Weak regularization by degenerate Lévy noise and its applications

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Régularisation faible par un bruit de Lévy dégénéré et applications

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1 Estimations de Schauder pour des équations de Kolmogorov Stables Dégénérées

Nous présentons dans ce chapitre des estimations globales de type Schauder pour une chaîne d'équations intégro-différentielles partielles (EIDP) dirigées par un opérateur dégénéré stable de type Ornstein-Uhlenbeck qui peut être perturbé, pour le terme d'ordre un, par une composante déterministe, lorsque les coefficients appartiennent à des espaces de Hölder anisotropes convenables. Notre approche suit une méthode perturbative basée sur des développements *parametrix* progressifs et, en raison des faibles propriétés de régularisation sur les composantes dégénérées et de certaines limites d'intégrabilité liées à l'indice de stabilité, elle exploite également des résultats de dualité entre les espaces de Besov. En particulier, notre méthode s'applique également à certains cas sur-critiques. Grâce à ces estimations, nous sommes en mesure de montrer en plus le caractère bien posé de l'EIDP considérée dans un espace fonctionnel convenable. Ce travail a donné lieu à une publication [Mar20] dans le *Bulletin des Sciences Mathématiques*.

2 Estimations de Schauder pour des Opérateurs de Lévy dégénérés des Type Ornstein-Uhlenbeck

Nous établissons des estimations de Schauder globales pour des équations intégro-différentielles partielles (EIDP) dirigées par un opérateur de Lévy éventuellement dégénéré de type Ornstein-Uhlenbeck, à la fois dans le contexte elliptique et dans le contexte parabolique, en utilisant des espaces de Hölder anisotropes convenables. Les opérateurs que nous pouvons considérer s'écrivent comme la somme d'une partie linéaire d'ordre un et d'un opérateur de Lévy qui est comparable, dans un sens convenable, à un opérateur stable qui peut être tronqué. La classe d'opérateurs considérée comprend par exemple les opérateurs stables relativistes, tempérés, stratifiés ou de type Lamperti. Notre méthode ne suppose ni la symétrie de l'opérateur de Lévy, ni l'invariance des dilatations pour la partie linéaire de l'opérateur. Grâce à ces estimations, nous pouvons également obtenir le caractère bien posé de l'équation considérée dans un espace fonctionnel convenable. Dans la dernière section, nous étendons certains de ces résultats à des opérateurs plus généraux qui comportent également une composante non linéaire d'ordre un dépendante de l'espace et du temps. Ce travail a été publié dans le *Journal of Mathematical Analysis* *and Applications* (cf. [Mar21]).

3 Caractère bien posé faible pour des EDS de Kolmogorov dirigées par des processus de Lévy

Dans ce chapitre, nous étudions les effets de la propagation d'un bruit de Lévy non dégénéré à travers une chaîne d'équations différentielles déterministes dont les coefficients sont continus Hölder et vérifient une condition de Hörmander faible. En particulier, on suppose la non-dégénérescence de la dérive par rapport aux composantes qui transmettent le bruit. Pour certaines dynamiques spécifiques, nous caractérisons davantage à travers des contre-exemples convenables, les exposants de régularité quasi-optimaux qui assurent le caractère faiblement bien posé pour l'équation différentielle stochastique (EDS) associée. Comme corollaire de notre approche, nous obtenons également des estimations de type Krylov pour la densité des solutions faibles pour l'EDS considérée. Rédigé en collaboration avec Stéphane Menozzi, ce chapitre est désormais disponible en pré-publication (cf. [MM21]).

4 Sur les constantes optimales dans les estimations de Sobolev et Schauder pour des équations dégénérées de Kolmogorov

Dans ce chapitre, nous considérons un opérateur d'Ornstein-Uhlenbeck de type Kolmogorov de la forme $L = \mathrm{Tr}(BD^2) + \langle Ax, D\rangle$, où A, B sont des matrices qui vérifient une condition de Kalman équivalente à la condition d'hypoellipticité. En particulier, nous montrons le résultat suivant : les estimations de Schauder ou de Sobolev associées au problème parabolique de Cauchy correspondant restent valables, et avec la même constante, également pour le problème parabolique de Cauchy associé à une perturbation du second ordre de L, c'est-à-dire pour $L + \mathrm{Tr}(S(t)D^2)$, où $S(t)$ est une matrice $N \times N$ *définie non négative* qui dépend continûment du temps t. Notre approche est basée sur une technique de perturbation par processus de Poisson initialement introduite par Krylov et Priola dans [KP17]. Ce chapitre a été réalisé en collaboration avec Stéphane Menozzi et Enrico Priola et a donné lieu à une pré-publication (cf. [MMP21]).

Chapitre 1

Introduction

1 Le modèle considéré

Cette thèse de doctorat porte sur l'étude des phénomènes de régularisation par bruit dégénéré de type Lévy pour des chaînes d'équations différentielles éventuellement mal posées. En particulier, notre objectif principal est de déterminer, sous des hypothèses convenables sur le système dégénéré, quelle est la régularité minimale de Hölder sur les coefficients qui assure le caractère bien posé de la dynamique stochastique associée.

Plus précisément, pour un "grand" espace $\mathbb{R}^N$ et un "petit" espace $\mathbb{R}^d$ ($d \le N$) fixés, on s'intéresse aux équations différentielles de la forme suivante :

$$dX_t \;=\; F(t, X_t)dt + B\sigma(t, X_{t-})dZ_t, \quad t \ge 0 \tag{1.1}$$

où $\{Z_t\}_{t\ge0}$ est un processus de Lévy d-dimensionnel sur un espace de probabilité filtré $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ et la dérive $F\colon [0, +\infty) \times \mathbb{R}^N \to \mathbb{R}^N$ et la matrice de diffusion $\sigma\colon [0, +\infty) \times \mathbb{R}^N \to \mathbb{R}^d \otimes \mathbb{R}^d$ sont des fonctions Hölder continues en espace, uniformément dans le temps.
Ci-dessus, le processus $\{Z_t\}_{t\ge0}$ est transmis à travers la matrice de diffusion σ (grâce à une propriété d'ellipticité uniforme) sur $\mathbb{R}^d$ puis immergé dans le grand espace $\mathbb{R}^N$ à travers la matrice B dans $\mathbb{R}^N \otimes \mathbb{R}^d$ laquelle on suppose, sans perte de généralité, que $\operatorname{rank} B = d$.

Une analyse détaillée du comportement bien posé des chaînes dégénérées stochastiques de la forme (1.1) est non seulement importante pour son intérêt mathématique intrinsèque mais surtout pour les innombrables contextes scientifiques dans lesquels ce type de dynamique est utilisé comme modèle.
Pour n'en citer que quelques-unes, les *diffusions cinétiques dégénérées non locales* de la forme suivante :

$$\begin{cases} dX^1_t &= F_1(t, X^1_t, X^2_t)dt + dZ_t \\ dX^2_t &= F_2(t, X^1_t, X^2_t)dt, \end{cases} \tag{1.2}$$

correspondant à l'Equation (1.1) avec seulement deux composantes ($N = 2d$), $\sigma = 1$ et $B = (I_{d\times d}, 0_{d\times d})^t$, sont souvent utilisés en mécanique hamiltonienne pour les régimes

turbulents ou dans des modèles prenant en compte des phénomènes de diffusion anormaux tels que, par exemple, la variation de chaleur entre deux matériaux différents en contact (cf. [BBM01, EPRB99]). En fait, une dynamique de la forme (1.2) avec $F_2(t, x_1, x_2) = x_1$ peut être utilisée pour décrire la dynamique de position/vitesse d'une particule en mouvement dans un système particulier, où seule la composante de vitesse X_t^1 est perturbée aléatoirement (par exemple en raison d'effets turbulents dus à l'air dans des régimes élevés à grande vitesse) tandis que la variable de déplacement X_t^2 perçoit le bruit aléatoire uniquement par sa dépendance vis-à-vis de la première composante. Ce type de modèles cinétiques dégénérés apparaît également comme une limite de diffusion pour les équations de Boltzmann linéarisées si la fonction d'équilibre est supposée être une distribution de type Lévy ([Ale12, MMM11, Mel16, CZ18]). Voir aussi [CPKM05] pour une application à la loi d'échelle de Richardson sur la turbulence.
Une autre utilisation naturelle des chaînes dégénérées cinétiques de type Lévy apparaît en finance et, en particulier, dans la modélisation de l'évolution temporelle du prix des *options asiatiques*, un type particulier de produit dérivé "exotique" (cf. [BNS01, Bro01, JYC09, BKH10]). Il s'agit d'options dont le payoff, c'est-à-dire combien le titulaire de l'option gagne à une date d'expiration donnée, dépend de la moyenne des valeurs du sous-jacent pendant toute la durée de vie du contrat et non de la valeur du sous-jacent à la fermeture de celui-ci, comme dans la plupart des options européennes courantes.
Cependant, tous les exemples cités ci-dessus ne portent que sur le cas cinétique, lorsque l'équation (1.3) est composée de seulement deux composantes ($N = 2d$). Dans le cadre plus général ($N = nd$), des modèles du type :

$$\begin{cases} dX_t^1 &= F_1(t, X_t^1, \dots, X_t^n)dt + dZ_t \\ dX_t^2 &= F_2(t, X_t^1, \dots, X_t^n)dt; \\ dX_t^3 &= F_3(t, X_t^2, \dots, X_t^n)dt; \\ \vdots & \\ dX_t^n &= F_n(t, X_t^{n-1}, X_t^n)dt, \end{cases} \tag{1.3}$$

apparaissent par exemple en sismologie lorsque l'on considère la propagation d'une onde de choc à travers des structures de matériaux différents. De plus, la dynamique de la forme dans (1.3) est souvent utilisée pour représenter des oscillateurs élasto-plastiques inter-connectés, c'est-à-dire des systèmes de ressorts reliés entre eux, où une perturbation aléatoire n'est appliquée qu'au premier (cf. Figure 1). A ce propos, voir par exemple [BT08, BMPT09] dans un contexte diffusif.
Plus généralement, les modèles qui considèrent des bruits de type Lévy apparaissent plus polyvalents et réalistes puisqu'ils permettent la présence de sauts aléatoires dans la dynamique, contrairement au cas brownien.

Pour mettre en évidence le contexte plus général dans lequel s'inscrit notre problème spécifique, nous commençons cette thèse par une brève présentation de la théorie de la régularisation par le bruit.

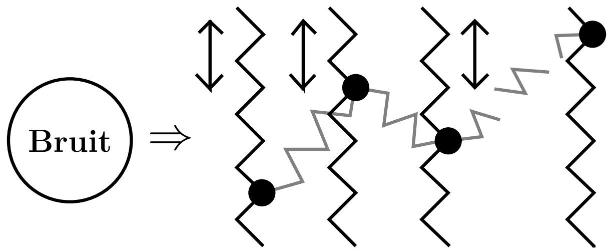

FIGURE 1.1 – Oscillateurs élasto-plastiques interconnectés de [DM10].

2 Régularisation par le bruit

Pour un point initial fixé x dans $\mathbb{R}^N$, la théorie classique de Cauchy-Lipschitz assure qu'une équation différentielle ordinaire du type :

$$\begin{cases} dX_t \;=\; F(t, X_t)dt; \\ X_0 \;=\; x, \end{cases} \tag{2.4}$$

admet une solution globale unique si la dérive F est suffisamment régulière, c'est-à-dire Lipschitz continue et croît au plus linéairement.
Par la suite, ce résultat classique a été étendu par plusieurs auteurs sous des conditions de plus en plus faibles pour la régularité de la dérive F. A cet égard, on rappelle la fameuse théorie des flots de DiPerna-Lions dans [DL89] où sont considérées des dérives seulement faiblement Lipschitz continues (i.e. dans les espaces de Sobolev) et la généralisation de ce résultat aux dérives à variation bornée (cf. [Amb04, CDL08]). Cependant, tous les travaux cités ci-dessus partagent une sorte de condition de bornitude sur les dérivées de F qui, au moins dans le cas de dérives homogènes dans le temps, peut s'écrire sous la forme $\operatorname{div} F$ in $L^\infty(\mathbb{R}^N)$.
L'affaiblissement de cette condition sur la divergence permet immédiatement l'émergence de modèles spécifiques qui réfutent le caractère bien posé de l'équation (2.4).
Un contre-exemple classique est donné par le modèle dit de *Peano* obtenu à partir de (2.4) en imposant $d = 1$ et $F(s, x) = \operatorname{sgn}(x)|x|^\beta$ pour une certaine valeur β dans $(0, 1)$. En fait, il n'est pas difficile de montrer que cette équation, avec point initial $x = 0$, a un nombre infini de solutions de la forme :

$$X_t \;=\; (1-\beta)^{\frac{1}{1-\beta}} \mathbb{1}_{[T_0,+\infty)}(t)\,(t-T_0)^{\frac{1}{1-\beta}}\,, \quad t \geq 0, \tag{2.5}$$

où $T_0 > 0$. Intuitivement, la présence de la singularité à l'origine du système, où la dérive F n'est pas Lipschitz régulière, permet de piéger les solutions en ce point pour une durée quelconque. Ce type de phénomène est souvent appelé *bifurcation* des flux.
Nous nous sommes concentrés ici uniquement sur un contre-exemple avec une dérive Hölder régulière et uniquement sur le problème de la non-unicité des solutions car ce sera le contexte que nous considérerons. Des phénomènes différents et plus variés peuvent effectivement se produire (cf. [Fla11, DL89]), incluant, par exemple, la *coalescence* des flux, c'est-à-dire la collision de solutions à partir de points initiaux différents, voire l'inexistence de solutions.

Régularisation par bruit Brownien non-dégénéré. La situation change radicalement si un bruit aléatoire est ajouté au système, c'est-à-dire si l'on considère, au lieu de l'équation déterministe (2.4), sa contrepartie stochastique donnée par

$$\begin{cases} dX_t \; = \; F(t, X_t)dt + \sigma(t, X_t)dW_t; \\ X_0 \; = \; x, \end{cases} \tag{2.6}$$

où $\{W_s\}_{s\geq 0}$ est un mouvement Brownien sur $\mathbb{R}^N$. En effet, la présence d'un bruit aléatoire "suffisamment" intense peut effectivement restaurer le caractère bien posé du système. Pour la dynamique de Peano stochastique avec bruit évanescent (cf. Equation (2.6) avec $\sigma = \epsilon$ petit et $F(t, x) = \text{sgn}(x)|x|^\beta$), ce phénomène a été mis en évidence par Delarue et Flandoli dans [DF14] (voir aussi [BB81]) : en peu de temps, les fluctuations de bruit dominent le système de sorte que la solution peut échapper à la singularité en zéro, tandis qu'en longtemps, la dérive F domine le bruit et force alors la solution à fluctuer autour d'une des deux solutions extrêmes (cf. Équation (2.5) avec $T_0 = 0$). Il est donc important de mettre en évidence comment, en peu de temps, il existe une forte compétition entre l'irrégularité de la dérive et les fluctuations moyennes du bruit.
Nous soulignons, cependant, que dans le contexte stochastique, un soin particulier doit être apporté à ce que l'on entend exactement par unicité de solution pour l'équation (2.6).
Une solution *faible* de la dynamique stochastique dans (2.6) sera une paire de processus $\{(X_t, W_t)\}_{t\geq 0}$ où $\{W_t\}_{t\geq 0}$ est un un mouvement Brownien sur une base stochastique $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ pendant $\{X_t\}_{t\geq 0}$ vérifie l'équation (2.6) par rapport au bruit W_t. Intuitivement, le mouvement Brownien $\{W_t\}_{t\geq 0}$ est ici construit et fait lui-même partie de la notion de solution. On dira alors que l'unicité au sens *faible* est vraie pour l'Équation (2.6) quand à chaque paire de solutions faibles $\{(X^1_t, W^1_t)\}_{t\geq 0}$ et $\{(X^2_t, W^2_t)\}_{t\geq 0}$, les lois marginales des processus $\{X^1_t\}_{t\geq 0}$ et$\{X^2_t\}_{t\geq 0}$ coïncident. Nous soulignons en particulier que cette notion d'unicité ne nécessite même pas que les deux solutions soient définies sur la même base stochastique.
Par contre, une solution $\{X_t\}_{t\geq 0}$ de l'Équation (2.6) est appelée *fort* si, étant donné un bruit $\{W_t\}_{t\geq 0}$, X_t vérifie l'équation donnée par rapport à W_t et est adaptée à la filtration naturelle de $\{W_t\}_{t\geq 0}$. Intuitivement, une solution forte $\{X_t\}_{t\geq 0}$ peut être vue à tout instant t comme une fonctionnelle de bruit W_t donnée en entrée du système.
Une méthode habituelle pour établir la forte unicité des dynamiques stochastiques telles que (2.6), c'est-à-dire, lorsqu'une solution forte est unique pour un bruit W_t donné, c'est exploiter le théorème de Yamada-Watanabe (cf. [YW71]) qui assure le caractère bien posé au sens fort de dynamique à partir de l'existence de solutions faibles et d'unicité de trajectoire, c'est-à-dire lorsque deux processus de solution sont indiscernables.

Dans un contexte Brownien, il a été montré qu'une dynamique stochastique non-dégénérée telle que (2.6) est bien posè au sens faible dès que la dérive F est mesurable et bornée et que la la matrice de diffusion σ est uniformément elliptique et seulement continue dans l'espace. Ce résultat a été obtenu par Stroock et Varadan [SV79] à travers la célèbre théorie du *problème de martingale*.
Ces dernières années, une grande attention a également été accordée à des dynamiques au bruit additif (i.e. $\sigma = 1$) et dérives de type distributionnel dans l'espace, en particulier

pour leur connexion avec certains modèles physiques en théorie des matériaux (cf. [AKQ14, Bro86]). Un premier résultat à cet égard peut être trouvé dans [FIR17] où une dérive F est considérée comme in-homogène dans le temps et appartenant à un espace de Hölder d'indice négatif mais strictement supérieur à $-1/2$. Pour une présentation plus précise de ces espaces de Hölder négatifs, le lecteur est renvoyé à [FH14], Section 13. Par la suite, Delarue et Diel dans [DD16] ont prouvé que le même résultat est valable même si l'on considère des indices de régularité qui ne sont supérieurs qu'à $-2/3$, mais uniquement pour la dynamique unidimensionnelle. Ce travail a finalement été étendu au cas multidimensionnel en [CC18]. Nous mentionnons également les travaux de Bass et Chen [BC03] où une dynamique au bruit additive et dérive indépendante du temps et appartenant à la classe Kato est considérée.

Quant à l'unicité de trajectoire pour les solutions de l'Équation (2.6), le premier résultat a été obtenu par Zvonkin dans [Zvo74] pour une dynamique unidimensionnelle à dérive F mesurable, bornée et une matrice de diffusion σ Hölder continue d'indice de régularité strictement supérieur à $1/2$. Ce résultat a ensuite été étendu par Veretennikov [Ver81] au cas multi-dimensionnel pour une matrice de diffusion Lipschitz continue. Comme mentionné précédemment, ces travaux sur l'unicité de trajectoire permettent alors de montrer l'existence de solutions fortes pour la dynamique, à travers le théorème de Yamada-Watanabe. Nous mentionnons également l'approche plus directe de la construction de solutions fortes sous les mêmes hypothèses sur la dérive ci-dessus, obtenue en [MBP10, MPMBN$^+$13] par le calcul de Malliavin.
Par la suite, Krylov et Röckner dans [KR05] et Zhang dans [Zha13b] ont montré le caractère bien posé au sens fort pour la dynamique stochastique comme dans (2.6) lorsque la dérive est seulement intégrable, c'est-à-dire si F appartient à $L^p(0,T;L^q(\mathbb{R}^N))$ sous la condition de Prodi-Serrin : $\frac{N}{q}+\frac{2}{p}<1$, $p\geq 2, q\geq 2$, respectivement à bruit additif et bruit multiplicatif avec matrice de diffusion σ dans les espaces de Sobolev. Voir aussi à ce sujet les travaux récents [Kry21] e [RZ21, Nam20] dans lequel le cas critique additif est confronté (i.e. $\frac{N}{q}+\frac{2}{p}=1$) respectivement pour des dérives homogènes et inhomogènes dans le temps. Soulignons cependant que dans [Nam20] une notion d'intégrabilité de type Lorentz est considérée, légèrement plus forte que l'habituelle de Lebesgue. Voir la Définition 2.1 dans l'article cité pour plus de détails. Enfin, nous citons le travail de Fedrizzi et Flandoli [FF11], où dans les mêmes conditions de Krylov et Röckner, la dépendance continue des solutions X_t sur la condition initiale x est montré et les travaux de Zhang *al.*[Zha15, XZ16] où la différentiabilité au sens faible (c'est-à-dire, dans les espaces de Sobolev) est analysée, toujours par rapport à la condition initiale.

Les travaux énumérés ci-dessus illustrent intuitivement ce qu'on appelle, suivant la terminologie de Flandoli dans [Fla11], un phénomène de *régularisation par le bruit* : lorsqu'une équation différentielle déterministe est mal posè (parce que l'existence ou l'unicité de la solution échoue) alors que la dynamique stochastique corrélée est bien posè dans un sens faible ou fort. Nous suggérons au lecteur intéressé de consulter la monographie [Fla11] où une analyse plus générale du sujet est présentée.

Nous nous sommes concentrés pour l'instant sur la dynamique stochastique *non dégénérée*, c'est-à-dire lorsque le bruit est de la même taille que le système sous-jacent il agit (i.e. lorsque $N=d$ dans (1.1)).

Cependant, nous soulignons que cette condition n'est en fait pas toujours vérifiée dans de nombreux cas pratiques. Considérons, par exemple, en mécanique hamiltonienne, les équations de Langevin avec une perturbation sur la composante vitesse, ou en finance, dans l'analyse des options asiatiques. Il a également été souligné ([dCN10, Woy01]) que dans de nombreuses applications pratiques, les fluctuations aléatoires dans les systèmes complexes réels sont en effet souvent de nature non gaussienne.
Le point de départ et la motivation principale de ce travail était donc d'analyser les phénomènes de régularisation par le bruit introduits ci-dessus, pour des chaînes d'équations différentielles ordinaires lorsque la perturbation aléatoire n'est plus un mouvement Brownien mais un processus de Lévy plus général avec des propriétés convenables et qui agit, si possible, uniquement sur certains des composants du système (i.e. si $d < N$ dans (1.1)). Par manque de temps, nous avons décidé de nous concentrer uniquement sur la caractérisation au sens faible de cette dynamique. Nous présentons maintenant brièvement un aperçu des principaux résultats déjà connus dans ce domaine.

Régularisation faible pour un bruit stable non-dégénéré. Le caractère bien posé dans un sens faible pour des dynamiques stochastiques de la forme suivante :

$$X_t \,=\, x + \int_0^t F(X_s)ds + Z_t, \quad t \geq 0, \tag{2.7}$$

où $\{Z_t\}_{t\geq 0}$ est un processus α-stable symétrique sous $\mathbb{R}^N$, a été largement étudiée au cours des dernières décennies. Une des premières contributions au cas unidimensionnel est donnée par les travaux de Tanaka *et al.* [TTW74] où l'unicité en droit est prouvée par l'Équation (2.7) lorsque la dérive F est bornée et continue et que le symbole de Lévy Phi associé à $\{Z_t\}_{t\geq 0}$ vérifie des conditions naturelles à l'infini : $\Re\Phi(\xi)^{-1} \,=\, O(1/|\xi|)$ si $|\xi| \to \infty$. Ensuite, une extension au cas multidimensionnel a été obtenue dans [Kom83] en supposant que la dérive F est continue et bornée et que la loi de $\{Z_t\}_{t\geq 0}$ admet une densité avec par rapport à la mesure de Lebesgue sur $\mathbb{R}^N$. Une dynamique stochastique comme dans (2.7) pour des dérives F dans des espaces L^p adéquats a également été considérée dans [Jin18].
A notre connaissance, les premiers travaux traitant des modèles à dérive distributionnelle dans l'espace et des bruits α -stable est [ABM20] dans le cas unidimensionnel, où la dérive F est homogène dans le temps et appartenant à un espace de Hölder (négatif) d'indice strictement supérieur à $(1 - \alpha)/2$.
On cite aussi les travaux quasi simultanés [LZ19] et [CdRM20a] où une dérive F sur des espaces de Besov généraux est considérée, dans des conditions convenables sur les paramètres, qui est respectivement homogène et inhomogène en temps.
Les résultats présentés ci-dessus sont basés sur une formulation de type Young pour donner un sens à la dynamique. Au-delà du régime de Young, on cite plutôt [KP20] où des techniques telles que les produits paracontrôlés sont exploitées (cf. [GIP15]) pour obtenir le caractère faiblement bien posé pour des dynamiques dirigées par des dérives inhomogènes dans le temps et appartenant à des espaces de Hölder négatifs avec un indice de régularité strictement supérieur à $(2 - 2\ alpha)/3$.

Enfin, nous soulignons que les travaux ci-dessus se concentrent tous sur le cas α-stable *sub-critical*, c'est-à-dire, lorsque $\alpha > 1$. En fait, nous soulignons que si $\alpha \leq 1$, les dynamiques stochastiques de la forme (2.7) sont beaucoup plus difficiles à traiter puisque le bruit ne domine pas le système en peu de temps. A cet égard, nous citons les travaux [Zha19] et [CdRMP20b] où les auteurs considèrent respectivement que $\alpha < 1$, $(1-\alpha)$-Hölder continue et $\alpha = 1$, continuez.

Régularisation faible par bruit dégénéré. Dans tous les résultats présentés ci-dessus, le bruit a joué un rôle fondamental, permettant de régulariser les coefficients de chaque composant du système. Il est donc clair que dans un contexte de bruit *dégénéré*, c'est-à-dire lorsque la perturbation aléatoire n'agit directement que sur certaines parties de la dynamique, la situation semble d'emblée beaucoup plus délicate. Considérons par exemple le modèle classique d'une chaîne de n équations déterministes ordinaires sur $\mathbb{R}^d$ où seule la première est perturbée par une diffusion Brownienne :

$$\begin{cases} dX_t^1 &= F_1(t, X_t^1, \dots, X_t^n)dt + \sigma(t, X_t^1, \dots, X_t^n)dW_t; \\ dX_t^2 &= F_2(t, X_t^1, \dots, X_t^n)dt; \\ \vdots & \\ dX_t^n &= F_n(t, X_t^{n-1}, X_t^n)dt. \end{cases} \tag{2.8}$$

Afin d'obtenir également un phénomène de régularisation par bruit dans ce cas, il faut alors que le bruit W_t agissant uniquement sur la première ligne se propage dans le système, atteignant ainsi toutes les composantes du modèle. Les hypothèses classiques qui assurent ce type de phénomène sont l'ellipticité uniforme de la matrice de diffusion σ (cf. [**UE**] dans la Section 6) qui garantit la préservation du bruit sur $\mathbb{R}^d$, et la condition dite de Hörmander (cf. [Hör67]) pour l'hypoellipticité du système.

Sous une condition de Hörmander *forte*, c'est-à-dire lorsque les champs de vecteurs diffusifs et leurs commutateurs génèrent de l'espace, les principaux travaux ont été obtenus dans le domaine de la diffusion par Kusuoka et Stroock [KS84, KS85, KS87], exploitant des techniques de calcul de Malliavin.

Dans la littérature concernant les modèles dégénérés de type (2.8), différents auteurs ont au contraire supposé que chaque composante de la dérive F est différentiable par rapport à sa première composante et que le gradient résultant est non singulier, c'est-à-dire qu'il est supposé que $D_{x_{i-1}}F_i$ (avec $i = 2, \dots, n$) a le rang maximum. Cette hypothèse de non-dégénérescence pour la sous-diagonale de la matrice jacobienne de F est la raison pour laquelle ce type de condition a souvent été appelé de type Hörmander *faible*. En termes simples, en considérant une mollification de la coefficients si nécessaire, la dérive F est en fait nécessaire pour générer l'espace $\mathbb{R}^N$ à travers les commutateurs de Lie.
Un des premiers travaux à considérer ce type de condition a été [Men11] où l'auteur a montré que (2.8) est bien posè au sens faible en supposant que la dérive F est Lipschitz continue et la matrice de diffusion est Hölder continue. Ce résultat a ensuite été étendu dans [Men18] aux matrices de diffusion qui ne sont que continues dans l'espace.
Pour la caractérisation bien posée au sens faible dans un contexte cinétique, correspondant à la dynamique dans (2.8) pour $n = 2$, nous citons Zhang [Zha18] où sont considérées des dérivées F semi-linéaires come telles que $F_2(x_1, x_2) = Ax_1$ sous des conditions d'intégrabilité locale pour F_1 et la continuité du coefficient de diffusion σ. Encore une

fois dans le cas de deux oscillateurs sous une condition de Hörmander faible, Chaudru de Raynal a montré dans [CdR18] le caractère bien posé au sens faible de l'équation dès que la dérive F est Hölder continue par rapport à la variable dégénérée x_2 avec un indice de régularité strictement supérieur à 1/3. Dans ce travail, il est également montré, par des contre-exemples convenables, que le seuil 1/3 pour la régularité de Hölder sur F est en fait "presque" optimal. Ce résultat a ensuite été étendu dans [CdRM20b] au cas plus général des n oscillateurs (cf. Équation (2.8)).
Intuitivement, ce seuil minimum pour la régularité de Hölder sur la dérive peut s'expliquer comme le prix à payer pour équilibrer la dégénérescence du bruit. En se référant à la discussion précédente sur les modèles de type Peano dans [DF14] (cf. Équation (2.5)), ce seuil est lié au fait que les fluctuations de bruit ne sont pas assez fortes pour éloigner la solution de la singularité à l'origine si la dérive est trop irrégulière.

A notre connaissance, il n'y a pas beaucoup de travaux qui traitent du caractère bien posé au sens faible pour les dynamiques avec bruit dégénéré et aux sauts, c'est à dire quand dans (2.8) on remplace le mouvement Brownien $\{W_t\}_{t\geq 0}$ avec un processus $\{Z_t\}_{t\geq 0}$ α-stable. Dans le cadre de la théorie de la régularisation par le bruit (i.e. lorsque la dynamique déterministe associée est mal posée), nous ne citons que [HM16] où les auteurs ont montré le caractère faible bien posé pour une version linéarisée de la dynamique dans (2.8) avec $F(t,x) = Ax$ et un coefficient de diffusion continue σ Hölder, sous certaines limitations de taille d, N.

3 Unicité en loi pour les chaînes dégénérées

A partir du travail [SV79] dans le cas diffusif non-dégénéré, le lien entre les solutions du *problème de martingale* et les solutions faibles (au sens probabiliste) de la dynamique stochastique (1.1) est bien connu. En un sens, cette méthode définit le processus $\{X_t\}_{t\geq 0}$ via son générateur infinitésimal L_t. Pour pouvoir l'introduire correctement dans notre contexte aux sauts, soit $\mathcal{D}([0,\infty);\mathbb{R}^N)$ d'abord l'espace de tous les chemins entre $[0,\infty)$ a $\mathbb{R}^N$ qui sont càdlàg, c'est-à-dire des chemins continus vers la droite et avec une limite finie vers la gauche. Skorokhod a montré dans [Sko56] qu'un tel espace peut être doté d'une métrique naturelle telle qu'il devienne un espace métrique séparable. On peut alors penser à $\mathcal{D}([0,\infty);\mathbb{R}^N)$ comme un espace Borélien mesurable et considérer des mesures de probabilité sur celui-ci.
On introduit aussi le processus canonique, ou valeur ponctuelle, $\{y_t\}_{t\geq 0}$ associé à l'espace $\mathcal{D}([0,\infty);\mathbb{R}^N)$ donné par

$$y_t(\omega) \;=\; \omega(t), \quad \omega \in \mathcal{D}([0,\infty);\mathbb{R}^N).$$

Pour plus de détails, voir par exemple [EK86] o [Bas11].
Fixe encore x dans $\mathbb{R}^N$, une mesure de probabilité $\mathbb{P}$ sur $\mathcal{D}([0,\infty);\mathbb{R}^N)$ est une solution au problème de martingale (associée au générateur infinitésimal L_t) de point de départ x si :

— $\mathbb{P}\,(y_0 = x) \;=\; 1$;

— pour chaque fonction ϕ dans le domaine $\mathrm{dom}\big(\partial_t + L_t\big)$, le processus

$$\Big\{\phi(t, y_t) - \phi(0, x) - \int_0^t \Big(\partial_s + L_s\Big)\phi(s, y_s)\, ds\Big\}_{t\geq 0} \tag{3.9}$$

est une $\mathbb{P}$-martingale par rapport à la filtration naturelle du processus canonique y_t.

Une troisième caractérisation possible du processus $\{X_t\}_{t\geq 0}$ est obtenue grâce à sa loi marginale μ_t :

$$\begin{cases} \partial_t \mu_t \,=\, L^*_t \mu_t; \\ \mu_0 \,=\, x, \end{cases} \tag{3.10}$$

où L^*_t représente (formellement) l'opérateur ajouté de L_t. La dynamique ci-dessus est généralement appelée *Equation progressive de Fokker-Plank.* Rappelons qu'une famille continue de mesures $\{\mu_t\}_{t\geq 0}$ est une solution de (3.10) si pour chaque fonction test ϕ dans $C^\infty_c(\mathbb{R}^N)$,

$$\partial_t \int_{\mathbb{R}^N} \phi(y)\, d\mu_t(y) \,=\, \int_{\mathbb{R}^N} L_t\phi(y)\, d\mu_t(y)$$

dans un sens distributionnel en temps t et la condition initiale exige que la mesure μ_t converge (dans un sens convenable) vers la masse de Dirac δ_x en x. Pour une présentation plus exhaustive de ce sujet, nous renvoyons le lecteur à la monographie [BKRS15].

Une analyse approfondie de ces phénomènes au cours des années (cf. [SV79, EK86, Kur98, Kur11]) a maintenant révélé que les trois modes décrits ci-dessus (équation stochastique, problème de martingale et équation de Fokker- Plank) sont, sous des conditions minimales sur les coefficients, effectivement équivalentes en spécifiant une diffusion aux sauts dans le sens que où l'existence et/ou l'unicité pour l'un implique également l'existence et/ou l'unicité dans les deux autres cas.

Nous nous intéressons maintenant à la question de l'unicité de la solution pour le problème de martingale associé à l'opérateur $\partial_t + L_t$ qui, comme nous le verrons plus loin, est la motivation principale de l'analyse menée dans les deux premiers chapitres de cette thèse.

Soient $\mathbb{P}_1$, $\mathbb{P}_2$ deux mesures sur l'espace de Skorokhod $\mathcal{D}([0,\infty);\mathbb{R}^N)$ et solutions du problème de martingale avec le point de départ x dans $\mathbb{R}^N$. A partir de la définition du problème de martingale dans (3.9), il est logique d'introduire le problème de Cauchy associé à L_t avec condition terminale nulle. En particulier, fixe un instant final $T > 0$ et une source $f\colon [0,T] \times \mathbb{R}^N \to \mathbb{R}$ dans une classe des fonctions $\mathcal{F}$ suffisamment "riche", on considère l'équation aux dérivées partielles suivante avec une condition terminale de Cauchy :

$$\begin{cases} \partial_t u(t,x) + L_t u(t,x) \,=\, f(t,x) & \text{sur } [0,T) \times \mathbb{R}^N; \\ u(T,x) \,=\, 0 & \text{sur } \mathbb{R}^N. \end{cases} \tag{3.11}$$

Supposons pour l'instant qu'une solution $u\colon [0,T] \times \mathbb{R}^N \to \mathbb{R}$ au problème de Cauchy ci-dessus existe. De plus, si u est suffisamment régulier, il résulte immédiatement de (3.9) que

$$\Big\{u(t, y_t) - \int_0^t f(s, y_s)\, ds - u(0,x)\Big\}_{t\in[0,T]} \tag{3.12}$$

est un $\mathbb{P}_i$-martingale pour chaque $i \in \{1, 2\}$, où, rappelons-le, nous avons noté $\{y_t\}_{t\geq 0}$ le processus canonique sur $\mathcal{D}([0, T]; \mathbb{R}^N)$.
Prenant maintenant la valeur attendue à l'instant final T ci-dessus, on peut exploiter la propriété de martingale dans (3.12) et le système (3.11) vérifié par u (en particulier, $u(T, \cdot) = 0$) pour relier les deux solutions au problème comme suit :

$$\mathbb{E}^{\mathbb{P}_1}\left[\int_0^T f(s, y_s)\, ds\right] \;=\; u(0, x) \;=\; \mathbb{E}^{\mathbb{P}_2}\left[\int_0^T f(s, y_s)\, ds\right],$$

où $\mathbb{E}^{\mathbb{P}_i}$ représente la valeur attendue par rapport à la mesure de probabilité $\mathbb{P}_i$. Si la classe de fonctions $\mathcal{F}$ considérée est assez riche, alors on peut conclure que la loi marginale du processus canonique $\{y_t\}_{t\geq 0}$ est la même sous les deux mesures considérées, à chaque instant fixe t. En exploitant des techniques de probabilités conditionnelles régulières, il est alors possible de montrer que le processus $\{y_t\}_{t\geq 0}$ a les mêmes distributions de dimension finie par rapport aux deux mesures de probabilité, c'est-à-dire $\mathbb{P}_1 = \mathbb{P}_2$ (cf. Théorème 4.4.2 dans [EK86]) et donc, que la solution du problème de martingale associée à l'opérateur $\partial_t + L_t$ est unique.
La principale difficulté dans le raisonnement ci-dessus est l'hypothèse de régularité sur la solution u du problème de Cauchy (3.11), nécessaire pour conclure dans l'équation (3.12). En fait, même si on considérait une source f lisse dans $C^\infty_c([0, T] \times \mathbb{R}^N)$, il ne serait pas forcément vrai que la solution u soit aussi lisse, en raison de la faible régularité des coefficients F et σ considérés, pour notre modèle, uniquement Hölder continues dans l'espace.
Lorsque le bruit est additif ($\sigma = 1$) et les coefficients sont assez réguliers, la régularité de la solution u peut être obtenue, par exemple, en exploitant des techniques de flux stochastique (cf. [Kun19]). De plus, dans ce cas, le raisonnement usuel permet de démontrer le caractère bien posé au sens fort de la dynamique stochastique considérée.
Pour appliquer le raisonnement ci-dessus dans notre contexte multiplicatif et sous l'hypothèse d'une régularité minimale, il faudra plutôt d'abord approximer les coefficients qui apparaissent dans l'opérateur L_t avec une séquence de fonctions lisses, par exemple, via une méthode de mollification. Il sera alors possible d'appliquer la méthode décrite ci-dessus pour ces coefficients suffisamment réguliers et de conclure que les deux solutions du problème de martingale "régularisé" sont égales. Enfin, pour retrouver la dynamique initialement envisagée, nous aurons besoin d'une théorie analytique "convenable" associée à l'opérateur $\partial_t + L_t$ afin de vérifier la convergence des solutions par rapport au paramètre d'approximation.
Une littérature abondante sur ce sujet (cf. [Pri09, CdR17, CdRHM18b, FGP10]), a montré qu'un premier pas dans cette direction consiste à établir des estimations particulières, appelées *Schauder*, qui permettent de vérifier les solutions approchées du problème de Cauchy (3.11) à partir des coefficients régularisés, sur un espace fonctionnel convenable. De plus, il est également possible de prouver au moyen d'arguments de compacité, qu'en fait les estimations sont également valables pour la limite des solutions approchées.

3.1 Estimations de Schauder pour des systèmes dégénérés

Comme expliqué à la fin de la section précédente, nous nous intéressons maintenant à une analyse détaillée d'une équation de Kolmogorov non locale qui peut s'écrire comme suit :

$$\begin{cases} \partial_t u(t,x) + L_t u(t,x) \;=\; f(t,x) & \text{sur } [0,T) \times \mathbb{R}^N; \\ u(T,x) \;=\; u_T(x) & \text{sur } \mathbb{R}^N, \end{cases} \tag{3.13}$$

où la source $f\colon [0,T) \times \mathbb{R}^N \to \mathbb{R}$ et la condition terminale $u_T\colon \mathbb{R}^N \to \mathbb{R}$ sont des fonctions suffisamment régulières. Nous étudierons cette équation en temps fini, c'est-à-dire pour chaque t dans $[0,T]$ à une valeur finale fixe $T > 0$. Ci-dessus, L_t représente l'opérateur intégro-différentiel dépendant du temps qui peut être vu comme le générateur infinitésimal associé à la diffusion $\{X_t\}_{t\geq 0}$ solution de (1.1), ou tel que

$$L_t \;=\; \langle F(t,x), D_x \rangle + \mathcal{L}_t, \qquad \text{su } [0,T) \times \mathbb{R}^N, \tag{3.14}$$

où $\mathcal{L}_t$ est le générateur infinitésimal associé au processus $\{B\sigma Z_t\}_{t\geq 0}$. Nous soulignons également que de façon similaire à la dynamique stochastique dont il est issu, ce système est dégénéré en ce sens que la composante principale $\mathcal{L}_t$ de l'opérateur L_t ne considère que certaines directions (associées au matrice B dans (1.1)) de l'espace $\mathbb{R}^N$ en dessous.

