $\mathcal{P}(\mathbb{R}^4)$, where P is the limit of the Diaconis-Freedman measure. Combining with (19.60) we may apply the improved Fatou Lemma of [**48**, Theorem 1.1] to (19.56), obtaining

$$\liminf_{N\to\infty} \int_{\mathcal{P}(\mathbb{R}^4)} \mathcal{E}_V\left[\mathrm{Ave}\left[\mu\right]\right] \mathrm{d}Q_N(\mu) \geq \int_{\mathcal{P}(\mathbb{R}^4)} \mathcal{E}_V\left[\mu\right] \mathrm{d}P(\mu).$$

Combining with (19.55) and letting $\sigma \to 0$ after $N \to \infty$ we conclude

$$e^{\mathrm{TF}} \geq \int_{\mathcal{P}(\mathbb{R}^4)} \mathcal{E}_V\left[\mu\right] \mathrm{d}P(\mu).$$

Recall that P is concentrated on probability measures satisfying $\mu \leq (2\pi)^{-2}$ and that e^{TF} is the infimum of $\mathcal{E}_V\left[\mu\right]$ over those. Hence P must be concentrated on the unique minimizer of $\mathcal{E}_V\left[\mu\right]$, and this concludes the proof.

CHAPTER 20

Appendix for the almost-fermionic case

1. Properties of the Vlasov functional

In this appendix we etablish some of the fundamental properties of the functional $\mathcal{E}_V$ and some useful bounds on the vector potential $\mathbf{A}^R[\rho]$ associated to a measure ρ. Note that sometimes use the same letter C for different constants.

LEMMA 20.1 (**Lower semicontinuity of $\mathcal{E}_V$**).
Let $(\mu_n)_{n\geq 0}$ be a sequence of positive measures on $\mathbb{R}^4$ satisfying

$$0 \leq \mu_n \leq (2\pi)^{-2}, \quad \int_{\mathbb{R}^4} \mu_n \leq C$$

with C independent of N. If μ_n converges to μ as measures we have

$$\liminf_{n\to\infty} \mathcal{E}_V[\mu_n] \geq \mathcal{E}_V[\mu]. \tag{20.1}$$

PROOF. We recall that the marginal in p of μ_n is

$$\rho_n(x) = \int_{\mathbb{R}^2} \mu_n(x,p)\mathrm{d}p. \tag{20.2}$$

It follows from applying the bathtub principle [**92**, Theorem 1.14] in the p variable that

$$\mathcal{E}_V[\mu_n] \geq \mathcal{E}_V^{\mathrm{TF}}[\rho_n] := 2\pi \int_{\mathbb{R}^2} \rho_n^2(x)\mathrm{d}x + \int_{\mathbb{R}^2} V(x)\rho_n(x)\mathrm{d}x.$$

Hence we may assume that $\|\rho_n\|_{L^2}$ is uniformly bounded and that $\rho_n \rightharpoonup \rho$ weakly in $L^2\left(\mathbb{R}^2\right)$. We then deduce from the weak Young inequality [**92**, Chapter 4] that

$$\left\|\mathbf{A}^2[\rho_n]\rho_n\right\|_{L^1} \leq C \left\|\nabla^\perp w * \rho_n\right\|_{L^\infty}^2 \leq C \|\rho_n\|_{L^2}^2 \left\|\nabla^\perp w\right\|_{L^{2,w}}^2 \leq C.$$

We recall that the weak $L^p(\mathbb{R}^d)$ space is the set of functions

$$L^{w,2} = \left\{ f \mid \mathrm{Leb}\left(\left\{x \in \mathbb{R}^d : |f(x)| > \lambda\right\}\right) \leq \left(\frac{C}{\lambda}\right)^p \right\}$$

with the associated norm

$$\|f\|_{L^{w,2}} = \sup_{\lambda>0} \lambda\, \mathrm{Leb}\left(x \in \mathbb{R}^d,\ |f(x)| > \lambda\right)^{1/2}$$

where Leb is the Lebesgue measure. We expand the energy

$$\mathcal{E}_V[\mu_n] = \int_{\mathbb{R}^4} \left(\left(p^{\mathbf{A}}\right)^2 + 2\beta_0 p^{\mathbf{A}} \cdot \mathbf{A}[\rho_n] + \beta_0^2 \mathbf{A}^2[\rho_n] + V\right) \mu_n(x,p)\mathrm{d}x\mathrm{d}p \tag{20.3}$$

and use that $ab \leq \frac{a^2}{2\sigma} + \frac{\sigma b^2}{2}$ to get

$$2\left(p^{\mathbf{A}}\right) \cdot \mathbf{A}\left[\rho_n\right] \geq -\sigma |p^{\mathbf{A}}|^2 - \frac{1}{\sigma} \left|\mathbf{A}\left[\rho_n\right]\right|^2$$

and hence

$$\mathcal{E}_V\left[\mu_n\right] + C \left\|\mathbf{A}^2\left[\rho_n\right] \rho_n\right\|_{L^1} \geq C \int_{\mathbb{R}^4} \left(|p^{\mathbf{A}}|^2 + V\right) \mathrm{d}\mu_n.$$

Thus (μ_n) is tight and, up to extraction, converges strongly in $L^1\left(\mathbb{R}^4\right)$. In order to obtain the convergence of $\mathbf{A}^2[\rho_n]\mu_n$ we write

$$\begin{aligned}\rho_n(x_1)\rho_n(x_2)\mu_n(x_3,p_3) - \rho(x_1)\rho(x_2)\mu(x_3,p_3) &= \left(\rho_n(x_1) - \rho(x_1)\right)\left(\rho_n(x_2) - \rho(x_2)\right)\mu_n(x_3,p_3) \\ &+ \left(\rho_n(x_1) - \rho(x_1)\right)\rho(x_2)\mu_n(x_3,p_3) \\ &+ \rho(x_1)\left(\rho_n(x_2) - \rho(x_2)\right)\mu_n(x_3,p_3) \\ &- \rho(x_1)\rho(x_2)\left(\mu_n(x_3,p_3) - \mu(x_3,p_3)\right) \quad (20.4)\end{aligned}$$

and treat each resulting term separately. The first term yields

$$\left\|\mathbf{A}^2[\rho_n - \rho]\mu_n\right\|_{L^1} \leq \|\mu_n\|_{L^1} \left\|\nabla^\perp w * (\rho_n - \rho)\right\|_{L^\infty}^2 \leq \|\rho_n - \rho\|_{L^2}^2 \left\|\nabla^\perp w\right\|_{L^{2,w}}^2$$

using that $||\mu_n||_{L^1} \leq C$ and the weak Young inequality. For the second term of (20.4)

$$\|\mathbf{A}[\rho_n - \rho]\mathbf{A}[\rho]\mu_n\|_{L^1} \leq \|\mathbf{A}[\rho]\|_{L^\infty} \left\|\nabla^\perp w * (\rho_n - \rho)\right\|_{L^\infty} \leq \|\rho\|_{L^2} \|\rho_n - \rho\|_{L^2} \left\|\nabla^\perp w\right\|_{L^{2,w}}^2$$

and for the last one

$$\left\|\mathbf{A}^2[\rho](\mu_n - \mu)\right\|_{L^1} \leq \|\mu_n - \mu\|_{L^1} \left\|\mathbf{A}^2[\rho]\right\|_{L^\infty} \leq \|\mu_n - \mu\|_{L^1} \left\|\nabla^\perp w\right\|_{L^{2,w}}^2 \|\rho\|_{L^2}$$