En particulier, les deux premiers chapitres de cette thèse s'attacheront à établir les estimations de Schauder associées aux solutions du système dégénéré (3.13).
Une caractéristique fondamentale de ces estimations est de mettre en évidence combien une solution u du problème de Cauchy (3.13) gagne en termes de régularité par rapport à la source f donnée par le système, notion généralement appelé *bootstrap parabolique* (ou effets régularisant, comme on le verra ci-dessous) associé à l'opérateur L_t.
Nous illustrons d'abord ce phénomène dans un contexte Gaussien non dégénéré, c'est-à-dire lorsque dans (3.14) l'opérateur $\mathcal{L}_t$ coïncide avec la matrice Hessienne D^2_x agissant sur l'espace ensemble $\mathbb{R}^N$. Dans ce cas et pour des coefficients bornés et convenablement réguliers (i.e. F in $C^{\frac{\beta}{2},\beta}$ avec les notations ci-dessous), Friedman dans [Fri64] et Krylov dans [Kry96] a montré le contrôle suivant :

$$\|u\|_{C^{\frac{2+\beta}{2},2+\beta}} \;\leq\; C\|f\|_{C^{\frac{\beta}{2},\beta}}, \tag{3.15}$$

où $\|\cdot\|_{C^{\gamma,\gamma'}}$ désigne la norme classique de Hölder sous l'espace-temps $[0,T)\times\mathbb{R}^N$ d'indices γ dans le temps et γ' dans l'espace. Les estimations de Schauder dans (3.15) suggèrent alors que chaque solution u au problème est en fait Hölder régulier d'ordre $\frac{2+\beta}{2}$ dans le temps et ordre $2+\beta$ dans l'espace si nous supposons que la source f n'est que $\frac{\beta}{2}$-Hölder continue dans le temps et β-Hölder continue dans l'espace.
Nous soulignons également que c'est précisément le caractère borné des coefficients dans l'espace et leur régularité dans le temps qui nous permet de considérer le bootstrap parabolique également par rapport à la variable de temps. Dans le cas où les coefficients ne sont pas bornés dans l'espace mais au maximum de croissance linéaire, Krylov et Priola dans [KP10] ont montré la variante suivante des estimations de Schauder pour le problème :

$$\|u\|_{L^\infty(C^{2+\beta})} \;\leq\; C\|f\|_{L^\infty(C^\beta)}, \tag{3.16}$$

où $\|\cdot\|_{L^\infty(C^\gamma)}$ représente la norme de Hölder d'ordre γ dans l'espace, uniformément dans le temps. En pratique, ce type d'estimation ne considère à la place que le gain de régularité dans l'espace, d'ordre 2 dans ce cas, de la solution u par rapport à la source

f. Nous soulignons cependant qu'au moins dans le cas uniformément elliptique, il est possible de déduire ultérieurement la régularité en temps pour la solution u.

Une application immédiate des estimations de Schauder est l'unicité de la solution du Problème de Cauchy (3.13) dans l'espace fonctionnel considéré. En fait, soit u_1, u_2 deux solutions du système considéré. En exploitant la linéarité du problème, nous pouvons voir que leur différence $u_1 - u_2$ est alors la solution du problème de Cauchy :

$$\begin{cases} (\partial_t + L_t)\,(u_1 - u_2)(t,x) \,=\, 0 \\ (u_1 - u_2)(T,x) \,=\, 0. \end{cases}$$

Les estimations de Schauder présentées, par exemple, dans (3.16) avec $f = 0$ impliquent maintenant la vérification suivante :

$$\|u_1 - u_2\|_{L^\infty(C^{2+\beta})} \,\leq\, 0,$$

ce qui conduit immédiatement à l'égalité des deux solutions dans l'espace fonctionnel considéré.
L'étude des problèmes de Cauchy tels que (3.13) est également fondamentale pour l'analyse de la dynamique stochastique associée (1.1). En plus de la démonstration de l'unicité du problème de martingale comme expliqué à la fin de la section précédente, nous soulignons que les estimations de Schauder sont souvent utilisées dans la transformation de Zvonkin pour déterminer le caractère fortement bien posé de l'équation stochastique associée. Pour plus de détails, voir [Zvo74, Ver81] dans un contexte diffusif non-dégénéré respectivement mono et multidimensionnel ou [CdRHM18b, HWZ20] dans le cas dégénéré, respectivement diffusif et α-stable.
Ces estimations apparaissent également en lien avec certaines équations aux dérivées partielles stochastiques (SPDE). Par exemple, nous citons l'application à l'équation de transport stochastique présentée dans [FGP10], où les estimations de Schauder comme dans (3.16) sont utilisées pour prouver l'existence d'un flux stochastique différentiable pour l'équation caractéristique stochastique.
Enfin, nous mentionnons qu'un exemple typique de système dégénéré de la forme (3.13) est l'équation cinétique stable suivante :

$$\partial_t u(t,x) \,=\, \mathcal{L}^\alpha u(t,x) - x_1 \cdot \nabla_{x_2} u(t,x) + f(t,x), \quad \text{su } \mathbb{R}^{2d},$$

où $x = (x_1, x_2)$ dans $\mathbb{R}^{2d}$ et $\mathcal{L}^\alpha$ est un opérateur stable agissant uniquement sur x_1, qui apparaît naturellement dans l'étude de l'équation linéarisée de Boltzmann (cf. [Vil02, CZ18]).

Résumons maintenant brièvement les principaux travaux concernant les estimations de Schauder pour des systèmes paraboliques dégénérés.
Dans un contexte diffusif gaussien, c'est-à-dire lorsque dans (3.13) l'opérateur L_t peut être réécrit comme

$$L_t \,=\, \langle F(t,x), D_x\rangle + \frac{1}{2}\mathrm{tr}\left(Ba(t,x)B^*D_x^2\right) \,=:\, \langle F(t,x), D_x\rangle + \mathcal{L}_t,$$

pour une certaine matrice de diffusion $a(t,x)$ dans $\mathbb{R}^d \otimes \mathbb{R}^d$, Lunardi dans [Lun97] a été la première à établir des estimations de Schauder pour les équations de Kolmogorov

de type Ornstein-Uhlenbeck, c'est-à-dire $a(t,x) = 1$ et $F(t,x) = Ax$ avec un certain structure. Ce résultat a été atteint en exploitant les anisotropes Hölder, où l'indice de Hölder dépend de la direction spatiale considérée, précisément pour contrôler les différentes échelles de régularité provoquées par la dégénérescence du système. Voir la section suivante pour une explication plus exhaustive de ce phénomène et une définition précise de ces espaces anisotropes.
Par la suite, dans [Lor05] et [Pri09], les auteurs ont obtenu des estimations de type Schauder pour des équations hypoelliptiques de Kolmogorov dont la dérive n'est partiellement non-linéaire que le long des composantes dans lesquelles l'opérateur principal $\mathcal{L}_t$ est non dégénéré, c'est-à-dire :

$$F(x) \,=\, Ax + \begin{pmatrix} \tilde{F}_1(x) \\ 0_{(N-d),d} \end{pmatrix}.$$

Enfin, le cas diffusif totalement non-linéaire et à matrice de diffusion non-homogène dans le temps et dans l'espace a été abordé pour la première fois dans [CdRHM18a]. Leur résultat a été atteint dans des conditions de régularité minimale et de non-dégénérescence sur les coefficients F et a, ce qui conduit à des hypothèses de type Hörmander faible pour le système. La méthode de prouve dans [CdRHM18a] c'est basée sur une approche perturbative progressive que nous adapterons et exploiterons également par la suite.
Enfin, nous citons les travaux de Di Francesco et Polidoro [DFP06], où des estimations locales de Schauder pour un système dégénéré linéaire avec coefficient de diffusion sont obtenues en supposant une notion différente de continuité pour le coefficient de diffusion a qui, en dans un certain sens, il prend également en compte le transport associé à la dérive linéaire.

Ces derniers temps, les estimations de Schauder pour les opérateurs non locaux, en particulier de type stable, ont suscité un grand intérêt dans la communauté mathématique (cf. [BK15, Bas09, CdRMP20a, DK13, FRRO17, IJS18, Pri12, ROS16, ZZ18]). Cependant, la plupart des travaux actuels se concentrent uniquement sur le cas non dégénéré. A notre connaissance, le seul travail existant traitant du cas dégénéré non local est [HWZ20], où Zhang et ses collaborateurs prouvent des estimations de Schauder pour un système cinétique stable dégénéré (quand $N = 2d$ et $\operatorname{rank} B = d$ dans (3.13)). Leur méthode est basée sur une généralisation des décompositions de Littlewood-Paley déjà exploitées dans d'autres travaux du même auteur (cf. [ZZ18]) au contexte anisotrope naturel pour les systèmes dégénérés.

Dans cette thèse, nous analyserons en détail deux cas particuliers qui généralisent les résultats exposés ci-dessus :

- un système dégénéré à dérive non-linéaire dans lequel la composante principale de l'opérateur est de type α-stable. Ce modèle sera présenté en détail dans la Section 4 ;
- un système dégénéré dirigé par un opérateur d'Ornstein-Uhlenbeck de type Lévy. Nous renvoyons à la Section 5 pour une présentation plus détaillée du modèle.

3.2 Géométrie anisotrope de la dynamique dégénérée

Dans cette section, nous nous concentrons sur la compréhension de l'espace fonctionnel convenable pour établir nos estimations de Schauder. Comme indiqué précédemment, les estimations de Schauder dans le cas non-dégénéré sont généralement exprimées par rapport à des espaces de Hölder "habituels". Nous voulons donc obtenir un espace de Hölder par rapport à une nouvelle distance adaptée à notre structure dégénérée. En particulier, à la fin de cette section, nous construirons quelques espaces de Hölder avec des multi-indices de régularité en fonction de la coordonnée considérée.

Pour faire comprendre au lecteur comment cette structure anisotrope de la chaîne dégénérée apparaît naturellement, nous présenterons deux approches possibles : une analytique, basée sur des opérateurs de dilatation multi-échelles, et une plus probabiliste, qui exploite plutôt les échelles pour le temps caractéristique du processus, solution de l'équation stochastique associée. Aussi, pour rendre cette affirmation le plus claire que possible, nous allons nous concentrer sur un exemple linéaire avec seulement deux composants ($n = 2$).

D'un point de vue analytique, on s'intéresse à l'opérateur de Kolmogorov de type α-stable pour un certain $\alpha \in (0, 2)$:

$$L^{\mathrm{K}} \,:=\, \Delta^{\frac{\alpha}{2}}_{x_1} + x_1 \cdot \nabla_{x_2} \;\text{ sous } \mathbb{R}^{2d}$$

où $x = (x_1, x_2)$ est un point dans $\mathbb{R}^{2d}$. Ci-dessus, l'opérateur $\Delta^{\frac{\alpha}{2}}_{x_1}$ représente le Laplacien fractionnaire d'ordre $\alpha/2$ par rapport à la variable x_1, donné par

$$\Delta^{\frac{\alpha}{2}}_{x_1}\phi(x_1, x_2) \,=\, \text{p.v.} \int_{\mathbb{R}^d} [\phi(x_1 + z, x_2) - \phi(x_1, x_2)] \frac{dz}{|z|^{d+\alpha}} \tag{3.17}$$

pour toute fonction $\phi \colon \mathbb{R}^{2d} \to \mathbb{R}$ assez régulière. Dans le cas diffusif, c'est quand $\Delta^{\frac{\alpha}{2}}_{x_1}$ devient le Laplacien classique Δ_{x_1} agissant sous x_1 :

$$\Delta_{x_1}\phi(x_1, x_2) \,=\, \sum_{i=1}^{d} \partial^2_{x_1^i}\phi(x_1, x_2),$$

l'équation associée à cet opérateur a été analysée précisément par Kolmogorov [Kol34] et a été le premier exemple qui a inspiré la théorie de l'hypoellipticité de Hörmander [Hör67].

Pour comprendre comment les différentes composantes de la dynamique se comportent les unes par rapport aux autres, nous introduisons un opérateur de dilatation $\delta \colon [0, \infty) \times \mathbb{R}^{2d} \to [0, \infty) \times \mathbb{R}^{2d}$ donnè par :

$$\delta(t, x) \,:=\, (\delta_0 t, \delta_1 x_1, \delta_2 x_2).$$

Les valeurs exactes de $\delta_0, \delta_1, \delta_2$ sont alors à déterminer pour que la dilatation δ soit invariante sous la dynamique

$$\partial_t u(t, x) + L^{\mathrm{K}} u(t, x) \,=\, 0 \quad \text{ sous } (0, \infty) \times \mathbb{R}^{2d}, \tag{3.18}$$

en ce sens qu'il transforme les solutions de l'équation ci-dessus en d'autres solutions de la même.
L'idée d'une dilatation δ qui permet de résumer le comportement multi-échelle, ou anisotrope, de la dynamique dégénérée considérée a été initialement introduite par Lanconelli et Polidoro ([LP94]) pour l'analyse de l'équation diffusive de Kolmogorov. Depuis lors, il est devenu un outil commun pour analyser la géométrie anisotrope pour les équations dégénérées, comme en témoigne la riche littérature dans laquelle il apparaît ([Lun97, HMP19, HWZ20, DFP05, MPP02]). En exploitant maintenant la propriété d'échelle du Laplacien fractionnaire, nous pouvons voir que :

$$\begin{aligned}\Big(\partial_t + L^{\mathrm{K}}\Big)\,(u\circ\delta) \;&=\; \delta_0(\partial_t u\circ\delta) + \delta_1^{\alpha}(\Delta_{x_1}^{\frac{\alpha}{2}} u\circ\delta) + \delta_2\,(x_1\cdot(\nabla_{x_2} u\circ\delta))\\ &=\; \delta_0(\partial_t u\circ\delta) + \delta_1^{\alpha}(\Delta_{x_1}^{\frac{\alpha}{2}} u\circ\delta) + \delta_1^{-1}\delta_2\,([x_1\cdot\nabla_{x_2} u]\circ\delta) \;=\; 0,\end{aligned}$$

où, nous soulignons, nous avons dénoté $[x_1\cdot\nabla_{x_2}u]\circ\delta(t,x) := \delta_1 x_1\cdot\nabla_{x_2}u(\delta(t,x))$. Afin d'obtenir l'homogénéité dans les termes ci-dessus, il est alors naturel de considérer, pour chaque $\lambda > 0$ fixé, l'opérateur de dilatation δ_λ suivant :

$$\delta_\lambda(t,x_1,x_2) := (\lambda^{\alpha}t, \lambda x_1, \lambda^{1+\alpha}x_2). \tag{3.19}$$

En particulier, on note que, pour notre choix de δ_λ, il retient que

$$\Big(\partial_t + L^{\mathrm{K}}\Big)u = 0 \implies \Big(\partial_t + L^{\mathrm{K}}\Big)(u\circ\delta_\lambda) = 0.$$

En résumé, l'émergence de ce phénomène multi-échelle est due essentiellement à la structure particulière de la dynamique considéré, constitué d'une partie principale $\Delta_{x_1}^{\alpha/2}$ qui procure un effet régularisant de ordre α uniquement sur la première composante et à partir d'un terme de transport $x_1\cdot\nabla_{x_2}$ qui permet à cet effet de également être transmis au deuxième composant, bien qu'avec une intensité plus faible (d'ordre $\alpha/(1+\alpha)$), comme le montre le bootstrap parabolique associé aux estimations de Schauder que nous avons considérées.
D'un point de vue plus probabiliste, les échelles apparaissant dans l'opérateur de dilatation δ peuvent être associées aux exposants temporels caractéristiques d'un processus α-stable et à son intégrale temporelle. En fait, le temps caractéristique d'un processus stochastique multidimensionnel peut aider à expliquer la relation entre les vitesses des différentes composantes du processus lui-même.
On commence par considérer la contrepartie stochastique du système (3.18) donnée par l'équation suivante :

$$\begin{cases} dX^1_t \;=\; dZ_t,\\ dX^2_t \;=\; X^1_t dt, \quad t\geq 0,\end{cases} \tag{3.20}$$

où, pour simplifier, nous supposons que la solution commence à l'origine au temps initial. Cette équation stochastique est associée à l'opérateur de Kolmogorov introduit précédemment en ce sens que $\mathcal{L}^{\mathrm{K}}$ est le générateur infinitésimal du processus X_t.
La dynamique stochastique (3.20) peut maintenant être résolu explicitement via une intégration en temps :

$$X_t \;=\; (X^1_t, X^2_t) \;=\; \Big(Z_t, \int_0^t Z_s\, ds\Big). \tag{3.21}$$

Si on vérifie maintenant les temps caractéristiques associés aux deux composantes du processus X_t, on constate qu'ils sont donnés par $(t^{\frac{1}{\alpha}}, t^{1+\frac{1}{\alpha}})$. En fait, le processus stable est connu pour être α-scalable et son intégrale temporelle ajoute simplement un ordre de plus. Nous avons en fait trouvé les mêmes échelles que celles affichées dans l'opérateur de propagation δ, bien que dans ce cas, redimensionnées par rapport à l'heure actuelle. Dans le cas diffusif ($\alpha = 2$), ce comportement multi-échelle du processus de solution X_t est encore plus clair. En effet, grâce à l'existence de moments seconds finis, il est possible de traduire le raisonnement décrit ci-dessus par rapport à la matrice de covariance du processus. Si on remplace dans (3.20) et dans (3.21) le processus α-stable Z_t par un mouvement Brownien W_t, on obtient immédiatement que la solution $X_t = (X^1_t, X^2_t)$ est un processus Gaussien à moyenne nulle et K_t covariance dans $\mathbb{R}^{2d} \otimes \mathbb{R}^{2d}$ donné par

$$K_t = \begin{pmatrix} tI_{d\times d} & \frac{t^2}{2}I_{d\times d} \\ \frac{t^2}{2}I_{d\times d} & \frac{t^3}{3}I_{d\times d} \end{pmatrix}.$$

L'équivalence, en termes de formes quadratiques associées, entre la matrice de covariance K_t et une diagonale a ensuite été montrée dans [KMM10] :

$$\sqrt{K_t} \asymp \begin{pmatrix} t^{\frac{1}{2}}I_{d\times d} & 0_{d\times d} \\ 0_{d\times d} & t^{\frac{3}{2}}I_{d\times d} \end{pmatrix},$$

où pour deux matrices A, B dans $\mathbb{R}^{nd} \otimes \mathbb{R}^{nd}$, la notation $A \asymp B$ indique qu'il existe une constante $C \geq 1$ telle que $C^{-1}|A\xi|^2 \leq |B\xi|^2 \leq C|A\xi|^2$ pour tous ξ dans $\mathbb{R}^{nd}$. Cette propriété a été appelée par les auteurs, propriété de *bonne échelle*. Voir aussi la définition 3.2 dans [DM10] pour une extension à la chaîne générale ($n > 2$). Il est maintenant clair comment on peut alors retrouver sur la diagonale de la matrice d'échelle les mêmes échelles montrées précédemment dans le cas diffusif.

On veut maintenant introduire une distance parabolique $\mathbf{d}_P$ sur $[0, \infty) \times \mathbb{R}^N$ qui soit homogène par rapport à la structure multi-échelle de la dynamique considérée, au sens cette :

$$\mathbf{d}_P\Big(\delta_\lambda(t,x); \delta_\lambda(s,x')\Big) = \lambda \mathbf{d}_P\Big((t,x);(s,x')\Big).$$

Un choix naturel pour cette distance est alors donné par :

$$\mathbf{d}_P\left((t,x),(s,x')\right) = |t-s|^{\frac{1}{\alpha}} + |x_1 - x'_1| + |x_2 - x'_2|^{\frac{1}{1+\alpha}}.$$

La distance d_P qui vient d'être introduite peut être vue comme une généralisation naturelle de la distance parabolique classique (cf. [Kry96, Fri64]) à la structure multi-échelle de notre dynamique stable dégénérée. Comme nous n'exploiterons presque toujours que la partie spatiale de cette distance par la suite, nous introduisons également pour référence future :

$$\mathbf{d}(x,x') := |x_1 - x'_1| + |x_2 - x'_2|^{\frac{1}{1+\alpha}}. \tag{3.22}$$

Pour plus de détails sur les métriques homogènes, voir aussi le livre de Stein [Ste93].

Nous sommes enfin prêts à introduire l'espace fonctionnel adapté à notre propos : un espace anisotrope de Hölder $C^\beta_d(\mathbb{R}^{2d})$ associé à la distance $\mathbf{d}_P$, dans le sens que il est

homogène par rapport aux δ_λ opérateurs de dilatation définis dans (3.19). Concrètement, la semi-norme anisotrope $\|\cdot\|_{C^\beta_d}$ sera en fait considérée composante par composante. En fait, pour une coordonnée fixée, on va calculer la norme de Hölder long de cette direction particulière, mais avec un indice de régularité redimensionné selon l'ordre donné par l'opérateur de dilatation δ_λ in (3.19), uniformément dans le temps et par rapport aux autres coordonnées. Enfin, nous additionnerons toutes les contributions dérivant des différentes composantes.
Plus précisément, on peut introduire pour une fonction $\phi\colon \mathbb{R}^{2d} \to \mathbb{R}$ et un point z dans $\mathbb{R}^{2d}$, la fonction $\Pi^1_z\phi\colon \mathbb{R}^d \to \mathbb{R}$ donné par

$$\Pi^1_z\phi(x_1) \,:=\, \phi(z_1 + x_1, z_2).$$

Une notation similaire est également introduite pour $\Pi^2_z\phi$.
Étant donné un temps final $T > 0$, on définit l'espace *homogène* $L^\infty(0,T;C^\beta_d(\mathbb{R}^{2d}))$ comme la famille de les fonctions Boréliennes $\phi\colon [0,T] \times \mathbb{R}^{2d} \to \mathbb{R}$ telles que la semi-norme de Hölder anisotrope suivante est finie :

$$\|\phi\|_{L^\infty(C^\beta_d)} \,:=\, \sup_{t,z}\Big([\Pi^1_z\phi(t,\cdot)]_{C^\beta(\mathbb{R}^d)} + [\Pi^2_z\phi(t,\cdot)]_{C^{\frac{\beta}{1+\alpha}}(\mathbb{R}^d)}\Big) \asymp \sup_{t,x,x'} \frac{|\phi(t,x) - \phi(t,x')|}{\mathbf{d}(x,x')}.$$

Choisi α dans $(0,2)$ tel que $\alpha + \beta$ soit dans $(1,2)$, on peut alors définir aussi l'espace de Hölder anisotrope $L^\infty(0,T;C^{\alpha+\beta}_d(\mathbb{R}^{2d}))$ d'ordre $\alpha + \beta$ par rapport à la semi-norme suivante :

$$\begin{aligned}\|\phi\|_{L^\infty(C^{\alpha+\beta}_d)} \,:=\,& \|D_{x_1}\phi\|_{L^\infty} \\ &+ \sup_{t,z}\Big([\Pi^1_z D_{x_1}\phi(t,\cdot)]_{C^{\alpha+\beta-1}(\mathbb{R}^d)} + [\Pi^2_z\phi(t,\cdot)]_{C^{\frac{\alpha+\beta}{1+\alpha}}(\mathbb{R}^d)}\Big)\end{aligned} \tag{3.23}$$

où, on rappelle que $\|\cdot\|_{L^\infty}$ représente la norme uniforme sur $[0,T]\times\mathbb{R}^{2d}$. Si on a $\alpha+\beta > 2$, une extension naturelle aux dérivées du second ordre est alors nécessaire.
Puisque la source f et la condition terminale u_T seront considérées comme bornées dans notre modèle, nous introduisons également la version inhomogène (avec a b ci-dessous) des espaces de Hölder qui viennent d'être décrits, en ajoutant simplement la norme uniforme fonction elle-même. Par exemple,

$$\|\phi\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \,:=\, \|\phi\|_{L^\infty} + \|\phi\|_{L^\infty(C^{\alpha+\beta}_d)}.$$

Enfin, on dira qu'une fonction ϕ est dans $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{2d})$ si ϕ est indépendant du temps et sa norme de Hölder anisotrope est finie. Ce sera le cas, par exemple, de la condition terminale u_T.

4 Estimations de Schauder pour un système dégénéré non linéaire de type stable

Nous résumons dans cette section les résultats présentés au chapitre 2 du présent travail, puis publiés dans *Bulletin des Sciences Mathématiques*. Notre but est d'établir des estimations de Schauder optimales, au sens de régularité minimale sur les coefficients, pour les solutions d'une équation parabolique intégro-différentielle partielle dégénérée sur $\mathbb{R}^{nd}$. La dégénérescence dans ce contexte vient du fait que la partie principale de l'opérateur, de type α-stable, n'agit que sur les premiers d composants du système.

En particulier, étant donné une source $f\colon [0,T]\times\mathbb{R}^{nd}\to\mathbb{R}$ et une condition terminale $u_T\colon\mathbb{R}^{nd}\to\mathbb{R}$, on est intéressé à un problème de Cauchy de la forme suivante :

$$\begin{cases}\partial_t u(t,x)+\langle Ax+F(t,x),D_xu(t,x)\rangle+\mathcal{L}_\alpha u(t,x) \,=\, -f(t,x) & \text{sur } [0,T)\times\mathbb{R}^{nd};\\ u(T,x)\,=\,u_T(x) & \text{sur } \mathbb{R}^{nd}.\end{cases} \tag{4.1}$$

où $x:=(x_1,\dots,x_n)$ est un point dans $\mathbb{R}^{nd}$ avec chaque x_i dans $\mathbb{R}^d$ et $\langle\cdot,\cdot\rangle$ représente le produit scalaire sur $\mathbb{R}^{nd}$. Ci-dessus, $F\colon[0,T]\times\, R^{nd}\to\mathbb{R}^{nd}$ est une fonction suffisamment régulière et A est un matrice dans $\mathbb{R}^{nd}\otimes\mathbb{R}^{nd}$ sur lequel nous imposerons des conditions convenables.
L'opérateur $\mathcal{L}_\alpha$ considéré est le générateur d'un processus α-stable symétrique et non-dégénéré qui n'agit que sur la première composante x_1 du système. Plus précisément, l'opérateur $\mathcal{L}_\alpha$ peut être représenté, pour toute fonction assez régulière $\phi\colon[0,T]\times\mathbb{R}^{nd}\to\mathbb{R}$, dans le forme suivante :

$$\mathcal{L}_\alpha\phi(t,x)\,:=\,\text{p.v.}\int_{\mathbb{R}^d}\Big[\phi(t,x+By)-\phi(t,x)\Big]\,\nu_\alpha(dy), \tag{4.2}$$

où $B:=(I_{d\times d},0_{d\times d},\dots,I_{d\times d})^*$ est la matrice d'immersion de $\mathbb{R}^d$ dans $\mathbb{R}^{nd}$ et ν_α est la mesure de Lévy symétrique sur $\mathbb{R}^d$ associée à un processus α-stable. Rappelons maintenant que le symbole de Lévy Φ associé à l'opérateur $\mathcal{L}_\alpha$ (ou plus exactement, le processus qu'il génère) est normalement défini par la formule de Lévy-Khintchine qui, dans notre cas stable, on peut l'écrire de la façon suivante (cf. [Sat13]) :

$$\Phi(p)\,=\,-\int_{\mathbb{S}^{d-1}}|p\cdot s|^\alpha\,\mu(ds),$$

où "$\cdot$ " représente le produit scalaire sur le " petit " espace $\mathbb{R}^d$. Ci-dessus, μ est une mesure sur la sphère $\mathbb{S}^{d-1}$ habituellement appelée la mesure *spectral* (ou sphérique) associée à ν_α, au sens qu'un passage en coordonnées polaires $y=\rho s$, où $(\rho,s)\in(0,\infty)\times\mathbb{S}^{d-1}$, permet de décomposer le Lévy mesure stable comme

$$\nu_\alpha(dy)\,:=\,C_\alpha\mu(ds)\frac{d\rho}{\rho^{1+\alpha}}. \tag{4.3}$$

Pour une preuve de ce fait, voir par exemple le Théorème 14.3 dans [Sat13]. En particulier, nous imposerons que la mesure de Lévy stable ν_α soit symétrique et *non-dégénérée* dans le sens où sa mesure sphérique μ vérifie la condition suivante :

[ND] Il existe une constante $\eta\geq1$ telle que pour chaque p dans $\mathbb{R}^d$,

$$\eta^{-1}|p|^\alpha\,\leq\,\int_{\mathbb{S}^{d-1}}|p\cdot s|^\alpha\,\mu(ds)\,\leq\,\eta|p|^\alpha. \tag{4.4}$$

Comme nous le verrons plus loin, cette condition implique notamment l'existence d'une solution fondamentale pour l'opérateur $\mathcal{L}_\alpha$, puisque la transformée de Fourier du

processus $\{Z_t\}_{t\geq 0}$ α-stable associé à $\mathcal{L}_\alpha$ est puis intégrable.
Il est également important de noter que la famille des mesures spectrales non-dégénérées (au sens ci-dessus) est très riche et variée. En fait, la condition [**ND**] est vérifiée, par exemple, de la mesure de Lebesgue sur la sphère, correspondant au Laplacien fractionnaire "habituel" :

$$\mathcal{L}_\alpha\phi(t,x) \,:=\, \Delta^{\frac{\alpha}{2}}_{x_1}\phi(t,x) \,=\, \text{p.v.}\int_{\mathbb{R}^d}\left[\phi(t,x_1+z,x_2,\dots,x_n)-\phi(t,x)\right]\frac{dz}{|z|^{d+\alpha}},$$

mais aussi de cas très singuliers (par rapport à la mesure de Lebesgue sur la sphère), comme par exemple la somme des masses de Dirac le long des coordonnées canoniques, correspondant à l'opérateur suivant :

$$\mathcal{L}_\alpha\phi(t,x) \,=\, \sum_{i=1}^{d}\Delta^{\frac{\alpha}{2}}_{x^i_1}\phi(t,x) \tag{4.5}$$

où, rappelez-vous, $x_1=(x^1_1,\dots,x^d_1)$ est un point dans $\mathbb{R}^d$ et $\Delta^{\frac{\alpha}{2}}_{x^i_1}$ représente le Laplacien fractionnaire “habituel” scalaire appliqué à la variable x^i_1. x^i_1. Ce type d'opérateur est normalement appelé un laplacien fractionnaire cylindrique et dans ce cas, la mesure Lévy ν_α associée est concentrée sur les axes $\{x_1=0\}\cup\dots\cup\{x_d=0\}$. Pour plus de détails, voir par exemple l'Équation (1.2) dans [BC06] .
Enfin, nous mentionnons que dans la littérature la condition [**ND**] pour la mesure de Lévy ν_α apparaît souvent aussi dans les formulations suivantes :

— (Support minimum) le support de la mesure sphérique μ n'est contenu dans aucun sous-espace linéaire propre de $\mathbb{R}^d$;
— (Condition de Picard) Il existe une constante $C:=C(\alpha)$ qui pour chaque $\rho>0$, u dans $\mathbb{S}^{d-1}$, elle tient que

$$\int_{\{|u\cdot y|\}\leq\rho}|u\cdot y|^2\nu_\alpha(dy) \,\geq\, C\rho^{2-\alpha}.$$

Pour plus de détails et une preuve d'équivalence entre les conditions, voir par exemple [Pic96, Szt10b, Pri12]. Comme déjà expliqué en introduction, l'analyse des dynamiques dégénérées où l'effet régularisant n'agit directement que sur un sous-espace ($\mathbb{R}^d$) de l'espace total considéré ($\mathbb{R}^{nd}$), nécessite quelques hypothèses sur système considéré Pour que la régularisation soit effectivement transmise dans tout le système. Dans le cas diffusif ($\alpha=2$), c'est-à-dire lorsque $\mathcal{L}_\alpha=\Delta_{x_1}$, une condition naturelle est donnée par l'hypoellipticité de Hörmander (cf.[Hör67]) qui nécessite, au moins formellement, que les $n-1$ commutateurs de Lie itérés associés à ∂_{x_1} et $\langle Ax,D_x\rangle$ génèrent tout l'espace. Dans notre cas non-local , bien qu'il ne semble pas y avoir de théorème de Hörmander au sens général [KT01], certaines hypothèses naturelles (conditions [**H**] et **ND** ci-dessous) assurent que le semi-groupe de Markov associé à l'opérateur

$$L^{\text{ou}} \,:=\, \mathcal{L}_\alpha+\langle Ax,D_x\rangle$$

admet une densité suffisamment régulière [PZ09].
A cet effet, nous renvoyons également aux travaux de Cass [Cas09] où une extension,

bien qu'encore incomplète, du résultat de Hörmander au cas non local est présentée, dans des conditions très générales.
En pratique, on imposera une forme particulière à la matrice A qui assure l'hypoellitícité du système :

[H] la matrice A a la structure sous-diagonale suivante :

$$A := \begin{pmatrix} 0_{d\times d} & \dots & \dots & \dots & 0_{d\times d} \\ A_{2,1} & 0_{d\times d} & \dots & \dots & 0_{d\times d} \\ 0_{d\times d} & A_{3,2} & 0_{d\times d} & \dots & 0_{d\times d} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{d\times d} & \dots & 0_{d\times d} & A_{n,n-1} & 0_{d\times d} \end{pmatrix} \tag{4.6}$$

et les éléments $A_{i,i-1}$ dans $\mathbb{R}^d \otimes \mathbb{R}^d$ ont tous un rang maximum d.

La structure spécifique choisie pour la matrice A apparaît également naturelle (cf. [LP94]) car elle est invariante pour les dilatations δ_λ (définies dans (3.19) pour $n = 2$) intrinsèque à notre dynamique dégénérée, en ce sens que

$$e^{tA} = \mathbb{M}_t e^A \mathbb{M}_t^{-1}, \tag{4.7}$$

où $\mathbb{M}_t$ est une matrice dans $\mathbb{R}^{nd} \otimes \mathbb{R}^{nd}$ donné par

$$\Big[\mathbb{M}_t\Big]_{i,j} := \begin{cases} t^{i-1} I_{d\times d}, & \text{si } i = j; \\ 0_{d\times d}, & \text{autrement.} \end{cases} \tag{4.8}$$

Intuitivement, $\mathbb{M}_t$ prend en compte la structure multi-échelle associée à la distance d par rapport au temps caractéristique $t^{\frac{1}{\alpha}}$. La décomposition en (4.7) peut être facilement obtenue à partir de la définition de la matrice exponentielle et de l'identité $\mathbb{M}_t A \mathbb{M}_t^{-1} = tA$. Cependant, nous soulignons que notre modèle ne considère qu'une seule structure spécifique particulière parmi celles éventuellement incluses dans la théorie générale de l'hypoellipticité développée par Hörmander. En fait, la non-dégénérescence des éléments sous-diagonaux dans la matrice A nécessite, à chaque niveau de la chaîne, d'exploiter une seule parenthèse de Lie supplémentaire pour générer la direction correspondante. C'est précisément cette propriété qui permet la transmission des effets stabilisateurs α-stable à chaque composant de la chaîne, comme expliqué dans la section précédente. Enfin, nous soulignons que dans notre contexte non-linéaire, la condition "classique" de Hörmander (cf. [Hör67]) ne peut être considérée, du fait de la régularité faible de la dérive déterministe F que nous allons supposer, seul Hölder continue dans l'espace. En particulier, cela nous empêche de calculer explicitement les commutateurs nécessaires dans la condition de Hörmander.

La géométrie anisotrope associée à notre système différentiel dégénéré sur $\mathbb{R}^{nd}$ peut facilement être comprise comme une extension sur n vu de celle introduite dans la section précédente dans le cas cinétique ($n = 2$), en le sens que, par exemple, la distance spatiale $\mathbf{d}$ est maintenant défini par

$$\mathbf{d}(x, x') = \sum_{i=1}^{n} |(x - x')_i|^{\frac{1}{1+\alpha(i-1)}}, \quad x, x' \in \mathbb{R}^{nd}. \tag{4.9}$$

En particulier, une fonction ϕ dans $C^\beta_d(\mathbb{R}^{nd})$ sera $\beta/(1+\alpha(i-1))$-Hölder continue par rapport à sa variable x_i, uniformément dans les autres variables x_j $(j \neq i)$.
La fonction $F = (F_1, \dots, F_n)$ peut être comprise comme une perturbation (éventuellement non linéaire) de la dérive Ax dans le modèle dégénéré d'Ornstein-Uhlenbeck L^{ou}. Nous soulignons également que, même plus tard, la non-linéarité du système se référera à la forme de la dérive. En fait, l'opérateur perturbé $L^{\text{ou}} + \langle F(t,x), D_x\cdot\rangle$ reste de toute façon linéaire. Nous imposerons une structure particulière à cette perturbation F de telle sorte qu'elle ne détruise pas l'hypoellipticité du système considéré. Nous supposerons également un certain degré de régularité sur la dérive déterministe F, nécessaire à notre propos.

[R] pour chaque niveau i dans $[\![1,n]\!]$, F_i ne dépend que du temps et des dernières $n-(i-1)$ variables, soit $F_i(t, x_i, \dots, x_n)$. De plus, F_i appartient à l'espace $L^\infty(0,T; C^{\gamma_i+\beta}_d(\mathbb{R}^{nd}))$, où

$$\gamma_i := \begin{cases} 1+\alpha(i-2), & \text{si } i > 1; \\ 0, & \text{si } i = 1. \end{cases} \tag{4.10}$$

Nous soulignons déjà qu'aucune condition de bornage n'a été imposée sur la dérive F mais seulement une régularité de Hölder avec des multi-indices croissants. Ce sera en fait l'une des principales difficultés à rencontrer dans notre méthode de démonstration. Nous soulignons également que, contrairement au cas non dégénéré (cf. [CdRMP20a]), il faut ici imposer une régularité supplémentaire croissante sur les composantes dégénérées $(i > 1)$ de la perturbation F_i, représentée à partir du paramètre γ_i ci-dessus. Cependant, cette hypothèse paraît naturelle si l'on pense que, du fait de la structure dégénérée de notre système, l'effet régularisant de l'opérateur α-stable $\mathcal{L}_\alpha$ qui n'agit que sur la première composante fragilise en descendant le long de la chaîne. En un certain sens, la régularité supplémentaire sur F apparaît comme le prix à payer pour rééquilibrer les singularités croissantes dans le temps qui apparaissent le long de la chaîne.

Enfin, il faut imposer quelques limitations aux valeurs possibles de l'indice de stabilité α en $(0,2)$ et de celui de Hölder régularité β en $(0,1)$. particulier,

[P] $\alpha + \beta < 2$ et si $\alpha < 1$, alors il est vrai que

$$\beta < \alpha, \quad \alpha + \beta > 1, \quad 1 - \alpha < \frac{\alpha - \beta}{1 + \alpha(n-1)}.$$

Quelques considérations sur ces limitations dans le cas super-critique $(\alpha < 1)$ sont maintenant nécessaires. La condition $\beta < \alpha$ reflète essentiellement la propriété d'intégrabilité faible (strictement inférieure à α) pour un processus α-stable, éventuellement non-isotrope.
La condition $\alpha + \beta > 1$ semble naturellement donner un sens ponctuel au gradient de la solution u par rapport à la variable non-dégénérée x_1. A cet égard, nous citons également les travaux de Tanaka *et al.* [TTW74] où il est montré que le caractère bien posé au sens faible, propriété strictement inter-connectée avec les estimations de Schauder en cause ici, peut échoue déjà pour une dynamique stochastique non-dégénérée avec un bruit additif stable sur $\mathbb{R}$ lorsque $\alpha + \beta < 1$, où α est l'indice de stabilité du processus et β la régularité de Hölder pour la dérive déterministe. Ce contre-exemple peut être compris

comme une généralisation stochastique de l'exemple de Peano, montré dans (2.5).
Cette dernière hypothèse est une limitation technique et semble être essentiellement associée à notre méthode de prouve perturbative.
Enfin, nous soulignons que dans le cas sous-critique, lorsque $\alpha \geq 1$, ces conditions sont toujours vérifiées.

Compte tenu de la régularité faible assumée par les coefficients, il n'est possible de considérer la Dynamique (4.1) que dans un sens distributionnel. En effet, la régularité attendue (à travers le *bootstrap* parabolique en espace donné par les estimations de Schauder) pour une solution u au sens classique du problème est dans l'espace $L^\infty(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^N))$, pas suffisant pour donner un sens ponctuel au gradient $D_x u$ par rapport aux variables dégénérées et donc donner un sens classique à l'équation.
Au lieu de cela, nous exploiterons ici deux autres notions de solution possibles, plus adaptées à notre propos. Comme mentionné précédemment, par solution faible de l'équation (4.1) on entendra essentiellement une solution au sens des distributions, c'est-à-dire une fonction $u\colon [0,T]\times\mathbb{R}^{nd}\to\mathbb{R}$ tel que pour chaque fonction test ϕ (fonction lisse à support compact) sur $(0,T]\times\mathbb{R}^{nd}\to\mathbb{R}$, il retient que

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^{nd}}\Big(-\partial_t+(L_t)^*\Big)\phi(t,y)u(t,y)\,dy+\int_{\mathbb{R}^{nd}}u_T(y)\phi(T,y)\,dy \\ = -\int_0^T\int_{\mathbb{R}^{nd}}\phi(t,y)f(t,y)\,dy,\end{aligned}$$

où $(L_t)^*$ représente (formellement) l'opérateur ajouté de L_t donné par

$$L_t \;=\; \langle Ax+F(t,x),D_x\rangle+\mathcal{L}_\alpha \quad \text{su} \quad \mathbb{R}^{nd}.$$

Au lieu de cela, une solution *mild* du problème (4.1) (au sens de Stroock et Varadhan [SV79]) sera une fonction $u\colon [0,T]\times\mathbb{R}^{nd}\to\mathbb{R}$ obtenu comme limite dans un espace fonctionnel adéquat de la séquence de solutions classiques pour des versions régularisées du problème de Cauchy considéré. Pour plus de détails, voir Definition 2.2 au Chapitre 2 ou le livre [Kol11].

Les principaux résultats du Chapitre 2 peuvent maintenant être résumés dans le théorème suivant :

Théorème 4.1. *Sous les hypothèses décrites ci-dessus, soit f dans $L^\infty(0,T;C^{\beta}_{b,d}(\mathbb{R}^{nd}))$ et u_T dans $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})$. Il existe une unique solution mild et faible $u\colon [0,T]\times\mathbb{R}^{nd}\to\mathbb{R}$ du problème de Cauchy* (4.1). *De plus, u appartient à $L^\infty(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd}))$ et il existe une constante C, indépendant de f et u_T, tel que*

$$\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \;\leq\; C\Big[\|f\|_{L^\infty(C^{\beta}_{b,d})}+\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big]. \tag{4.11}$$

En conclusion de cette section, nous soulignons enfin qu'il est possible, avec de petites modifications aux arguments exposés ci-dessous, de considérer une dynamique avec une dérive déterministe totalement non-linéaire ou un coefficient de diffusion inhomogène dans l'espace-temps dans un espace fonctionnel adéquat, comme expliqué à la fin du Chapitre 2.

4.1 Guide à l'épreuve

Nous présentons ici brièvement la méthode de l'épreuve que nous avons utilisée pour déterminer les estimations de Schauder (4.11) sous les hypothèses introduites dans la section précédente.

Les principales difficultés à rencontrer dans notre contexte seront liées à la dégénérescence de l'opérateur $\mathcal{L}_\alpha$ α-stable qui n'agit que sur la première composante et à la non-limitation de la perturbation F. On se souvient aussi qu'on veut obtenir des estimations de Schauder sous les régularités minimales de Hölder pour les coefficients de l'équation. En particulier, on ne pourra pas se fier aux dérivées le long des composantes dégénérées. Pour ce faire, il faudra exploiter les propriétés de dualité entre les espaces de Hölder et ceux de Besov et notamment, établir des contrôles délicats en accord avec ces derniers.

Notre approche est basée sur une méthode perturbative dite *technique de la parametrix*, initialement introduite par Levi [Lev07] pour l'analyse d'équations aux dérivées partielles linéaires elliptiques d'ordre pair à coefficients variables. Dans le domaine diffusif non-dégénéré, nous citons plutôt les travaux de Friedman [Fri64] et de McKean et Singer [MS67] qui exploitent cette technique pour obtenir des estimations de type Aronson (cf. [Aro59, Aro67]) sur la solution fondamentale du système, respectivement sous les conditions de Hölder régularité dans l'espace et le temps ou de mesurabilité dans le temps et Hölder continuité dans l'espace, des coefficients.
Dans un contexte dégénéré plus proche du présent mais toujours de type Gaussien, cette méthode a ensuite été exploitée dans [CdRHM18a] pour obtenir des estimations de Schauder pour une chaîne diffusive dégénérée.
Enfin, nous rappelons que Hadamard dans [Had32, Had64] a étendu cette technique également à l'analyse des équations hyperboliques.
Une autre méthode, plus classique, pour obtenir des estimations de Schauder est de procéder en exploitant des contrôles a priori sur la solution fondamentale associée au problème. L'existence et l'unicité d'une solution pour l'équation considérée, dans un espace fonctionnel convenable, ne sont dans ce cas abordées qu'à un moment ultérieur. Nous renvoyons à [Fri64] et à [Kry96] pour une présentation claire de cette approche a priori ou à [KP10] pour une extension au cas diffusif non-dégénéré à coefficients non-bornés.

L'opérateur proxy congelé

L'élément crucial dans la méthode de la parametrix que nous avons considérée est de choisir un opérateur *proxy* adapté à l'équation d'intérêt, c'est-à-dire, un opérateur $\tilde{L}_t$ dont les propriétés (existence et comportement de la densité ou du semi-groupe de Markov associé) sont connus et proche, dans un certain sens, de l'opérateur original L_t. On peut alors appliquer un développement du premier ordre, comme une formule de type Duhamel, pour la solution de l'équation (4.1) autour de l'opérateur proxy choisi. En tirant parti des propriétés connues de l'opérateur proxy, nous pourrons enfin obtenir un contrôle convenable de l'erreur d'expansion.

Dans le cas de coefficients bornés, un choix commun pour le proxy est donné par l'opérateur d'origine avec des coefficients constants. Lorsque l'on considère une perturbation F potentiellement illimitée comme dans notre cas, il est plutôt naturel (cf. [KP10, CdRMP20a]) utilisent un opérateur faisant intervenir un terme du premier ordre non-nul, tel que celui associé au flux déterminé par le terme de transport $Ax + F$:

$$\begin{cases} d\theta_{\tau,s}(\xi) \;=\; \Big[A\theta_{\tau,s}(\xi) + F(s,\theta_{\tau,s}(\xi))\Big]ds \quad \text{on } [\tau,T]; \\ \theta_{\tau,\tau}(\xi) \;=\; \xi, \end{cases} \tag{4.12}$$

où (τ,ξ) dans $[0,T]\times\mathbb{R}^N$ sont deux paramètres "congelé", respectivement dans le temps et dans l'espace, dont la valeur exacte sera choisie ultérieurement.
Nous notons immédiatement cependant que la solution de la dynamique ci-dessus peut ne pas être unique, puisque la dérive F n'est que Hölder continue dans l'espace. Pour cette raison, nous choisirons un flux particulier, noté $\theta_{\tau,s}(\xi)$, et nous le considérerons fixé à partir de maintenant. L'opérateur proxy que nous avons choisi est alors obtenu en gelant le L_t d'origine le long du flux $\theta_{\tau,t}(\xi)$ fixé :

$$\tilde{L}^{\tau,\xi}_t \;=\; \mathcal{L}_\alpha + \langle Ax + F(t,\theta_{\tau,t}(\xi)), D_x\rangle.$$

On peut maintenant introduire le problème de Cauchy "congelé" associé au proxy choisi :

$$\begin{cases} \Big(\partial_t + \tilde{L}^{\tau,\xi}_t\Big)\,\tilde{u}^{\tau,\xi}(t,x) \;=\; -f(t,x) & \text{sur } [0,T)\times\mathbb{R}^{nd}; \\ \tilde{u}^{\tau,\xi}(T,x) \;=\; u_T(x) & \text{sur } \mathbb{R}^{nd}. \end{cases} \tag{4.13}$$

Pour déterminer les propriétés de l'opérateur proxy gelé $\tilde{L}^{\tau,\xi}$, nous introduisons maintenant la dynamique stochastique associée. Pour un point de départ (t,x) dans $[0,T]\times\mathbb{R}^{nd}$ et d'un processus $\{Z_s\}_{s\geq t}$ α-stable sur $\mathbb{R}^d$ avec mesure de Lévy donnée par ν_α fixés, on s'intéresse à

$$\begin{cases} \tilde{X}^{\tau,\xi}_s \;=\; \Big[A\tilde{X}^{\tau,\xi}_s + F(s,\theta_{\tau,s}(\xi))\Big]\,ds + BdZ_s; \quad s>t \\ \tilde{X}^{\tau,\xi}_t \;=\; x. \end{cases}$$

Une intégration explicite via la fonction exponentielle matricielle permet alors de réécrire la dynamique de la manière suivante :

$$\begin{aligned} \tilde{X}^{\tau,\xi}_s \;&=\; e^{A(s-t)}x + \int_t^s e^{A(s-u)}F(u,\theta_{\tau,u}(\xi))\,du + \int_t^s e^{A(s-u)}BdZ_u \\ &=:\; \tilde{m}^{\tau,\xi}_{t,s}(x) + \int_t^s e^{A(s-u)}BdZ_u \end{aligned} \tag{4.14}$$

Grâce à la symétrie de Z_t, on peut considérer le terme de transport $\tilde{m}^{\tau,\xi}_{t,s}(x)$ comme la "moyenne" du processus congelé $\tilde{X}^{\tau,\xi}_s$ affine au point de départ x (bien que lorsque $\alpha<1$, la valeur moyenne de ce processus ne soit pas définie) ou au moins comme le longue valeur que le processus fluctue.
L'identité ci-dessus est maintenant cruciale pour montrer que la convolution stochastique

$$\Lambda_{t,s} \;:=\; \int_t^s e^{A(s-v)}BdZ_v$$

est elle-même un processus α-stable, symétrique et non dégénéré sur le " gros " espace $\mathbb{R}^{nd}$ mais re-dimensionné selon la structure anisotrope du système. En effet, en passant sur les espaces de Fourier, on note que

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(i\langle p,\Lambda_{t,s}\rangle\Big)\Big] &= \exp\Big(-\int_t^s\int_{\mathbb{S}^{d-1}}\Big|\langle p,e^{(s-u)A}B\varsigma\rangle\Big|^\alpha\,\mu(d\varsigma)du\Big)\\ &=: \exp\Big(\Phi_{\Lambda_{t,s}}(p)\Big),\end{aligned}\tag{4.15}$$

ou alors $\Phi_{\Lambda_{t,s}}$ est le symbole de Lévy associé à la variable aléatoire $\Lambda_{t,s}$ à chaque instant fixe $t<s$. Changer la variable $v=(s-u)/(s-t)$ nous permet de la réécrire comme suit

$$\begin{aligned}\Phi_{\Lambda_{t,s}}(p) &= (t-s)\int_0^1\int_{\mathbb{S}^{d-1}}|\langle p,e^{(s-t)vA}B\varsigma\rangle|^\alpha\,\mu(d\varsigma)dv\\ &(t-s)\int_0^1\int_{\mathbb{S}^{d-1}}|\langle \mathbb{M}_{s-t}p,e^{vA}B\varsigma\rangle|^\alpha\,\mu(d\varsigma)dv,\end{aligned}$$

exploitant dans la dernière étape la décomposition de $e^{A(s-t)v}$ donnée dans (4.7) :

$$e^{(s-t)vA}B \;=\; \mathbb{M}_{s-t}e^{vA}B,$$

où la matrice d'échelle $\mathbb{M}_{s-t}$ a été définie dans (4.8). De plus, nous pouvons maintenant re-normaliser le terme dans le produit scalaire :

$$\begin{aligned}\Phi_{\Lambda_{t,s}}(p) &= (t-s)\int_0^1\int_{\mathbb{S}^{d-1}}|\langle \mathbb{M}_{s-t}p,\frac{e^{vA}B\varsigma}{|e^{vA}B\varsigma|}\rangle|^\alpha\,|e^{vA}B\varsigma|^\alpha\mu(d\varsigma)dv\\ &=: (t-s)\int_{[0,1]\times\mathbb{S}^{d-1}}|\langle \mathbb{M}_{s-t}p,l(v,\varsigma)\rangle|^\alpha\,m_\alpha(dv,d\varsigma)\end{aligned}$$

où $m_\alpha(dv,d\varsigma):=|e^{vA}B\varsigma|^\alpha\mu(d\varsigma)dv$ est une mesure produit sur $[0,1]\times\mathbb{S}^{d-1}$ et $l\colon[0,1]\times\mathbb{S}^{d-1}\to\mathbb{S}^{nd-1}$ est la fonction *lift* définie par

$$l(v,\varsigma) \;:=\; \frac{e^{vA}B\varsigma}{|e^{vA}B\varsigma|}.$$

En particulier, on définit avec $\mu_S:=\mathrm{Sym}(l_*(m_\alpha))$ la version symétrisée de la mesure image de m_α par l. Donc :

$$\mathbb{E}\Big[\exp\Big(i\langle p,\Lambda_{t,s}\rangle\Big)\Big] = \exp\Big(-(s-t)\int_{\mathbb{S}^{d-1}}|\langle\mathbb{M}_{s-t}p,\eta\rangle|^\alpha\,\mu_S(d\eta)\Big).\tag{4.16}$$

Si on note maintenant $\{S_u\}_{u\geq0}$ le processus de Lévy sur $\mathbb{R}^{nd}$ dont le symbole de Lévy Φ_S est donné par

$$\Phi_S(p) \;=\; -\int_{S^{nd-1}}|\langle p,\eta\rangle|^\alpha\mu_S(d\eta),$$

on obtient enfin l'identité juridique suivante :

$$\Lambda_{s,t} \overset{(\text{loi})}{=} \mathbb{M}_{s-t}S_t.\tag{4.17}$$

A ce stade, nous pensons qu'il faut souligner que, même si la mesure spectrale μ_S est en réalité non-dégénérée au sens de [**ND**], elle est extrêmement singulière par rapport à la mesure de Lebesgue sur la sphère $\mathbb{S}^{nd-1}$. En fait, on constate d'après la construction de la mesure μ_S que son support est donné par l'image support de $|e^{vA}B\varsigma|^\alpha\mu(d\varsigma)dv$ à travers la fonction lift $(v,\varsigma)\mapsto e^{vA}B\varsigma/|e^{vA}B\varsigma|$ sur $\mathbb{S}^{nd-1}$. En supposant aussi que le support de la mesure spectrale μ associée au processus $\{Z_t\}_{t\geq 0}$ est la sphère $\mathbb{S}^{d-1}$, on note que le dimension du support de μ_S ne sera que $d-1+1=d$.

Sous la condition de non-dégénéescence [**ND**], on sait alors que le processus $\{S_t\}_{t\geq 0}$ admet une densité régulière $p_S(t,z)$. Les équations (4.14) et (4.17) impliquent maintenant l'existence d'une densité $\tilde{p}^{\tau,\xi}$ associée à l'opérateur congelé $\tilde{L}^{\tau,\xi}_t$ donné par

$$\tilde{p}^{\tau,\xi}(t,s,x,y) \;=\; \frac{1}{\det(\mathbb{M}_{s-t})}p_S\Big(s-t,\mathbb{M}^{-1}_{s-t}(y-\tilde{m}^{\tau,\xi}_{t,s}(x))\Big). \tag{4.18}$$

En particulier, les estimations sur la densité du processus congelé $\tilde{X}^{\tau,\xi}_s$ seront obtenues à partir de celles sur la densité de $\{S_u\}_{u\geq 0}$. Pour être complet, nous introduisons également le semi-groupe de Markov congelé $\Big\{\tilde{P}^{\tau,\xi}_{t,s}\Big\}_{t\leq s}$ donné par

$$\tilde{P}^{\tau,\xi}_{t,s}\phi(x) \;:=\; \int_{\mathbb{R}^{nd}}\tilde{p}^{\tau,\xi}(t,s,x,y)\phi(y)\,dy,$$

pour chaque fonction $\phi\colon\mathbb{R}^{nd}\to\mathbb{R}$ suffisamment régulière. Nous voulons maintenant analyser quelles propriétés de régularité cette densité gelée $\tilde{p}^{\tau,\xi}(t,s,x,\cdot)$ possède. Notons tout d'abord que les dérivées spatiales peuvent être contrôlées par une autre densité mais au prix de singularités temporelles supplémentaires. Pour chaque i dans $[\![1,n]\!]$ et $k\in\{1,2\}$, en fait c'est vrai que

$$\Big|D^k_{x_i}\tilde{p}^{\tau,\xi}(t,s,x,y)\Big| \;\leq\; C\frac{\bar{p}\Big(s-t,\mathbb{M}^{-1}_{s-t}(y-\tilde{m}^{\tau,\xi}_{t,s}(x))\Big)}{\det(\mathbb{M}_{s-t})}(s-t)^{-k\frac{1+\alpha(i-1)}{\alpha}}. \tag{4.19}$$

où $\bar{p}(t,\cdot)$ est une densité sur $\mathbb{R}^{nd}$ avec des propriétés convenables. En termes simples, dériver la densité congelé $\tilde{p}^{\tau,\xi}(t,s,x,\cdot)$ induit une singularité en temps dont l'intensité dépend des échelles intrinsèques du système liées à la direction de la dérivation. L'élément crucial et récurrent dans notre analyse consistera à rééquilibrer ces singularités dans le temps à travers les régularités spatiales de la densité $\bar{p}(t,\cdot)$.
En particulier, nous montrerons qu'il a un effet régularisant l'espace d'ordre α, en ce sens que, pour chaque γ dans $[0,\alpha)$, il tient que

$$\int_{\mathbb{R}^{nd}}\frac{\bar{p}\Big(s-t,\mathbb{M}^{-1}_{s-t}(y-\tilde{m}^{\tau,\xi}_{t,s}(x))\Big)}{\det(\mathbb{M}_{s-t})}\mathbf{d}^\gamma(y,\tilde{m}^{\tau,\xi}_{t,s}(x))\,dy \;\leq\; C(s-t)^{\frac{\gamma}{\alpha}}, \tag{4.20}$$

où, rappelons-le, la distance homogène $\mathbf{d}$ a été définie dans (3.22) dans le cas $n=2$ puis naturellement étendue à notre contexte plus général dans (4.9).
On peut maintenant mettre en évidence une des principales différences entre le cas diffusif, considéré par exemple dans [CdRHM18a], et le cas α-stable analysé ici, caractérisé par un effet régularisant strictement borné par l'ordre de α-stabilité de l'opérateur. En termes stochastiques, ce problème est essentiellement lié aux propriétés d'intégrabilité plus faibles associées aux processus de Lévy par rapport au mouvement Brownien.
Les propriétés présentées dans (4.19) et (4.20) peuvent être enfin résumées par l'effet

régularisant associé à l'opérateur proxy $\tilde{L}_t^{\tau,\xi}$ dans le contrôle fondamental suivant sur son semi-groupe congelé :

$$\left|D_{x_i}^k \tilde{P}_{t,s}^{\tau,\xi}\phi(x)\right| \le C\|\phi\|_{C_d^\gamma}(s-t)^{\frac{\gamma}{\alpha}-k\frac{1+\alpha(i-1)}{\alpha}}, \quad \forall \phi \in C_d^\gamma(\mathbb{R}^{nd}). \tag{4.21}$$

Les contrôles tels que (4.21) sont obtenus à partir des propriétés régulatrices de la densité (4.19), en utilisant des techniques d'annulation. En notant que

$$\int_{\mathbb{R}^{nd}} D_{x_i}\tilde{p}^{\tau,\xi}(t,s,x,y)\,dy \;=\; 0, \tag{4.22}$$

pour chaque couple de paramètres gelés (τ,ξ), l'idée sous-jacente est d'ajouter un terme à l'intérieur de l'intégrale et ainsi d'exploiter la régularité de Hölder de la fonction ϕ, soit :

$$\begin{aligned}
\left|D_{x_i}^k \tilde{P}_{t,s}^{\tau,\xi}\phi(x)\right| &= \int_{\mathbb{R}^{nd}} D_{x_i}^k\tilde{p}^{\tau,\xi}(t,s,x,y)\phi(y)\,dy \\
&= \int_{\mathbb{R}^{nd}} D_{x_i}^k\tilde{p}^{\tau,\xi}(t,s,x,y)\left[\phi(y)-\phi(\tilde{m}_{t,s}^{\tau,\xi}(x))\right]\,dy \\
&\le \|\phi\|_{C_d^\gamma}\int_{\mathbb{R}^{nd}} \left|D_{x_i}^k\tilde{p}^{\tau,\xi}(t,s,x,y)\right| d^\gamma(y-\tilde{m}_{t,s}^{\tau,\xi}(x))\,dy \\
&\le C\|\phi\|_{C_d^\gamma}(s-t)^{\frac{\gamma}{\alpha}-k\frac{1+\alpha(i-1)}{\alpha}}.
\end{aligned} \tag{4.23}$$

En exploitant soigneusement les effets de régularisation dans l'espace afin d'équilibrer les singularités dans le temps dues aux dérivées de densité (cf. Control (4.19)), il est maintenant possible de montrer que les estimations de Schauder sont vraies pour les solution $\tilde{u}^{\tau,\xi}$ de la dynamique congelé (4.13) :

$$\|\tilde{u}^{\tau,\xi}\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \;\le\; C\Big[\|f\|_{L^\infty(C_{b,d}^{\beta})} + \|u_T\|_{C_{b,d}^{\alpha+\beta}}\Big]. \tag{4.24}$$

En particulier, ces estimations sont valables pour toute valeur *fixe* des paramètres (τ,ξ) gel et la constante C ci-dessus est indépendante d'eux.
Grâce à la stabilité donnée par les contrôles ci-dessus, il est également possible de représenter l'unique solution $\tilde{u}^{\tau,\xi}$ du problème de Cauchy congelé (4.13) en termes de semi-groupe Markovien $\tilde{P}_{t,s}^{\tau,\xi}$:

$$\tilde{u}^{\tau,\xi}(t,x) \;=\; \tilde{P}_{t,T}^{\tau,\xi}u_T(x) + \int_t^T \tilde{P}_{t,s}^{\tau,\xi}f(s,x)\,ds. \tag{4.25}$$

On veut maintenant appliquer la méthode perturbative de manière à obtenir des estimations de Schauder pour une solution u de l'équation de départ à partir de celles qui viennent d'être obtenues dans (4.24) pour la solution $\tilde{u}^{\tau,\xi}$ du problème gelé associé à l'opérateur proxy.

Expansion de type Duhamel

Au moins formellement (en pratique, par régularisation des coefficients à l'aide de la définition de solution mild), la dynamique d'origine (4.1) peut être réécrite autour de l'opérateur $\tilde{L}_\alpha^{\tau,\xi}$ congelé comme suit :

$$\begin{cases} \left(\partial_t + \tilde{L}_t^{\tau,\xi}\right) u(t,x) \,=\, -f(t,x) - \left(L_t - \tilde{L}_t^{\tau,\xi}\right) u(t,x), & \text{on } (0,T)\times\mathbb{R}^{nd}; \\ u(T,x) \,=\, u_T(x) & \text{on } \mathbb{R}^{nd}. \end{cases}$$

L'unicité de solution pour le problème congelé (4.13) et la représentation de $\tilde{u}^{\tau,\xi}$ dans (4.25) implique maintenant la formule de Duhamel suivante, qui correspond à un développement du parametrix de premier ordre :

$$u(t,x) \,=\, \tilde{u}^{\tau,\xi}(t,x) + \int_t^T \tilde{P}_{t,s}^{\tau,\xi} R^{\tau,\xi}(s,x)\, ds, \tag{4.26}$$

où $R^{\tau,\xi}$ est le reste donné par

$$R^{\tau,\xi}(t,x) \,:=\, \langle F(t,x) - F(t,\theta_{\tau,t}(\xi)), D_x u(t,x)\rangle. \tag{4.27}$$

Puisque le contrôle convenable (4.24) pour la solution congelée $\tilde{u}^{\tau,\xi}$ a déjà été démontré, la représentation Duhamel (4.26) indique que pour obtenir les estimations de Schauder pour u, le terme principal qui reste à étudier est le reste

$$\int_t^T \tilde{P}_{t,s}^{\tau,\xi} R^{\tau,\xi}(s,x)\, ds, \tag{4.28}$$

qui représente précisément l'erreur d'expansion le long du proxy.

Jusqu'à présent, les paramètres congelés (τ,ξ) ont été considérés comme fixes mais libres. Ils seront désormais choisis de manière convenable en fonction du contrôle que vous souhaitez établir. En particulier, dans cet article, nous suivrons une approche du paramètre de type *progressif*, dans le sens où nous imposerons $(\tau,\xi) = (t,x)$ et donc le flux $\theta_{\tau,s}(\xi)$ donné dans (1.5) se déplacera progressivement du point de départ (t,x) vers (s,y), où y est la variable d'intégration dans la densité congelée. Cette méthode a été largement utilisée par Friedman [Fri64] et Il'in et al. [IKO62] pour obtenir des contrôles précis sur les dérivées de la solution fondamentale pour l'équation de la chaleur ou par Chaudru de Reynal dans [CdR17] pour en déduire le caractère bien posé au sens fort pour une chaîne diffusive dégénérée de type cinétique. En particulier, la méthode progressive permet de mieux exploiter les techniques d'annullation dans (4.23) que nous avons vues comme fondamentales pour le contrôle des dérivées de la densité gelée $\tilde{p}^{\tau,\ xi}(t,s,x,y)$.
Pour être complet, nous mentionnons qu'il existe également une méthode parametrix de type *retrograde* (en fixant $(\tau,\xi) = (s,y)$), introduite par McKean et Singer dans [MS67]. Nous soulignons cependant que dans ce cas, la densité $\tilde{p}^{\tau,\xi}(t,s,x,y)$ congelée dans $\xi = y$ n'est plus une vraie densité par rapport à la variable y, puisque le paramètre de gel joue également le rôle de variable d'intégration. Cela rend les effets de régularisation présentés dans (4.21) pour le semi-groupe de Markov beaucoup plus difficiles à établir. Pour plus de détails sur la méthode perturbative à parametrix rétrograde, nous nous référons à la Section 6 du présent travail.
Enfin, rappelons que dans le cas de coefficients bornés, un choix naturel est donné par le flux trivial $\theta_{\tau,s}(\xi) = \xi$.

Control de l'erreur d'expansion

Pour conclure la méthode perturbative, il faut enfin montrer que l'erreur d'expansion dans la formule de Duhamel apporte une petite contribution aux estimations de Schauder (4.24) pour la solution congelée $\tilde{u}^{\tau,\xi}$. Comme mentionné au début de cette section, ce contrôle sera le plus délicat à établir, en raison de la faible régularité de la dérive F le long des composantes dégénérées x_i $(i > 1)$. Pour montrer brièvement quelles sont réellement les difficultés, nous allons nous concentrer sur l'établissement d'un contrôle de norme uniforme sur le terme dans (4.28). Les vérifications ultérieures de la norme uniforme pour la dérivée et celles par rapport à la semi-norme de Hölder sont encore plus longues à établir bien qu'elles partagent toujours la même approche.
Commençons par décomposer le reste par rapport à les composants du système :

$$\Big|\int_t^T \tilde{P}^{\tau,\xi}_{t,s} R^{\tau,\xi}(s,x)\,ds\Big| = \Big|\sum_{j=1}^n \int_t^T \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\cdot D_{y_j}u(s,y)\,dyds\Big|, \tag{4.29}$$

où nous avons appelé, pour simplicité

$$\Delta^{\tau,\xi}F_j(s,y) := F_j(s,y) - F_j(s,\theta_{\tau,s}(\xi)), \quad j \in [\![1,n]\!].$$

En rappelant que la solution u est dérivable par rapport à la composante non-dégénérée $(j = 1)$, la première contribution de la sommation peut être contrôlée en exploitant à nouveau les effets régularisants de la densité congelée dans (4.19) :

$$\begin{aligned}
\Big|\int_t^T \int_{\mathbb{R}^{nd}} &\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_1(s,y)\cdot D_{y_1}u(s,y)\,dyds\Big| \\
&\le C\|D_{y_1}u\|_\infty \|F_1\|_{C^\beta_d} \int_t^T \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t,s,x,y) d^\beta\Big(y,\theta_{\tau,s}(\xi)\Big)\,dyds \\
&\le C\|D_{y_1}u\|_\infty \|F_1\|_{C^\beta_d} \int_t^T (s-t)^{\frac{\beta}{\alpha}}\,ds \\
&\le C\|D_{y_1}u\|_\infty \|F_1\|_{C^\beta_d} (T-t)^{\frac{\alpha+\beta}{\alpha}}.
\end{aligned}$$

En particulier, on note qu'imposer $(\tau,\xi) = (t,x)$ est le choix naturel pour équilibrer la différence entre les tendances déterministes

$$|F_1(s,y) - F_1(s,\theta_{\tau,s}(\xi))|$$

et la structure anisotrope de la densité congelée :

$$\frac{1}{\det(\mathbb{M}_{s-t})}\bar{p}\Big(s-t, \mathbb{M}^{-1}_{s-t}(y - \tilde{m}^{\tau,\xi}_{t,s}(x))\Big)$$

et pouvoir ainsi exploiter les propriétés régularisantes multi-échelles des densités congelées dans (4.20). En effet, il est facile de vérifier à partir des dynamiques (4.12) et (4.14) que pour notre choix de paramètres gelés il tient que $\tilde{m}^{t,x}_{t,s}(x) = \theta_{t,s}(x)$.

Nous pouvons maintenant nous concentrer sur les contributions dégénérées $(j > 1)$ dans la décomposition en (4.29). Puisque u n'est pas dérivable en y_j si $j > 1$ (en fait ce n'est que $(\alpha+\beta)/(1+\alpha(j-1))$-Hölder continue dans cette variable), nous commençons par déplacer la dérivée vers les autres termes par intégration par parties :

$$\Big|\int_t^T\int_{\mathbb{R}^{nd}} D_{y_j}\cdot\Big\{\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\Big\}u(s,y)\,dyds\Big|. \tag{4.30}$$

L'idée serait alors de contrôler $D_{y_j}\cdot\Big\{\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\Big\}$ en exploitant la régularité de la solution u dans $L^\infty(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd}))$. A priori, cependant, cette contrôle ne semble pas immédiate puisque le terme $\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)$ n'est pas dérivable en y_j en raison de la faible régularité de la dérive F.
Pour l'obtenir, il faudra en effet appliquer un raisonnement de dualité sur les espaces de Besov. En fait, rappelez-vous que pour un $\tilde{\gamma}$ générique dans $\mathbb{R}$, l'identification suivante est vraie

$$C^{\tilde{\gamma}}_b(\mathbb{R}^d) \;=\; B^{\tilde{\gamma}}_{\infty,\infty}(\mathbb{R}^d), \tag{4.31}$$

où pour p,q dans $[1,\infty]$, $B^{\tilde{\gamma}}_{p,q}(\mathbb{R}^{nd})$ représente un espace de Besov dans $\mathbb{R}^{nd}$ avec indexes $(\tilde{\gamma},p,q)$. De plus, il est connu (voir par exemple la Proposition 3.6 dans [LR02]) que $B^{\tilde{\gamma}}_{\infty,\infty}(\mathbb{R}^d)$ et $B^{-\tilde{\gamma}}_{1,1}(\mathbb{R}^d)$ sont en dualité, dans le sens où

$$\Big|\int_{\mathbb{R}^d} fg\,dx\Big| \;\le\; C\|f\|_{B^{\tilde{\gamma}}_{\infty,\infty}}\|g\|_{B^{-\tilde{\gamma}}_{1,1}}, \tag{4.32}$$

pour chaque f dans $B^{\tilde{\gamma}}_{\infty,\infty}(\mathbb{R}^d)$ et chaque g dans $B^{-\tilde{\gamma}}_{1,1}(\mathbb{R}^d)$.
Nous rappelons également qu'il existe différentes manières de définir de tels espaces de Besov (par module de continuité, décomposition de Littlewood-Paley, etc.) mais la caractérisation thermique, par convolution avec un noyau thermique fractionnaire, semble être la plus naturelle pour notre propos. Pour une analyse plus détaillée sur le sujet, nous vous suggérons de consulter la Section 2.6.4 dans [Tri92].
On définit alors l'espace de Besov avec les indices $(\tilde{\gamma},p,q)$ sur $\mathbb{R}^d$ comme :

$$B^{\tilde{\gamma}}_{p,q}(\mathbb{R}^d) := \{f\in\mathcal{S}'(\mathbb{R}^d)\colon \|f\|_{\mathcal{H}^{\tilde{\gamma}}_{p,q}} < +\infty\},$$

où $\mathcal{S}(\mathbb{R}^d)$ est la classe de Schwartz sur $\mathbb{R}^d$. La norme $\|\cdot\|_{\mathcal{H}^{\tilde{\gamma}}_{p,q}}$ est alors donnée par

$$\|f\|_{\mathcal{H}^{\tilde{\gamma}}_{p,q}} \;:=\; \|(\phi_0\hat{f})^\vee\|_{L^p} + \Big(\int_0^1 v^{(1-\frac{\tilde{\gamma}}{\alpha})q}\|\partial_v p_h(v,\cdot)*f\|^q_{L^p}\frac{dv}{v}\Big)^{\frac{1}{q}}, \tag{4.33}$$

où ϕ_0 est une fonction test sur $C^\infty_c(\mathbb{R}^d)$ telle que $\phi_0(0)\neq 0$ et p_h est le noyau de chaleur α -stable isotrope sur $\mathbb{R}^d$, c'est-à-dire la densité sur $\mathbb{R}^d$ dont le symbole de Lévy est équivalent à $-|\lambda|^\alpha$.
Le principal avantage de la caractérisation thermique est justement celui de permettre le passage de la dérivée D_{y_j} sur la densité isotrope supplémentaire $p_h(s-t,y)$ et donc d'exploiter la régularité de Hölder de la dérive F_j.

On peut alors exploiter l'identification dans (4.31) et la propriété de dualité dans (4.32) le long de la composante y_j pour estimer la quantité (4.30) :

$$\begin{aligned}&\Big|\int_t^T\int_{\mathbb{R}^{nd}} D_{y_j}\cdot\Big\{\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\Big\}u(s,y)\,dyds\Big|\\ &\le\; \|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\int_t^T\int_{\mathbb{R}^{(n-1)d}}\Big\|y_j\to D_{y_j}\cdot\Big\{\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\Big\}\Big\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}dy_{\smallsetminus j}ds,\end{aligned}$$

où nous avons noté, pour simplifier, par $y_{\smallsetminus j}$ la variable dans $\mathbb{R}^{d(n-1)}$ sans la composante y_j.
A ce stade, il resterait alors à contrôler correctement l'intégrale de la norme de Besov ci-dessus. En particulier, dans la Section 5.1 du Chapitre 2 nous montrerons l'estimation optimale suivante :

$$\begin{aligned}\int_{\mathbb{R}^{(n-1)d}} \Big\| y_j \to D_{y_j} \cdot \Big\{ \tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\cdot)\Delta^{\tau,\xi}F_j(s,y)\Big\}\Big\|_{B^{-(\alpha_j+\beta_j)}_{1,1}} dy_{\smallsetminus j} \\ \le C\|F_j\|_{L^\infty(C^{\gamma_j+\beta}_d)}(s-t)^{\frac{\beta}{\alpha}},\end{aligned}$$

où, nous nous souvenons, nous avons choisi $(\tau,\xi)=(t,x)$.

Sujet circulaire et conclusion du test

En exploitant les différents contrôles résumés ci-dessus, nous pourrons enfin montrer, à partir de la formule de Duhamel dans (4.26), l'estimation suivante pour une solution u dans $L^\infty(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd}))$ de la dynamique d'origine (4.1) :

$$\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \le C\Big[\|u_T\|_{C^{\alpha+\beta}_{b,d}} + \|f\|_{L^\infty(C^{\beta}_{b,d})}\Big] + C\sup_i \|F_i\|_{L^\infty(C^{\gamma_i+\beta}_d)}\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}, \quad (4.34)$$

où la constante C est indépendante de f, u_T et F.
Nous soulignons en particulier que l'étude de la norme complète associée nous fera perdre, comme dans (4.34), la dépendance à $(s-t)$ en peu de temps et ne nous permettra pas d'appliquer directement une argument de type circulaire pour déplacer le terme $\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}$ à gauche du contrôle ci-dessus.
Si on suppose alors que la norme de Hölder pour la dérive F est suffisamment petite, c'est-à-dire, par exemple, telle que

$$C\sup_i \|F_i\|_{L^\infty(C^{\gamma_i+\beta}_d)} \le \frac{1}{2}, \quad (4.35)$$

nous pouvons utiliser un argument circulaire pour conclure que les estimations de Schauder sont également valables pour u :

$$\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \le 2C\Big[\|u_T\|_{C^{\alpha+\beta}_{b,d}} + \|f\|_{L^\infty(C^{\beta}_{b,d})}\Big]. \quad (4.36)$$

Dans le cas général, il faudra appliquer au départ une procédure de remise à l'échelle des coefficients afin d'imposer une condition similaire à (4.35).

Enfin, nous soulignons que la procédure décrite dans cette section ne peut être appliquée efficacement que si l'intervalle de temps considéré est suffisamment petit. Intuitivement, cela semble naturel puisque l'erreur d'expansion à contrôler, sur laquelle est basée la méthode perturbative, nécessite que l'opérateur original $\mathcal{L}_\alpha$ et celui proxy $\mathcal{L}^{\tau,\xi}_\alpha$ ne sont pas trop éloignés les uns des autres. Pour obtenir ensuite les estimations de Schauder pour un temps final arbitraire mais fini, nous devrons alors itérer le raisonnement ci-dessus plusieurs fois sur chaque intervalle de temps suffisamment petit.

5 Estimations de Schauder pour un système dégénéré linéaire de type Lévy

Nous présentons maintenant brièvement les résultats présentés dans le chapitre 3, publié plus tard dans *Journal of Mathematical Analysis and Applications*. Bien que nous nous limitions ici à une dynamique linéaire, ce travail peut être compris comme une extension du chapitre précédent à plusieurs points de vue.
Étant donné un espace “grand” $\mathbb{R}^N$, nous nous intéressons à une analyse de l'opérateur d'Ornstein-Uhlenbeck suivant :

$$L^{\text{ou}} := \mathcal{L} + \langle Ax, D_x \rangle \quad \text{su } \mathbb{R}^N, \tag{5.1}$$

où $\langle \cdot, \cdot \rangle$ désigne le produit scalaire euclidien sur $\mathbb{R}^N$, A est une matrice dans $\mathbb{R}^N \otimes \mathbb{R}^N$ et $\mathcal{L}$ est un opérateur de Lévy éventuellement dégénéré, dans le sens où il ne pourrait agir de manière non-dégénérée que sur un sous-espace de $\mathbb{R}^N$.
Plus précisément, étant donné un entier $d \leq N$ et une matrice B dans $\mathbb{R}^N \otimes \mathbb{R}^d$ telle que $\text{rank}(B) = d$, l'opérateur $\mathcal{L}$ peut être représenté pour toute fonction assez régulier $\phi \colon \mathbb{R}^N \to \mathbb{R}$ à travers

$$\begin{aligned}\mathcal{L}\phi(x) := \frac{1}{2}\text{Tr}\Big(B\Sigma B^* D_x^2\phi(x)\Big) + \langle Bb, D_x\phi(x)\rangle \\ + \text{p.v.}\int_{\mathbb{R}_0^d}\Big[\phi(x+Bz) - \phi(x) - \langle D_x\phi(x), Bz\rangle \mathbb{1}_{B(0,1)}(z)\Big]\,\nu(dz),\end{aligned} \tag{5.2}$$

où b est un vecteur dans $\mathbb{R}^d$, Σ est une matrice symétrique, définie non-négative dans $\mathbb{R}^d \otimes \mathbb{R}^d$ et ν est une mesure de Lévy sur $\mathbb{R}_0^d$.
Nous soulignons déjà que pour ce modèle nous n'assumerons plus la *symétrie* de la mesure ν comme nous l'avons fait dans le chapitre 2. C'est pour cette raison même qu'on ne peut pas maintenant supprimer le terme de premier ordre $\langle D_x\phi(x), Bz\rangle$ dans la définition de l'opérateur $\mathcal{L}$ dans (5.2) (cf. Équation (4.2) dans la section précédente).

Nous nous intéressons ici à établir le caractère bien posé et les estimations de Schauder associées pour des équations elliptiques et paraboliques faisant intervenir l'opérateur L^{ou} sous des conditions *minimal* de régularité de Hölder sur les coefficients. En particulier, pour $\lambda > 0$ fixé, nous considérerons l'équation elliptique suivante :

$$\lambda u(x) - L^{\text{ou}}u(x) = g(x), \quad x \in \mathbb{R}^N, \tag{5.3}$$

et, pour un temps final fixe $T > 0$, le problème de Cauchy suivant :

$$\begin{cases}\partial_t u(t,x) = L^{\text{ou}}u(t,x) + f(t,x), & (t,x) \in (0,T) \times \mathbb{R}^N; \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N,\end{cases} \tag{5.4}$$

où f, g, u_0 sont des fonctions données. Puisque notre but est notamment d'établir des résultats de régularité optimale, nous allons fixer un $\beta \in (0,1)$ et supposer, dans le problème elliptique (1.5), que la source g appartient à l'espace de Hölder anisotrope $C^{\beta}_{b,d}(\mathbb{R}^N)$ tandis que pour le cas parabolique (5.4), que u_0 est dans $C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)$ et f in

$L^\infty(0,T;C^\beta_{b,d}(\mathbb{R}^N))$. Une définition exacte des espaces de Hölder anisotrope $C^\gamma_{b,d}(\mathbb{R}^N)$ dans ce contexte ne sera donnée que plus tard, lorsque nous introduirons les conditions sur le système.