For the cross term of (20.3), $p^{\mathbf{A}}.\mathbf{A}\left[\rho_n\right]\mu_n$ we observe that

$$\|\mathbf{A}[\rho - \rho_n]\|_{L^2} \leq \|\rho - \rho_n\|_{L^1} \left\|\nabla^\perp w\right\|_{L^{2,w}}$$

so $\mathbf{A}[\rho_n]$ converges strongly in L^2. We have $||\left(p^{\mathbf{A}}\right)^2 \mu_n||_{L^1} \leq C$ so $p^{\mathbf{A}}\mu_n$ converges weakly in L^2 and by weak-strong convergence we deduce that the cross term converges. We conclude using Fatou's lemma for V and the kinetic term

$$\begin{aligned}\liminf_{n\to\infty} \int_{\mathbb{R}^4} |p + \mathbf{A}_e(x)|^2 \mu_n(x,p)\mathrm{d}x\mathrm{d}p &\geq \int_{\mathbb{R}^4} |p + \mathbf{A}_e(x)|^2 \mu(x,p)\mathrm{d}x\mathrm{d}p \\ \liminf_{n\to\infty} \int_{\mathbb{R}^2} V(x)\rho_n(x)\mathrm{d}x &\geq \int_{\mathbb{R}^2} V(x)\rho(x)\mathrm{d}x\end{aligned}$$

□

LEMMA 20.2 (**Measures $m_{\Psi_N}^{(k)}$ are tight**).
Taking any normalized sequence $\{\Psi_N\}$ of N-particles wave functions satisfying: $\left\langle \Psi_N, H_N^R \Psi_N \right\rangle = O(N)$. The associated Husimi function $m_{\Psi_N}^{(k)}$ is tight for all k:

$$\lim_{K\to\infty} \limsup_{N\to\infty} \int_{B^c(0,K)} m_{\Psi_N}^{(k)}(x_1,...,x_k,p_1,...,p_k)\mathrm{d}x_1...\mathrm{d}x_k\mathrm{d}p_1...\mathrm{d}p_k = 0 \quad (20.5)$$

PROOF. We have:

$$e_{\mathrm{TF}} + o_N(1) \geq C' \inf_{|x|^2+|p|^2 \geq K} \left(p^2 + V_+(x)\right) \left(1 - \int_{B(0,K)} m^{(1)}_{\Psi_N}(x_1, p_1)\mathrm{d}x_1\mathrm{d}p_1\right) \tag{20.6}$$

which gives that $m^{(1)}_{\Psi_N}$ is tight.
For $k = 2$ we call $B = B(0, K) \subset \mathbb{R}^4$ and observe that:

$$\mathbb{R}^4 \times \mathbb{R}^4 = (B \times B) \cup (B \times B^c) \cup (B^c \times B) \cup (B^c \times B^c) \tag{20.7}$$

We can deduce that:

$$m^{(2)}_{\Psi_N}(B \times B) \geq 1 - 2 \int_{B^c(0,K)} m^{(1)}_{\Psi_N} \tag{20.8}$$

because $m^{(2)}_{\Psi_N}$ is symmetric. The tightness of $m^{(1)}_{\Psi_N}$ allows us to conclude that $m^{(2)}_{f,\Psi_N}$ is tight to. The tightness of all other marginals follows. □

LEMMA 20.3 (**Existence of minimizers**).
There exists a minimizer for the problem

$$e^{\mathrm{TF}} = \inf \left\{ \mathcal{E}_V[\mu] \mid 0 \leq \mu \leq (2\pi)^{-2} \mid \int_{\mathbb{R}^4} \mu = 1 \right\}$$

PROOF. We consider a minimizing sequence $(\mu_n)_n$ converging to a candidate minimizer μ_∞. By Lemma (20.1) we have

$$e^{\mathrm{TF}} = \inf_{n\to\infty} \mathcal{E}_V\left[\mu_n\right] \geq \mathcal{E}_V\left[\mu_\infty\right].$$

We also found during the previous proof that $(\mu_n)_n$ must be tight, hence

$$\int_{\mathbb{R}^4} \mu_\infty = 1.$$

□

LEMMA 20.4 (**Convergence to $\mathcal{E}_V$ when $R \to 0$**).
For any measure $\mu \leq (2\pi)^{-2}$ such that $\int_{\mathbb{R}^4} \mu \leq C$ we have that

$$\left|\mathcal{E}^R_V[\mu] - \mathcal{E}^0_V[\mu]\right| \leq CR\left(\mathcal{E}^0_V[\mu] + \mathcal{E}^R_V[\mu]\right)$$

where

$$\mathcal{E}^R_V[\mu] = \int_{\mathbb{R}^4} \left(p^{\mathbf{A}} + \beta_0 \mathbf{A}^R\left[\mu\right](x)\right)^2 + V(x)\mathrm{d}\mu$$

PROOF.

$$\begin{aligned}
\left|\mathcal{E}^R_V[\mu] - \mathcal{E}^0_V[\mu]\right| &= \left| \left\| \left(p^{\mathbf{A}} + \beta_0 \mathbf{A}^R\left[\mu\right](x)\right)\sqrt{\mu}\right\|^2_{L^2} - \left\| \left(p^{\mathbf{A}} + \beta_0 \mathbf{A}^0\left[\mu\right](x)\right)\sqrt{\mu}\right\|^2_{L^2} \right| \\
&\leq \left| \left\| \left(p^{\mathbf{A}} + \beta_0 \mathbf{A}^R\left[\mu\right](x)\right)\sqrt{\mu}\right\|_{L^2} - \left\| \left(p^{\mathbf{A}} + \beta_0 \mathbf{A}^0\left[\mu\right](x)\right)\sqrt{\mu}\right\|_{L^2} \right| \\
&\qquad \cdot \left(\mathcal{E}^R_V\left[\mu\right]^{1/2} + \mathcal{E}_V\left[\mu\right]^{1/2}\right) \\
&\leq \left\| \left(\mathbf{A}^R\left[\mu\right] - \mathbf{A}\left[\mu\right]\right)\sqrt{\mu}\right\|_{L^2} \left(\mathcal{E}^R_V\left[\mu\right]^{1/2} + \mathcal{E}_V\left[\mu\right]^{1/2}\right)
\end{aligned}$$

where we have used the triangle inequality. Moreover we have that

$$\begin{aligned}\left\|\left(\mathbf{A}^R[\mu]-\mathbf{A}[\mu]\right)^2\rho\right\|_{L^1}^{1/2} &\leq \|\rho\|_{L^1}^{1/2}\left\|\left(\nabla^\perp w_R-\nabla^\perp w_0\right)*\rho\right\|_{L^\infty}\\ &\leq \|\rho\|_{L^2}\left\|\nabla^\perp w_R-\nabla^\perp w_0\right\|_{L^{2,w}}\end{aligned}$$

by the weak Young inequality and because

$$\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{1}{|x|}\chi(u)\mathrm{d}u-\int_{B(0,R)}\frac{1}{|x|}\chi(u)\mathrm{d}u\right)\mathrm{d}x\leq\int_{B(0,2R)}\frac{1}{|x|}\mathrm{d}x=CR$$