Une autre différence par rapport au Chapitre 2 est que pour ce modèle linéaire, nous supposerons seulement que A et B vérifient une condition de type Hörmander faible, appelée du *rang de Kalman*, ce qui assure l'hypoellipticité du système et que pour un certain $\alpha < 2$, l'opérateur de Lévy $\mathcal{L}$ est assimilable, dans un sens convenable, à un opérateur α-stable éventuellement tronqué et non dégénéré sur le même sous-espace ($B\mathbb{R}^N \sim \mathbb{R}^d$) de $\mathbb{R}^N$.
Plus précisément, la mesure de Lévy associée à la partie integro-différentielle de l'opérateur $\mathcal{L}$ sera contrôlée par le bas avec la mesure de Lévy d'un opérateur α-stable éventuellement tronqué. Rappelant de (4.3) que chaque mesure de Lévy α-stable ν_α peut être décomposée en une partie sphérique μ et une partie radiale $r^{(1+\alpha)}dr$, on imposera notamment que la mesure ν vérifie la condition suivante, appelée usuellement de *domination stable* :

[**DS**] il existe $r_0 > 0$, α dans $(0,2)$ et une mesure μ finie et non-dégénérée (au sens de (4.4)) sur la sphère $\mathbb{S}^{d-1}$ tel que

$$\nu(C) \;\ge\; \int_0^{r_0}\int_{\mathbb{S}^{d-1}} \mathbb{1}_C(r\theta)\,\mu(d\theta)\frac{dr}{r^{1+\alpha}}, \quad C\in\mathcal{B}(\mathbb{R}^d_0). \tag{5.5}$$

Intuitivement, la condition [**DS**] assure l'existence d'une densité associée à l'opérateur $\mathcal{L}$ avec des propriétés régulatrices d'ordre (au moins) α. En fait, ce sont précisément les petits sauts (de petit rayon r_0) associés à la mesure de Lévy qui permettent, s'ils sont suffisamment intenses, de générer une densité pour le processus associé. Sous la condition [**DS**], on sait notamment que les contributions associées aux petits sauts de ν sont contrôlées par le bas avec celles d'une mesure α-stable non-dégénérée, dont la continuité absolue est bien connu dans ce contexte.
En clair, la condition de non-dégénérescence [**ND**] assumée dans le chapitre précédent peut être comprise ici comme un cas particulier de [**DS**] lorsque $r_0 = +\infty$ e détient une égalité dans (5.5). En particulier, nous soulignons que la classe des opérateurs de Lévy sur $\mathbb{R}^d$ vérifiant [**DS**] est très riche et variée et comprend certains de ceux de type quasi-stable qui ne pouvaient pas être considérés auparavant dans la littérature. Un exemple possible sur $\mathbb{R}^2$ est donné par le Laplacien fractionnaire relativiste $\Delta^{\alpha/2}_{\text{rel}}$ agissant uniquement sur la première composante, qui est définie par :

$$\Delta^{\alpha/2}_{\text{rel}}\phi(x) \;:=\; \text{p.v.}\int_{\mathbb{R}}\Big[\phi\begin{pmatrix}x_1+z\\ x_2\end{pmatrix}-\phi\begin{pmatrix}x_1\\ x_2\end{pmatrix}\Big]\frac{1+|z|^{\frac{d+\alpha-1}{2}}}{|z|^{d+\alpha}}e^{-|z|}\,dz,$$

où $x=(x_1,x_2)$ est dans $\mathbb{R}^2$. Ce type d'opérateur, aussi appelé opérateur relativiste de Schrödinger, est souvent considéré pour ses liens avec l'étude de la stabilité relativiste de la matière. Pour plus de détails, voir par exemple [BB99, CMS90, Fef86, Lie90] et les références à l'intérieur.
Comme vu précédemment, des conditions telles que [**DS**] assurent un effet régularisant d'ordre (minimum) α associé à l'opérateur $\mathcal{L}$ qui n'agit cependant que sur un sous-espace de $\mathbb{R}^N$. Pour obtenir un effet global sur tout l'espace $\mathbb{R}^N$, il faut alors que cette propriété

se diffuse dans tout le système. Pour cette raison, nous supposerons que les matrices A, B vérifient la *condition de Kalman* suivante :
[K] on a que $N = \mathrm{rank}\Big[B, AB, \dots, A^{N-1}B\Big]$,
où $\Big[B, AB, \dots, A^{N-1}B\Big]$ est la matrice dans $\mathbb{R}^N \otimes \mathbb{R}^{dN}$ dont les colonnes sont données par $B, AB, \dots, A^{N-1}B$. On sait (voir par exemple [Zab92]) qu'il existe une équivalence entre la condition [**K**] et l'énoncé suivant :

$$\det K_t \,:=\, \det \int_0^t e^{sA}BB^*e^{sA^*}\, ds \,>\, 0, \quad \forall t > 0. \tag{5.6}$$

Au moins dans le cas diffusif ($\alpha = 2$ et $\mathcal{L} = BB^*\Delta_x$), K_t est la matrice de covariance pour le processus solution de la dynamique stochastique associée. A son tour, l'équation (5.6) peut être montrée équivalente à l'hypoellipticité au sens de Hörmander ([Hör67]) de l'opérateur d'Ornstein-Uhlenbeck L^{ou}, qui assure notamment l'existence et la régularité d'une solution au sens distributionnel de l'équation

$$L^{\mathrm{ou}}u(x) \,=\, \mathcal{L}u(x) + \langle Ax, Du(x)\rangle \,=\, \phi(x), \quad x \text{ in } \mathbb{R}^N$$

pour chaque fonction $\phi\colon \mathbb{R}^N \to \mathbb{R}$ suffisamment régulière. Voir aussi le livre d'Ishikawa [Ish16], Chapitre 3.6, pour plus de détails sur le cas non-dégénéré.
Signalons enfin que la condition [**K**] est bien connue en théorie des contrôles. En fait, elle a été introduite par Kalman (cf. [Kal60a, Kal60b]) comme condition suffisante (alors effectivement équivalente) pour la *contrôlabilité à partir de zéro* des systèmes linéaires du type :

$$\dot{x}_t \,=\, Ax_t + Bu_t, \tag{5.7}$$

c'est-à-dire, que, pour chaque état final x dans $\mathbb{R}^N$, il existe un contrôle $t \mapsto u_t$ in $\mathbb{R}^d$ tel que $t \mapsto x_t$ est solution de (5.7), commence à 0 et atteint x en un temps fini. Pour plus de détails sur ce sujet, voir, par exemple, [KHN63] ou [Zab92].

Grâce à la condition [**K**], on peut maintenant expliquer plus précisément la distance anisotrope $\mathbf{d}$ et les espaces de Hölder associé $C^\beta_{b,d}(\mathbb{R}^N)$ dans ce contexte.
Pendant ce temps, nous fixons n comme le plus petit entier tel que la condition de Kalman [**K**] soit vérifiée, c'est-à-dire :

$$n \,=\, \min\{r \in \mathbb{N} \colon N \,=\, \mathrm{rank}\Big[B, AB, \dots, A^{r-1}B\Big]\}.$$

D'un point de vue plus probabiliste, c'est-à-dire en considérant la dynamique stochastique suivante :

$$dX_t \,=\, AX_tdt + BdZ_t, \quad t \geq 0$$

où $\{Z_t\}_{t\geq 0}$ est un processus de Lévy sur $\mathbb{R}^d$ avec le triplet de Lévy (b, Σ, ν), l'entier n peut être compris comme le nombre minimum des applications de A qui permettent de transmettre le bruit, situé sur $B\mathbb{R}^N$, à tout l'espace en dessous de $\mathbb{R}^N$.
Plus exactement, fixe i dans $[\![1, n]\!]$, on définit V_i comme l'espace image atteint par les premiers $i-1$ commutateurs itérés entre A et B :

$$V_i \,:=\, \begin{cases} \mathrm{Im}(B), & \text{si } i = 1, \\ \bigoplus_{k=1}^i \mathrm{Im}(A^{k-1}B), & \text{autrement.} \end{cases} \tag{5.8}$$

Puisque clairement $V_1 \subset V_2 \subset \dots V_n = \mathbb{R}^N$, il est maintenant logique de désigner

$$W_i := \begin{cases} V_1, & \text{si } i = 1, \\ (V_{i-1})^\perp \cap V_i, & \text{autrement.} \end{cases}$$

Intuitivement, chaque espace W_i caractérise combien une application ultérieure du commutateur sur V_i permet d'ajouter en termes d'espace couvert.
Enfin, on peut introduire les projections orthogonales $E_i \colon \mathbb{R}^N \to \mathbb{R}^N$ à partir de $\mathbb{R}^N$ sur W_i. En notant que $\dim E_1(\mathbb{R}^N) = \dim B\mathbb{R}^N = d$, il est logique de définir $d_1 := d$ et d'écrire

$$d_i := \dim E_i(\mathbb{R}^N), \quad \text{si } i > 1. \tag{5.9}$$

En considérant un changement de variables si nécessaire, on supposera désormais que l'espace $\mathbb{R}^N$ est décomposable en $x = (x_1, \dots, x_n)$ tel que $E_i x = x_i$ et x_i est dans $\mathbb{R}^{d_i}$, pour chaque i dans $[\![1, n]\!]$. Cette décomposition explicite permet maintenant d'étendre facilement à ce contexte la mesure anisotrope $\mathbf{d}$, définie comme dans (4.9), et les espaces de Hölder associés à l'opérateur de dilatation

$$\delta_\lambda(t, x_1, \dots, x_n) = (\lambda^\alpha, \lambda x_1, \dots, \lambda^{1+\alpha(i-1)} x_n). \tag{5.10}$$

En particulier, une fonction ϕ dans $C^\gamma_d(\mathbb{R}^N)$ sera telle que $x_i \to \phi(x_1, \dots, x_i, \dots, x_n)$ est dans $C^{\frac{\gamma}{1+\alpha(i-1)}}(\mathbb{R}^{d_i})$, uniformément dans les autres variables x_j, $(j \neq i)$.

Enfin, nous mentionnons que la structure linéaire analysée ici est plus générale que celle considérée (sur la matrice A) dans le chapitre précédent. En effet, il a été montré par Lanconelli et Polidoro dans [LP94] que la condition de Kalman [**K**] sur A et B impose la forme suivante sur les matrices :

$$B = \begin{pmatrix} B_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{et} \quad A = \begin{pmatrix} * & * & \dots & \dots & * \\ A_2 & * & \ddots & \ddots & \vdots \\ 0 & A_3 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ 0 & \dots & 0 & A_n & * \end{pmatrix} \tag{5.11}$$

où B_0 est une matrice non-dégénérée dans $\mathbb{R}^{d_1} \otimes \mathbb{R}^{d_1}$, A_i est une matrice dans $\mathbb{R}^{d_i} \otimes \mathbb{R}^{d_{i-1}}$ tel que $\text{rank}(A_i) = d_i$ pour chaque i dans $[\![2, n]\!]$ et les éléments $*$ peuvent être non-nuls. Aussi, $d_1 \geq d_2 \geq \dots \geq d_n \geq 1$.
De plus, la présence d'éléments $*$ non-nuls dans la matrice A ajoute une difficulté supplémentaire à notre méthode de preuve. En effet, il a été montré (cf. [LP94]) que la matrice A est invariante sous les dilatations δ_λ (définies dans (5.10)) si et seulement si A a des éléments non-nuls sur la sous-diagonale uniquement. En particulier, les décompositions explicites comme dans (4.7) ne seront désormais plus disponibles.

Les hypothèses [**DS**] et [**K**] décrites ci-dessus permettent de montrer les estimations de Schauder associées à l'opérateur d'Ornstein-Uhlenbeck L^{ou} . Par souci de cohérence, nous soulignons également qu'à l'instar du chapitre précédent, nous ne considérerons que des solutions *faibles* pour le problème elliptique ou parabolique au sens des distributions. Nous pouvons maintenant résumer les principaux résultats obtenus à la fois dans le contexte elliptique et dans le contexte parabolique.

Théorème 5.1 (Cas Elliptique)**.** *Étant donné $\lambda > 0$, soit g dans $C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)$. Il existe une unique solution faible u de l'équation elliptique* (5.3)*. De plus, u appartient à $C^{\beta}_{b,d}(\mathbb{R}^N)$ et il existe une constante $C > 0$ telle que*

$$\|u\|_{C^{\alpha+\beta}_{b,d}} \le C\Big(1+\frac{1}{\lambda}\Big)\|g\|_{C^{\beta}_{b,d}}. \tag{5.12}$$

Comme dans le cas parabolique, les estimations de Schauder dans un contexte elliptique sont souvent utilisées en relation avec la dynamique stochastique associée. Par exemple dans [CC07], de telles estimations pour les opérateurs dégénérés du second ordre sur des domaines non-lisses sont un outil fondamental pour obtenir l'unicité du problème de martingale corrélé. Dans un cas α-stable sous-critique ($\alpha \ge 1$) non-dégénéré, nous citons également les estimations globales obtenues par Priola dans [Pri12] et dans [Pri18] et leurs applications respectives à le caractère fortement bien posé et l'unicité de type Davie pour l'équation stochastique associée.

Théorème 5.2 (Cas Parabolique)**.** *Étant donné $T > 0$ et β dans $(0,1)$, soit u_0 dans $C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)$ e f en $L^\infty\Big(0,T;C^{\beta}_{b,d}(\mathbb{R}^N)\Big)$. Il existe une unique solution faible u du Problème de Cauchy* (5.4)*. De plus, u appartient à $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)\Big)$ et il existe une constante $C := C(T) > 0$ tel que*

$$\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \le C\Big[\|u_0\|_{C^{\alpha+\beta}_{b,d}} + \|f\|_{L^\infty(C^{\beta}_{b,d})}\Big]. \tag{5.13}$$

Dans le cas parabolique, les résultats présentés dans cette section peuvent aussi être naturellement étendus aux problèmes de Cauchy avec des coefficients dépendants du temps A_t, B_t, sous des hypothèses naturelles supplémentaires, telles que la bornage de la matrice A_t et l'uniforme ellipticité de B_t sur le petit espace $\mathbb{R}^d$. Nous renvoyons à la Section 6 du Chapitre 3 pour plus de détails.

5.1 Guide à l'épreuve

Inspirés par les travaux de Priola [Pri09], où des estimations de Schauder analogues sont prouvées dans le cas diffusif dégénéré, nous avons décidé de suivre une approche avec la théorie des semi-groupes pour notre problème. Cette méthode, introduite à l'origine par Da Prato et Lunardi en [DPL95], consiste à établir des contrôles a priori sur le semi-groupe P_t associé à l'opérateur d'Ornstein-Uhlenbeck L^{ou} sur des espaces fonctionnels convenables et à en déduire les estimations parabolique par une représentation mild (ou Duhamel) de la solution u par rapport au semi-groupe (cf. Équation (4.25)).
Nous soulignons déjà que cette approche dans le contexte parabolique ne se focalise cependant que sur la régularité dans l'espace pour les solutions u. Cela se reflète aussi, par exemple, sur notre définition des espaces de Hölder anisotropes $L^\infty(0,T;C^{\gamma}_{b,d}(\mathbb{R}^N))$, uniformes seulement en temps. En particulier, les estimations de Schauder paraboliques dans (5.13) ne montrent aucun effet bootstrap en temps par rapport à la condition initiale u_0.
Enfin, nous mentionnons qu'une autre méthode possible pour l'analyse des opérateurs d'Ornstein-Uhlenbeck tels que L^{ou} a été introduite par Manfredini dans [Man97] et

exploite un raisonnement plus abstrait en termes de Lie groupes associés au système différentiel. En particulier, les estimations de Schauder dans ce contexte sont construites par rapport à des espaces Hölder intrinsèques, en ce sens qu'elles prennent en compte la régularité conjointe entre l'espace et le temps des fonctions impliquées. Pour plus de détails sur le sujet, voir par exemple [Pas03].

Comme nous l'avons déjà dit, l'élément fondamental de notre méthode consiste en une analyse a priori des propriétés du semi-groupe Markovien $\{P_t\}_{t\geq 0}$ (formellement) associé à l'opérateur d'Ornstein-Uhlenbeck L^{ou}. Après cela, la formule de la transformée de Laplace va nous permettre d'écrire chaque solution u du problème elliptique (1.5) en fonction de P_t :

$$u(x) \;=\; \int_0^\infty e^{-\lambda t}\big[P_t g\big](x)\,dt \;=:\; \int_0^\infty e^{-\lambda t} P_t g(x)\,dt. \tag{5.14}$$

Dans le cas parabolique, il sera possible d'utiliser la formule de variation des constantes pour montrer une représentation analogue pour une solution u du problème de Cauchy (5.4) :

$$u(t,x) \;=\; P_t u_0(x) + \int_0^t \big[P_{t-s} f(s,\cdot)\big](x)\,ds \;=:\; P_t u_0(x) + \int_0^t P_{t-s} f(s,x)\,ds. \tag{5.15}$$

Pour établir des estimations telles que celles de Schauder pour une solution u (aussi bien dans le cas elliptique que dans le cas parabolique), il est alors clair qu'il est d'abord nécessaire d'obtenir des contrôles similaires sur le semi-groupe associé P_t. Surtout, nous nous intéressons à comprendre comment l'opérateur P_t se comporte sur les espaces de Hölder anisotropes $C^\gamma_{b,d}(\mathbb{R}^N)$ considérés par nous.
Nous soulignons également que les techniques habituelles pour obtenir ce type de commandes dans le contexte Gaussien sont cependant difficiles à étendre à notre contexte non-local. Par exemple, des stratégies de preuve comme dans [Lun97], par des formules explicites sur la densité du semi-groupe, dans [Lor05] ou [Sai07] dans le cas $n = 2$, par des combinaisons des estimations a priori de Bernstein-type avec des méthodes d'interpolation ou dans [Pri09], par calcul de Malliavin pour les représentations probabilistes du semi-groupe P_t, ne peuvent plus être suivies ici, principalement en raison de la nature non-locale ou de la faible intégrabilité associée à l'opérateur $\mathcal{L}$.
Pour surmonter cette difficulté, nous allons plutôt exploiter une méthode de type *perturbative* qui nous permettra de considérer l'opérateur de Lévy $\mathcal{L}$ comme une perturbation, dans un sens convenable, d'un opérateur α-stable dont les propriétés sont bien connues. Enfin, nous mentionnons que ces techniques de décomposition ont été introduites à l'origine dans [SSW12] dans le cadre de l'étude des propriétés de couplage pour les processus de Lévy et dans [SW12], en relation avec la généralisation de certains théorèmes de Liouville-type pour les opérateurs d'Ornstein-Uhlenbeck non-locaux.

Régularisation des propriétés associées à l'opérateur de Ornstein-Uhlenbeck

Comme déjà vu dans la section précédente, pour déterminer les propriétés sur le semi-groupe associé à l'opérateur L^{ou}, il convient de considérer dans un premier temps son homologue stochastique. Étant donné un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$, on introduit alors le processus de Lévy $\{Z_t\}_{t\geq 0}$ déterminé (en loi) par la suite symbole de Lévy :

$$\Phi(p) \,=\, ib\cdot p - \frac{1}{2}p\cdot\Sigma p + \int_{\mathbb{R}^d_0}\Big(e^{ip\cdot z} - 1 - ip\cdot z\mathbb{1}_{B(0,1)}(z)\Big)\,\nu(dz), \quad p\in\mathbb{R}^d,$$

où, rappelons-le, le triplet (b,Σ,ν) est le même que celui qui apparaît dans la définition de l'opérateur $\mathcal{L}$ et, en particulier, la mesure de Lévy ν vérifie l'hypothèse de domination stable [**DS**]. Soulignons maintenant que ce processus $\{Z_t\}_{t\geq 0}$ est associé à l'opérateur $\mathcal{L}$ au sens où le générateur infinitésimal du processus dégénéré $\{BZ_t\}_{t\geq 0}$ étendu sur $\mathbb{R}^N$ est alors donné par $\mathcal{L}$.
Pour un point x dans$\mathbb{R}^N$ fixé, on s'intéresse alors au processus d'Ornstein-Uhlenbeck $\{X_t\}_{t\geq 0}$ sur $\mathbb{R}^N$ dirigé par BZ_t, c'est la seule solution (au sens fort) de l'équation différentielle stochastique suivante :

$$\begin{cases} dX^x_t \,=\, AX^x_t dt + BZ_t, \quad t>0; \\ X^x_0 \,=\, x. \end{cases}$$

En intégrant directement l'équation ci-dessus à travers la fonction exponentielle matricielle $e^{(t-s)A}$, il est également possible de montrer une représentation explicite du processus $\{X_t\}_{t\geq 0}$:

$$X^x_t \,=\, e^{tA}x + \int_0^t e^{(t-s)A}B\,dZ_s, \qquad t\geq 0.$$

Le semi-groupe de Markov associé à $\{X^x_t\}_{t\geq 0}$ est maintenant défini comme la famille des contractions linéaires $\{P_t\colon t\geq 0\}$ sur $C_b(\mathbb{R}^N)$, l'espace des fonctions continues et bornées sur $\mathbb{R}^N$ à valeurs réelles, tel que

$$P_t\phi(x) \,=\, \mathbb{E}\Big[\phi(X^x_t)\Big], \quad x\in\mathbb{R}^N. \tag{5.16}$$

Enfin, nous mentionnons que le semi-groupe P_t est généré par l'opérateur L^{ou} dans le sens où son générateur infinitésimal coïncide avec L^{ou} sur l'espace de fonction test $C^\infty_c(\mathbb{R}^N)$.

Des raisonnements sur les espaces de Fourier, similaires à ceux développés dans la section précédente dans (4.15)-(4.16), permettent de montrer dans ce cas que la partie aléatoire de X_t vérifie encore l'hypothèse de domination stable [**DS**] sur $\mathbb{R}^N$, même si redimensionnée selon la structure anisotrope de la dynamique (cf. matrice $\mathbb{M}_t$ dans (4.8)). De la même manière que (4.17), vous pouvez en fait obtenir que

$$X_t \overset{\text{(loi)}}{=} e^{tA}x + \mathbb{M}_t S^t_t, \tag{5.17}$$

où pour chaque paramètre fixe t, $\{S^t_u\}_{u\geq 0}$ est un processus de Lévy sur $\mathbb{R}^N$ avec des propriétés convenables. Surtout, sa mesure de Lévy $\tilde{\nu}^t$ vérifie à nouveau l'hypothèse de domination stable [**DS**] étendue sur $\mathbb{R}^N$, à savoir :

$$\tilde{\nu}^t(C) \,\geq\, \int_0^{R_0}\int_{\mathbb{S}^{N-1}} \mathbb{1}_C(r\theta)\,\tilde{\mu}^t(d\theta)\frac{dr}{r^{1+\alpha}} \,=:\, \tilde{\nu}^t_\alpha(C), \quad C\in\mathcal{B}(\mathbb{R}^N_0), \tag{5.18}$$

pour un certain $R_0 > 0$ et une famille $\{\tilde{\mu}^t \colon t \geq 0\}$ de mesures finies et non-dégénérées sur la sphère $\mathbb{S}^{N-1}$.
Soulignons maintenant que la dépendance au paramètre t pour le processus $\{S^t_u\}_{u\geq 0}$ apparaît précisément à cause des éléments non-nuls $*$ dans la représentation de A donnée dans l'Équation (5.11). Comme déjà souligné précédemment, la matrice A dans ce cas n'est plus invariante sous les opérateurs d'expansion anisotropes δ_λ donnés dans (5.10). En effet, les décompositions comme dans (4.7), exploitées dans la section précédente pour ce type de résultats, ne sont plus valables mais doivent être modifiées avec des versions "approchées" de la forme :

$$e^{tA} \,=\, \mathbb{M}_t R_t \mathbb{M}_t^{-1}, \quad t \ll 1,$$

où R_t est une matrice localement bornée et non-dégénérée (dépendante du temps) en $\mathbb{R}^N \otimes \mathbb{R}^N$. Nous mentionnons également que c'est l'une des principales raisons pour lesquelles les contrôles a priori que nous voulons établir ne seront valables que dans un petit intervalle de temps.

En termes plus analytiques, l'identité dans (5.17) suggère que le semi-groupe généré par l'opérateur d'Ornstein-Uhlenbeck L^{ou} coïncide avec un semi-groupe non dégénéré même si "multiplié" par la matrice $\mathbb{M}_t$ qui prend en compte la dégénérescence originelle de l'opérateur considéré. Plus précisément, on peut établir, à partir de l'identité en loi dans (5.17), une première représentation du semi-groupe Markovien P_t :

$$P_t\phi(x) \,=\, \int_{\mathbb{R}^N} \phi(e^{tA}x + \mathbb{M}_t y)\, \mathbb{P}_{S^t_t}(dy), \quad \phi \in C_b(\mathbb{R}^N), \tag{5.19}$$

où, pour toute variable aléatoire X, $\mathbb{P}_X$ désigne la loi de X.
Pour déterminer les propriétés régularisantes associées au semi-groupe P_t, il est maintenant clair qu'une analyse plus approfondie de la mesure $\mathbb{P}_{S^t_t}$ est nécessaire.
Tout d'abord, on note que l'hypothèse [**DS**] permet de voir le processus de Lévy $\{S^t_u\}_{u\geq 0}$ comme une *perturbation* du processus α-stable éventuellement tronqué $\{Y^t_u\}_{t\geq 0}$, associé à la mesure de Lévy $\tilde{\nu}^t_\alpha$ définie dans (5.18). Des considérations sur la mesure $\mathbb{P}_{S^t_t}$ peuvent alors être exprimées à partir des propriétés bien mieux connues de la mesure stable.
En réalité, nous pousserons encore plus loin cette méthode perturbative, en ne considérant au lieu du processus stable *complet* que la contribution associée à ses petits sauts qui, comme on le sait, permettent l'existence de la densité de processus. Cette étape supplémentaire permettra d'obtenir des contrôles encore plus précis sur cette densité et ses dérivées dans l'espace.
Plus précisément, soit $(\tilde{\Sigma}^t, \tilde{b}^t, \tilde{\nu}^t)$ le triplet de Lévy associé au processus $\{S^t_u\}_{u\geq 0}$ à chaque instant fixe t, qui, rappelons-le, caractérise le symbole de Lévy Φ_{S^t} par la formule de Lévy-Khintchine sur $\mathbb{R}^N$:

$$\Phi_{S^t}(\xi) \,:=\, i\langle \tilde{b}^t, \xi\rangle - \frac{1}{2}\langle \xi, \tilde{\Sigma}^t \xi\rangle + \int_{\mathbb{R}^d_0} \Bigl(e^{i\langle \xi, z\rangle} - 1 - i\langle \xi, z\rangle \mathbb{1}_{B(0,1)}(z)\Bigr)\, \tilde{\nu}^t(dz).$$

Dans un intervalle suffisamment petit, en supposant fondamentalement que $t^{\frac{1}{\alpha}} < R_0 \wedge 1$, on peut tronquer la mesure de Lévy $\tilde{\nu}^t_\alpha$ al typique temps caractéristique associé au processus α-stable au temps t, c'est-à-dire qu'on peut tronquer au temps $t^{1/\alpha}$. En particulier, nous introduisons maintenant le symbole de Lévy Φ_t^{tr} associé aux petits sauts du processus α-stable $\{Y_u^t\}_{u\geq 0}$, défini par

$$\Phi_t^{\mathrm{tr}}(\xi) \,:=\, \int_{|z|\leq t^{\frac{1}{\alpha}}} \left[e^{i\langle \xi, z\rangle} - 1 - i\langle \xi, z\rangle\right] \tilde{\nu}_\alpha^t(dz).$$

Il faudra aussi considérer le terme reste qui apparaît, intuitivement, de l'erreur d'avoir considéré le processus $\{S_u^t\}_{u\geq 0}$ comme une perturbation du processus α-stable tronqué $\{Y_u^t\}_{u\geq 0}$. Plus précisément, on introduit maintenant le symbole de Lévy Φ_t^{err} défini par

$$\Phi_t^{\mathrm{err}}(\xi) := \Phi_{S^t}(\xi) - \Phi_t^{\mathrm{tr}}(\xi), \quad \xi \in \mathbb{R}^N,$$

et associé au triplet $(\tilde{\Sigma}^t, \tilde{b}^t, \tilde{\nu}^t - \mathbb{1}_{B(0,t^{1/\alpha})}\tilde{\nu}_\alpha^t)$.
Soulignons que c'est précisément la condition de domination stable [**DS**] sur $\{S_u^t\}_{u\geq 0}$ qui nous permet de conclure que Φ_t^{err} est en fait un symbole de Lévy, puisqu'il assure la positivité de la mesure $\tilde{\nu}^t - \mathbb{1}_{B(0,t^{1/\alpha})}\tilde{\nu}_\alpha^t$ associé au terme de reste.
On note maintenant $\{\mathbb{P}_t^{\mathrm{tr}}\}_{t\geq 0}$ et $\{\pi_t\}_{t\geq 0}$ respectivement le familles de probabilités infiniment divisibles associées aux symboles de Lévy Φ_t^{tr} et Φ_t^{err}. A partir de la représentation en fonction caractéristique des mesures de probabilité, on peut alors désintégrer la probabilité $\mathbb{P}_{S_t^t}$ comme suit :

$$\mathbb{P}_{S_t^t} \,=\, \mathbb{P}_t^{\mathrm{tr}} * \pi_t, \quad t > 0, \tag{5.20}$$

où $*$ représente l'opération de convolution entre les mesures de probabilité.

A partir des considérations ci-dessus, nous pouvons maintenant nous concentrer sur la famille de mesures $\{\mathbb{P}_t^{\mathrm{tr}}\}_{t\geq 0}$ qui, rappelons-le, sont associées à la petits sauts d'un processus α-stable.
En exploitant des résultats connus sur le symbole de Lévy associé à un processus α-stable non-dégénéré, comme la condition de Hartman-Wintner et quelques hypothèses de contrôlabilité dans les espaces de Fourier, on peut obtenir l'existence d'une densité régulière dans l'espace pour mesure de probabilité $\mathbb{P}_t^{\mathrm{tr}}$ et des contrôles convenables sur ses dérivées, au moins dans un intervalle de temps suffisamment petit. Plus précisément, nous montrerons qu'il existe un temps final $T_0 := T_0(N) > 0$ tel que sur $(0, T_0]$, la probabilité $\mathbb{P}_t^{\mathrm{tr}}$ admet une densité $p^{\mathrm{tr}}(t, \cdot)$ qui est 3-fois dérivable avec dérivée continue sur $\mathbb{R}^N$. De plus, pour chaque k dans $[\![0, 3]\!]$, c'est-à-dire

$$|\partial_y^k p^{\mathrm{tr}}(t, y)| \,\leq\, Ct^{-\frac{N+k}{\alpha}}\left(1 + \frac{|y|}{t^{1/\alpha}}\right)^{-(N+3)} \;=:\, Ct^{-\frac{k}{\alpha}}\bar{p}(t, y), \tag{5.21}$$

pour chaque (t, y) dans $(0, T_0] \times \mathbb{R}^N$, où $C > 0$ est une constante dépendante uniquement de N.
Nous soulignons cependant qu'au prix de réduire encore l'intervalle en temps, il est possible de montrer que la densité $p^{\mathrm{tr}}(t, \cdot)$ est encore plus régulière et la densité associée $\bar{p}(t, \cdot)$ a des propriétés régularisantes plus fortes mais que le choix dans (5.21) est le minimum pour notre propos. En particulier, l'exposant $N + 3$ dans la densité $\bar{p}(t, \cdot)$ est le minimum nécessaire pour intégrer les contributions associées aux fonctions ϕ dans $C^\beta_{b,d}(\mathbb{R}^N)$ d'indice $\beta < 1 + \alpha < 3$ (cf. Équation (5.27) ci-dessous).

Grâce à l'identité (5.20) nous pouvons maintenant réécrire la représentation pour le semi-groupe Markovien P_t dans (5.19) comme suit :

$$
\begin{aligned}
P_t\phi(x) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi\Big(e^{tA}x+\mathbb{M}_t(y_1+y_2)\Big)p^{\mathrm{tr}}(t,y_1)\,dy_1\pi_t(dy_2)\\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi\Big(y_1+\mathbb{M}_t y_2\Big)\frac{p^{\mathrm{tr}}(t,\mathbb{M}_t^{-1}(y_1-e^{tA}x))}{\det \mathbb{M}_t}\,dy_1\pi_t(dy_2). \quad (5.22)
\end{aligned}
$$

De plus, la forme explicite de la densité $\bar{p}(t,\cdot)$ et les contrôles associés dans (5.21) permettent facilement d'exhiber les effets régularisants associés à la densité “partielle” $p^{\mathrm{tr}}(t,\cdot)$, mais uniquement le long de la composante y_1. Plus précisément, fixe γ dans $(0,3)$, k dans $[\![0,2]\!]$, i dans $[\![1,n]\!]$ et t assez petit , nous avons que

$$
\int_{\mathbb{R}^{nd}} \frac{|D^k_{x_i}p^{\mathrm{tr}}(t,\mathbb{M}_t^{-1}(y_1-e^{tA}x))|}{\det \mathbb{M}_t}\mathbf{d}^\gamma(y_1,e^{tA}x)\,dy_1 \;\le\; C(s-t)^{\frac{\gamma}{\alpha}-k\frac{1+\alpha(i-1)}{\alpha}}. \qquad (5.23)
$$

A ce stade, il est clair que la principale différence avec les techniques présentées dans la Section 4 (pour les estimations de Schauder sur le proxy) consiste dans le fait qu'ici on ne pourra exploiter que les effets régularisants sur la variable y_1 , associée aux contributions des petits sauts du processus α-stable tronqué, puisque nous ne connaissons pas de propriétés particulières du terme de reste (associé à la variable d'intégration y_2). En fait, on sait seulement que la probabilité π_t est de mesure totale finie, uniformément en t.
Si cette particularité introduit certainement d'autres difficultés supplémentaires dans certains passages de la prouve, par exemple sur les techniques d'annulation pour les contrôles sur les normes de Hölder, nous soulignons qu'exploiter uniquement les effets de régularisation associés aux petits sauts permet d'éviter certaines hypothèses ici (cf . hypothèse [$\mathbf{P}$]) sur les paramètres α, β nécessaires dans le Chapitre 2 précisément parce que nous avons considéré la densité associée à un processus α-stable complet.

Contrôles sur le semi-groupe de Markov associé

Les effets régularisants de la densité $p^{\mathrm{tr}}(t,\cdot)$ montrés dans (5.23) impliquent désormais naturellement les premières contrôles sur P_t et ses dérivées lorsque le semi-groupe agit sur l'espace $C_b(\mathbb{R}^N)$. Plus précisément, il est vrai que :

$$
\|D^k_{x_i}P_t\phi\|_\infty \;\le\; C\|\phi\|_\infty\Big(1+t^{-k\frac{1+\alpha(i-1)}{\alpha}}\Big), \quad t>0, \qquad (5.24)
$$

où k est dans $[\![0,3]\!]$, i dans $[\![1,n]\!]$ et ϕ une fonction dans $C_b(\mathbb{R}^N)$.
En particulier, à partir des estimations ci-dessus, il est maintenant possible de montrer la continuité du semi-groupe P_t, à temps fixe $t>0$, en tant qu'opérateur sur l'espace $C_b(\mathbb{R}^N)$ avec des valeurs dans le même ou dans l'espace de Zygmund-Hölder anisotrope $C^3_{b,d}(\mathbb{R}^N)$, une généralisation naturelle aux indices entiers des espaces de Hölder anisotrope considérés par nous. Ce résultat sera ensuite étendu aux espaces de Hölder anisotropes en notant que chaque espace $C^\gamma_{b,d}(\mathbb{R}^N)$ (avec γ dans $(0,3)$) peut être vu comme un espace d'interpolation entre $C_b(\mathbb{R}^N)$ et $C^3_{b,d}(\mathbb{R}^N)$. En effet, nous allons exploiter l'identité suivante :

$$
\Big(C_b(\mathbb{R}^N),C^3_{b,d}(\mathbb{R}^N)\Big)_{\gamma,\infty} \;=\; C^\gamma_{b,d}(\mathbb{R}^N), \qquad (5.25)
$$

avec l'équivalence entre les normes respectives, où étant donnés deux espaces de Banach X, Y, le symbole $(X, Y)_{\gamma,\infty}$ représente généralement l'espace de *interpolation réelle* à norme infinie entre X et Y. Pour plus de détails sur le sujet en général, nous renvoyons aux livres de Triebel [Tri92] et Lunardi [Lun18] ou le Théorème 2.2 dans [Lun97] dans un contexte diffusif anisotrope.
Grâce à des techniques d'interpolation similaires à celles dans (5.25), nous montrerons alors que le semi-groupe P_t est continu comme opérateur sur $C_b(\mathbb{R}^N)$ avec des valeurs dans l'espace de Hölder anisotrope $C^\gamma_{b,d}(\mathbb{R}^N)$ pour chaque γ dans $[0, 1+\alpha)$:

$$\|P_t\|_{\mathcal{L}(C_b, C^\gamma_{b,d})} \le C\Big(1 + t^{-\frac{\gamma}{\alpha}}\Big), \quad t > 0, \tag{5.26}$$

où $\|\cdot\|_{\mathcal{L}(X,Y)}$ désigne la norme usuelle de l'opérateur entre deux espaces de Banach génériques X et Y.

Pour l'instant, nous n'avons considéré que le comportement du semi-groupe P_t sur l'espace $C_b(\mathbb{R}^N)$. Pour étendre cette analyse également sur $C^\beta_{b,d}(\mathbb{R}^N)$, la première étape consiste à présenter des contrôles similaires à ceux dans (5.24) lorsque P_t agit à la place sur les fonctions ϕ qui sont Hölder régulières. De la même manière que le chapitre précédent, nous utiliserons dans ce cas quelques techniques de annulation pour exploiter la régularité de la fonction ϕ et ainsi obtenir les contrôles souhaités. Cependant, la principale difficulté dans ce cas sera liée au fait que les effets de régularisation relatifs à la densité p^{tr} ne se feront que par rapport à la variable “bonne” y_1. Nous devrons alors considérer des techniques de annulation *partielle* qui nous permettent d'isoler uniquement les composantes de ϕ le long de y_1.
Plus précisément et en s'attardant uniquement sur le cas $\beta < 1$, $k = 1$ pour plus de simplicité, l'idée fondamentale sera d'exploiter, à l'instar de (4.23) que

$$\int_{\mathbb{R}^N} \phi(\mathbb{M}_t y_2 + e^{tAx}) D_{x_i} \int_{\mathbb{R}^N} \frac{p^{\mathrm{tr}}(t, \mathbb{M}_t^{-1}(y_1 - e^{tA}x))}{\det \mathbb{M}_t}\, dy_1 \pi_t(dy_2) \;=\; 0,$$

pour ajouter le terme $\phi(\mathbb{M}_t y_2 + e^{tAx})$ dans le contrôle de $|D_{x_i} P_t|$ et conclure comme dans (4.23) grâce à effet régularisant associé à la densité $p^{\mathrm{tr}}(t, \cdot)$ donnée dans (5.23).
Un raisonnement similaire à celui évoqué maintenant va nous permettre de montrer en particulier que pour chaque β dans $[0, 3)$, i dans $[\![1, n]\!]$ et k dans $[\![0, 3]\!]$, c'est-à-dire

$$\|D^k_{x_i} P_t \phi\|_\infty \le C\|\phi\|_{C^\beta_{b,d}}\Big(1 + t^{\frac{\beta}{\alpha} - k\frac{1+\alpha(i-1)}{\alpha}}\Big), \quad t > 0, \tag{5.27}$$

Exploitant toujours des techniques d'interpolation similaires à celles de (5.25) à partir des contrôles sur les dérivées de $P_t\phi$ dans (5.27), nous pourrons enfin montrer la continuité du semi-groupe P_t comme opérateur entre les espaces de Hölder anisotropes. Plus précisément, nous obtiendrons que pour chaque $\beta < \gamma$ dans $[0, 1+\alpha)$, le contrôle suivante est vérifiée :

$$\|P_t\|_{\mathcal{L}(C^\beta_{b,d}, C^\gamma_{b,d})} \le C\Big(1 + t^{\frac{\beta-\gamma}{\alpha}}\Big), \quad t > 0. \tag{5.28}$$

Ces estimations semblent être nouvelles dans le contexte de Lévy dégénéré et d'intérêt indépendant des finalités que nous considérons ici.

Nous soulignons également que la possibilité d'un effet régularisant indépendant de l'ordre α dans ce cas (contrairement à la section précédente) reflète essentiellement le fait que dans ce cas la densité n'est associée qu'à la contribution des petits sauts, tandis que ce sont les queues, corrélées aux grands sauts du processus, à imposer les conditions d'intégrabilité vues précédemment.

Estimations de Schauder dans le cas elliptique

Une fois que les contrôles a priori nécessaires sur le semi-groupe P_t ont été montrées, les estimations de Schauder dans le contexte elliptique (5.12) et dans le contexte parabolique (5.13) sont alors obtenues à partir des représentations relatives de la solution u en fonction du semi-groupe Markovien P_t (cf. Équation (5.14) et (5.15)).
Pour donner au lecteur une idée de la méthode que nous avons suivie, nous ne présentons maintenant brièvement que le cas elliptique.
Étant donnée une solution u de l'Équation elliptique (5.3), on sait d'après la formule de Laplace dans (5.14) qu'il faut alors contrôler le terme suivant :

$$u(t,x) \,=\, \int_0^\infty e^{-\lambda t}\Big(P_t g\Big)(z)\,dt,$$

dans l'espace de Hölder anisotrope d'ordre $\alpha+\beta$ par rapport à la norme de g dans $C^\beta_{b,d}(\mathbb{R}^N)$.
Pour ce type de problème, il est en fait commode d'exploiter une norme équivalente à celle de Hölder introduite dans (3.23) qui ne nécessite pas de considérer les dérivées suivant chaque direction mais seulement les différences finies d'ordre 3 pour la fonction u. Plus précisément, on introduit par un point de départ x_0 dans $\mathbb{R}^N$ et z dans $E_i(\mathbb{R}^N)$,

$$\Delta^3_{x_0}\phi(z) \,:=\, \phi(x_0+3z)-3\phi(x_0+2z)+3\phi(x_0+z)-\phi(x_0).$$

il a été montré en effet dans [Lun97] qu'une fonction ϕ est dans $C^\gamma_b(E_i(\mathbb{R}^N))$ si et seulement si

$$\sup_{x_0\in\mathbb{R}^N}\;\sup_{z\in E_i(\mathbb{R}^N);z\neq 0}\frac{\Big|\Delta^3_{x_0}\phi(z)\Big|}{|z|^\gamma} \,<\, \infty.$$

Pour conclure, il faut alors montrer que pour chaque i fixé dans $[\![1,n]\!]$, il tient que

$$|\Delta^3_{x_0}u(z)| \,=\, \left|\int_0^\infty e^{-\lambda t}\Delta^3_{x_0}\Big(P_t g\Big)(z)\,dt\right| \,\le\, C\|g\|_{C^\beta_{b,d}}|z|^{\frac{\alpha+\beta}{1+\alpha(i-1)}}, \tag{5.29}$$

pour une certaine constante $C>0$ indépendante de x_0 dans $\mathbb{R}^N$ et z dans $E_i(\mathbb{R}^N)$.

Comme cela arrive souvent dans la preuve des estimations de norme de Hölder, il faut d'abord diviser l'analyse selon trois régimes possibles, par rapport au rapport entre le point spatial z dans $E_i(\mathbb{R}^N)$ et le temps t à l'échelle intrinsèque du système le long de la i -ième direction considérée.
D'une part, le régime *macroscopique* apparaîtra lorsque $|z|\ge 1$ et sera le plus simple à manipuler. En fait, il suffira de montrer la borne de la solution u à partir de celle de g, puisque dans ce cas, $\|g\|_\infty \le \|g\|_\infty|z|^\gamma$ pour chaque $\gamma>0$.
Par contre, on dira qu'on est dans un régime *hors diagonale* si $t^{\frac{1+\alpha(i-1)}{\alpha}} \le |z| \le 1$. Dans

ce cas, la distance spatiale le long de la i-ième composante sera supérieure au temps caractéristique associé. Enfin, un régime *diagonal* apparaîtra lorsque $t^{\frac{1+\alpha(h-1)}{\alpha}} \geq |z|$ et le point spatial est à la place plus petit que intensité typique du temps caractéristique. En particulier, nous désignerons par t_0 le temps de transition entre les régimes diagonal et hors diagonale, par rapport aux échelles de dilatation δ_λ (définies dans (5.10)) le long de la i-ième direction du système, c'est-à-dire :

$$t_0 \;:=\; |z|^{\frac{\alpha}{1+\alpha(i-1)}}.$$

Comme déjà mentionné, le contrôle recherché dans le régime macroscopique ($|z| \geq 1$) suit immédiatement de la propriété de contraction du semi-groupe P_t sur $C_b(\mathbb{R}^N)$:

$$\left|\int_0^\infty e^{-\lambda t}\Delta^3_{x_0}\Big(P_t g\Big)(z)\,dt\right| \;\leq\; 3\int_0^\infty e^{-\lambda t}\|P_t g\|_\infty\,dt \;\leq\; C\|g\|_\infty|z|^{\frac{\alpha+\beta}{1+\alpha(i-1)}}. \tag{5.30}$$

Pour analyser séparément le regime diagonale et hors diagonale, on décomposera, comme dans [Pri09], le terme $\Delta^3_{x_0}u(z)$ en deux composantes $R_1(z) + R_2(z)$, où

$$\begin{aligned} R_1(z) \;&:=\; \int_0^{t_0} e^{-\lambda t}\Delta^3_{x_0}\Big(P_t g\Big)(z)\,dt; \\ R_2(z) \;&:=\; \int_{t_0}^{\infty} e^{-\lambda t}\Delta^3_{x_0}\Big(P_t g\Big)(z)\,dt. \end{aligned}$$

Dans le cas hors diagonale associé à la première composante R_1, nous devrons éventuellement intégrer sur $[0, t_0]$ et il faudra donc faire attention à ne pas introduire de singularités en temps qui ne sont pas alors intégrable. En pratique, on va exploiter la continuité du semi-groupe P_t sur l'espace $C^\beta_{b,d}(\mathbb{R}^N)$ lui-même pour contrôler le terme R_1 comme suit :

$$\begin{aligned} |R_1(z)| \;&\leq\; \int_0^{t_0} |\Delta^3_{x_0}\Big(P_t g\Big)(z)|\,dt \\ &\leq\; |z|^{\frac{\beta}{1+\alpha(h-1)}}\int_0^{t_0} \|P_t g\|_{C^\beta_{b,d}}\,dt \\ &\leq\; C\|g\|_{C^\beta_{b,d}}|z|^{\frac{\alpha+\beta}{1+\alpha(h-1)}}. \end{aligned} \tag{5.31}$$

En revanche, le régime diagonal relatif à la contribution R_2 ne posera pas de problèmes d'intégrabilité dans le temps. En effet, puisque le point z est, dans ce cas, proche de zéro par rapport aux échelles de temps caractéristiques, il est logique d'appliquer itérativement un développement de Taylor sur $\Delta^3_{x_0}\Big(P_t\phi\Big)$ de sorte qu'une dérivée du troisième ordre apparaisse le long de la i-ième composante concernée. Plus précisément,

$$\begin{aligned} &\left|\Delta^3_{x_0}\Big(P_t g\Big)(z)\right| \\ &\quad= \left|\int_0^1 \langle D_{x_i}P_t g(x_0+\lambda z) - 2D_{x_i}P_t g(x_0+z+\lambda z) + D_{x_i}P_t g(x_0+2z+\lambda z), z\rangle\,d\lambda\right| \\ &\quad\leq \left|\int_0^1\int_0^1 \langle\Big[D^2_{x_i}P_t g(x_0+(\lambda+\mu)z) - D^2_{x_i}P_t g(x_0+z+(\lambda+\mu)z)\Big]z, z\rangle\,d\lambda d\mu\right| \\ &\quad\leq \left|\int_0^1\int_0^1\int_0^1 \langle\Big[D^3_{x_i}P_t g(x_0+(\lambda+\mu+\nu)z)\Big](z,z), z\rangle\,d\lambda d\mu d\nu\right|, \end{aligned}$$

où ci-dessus, nous avons identifié l'opérateur différentiel $D^3_{x_i}P_t\phi$ avec le 3-tenseur associé. Les contrôles a priori dans (5.27) sur les dérivées du semi-groupe P_t impliquent maintenant que

$$\left|\Delta^3_{x_0}\big(P_tg\big)(z)\right| \leq C\|D^3_{x_i}P_tg\|_\infty|z|^3 \leq C\|g\|_{C^\gamma_{b,d}}\Big(1+t^{\frac{\gamma-3(1+\alpha(i-1))}{\alpha}}\Big)|z|^3.$$

Nous pouvons alors également conclure dans le cas diagonal que le contrôle suivante est vérifiée :

$$\begin{aligned}
|R_2(z)| &\leq \int_{t_0}^\infty e^{-\lambda t}|\Delta^3_{x_0}\big(P_tg\big)(z)|\,dt \\
&\leq C\|g\|_{C^\beta_{b,d}}|z|^3\int_{t_0}^\infty e^{-\lambda t}\Big(1+t^{\frac{\beta-3(1+\alpha(i-1))}{\alpha}}\Big)\,dt \\
&\leq C\|g\|_{C^\beta_{b,d}}|z|^3\Big(\lambda^{-1}+|z|^{\frac{\alpha+\beta-3(1+\alpha(i-1))}{1+\alpha(i-1)}}\Big) \\
&\leq C\|g\|_{C^\beta_{b,d}}|z|^{\frac{\alpha+\beta}{1+\alpha(i-1)}},
\end{aligned} \tag{5.32}$$

où, dans la dernière étape, nous avons à nouveau exploité que $|z|\leq 1$.

Enfin, en réunissant les contributions (5.30), (5.31) et (5.32) associées aux trois régimes considérés, on peut conclure que l'Équation (5.29) est vrai et qu'en particulier, les estimations de Schauder (5.12) dans le cas elliptique sont valides.

Les estimations de Schauder dans le cas parabolique seront obtenues en suivant une procédure similaire. Nous soulignons également que ce type de décomposition en régimes diagonaux et hors-diagonaux apparaît, quoique sous une forme moins explicite, également dans les contrôles de Hölder pour les estimations de Schauder du Chapitre 2.

Enfin, nous voulons mentionner que nous aurions pu utiliser la méthode présentée ici également au Chapitre 2 pour montrer les estimations de Schauder (4.24) pour la solution $\tilde{u}^{\tau,\xi}$ associé à l'opérateur proxy, sous la condition plus générale de domination stable[**DS**]. Cependant, pour obtenir les estimations dans (4.11) sur la solution u du système non-linéaire d'origine (4.1) grâce à la méthode perturbative décrite ci-dessus, la partie délicate aurait été de prouver, sous les hypothèses plus générales, l'indépendance effective de la constante par rapport aux paramètres de congélation (τ,ξ) utilisés. Cette difficulté apparaîtra encore plus clairement dans les hypothèses du modèle de la section suivante, où nous considérerons en plus un bruit multiplicatif.

6 Caractère bien posé faible pour des chaînes stochastiques dirigées par des processus de type Lévy

Nous présentons maintenant brièvement les principaux éléments du Chapitre 4 de cette thèse. Rédigé en collaboration avec mon directeur de thèse, Professeur Stéphane Menozzi, ce travail est récemment paru en pré-publication [MM21].

Nous voulons étudier ici l'effet de la propagation d'un bruit de type Lévy non-dégénéré à travers une chaîne d'oscillateurs interconnectés, où le bruit précité n'agit que sur le

premier d'entre eux, comme illustré sur la figure 1.
Plus précisément, on s'intéresse à une dynamique stochastique sur $\mathbb{R}^N$ de la forme :

$$\begin{cases} dX_s \,=\, G(s,X_s)ds + B\sigma(s,X_{s-})dZ_s, \quad s \geq t, \\ X_t \,=\, x, \end{cases} \tag{6.1}$$

pour un certain point de départ (t,x) dans $[0,\infty) \times \mathbb{R}^N$, où la dérive déterministe $G\colon [0,\infty) \times \mathbb{R}^N \to \mathbb{R}^N$ et la matrice de diffusion $\sigma\colon [0,\infty) \times \mathbb{R}^N \to \mathbb{R}^d \otimes \mathbb{R}^d$ sont deux coefficients donnés.
Dorénavant, on dira que l'équation dans (6.1) est *dégénéré* dans le sens suivant : le bruit de type Lévy $\{Z_s\}_{s\geq 0}$ n'agit initialement que sur le "petit" espace $\mathbb{R}^d$. Le caractère non dégénéré de la matrice σ garantit alors que le bruit associé à l'intégrale stochastique $\int_0^t \sigma(s,X_{s-})dZ_s$ s'étale sur tout l'espace $\mathbb{R}^d$. Enfin, ce dernier agit sur le "grand" espace $\mathbb{R}^N$ à travers la matrice d'immersion B dans $\mathbb{R}^N \otimes \mathbb{R}^d$, donnée par :

$$B := \begin{pmatrix} I_{d\times d}, & 0_{d\times(N-d)} \end{pmatrix}^t.$$

En particulier, on supposera que $\{Z_s\}_{s\geq 0}$ est un processus de saut pur, c'est-à-dire que $\Sigma = 0$ dans le triplet de Lévy (b,Σ,ν) associé au processus.
Pour souligner la dépendance effective au point de départ choisi (t,x), on indiquera désormais avec $\{X_s^{t,x}\}_{s\geq 0}$ un processus solution générique de la dynamique stochastique dans (6.1) étendue, par commodité, jusqu'à l'instant zéro, c'est à dire en imposant $X_s^{t,x} = x$ si s est dans $[0,t]$.

Dans [CdRM20b], les auteurs ont caractérisé pour une chaîne dégénérée comme dans (6.1) mais perturbé par un mouvement Brownien $\{Z_s\}_{s\geq 0}$, la régularité de Hölder minimale sur la dérive G qui assure le caractère faiblement bien posé pour la dynamique stochastique considérée. Le but initial de ce travail était d'étendre ce résultat à la dynamique dans (6.1) dont le bruit était seulement de Lévy, sous la même hypothèse de domination stable [**DS**] présentée dans la section précédente (cf. Équation (5.5)).
Nous n'avons en fait réussi à obtenir qu'une généralisation partielle des résultats dans [CdRM20b]. En fait, nous montrerons plus loin l'existence d'un seuil optimal pour la régularité de Hölder pour la dérive G, mais uniquement pour un type particulier de structure déterministe diagonale. De plus, la méthode de preuve perturbative que nous avons suivie, à travers un proxy rétrograde, nécessitait en pratique de renforcer la condition de domination stable [**DS**] et d'ajouter quelques hypothèses supplémentaires. Nous résumons brièvement dans cette section les raisons naturelles qui nous ont conduits à des telles considérations.

La structure déterministe de la dynamique stochastique (cf. (6.1) pour $\sigma = 0$) sera similaire à celle présentée dans les deux premiers chapitres. En particulier, nous supposerons à nouveau que nous pouvons décomposer le "grand" espace $\mathbb{R}^N$ en n sous-espaces $\mathbb{R}^{d_i}$, $i \in [\![1,n]\!]$ tel que $d_1 = d$ e $d_1 + \cdots + d_n = N$, comme indiqué dans la Section 5. De plus, la dérive déterministe G admettra, par rapport à cette décomposition, une structure particulière "au-dessus de la diagonale" et ses éléments sur la sous-diagonale seront considérés comme non-dégénérés et linéaires. En pratique, on imposera que G ait la forme suivante :

$$G(s,x) \,:=\, A_s x + F(s,x), \tag{6.2}$$

où $A\colon [0,\infty) \to \mathbb{R}^N \otimes \mathbb{R}^N$ et $F\colon [0,\infty) \times \mathbb{R}^N \to \mathbb{R}^N$ sont deux fonctions telles que

[H] — pour chaque niveau i dans $[\![1,n]\!]$, F_i ne dépend que du temps et des dernières $n-(i-1)$ variables, c'est-à-dire $F_i(s,x_i,\dots,x_n)$;

— $A\colon [0,\infty) \to \mathbb{R}^N \otimes \mathbb{R}^N$ est bornée et

$$[A_s]^{i,j} \;=\; \begin{cases} \text{est non-dégénérée, uniformément dans } s, & \text{si } j = i-1; \\ 0, & \text{si } j < i-1. \end{cases}$$

Comme déjà mentionné ci-dessus, cette hypothèse dans le cas linéaire avec bruit additif (i.e. $F=0$ et $\sigma=1$) peut être comprise comme une condition de type Hörmander, ou de manière équivalente de rang de Kalman [**K**], qui assure l'hypoelliticité du générateur infinitésimal associé au processus $\{X^{t,x}_s\}_{s\ge 0}$ solution de l'Équation (6.1).

Sous l'hypothèse [**H**], on note alors que la matrice A peut être réécrite comme une version “dépendante du temps” de celle apparue dans la Section 5, Équation (5.11). De plus, cette condition permet de représenter la dynamique stochastique dans (6.1) sous la forme suivante, plus explicite :

$$\begin{cases} dX^1_t \;=\; \Big[A^{1,1}_t X^1_t + \cdots + A^{1,n}_t X^n_t + F_1(t,X^1_t,\dots,X^n_t)\Big]\, dt + \sigma(t,X^1_{t-},\dots,X^n_{t-})dZ_t, \\ dX^2_t \;=\; \Big[A^{2,1}_t X^1_t + \cdots + A^{2,n}_t X^n_t + F_2(t,X^2_t,\dots,X^n_t)\Big]\, dt, \\ dX^3_t \;=\; \Big[A^{3,2}_t X^2_t + \cdots + A^{3,n}_t X^n_t + F_3(t,X^3_t,\dots,X^n_t)\Big]\, dt, \\ \vdots \\ dX^n_t \;=\; \Big[A^{n,n-1}_t X^{n-1}_t + A^{n,n}_t X^n_t + F_n(t,X^n_t)\Big]\, dt, \end{cases}$$

où nous avons décomposé $X_t = (X^1_t,\dots,X^n_t)$ tel que X^i_t est dans $\mathbb{R}^{d_i}$, pour chaque i dans $[\![1,n]\!]$.

Puisque l'objectif premier de ce chapitre sera de déterminer la régularité de Hölder optimale sur la dérive F qui assure le caractère bien posé de la dynamique stochastique et que de tels seuils minimaux ne feront intervenir que les composantes dégénérées ($i>1$) de F, nous pouvons maintenant énoncer séparément les conditions sur les autres coefficients du système. En particulier, nous supposerons que :

[R] ils existent un indice β^1 dans $(0,1)$ et une constante $K>0$ telle que

— $\sigma(t,\cdot)$ est β^1-Hölder continue, uniformément dans t ;

— $F_1(t,x)$ est β^1-Hölder continue, uniformément dans t ;

— pour chaque i dans $[\![1,n]\!]$, nous avons que

$$|F_i(t,0)| \;\le\; K, \quad t\in[0,T].$$

Pour qu'un effet régularisant minimum (d'ordre α) soit véhiculé par le bruit, on supposera que $\{Z_t\}_{t\ge 0}$ est attribuable à un processus α-stable “tempéré”. Nous soulignons, cependant, que cette classe inclut mais n'inclut pas seulement les processus α-stables tempérés classiques. Plus précisément, nous imposerons que la mesure de Lévy ν associée au processus $\{Z_s\}_{s\ge 0}$ est *symétrique* et

[ND'] il existe une fonction $Q\colon \mathbb{R}^d \to \mathbb{R}$ Borel mesurable telle que

- Q est positive et bornée, i.e. $Q \geq 0$ et $\sup_{z\in\mathbb{R}^d} Q(z) < \infty$;
- Q est loin de zéro et Lipschitz régulier dans un voisinage de l'origine, c'est-à-dire qu'il existe $r_0 > 0$ et $c > 0$ telles que $Q(z) \geq c$ et Q Lipschitz continue dans $B(0, r_0)$;
- ils existent $\alpha \in (1, 2)$ et une mesure μ finie et non-dégénérée (au sens de (4.4)) sur $\mathbb{S}^{d-1}$ telle que

$$\nu(\mathcal{A}) \,=\, \int_0^\infty \int_{\mathbb{S}^{d-1}} \mathbb{1}_{\mathcal{A}}(\rho s) Q(\rho s)\, \mu(ds) \frac{d\rho}{\rho^{1+\alpha}}, \quad \mathcal{A} \in \mathcal{B}(\mathbb{R}^d_0).$$

En gardant à l'esprit la décomposition classique de la mesure de Lévy associée à un processus α-stable donné, par exemple, dans (4.3), la condition de *non-dégénérescence* [**ND'**] impose intuitivement que la mesure de Lévy de $\{Z_t\}_{t\geq 0}$ est absolument continue par rapport à celle d'un processus α-stable non-dégénéré et que sa dérivée de Radon-Nikodym est donnée par une fonction de "tempérage" Q avec des propriétés convenables. Clairement, une possibilité naturelle est donnée par $Q = c$ constant, c'est-à-dire lorsque le processus $\{Z_t\}_{t\geq 0}$ peut être remonté à un processus symétrique α-stable habituel. Cependant, il est à noter que la classe de bruits considérée ici comprend également plusieurs processus *quasi-stable* dont, par exemple, le processus α-stable relativiste symétrique ou celui de Lamperti (voir Chapitre 4, Section 1.1 pour plus de détails).

Une hypothèse naturelle lorsque l'on considère les équations dépendantes des bruits multiplicatifs est la *ellipticité uniforme* de la composante non-dégénérée associée à la matrice de diffusion $\sigma(t, x)$ en chaque point de l'espace-temps fixe. En effet, on supposera que :

[UE] il existe une constante $\eta > 1$ telle que pour chaque $s \geq 0$ et chaque x dans $\mathbb{R}^N$, nous avons que

$$\eta^{-1}|\xi|^2 \,\leq\, \sigma(s, x)\xi \cdot \xi \,\leq\, \eta|\xi|^2, \quad \xi \in \mathbb{R}^d,$$

où "$\cdot$", rappelez-vous, indique le produit scalaire sur le "petit" espace $\mathbb{R}^d$.
Intuitivement, l'hypothèse d'uniforme ellipticité garantit que le coefficient de diffusion σ préserve efficacement le bruit Z_s sur tout l'espace $\mathbb{R}^d$, uniformément dans le temps et dans l'espace.

Lorsque l'on considère la dynamique (6.1) régularisée par un bruit multiplicatif, c'est-à-dire en présence d'une matrice de diffusion σ qui dépend également de x, nous devrons de plus supposer que la mesure ν est absolument continue par rapport à la mesure de Lebesgue sur $\mathbb{R}^d$ et que sa dérivée de Radon-Nykodim est Lipschitz régulière. Plus précisément, nous imposerons la condition suivante :

[AC] si $\sigma(t, \cdot)$ non est pas constante pour un certain $t \geq 0$, alors il existe une fonction Lipschitz continue $g\colon \mathbb{S}^{d-1} \to \mathbb{R}$ telle que

$$\nu(dz) \,=\, Q(z) \frac{g\left(\frac{z}{|z|}\right)}{|z|^{d+\alpha}} dz.$$

Bien que la condition [**AC**] réduise significativement la classe des mesures de Lévy, et donc des processus, que l'on peut inclure comme bruit dans l'Équation stochastique (6.1), nous soulignons que cette hypothèse semble être nécessaire dans notre contexte, au moins par rapport à l'approche envisagée (cf. Équation (6.32) ci-dessous) et naturelle, puisqu'elle est apparue dans d'autres travaux antérieurs qui traitaient de sujets similaires (cf. [HM16, FKM21]).
Cependant, soulignons qu'au moins dans le cas additif, ou plus généralement, lorsque la matrice de diffusion σ ne dépend pas de l'espace, la condition de non-dégénérescence [**ND**], considéré dans le Chapitre 2 pour les estimations de Schauder dans le contexte stable, peut être compris comme un cas particulier de [**ND**'] supposé ici. En particulier, également dans ce cas nous n'avons imposé aucune régularité, dans le cas additif, à la mesure de Lévy ν du processus $\{Z_t\}_{t\geq 0}$ qui, a priori, pourrait avoir un support très singulier sur l'espace $\mathbb{R}^d$. Par exemple, notre modèle nous permet de considérer comme du bruit $\{Z_t\}_{t\geq 0}$ un processus cylindrique α-stable, correspondant au générateur infinitésimal présenté en (4.5), dont la mesure spectrale a un support concentré uniquement sur les axes de $\mathbb{R}^d$.

Un exemple est donné par la mesure spectrale associée au processus α-stable cylindrique, dont le générateur infinitésimal a été présenté dans (4.5), qui a en fait un support concentré uniquement sur les axes de $\mathbb{R}^d$.

Nous pouvons maintenant résumer les principaux résultats que nous allons prouver dans le Chapitre 4. On commence par montrer que sous des conditions de régularité de Hölder minimale sur la dérive déterministe F, il est possible de montrer le caractère bien posé au sens faible de l'équation stochastique en (6.1).

Théorème 6.1. *Pour chaque j dans $[\![2,n]\!]$, soit β^j un indice dans $(0,1)$ telle que*
- *$x_j \to F_i(t,x_i,\dots,x_j,\dots,x_n)$ est β^j-Hölder continue, uniformément dans le temps et dans les autres variables spatiales, pour chaque i dans $[\![1,j]\!]$.*

Alors l'Équation stochastique (6.1) est bien posée dans un sens faible si

$$\beta^j > \frac{1+\alpha(j-2)}{1+\alpha(j-1)}, \quad j \geq 2. \tag{6.3}$$

Pour obtenir ce résultat, nous allons exploiter l'équivalence, expliquée dans la Section 3, entre le caractère bien posé au sens faible de la dynamique stochastique et le caractère bien posé relative pour le problème de martingale associé au opérateur $\partial_s + L_s$, où L_s est (formellement) le générateur infinitésimal du processus $\{X^{t,x}_s\}_{s\geq 0}$ solution de (6.1). Plus précisément, en rappelant que le triplet de Lévy associé au processus $\{Z_s\}_{s\geq 0}$ est, pour notre modèle, $(b,0,\nu)$, l'opérateur L_s peut être représenté pour chaque fonction ϕ assez régulière à travers

$$\begin{aligned} L_s\phi(s,x) &:= \langle G(s,x), D_x\phi(x)\rangle + \mathcal{L}_s\phi(s,x) \\ &:= \langle A_s x + F(s,x), D_x\phi(x)\rangle + \text{p.v.}\int_{\mathbb{R}^d}\Big[\phi(x+B\sigma(s,x)z)-\phi(x)\Big]\,\nu(dz), \end{aligned} \tag{6.4}$$

où, par commodité, nous avons absorbé le terme faisant intervenir b au sein de l'expression de F et, de façon similaire au Chapitre 2, nous avons exploité la symétrie de la mesure de Lévy ν pour supprimer le terme du premier ordre $\langle D_x\phi(x), B\sigma(t,x)z\rangle$ dans l'intégrale.

Notre méthode de preuve nécessitera également, comme étape intermédiaire pour obtenir le caractère bien posé, d'exhiber un type particulier d'estimations, appelées estimations de Krylov, pour le processus solution de la dynamique dans (6.1). Ces contrôles doivent leur nom à N.V. Krylov qui les a d'abord montré dans [Kry71] pour les diffusions de Itô. Depuis lors, ils sont devenus un outil polyvalent pour de nombreux domaines différents, de la démonstration d'une caractère bien posé en sens forte ou faible pour la dynamique stochastique aux applications pour la théorie du contrôle ou les problèmes de filtration non-linéaire.
Dans un contexte multiplicatif α-stable ($\alpha > 1$) non-dégénéré où de telles estimations sont exploitées, on cite, par exemple, [Kur08] dans lequel l'existence de solutions faibles est montrée pour une dynamique stochastique unidimensionnel à bruit multiplicatif et dérive mesurable et bornée, ou [Zha13a], où le caractère bien posé (au sens fort) d'une équation stochastique à bruit additif dirigée par une dérive singulière dans ses espaces de Sobolev convenables est démontré (sous des conditions similaires à (6.5) pour $n = 1$). Pour d'autres travaux similaires dans le domaine multiplicatif non dégénéré, voir aussi les ouvrages suivants : [AP77, Mel83, LM76].
Pour énoncer précisément les estimations de type Krylov dans notre contexte, cependant, nous devrons imposer certaines conditions sur les indices d'intégrabilité sur l'espace $L^p(0,T;L^q(\mathbb{R}^N))$ des fonctions f considérées. Intuitivement, ce seuil garantira l'intégrabilité nécessaire à notre propos vis-à-vis des échelles intrinsèques données par le caractère dégénéré du système considéré.
Pour simplifier, nous dirons que deux nombres réels p, q dans $(1,+\infty)$ vérifient la condition d'intégrabilité $(\mathscr{C})$ si :

$$\Big(\frac{1-\alpha}{\alpha}N + \sum_{i=1}^{n} id_i\Big)\frac{1}{q} + \frac{1}{p} \,<\, 1. \tag{$\mathscr{C}$}$$

Ce seuil devient en fait plus clair si l'on considère le cas homogène, c'est-à-dire lorsque tous les composants $\mathbb{R}^{d_i}$ du système ont la même dimension ($d_i = d$ et $N = nd$). En fait, dans ce cas, la condition $(\mathscr{C})$ peut alors être réécrite comme

$$\left(\frac{2+\alpha(n-1)}{\alpha}\right)\frac{nd}{q} + \frac{2}{p} \,<\, 2. \tag{6.5}$$

Sous cette forme, ce seuil peut plus naturellement être compris comme une extension des conditions imposées dans [CdRM20b], pour obtenir le même type d'estimations dans le cas diffusif dégénéré ($\alpha = 2$). On mentionne aussi que la condition $(\mathscr{C})$ apparaît aussi dans [KR05] dans un contexte diffusif non-dégénéré ($\alpha = 2$ et $n = 1$) comme l'hypothèse (d'intégrabilité sur f) nécessaire pour intégrer une fonction f contre la densité Gaussienne (cf. Équation (3.2) dans la preuve du Lemme 3.2 dans [KR05]).

Théorème 6.2. *Sous les hypothèses du Théorème 6.1, soit $T > 0$ et p, q dans $(1,+\infty)$ tel que la condition $(\mathscr{C})$ soit valide. Alors, il existe une constante $C := C(T,p,q)$ telle que pour tout f dans $L^p\Big(0,T;L^q(\mathbb{R}^N)\Big)$,*

$$\left|\mathbb{E}\Big[\int_t^T f(s,X_s^{t,x})\,ds\Big]\right| \,\le\, C\|f\|_{L^p_tL^q_x}, \quad (t,x)\in[0,T]\times\mathbb{R}^N, \tag{6.6}$$

où $\{X_s^{t,x}\}_{s\geq 0}$ *est la seule solution faible de la dynamique* (6.1) *avec la condition initiale* (t,x).

Ce type d'estimation souligne également que le processus solution $\{X_s\}_{s\geq 0}$ possède en fait une densité avec des propriétés d'intégrabilité convenables jusqu'à un certain seuil déterminé par la condition $(\mathscr{C})$.

Dans [CdR18], Chaudru de Reynal a caractérisé, à travers des contre-exemples convenables, la régularité de Hölder optimale pour le caractère faiblement bien posé d'une équation stochastique cinétique à bruit diffusif dégénéré (cf. Équation (6.1) avec $\alpha = 2$ et $n = 2$). Grâce à une extension de tels contre-exemples à Peano, nous serons également en mesure d'exhiber un résultat de non-unicité pour la chaîne dégénérée dans (6.1). Rappelons que $\{e_i \colon i \in [\![1, n]\!]\}$ est la base canonique sur l'espace $\mathbb{R}^N$.

Théorème 6.3. *Donnés* j *dans* $[\![2, n]\!]$ *et* i *dans* $[\![2, j]\!]$, *il existe* $F(t,x) = e_i sgn(x_j)|x_j|^{\beta_i^j}$ *avec*
$$\beta_i^j < \frac{1+\alpha(i-2)}{1+\alpha(j-1)},$$
où l'unicité en loi échoue pour la dynamique stochastique (6.1).

Contrairement à la chaîne Gaussienne dégénérée analysée dans [CdRM20b], il n'a cependant pas été possible de montrer dans ce cas que les régularités de Hölder sur les coefficients F_i sont en réalité le minimum nécessaire pour déterminer le caractère bien posé de la dynamique stochastique. Comme expliqué plus loin, ce problème est intrinsèquement lié au caractère stable du processus $\{Z_s\}_{s\geq 0}$, au caractère dégénéré de la dynamique dans (6.1) et en particulier, à la géométrie, éventuellement très singulière, de la mesure spectrale associée au processus proxy sous-jacent, entendu comme le modèle linéarisé (i.e. le processus de type Ornstein-Uhlenbeck) autour duquel nous allons "développer".
Cependant, nous soulignons qu'au moins pour la chaîne dégénérée perturbée par une dérive non-linéaire F uniquement sur la diagonale, nos résultats présentent les seuils optimaux souhaités. Considérons maintenant un système de la forme :

$$\begin{cases} dX_t^1 &= F_1(t, X_t^1, \dots, X_t^n)dt + \sigma(t, X_{t-}^1, \dots, X_{t-}^n)dZ_t, \\ dX_t^2 &= [A_t^2 X_t^1 + F_2(t, X_t^2)]\, dt, \\ dX_t^3 &= [A_t^3 X_t^2 + F_3(t, X_t^3)]\, dt, \\ \vdots \\ dX_t^n &= \left[A_t^n X_t^{n-1} + F_n(t, X_t^n)\right] dt, \end{cases} \tag{6.7}$$

c'est-à-dire, de telle sorte que la fonction F ne dépende que du niveau courant de la chaîne. On constate alors que les Théorèmes 6.1 et 6.3 présentent ensemble une caractérisation (presque) complète de le caractère bien posé faible pour des dynamiques stochastiques dégénérées telles que (6.7), par rapport à la régularité des Hölder de leurs coefficients. En fait, on sait que :

— si $\beta^j > \frac{1+\alpha(j-2)}{1+\alpha(j-1)}$ pour chaque $j \geq 0$, le caractère bien posé faible de la dynamique dans (6.1) dérive du Théorème 6.1 ;

— si il existe $j \geq 2$ telle que $\beta^j = \beta^j_j < \frac{1+\alpha(j-2)}{1+\alpha(j-1)}$, il y a des contre-exemples (cf. Théorème 6.3) pour lesquels l'unicité en loi échoue pour la dynamique en (6.1).

Enfin, nous mentionnons que le cas critique, associé aux exposants

$$\overline{\beta}^j_j = \frac{1+\alpha(j-2)}{1+\alpha(j-1)}, \quad j \in [\![2,n]\!],$$

reste à étudier et apparaît comme un problème délicat déjà dans un contexte brownien cinétique (cf. [Zha18]).

6.1 Guide à l'épreuve

Nous avons décidé de suivre ici une approche perturbative *rétrograde* telle qu'initialement introduite par McKean-Singer dans [MS67] dans un contexte diffusif non-dégénéré puis étendue au cas dégénéré avec dérive illimitée dans [DM10, Men18]. Cette terminologie vient du fait que le processus proxy sous-jacent sera associé à un flux rétrograde dans le temps, c'est-à-dire qu'on mettra $(\tau, \xi) = (s, y)$ dans la dynamique dans (4.12). Cette méthode s'est avérée particulièrement utile dans l'analyse de l'unicité faible dans un contexte dégénéré sur $L^p_t - L^q_x$ espaces (cf. [CdRM20b] dans le cas Brownien). En fait, elle nécessite heuristiquement de n'obtenir que des estimations sur les gradients (au sens faible) des solutions du problème de Cauchy associé, de manière à appliquer la technique d'inversion d'opérateurs, telle qu'exploitée à l'origine dans [SV79] dans le cas diffusif.

Malgré ce qui a été suggéré initialement dans la Section 3, nous avons finalement décidé de ne pas exploiter les estimations de Schauder, montrées dans les deux sections précédentes, pour prouver l'unicité en loi de la dynamique stochastique dans (6.1). Nous sommes cependant convaincus que l'approche perturbative par le proxy progressif présentée dans la Section 4 aurait pu être étendue ici pour présenter des estimations de Schauder pour la classe de processus considérée maintenant. A partir d'eux, nous aurions effectivement pu démontrer le caractère bien posé de l'équation stochastique au sens faible, par un raisonnement de type Zvonkin. Cette méthode est, dans l'imaginaire collectif, étroitement liée à la preuve de le caractère bien posé forte de la dynamique stochastique mais nous soulignons qu'elle a aussi été utilisée pour prouver le caractère bien posé au sens faible pour les chaînes dégénérées, comme fait, pour exemple , dans [CdR18]. C'est ce dernier type de raisonnement que nous appellerons plus loin raisonnement de type Zvonkin. Cependant, cette méthode apparaît d'emblée très longue et compliquée puisqu'elle aurait nécessité d'étendre les estimations de Schauder, présentées dans (4.11) sous la régularité optimale attendue dans $C^\beta_{b,d}(\mathbb{R}^N)$ sur la source f, vers un espace de Hölder plus générique avec des indices de régularité sans rapport, du type $C^\gamma_{b,d}(\mathbb{R}^N)$ avec $\gamma = (\gamma_1, \dots, \gamma_d)$ dans $\mathbb{N}^d$, juste à cause de la méthode de Zvonkin. En particulier, il aurait fallu établir des estimations ponctuelles sur les dérivées du premier ordre de la solution du problème de Cauchy associé également par rapport aux composantes dégénérées du système. Pour ce faire, il aurait fallu avant tout étendre notre raisonnement de dualité sur les espaces de Besov également par rapport aux dérivées dégénérées et considérer une source f avec la même régularité que la dérive F dans un espace $C^\gamma_{b,d}(\mathbb{R}^N)$ avec multi-indices de régularité. Un autre avantage possible de la méthode perturbative rétrograde est qu'elle permet d'obtenir facilement, en tant qu'étape intermédiaire de notre épreuve, les estimations de

Krylov dans (6.6) sur le processus $\{X_s^{t,x}\}_{s\geq 0}$ solution de la dynamique stochastique dans (1.3). Ce type de contrôle semble être nouveau pour les chaînes stochastiques dégénérées dirigées par des bruits quasi-stables et aurait été très difficile à obtenir à partir des estimations de Schauder.
Enfin, nous soulignons que notre méthode, en comparaison avec l'approche par la transformation de Zvonkin comme dans [CdR18], permet d'obtenir une analyse plus précise de la chaîne dégénérée au moins le long de la première composante déterministe, dans le sens où l'on peut souligner ici que des seuils minimaux sur la régularité spatiale de F_1 ne sont pas requis. Intuitivement, la méthode via la transformation de Zvonkin oblige à mettre comme source chaque composant F_i de la dérive. Ceci conduit notamment aux mêmes seuils optimales globales sur la régularité de Hölder pour F à chaque niveau de la chaîne (cf. Équation (6.3) avec $j \geq 1$). Comme montré par exemple dans [CdR17, FFPV17, CdRM20b] dans le cas diffusif, cette approche semble être plus adéquate quand on veut exhiber le caractère bien posé *fort* de la dynamique stochastique.

Pour les besoins de cette présentation, nous nous limiterons à considérer une matrice A_t dans (6.2) indépendante du temps et telle que seuls ses éléments dans la sous-diagonale sont non-nuls, c'est-à-dire que nous allons supposons que A est donné dans (4.6).
En fait, comme expliqué dans la section précédente, les difficultés supplémentaires données par la présence d'éléments non-nuls au-dessus de la sous-diagonale peuvent être facilement résolues par un raisonnement en peu de temps. De plus, l'éventuelle dépendance au temps nécessiterait essentiellement d'introduire, au lieu de la matrice exponentielle $e^{A(s-t)}$, la résolvante $\mathcal{R}_{s,t}$ associée à la matrice A_t. En pratique, $\mathcal{R}_{s,t}$ est la matrice en $\mathbb{R}^N \otimes \mathbb{R}^N$ solution dépendante du temps de l'équation différentielle matricielle suivante :

$$\begin{cases} \partial_s \mathcal{R}_{s,t} \ = \ A_s \mathcal{R}_{s,t}, \quad s \in [t,T]; \\ \mathcal{R}_{t,t} \ = \ \mathrm{Id}_{N\times N}\, . \end{cases}$$

Enfin, nous mentionnons que des décompositions similaires à celles présentées dans (4.7), se sont également avérées valables pour la résolvante $\mathcal{R}_{s,t}$. Voir par exemple [HM16], Lemmes 5.1 et 5.2 ou [DM10], Proposition 3.7.