Using that the minimizer in p

$$\mu^{\mathrm{TF}}=(2\pi)^{-2}\mathbb{1}\left(|p^{\mathbf{A}}+\mathbf{A}[\rho]|\leq\sqrt{4\pi\rho}\right)$$

is explicit we have

$$\mathcal{E}_V[\mu]\geq\mathcal{E}_{\mathrm{TF}}(\rho)=2\pi\int_{\mathbb{R}^2}\rho^2(x)\mathrm{d}x+\int_{\mathbb{R}^2}V(x)\rho(x)\mathrm{d}x$$

and the minimization problem is now formulated in terms of

$$\rho(x)=\int_{\mathbb{R}^2}\mu(x,p)\mathrm{d}p.$$

This gives

$$\mathcal{E}_V[\mu]\geq C\|\rho\|_{L^2}^2$$

and concludes the proof. □

Lemma 20.5 (**Dependence on the upper perturbed constraint**).
The infimum of $\mathcal{E}_V$ does not depend on ε nor γ at first order

$$\inf\left\{\mathcal{E}_V[\mu]\mid 0\leq\mu\leq\frac{1+\varepsilon}{(2\pi)^2},\ \int\mu=1-\gamma\right\}\geq(1-2\varepsilon)(1-\gamma)\,e_{\mathrm{TF}}$$

Proof. We calculate the infimum with the Bathtub principle [**92**, Theorem 1.14]. This infimum is achieved for

$$\mu=\mu_\varepsilon^{\mathrm{TF}}=(2\pi)^{-2}(1+\varepsilon)\mathbb{1}\left(|p^{\mathbf{A}}+\beta_0\mathbf{A}[\rho]|^2\leq s(x)\right)$$

where $s(x)=\frac{4\pi\rho}{1+\varepsilon}$ because

$$\int_{\mathbb{R}^2}\mu^{\mathrm{TF}}(x,p)\mathrm{d}p=\rho(x)=\int_{\mathbb{R}^2}(1+\varepsilon)\mathbb{1}\left(|p^{\mathbf{A}}+\beta_0\mathbf{A}[\rho]|\leq s(x)\right)\mathrm{d}p=\pi(1+\varepsilon)s(x)$$

So, evaluating the energy

$$\mathcal{E}_V[\mu_\varepsilon^{\mathrm{TF}}]=\mathcal{E}_{\mathrm{TF}}^\varepsilon[\rho]=2\pi\int_{\mathbb{R}^2}\frac{\rho^2(x)}{(1+\varepsilon)^2}\mathrm{d}x+\int_{\mathbb{R}^2}\frac{V(x)\rho(x)}{(1+\varepsilon)}\mathrm{d}x \tag{20.9}$$

but we can see that

$$\mathcal{E}_{\mathrm{TF}}^\varepsilon[\rho]\geq\mathcal{E}_{\mathrm{TF}}[\rho](1-2\varepsilon) \tag{20.10}$$

where $\mathcal{E}_{\mathrm{TF}}[\rho]$ is given in (16.21). □

2. Bounds for $\mathbf{A}^R$

LEMMA 20.6 (**Bounds linked to $\mathbf{A}^R$**).
All second-order directional derivative of the function

$$w_R(u_1, u_2) : \mathbb{R}^2 \to \mathbb{R}^2$$

are bounded in absolute value by the radial derivative:

$$\left|\partial_{u_i}\partial_{u_j} w_R(\vec{u})\right| \leq C \left|\partial_u^2 w_R(u)\right| \tag{20.11}$$

for any $(i,j) \in \{1,2\}^2$. We also have the estimates

$$\left\|\Delta\nabla^\perp w_R\right\|_{L^\infty} \leq \frac{C}{R^3} \tag{20.12}$$

$$\left\|\nabla^\perp w_R\right\|_{L^\infty} \leq \frac{C}{R} \tag{20.13}$$

$$\left\|\Delta\nabla^\perp w_R\right\|_{2} \leq \frac{C}{R^2} \tag{20.14}$$

$$\left\|\nabla\left(\nabla^\perp w_R(u)\right)^i\right\|_{L^\infty} \leq \frac{C}{R^2} \tag{20.15}$$

$$\left\|\nabla\left(\nabla^\perp w_R(u).\nabla^\perp w_R(v)\right)\right\|_{L^\infty} \leq \frac{C}{R^3} \tag{20.16}$$

PROOF. Recall that

$$w_R(x) = \left(\log|\,.\,| * \chi_R\right)(x)$$

with χ_R defined as in (16.8). We call u_1 and u_2 the two components of the vector $\vec{u}$ and u its norm. Using Newton's theorem [**92**, Theorem 9.7] we write

$$w_R(u) = \int_{\mathbb{R}^2} \log|\vec{u} - \vec{v}|\chi_R(v)\mathrm{d}v = 2\pi\log(u)\int_0^u \chi_R(r)r\mathrm{d}r + 2\pi\int_u^{+\infty} r\log(r)\chi_R(r)\mathrm{d}r$$

hence

$$\begin{aligned}
\partial_u w_R(u) &= \frac{1}{u}\int_{B(0,u)} \chi_R(v)\mathrm{d}v \\
\partial_u^2 w_R(u) &= -\frac{1}{u^2}\int_{B(0,u)} \chi_R(v)\mathrm{d}v + 2\pi\chi_R(u) \qquad (20.17)\\
\partial_u^3 w_R(u) &= \frac{1}{u^3}\int_{B(0,u)} \chi_R(v)\mathrm{d}v + C\frac{\chi_R(u)}{u} + C\phi_R(u) \qquad (20.18)
\end{aligned}$$

where $\phi_R(u) = \partial_u\chi_R(u)$ is bounded with compact support. We observe that regardless of whether u is smaller or greater than $2R$ we have

$$|\partial_u w_R(u)| \leq \frac{C}{R} \text{ and } \left|\partial_u^2 w_R(u)\right| \leq \frac{C}{R^2} \tag{20.19}$$

and get (20.13). Moreover

$$\nabla^\perp w_R(u_1, u_2) = \nabla^\perp w_R(u) = \frac{\partial_u w_R(u)}{u}\vec{u}^\perp \tag{20.20}$$

We compute the two derivatives of the second component of (20.20)

$$\left|\partial_{u_1}(\frac{u_1}{u}\partial_u w_R)\right| = \left|\frac{\partial_u w_R}{u} + \frac{u_1^2}{u^2}\partial_u^2 w_R - \frac{u_1^2}{u^3}\partial_u w_R\right| \leq \frac{C}{u^2}\int_{B(0,u)} \chi_R(v)\mathrm{d}v + C\chi_R(u) \quad (20.21)$$

$$\left|\partial_{u_2}(\frac{u_1}{u}\partial_u w_R)\right| = \left|-\frac{u_1u_2}{u^3}\partial_u w_R(u) + \frac{u_1u_2}{u^2}\partial_u^2 w_R(u)\right| \leq \frac{C}{u^2}\int_{B(0,u)} \chi_R(v)\mathrm{d}v + C\chi_R(u) \quad (20.22)$$

we do the same with the first component of $\nabla^\perp w_R$ and get (20.11). If we differentiate once again we get

$$\left|\partial_{u_1}^2(\frac{u_1}{u}\partial_u w_R)\right| \leq \frac{C}{u^3}\int_{B(0,u)} \chi_R(v)\mathrm{d}v + C\frac{\chi_R(u)}{u} + C\phi_R(u)$$

which is also the case for the other component and derivative. We deduce

$$\left\|\Delta\nabla^\perp w_R\right\|_{L^\infty} \leq \frac{C}{R^3}$$

We can also compute

$$\left\|\Delta\nabla^\perp w_R\right\|_{L^2}^2 \leq \frac{C}{R^4}$$

which gives (20.14). To get (20.15) we combine (20.19) and (20.22). For the third inequality (20.16) we expand a little bit our first expression

$$\nabla^\perp w_R(u).\nabla^\perp w_R(v) = (u_1v_1 + u_2v_2)\frac{\partial_u w_R(u)\partial_v w_R(v)}{uv}$$

so the first component of the gradient is

$$\partial_{u_1}\nabla^\perp w_R(u).\nabla^\perp w_R(v) = v_1\frac{\partial_u w_R(u)\partial_v w_R(v)}{uv} + \frac{u_1}{u}\partial_u\left[\frac{\partial_u w_R(u)\partial_v w_R(v)}{uv}\right](u_1v_1 + u_2v_2)$$

Now

$$\left\|\nabla\left(\nabla^\perp w_R(u).\nabla^\perp w_R(v)\right)\right\|_{L^\infty}^2 = \sum_{x=u_1,u_2,v_1,v_2}\left(\partial_x\nabla^\perp w_R(u).\nabla^\perp w_R(v)\right)^2$$

The rest of the proof consists of the computation of the above term using basic inequalities. □

3. Computations for squeezed coherent states

We give for completeness three proofs that we skipped in the main text.

Proof of Lemma 16.4.
For any $u \in L^2(\mathbb{R}^2)$,

$$\begin{aligned}\langle F_{x,p}, u\rangle &= \int_{\mathbb{R}^2} F_{\hbar_x}(y-x)u(y)e^{-\mathrm{i}\frac{p.y}{\hbar}}\mathrm{d}y \\ &= 2\pi\hbar\mathcal{F}_\hbar\left[F_{\hbar_x}(\cdot - x)u(\cdot)\right](p) = 2\pi\hbar\left(\mathcal{F}_\hbar\left[F_{\hbar_x}(\cdot - x)\right] * \mathcal{F}_\hbar\left[u(\cdot)\right]\right)(p) \quad (20.23) \\ &= \int_{\mathbb{R}^2} G_{\hbar_p}(k-p)\mathcal{F}_\hbar[u](k)e^{-\mathrm{i}\frac{k.x}{\hbar}}\mathrm{d}k = 2\pi\hbar\mathcal{F}_\hbar\left[G_{\hbar_p}(\cdot - p)\mathcal{F}_\hbar[u]\right](x). \quad (20.24)\end{aligned}$$

It follows that

$$\begin{aligned}\frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\langle\psi,P_{x,p}\psi\rangle\ \mathrm{d}x\mathrm{d}p &= \frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|\langle F_{x,p},u\rangle|^2\mathrm{d}x\mathrm{d}p\\ &= \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\left|\mathcal{F}_\hbar\left[F_{\hbar_x}(\cdot-x)u(\cdot)\right](p)\right|^2\mathrm{d}p\right)\mathrm{d}x\\ &= \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\left|F_{\hbar_x}(y-x)u(y)\right|^2\mathrm{d}y\right)\mathrm{d}x\\ &= ||F_{\hbar_x}||^2_{L^2(\mathbb{R}^2)}||u||^2_{L^2(\mathbb{R}^2)} = ||u||^2_{L^2(\mathbb{R}^2)}.\end{aligned}$$

□

Fourier transform of F.

We need to calculate the Fourier Transform of the Gaussian

$$G_{\hbar_p}(p) = \frac{1}{2\pi\hbar}\frac{\sqrt{\hbar_x}^2}{\sqrt{\hbar_x}}\frac{1}{\sqrt{\pi}}\left(\int_{\mathbb{R}}e^{-\frac{u^2}{2}-\frac{\mathrm{i}pu}{\sqrt{\hbar_p}}}\mathrm{d}u\right)^2 = \frac{1}{2\pi\sqrt{\hbar_p}}\frac{1}{\sqrt{\pi}}G_1^2(p)$$

with

$$G_1(p) = \int_{\mathbb{R}}e^{-\frac{u^2}{2}-\frac{\mathrm{i}pu}{\sqrt{\hbar_p}}}\mathrm{d}u$$

but we also have

$$\frac{\mathrm{d}G_1(p)}{\mathrm{d}p} = -\frac{p}{\hbar_p}G_1(p)$$

which gives the result. □

Proof of Lemma 18.2. We use (20.24) and write for every fixed $y\in\mathbb{R}^{2(N-k)}$

$$\begin{aligned}&\left\langle F_{x_1,p_1}(\cdot)\otimes...\otimes F_{x_k,p_k}(\cdot),\Psi_N(\cdot,z)\right\rangle_{L^2(\mathbb{R}^{2k})}\\ =&(2\pi\hbar)^{2k}\mathcal{F}_\hbar\left[F_{x_1,0}(\cdot)\otimes...\otimes F_{x_k,0}(\cdot)\Psi_N(\cdot,z)\right](p_1,...,p_k)\\ =&(2\pi\hbar)^{2k}\mathcal{F}_\hbar\left[G_{0,p_1}(\cdot)\otimes...\otimes G_{0,p_k}(\cdot)\mathcal{F}_\hbar\left[\Psi_N\right](\cdot,z)\right](x_1,...,x_k).\end{aligned}$$

Next we sum over the p_j's using (16.32):

$$\begin{aligned}&\frac{1}{(2\pi)^{2k}}\int_{\mathbb{R}^{2k}}m^{(k)}_{\Psi_N}(x_1,p_1,...,x_k,p_k)\mathrm{d}p_1...\mathrm{d}p_k\\ =&\frac{k!}{(2\pi)^{2k}}\binom{N}{k}\int_{\mathbb{R}^{2k}}\mathrm{d}p_1...\mathrm{d}p_k\int_{\mathbb{R}^{2(N-k}}\left|\left\langle F_{x_1,p_1}(\cdot)\otimes...\otimes F_{x_k,p_k}(\cdot),\Psi_N(\cdot,z)\right\rangle\right|^2\mathrm{d}z\\ =&k!\binom{N}{k}\int_{\mathbb{R}^{2k}}\mathrm{d}p_1...\mathrm{d}p_k\int_{\mathbb{R}^{2(N-k)}}\left|\mathcal{F}_\hbar\left[F_{x_1,0}(\cdot)\otimes...\otimes F_{x_k,0}(\cdot)\Psi_N(\cdot,z)\right](p_1,...,p_k)\right|^2\mathrm{d}z\\ =&k!\binom{N}{k}\int_{\mathbb{R}^{2k}}\left|F_{\hbar_x}(y_1-x_1)...F_{\hbar_x}(y_k-x_k)\Psi_N(y)\right|^2\mathrm{d}y_1...\mathrm{d}y_N\\ =&k!\hbar^{2k}\rho^{(k)}_{\Psi_N}*\left(|F_{\hbar_x}|^2\right)^{\otimes k}(x_1,...,x_k).\end{aligned}$$

which gives (18.5). The proof of (18.6) is similar. □

Part 5

Improvement of the almost-fermionic theorem assumptions

CHAPTER 21

Almost-fermionic limit with a Lieb-Thirring inequality for extended anyons

We present here the proof of the result mentioned in Remark 16.3. The assumption $R = N^{-\eta}$ with $0 \leq \eta \leq 1/4$ of Theorem 16.1 is unsatisfying in view of the description of the quasi-particles of the FQHE (see Remark 16.2). Is this context particles should not overlap. We would thus like to relax the assumption on η until $\eta > 1/2$. Here is presented a method allowing any $\eta < 1/3$ assuming the valididy of a Lieb-Thirring inequality for extended anyons.