Méthode de la parametrix rétrogradée

Comme déjà expliqué dans la section 4, l'élément crucial de la méthode perturbative consiste à choisir soigneusement un opérateur proxy convenable avec des propriétés et des contrôles connues, autour duquel développer le générateur infinitésimal L_s, au détriment d'une erreur d'extension supplémentaire à vérifier.
Lorsque la dérive F est suffisamment régulière, par exemple globalement Lipschitz continue, il a été montré par exemple dans [DM10, Men11, Men18] qu'une proxy adéquate est donnée par la linéarisation de la dynamique stochastique (6.1) autour du flot déterministe associé à la dynamique (i.e. quand $\sigma = 0$ dans (6.1)), ce qui a conduit, dans les travaux ci-dessus dans un contexte Brownien dégénéré, à considérer un processus Gaussien à plusieurs échelles comme proxy. La généralisation naturelle au contexte considéré ici conduira plutôt à choisir comme proxy un processus multi-échelles de type Lévy dont le symbole sera dépendant du temps.
Plus précisément, pour les paramètres de congélation (s, y) fixés dans $[0, T] \times \mathbb{R}^N$, soit

$\theta_{t,s}(y)$ l'une des solutions possibles de l'équation suivante :

$$\theta_{t,s}(y) \;=\; y - \int_t^s \Big[A\theta_{u,s}(y) + F(u,\theta_{u,s}(y))\Big]\, du. \tag{6.8}$$

Puisque le point de congélation y sera alors aussi la variable d'intégration (voir par exemple la définition du noyau de Green dans (6.18)), il est important de souligner tout de suite que est toujours possible, parmi les flots possibles $\theta_{t,s}(y)$ solution de (6.8), de choisir un qui soit mesurable par rapport à (s,y) dans $[0,T]\times\mathbb{R}^N$ (cf. Lemme 2.13 dans le Chapitre 4). Nous supposerons à partir de maintenant que nous n'avons corrigé que cette version de $\theta_{t,s}(y)$.

La prochaine étape sera d'introduire la dynamique stochastique *linéarisée* autour du flot rétrograde $\theta_{t,s}(y)$. Plus précisément, nous considérerons pour chaque point de départ (t,x) dans $[0,s]\times\mathbb{R}^N$, le processus proxy $\{\tilde{X}^{t,x,s,y}_u\}_{u\ge 0}$ solution de l'équation stochastique suivante :

$$\begin{cases} d\tilde{X}^{t,x,s,y}_u \;=\; \Big[A\tilde{X}^{t,x,s,y}_u + \tilde{F}^{s,y}_u\Big]\, du + B\tilde{\sigma}^{s,y}_u\, dZ_u, \quad u\in[t,T], \\ \tilde{X}^{t,x,s,y}_t \;=\; x, \end{cases} \tag{6.9}$$

où nous avons noté pour simplifier, $\tilde{F}^{s,y}_u := F(u,\theta_{u,s}(y))$ et $\tilde{\sigma}^{s,y}_u := \sigma(u,\theta_{u,s}(y))$. Un calcul direct permet alors d'obtenir une représentation intégrale du processus proxy :

$$\tilde{X}^{t,x,s,y}_s \;=\; \tilde{m}^{s,y}_{s,t}(x) + \int_t^s e^{A(s-u)}B\tilde{\sigma}^{s,y}_u dZ_u, \tag{6.10}$$

où, de manière similaire à (4.14), le terme de transport congelé $\tilde{m}^{s,y}_{s,t}(x)$ est donné par

$$\tilde{m}^{s,y}_{s,t}(x) \;=\; e^{A(s-t)}x + \int_t^s e^{A(s-u)}\tilde{F}^{s,y}_u\, du.$$

Des raisonnements sur les espaces de Fourier, similaires à ceux développés dans la Section 4 dans (4.15)-(4.16), permettent de montrer aussi dans ce cas l'identité fondamentale suivante en loi :

$$\tilde{X}^{t,x,s,y}_s \overset{(\text{loi})}{=} \tilde{m}^{s,y}_{s,t}(x) + \mathbb{M}_{s-t}\tilde{S}^{s,y}_{s-t}, \tag{6.11}$$

où pour chaque paramètre de congélation (s,y) fixé, $\{\tilde{S}^{s,y}_u\}_{u\ge 0}$ est attribuable à un processus α-stable non-dégénéré sur $\mathbb{R}^N$ au sens indiqué dans [**ND**']. Nous soulignons également que la dépendance aux paramètres de congélation, une difficulté cruciale dans cette partie de la preuve, est essentiellement liée à la présence de la matrice de diffusion congelée $\tilde{\sigma}^{s,y}$ dans la convolution stochastique dans (6.10). En fait, cette dépendance disparaîtrait dans le cas d'un bruit additif (i.e. $\sigma(t,x)=1$) ou, plus généralement, pour un coefficient de diffusion homogène dans l'espace.
Comme déjà expliqué dans la Section 4, la non-dégénérescence de la mesure spectrale du processus $\{\tilde{S}^{s,y}_u\}_{u\ge 0}$ assure notamment l'existence d'une densité $p_{\tilde{S}^{s,y}}(u,\cdot)$ suffisamment régulière (dans l'espace) pour ce processus. L'identité dans (6.11) implique alors que la fonction suivante :

$$\tilde{p}^{s,y}(t,s,x,y) \;:=\; \frac{p_{\tilde{S}^{s,y}}(s-t,\mathbb{M}^{-1}_{s-t}(y-\tilde{m}^{s,y}_{t,s}(x)))}{\det\mathbb{M}_{s-t}},$$

est plutôt la "densité" associée au processus proxy $\{\tilde{X}_s^{t,x,s,y}\}_{s\geq 0}$ congelé au point final (s, y).

De la même manière que dans les sections précédentes, nous nous concentrerons ensuite sur la détermination des propriétés régularisantes associées à $\tilde{p}^{s,y}(t, s, x, y)$. En particulier, nous montrerons que les dérivées de la "densité" congelée sont contrôlées d'en haut par une autre densité au prix de singularités supplémentaires dans le temps et surtout, que ce contrôle tient *uniformément* dans les paramètres de congélation (s, y). Plus précisément, il vaudra pour chaque k dans $[\![0, 2]\!]$ et pour chaque i dans $[\![1, n]\!]$, qui :

$$|D^k_{x_i}\tilde{p}^{s,y}(t, s, x, y)| \leq C\frac{\overline{p}\left(1, \mathbb{T}^{-1}_{s-t}(y - \tilde{m}^{s,y}_{s,t}(x))\right)}{\det \mathbb{T}_{s-t}}(s - t)^{-k\frac{1+\alpha(i-1)}{\alpha}}, \tag{6.12}$$

où nous avons noté pour simplifier $\mathbb{T}_u = u^{\frac{1}{\alpha}}\mathbb{M}_u$. Ce type de contrôle peut essentiellement être vu comme un analogue de celui obtenu dans (4.19), où pour plus de commodité nous avons déjà re-dimensionné le système à l'unité de temps $t = 1$, en utilisant le α -autosimilarité associée à la densité α -stable $\overline{p}(u, \cdot)$, c'est à dire :

$$\overline{p}(u, z) \;=\; u^{-N/\alpha}\overline{p}(1, u^{-1/\alpha}z), \quad (u, z) \in [0, \infty) \times \mathbb{R}^N.$$

Puisque le point de congélation y servira aussi plus tard comme variable d'intégration (cf. définition de $\tilde{G}_\epsilon f$ dans (6.18)), nous soulignons en particulier ce qu'il est important d'obtenir un contrôle d'en haut avec une densité indépendante de ce paramètre.
Les estimations sur les dérivées de densité comme dans (6.12) ou dans (4.19) sont souvent obtenues grâce à la décomposition de Itô-Lévy de la variable aléatoire $\tilde{S}^{s,y}_u$ au temps caractéristique stable correspondant. Plus précisément, soit $\tilde{M}^{s,y}_u$ et $\tilde{N}^{s,y}_u$ les deux variables aléatoires indépendantes associées aux contributions pour les petits et grands sauts du processus $\{\tilde{S}^{s,y}_u\}$ tronqué à l'instant $u^{1/\alpha}$.
Cette troncature permettra notamment de réécrire la densité $p_{\tilde{S}^{s,y}}(u, z)$ de $\tilde{S}^{s,y}_u$ comme suit :

$$p_{\tilde{S}^{s,y}}(u, z) \;=\; \int_{\mathbb{R}^N} p_{\tilde{M}^{s,y}}(u, z - w)\, P_{\tilde{N}^{s,y}_u}(dw) \tag{6.13}$$

où $p_{\tilde{M}^{s,y}}(u, \cdot)$ est la densité générée par $\tilde{M}^{s,y}_u$ et $P_{\tilde{N}^{s,y}_u}$ est la loi du $\tilde{N}^{s,y}_u$.
Tandis que des arguments tels que ceux développés dans la Section 5 ci-dessus (cf. Équation (5.21)) sur la densité tronquée p^{tr} peuvent également être appliqués dans ce cas conduisant aux estimations suivantes :

$$\left|D^k_z p_{\tilde{M}^{s,y}}(u, z)\right| \;\leq\; Cu^{-(N+k)/\alpha}\left(\frac{u^{1/\alpha}}{u^{1/\alpha} + |z|}\right)^{N+3} \;=:\; Cu^{-\frac{k}{\alpha}}p_{\bar{M}}(u, z), \tag{6.14}$$

où C est indépendant des paramètres de congélation (s, y) ; il sera beaucoup plus délicat d'obtenir un contrôle uniforme sur la mesure de probabilité $P_{\tilde{N}^{s,y}_u}(dy)$ comme :

$$P_{\tilde{N}^{s,y}_u}(\mathcal{A}) \;\leq\; C\overline{P}_u(\mathcal{A}), \quad \mathcal{A} \in \mathcal{B}(\mathbb{R}^N), \tag{6.15}$$

où $\{\overline{P}_u\}_{u\geq 0}$ est une famille de mesures de probabilité qui conserve les mêmes propriétés d'intégration de la queue d'un processus α-stable.
Cette difficulté est la principale raison pour laquelle on n'a pas pu considérer, comme dans

[CdRM20b], une dérive totalement non-linéaire, c'est-à-dire $G_i(t,x) = G_i(t,x_{i-1},\dots,x_n)$ avec une dépendance non-linéaire sur la variable de transmission de bruit x_{i-1}, mais seulement une version semi-linéaire de G, donnée dans (6.2). En fait, le modèle plus général analysé dans [CdRM20b] aurait nécessité notamment de linéariser la dynamique stochastique (6.1) le long de la sous-diagonale de la matrice Jacobienne de G congelée dans (s,y), ou de considérer le processus proxy suivant :

$$d\tilde{X}^{t,x,s,y}_u \;=\; \Big[\tilde{A}^{s,y}_u \tilde{X}^{t,x,s,y}_u + \tilde{F}^{s,y}_u\Big]\, du + B\tilde{\sigma}^{s,y}_u \, dZ_u,$$

où $\tilde{A}^{s,y}_u$ est une matrice dépendante des paramètres de congélation telle que

$$[\tilde{A}^{s,y}_u]_{i,j} \;=\; \begin{cases} D_{x_{i-1}} G_i(s,\theta_{t,s}(y)), & \text{si } j = i-1, \\ 0, & \text{autrement.} \end{cases}$$

Pour ce type de modèle, nous n'avons pas pu réellement montrer un contrôle comme dans (6.15).
Nous soulignons également que c'est toujours pour cette raison que nous ne pourrions pas considérer une mesure de Lévy ν associée au processus $\{Z_t\}_{t\ge 0}$ comme étant asymétrique, comme cela a été fait par exemple dans le Chapitre 3 de la présente thèse. En effet, la méthode perturbative à parametrix rétrograde nécessite des propriétés de régularisation plus délicates associées aux opérateurs impliqués et surtout une certaine compatibilité entre le proxy et le processus d'origine.
Intuitivement, la symétrie de la mesure de Lévy sous-jacente n'est pas une contrainte forte pour la contribution dans (6.14) associée aux petits sauts et cela a en effet permis dans la section précédente de considérer également des opérateurs asymétriques. Dans notre modèle, qui nécessite au contraire un effet global de régularisation pour la densité, il semble naturel d'imposer la symétrie de la mesure spectrale correspondante pour le contrôle des queues du processus, comme on le verra dans l'épreuve pour l'estimation en (6.15) (cf. Lemme 5.2 dans le Chapitre 4).

Propriétés analytiques de la densité congelée le long de la condition terminale

Soulignons maintenant qu'il n'est pas immédiat de déterminer pour quel type de problème de Cauchy la "densité" $\tilde{p}^{s,y}(t,s,x,y)$ congelée au point final (s,y) est en effet une solution fondamentale. En effet, la présence du point de congélation y également comme variable d'intégration (par exemple dans (6.18)) rend beaucoup plus délicat la détermination de ses propriétés analytiques et surtout, de prouver sa convergence vers la masse de Dirac δ_x lorsque le temps t tend vers zéro.
Soit $\tilde{L}^{s,y}_t$ le générateur infinitésimal associé au processus congelé $\{\tilde{X}^{s,y,t,x}_s\}_{s\ge 0}$. Plus précisément, on écrit pour chaque fonction $\phi\colon \mathbb{R}^N \to \mathbb{R}$ assez régulière, qui :

$$\begin{aligned} \tilde{L}^{s,y}_t \phi(x) &:= \langle Ax + \tilde{F}^{s,y}_t, D_x\phi(x)\rangle + \tilde{\mathcal{L}}^{s,y}_t\phi(x) \\ &:= \langle Ax + \tilde{F}^{s,y}_t, D_x\phi(x)\rangle + \text{p.v.}\int_{\mathbb{R}^d_0} \Big[\phi(x+B\tilde{\sigma}^{s,y}_t w) - \phi(x)\Big]\,\nu(dw), \end{aligned} \tag{6.16}$$

où, rappelons-le, nous avons dénoté $\tilde{F}^{s,y}_t := F(t,\theta_{t,s}(y))$ et $\tilde{\sigma}^{s,y}_t := \sigma(t,\theta_{t,s}(y))$.
Si nous définissons maintenant le paramètre de congélation y et changeons uniquement

la variable d'intégration dans $\tilde{p}^{s,y}(t,s,x,\cdot)$, la fonction devient une densité à tous les effets et il n'est donc pas difficile de montrer, par un calcul direct, que :

$$\left(\partial_t + \tilde{L}^{s,y}_t\right)\tilde{p}^{s,y}(t,s,x,z) \,=\, 0, \quad (t,z)\in[0,s)\times\mathbb{R}^N, \tag{6.17}$$

pour chaque (s,x,y) dans $[0,T]\times\mathbb{R}^{2N}$ fixé.

Pour déterminer quel type de système parabolique est résolu par la densité congelée $(s,y)\mapsto \tilde{p}^{s,y}(t,s,x,y)$, nous considérons maintenant un opérateur $\tilde{G}_\epsilon$ qui peut être compris comme le noyau de Green associé à la “densité” $\tilde{p}^{s,y}(t,s,x,y)$ et situé à l'extérieur du temps initial t. En particulier, étant donné $\epsilon>0$ assez petit pour nos besoins,

$$\tilde{G}_\epsilon f(t,x) \,:=\, \int_{t+\epsilon}^{T}\int_{\mathbb{R}^N}\tilde{p}^{s,y}(t,s,x,y)f(s,y)\,dyds, \quad (t,x)\in[0,T)\times\mathbb{R}^N, \tag{6.18}$$

pour toute fonction $f\colon[0,T)\times\mathbb{R}^N\to\mathbb{R}$ suffisamment régulière et à support compact. L'expression dans (6.18) est bien définie, car on sait que la densité gelée $\tilde{p}^{s,y}(t,s,x,y)$ est mesurable en (s,y).
La localisation en ϵ pour éloigner l'intégrale de t est fondamentale puisqu'elle assure l'effet régularisant recherché pour le noyau de Green $\tilde{G}_\epsilon$. En fait, nous soulignons que, dans le cas limite (i.e. $\epsilon\to0$), la régularité de la fonction f n'est pas une condition suffisante pour dériver la régularité pour le noyau de Green $\tilde{G}_0f$. Cette difficulté supplémentaire provient de la dépendance du proxy vis-à-vis de la variable d'intégration y.

Si nous introduisons maintenant également la quantité suivante :

$$\tilde{M}_\epsilon f(t,x) \,:=\, \int_{t+\epsilon}^{T}\int_{\mathbb{R}^N}\tilde{L}^{s,y}_t\tilde{p}^{s,y}(t,s,x,y)f(s,y)\,dyds, \quad (t,x)\in[0,T)\times\mathbb{R}^N,$$

enfin on peut déduire de l'équation (6.17) que le pseudo-noyau de Green $\tilde{G}_\epsilon$ résout l'équation parabolique suivante :

$$\partial_t\tilde{G}_\epsilon f(t,x)+\tilde{M}_\epsilon f(t,x) \,=\, -I_\epsilon f(t,x), \quad (t,x)\in[0,T)\times\mathbb{R}^N. \tag{6.19}$$

Encore une fois, la localisation dans ϵ était cruciale pour obtenir (6.19) directement à partir de l'équation (6.17) en utilisant des arguments de convergence dominés classiques. Ci-dessus, l'opérateur I_ϵ pour chaque fonction $f\colon[0,T)\times\mathbb{R}^N\to\mathbb{R}$ assez régulière peut être représenté par :

$$I_\epsilon f(t,x) \,:=\, \int_{\mathbb{R}^N} f(t+\epsilon,y)\mathbb{1}_{[0,T-\epsilon]}(t)\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)\,dy. \tag{6.20}$$

Nous soulignons maintenant que I_ϵ peut être compris comme une version localisée a l'extérieur de t de l'opérateur d'identité. En particulier, on obtiendra dans le Chapitre 4 (cf. Lemmes 2.18 et 2.19), la convergence vers la masse de Dirac concentrée en (t,x) sur les espaces fonctionnels que nous considérons :

$$\lim_{\epsilon\to0}\|I_\epsilon f-f\|_\infty \,=\, 0, \quad \lim_{\epsilon\to0}\|I_\epsilon f-f\|_{L^p_tL^q_x} \,=\, 0. \tag{6.21}$$

Bien qu'à première vue les propriétés de convergence ci-dessus semblent immédiates, nous soulignons que la présence de la variable d'intégration y également comme paramètre de gel empêche d'obtenir les estimations dans (6.21) directement de la convergence en loi du processus congelé $\{\tilde{X}^{s,y,t,x}_s\}_{s\geq0}$ vers la masse de Dirac (cf. Équation (6.17)).

Unicité du problème de la martingale et estimations de Krylov associées

Nous pouvons maintenant présenter les principales étapes de notre démonstration de l'unicité du problème de martingale associé à $\partial_s + \mathcal{L}_s$. Comme mentionné précédemment, la théorie analytique associée au processus proxy expliqué ci-dessus sera l'outil fondamental de notre méthode d'épreuve.
Nous exhiberons dans un premier temps les estimations de type Krylov dans (6.6) en utilisant la méthode perturbative avec le proxy rétrograde, en supposant cependant que les indices p, q sont suffisamment grands mais finis.
Plus précisément, étant donné une fonction f assez régulière pour notre propos et une solution $\{X^{t,x}_s\}_{s\geq 0}$ de la dynamique stochastique dans (6.1), la première étape de notre méthode consiste à appliquer la formule de Itô au noyau de Green congelé $\tilde{G}_\epsilon f$ par rapport au processus $\{X^{t,x}_s\}_{s\geq 0}$:

$$\mathbb{E}\left[\tilde{G}_\epsilon f(t,x) + \int_t^T (\partial_s + L_s)\tilde{G}_\epsilon f(s, X^{t,x}_s) ds\right] \,=\, 0. \tag{6.22}$$

En rappelant que le pseudo-noyau de Green $\tilde{G}_\epsilon f$ résout l'équation en (6.19), on peut réécrire (6.22) comme suit :

$$\begin{aligned}\mathbb{E}\Big[\int_t^T \int_{\mathbb{R}^N} I_\epsilon f(s, X^{t,x}_s)\, ds\Big] \,&=\, \tilde{G}_\epsilon f(t,x) + \mathbb{E}\Big[\int_t^T \Big[L_s \tilde{G}_\epsilon f - \tilde{M}_\epsilon f\Big](s, X^{t,x}_s)\, ds\Big] \\ &=: \, \tilde{G}_\epsilon f(t,x) + \mathbb{E}\Big[\int_t^T \tilde{R}_\epsilon f(s, X^{t,x}_s)\, ds\Big].\end{aligned} \tag{6.23}$$

Pour obtenir les estimations de Krylov dans (6.6) à partir de l'équation ci-dessus, nous devrons présenter des contrôles telles que :

$$\|\tilde{G}_\epsilon f\|_\infty \,\leq\, C\|f\|_{L^p_t L^q_x}, \quad \|\tilde{R}_\epsilon f\|_\infty \,\leq\, C\|f\|_{L^p_t L^q_x}. \tag{6.24}$$

En particulier, la nécessité d'imposer d'abord un seuil minimum sur les indices p, q, est justement due au contrôle précis du terme du reste $\tilde{R}_\epsilon f$, valable uniquement pour p et q suffisamment grands.
Après avoir exposé des estimations comme dans (6.24), on peut alors montrer à partir de l'Équation (6.23) que

$$\left|\mathbb{E}\left[\int_t^T I_\epsilon f(s, X^{t,x}_s)\, ds\right]\right| \,\leq\, C\|f\|_{L^p_t L^q_x}.$$

Exploitant maintenant les propriétés de convergence ponctuelle de I_ϵ dans (6.21) pour faire tendre ϵ vers zéro, et un raisonnement d'approximation régulier sur l'espace des fonctions $L^p(0,T; L^q(\mathbb{R}^N))$, il sera enfin possible de conclure que des estimations de Krylov de la forme :

$$\left|\mathbb{E}\left[\int_t^T f(s, X^{t,x}_s)\, ds\right]\right| \,\leq\, C\|f\|_{L^p_t L^q_x}, \tag{6.25}$$

sont valables pour chaque f dans $L^p(0,T; L^q(\mathbb{R}^N))$, bien que sous la condition supplémentaire que p et q soient suffisamment grands.

Le résultat "partiel" obtenu ci-dessus est cependant suffisant pour montrer l'unicité du problème de martingale associé à l'opérateur $\partial_s + L_s$. A travers un raisonnement de dualité sur les espaces $L^p_t - L^q_x$, l'estimation dans (6.25) impliquera notamment l'existence d'une "densité" pour le processus $\{X^{t,x}_s\}_{s\geq 0}$ solution de la dynamique stochastique (6.1) avec le point de départ (t,x), défini seulement pour presque tout (s,y).
Appelée $p(t,s,x,y)$ cette densité, il suivi facilement de l'Équation (6.23) qui

$$\tilde{G}_\epsilon f(t,x) \,=\, \int_t^T \int_{\mathbb{R}^N} (I_\epsilon - \tilde{R}_\epsilon) f(s,y) p(t,s,x,y)\, dyds.$$

A ce stade, nous utiliserons la technique d'inversion de l'opérateur $I_\epsilon - \tilde{R}_\epsilon$ sur l'espace fonctionnel $L^p\Big(0,T;L^q(\mathbb{R}^N)\Big)$. Pour ce faire, il faudra notamment exhiber des estimations sur le terme de reste $\tilde{R}_\epsilon$ de la forme suivante :

$$\|\tilde{R}_\epsilon\|_{L^p_tL^q_x} \,\leq\, C_T \|f\|_{L^p_tL^q_x}, \tag{6.26}$$

pour une certaine constante $C_T > 0$ indépendante de ϵ et telle que $C_T \to 0$ lorsque T tend vers zéro.
En choisissant un intervalle de temps suffisamment petit, on peut alors supposer que la norme de $\tilde{R}_\epsilon$ comme opérateur sur $L^p\Big(0,T;L^q(\mathbb{R}^N)\Big)$ est plus petit de 1 et donc, que l'opérateur $I_\epsilon - \tilde{R}_\epsilon$ est en réalité inversible :

$$\mathbb{E}\left[\int_t^T f(s,X^{t,x}_s)\, ds\right] \,=\, \tilde{G}_\epsilon \circ (I_\epsilon - \tilde{R}_\epsilon)^{-1} f(t,x).$$

Exploitant à nouveau les propriétés de convergence de l'opérateur I_ϵ dans (6.21), cette fois dans l'espace $L^p\Big(0,T;L^q(\mathbb{R}^N)\Big)$, nous pouvons conclure que

$$\mathbb{E}\left[\int_t^T f(s,X^{t,x}_s)\, ds\right] \,=\, \tilde{G} \circ (I - \tilde{R})^{-1} f(t,x).$$

L'identité ci-dessus implique immédiatement l'unicité du problème de martingale sur un intervalle de temps suffisamment petit. Un argument de chaînage temporel nous permettra d'étendre à l'unicité globale pour le problème de martingale, concluant ainsi la preuve du théorème 6.1. Pour plus de détails sur ce type de localisation, voir pour exemple [EK86] Section 4.6.

Nous montrerons seulement plus tard que les estimations de Krylov dans (6.6) sont en fait valables pour chaque paire (p,q) qui vérifie la condition d'intégrabilité $(\mathscr{C})$, par un raisonnement de mollification stochastique.
Plus précisément, nous allons fixer un processus α-stable isotrope $\{\overline{Z}_s\}_{s\geq 0}$ et un petit paramètre de régularisation δ et considérer une version régularisée du processus $\{X^{t,x}_s\}_{s\geq 0}$ solution de la dynamique stochastique dans (6.1), donnée par

$$\overline{X}^{t,x,\delta}_s \,:=\, X^{t,x}_s + \delta \mathbb{M}_{s-t} \overline{Z}_{s-t}. \tag{6.27}$$

Intuitivement, l'idée de régulariser le processus solution $\{X^{t,x}_s\}_{s\geq 0}$ sert à obtenir, au niveau des densités associées, une contrôlabilité dans l'espace fonctionnel dual de $L^p_t - L^q_x$

jusqu'au seuil souhaité (cf. condition d'intégrabilité ($\mathscr{C}$)).
En effet, si on note avec p', q' les exposants conjugués de p et q respectivement, on sait d'après les estimations “partielles” de Krylov dans (6.25) que la densité $p(t,s,x,y)$ a une norme finie dans l'espace $L^{p'}\left(0,T,L^{q'}(\mathbb{R}^N)\right)$ pour des valeurs de p,q assez grand.
De plus, si on appelle avec $p^\delta(t,s,x,\cdot)$ la densité associée à la variable aléatoire $\overline{X}^{t,x,\delta}_s$, l'identité dans (6.27) implique immédiatement que

$$p^\delta(t,s,x,y) \;=\; \Big[p(t,s,x,\cdot) * q^\delta(s-t,\cdot)\Big](y), \tag{6.28}$$

où $q^\delta(t,\cdot)$ représente la densité associée au processus $\{\delta\mathbb{M}_s\overline{Z}_s\}_{s\geq 0}$.
En exploitant maintenant des inégalités de convolution dans (6.28), nous obtiendrons en particulier que pour chaque paire (p,q) qui vérifie la condition ($\mathscr{C}$), la quantité $\|p^\delta\|_{L^{p'}_t L^{q'}_x}$ est fini, bien que possiblement explosif pour δ tendant vers zéro.
Ce contrôle nous permettra de prouver que le processus mollifié $\overline{X}^{t,x,\delta}_s$ vérifie les estimations de Krylov dans (6.6) pour tous les indices p, q dans l'intervalle concerné mais pour une constante C éventuellement dépendante du paramètre du régularisation δ (et explosive par rapport à ce paramètre).
Reproduisant l'analyse perturbative réalisée dans la première partie de la preuve, nous montrerons enfin que les contrôles sur la densité mollifié $p^\delta(t,s,x,y)$, et donc la constante dans les estimations de Krylov, ne dépendent pas réellement de δ.
Enfin, en faisant tendre δ vers zéro, nous pourrons exhiber les estimations de Krylov dans les conditions requises sur p, q pour le processus $X^{t,x}_s$ solution de la dynamique stochastique originale (6.1).

Sur les contrôles fondamentales du noyau de Green et du terme de reste

Comme nous venons de le voir, notre méthode repose en fait sur des estimations fondamentales sur le noyau de Green $\tilde{G}_\epsilon f$ et sur le terme de reste $\tilde{R}_\epsilon f$ donné dans (6.24) et (6.26). La preuve de ces estimations sera assez longue et complexe et remplira, avec la preuve de la convergence de I_ϵ dans (6.21), une grande partie de la section technique du Chapitre 4.
Comme nous ne pensons pas qu'il soit possible de les résumer pour cette introduction de manière cohérente pour le lecteur, nous avons plutôt décidé de ne mettre en évidence que quelques-uns des passages les plus saillants du raisonnement qui nous permettent de montrer la nécessité de certaines des hypothèses nous avons fait.

Par exemple, les seuils dans la condition d'intégrabilité ($\mathscr{C}$) apparaissent évidents dans la preuve des contrôles ponctuels dans (6.24) sur le noyau de Green $\tilde{G}_\epsilon$.
En fait, en rappelant la définition de $\tilde{G}_\epsilon f$ donnée dans (6.18), on peut dans un premier temps casser l'intégrale par une inégalité de Hölder :

$$\begin{aligned}|\tilde{G}_\epsilon f(t,x)| \;&\leq\; C\|f\|_{L^p_t L^q_x}\left(\int_{t+\epsilon}^T\left(\int_{\mathbb{R}^N}|\tilde{p}^{s,y}(t,s,x,y)|^{q'}\,dy\right)^{\frac{p'}{q'}}ds\right)^{\frac{1}{p'}} \\ &=: C\|f\|_{L^p_t L^q_x}\left(\int_{t+\epsilon}^T\left(|\mathcal{I}(t,s,x)|\right)^{\frac{p'}{q'}}ds\right)^{\frac{1}{p'}},\end{aligned} \tag{6.29}$$

où, rappelons-le, nous avons dénoté respectivement avec p', q' les exposants conjugués de p et q.
Profitant des contrôles sur la densité $\tilde{p}^{s,y}(t,s,x,\cdot)$ dans (6.12), il est désormais possible de contrôler le terme $\mathcal{J}(t,s,x)$ comme suit :

$$|\mathcal{J}(t,s,x)| \le C\,(\det \mathbb{T}_{s-t})^{-q'} \int_{\mathbb{R}^N} \Big[\bar{p}\left(1, \mathbb{T}_{s-t}^{-1}(y-\tilde{m}_{s,t}^{s,y}(x))\right)\Big]^{q'} dy \le C\,(\det \mathbb{T}_{s-t})^{1-q'}.$$

Il s'ensuit alors que

$$|\tilde{G}_\epsilon f(t,x)| \le C\|f\|_{L_t^p L_x^q} \left(\int_{t+\epsilon}^T (\det \mathbb{T}_{s-t})^{(1-q')\frac{p'}{q'}}\, ds\right)^{\frac{1}{p'}}. \tag{6.30}$$

Pour conclure, il faut montrer que l'intégrale en temps est finie. A partir de la définition $\mathbb{T}_t := t^{1/\alpha}\mathbb{M}_t$, on note alors que

$$\det \mathbb{T}_{s-t} = (s-t)^{\frac{1}{\alpha}+\sum_{i=1}^n d_i(i-1)} = (s-t)^{\sum_{i=1}^n d_i \frac{1+\alpha(i-1)}{\alpha}}.$$

Puisque $(1-q')\frac{p'}{q'} = -\frac{p'}{q}$, on peut enfin conclure que l'intégrale en temps dans (6.30) est finie si

$$\Big(\sum_{i=1}^n d_i \frac{1+\alpha(i-1)}{\alpha}\Big)\frac{p'}{q} < 1 \Leftrightarrow \Big(\sum_{i=1}^n d_i \frac{1+\alpha(i-1)}{\alpha}\Big)\frac{1}{q} + \frac{1}{p} < 1.$$

L'inégalité à droite est précisément la condition d'intégrabilité ($\mathscr{C}$) que nous avons supposée dans le Théorème 6.2. Intuitivement, on peut alors expliquer le seuil en ($\mathscr{C}$) comme la condition d'intégrabilité temporelle nécessaire pour contrôler les différentes échelles intrinsèques associées au système dégénéré, lorsque l'on considère les estimations de normes $L_t^p - L_x^q$.

On se concentre maintenant sur le contrôle ponctuel du terme de reste $\tilde{R}_\epsilon$ donné dans (6.24). A partir de la définition dans (6.23), on peut décomposer $\tilde{R}_\epsilon$ en deux parties :

$$\begin{aligned}\tilde{R}_\epsilon f(t,x) &= \int_{t+\epsilon}^T \int_{\mathbb{R}^N} (\mathcal{L}_t - \tilde{\mathcal{L}}_t^{s,y})\tilde{p}^{s,y}(t,s,x,y) f(s,y)\, dyds \qquad (6.31)\\ &\quad + \int_{t+\epsilon}^T \int_{\mathbb{R}^N} \langle F(t,x) - \tilde{F}_t^{s,y}, D_x \tilde{p}^{s,y}(t,s,x,y)\rangle f(s,y)\, dyds \\ &=: \tilde{R}_\epsilon^0 f(t,x) + \tilde{R}_\epsilon^1 f(t,x),\end{aligned}$$

où, rappelons-le, les opérateurs $\mathcal{L}_s$ et $\tilde{\mathcal{L}}_s^{s,y}$ sont la composante non-locale du générateur infinitésimal original et du générateur congelé, défini dans (6.4) et (6.16), respectivement. Nous soulignons, en particulier, que le premier terme de la décomposition $\tilde{R}_\epsilon^0 f$ n'est non-nul que si la matrice de diffusion $\sigma(t,\cdot)$ n'est pas constante dans l'espace. Contrôler directement la différence des deux générateurs dans $\tilde{R}_\epsilon^0 f$, donnée par :

$$\begin{aligned}&(\mathcal{L}_t - \tilde{\mathcal{L}}_t^{s,y})\tilde{p}^{s,y}(t,s,\cdot,y)(x) \\ &\quad = \int_{\mathbb{R}_0^d} [\tilde{p}^{s,y}(t,s,x+B\sigma(t,x)z,y) - \tilde{p}^{s,y}(t,s,x+B\tilde{\sigma}_t^{s,y}z,y)]\,\nu(dz) \qquad (6.32)\end{aligned}$$

apparaît d'emblée délicat, d'autant plus qu'on ne peut pas directement exploiter la régularité de la densité congelée $\tilde{p}^{s,y}(t,s,x,y)$ dans x, puisque ses effets régularisant dans la variable y seront nécessaire plus tard pour estimer l'intégrale externe (par rapport à y). Dans le cas où la condition de continuité absolue [**AC**] est vérifiée, on sait au contraire que

$$\nu(dz) \;=\; Q(z)\frac{g\left(\frac{z}{|z|}\right)}{|z|^{d+\alpha}}dz,$$

pour une certaine fonction Lipschitz continue g sur $\mathbb{S}^{d-1}$.
De plus, la condition d'ellipticité uniforme [**UE**] implique notamment que $\det\sigma(t,x)\neq 0$. En supposant alors, sans perte de généralité, que $\det\sigma(t,x)>0$, les changements de variable $\tilde{z}=\sigma(t,x)z$ dans l'intégrale à l'intérieur de $\mathcal{L}_s$ et $\tilde{z}=\tilde{\sigma}_t^{s,y}z$ dans celui pour $\tilde{\mathcal{L}}_s^{s,y}$ permettent de réécrire la différence des générateurs infinitésimaux de la manière suivante :

$$\left(\mathcal{L}_t-\tilde{\mathcal{L}}_t^{s,y}\right)\tilde{p}^{s,y}(t,s,\cdot,y)(x) \;=\; \int_{\mathbb{R}_0^d}\left[\tilde{p}^{s,y}(t,s,x+Bz,y)-\tilde{p}^{s,y}(t,s,x,y)\right]\tilde{H}_{t,x}^{s,y}(z)\,\frac{dz}{|z|^{d+\alpha}},$$

où nous avons noté, pour simplifier,

$$\tilde{H}_{t,x}^{s,y}(z) := Q(\sigma^{-1}(t,x)z)\frac{g\left(\frac{\sigma^{-1}(t,x)z}{|\sigma^{-1}(t,x)z|}\right)}{\det\sigma(t,x)|\sigma^{-1}(t,x)\frac{z}{|z|}|^{d+\alpha}}-Q((\tilde{\sigma}_t^{s,y})^{-1}z)\frac{g\left(\frac{(\tilde{\sigma}_t^{s,y})^{-1}z}{|(\tilde{\sigma}_t^{s,y})^{-1}z|}\right)}{\det\tilde{\sigma}_t^{s,y}|(\tilde{\sigma}_t^{s,y})^{-1}\frac{z}{|z|}|^{d+\alpha}}$$

Intuitivement, la condition [**AC**] permet de reporter l'erreur à estimer sur les fonctions tempérant g,Q, de manière à pouvoir exploiter leurs propriétés.
On mentionne aussi que pour le contrôle de la différence entre les opérateurs infinitésimaux $\mathcal{L}_t-\tilde{\mathcal{L}}_t^{s,y}$ il sera indispensable que $\tilde{H}_{t,x}^{s,y}$ est pair, grâce à la symétrie de la mesure de Lévy ν, car il nous permettra d'ajouter les termes d'ordre premier, nécessaires aux développements de Taylor, à n'importe quelle niveau de coupe.

En contrôlant le terme d'erreur $\tilde{R}_\epsilon^1 f$ dans (6.31), il est au contraire possible de montrer, au moins heuristiquement, que les seuils sur les régularités de Hölder pour F date dans le Théorème 6.3 de non-unicité, sont en réalité naturelles et qu'il serait donc fiable d'obtenir une caractérisation (presque) optimale de le caractère bien posé de la dynamique stochastique (6.1) en terme de la régularité des coefficients.
Supposons alors, comme dans le théorème cité ci-dessus, que $x_j\to F_i(t,x)$ soit β_i^j-Hölder continue, uniformément dans le temps et dans les autres variables spatiales.
Un calcul récurrent auquel nous devrons faire face en estimant $\tilde{R}_\epsilon^1 f$ sera de montrer que une quantité de la forme suivante :

$$\tilde{R}_\epsilon^i f(t,x) \;:=\; \int_{\mathbb{R}^N}\left[F_i(s,x)-F_i(s,\theta_{t,s}(y))\right]D_{x_i}\tilde{p}^{s,y}(t,s,x,y)\,dy$$

engendre une singularité intégrable dans le temps.
En exploitant les propriétés sur la densité gelée $\tilde{p}^{s,y}(t,s,x,y)$ dans (6.12), on peut alors

écrire que :

$$
\begin{aligned}
|\tilde{R}^i_\epsilon f(t,x)| &= \sum_{j=i}^n \int_{\mathbb{R}^N} \frac{\left|(x-\theta_{t,s}(y))_j\right|^{\beta^j_i}}{(s-t)^{\frac{1+\alpha(i-1)}{\alpha}}} \frac{\bar{p}(1,\mathbb{T}^{-1}_{s-t}(y-\tilde{m}^{s,y}_{s,t}(x))}{\det \mathbb{T}_{s-t}}\, dy \\
&= \sum_{j=i}^n (s-t)^{-\zeta^j_i} \int_{\mathbb{R}^N} \left(\frac{\left|(x-\theta_{t,s}(y))_j\right|}{(s-t)^{\frac{1+\alpha(i-1)}{\alpha}}}\right)^{\beta^j_i} \frac{\bar{p}(1,\mathbb{T}^{-1}_{s-t}(y-\tilde{m}^{s,y}_{s,t}(x))}{\det \mathbb{T}_{s-t}}\, dy \\
&\le C\sum_{j=i}^n (s-t)^{-\zeta^j_i} \int_{\mathbb{R}^N} \left|\mathbb{T}^{-1}_{s-t}\left(x-\theta_{t,s}(y)\right)\right|^{\beta^j_i} \frac{\bar{p}(1,\mathbb{T}^{-1}_{s-t}(y-\tilde{m}^{s,y}_{s,t}(x))}{\det \mathbb{T}_{s-t}}\, dy
\end{aligned}
$$

où nous avons défini, par commodité,

$$
\zeta^j_i := \frac{1+\alpha(i-1)}{\alpha} - \beta^j \frac{1+\alpha(j-1)}{\alpha}.
$$

En attendant, on note que le choix de geler au point final $(\tau,\xi)=(s,y)$ est nécessaire pour avoir une homogénéité entre la différence des dérives en $x-\theta_{t,s}(y)$ et l'argument de la densité en $y-\tilde{m}^{s,y}_{s,t}(x)$. En particulier, il est vrai que

$$
y-\tilde{m}^{s,y}_{t,s}(x) = \tilde{m}^{s,y}_{t,s}(y)-x = \theta_{t,s}(y)-x.
$$

En supposant pour le moment qu'à partir des propriétés régularisantes associées à la densité $\bar{p}(t,\cdot)$, on peut obtenir le contrôle suivant :

$$
\int_{\mathbb{R}^N} \left|\mathbb{T}^{-1}(\theta_{t,s}(y)-x)\right|^{\beta^j_i} \frac{\bar{p}(1,\mathbb{T}^{-1}(\theta_{t,s}(y)-x))}{\det \mathbb{T}_{s-t}}\, dy < +\infty, \tag{6.33}
$$

il s'ensuit immédiatement que

$$
\begin{aligned}
|\tilde{R}^i_\epsilon f(t,x)| &\le C\sum_{j=i}^n (s-t)^{\zeta^j_i} \int_{\mathbb{R}^N} \left|\mathbb{T}^{-1}_{s-t}\left(x-\theta_{t,s}(y)\right)\right|^{\beta^j_i} \frac{\bar{p}(1,\mathbb{T}^{-1}_{s-t}(x-\theta_{t,s}(y))}{\det \mathbb{T}_{s-t}}\, dy \\
&\le C\sum_{j=i}^n (s-t)^{-\zeta^j_i}.
\end{aligned}
$$

Car alors le module de $\tilde{R}^i_\epsilon f$ donne une singularité intégrable en temps, les seuils *naturels* sur la régularité de Hölder sur F_i doivent être donnés par :

$$
\zeta^j_i < 1 \Leftrightarrow \beta^j_i > \frac{1+\alpha(i-2)}{1+\alpha(j-1)}. \tag{6.34}
$$

Si $\alpha=2$, c'est-à-dire dans un contexte Brownien, ces seuils se retrouvent effectivement dans le travail [CdRM20b].