1. Lieb-Thirring inequality for extended anyons

In this Chapter, we shall assume a Lieb-Thirring inequality for extended anyons

THEOREM 21.1. *For any $\Psi_N \in L^2_{\text{asym}}\left(\mathbb{R}^{2N}\right)$ and $\alpha \in]0,2[\setminus\{1\}$ there exists an uniformly bounded below constant $C_{\alpha,R,N}$ such that*

$$\left\langle \Psi_N, \sum_{j=1}^{N} \left(-\mathrm{i}\nabla_j + \alpha \mathbf{A}^R(\mathbf{x}_j)\right)^2 \Psi_N \right\rangle \geq C_{\alpha,R,N} \int_{\mathbb{R}^2} \rho_{\Psi_N}(\mathbf{x})^2 \mathrm{d}x \tag{21.1}$$

Recall that

$$\mathbf{A}^R(\mathbf{x}_j) = \sum_{k=1}^{N} \nabla^{\perp} w_R(\mathbf{x}_j - \mathbf{x}_k) \tag{21.2}$$

is the anyonic vector potential defined in (13.6) and that

$$\rho_{\Psi_N}(\mathbf{x}) = N \int_{\mathbb{R}^{2(N-1)}} \left|\Psi_N\left(\mathbf{x}, \mathbf{x}_2, ..., \mathbf{x}_N\right)\right|^2 \mathrm{d}\mathbf{x}_2 ... \mathrm{d}\mathbf{x}_N \tag{21.3}$$

is the one-body density.

2. Improved results

THEOREM 21.2 (**Improved convergence of the energy and associated states**). *We consider N extended anyons of radius $R = N^{-\eta}$ and statistical parameter $\alpha = \beta/N$ in an external potential V. Under Assumption* (16.1) *and setting*

$$0 < \eta < \frac{1}{3}$$

we have, in the parameter regime (16.5)

$$\boxed{\lim_{N\to\infty} \frac{E^R(N)}{N} = e_{\text{TF}}}$$

where the many-anyon ground state energy $E^R(N)$ and the Thomas-Fermi energy are defined in (16.12) *and* (16.22)*, respectively.*
Moreover, let $\{\Psi_N\} \subset L^2_{asym}(\mathbb{R}^{2N}) = \bigwedge^N(\mathbb{R}^2)$ be any L^2-normalized sequence such that

$$\langle \Psi_N, H_N \Psi_N \rangle = E^R(N) + o(N)$$

in the limit $N \to \infty$ with (16.5)*. For any choice of $\hbar_x, \hbar_p \to 0$, related by* (16.26)*, and under Assumption 16.1, the Husimi functions of Ψ_N converge to those of the Vlasov minimizer*

$$m^{(k)}_{\Psi_N}(x_1, p_1, \dots, x_k, p_k) \to \prod_{j=1}^{k} m_{\rho^{\mathrm{TF}}}(p_j, x_j)$$

weakly as measure for all $k \geq 1$, i.e.

$$\int_{\mathbb{R}^{4k}} m^{(k)}_{\Psi_N} \phi_k \to \int_{\mathbb{R}^{4k}} (m_{\rho^{\mathrm{TF}}})^{\otimes k} \phi_k$$

for any bounded continous function $\phi_k \in C_b\left(\mathbb{R}^{4k}\right)$.

Consequently we also have the convergence of the k-particles marginals in position and momentum

$$\binom{N}{k}^{-1} \rho^{(k)}_{\Psi_N}(x_1, \dots, x_k) \to \prod_{j=1}^{k} \rho^{\mathrm{TF}}(x_j)$$

$$\binom{N}{k}^{-1} t^{(k)}_{\Psi_N}(p_1, \dots, p_k) \to \prod_{j=1}^{k} t^{\mathrm{TF}}(x_j)$$

weakly as measures. For the Wigner function $\mathcal{W}^{(k)}_{\Psi_N}$ defined in (11.58)*, we have the same limit*

$$\int_{\mathbb{R}^{4k}} \mathcal{W}^{(k)}_{\Psi_N} \phi \to \int_{\mathbb{R}^{4k}} \left(m_{\rho^{\mathrm{TF}}}\right)^{\otimes k} \phi \tag{21.4}$$

for every test function ϕ such that

$$\partial^{\alpha_1}_{x_1} ... \partial^{\alpha_k}_{x_k} \partial^{\beta_1}_{p_1} ... \partial^{\beta_k}_{p_k} \phi \in L^\infty\left(\mathbb{R}^{4k}\right), \tag{21.5}$$

$$\max\left(\alpha_j, \beta_j\right) \leq 1. \tag{21.6}$$

We derive new estimates for (21.1) in order to prove 21.2.

3. New estimates and proof of the improved convergence

We will use (21.1) to improve the a priori estimates 18.1 and the control of the error in Lemma 18.4. To this end we first establish the following lemma

Lemma 21.3 (**Bound on two-body interactions**).
For any $\Psi_N \in L^2_{\mathrm{asym}}\left(\mathbb{R}^{2N}\right)$ such that

$$\left\langle \Psi_N, H^R_N \Psi_N \right\rangle \leq CN \tag{21.7}$$

with $R = N^{-\eta}$, $\eta < 1/2$ and any potential $w \in L^2\left(\mathbb{R}^2\right)$ we have that

$$\int_{\mathbb{R}^4} w\left(\mathbf{x} - \mathbf{y}\right) \rho^{(2)}_{\Psi_N}\left(\mathbf{x}, \mathbf{y}\right) \leq \|w\|_{L^2}\left(N^2 + o\left(N^2\right)\right) \tag{21.8}$$

where $\rho^{(2)}_{\Psi_N}$ *is the second reduced density matrix defined in* (16.35).

We will apply this to $w = \nabla^\perp w_R$ obtaining the estimate

$$\int_{\mathbb{R}^4} \nabla^\perp w_R(\mathbf{x}-\mathbf{y})\, \rho^{(2)}_{\Psi_N}(\mathbf{x},\mathbf{y}) \leq N^2 \log N\,(1+o_N(1)) \tag{21.9}$$

instead of

$$\int_{\mathbb{R}^4} \nabla^\perp w_R(\mathbf{x}-\mathbf{y})\, \rho^{(2)}_{\Psi_N}(\mathbf{x},\mathbf{y}) \leq \frac{N^2}{R} \tag{21.10}$$

We will use (21.9) to improve several other bounds in our proof.

Proof. We introduce the kinetic operator

$$K_j := -\mathrm{i}\hbar\nabla_j + \alpha \mathbf{A}^R(\mathbf{x}_j). \tag{21.11}$$

Note that (21.7) implies that

$$\left\langle \Psi_N, \sum_{j=1}^N K_j^2 \Psi_N \right\rangle \leq CN + o(N). \tag{21.12}$$

Using the method of [**147**, Proof of Proposition 3.5] we define for any $\varepsilon > 0$ (to be fixed later) the N-body Hamiltonian

$$\widetilde{H}_N := \varepsilon \sum_{j=1}^N K_j^2 - \frac{1}{N} \sum_{1\leq k<l\leq N} w(\mathbf{x}_k - \mathbf{x}_l) \tag{21.13}$$

Following Lévy-Leblond [**79**] we use that

$$\left\langle \Psi_N, \sum_{j=1}^N K_j^2 \Psi_N \right\rangle = \frac{N}{N-1} \left\langle \Psi_N, \sum_{j=2}^N K_j^2 \Psi_N \right\rangle \tag{21.14}$$

$$\left\langle \Psi_N, \frac{1}{N} \sum_{1\leq k<l\leq N} w(\mathbf{x}_k - \mathbf{x}_l)\, \Psi_N \right\rangle = \left\langle \Psi_N, \frac{1}{2} \sum_{l=2}^N w(\mathbf{x}_1 - \mathbf{x}_l)\, \Psi_N \right\rangle \tag{21.15}$$

to replace the two terms of (21.13) and get

$$\left\langle \Psi_N, \widetilde{H}_N \Psi_N \right\rangle = \int_{\mathbb{R}^2} \left\langle \Psi_N, \widetilde{H}_{N-1} \Psi_N \right\rangle (\mathbf{x}_1)\, \mathrm{d}\mathbf{x}_1 \tag{21.16}$$

with the $(N-1)$-body Hamiltonian

$$\widetilde{H}_{N-1}(\mathbf{x}_1) = \frac{N}{N-1} \varepsilon \sum_{j=2}^N K_j^2 - \frac{1}{2} \sum_{l=2}^N w(\mathbf{x}_1 - \mathbf{x}_l) \tag{21.17}$$

The operator K_j^2 still contains the variable $\mathbf{x}_1$. We then isolate it

$$K_j^2 = \left(-\mathrm{i}\hbar\nabla_j + \alpha \sum_{k=2}^N \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) + \alpha \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_1) \right)^2 \tag{21.18}$$

$$\geq \frac{1}{2} \left(-\mathrm{i}\hbar\nabla_j + \alpha \sum_{k=2}^N \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) \right)^2 - \frac{\alpha^2}{8} \left| \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_1) \right|^2 \tag{21.19}$$

For $\Psi_{N-1} \in L^2_{\mathrm{asym}}\left(\mathbb{R}^{2(N-1)}\right)$ we use (21.1) to obtain that

$$\left\langle \Psi_{N-1}, \sum_{j=2}^{N} K_j^2 \Psi_{N-1} \right\rangle \geq C\hbar^2 \int_{\mathbb{R}^2} \rho^2_{\Psi_{N-1}}(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{21.20}$$

$$- \frac{C'}{N} \int_{\mathbb{R}^4} \rho_{\Psi_{N-1}}(\mathbf{x}_2) \left|\nabla^\perp w_R(\mathbf{x}_2 - \mathbf{x}_1)\right|^2 \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2 \tag{21.21}$$

$$\geq \left(\frac{C}{N} - \frac{C'}{2N\delta}\right) \int_{\mathbb{R}^2} \rho^2_{\Psi_{N-1}}(\mathbf{x})\,\mathrm{d}\mathbf{x} - \frac{\delta C'}{2N} \left\|\nabla^\perp w_R\right\|_4^4 \tag{21.22}$$

$$\geq \frac{C}{2N} \int_{\mathbb{R}^2} \rho^2_{\Psi_{N-1}}(\mathbf{x})\,\mathrm{d}\mathbf{x} - \frac{C'^2}{2CNR^2} \tag{21.23}$$

where we chose $\delta = C'/C$ and used that $\left\|\nabla^\perp w_R\right\|_4^4 \leq \frac{C}{R^2}$. We then obtain that

$$\left\langle \Psi_{N-1}, \widetilde{H}_{N-1}\Psi_{N-1} \right\rangle(\mathbf{x}_1) \geq \frac{C\varepsilon}{N} \int_{\mathbb{R}^2} \rho^2_{\Psi_{N-1}}(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{21.24}$$

$$- \frac{1}{2} \int_{\mathbb{R}^2} \rho_{\Psi_{N-1}}(\mathbf{x}_2)\, w(\mathbf{x}_1 - \mathbf{x}_2) - \frac{C\varepsilon}{NR^2} \tag{21.25}$$

$$\geq \left(\frac{C\varepsilon}{N} - \frac{1}{4\eta}\right) \int_{\mathbb{R}^2} \rho^2_{\Psi_{N-1}}(\mathbf{x})\,\mathrm{d}\mathbf{x} - \frac{\eta}{4} \|w\|_{L^2}^2 - \frac{C\varepsilon}{NR^2} \tag{21.26}$$

where we used Young's inequality for any $\eta > 0$ in the last term of (21.25)

$$ab \leq \frac{a^2}{2\eta} + \frac{\eta b^2}{2} \tag{21.27}$$

We pick

$$\varepsilon = \frac{N}{4C\eta}$$

and use (21.16) to get

$$\frac{N}{4C\eta} \left\langle \Psi_N, \sum_{j=1}^{N} K_j^2 \Psi_N \right\rangle - \frac{1}{N} \int_{\mathbb{R}^4} w(\mathbf{x}_1 - \mathbf{x}_2)\, \rho^{(2)}_{\Psi_N}(\mathbf{x}_1, \mathbf{x}_2)\,\mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2 \geq -\frac{\eta}{4} \|w\|_{L^2}^2 - \frac{C}{\eta R^2} \tag{21.28}$$

Inserting (21.7) implies that

$$\frac{1}{N} \int_{\mathbb{R}^4} w(\mathbf{x}_1 - \mathbf{x}_2)\, \rho^{(2)}_{\Psi_N}(\mathbf{x}_1, \mathbf{x}_2)\,\mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2 \leq \frac{CN^2}{\eta} + C\eta \|w\|_{L^2}^2 + \frac{C}{\eta R^2} \tag{21.29}$$

where choosing $\eta = N/\|w\|_{L^2}$ gives the result. □

We can now upgrate our a priori estimates from Lemma 18.1 in Chapter 18.

LEMMA 21.4 (**Improved a priori estimates**).
For any $\Psi_N \in L^2_{\mathrm{asym}}\left(\mathbb{R}^{2N}\right)$ such that

$$\left\langle \Psi_N, H_N^R \Psi_N \right\rangle \leq CN \tag{21.30}$$

we have

$$\left\langle \Psi_N, \left(\sum_{j=1}^{N} -\Delta_j\right) \Psi_N \right\rangle \leq \frac{CN^2}{R} \log(N)\,(1 + o_N(1)) \tag{21.31}$$

and

$$\int_{\mathbb{R}^2} \left(\frac{\rho^{(1)}_{\Psi_N}}{N} \right)^2 \leq C \log N \left(1 + o_N(1)\right). \tag{21.32}$$

Moreover

$$\int_{\mathbb{R}^2} V(\mathbf{x}) \rho^{(1)}_{\Psi_N}(\mathbf{x}) \mathrm{d}\mathbf{x} \leq CN \tag{21.33}$$

The new bounds are better than the former. We have indeed

$$\frac{CN^2}{R} \log(N) \ \textbf{instead of} \ \frac{CN^2}{R^2}$$

for (21.31) and

$$C \log N \ \textbf{instead of} \ \frac{C}{R^2}$$

for (21.32).