Le type de contrôles montré ci-dessus nous permet également de montrer clairement pourquoi, dans un premier temps, nous devrons supposer que les indices p, q sont suffisamment grands dans l'estimation ponctuelle du terme de reste $\tilde{R}_\epsilon f$.

En fait, un raisonnement similaire à celui de (6.29) permet de contrôler le terme $\tilde{R}^1_\epsilon f$ associé à la dérive F dans (6.31) comme suit :

$$|\tilde{R}^1_\epsilon f(t,x)| \,\le\, C\|f\|_{L^p_t L^q_x} \sum_{j=i}^n \left(\int_t^T (s-t)^{-\zeta_i^j p'} \, (\det \mathbb{T}_{s-t})^{-\frac{p'}{q}} \, ds \right)^{\frac{1}{p'}},$$

où, rappelez-vous, p' et q' sont respectivement les exposants conjugués de p et q. Ensuite, pour obtenir une quantité intégrable dans l'intégrale en temps il faudra imposer un seuil minimum sur p et q pour que p' et q' soient suffisamment petits pour s'assurer que

$$\zeta_i^j p' + \Big(p' - \frac{p'}{q'}\Big)\Big(\sum_{i=1}^n d_i \frac{1+\alpha(i-1)}{\alpha}\Big) \,<\, 1.$$

En conclusion, nous expliquons maintenant brièvement pourquoi nous n'avons pas pu obtenir le résultat souhaité, c'est-à-dire le caractère faiblement bien posé de la dynamique stochastique (6.1) par rapport aux seuils naturels dans (6.34) sur la régularité des dérives F, mais à la place nous avons dû supposer la même régularité β^j pour chaque composante F_i le long de la variable x_j.
L'élément crucial réside précisément dans le fait de pouvoir prouver l'Équation (6.33) dans le contrôle ponctuel du terme de reste $\tilde{R}^1_\epsilon f$. Nous soulignons immédiatement que pour notre modèle, la faible régularité du flot $y \to \theta_{t,s}(y)$ empêche de dériver directement (6.33) par le changement de variables $\tilde{y} = \theta_{t,s}(y) - x$.
En fait, l'approche la plus naturelle, exploitée par exemple aussi dans [CdRM20b], consiste à déplacer le flot sur la variable x par une propriété de Lipschitz "approximative" du type :

$$|\mathbb{T}^{-1}_{s-t}(\theta_{t,s}(y) - x)| \,\asymp (1 + |\mathbb{T}^{-1}_{s-t}(y - \theta_{s,t}(x))|). \tag{6.35}$$

Le principal problème dans notre cas est que, cependant, nous n'avons pas été en mesure d'établir, en toute généralité, que :

$$\overline{p}(1, \mathbb{T}^{-1}_{s-t}(\theta_{t,s}(y) - x)) \le C\check{p}(1, \mathbb{T}^{-1}_{s-t}(y - \theta_{s,t}(x)), \tag{6.36}$$

pour une certaine densité $\check{p}$ qui a les mêmes propriétés régulatrices que $\bar{p}$.
Nous mentionnons que cette difficulté est intrinsèquement liée à la nature α-stable dégénérée de notre système. En fait, un contrôle comme dans (6.36) est absolument direct dans le cas Gaussien à partir de l'expression explicite de la densité $\overline{p}$ et du contrôle en (6.35).
Pour obtenir une estimation ponctuelle, comme dans (6.36), la difficulté dans le cas stable consiste justement à pouvoir obtenir une description suffisamment précise du comportement des queue (associées aux grands sauts) qui, comme on le sait , sont associés à la géométrie de la mesure spectrale correspondante. On cite à ce propos les travaux de Watanabe [Wat07] dans le cas stable et celui de Sztonyk [Szt10a] pour une extension au cas stable tempéré.
En particulier, la partie délicate apparaît lorsque l'on considère le comportement de la mesure de Poisson (connecté aux grands sauts) associée à la densité $\bar{p}$ en régime hors diagonale, c'est-à-dire si $|\mathbb{T}^{-1}_{s-t}(\theta_{t,s}(y) - x)| > K$. Dans ce cas, on aurait de (6.13), (6.14)

et (6.15), que :

$$\begin{aligned}\overline{p}(1,\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)) &\le C\int_{\mathbb{R}^N}\frac{1}{(1+|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)-w|)^{N+3}}P_{\bar{N}_1}(dw)\\ &\le C\int_0^1 P_{\bar{N}_1}(\{w\in\mathbb{R}^N:(1+|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)-w|)^{-(N+3)}|>u\})du\\ &\le C\int_0^1 P_{\bar{N}_1}(B(\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x),u^{-1/(N+3)})du. \end{aligned}\tag{6.37}$$

Rappelant des arguments sous (4.17) que la mesure spectrale prend en charge sur $\mathbb{S}^{N-1}$ du processus $\{\tilde{S}_u^{s,y}\}_{u\geq 0}$ a en fait la dimension d, Watanabe dans [Wat07] Lemme 3.1 a montré qu'il existe une constante $C>0$ telle que, pour chaque z dans $\mathbb{R}^N$ et $r>0$:

$$P_{\bar{N}_1}(B(z,r))\le Cr^{d+1}(1+r^{\alpha})|z|^{-(d+1+\alpha)}.\tag{6.38}$$

En d'autres termes, la plus mauvaise décroissance dans l'estimation globale est précisément donnée par la taille du support de la mesure spectrale associée. Nous soulignons également que ces estimations sont en un certain sens optimales, au moins suivant certaines directions du système (cf. Lemme 3.1 dans [Wat07]). Sur ce point voir aussi [PT69]. En exploitant maintenant le contrôle dans (6.38) dans le contrôle principal en (6.37), on obtiendrait que

$$\begin{aligned}\overline{p}(1,\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)) &\le C|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|^{-(d+1+\alpha)}\int_0^1 u^{-\frac{d+1}{N+3}}(1+u^{-\frac{\alpha}{N+3}})du\\ &\le C(1+|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|)^{-(d+1+\alpha)}.\end{aligned}$$

La propriété de Lipschitz "approximative" dans (6.35) impliquerait finalement que

$$\begin{aligned}\overline{p}(1,\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)) &\le C(1+|\mathbb{T}_{s-t}^{-1}(y-\theta_{s,t}(x))|)^{-(d+1+\alpha)}\\ &=: C\check{p}(1,\mathbb{T}_{s-t}^{-1}(y-\theta_{s,t}(x))).\end{aligned}\tag{6.39}$$

Un tel contrôle serait en fait suffisant pour notre propos mais imposerait des conditions très fortes sur les dimensions d,n de l'espace pour que la densité $\check{p}(t,\cdot)$ soit intégrable. Ce phénomène est également apparu dans [HM16] et dans ce cas a imposé de limiter l'analyse quand $d=1,n=3$ pour démontrer le caractère bien posé du problème de martingale associé à une chaîne dégénérée avec une dérive linéaire et bruit multiplicatif stable isotrope.
Enfin, nous mentionnons que cette difficulté se poserait également dans le cas tempéré "classique", c'est-à-dire si l'on imposait des conditions supplémentaires à la fonction Q. L'avantage dans ce contexte aurait été de garder la fonction tempérant Q même dans la densité $\bar{p}$, de manière à exploiter les avantages du tempérament à l'infini, et ainsi récupérer les problèmes de concentration en (6.39). Comme analysé dans [Szt10a] Corollaire 6, nous aurions alors obtenu des contrôles de la forme :

$$\tilde{p}^{\tau,\xi}(t,s,x,y)\le C(1+|\mathbb{T}_{s-t}^{-1}(y-\theta_{s,t}(x))|)^{-(d+1+\alpha)}Q\left(|\mathbb{M}_{s-t}^{-1}(y-\theta_{s,t}(x))|\right)$$

qui améliorent nettement l'intégrabilité dans l'espace mais en même temps détériorent celle dans le temps, puisqu'il n'est plus possible d'exploiter la α-similarité de la densité

stable sous-jacente.
Ce genre de difficulté serait apparu même si on n'avait considéré que le cas tronqué, c'est-à-dire lorsque $Q(z) = \mathbb{1}_{B(0,r_0)}$ pour un certain $r_0 > 0$. Pour plus de détails, voir par exemple, [CKK08] dans un contexte non-dégénéré.

Pour résoudre cette difficulté, nous suivrons ensuite un raisonnement alternatif dans le Chapitre 4.
Rappelant que le changement des variables naturel $\tilde{y} := \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x)$ dans (6.33) n'est pas directement possible dans notre contexte car les coefficients ne sont pas assez réguliers, nous allons introduire un flot régularisé $\theta_{t,s}^{\delta}(y)$ associé à une version mollifié des coefficients impliqués. Puis, en appliquant le changement de variables souhaité (par rapport au flot régularisé), il sera possible d'obtenir l'estimation dans (6.33) par rapport à $\theta_{t,s}^{\delta}(y)$, et vérifier la différence entre les flots de la même manière que nous l'avons déjà fait pour établir la condition de Lipschitz approximative (cf. Équation (6.35)).
En conclusion, il ne restera que contrôler $\det(\nabla\theta_{t,s}^{\delta}(y))$, uniformément par rapport au paramètre de régularisation. Ce sera ce dernier contrôle qui nous obligera à renforcer le seuil naturel de régularité Hölder sur F en (6.34) et supposer que chaque composante F_i, ($i \in [\![2, n]\!]$) présentent la même régularité le long de la variable x_j ($j \in [\![2, n]\!]$), uniformément dans le temps et dans les autres variables spatiales (cf. Équation (6.3)).

Contre-exemples à Peano pour l'unicité dans un sens faible

Nous expliquons maintenant brièvement le raisonnement heuristique derrière la preuve du résultat de non-unicité (cf. Théorème (6.3)). L'idée est d'adapter les contre-exemples à Peano présentés dans (2.5) et également exploités dans [CdRM20b], à notre contexte de Lévy.
Si on veut tester par exemple le seuil β_i^j associé à l'exposant de Hölder critique pour la i-ième composante de la dérive F par rapport à la variable x_j (avec $j \geq i > 1$), on considérera le modèle suivant (pour $d_1 = \cdots = d_n = 1$ et $N = n$) :

$$\begin{cases} dX_t^1 \;=\; dZ_t, & \text{se } k = 1; \\ dX_t^k \;=\; X_t^{k-1}dt, & \text{se } k \in [\![2, i-1]\!]; \\ dX_t^i \;=\; X_t^{i-1}dt + \operatorname{sgn}(X_t^j)|X_t^j|^{\beta_i^j}dt, & \text{se } k = i; \\ dX_t^k \;=\; X_t^{k-1}dt, & \text{se } k \in [\![i+1, n]\!], \end{cases} \tag{6.40}$$

où $\{Z_t\}_{t\geq 0}$ est un processus α-stable symétrique réel. Il n'est pas difficile de noter que l'Équation (6.40) peut s'écrire sous la forme (6.1) en imposant $\sigma = 1$ et $G(t, x) = Ax + e_i \operatorname{sgn}(x_j)|x_j|^{\beta_i^j}$ où e_i est le i-ième élément de la base canonique sur $\mathbb{R}^N$ et A est la matrice dans $\mathbb{R}^N \otimes \mathbb{R}^N$ donnée par :

$$A := \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

Si on se concentre en particulier sur le i -ième composant de la dynamique (6.40) ci-dessus, il peut être réécrit sous forme intégrale comme :

$$X_t^j = \int_0^t \operatorname{sgn}\Big(I_t^{j-i}(X^j)\Big)\Big|I_t^{j-i}(X^j)\Big|^{\beta_i^j} dt + I_t^{i-1}(Z), \quad t \geq 0, \tag{6.41}$$

où, pour chaque chemin càdlàg $y\colon [0,\infty) \to \mathbb{R}$, la notation $I_t^k(y)$ représente l'intégrale itérée k-fois au temps t. Comme déjà expliqué ci-dessus, pour que la régularisation par le bruit se produise, il faut que, au moins en peu de temps, les fluctuations moyennes de la perturbation aléatoire dominent l'irrégularité de la dérive déterministe. Plus précisément pour notre modèle, nous pouvons alors comparer les fluctuations de bruit $I_t^{i-1}(Z)$ d'ordre $i-1+\frac{1}{\alpha}$ avec les solutions extrémales déterministes obtenues sans perturbation (i.e. $B=0$ ci-dessus), obtenant ainsi :

$$t^{i-1+\frac{1}{\alpha}} > t^{\frac{(j-1)\beta_i^j-1}{1-\beta_i^j}}.$$

Puisque cette condition doit être vérifiée pour t petit, nous pouvons alors conclure que la condition

$$i-1+\frac{1}{\alpha} < \frac{(j-1)\beta_i^j-1}{1-\beta_i^j} \;\Leftrightarrow\; \beta_i^j > \frac{1+\alpha(i-1)}{1+\alpha(j-2)}$$

est la relation heuristique qui garantit que le bruit domine le système et qu'une régularisation par le bruit se produit effectivement.

7 Sur les constantes optimales dans les estimations de Sobolev et Schauder pour les opérateurs de Kolmogorov dégénérés

Nous présentons maintenant brièvement les principaux résultats du Chapitre 5. Ce travail, rédigé en collaboration avec mes directeurs de thèse, Professeur Stéphane Menozzi et Professeur Enrico Priola, est paru récemment en pré-publication [MMP21].
Nous nous intéressons ici à étudier les effets d'une perturbation du second ordre sur un opérateur d'Ornstein-Uhlenbeck dégénéré diffusif. En particulier, nous voulons déterminer comment les constantes de certaines estimations "connues" pour cette classe d'opérateurs, telles que les estimations de Schauder présentées dans les Sections 4 et 5, dépendent en fait de la perturbation considérée. Notre méthode de prouve sera basée sur une transformation spatiale adéquate, qui permet d'annuler le terme de transport de premier ordre, et sur la méthode perturbative par processus de Poisson, introduite dans [KP17] et adaptée à notre contexte.

Plus précisément, étant donné un entier positif N, nous considérerons la famille suivante d'opérateurs d'Ornstein-Uhlenbeck diffusifs :

$$L^{\text{ou}} := \operatorname{Tr}(BD_z^2) + \langle Az, D_z\rangle, \quad \text{sur } \mathbb{R}^N, \tag{7.1}$$

où $\langle\cdot,\cdot\rangle$ désigne toujours le produit scalaire sur $\mathbb{R}^N$ et A, B sont deux matrices dans $\mathbb{R}^N \otimes \mathbb{R}^N$ telles que B est symétrique.

On supposera, comme au Chapitre 3, que les deux matrices A, B vérifient la condition du rang de Kalman qui assure l'hypoellipticité du système :

[K] il existe un entier non négatif n tel que

$$\mathrm{rank}[B, AB, \cdots, A^{n-1}B] \,=\, N,$$

où, rappelons-nous, $[B, AB, \dots, A^{n-1}B]$ est la matrice dans $\mathbb{R}^N \otimes \mathbb{R}^{Nn}$ dont les blocs sont $B, AB, \cdots A^{n-1}B$.

Comme déjà évoqué dans la Section 5, la condition de Kalman [**K**] permet de décomposer l'espace $\mathbb{R}^N$ en fonction de l'espace image atteint par les itérations successives des commutateurs entre A et B (cf. Équation (5.8)).
En supposant que $\mathrm{rank}(B) = d$ pour un certain $d > 0$, on sait en particulier qu'il existe $\{d_1, \dots, d_n\}$ entiers non-négatifs tels que $d_1 = d$, $\sum_{i=1}^n d_i = N$ et les deux matrices A, B sont réinscriptibles, après un éventuel changement de coordonnées, sous la forme plus explicite suivante (cf. Équation (5.11)) :

$$B \,=\, \begin{pmatrix} B_0 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} \quad \text{et} \quad A \,=\, \begin{pmatrix} * & * & \dots & \dots & * \\ A_2 & * & \ddots & \ddots & \vdots \\ 0 & A_3 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ 0 & \dots & 0 & A_n & * \end{pmatrix}$$

où B_0 est une matrice non-dégénérée dans $\mathbb{R}^d \otimes \mathbb{R}^d$, A_i est une matrice dans $\mathbb{R}^{d_i} \otimes \mathbb{R}^{d_{i-1}}$ tel que $\mathrm{rank}(A_i) = d_i$ pour chaque i dans $[\![2, n]\!]$ et les éléments $*$ peuvent être non-nuls.

On s'intéresse donc aux solutions du problème de Cauchy suivant :

$$\begin{cases} \partial_t u(t,z) \,=\, L^{\mathrm{ou}} u(t,z) + f(t,z) & \text{sur } (0,T) \times \mathbb{R}^N; \\ u(0,z) \,=\, 0, & \text{sur } \mathbb{R}^N. \end{cases} \tag{7.2}$$

Nous allons prouver l'existence et l'unicité de solutions bornées et régulières pour lÉquation (7.2) en supposant, comme dans [KP17], que la source f appartient à l'espace $B_b\Big(0,T; C^\infty_0(\mathbb{R}^N)\Big)$. Cet espace peut être compris comme la famille de fonctions mesurables, bornées dans le temps et lisses, supportées de manière compacte dans l'espace, uniformément dans le temps. Pour une définition précise de ces espaces, nous renvoyons le lecteur à la Section 1.2 du Chapitre 5.
Nous soulignons que nous ne pourrions pas considérer la source f dans une classe de fonctions plus "habituelle", telle que $C^\infty_c([0,T] \times \mathbb{R}^N)$ précisément parce que la méthode perturbative dans [KP17] en passant par les processus de Poisson, que nous exploiterons également, nous obligera à considérer des sources f dans (7.2) qui sont éventuellement discontinues dans le temps (cf. Section 2 dans [KP17]).

En raison de la faible régularité en temps de f, le système (7.2) ne sera compris qu'au sens intégral, c'est à dire une fonction bornée et continue $u\colon [0,T] \times \mathbb{R}^N \to \mathbb{R}$ sera une solution de l'Équation (7.2) si $u(t,\cdot)$ appartient à $C^2(\mathbb{R}^N)$ pour chaque t fixe et

$$u(t,x) \,=\, \int_0^t \Big[\mathrm{Tr}(BD^2_z u(s,z)) + \langle Az, D_z u(s,z)\rangle + f(s,z)\Big]\, ds. \tag{7.3}$$

Étant donné une fonction continue $t \mapsto S(t)$ telle que $S(t)$ soit une matrice symétrique et non-négative dans $\mathbb{R}^N \otimes \mathbb{R}^N$, on s'intéressera alors aussi au perturbation du second ordre associée à $S(t)$ pour l'opérateur d'Ornstein-Uhlenbeck, c'est-à-dire l'opérateur suivant :

$$L^{\text{ou},S}_t \,:=\, L^{\text{ou}} + \text{Tr}(S(t)D^2_z) \,=\, \text{Tr}\left([B + S(t)]\, D^2_z\right) + \langle Az, D_z\rangle, \qquad \text{sur } \mathbb{R}^N,$$

et au problème de Cauchy *perturbé* qui lui est associé :

$$\begin{cases}\partial_t u_S(t,z) \,=\, \text{Tr}\left([B + S(t)]\, D^2_z u_S(t,z)\right) + \langle Az, D_z u_S(t,z)\rangle + f(t,z); \\ u_S(0,z) \,=\, 0.\end{cases} \tag{7.4}$$

Dans cet article, nous nous concentrerons principalement sur deux types d'estimations pour les solutions u du Problème de Cauchy (7.2) : estimations de type Sobolev, c'est-à-dire contrôles en norme L^p sur la première composante non-dégénérée du gradient de la solution u et les estimations de Schauder par rapport aux espaces de Hölder avec multi-indice de régularité, déjà rencontrés dans les Sections 4 et 5 de cette thèse.
En particulier, Bramanti *et al.* ont montré dans [BCLP10], Théorème 3 que pour chaque p dans $(1,\infty)$, il existe une constante $C_p > 0$, indépendant de f, tel que :

$$\|B^{1/2}D^2 u\, B^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N)} \,\le\, C_p\|f\|_{L^p((0,T)\times\mathbb{R}^N)}, \tag{7.5}$$

Cependant, on remarque que les estimations dans (7.5) sont en fait prouvées dans [BCLP10] en supposant que la source f est lisse en espace et en temps. A travers quelques propriétés explicites sur le noyau de chaleur Gaussien sous-jacent, nous montrerons dans la Section 2.3 du Chapitre 5 qu'en effet ces estimations peuvent être étendues pour considérer également la classe des sources f dans notre contexte.

Sous la condition de Kalman **[K]**, Lunardi a plutôt montré dans [Lun97], Théorème 1.2, que pour chaque β dans $(0,1)$ il existe une constante C_β, indépendant de f, tel que :

$$\|u\|_{L^\infty((0,T),C^{2+\beta}_{b,d})} \,\le\, C_\beta\|f\|_{L^\infty((0,T),C^{\beta}_{b,d})}, \tag{7.6}$$

où, rappelez-vous, les espaces de Hölder anisotropes $C^\gamma_{b,d}(\mathbb{R}^N)$ sont définis exactement comme dans la Section 5.

Nous pouvons maintenant résumer les principaux résultats du Chapitre 5 dans le théorème suivant :

Théorème 7.1. *Soit f dans $B_b\left(0,T;C^\infty_0(\mathbb{R}^N)\right)$. Alors, il existe une unique solution intégrale u_S du Problème de Cauchy* (7.4) *tel que, pour chaque p dans* $(1,+\infty)$ *et β dans* $(0,1)$ *il retient que*

$$\|B^{1/2}D^2 u_S\, B^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N)} \,\le\, C_p\|f\|_{L^p((0,T)\times\mathbb{R}^N)} \tag{7.7}$$

$$\|u_S\|_{L^\infty(C^{2+\beta}_{b,d})} \,\le\, C_\beta\|f\|_{L^\infty(C^{\beta}_{b,d})}, \tag{7.8}$$

avec les ***mêmes*** *constantes C_p, C_β apparaissant respectivement dans* (7.5) *et* (7.6). *En particulier, les constantes C_p, C_β ne dépendent pas de la matrice $S(t)$.*

Au-delà de la propriété de préservation des constantes montrée dans le Théorème 7.1 ci-dessus, les estimations de Sobolev dans (7.7) semblent, au meilleur de notre connaissance, être nouvelles pour un opérateur comme $L_t^{\text{ou},S}$ et d'intérêt indépendant. Nous citons également à cet égard les travaux récents de Fornaro *et al.* [FMPS21] qui esquisse une description complète du spectre des opérateurs d'Ornstein-Uhlenbeck hypoelliptiques dans les espaces L^p.
Nous soulignons également que si nous considérons une matrice S indépendante du temps, des résultats analogues au Théorème 7.1 peuvent également être obtenus pour les estimations elliptiques correspondantes, en suivant la méthode indiquée dans le Corollaire 3.5 de [KP17].
Enfin, nous soulignons que dans le Chapitre 5 nous montrerons également que des estimations L^p plus générales, qui considèrent également les directions dégénérées, sont indépendantes des perturbations du second ordre, même si ce n'est que dans le cas des matrices A invariantes pour les dilatations (cf. Équation (4.6) dans la Section 4). Pour plus de détails sur ce sujet, voir Section 4 du Chapitre 5.

7.1 Guide à l'épreuve

Pour donner au lecteur une idée de la méthode d'épreuve que nous avons utilisée, nous illustrons maintenant brièvement les principales étapes de la preuve des estimations L^p dans (7.7). Comme déjà évoqué au début de la section, un outil fondamental sera une note (cf. [DPL95]) transformation de l'espace $\mathbb{R}^N$ (à chaque instant fixé) qui permettra justement de se débarrasser du terme de dérive $\langle Az, D_z u\rangle$ dans le problème de Cauchy (1.10).
Plus précisément, étant donné une solution bornée u du Problème de Cauchy (1.10), nous introduirons la fonction $v\colon [0,T]\times\mathbb{R}^N$, donnée par

$$v(t,z) \;:=\; u(t,e^{-tA}z).$$

En effet, en se souvenant que u résout le problème de Cauchy dans (1.10) et en notant que $u(t,z)=v(t,e^{tA}z)$, il n'est pas difficile à vérifier que

$$\begin{aligned} f(t,z) \;&=\; \partial_t u(t,z) - L^{\text{ou}}u(t,z)\\ &=\; v_t(t,e^{tA}z) + \langle Dv(t,e^{tA}z), Ae^{tA}z\rangle - \mathrm{Tr}\Big(e^{tA}Be^{tA^*}D^2v(t,e^{tA}z)\Big)\\ &\qquad - \langle Dv(t,e^{tA}z), Ae^{tA}z\rangle\\ &=\; v_t(t,e^{tA}z) - \mathrm{Tr}\Big(e^{tA}Be^{tA^*}D^2v(t,e^{tA}z)\Big). \end{aligned}$$

pour chaque (t,z) dans $(0,T)\times\mathbb{R}^N$. En dénotant pour simplifier $\tilde f(t,z) := f(t,e^{-tA}z)$, il résulte immédiatement des comptes ci-dessus que v est alors une solution du problème de Cauchy suivant :

$$\begin{cases} \partial_t v(t,z) \;=\; \mathrm{Tr}\Big(e^{tA}Be^{tA^*}D^2v(t,z)\Big) + \tilde f(t,z) & \text{on } (0,T)\times\mathbb{R}^N;\\ v(0,z) \;=\; 0 & \text{on } \mathbb{R}^N. \end{cases}$$

De plus, les estimations dans (7.5) peuvent être réécrites en termes de v comme

$$\|B^{1/2}e^{-tA^*}D^2v(t,e^{tA}\cdot)\,e^{tA}B^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N)} \;\le\; C_p\|\tilde f(t,e^{tA}\cdot)\|_{L^p((0,T)\times\mathbb{R}^N)}. \tag{7.9}$$

Par un changement de variables dans les intégrales et en dénotant $L^p((0,T)\times\mathbb{R}^N, m)$ l'espace usuel L^p par rapport à la mesure m donnée par

$$m(dt,dx) := \det(e^{-At})dtdx,$$

il est donc immédiat de constater que le contrôle dans (7.9) équivaut à l'estimation suivante :

$$\|B^{1/2}e^{tA^*}D^2v(t,\cdot)\,e^{tA}B^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N,m)} \le C_p\|\tilde{f}\|_{L^p((0,T)\times\mathbb{R}^N,m)}. \tag{7.10}$$

Nous pouvons maintenant nous concentrer sur le problème de Cauchy perturbé par la matrice $S(t)$:

$$\begin{cases}\partial_t w(t,z) + \mathrm{Tr}\Big(e^{tA}Be^{tA^*}D^2w(t,z)\Big) + \mathrm{Tr}\Big(e^{tA}S(t)e^{tA^*}D^2w(t,z)\Big) \;=\; \tilde{f}(t,z);\\ w(0,z) \;=\; 0.\end{cases} \tag{7.11}$$

A l'aide d'arguments probabilistes, nous montrerons notamment qu'il existe une unique solution $w\colon [0,T]\times\mathbb{R}^N \to \mathbb{R}$ du problème ci-dessus.

En adaptant maintenant certains des arguments présentés dans [KP17], nous pouvons déduire que les estimations en norme L^p dans (7.10) sont également valables pour la solution w du problème perturbé dans (7.11), quelle que soit la perturbation donnée par la matrice $S(t)$. Plus précisément, on obtiendra que

$$\|B^{1/2}e^{tA^*}D^2w(t,\cdot)\,e^{tA}B^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N,m)} \le C_p\|\tilde{f}(t,\cdot)\|_{L^p((0,T)\times\mathbb{R}^N,m)}, \tag{7.12}$$

tient avec la même constante C_p qui est apparue dans (7.10).
L'élément crucial de la preuve dans [KP17] est d'introduire une petite perturbation aléatoire sur la source f par un processus de Poisson convenable et d'analyser les propriétés associées à l'équation correspondante. Puis, en prenant la valeur moyen dans la formulation intégrale de l'équation, les contributions associées aux sauts de processus génèrent, pour une intensité convenable du processus de Poisson sous-jacent, un opérateur aux différences finies. En particulier, les estimations initiales restent conservées pour le système résolu par l'espérance de la solution et qui fait également intervenir l'opérateur aux différences finies. Enfin, des arguments de compacité nous permettent de conclure que les estimations initiales tiennent également à la limite, en échangeant l'opérateur de différence finie avec l'opérateur différentiel correspondant d'ordre deux.

Pour conclure, nous devrons alors revenir à notre modèle d'Ornstein-Uhlenbeck original, en appliquant la transformation inverse par rapport à la variable spatiale. En particulier, on introduira $\tilde{u}(t,z) := w(t, e^{tA}z)$ qui résout, par définition, l'équation suivante :

$$\begin{cases}\partial_t\tilde{u}(t,z) + L^{\mathrm{ou},S}_t\tilde{u}(t,z) \;=\; f(t,z), & \text{sur } (0,T)\times\mathbb{R}^N,\\ \tilde{u}(0,z) = 0, & \text{sur } \mathbb{R}^N.\end{cases}$$

Grâce à l'identité suivante :

$$D^2w(t,\cdot) = D^2[\tilde{u}(t,e^{-tA}\,\cdot)] = e^{-tA^*}D^2\tilde{u}(t,e^{-tA}\cdot)e^{-tA}$$

nous pouvons alors conclure à partir du Contrôle (7.12) que les estimations suivantes sont vérifiées :

$$\|B^{1/2}D^2\tilde{u}\,B^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N)} \;\leq\; C_p\|f\|_{L^p((0,T)\times\mathbb{R}^N)}.$$

En résumé, le raisonnement exposé ci-dessus nous permet en fait de construire une solution $\tilde{u}$ du problème de Cauchy dans (1.14) qui vérifie les estimations L^p dans (1.15) avec la *même* constante C_p qui est apparu dans des estimations similaires pour l'opérateur *proxy* L^{ou}. Enfin, notant que le principe du maximum est également valable par rapport à l'opérateur d'Ornstein-Uhlenbeck perturbé $L_t^{\text{ou},S}$, nous pouvons également montrer l'unicité de cette solution $\tilde{u}$.

En conclusion, nous soulignons que pour appliquer la méthode perturbative brièvement résumée ci-dessus, seules quelques propriétés spécifiques sont en fait requises sur les semi-normes sous-jacentes. Intuitivement, le raisonnement présenté dans [KP17] n'exploite que l'invariance translationnelle des semi-normes et une sorte de propriété de commutativité entre les normes (ou une fonction de la norme comme dans le cas L^p) et l'opérateur de la valeur attendue. En effet, il semble naturel que cette approche puisse ensuite être étendue à une classe beaucoup plus générale d'estimations sur autres espaces fonctionnels, comme par exemple les espaces de Besov. (cf. Équation (4.33)).
De plus, ce type de contrôles semble prometteur pour une analyse plus détaillée du caractère bien posé de certaines classes d'équations stochastiques corrélées. Ces aspects seront étudiés prochainement.

Chapter 2

Schauder estimates for degenerate stable Kolmogorov equations

Abstract: We provide here global Schauder-type estimates for a chain of integro-partial differential equations (IPDE) driven by a degenerate stable Ornstein-Uhlenbeck operator possibly perturbed by a deterministic drift, when the coefficients lie in some suitable anisotropic Hölder spaces. Our approach mainly relies on a perturbative method based on forward parametrix expansions and, due to the low regularizing properties on the degenerate variables and to some integrability constraints linked to the stability index, it also exploits duality results between appropriate Besov Spaces. In particular, our method also applies in some super-critical cases. Thanks to these estimates, we show in addition the well-posedness of the considered IPDE in a suitable functional space.

1 Introduction

For a fixed time horizon $T > 0$ and two integers n, d in $\mathbb{N}$, we are interested in proving global Schauder estimates for the following parabolic integro-partial differential equation (IPDE):

$$\begin{cases} \partial_t u(t,x) + \langle Ax + F(t,x), D_x u(t,x)\rangle + \mathcal{L}_\alpha u(t,x) \, = \, -f(t,x) & \text{on } [0,T]\times\mathbb{R}^{nd}; \\ u(T,x) \, = \, u_T(x) & \text{on } \mathbb{R}^{nd}. \end{cases} \tag{1.1}$$

where $x := (x_1, \dots, x_n)$ is in $\mathbb{R}^{nd}$ with each x_i in $\mathbb{R}^d$ and $\langle \cdot, \cdot \rangle$ represents the inner product on $\mathbb{R}^{nd}$. We consider a symmetric, α-stable operator $\mathcal{L}_\alpha$ acting non-degenerately only on the first d variables and a matrix A in $\mathbb{R}^{nd}\otimes\mathbb{R}^{nd}$ with the following sub-diagonal structure:

$$A := \begin{pmatrix} 0_{d\times d} & \dots & \dots & \dots & 0_{d\times d} \\ A_{2,1} & 0_{d\times d} & \dots & \dots & 0_{d\times d} \\ 0_{d\times d} & A_{3,2} & 0_{d\times d} & \dots & 0_{d\times d} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0_{d\times d} & \dots & 0_{d\times d} & A_{n,n-1} & 0_{d\times d} \end{pmatrix}. \tag{1.2}$$

We will assume moreover that it satisfies a Hörmander-like condition, allowing the smoothing effect of $\mathcal{L}_\alpha$ to propagate into the system.
Above, the source $f\colon [0,T]\times\mathbb{R}^{nd}\to\mathbb{R}$ and the terminal condition $u_T\colon \mathbb{R}^{nd}\to\mathbb{R}$ are assumed to be bounded and to belong to some suitable anisotropic Hölder space.
The additional drift term $F(t,x)=\big(F_1(t,x),\dots,F_n(t,x)\big)$ can be seen as a perturbation of the Ornstein-Ulhenbeck operator $\mathcal{L}_\alpha+\langle Ax,D_x\rangle$ and it has structure "compatible" with A, i.e. at level i, it depends only on the super diagonal entries:

$$F_i(t,x) \;:=\; F_i(t,x_i,\dots,x_n).$$

It may be unbounded but we assume it to be Hölder continuous with an index depending on the level of the chain.

Related results. A large literature on the topic of Schauder estimates in the α-stable non-local framework has been developed in the recent years (see e.g. Lunardi and Röckner [LR21] for an overview of the field), mainly in the non-degenerate setting and assuming that $\alpha\geq 1$, the so called sub-critical case. We mention for instance the stable-like setting, corresponding to time-inhomogeneous operators of the form

$$\begin{aligned}\bar{L}_t\phi(x) \;=\; \int_{\mathbb{R}^{nd}}\Big[\phi(x+y)-\phi(x)-\mathbb{1}_{1\leq\alpha<2}\langle y,D_x\rangle\Big]m(t,x,y)\frac{dy}{|y|^{d+\alpha}}\\ +\mathbb{1}_{1\leq\alpha<2}\langle F(t,x),D_xu(t,x)\rangle\end{aligned} \tag{1.3}$$

where the diffusion coefficient m is bounded from above and below, Hölder continuous in the spatial variable x and even in y if $\alpha=1$. Under these conditions and assuming the drift F to be bounded and Hölder continuous in space, Mikulevicius and Pragarauskas in [MP14] obtained parabolic Schauder type bounds on the whole space and derived from those estimates the well-posedness of the corresponding martingale problem. We notice however that for the super-critical case (when $\alpha<1$), the drift term in (1.3) is set to zero. This is mainly due to the fact that in the super-critical case, $\mathcal{L}_\alpha$ is of order α (in the Fourier space) and does not dominate the drift term F which is roughly speaking of order one.
In the non-degenerate, driftless framework (i.e. when $Ax+F=0$ and $n=1$ in (1.1)), Bass [Bas09] was the first to derive elliptic Schauder estimates for stable like operators. We can refer as well to the recent work of Imbert and collaborators [IJS18] concerning Schauder estimates for stable-like operator (1.3) with $\alpha=1$ and some related applications to non-local Burgers equations. Eventually, still in the driftless case, Ros-Oton and Serra worked in [ROS16] for interior and boundary elliptic-regularity in a general, symmetric α-stable setting, assuming that the Lévy measure ν_α associated with $\mathcal{L}_\alpha$ writes in polar coordinates $y=\rho s$, $(\rho,s)\in[0,\infty)\times\mathbb{S}^{d-1}$ as

$$\nu_\alpha(dy) \;=\; \tilde{\mu}(ds)\frac{d\rho}{\rho^{1+\alpha}}$$

where $\tilde{\mu}$ is a non-degenerate, symmetric measure on the sphere $\mathbb{S}^{d-1}$. Related to the above, we can mention also the associated work of Fernandez-Real and Ros-Oton [FRRO17] for parabolic equations.

In the elliptic setting, when $\alpha \in [1, 2)$ and $\mathcal{L}_\alpha$ is a non-degenerate, symmetric α-stable operator and for bounded Hölder drifts, global Schauder estimates were obtained by Priola in [Pri12] or in [Pri18] for respective applications to the strong well-posedness and Davie's uniqueness for the corresponding SDE. We notice furthermore that in the sub-critical case, elliptic Schauder estimates can be proven for more general, translation invariant, Lévy-type generators for following [Pri18] (see Section 6, and Remark 5 therein).

In the super-critical case, parabolic Schauder estimates were established by Chaudru de Raynal, Menozzi and Priola in [CdRMP20a] under similar assumptions to [ROS16]. An existence result is also provided therein.
We mention as well the work of Zhang and Zhao [ZZ18] who address through probabilistic arguments the parabolic Dirichlet problem for stable-like operators of the form (1.3) with a non-trivial bounded drift, i.e. getting rid of the indicator function for the drift. They also obtain interior Schauder estimates and some boundary decay estimates (see e.g. Theorem 1.5 therein).

As we have seen, most of the literature is focused on the non-degenerate case. In the degenerate diffusive setting, Lunardi [Lun97] was the first one to prove Schauder estimates for linear Kolmogorov equations under weak Hörmander assumptions, exploiting anisotropic Hölder spaces (where the Hölder index depends on the variable considered), in order exactly to control the multiple scales appearing in the different directions, due to the degeneracy of the system.
After, in [Lor05] and [Pri09], the authors established Schauder-like estimates for hypoelliptic Kolmogorov equations driven by partially nonlinear smooth drifts. On the other hand, let us also mention [CdRHM18a] where the authors first establish Schauder estimates for nonlinear Kolmogorov equations under some weak Hörmander-type assumption. Their method is based on a perturbative approach through proxies that we here adapt and exploit. In the degenerate, stable setting, we have to refer also to a recent work of Zhang and collaborators [HWZ20] who show Schauder estimates for the degenerate kinetic dynamics ($n = 2$ above) extending a method based on Littlewood-Paley decompositions already used in other works by Zhang (see e.g. [ZZ18]), to the degenerate, multi-scaled framework. Even with different approaches and frameworks, we consider here a generic d-level chain and we exploit thermic characterizations of Besov norms, our and their works bring to the same results in the intersecting cases, at least to the best of our knowledge. About a different but correlated argument, we mention that the L^p-maximal regularity for degenerate non-local Kolmogorov equations with constant coefficients was also obtained in [CZ19] for the kinetic dynamics ($n = 2$ above) and in [HMP19] for the general n-levels chain.

In the diffusive setting, Equation (1.1) appears naturally as a microscopic model for heat diffusion phenomena (see [RBT00]) or, in the kinetic case ($n = 2$), it can be naturally associated with speed/position (or Hamiltonian) dynamics where the speed component is noisy. It can be found in many fields of application from physics to finance, see for example [HN04] or [BPV01]. When noised by stable processes, it can be used to model the appearance of turbulence (cf. [CPKM05]) or some abnormal diffusion phenomena. Moreover, the Schauder estimates will be a fundamental first step in order to study the weak and strong well-posedness for the following stochastic differential equation (SDE):

$$\begin{cases} dX_t^1 \;=\; F_1(t, X_t^1, \dots, X_t^n)dt + dZ_t \\ dX_t^2 \;=\; A_{2,1}X_t^1 + F_2(t, X_t^2, \dots, X_t^n)dt \\ \vdots \\ dX_t^n \;=\; A_{n,n-1}X_t^{n-1} + F_n(t, X_t^n)dt \end{cases} \tag{1.4}$$

where Z_t is a symmetric, $\mathbb{R}^d$-valued α-stable process with non-degenerate Lévy measure ν_α on some filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. The complete operator $\mathcal{L}_\alpha + \langle Ax + F(t,x), D_x\rangle$ then corresponds to the infinitesimal generator of the process $\{X_t\}_{t\geq 0}$, solution of Equation (1.4).