Proof. We expand the Hamiltonian and use the Cauchy-Schwarz inequality for operators as in (18.4) to obtain

$$\mathrm{Tr}\left[\left(\left(p^{\mathbf{A}}\right)^2 + V \right) \gamma_N^{(1)} \right] \leq CN + 7\alpha^2 \left\langle \Psi_N, \sum_{j=1}^N |\mathbf{A}_j^R|^2 \Psi_N \right\rangle \tag{21.34}$$

where $\gamma_N^{(1)}$ is the 1-particle reduced density matrix

$$N \mathrm{Tr}_{2 \to N} \left[|\Psi_N\rangle \langle \Psi_N| \right].$$

We estimate the second term of the above using [**59**, Lemma 2.1]:

$$\left\langle \Psi_N, \sum_{j=1}^N \left|\mathbf{A}_j^R\right|^2 \Psi_N \right\rangle \leq \int_{\mathbb{R}^{3\times 2}} \nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2) \cdot \nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_3) \, \rho^{(3)}_{\Psi_N} \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2 \mathrm{d}\mathbf{x}_3 \tag{21.35}$$

$$+ \int_{\mathbb{R}^4} \left| \nabla^\perp w_R(\mathbf{x}_1 - \mathbf{x}_2) \right|^2 \rho^{(2)}_{\Psi_N} \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2 \tag{21.36}$$

$$\leq \frac{N}{R} \int_{\mathbb{R}^{2\times 2}} \nabla^\perp w_R(\mathbf{x} - \mathbf{y}) \, \rho^{(2)}_{\Psi_N}(\mathbf{x}, \mathbf{y}) \, \mathrm{d}\mathbf{x} \mathrm{d}\mathbf{y} + \frac{CN^2}{R^2} \tag{21.37}$$

where we used (20.13) to estimate the last term os the above inequality. We now use (21.8) combined with the tightness of $\rho^{(1)}_{\Psi_N}$ (same argument as in (20.2), but using (21.33))

$$\left\langle \Psi_N, \sum_{j=1}^N |\mathbf{A}_j^R|^2 \Psi_N \right\rangle \leq \frac{N}{R} \int_{B(0,K) \cup B^c(0,K) \times \mathbb{R}^2} \nabla^\perp w_R(\mathbf{x} - \mathbf{y}) \, \rho^{(2)}_{\Psi_N}(\mathbf{x}, \mathbf{y}) \, \mathrm{d}\mathbf{x} \mathrm{d}\mathbf{y} + \frac{CN^2}{R^2}$$

$$\leq \frac{N^3}{R} \left\| \mathbb{1}_{B(0,K)} \nabla^\perp w_R \right\|_{L^2} + \frac{N^2}{R^2} \int_{B^c(0,K)} \rho^{(1)}_{\Psi_N}(\mathbf{x}) \, \mathrm{d}\mathbf{x} + \frac{CN^2}{R^2} \tag{21.38}$$

$$\leq C \frac{N^3}{R} \log(N) \left(1 + o_N(1)\right). \tag{21.39}$$

where we used (21.8) to obtain the first term (21.38). Wee then used (20.13) and integrated over $\mathbf{y}$ to get the second. In the last we have chosen $K = N^{\kappa>3}$ combined with the tightness of $\rho^{(1)}_{\Psi_N}$. Recalling that $\alpha = N^{-1}$ we insert in (21.34) and deduce

$$\mathrm{Tr}\left[\left(\left(p^{\mathbf{A}}\right)^2 + V\right)\gamma_N^{(1)}\right] \leq \frac{CN}{R}\log(N)\left(1 + o_N(1)\right)$$

Now we use that

$$|-\mathrm{i}\hbar\nabla + \mathbf{A}_e|^2 \geq -\frac{\hbar^2}{2}\Delta - |\mathbf{A}_e|^2$$

and that $\hbar^2 = N^{-1}$ to obtain

$$\frac{CN}{R}\log(N) \geq \frac{C}{N}\left\langle \Psi_N, \left(\sum_{j=1}^N -\Delta_j\right)\Psi_N\right\rangle - \mathrm{Tr}\left[|\mathbf{A}_e|^2\gamma_N^{(1)}\right]$$

recalling that $V \geq 0$. But since $|\mathbf{A}_e|^2 \in L^\infty(\mathbb{R}^2)$, we deduce (21.31). We obtain (21.32) by using the Lieb-Thirring inequality [**104, 105**]

$$\mathrm{Tr}\left[-\Delta\gamma_N^{(1)}\right] \geq C\int_{\mathbb{R}^2}\left(\rho^{(1)}_{\Psi_N}(\mathbf{x})\right)^2 \mathrm{d}\mathbf{x}$$

combined with (21.31).

Finally (21.33) is a straightforward consequence of the positivity of the full kinetic energy (first term in (13.1)). □

We can now improve the estimates on some remainders we had in the first proof. To this end, we recall the Hausdorff-Young inequality [**92**][Theorem 5.7] and Plancherel's Theorem.

THEOREM 21.5 (**Hausdorff-Young inequality**).
Let $1 < p < 2$ and $f \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Then with $1/p + 1/p' = 1$

$$\|\mathcal{F}_1[f]\|_{L^{p'}} \leq C_{n,p}\|f\|_{L^p} \tag{21.40}$$

LEMMA 21.6 (**Plancherel theorem**).
For any function $W \in L^2(\mathbb{R}^2)$ such that $\Delta W \in L^2(\mathbb{R}^2)$ the n-convoluted product of W with $|F_{\hbar_x}|^2$:

$$W *_n |F_{\hbar_x}|^2 = W * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 * ... * |F_{\hbar_x}|^2 \tag{21.41}$$

converges to W in $L^2(\mathbb{R}^2)$ when $\hbar_x$ tends to 0. Precisely we have the estimate:

$$\left\|W *_n |F_{\hbar_x}|^2 - W\right\|_{L^2} \leq C_n\hbar_x\|\Delta W\|_{L^2} \tag{21.42}$$

PROOF. We first check that $W *_n |F_{\hbar_x}|^2$ is in $L^2(\mathbb{R}^2)$:

$$\left\|W *_n |F_{\hbar_x}|^2\right\|_{L^2} \leq \|W\|_2\left\||F_{\hbar_x}|^2\right\|_{L^2}^n = \|W\|_{L^2} \tag{21.43}$$

by Young's inequality. We calculate the 1-Fourier tranform of $F_{\hbar_x}$ recalling that $\hbar = \sqrt{\hbar_x}\sqrt{\hbar_p}$

$$G(p) = \mathcal{F}_1[F_{\hbar_x}](p) = \frac{1}{\sqrt{\hbar_p}\sqrt{\pi}}e^{-\frac{\hbar_x|p|^2}{2}}. \tag{21.44}$$

We can now apply the Plancherel theorem:

$$\left\|W *_n |F_{\hbar_x}|^2 - W\right\|_{L^2}^2 = \left\|\mathcal{F}_1\left[W *_n |F_{\hbar_x}|^2 - W\right]\right\|_{L^2}^2 \tag{21.45}$$

$$= \int_{\mathbb{R}^2} |\mathcal{F}_1[W](\mathbf{p})|^2 \left(e^{-n|\mathbf{p}|^2\hbar_x} - 1\right)^2 \mathrm{d}\mathbf{p} \tag{21.46}$$

$$= h_x^2 \int_{\mathbb{R}^2} \left||\mathbf{p}|^2 \mathcal{F}_1[W](\mathbf{p})\right|^2 \mathrm{d}\mathbf{p} + O\left(\hbar_x^4\right) \tag{21.47}$$

and we deduce:

$$\left\|W *_n |F_{\hbar_x}|^2 - W\right\|_{L^2}^2 \leq C_n \hbar_x^2 \|\Delta W\|_2^2 \tag{21.48}$$

by applying Plancherel's Theorem again. □

The estimates we need to improve are (18.13), (18.14) and (18.15).
Recall before starting that

$$\nabla^\perp w_R(\mathbf{x}) = \nabla^\perp (\log|\,.\,| * \chi_R)(\mathbf{x}) \tag{21.49}$$

with

$$\chi_R = \frac{1}{R^2}\chi\left(\frac{\cdot}{R}\right). \tag{21.50}$$

We then have

$$\left|\mathcal{F}_1\left[\nabla^\perp w_R\right](\mathbf{p})\right| = \frac{2\pi}{|\mathbf{p}|}\mathcal{F}_1[\chi_R](\mathbf{p}) = \frac{2\pi}{|\mathbf{p}|}\mathcal{F}_1[\chi](R\mathbf{p}) \tag{21.51}$$

with $\chi \in C^\infty(\mathbb{R}^2)$.
Estimate of (18.13):

$$\begin{aligned} &4\alpha \left|\mathrm{Tr}\left[\left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2\right)\cdot(-\mathrm{i}\hbar\nabla_1)\gamma_N^{(2)}\right]\right| \\ &\leq C\alpha \left\|(-\mathrm{i}\hbar\nabla)\sqrt{\gamma_N}\right\|_{\mathfrak{S}^2}\left[\int_{\mathbb{R}^4}\left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2\right)^2 \mathrm{d}\rho_N^{(2)}\right]^{1/2} \\ &\leq C\alpha\hbar\frac{N\sqrt{\log N}}{\sqrt{R}} N \left\|\left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2\right)\right\|_{L^4} \end{aligned} \tag{21.52}$$

where we used (21.31) to bound the kinetic energy term and applied Lemma (21.3) to the term with the second reduced density matrix. We now apply the Hausdorff-Young inequality Lemma (21.5) to the L^4 norm of above

$$\left\|\left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2\right)\right\|_{L^4} \leq \left\|\mathcal{F}_1\left[\nabla^\perp w_R\right](\mathbf{p})\left(1 - e^{-2|\mathbf{p}|^2\hbar_x}\right)\right\|_{L^{4/3}} \tag{21.53}$$

$$\leq C\left(\int_{\mathbb{R}^2}\left|\frac{|\mathbf{p}|^2}{|\mathbf{p}|}\mathcal{F}_1[\chi](R\mathbf{p})\right|^{4/3}\mathrm{d}\mathbf{p}\right)^{3/4} \tag{21.54}$$

$$\leq C\frac{\hbar_x}{R^{5/2}}\left(\int_{\mathbb{R}^2}\left||\mathbf{p}|\mathcal{F}_1[\chi](\mathbf{p})\right|^{4/3}\mathrm{d}\mathbf{p}\right)^{3/4} \tag{21.55}$$

$$\leq C\frac{\hbar_x}{R^{5/2}} \tag{21.56}$$

where we used (21.51).

Combining (21.56) and (21.52) with $\hbar_x = N^{-1+2\varepsilon}$, $\alpha = 1/N$ and $\hbar = N^{-1/2}$ we finally get

$$4\alpha \left| \mathrm{Tr}\left[\left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 \right) \cdot (-\mathrm{i}\hbar\nabla_1)\gamma_N^{(2)} \right] \right| \leq N^{3\eta+2\varepsilon-1/2} \tag{21.57}$$

Estimate of (18.14). We denote

$$\mathrm{I} = \mathbf{A}_e(\mathbf{x}_j) \cdot \nabla^\perp w_R(\mathbf{x}_j - \mathbf{x}_k) - \int_{\mathbb{R}^2} \mathbf{A}_e(\mathbf{x}_j - \mathbf{u}) \cdot \left(\nabla^\perp w_R * |F_{\hbar_x}|^2 \right)(\mathbf{x}_j - \mathbf{x}_k - \mathbf{u}) |F_{\hbar_x}(u)|^2 \mathrm{d}\mathbf{u}. \tag{21.58}$$

Then

$$\begin{aligned} 4\alpha \,\mathrm{Tr}\left[\mathrm{I}\gamma_N^{(2)} \right] &\leq C\alpha N^2 \left\| \mathbf{A}_e \right\|_{L^\infty} \left\| \nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}| \right\|_{L^2} \\ &\leq C\alpha N^2 \hbar_x \left\| \Delta \nabla^\perp w_R \right\|_{L^2} \\ &\leq \frac{CN\hbar_x}{R^2} \leq CN^{2\eta+2\varepsilon} \end{aligned} \tag{21.59}$$

where we used lemma (21.3) in the first line, the Plancherel formula (21.6) and then the estimate (20.12).

Estimate of (18.15). We denote

$$\begin{aligned} \mathrm{II} = {} & W_{123}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \\ & - \int_{\mathbb{R}^2} \left(\nabla^\perp w_R * |F_{\hbar_x}|^2 \right)(\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{u}) \cdot \left(\nabla^\perp w_R * |F_{\hbar_x}|^2 \right)(\mathbf{x}_1 - \mathbf{x}_3 - \mathbf{u}) |F_{\hbar_x}(\mathbf{u})|^2 \mathrm{d}\mathbf{u}. \end{aligned} \tag{21.60}$$

Then

$$\begin{aligned} 6\alpha^2 \,\mathrm{Tr}\left[\mathrm{II}\gamma_N^{(3)} \right] &\leq \frac{CN\alpha^2}{R} \left\| \left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 \right) \rho_{\gamma_N}^{(2)} \right\|_{L^1} \\ &\leq \frac{CN^3\alpha^2}{R} \left\| \nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 \right\|_{L^2} \\ &\leq C\alpha^2 \frac{N^3\hbar_x}{R} \left\| \Delta \nabla^\perp w_R \right\|_{L^2} \\ &\leq \frac{CN\hbar_x}{R^3} \leq CN^{3\eta+2\varepsilon} \end{aligned} \tag{21.61}$$

where we used [**59**, Lemma 2.1] and (21.42) combined with (20.12) to bound the last norm. Collecting all the previous estimates we have proved the following statement

PROPOSITION 21.7 (**Improved semi-classical bounds**).
Pick some $\varepsilon > 0$ and set

$$\sqrt{\hbar_x} = N^{-1/2+\varepsilon}, \quad \sqrt{\hbar_p} = N^{-\varepsilon}. \tag{21.62}$$

Pick a sequence Ψ_N of fermionic wave-functions such that

$$\left\langle \Psi_N, H_N^R \Psi_N \right\rangle \leq CN.$$

Under Assumption 16.1 we have

$$\frac{\left\langle \Psi_N, H_N^R \Psi_N \right\rangle}{N} \geq (1 - o_N(1))\, \mathcal{E}_{\mathrm{C}}^R \left[m_{\Psi_N}^{(3)} \right] - CN^{3\eta-1+2\varepsilon} - CN^{2\eta-1} - CN^{-2\varepsilon} \tag{21.63}$$

From there on the proof follows the same route as above (from Lemma (18.4) of Part 4) with the constraint $\eta < 1/3$ instead of $\eta < 1/4$.

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