Mathematical outline. In this work, we will establish global Schauder estimates for the solution of the IPDE (1.1) exploiting the perturbative approach firstly introduced in [CdRHM18a] to derive such estimates for degenerate Kolmogorov equations. Roughly speaking, the idea is to perform a first order parametrix expansion, such as a Duhamel-type representation, to a solution of the IPDE (1.1) around a suitable proxy. The main idea behind consists in exploiting this easier framework in order to subsequently obtain a tractable control on the error expansion. When applying such a strategy, we basically have two ways to proceed.
On the one hand, one can adopt a backward parametrix approach, as introduced by McKean and Singer [MS67] in the non-degenerate, diffusive setting. This technique has been extended to the degenerate Brownian case involving unbounded perturbation, and successfully exploited for handling the corresponding martingale problem in [CdRM20b]. Anyway, this approach does not seem very adapted to our framework especially because it does not allow to deal easily with point-wise gradient estimates which will, at least along the non-degenerate variable x_1, be fundamental to establish our result.
On the other hand, the so-called forward parametrix approach has been successfully used by Friedman [Fri64] or Il'in et al. [IKO62] in the non-degenerate, diffusive setting to obtain point-wise bounds on the fundamental solution and its derivatives for the corresponding heat-type equation or in [CdR17] to derive strong uniqueness for the associated SDE (1.4) (i.e. $n = 2$ with the previous notations). Especially, this approach is better tailored to exploit cancellation techniques that are crucial when derivatives come in, as opposed to the backward one.

The main difficulties to overcome in order to prove Schauder estimates in our framework will be linked to the degeneracy of the operator $\mathcal{L}_\alpha$ that acts only on the first d variables, as well as the unboundedness of the perturbation F. Concerning this second issue, let us also mention that Schauder estimates for unbounded non-linear drift coefficients in the non-degenerate diffusive setting were obtained under mild smoothness assumptions by Krylov and Priola [KP10] who heavily used an auxiliary, deterministic flow associated with the transport term in (1.1), i.e. for a fixed couple (t, x),

$$\begin{cases} \partial_s\theta_s(x) = A\theta_s(x) + F(s, \theta_s(x)); \quad \text{if } s > t \\ \theta_t(x) \;=\; x, \end{cases} \tag{1.5}$$

to precisely get rid of the unbounded terms.

The drawback of this approach is that we will need at first to establish Schauder estimates in a small time interval. This seems quite intuitive since the expansion along the chosen proxy on which the method relies is precisely designed for small times because it requires that the original operator and the proxy are "close" enough in a suitable sense. To obtain the result for an arbitrary but finite time, we will then iterate the reasoning, which is quite natural since Schauder estimates provide a sort of stability in the considered functional space. We are therefore far from the optimal constants for the Schauder estimates established in the non-degenerate, diffusive setting for time dependent coefficients by Krylov and Priola [KP17].

On the other hand, we want to establish the Schauder estimates in the sharpest possible Hölder setting for the coefficients of the IPDE (1.1). To do so, we will need to establish some subtle controls, in particular we have no true derivatives of the coefficients. This is the reason why we will heavily rely on duality results on Besov spaces (see Section 4.1 below, Chapter 3 in [LR02] or [Tri92] for a more complete survey of the argument). However, in contrast with the non-degenerate case (cf. [CdRMP20a]), we will need to ask for the perturbation F some additional regularity, represented by parameter γ_i in assumption [**R**] below, on the degenerate entries F_i $(i > 1)$. This assumption seems quite natural if we think that, due to the degenerate structure of the system (cf. Section 2.2 below), the more we descend on the chain, the lower the smoothing effect of $\mathcal{L}_\alpha$ will be. The additional smoothness on F can be then seen as the "price" to pay to re-equilibrate the increasing time singularities appearing along the chain.

Organization of the paper. The article is organized as follows. We state our precise framework and give our main results in the following Section 2. Section 3 is then dedicated to the perturbative approach which is the central argument to derive our estimates. In particular, we obtain therein some Schauder estimates for drifted operators along the inhomogeneous flow $\theta_{s,t}$ defined above in (1.5), as well as the key Duhamel representation for solutions. Since the arguments to show the Schauder estimates will be quite long and involved, we postpone the proofs of these results in the next Sections 4 and 5. The existence results are then established in Section 6. In the last Section 7, we are going to explain briefly how the perturbative approach presented before could be applied with slight modifications to prove Schauder-type estimates for a class of completely non-linear, locally Hölder continuous drifts with an additional "diffusion" coefficient.
Finally, the proof of some technical results concerning the stability properties of Hölder flows are postponed to the Appendix.

2 Setting and main results

2.1 Main operators considered

The operator $\mathcal{L}_\alpha$ we consider is the generator of a non-degenerate, symmetric, stable process and it acts only on the first d coordinates of the system. More precisely, $\mathcal{L}_\alpha$ can be represented for any sufficiently regular $\phi\colon [0,T]\times\mathbb{R}^{nd}\to\mathbb{R}$ as

$$\mathcal{L}_\alpha\phi(t,x) \,:=\, \text{p.v.}\int_{\mathbb{R}^d}\Big[\phi(t,x+By)-\phi(t,x)\Big]\,\nu_\alpha(dy), \quad \text{where} \;\; B \,:=\, \begin{bmatrix} I_{d\times d}\\ 0_{d\times d}\\ \vdots \\ 0_{d\times d}\end{bmatrix}$$

and ν_α is a symmetric, stable Lévy measure on $\mathbb{R}^d$ of order α that we assume to be non-degenerate in a sense that we are going to specify below.
Passing to polar coordinates $y=\rho s$ where $(\rho,s)\in[0,\infty)\times\mathbb{S}^{d-1}$, it is well-known (see for example Chapter 3 in [Sat13]) that the stable Lévy measure ν_α can be decomposed as

$$\nu_\alpha(dy) \,:=\, \tilde{\mu}(ds)\frac{d\rho}{\rho^{1+\alpha}}, \tag{2.6}$$

where $\tilde{\mu}$ is a symmetric measure on $\mathbb{S}^{d-1}$ which represents the spherical part of ν_α.
We remember now that the Lévy symbol associated with $\mathcal{L}_\alpha$ is defined through the Levy-Khitchine formula (see, for instance [Jac01]) as:

$$\Phi(p) \,:=\, \int_{\mathbb{R}^d}\Big[e^{ip\cdot y}-1\Big]\,\nu_\alpha(dy), \qquad \text{for any } p \text{ in } \mathbb{R}^d,$$

where "$\cdot$" represents the inner product on the smaller space $\mathbb{R}^d$. In the current symmetric setting, it can be rewritten (cf. Theorem 14.10 in [Sat13]) as

$$\Phi(p) \,=\, -\int_{\mathbb{S}^{d-1}}|p\cdot s|^\alpha\,\mu(ds), \tag{2.7}$$

where $\mu=C_{\alpha,d}\tilde{\mu}$ is usually called the spherical measure associated with ν_α . Following [Kol00], we then say that ν_α is non-degenerate if the associated Lévy symbol Φ is equivalent, up to some multiplicative constant, to $|p|^\alpha$. More precisely, we suppose that μ is non-degenerate if

[ND] there exists a constant $\eta\geq 1$ such that for any p in $\mathbb{R}^d$.

$$\eta^{-1}|p|^\alpha \,\leq\, \int_{\mathbb{S}^{d-1}}|p\cdot s|^\alpha\,\mu(ds) \,\leq\, \eta|p|^\alpha. \tag{2.8}$$

It is important to remark that such a condition does not restrict our model too much. Indeed, there are many different kind of spherical measures μ that are non-degenerate in the above sense, from the stable-like case, i.e. measures that are absolutely continuous with respect to the Lebesgue measure on $\mathbb{S}^{d-1}$, to very singular ones such that the spherical measure induced by the sum of Dirac masses along the canonical directions:

$$\sum_{i=1}^{d}(\partial^2_{x_k})^{\alpha/2}.$$

We can introduce now the complete Ornstein-Uhlenbeck operator L^{ou}, defined for any sufficiently regular $\phi\colon\mathbb{R}^{nd}\to\mathbb{R}$ as

$$L^{\text{ou}}\phi(x) \,:=\, \langle Ax, D_x\phi(x)\rangle+\mathcal{L}_\alpha\phi(x), \tag{2.9}$$

where A is the matrix in $\mathbb{R}^{nd}\times\mathbb{R}^{nd}$ defined in Equation (1.2). We assume that A satisfies the following Hörmander-like condition of non-degeneracy:

[H] $A_{i,i-1}$ is non-degenerate (i.e. it has full rank d) for any i in $[\![2,n]\!]$.

Above, $[\![2,n]\!]$ denotes the set of all the integers in the interval. It is well known (see for example [Sat13]) that under these assumptions, the operator L^{ou} generates a convolution Markov semigroup $\{P^{\text{ou}}_t\}_{t\geq 0}$ on $B_b(\mathbb{R}^{nd})$, the family of all the bounded and Borel measurable functions on $\mathbb{R}^{nd}$, defined by

$$\begin{cases} P^{\text{ou}}_t\phi(x) \, = \, \int_{\mathbb{R}^{nd}}\phi(x+y)\,\mu_t(dy); \\ P^{\text{ou}}_0\phi(x) \, = \, \phi(x), \end{cases}$$

where $\{\mu_t\}_{t>0}$ is a family of Borel probability measures on $\mathbb{R}^{nd}$. In particular, the function $P^{\text{ou}}_t\phi$ provides the classical solution to the Cauchy problem

$$\begin{cases} \partial_t u(t,x) + \mathcal{L}_\alpha u(t,x) + \langle Ax, D_x u(t,x)\rangle \, = \, 0 & \text{on } (0,\infty)\times\mathbb{R}^{nd}; \\ u(0,x) \, = \, \phi(x) & \text{on } \mathbb{R}^{nd}. \end{cases} \tag{2.10}$$

Moving to the stochastic counterpart if necessary, it is readily derived from [PZ09] that the semigroup $(P^{\text{ou}}_t)_{t\geq 0}$ admits a smooth density $p^{\text{ou}}(t,\cdot)$ with respect to the Lebesgue measure on $\mathbb{R}^{nd}$. Moreover, such a density p^{ou} has the following useful representation:

$$p^{\text{ou}}(t,x,y) \, = \, \frac{1}{\det \mathbb{M}_t} p_S\left(t, \mathbb{M}_t^{-1}\left(e^{At}x - y\right)\right), \tag{2.11}$$

where p_S is the density of $\{S_t\}_{t\geq 0}$, a stable process in $\mathbb{R}^{nd}$ whose Lévy measure satisfies the assumption [**ND**] above on $\mathbb{R}^{nd}$ and $\mathbb{M}_t$ is a diagonal matrix on $\mathbb{R}^{nd}\times\mathbb{R}^{nd}$ given by

$$\Big[\mathbb{M}_t\Big]_{i,j} \, := \, \begin{cases} t^{i-1}I_{d\times d}, & \text{if } i=j; \\ 0_{d\times d}, & \text{otherwise.} \end{cases} \tag{2.12}$$

We remark already that the appearance of the matrix $\mathbb{M}_t$ in Equation (2.11) and its particular structure reflect the multi-scaled structure of the dynamics considered (cf. Paragraph (2.2) below for a more precise explanation).
Moreover, the density p_S shows a useful property we will call the *smoothing effect* since it will be fundamental to reduce the singularities appearing when working with time integrals. Fixed γ in $[0,\alpha)$, there exists a constant $C:=C(\gamma)$ such that for any l in $[\![0,3]\!]$,

$$\int_{\mathbb{R}^{nd}} |y|^\gamma |D^l_y p_S(t,y)|\,dy \, \leq \, Ct^{\frac{\gamma-l}{\alpha}} \quad \text{for any } t>0. \tag{2.13}$$

These results can be proven following the arguments of Proposition 2.3 and Lemma 4.3 in [HMP19]. We will provide however a complete proof in the Appendix for the sake of completeness.

2.2 Intrinsic time scale and associated Hölder spaces

In this section, we are going to choose which is the most suitable functional space in which to state our Schauder estimates.

To answer this question, we need firstly to understand how the system typically behaves. We focus for the moment on the Ornstein-Uhlenbeck case:

$$\Bigl(\partial_t + L^{\mathrm{ou}}\Bigr)u(t,x) \;=\; -f(t,x) \qquad \text{on } (0,\infty)\times\mathbb{R}^{nd},$$

and search for a dilation operator $\delta_\lambda\colon (0,\infty)\times\mathbb{R}^{nd}\to(0,\infty)\times\mathbb{R}^{nd}$ that is invariant for the considered dynamics, i.e. a dilation that transforms solutions of the above equation into other solutions of the same equation.

Due to the structure of A and the α-stability of ν, we can consider for any fixed $\lambda>0$, the following

$$\delta_\lambda(t,x) := (\lambda^\alpha t, \lambda x_1, \lambda^{1+\alpha}x_2,\dots,\lambda^{1+\alpha(n-1)}x_n),$$

i.e. with a possible slight abuse of notation, $\Bigl(\delta_\lambda(t,x)\Bigr)_0 := \lambda^\alpha t$ and for any i in $[\![1,n]\!]$, $\Bigl(\delta_\lambda(t,x)\Bigr)_i := \lambda^{1+\alpha(i-1)}x_i$. It then holds that

$$\Bigl(\partial_t + L^{\mathrm{ou}}\Bigr)u = 0 \implies \Bigl(\partial_t + L^{\mathrm{ou}}\Bigr)(u\circ\delta_\lambda) = 0.$$

The previous reasoning suggests us to introduce a parabolic distance $\mathbf{d}_P$ that is homogenous with respect to the dilation δ_λ, so that $\mathbf{d}_P\Bigl(\delta_\lambda(t,x);\delta_\lambda(s,x')\Bigr) = \lambda\mathbf{d}_P\Bigl((t,x);(s,x')\Bigr)$. Precisely, following the notations in [HMP19], we set for any s,t in $[0,T]$ and any x,x' in $\mathbb{R}^{nd}$,

$$\mathbf{d}_P\Bigl((t,x),(s,x')\Bigr) \;:=\; |s-t|^{\frac{1}{\alpha}} + \sum_{j=1}^{n}|(x-x')_j|^{\frac{1}{1+\alpha(j-1)}}. \tag{2.14}$$

The idea of a dilation δ_λ that summarizes the multi-scaled behaviour of the dynamics was firstly introduced by Lanconelli and Polidoro in [LP94] for degenerate Kolmogorov equations in the diffusive setting. Since then, it has become a “standard” tool in the analysis of degenerate equations (see for example [Lun97], [HMP19] or [HWZ20]).

Since we will quite always use only the spatial part of the distance $\mathbf{d}_P$, we denote for simplicity

$$\mathbf{d}(x,y) \;=\; \sum_{j=1}^{n}|(x-x')_j|^{\frac{1}{1+\alpha(j-1)}}. \tag{2.15}$$

Technically speaking, $\mathbf{d}_P$ (and thus, $\mathbf{d}$) does not however induce a norm on $[0,T]\times\mathbb{R}^{nd}$ in the usual sense since it lacks of linear homogeneity. We remark anyhow again that for any $\lambda>0$, it precisely holds that $\mathbf{d}\Bigl(\delta_\lambda(t,x);\delta_\lambda(s,x')\Bigr) = \lambda\mathbf{d}\Bigl((t,x);(s,x')\Bigr)$. As it can be seen, $\mathbf{d}_P$ is an extension of the standard parabolic distance in the stable case, adapted to respect the multi-scaled nature of our dynamics. Indeed, the exponents appearing in (2.14) are those which make each space component homogeneous to the characteristic time scale $t^{1/\alpha}$.

The appearance of this kind of phenomena is due essentially by the particular structure of the matrix A (cf. Equation (1.1)) that allows the smoothing effect of $\mathcal{L}_\alpha$, acting only on the first variable, to propagate in the system, as it can be seen in the following lemma:

Lemma 2.1 (Scaling Lemma). *Let i be in $[\![1,n]\!]$. Then, there exist $\{C_j\}_{j\in[\![1,n]\!]}$ positive constants, depending only from A and i, such that*

$$D_{x_i}p^{ou}(t,x,y) \;=\; -\sum_{j=i}^{n}C_j t^{j-i}D_{y_j}p^{ou}(t,x,y)$$

for any $t > 0$ *and any* x, y *in* $\mathbb{R}^{nd}$.

Proof. Recalling the representation of p^{ou} in Equation (2.11), it is easy to see that

$$D_{x_i}p^{\text{ou}}(t,x,y) \;=\; \frac{1}{\det \mathbb{M}_t} D_z p_S(t,\cdot)\Big(\mathbb{M}_t^{-1}(e^{At}x-y)\Big)\mathbb{M}_t^{-1}D_{x_i}\Big[e^{At}x-y\Big].$$

Hence, in order to conclude, we need to show that

$$D_{x_i}\Big[e^{At}x-y\Big] \;=\; -\sum_{j=i}^{n} C_j t^{j-i} D_{y_j}\Big[e^{At}x-y\Big]. \tag{2.16}$$

To prove the above equality, we need to analyze more in depth the structure of the resolvent e^{At}. Recalling from Equation (1.2) that A has a sub-diagonal structure, we notice that for any i,j in $[\![1,n]\!]$,

$$\Big[e^{At}\Big]_{i,j} \;=\; \begin{cases} C_{i,j}t^{j-i}, & \text{if } j \geq i; \\ 0, & \text{otherwise,} \end{cases} \tag{2.17}$$

for a family of constants $\{C_{i,j}\}_{i,j\in[\![1,n]\!]}$ depending only from A. It then follows that for any x,y in $\mathbb{R}^{nd}$, it holds that

$$\Big[e^{At}x-y\Big]_i \;=\; \sum_{k=1}^{i} C_{i,k}t^{i-k}x_k - y_i. \tag{2.18}$$

Equation (2.16) then follows immediately. For a more detailed proof of this result, see also [HM16] or [HMP19]. □

We finally remark the link with the stochastic counterpart of equation (1.1). From a more probabilistic point of view, the exponents in Equation (2.14) can be related to the characteristic time scales of the iterated integrals of an α-stable process.

We are now ready to define the suitable Hölder spaces for our estimates. We start recalling some useful notations we will need below. Fixed k in $\mathbb{N}\cup\{0\}$ and β in $(0,1)$, we follow Krylov [Kry96], denoting the usual *homogeneous* Hölder space $C^{k+\beta}(\mathbb{R}^d)$ as the family of functions $\phi\colon \mathbb{R}^d \to \mathbb{R}$ such that

$$\|\phi\|_{C^{k+\beta}} \;:=\; \sum_{i=1}^{k} \sup_{|\vartheta|=i}\|D^\vartheta\phi\|_{L^\infty} + \sup_{|\vartheta|=k}[\mathbf{d}^\vartheta\phi]_\beta \;<\; \infty,$$

where

$$[\mathbf{d}^\vartheta\phi]_\beta \;:=\; \sup_{x\neq y}\frac{|\mathbf{d}^\vartheta\phi(x)-\mathbf{d}^\vartheta\phi(y)|}{|x-y|^\beta}.$$

Additionally, we are going to need the associated subspace $C^{k+\beta}_b(\mathbb{R}^d)$ of bounded functions in $C^{k+\beta}(\mathbb{R}^d)$, equipped with the norm

$$\|\cdot\|_{C^{k+\beta}_b} \;=\; \|\cdot\|_{L^\infty} + \|\cdot\|_{C^{k+\beta}}.$$

We can now define the anisotropic Hölder space with multi-index of regularity associated with the distance d. For sake of brevity and readability, we firstly define for a function $\phi\colon \mathbb{R}^{nd} \to \mathbb{R}$, a point z in $\mathbb{R}^{d(n-1)}$ and i in $[\![1,n]\!]$, the function

$$\Pi^i_z\phi\colon x \in \mathbb{R}^d \mapsto \phi(z_1,\dots,z_{i-1},x,z_{i+1},z_n) \in \mathbb{R},$$

with the obvious modifications if $i=1$ or $i=n$. Intuitively speaking, the function $\Pi^i_z\phi$ is the restriction of ϕ on its i-th d-dimensional variable while fixing all the other coordinates in z. The space $C^{k+\beta}_d(\mathbb{R}^{nd})$ is then defined as the family of all the function $\phi\colon \mathbb{R}^{nd} \to \mathbb{R}$ such that

$$\|\phi\|_{C^{k+\beta}_d} := \sum_{i=1}^n \sup_{z\in\mathbb{R}^{d(n-1)}} \|\Pi^i_z\phi\|_{C^{\frac{k+\beta}{1+\alpha(i-1)}}} < \infty. \tag{2.19}$$

The modification to the bounded subspace $C^{k+\beta}_{b,d}(\mathbb{R}^{nd})$ is straightforward.
Roughly speaking, the anisotropic norm works component-wise, i.e. we firstly fix a coordinate and then calculate the standard Hölder norm along that particular direction, but with index scaled according to the dilation of the system in that direction, uniformly over time and the other space components. We conclude summing the contributions associated with each component.
We highlight however that it is possible to recover the expected joint regularity for the partial derivatives, when they exist. In such a case, they actually turn out to be Hölder continuous in the pseudo-metric $\mathbf{d}$ with order one less than the function. (cf. Lemma 8.4 in the Appendix for the case $i=1$).

Since we are working with evolution equations, the functions we consider will quite often depend on time, too. For this reason, we denote by $L^\infty\Big(0,T;C^{k+\beta}_d(\mathbb{R}^{nd})\Big)$ (respectively, $L^\infty\Big(0,T;C^{k+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$) the family of functions $\psi\colon [0,T]\times\mathbb{R}^{nd} \to \mathbb{R}$ with finite $C^{k+\beta}_d$-norm (respectively, $C^{k+\beta}_{b,d}$-norm), uniformly in time. It is endowed with the following norm:

$$\|\phi\|_{L^\infty(C^{k+\beta}_d)} := \sup_{t\in[0,T]} \|\phi(t,\cdot)\|_{C^{k+\beta}_d}, \tag{2.20}$$

with a straightforward modification for the bounded subspace $L^\infty\Big(0,T;C^{k+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$.

2.3 Assumptions and main results

From this point further, we consider two fixed numbers α in $(0,2)$ and β in $(0,1)$ such that α will represent the index of stability of the operator $\mathcal{L}_\alpha$ while β will stand for the index of Hölder regularity of the coefficients.

We will assume the following:

[S] assumptions [**ND**] and [**H**] are satisfied and the drift $F=(F_1,\dots,F_n)$ is such that for any i in $[\![1,n]\!]$, F_i depends only on time and on the last $n-(i-1)$ components, i.e. $F_i(t,x_i,\dots,x_n)$;

[P] α is a number in $(0,2)$, β is in $(0,1)$ such that $\alpha+\beta \in (1,2)$ and if $\alpha<1$ (super-critical case),

$$\beta<\alpha, \quad 1-\alpha < \frac{\alpha-\beta}{1+\alpha(n-1)};$$

[R] Recalling the notations in (2.19)-(2.20), the source f is in $L^\infty(0,T;C^\beta_{b,d}(\mathbb{R}^{nd}))$, the terminal condition u_T is in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})$ and for any i in $[\![1,n]\!]$, the drift F_i belongs to $L^\infty(0,T;C^{\gamma_i+\beta}_d(\mathbb{R}^{nd}))$, where

$$\gamma_i := \begin{cases} 1+\alpha(i-2), & \text{if } i>1; \\ 0, & \text{if } i=1. \end{cases} \tag{2.21}$$

From now on, we will say that assumption [**A**] holds when the above conditions [**S**], [**P**] and [**R**] are in force.

Remark 2.1 (About the assumptions)**.** We remark that the constraints [**P**] we are imposing in the super-critical case ($\alpha<1$) seem quite natural for our system. The condition $\beta<\alpha$ reflects essentially the low integrability properties of the stable density p_S (cf. Equation (2.13)). Even if one is interested only on the fractional Laplacian case, i.e. $\mathcal{L}_\alpha=\Delta^{\alpha/2}$, such a condition cannot be dropped in general, since it does not refer to the integrability property of p_α and its derivatives but instead to those of its "projection" p_S on the bigger space $\mathbb{R}^{nd}$ (cf. Equation (2.11)).
The second condition $\alpha+\beta>1$ is necessary to give a point-wise definition of the gradient of a solution u with respect to the non-degenerate variable x_1. Moreover, there is a famous counterexample of Tanaka and his collaborators [TTW74] that shows that even in the scalar case, weak uniqueness (a direct consequence of Schauder estimates) may fail for the associated SDE if $\alpha+\beta$ is smaller than one.
The last assumption is indeed a technical constraint and it is necessary to work properly with the perturbation F at any level $i=1,\dots,n$. In particular, it seems the minimal threshold that allows us to exploit the smoothing effect of the density (see for example Equation (5.32) in the proof of Lemma 5.2 for more details). We conclude highlighting that these assumptions are always fulfilled if $\alpha\geq 1$ (sub-critical case).

At this stage, it should be clear that under our assumptions [**A**], IPDE (1.1) will be understood in a *distributional* sense. Indeed, we cannot hope to find a "classical" solution for Equation (1.1), since for such a function u in $L^\infty(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd}))$, the total gradient $D_x u$ is not defined point-wise.
Let us denote for any function $\phi\colon[0,T]\to\mathbb{R}^{nd}$ regular enough, the complete operator L_t as

$$L_t\phi(t,x) := \langle Ax+F(t,x), D_x u(t,x)\rangle + \mathcal{L}_\alpha u(t,x). \tag{2.22}$$

We will say that a function u in $L^\infty(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd}))$ is a distributional (or weak) solution of IPDE (1.1) if for any ϕ in $C^\infty_0((0,T]\times\mathbb{R}^{nd})$, it holds that

$$\int_0^T\int_{\mathbb{R}^{nd}}\Big(-\partial_t+L^*_t\Big)\phi(t,y)u(t,y)\,dy+\int_{\mathbb{R}^{nd}}u_T(y)\phi(T,y)\,dy \;=\; -\int_0^T\int_{\mathbb{R}^{nd}}\phi(t,y)f(t,y)\,dy, \tag{2.23}$$

where $\mathcal{L}^*_\alpha$ denotes the formal adjoint of L_t. On the other hand, denoting from now on,

$$\|F\|_H := \sup_{i\in[\![1,n]\!]}\|F_i\|_{L^\infty(C^{\gamma_i+\beta}_d)}, \tag{2.24}$$

we will quite often use the following other notion of solution:

Definition 2.2. A function u is a mild solution in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ of Equation (1.1) if for any triple of sequences $\{f_m\}_{m\in\mathbb{N}}$, $\{u_{T,m}\}_{m\in\mathbb{N}}$ and $\{F_m\}_{m\in\mathbb{N}}$ such that

— $\{f_m\}_{m\in\mathbb{N}}$ is in $C^\infty_b((0,T)\times\mathbb{R}^{nd})$ and f_m converges to f in $L^\infty\Big(0,T;C^{\beta}_{b,d}(\mathbb{R}^{nd})\Big)$;

— $\{u_{T,m}\}_{m\in\mathbb{N}}$ is in $C^\infty_b(\mathbb{R}^{nd})$ and $u_{T,m}$ converges to u_T in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})$;

— $\{F_m\}_{m\in\mathbb{N}}$ is in $C^\infty_b((0,T)\times\mathbb{R}^{nd};\mathbb{R}^{nd})$ and $\|F_m-F\|_H$ converges to 0,

there exists a sub-sequence $\{u_m\}_{m\in\mathbb{N}}$ in $C^\infty_b\Big((0,T)\times\mathbb{R}^{nd}\Big)$ such that

— u_m converges to u in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$;

— for any fixed m in $\mathbb{N}$, u_m is a classical solution of the following "regularized" IPDE:

$$\begin{cases} \partial_t u_m(t,x)+\mathcal{L}_\alpha u_m(t,x) & \\ \qquad\qquad + \langle Ax+F_m(t,x),D_xu_m(t,x)\rangle \;=\; -f_m(t,x), & \text{on } (0,T)\times\mathbb{R}^{nd};\\ u_m(T,x)\;=\;u_{T,m}(x) & \text{on } \mathbb{R}^{nd}. \end{cases} \tag{2.25}$$

We can now state our main result:

Theorem 2.3. *(Schauder Estimates) Let u be a mild solution in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ of IPDE* (1.1). *Under [**A**], there exists a constant $C:=C(T)$ such that*

$$\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\;\le\;C\Big[\|f\|_{L^\infty(C^{\beta}_{b,d})}+\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big]. \tag{2.26}$$

Associated with an existence result we will exhibit in Section 6, we will eventually derive the well-posedness for Equation (1.1).

Theorem 2.4. *Under [**A**], there exists a unique mild solution u of IPDE* (1.1) *belonging to $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$. Moreover, such a function u is a weak solution, too.*

In the following, we will denote for sake of brevity

$$\alpha_i\;:=\;\frac{\alpha}{1+\alpha(i-1)}\quad\text{and}\quad\beta_i\;:=\;\frac{\beta}{1+\alpha(i-1)}\quad\text{for any } i \text{ in } [\![1,n]\!]. \tag{2.27}$$

Clearly, these quantities were introduced to reflect exactly the relative scale of the system at every considered level i (cf. Section 2.2 above).
In the following, as well as in Theorem 2.3 above, C denotes a generic constant that may change from line to line but depending only on the parameters in assumption [**A**]. Other dependencies that may occur are explicitly specified.

3 Proof through perturbative approach

As already said in the introductive section, our method of proof relies on a perturbative approach introduced in [CdRHM18a] for the degenerate, Kolmogorov, diffusive setting. Roughly speaking, we will firstly choose a suitable proxy for the equation of interest, i.e. an operator whose associated semigroup and density are known and that is close enough to the original one:

$$\mathcal{L}_\alpha + \langle Ax + F(t,x), D_x\rangle.$$

Furthermore, we will exhibit suitable regularization properties for the proxy and in particular, we will show that it satisfies the Schauder estimates (2.26). This will be the purpose of Sub-section 3.1.
In Sub-section 3.2 below, we will then expand a solution u of IPDE (1.1) along the chosen proxy through a Duhamel-type formula and eventually show that the expansion error only brings a negligible contribution, so that the Schauder estimates still hold for u. Due to our choice of method, this will be possible only adding some more assumptions on the system. Namely, we will assume in addition to be in a small time interval, so that the proxy and the original operator do not differ too much.
The last Sub-section 3.3 will finally show how to remove the additional assumption in order to prove the Schauder estimates (Theorem 2.3) through a scaling argument.

3.1 Frozen semigroup

The crucial element in our approach consists in choosing wisely a suitable proxy operator along which to expand a solution u in $L^\infty(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd}))$ of IPDE (1.1). In order to deal with potentially unbounded perturbations F, it is natural to use a proxy involving a non-zero first order term associated with a flow representing the dynamics driven by $Ax+F$, the transport part of Equation (1.1) (see e.g. [KP10] or [CdRMP20a]).

Remembering that we assume F to be Hölder continuous, we know that there exists a solution of

$$\begin{cases} d\theta_{s,\tau}(\xi) \,=\, \Big[A\theta_{s,\tau}(\xi) + F(s,\theta_{s,\tau}(\xi))\Big]ds \quad \text{on } [\tau,T]; \\ \theta_{\tau,\tau}(\xi) \,=\, \xi, \end{cases}$$

even if it may be not unique. For this reason, we are going to choose one particular flow, denoted by $\theta_{s,\tau}(\xi)$, and consider it fixed throughout the work.
More precisely, given a freezing couple (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$, the flow will be defined on $[\tau,T]$ as

$$\theta_{s,\tau}(\xi) \,=\, \xi + \int_\tau^s \Big[A\theta_{v,\tau}(\xi) + F(v,\theta_{v,\tau}(\xi))\Big]\,dv. \tag{3.1}$$

We can now introduce the "frozen" IPDE on $(0,T)\times\mathbb{R}^{nd}$, associated with the chosen proxy:

$$\begin{cases} \partial_s\tilde{u}^{\tau,\xi}(s,x) + \mathcal{L}_\alpha\tilde{u}^{\tau,\xi}(s,x) + \langle Ax + F(s,\theta_{s,\tau}(\xi)), D_x\tilde{u}^{\tau,\xi}(s,x)\rangle \,=\, -f(s,x); \\ \tilde{u}^{\tau,\xi}(s,x) \,=\, u_T(x). \end{cases} \tag{3.2}$$

Noticing that the proxy operator $\mathcal{L}_\alpha + \langle Ax + F(s,\theta_{s,\tau}(\xi)), D_x\rangle$ can be seen as an Ornstein-Ulhenbeck operator with an additional time-dependent component $F(s,\theta_{s,\tau}(\xi))$, it is clear that under assumption [**A**], it generates a two parameters semigroups we will denote by $\{\tilde{P}^{\tau,\xi}_{s,t}\}_{t\le s}$. Moreover, it admits a density given by

$$\tilde{p}^{\tau,\xi}(t,s,x,y) \,=\, \frac{1}{\det\mathbb{M}_{s-t}}p_S\Big(s-t, \mathbb{M}^{-1}_{s-t}(y - \tilde{m}^{\tau,\xi}_{s,t}(x))\Big), \tag{3.3}$$

remembering Equation (2.11) for the definition of p_S and with the following notation for the "frozen shift" $\tilde{m}^{\tau,\xi}_{s,t}(x)$:

$$\tilde{m}^{\tau,\xi}_{s,t}(x) \,=\, e^{A(s-t)}x + \int_t^s e^{A(s-v)}F(v,\theta_{v,\tau}(\xi))\,dv. \tag{3.4}$$

We point out already the following important property of the shift $\tilde{m}^{\tau,\xi}_{s,t}(x)$:

Lemma 3.1. *Let $t<s$ in $[0,T]$ and x a point in $\mathbb{R}^{nd}$. Then,*

$$\tilde{m}^{\tau,\xi}_{s,t}(x) \,=\, \theta_{s,\tau}(\xi), \tag{3.5}$$

taking $\tau=t$ and $\xi=x$.

Proof. We start noticing that by construction, $\tilde{m}^{\tau,\xi}_{s,t}(x)$ satisfies

$$\tilde{m}^{\tau,\xi}_{s,t}(x) \,=\, x + \int_t^s \Big[A\tilde{m}^{t,x}_{v,t}(x) + F(v,\theta_{v,\tau}(\xi))\Big]\,dv.$$

It then holds that

$$|\tilde{m}^{t,x}_{s,t}(x)-\theta_{s,t}(x)| \,\le\, \int_t^s A|\tilde{m}^{t,x}_{v,t}(x)-\theta_{v,t}(x)|\,dv.$$

The above Equation (3.5) then follows immediately applying the Grönwall lemma. □

Moreover, we can extend the smoothing effect (2.13) of p_S to the frozen density $\tilde{p}^{\tau,\xi}$ through the representation (3.3):

Lemma 3.2 (Smoothing effects of the frozen density)**.** *Under [**A**], let ϑ,ϱ be two multi-indexes in $\mathbb{N}^n$ such that $|\varrho+\vartheta|\le 3$ and γ in $[0,\alpha)$. Then, there exists a constant $C:=C(\vartheta,\varrho,\gamma)$ such that*

$$\int_{\mathbb{R}^{nd}} |D^\varrho_y D^\vartheta_x \tilde{p}^{\tau,\xi}(t,s,x,y)|\mathbf{d}^\gamma\Big(y,\tilde{m}^{\tau,\xi}_{s,t}(x)\Big)\,dy \,\le\, C(s-t)^{\frac{\gamma}{\alpha}-\sum_{k=1}^n\frac{\vartheta_k+\varrho_k}{\alpha_k}} \tag{3.6}$$

for any $t<s$ in $[0,T]$, any x in $\mathbb{R}^{nd}$ and any frozen couple (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$. In particular, if $|\vartheta|\neq 0$, it holds for any ϕ in $C^\gamma_d(\mathbb{R}^{nd})$ that

$$\Big|D^\vartheta_x\tilde{P}^{\tau,\xi}_{s,t}\phi(x)\Big| \,\le\, C\|\phi\|_{C^\gamma_d}(s-t)^{\frac{\gamma}{\alpha}-\sum_{k=1}^n\frac{\vartheta_k}{\alpha_k}}. \tag{3.7}$$

Proof. Since p_S is the density of an α-stable process, we remember that the following α-scaling property

$$p_S(t,y) \,=\, t^{-\frac{nd}{\alpha}}p_S(1,t^{-\frac{1}{\alpha}}y) \tag{3.8}$$

holds for any $t>0$ and any y in $\mathbb{R}^{nd}$. Fixed i in $[\![1,n]\!]$, we then denote for simplicity

$$\mathbb{T}_{s-t} \,:=\, (s-t)^{\frac{1}{\alpha}}\mathbb{M}_{s-t}$$

and we calculate the derivative of $\tilde{p}^{\tau,\xi}$ with respect to x_i through

$$
\begin{aligned}
|D_{x_i}\tilde{p}^{\tau,\xi}(t,s,x,y)| &= \Big|\frac{1}{\det \mathbb{M}_{s-t}} D_{x_i}\Big[p_S\Big(s-t, \mathbb{M}_{s-t}^{-1}(\tilde{m}^{\tau,\xi}_{s,t}(x)-y)\Big)\Big]\Big| \\
&= \Big|\frac{1}{\det \mathbb{T}_{s-t}} D_{x_i}\Big[p_S\Big(1, \mathbb{T}_{s-t}^{-1}(\tilde{m}^{\tau,\xi}_{s,t}(x)-y)\Big)\Big]\Big| \\
&= \Big|\frac{1}{\det(\mathbb{T}_{s-t})}\langle D_z p_S\Big(1,\cdot\Big)(\mathbb{T}_{s-t}^{-1}(\tilde{m}^{\tau,\xi}_{s,t}(x)-y)); \mathbb{T}_{s-t}^{-1}D_{x_i}(\tilde{m}^{\tau,\xi}_{s,t}(x))\rangle\Big|,
\end{aligned}
$$

where in the second equality we exploited the α-scaling property in (3.8). From Equation (2.17) in the Scaling Lemma 2.1, we now notice that

$$
\begin{aligned}
\Big|\mathbb{T}_{s-t}^{-1}D_{x_i}(\tilde{m}^{\tau,\xi}_{s,t}(x))\Big| &= \Big|\mathbb{T}_{s-t}^{-1}D_{x_i}\Big(e^{A(t-s)}(x)\Big)\Big| \\
&= (s-t)^{-\frac{1}{\alpha}}\sum_{k=i}^{n} C_k (s-t)^{-(k-1)}(s-t)^{k-i} \\
&\le C(s-t)^{-\frac{1+\alpha(i-1)}{\alpha}},
\end{aligned}
$$

and we use it to show that

$$
|D_{x_i}\tilde{p}^{\tau,\xi}(t,s,x,y)| \le C(s-t)^{-\frac{1+\alpha(i-1)}{\alpha}}\frac{1}{\det(\mathbb{T}_{s-t})}\Big|D_z p_S\Big(1,\cdot\Big)(\mathbb{T}_{s-t}^{-1}(\tilde{m}^{\tau,\xi}_{s,t}(x)-y))\Big|.
$$

Similarly, if we fix j in $[\![1,n]\!]$, it holds that

$$
|D_{y_j}D_{x_i}\tilde{p}^{\tau,\xi}(t,s,x,y)| \le C(s-t)^{-\frac{1}{\alpha_i}-\frac{1}{\alpha_j}}\frac{1}{\det(\mathbb{T}_{s-t})}\Big|D^2_z p_S\Big(1,\cdot\Big)(\mathbb{T}_{s-t}^{-1}(\tilde{m}^{\tau,\xi}_{s,t}(x)-y))\Big|.
$$

It is then easy to show by iteration of the same argument that

$$
\begin{aligned}
&|D^{\varrho}_y D^{\vartheta}_x \tilde{p}^{\tau,\xi}(t,s,x,y)| \\
&\quad \le C(s-t)^{-\sum_{k=1}^{n}\frac{\varrho_k+\vartheta_k}{\alpha_k}}\frac{1}{\det(\mathbb{T}_{s-t})}\Big|D^{|\varrho+\vartheta|}_z p_S\Big(1,\cdot\Big)(\mathbb{T}_{s-t}^{-1}(\tilde{m}^{\tau,\xi}_{s,t}(x)-y))\Big|. \qquad (3.9)
\end{aligned}
$$

Control (3.6) immediately follows from the analogous smoothing effect for p_S (cf. Equation (2.13)) and the change of variables $z = \mathbb{T}_{s-t}^{-1}(\tilde{m}^{\tau,\xi}_{s,t}(x)-y)$. Indeed,

$$
\begin{aligned}
\int_{\mathbb{R}^{nd}} |D^{\varrho}_y D^{\vartheta}_x \tilde{p}^{\tau,\xi}(t,s,x,y)|\mathbf{d}^{\gamma}\Big(y,\tilde{m}^{\tau,\xi}_{s,t}(x)\Big)\,dy &\le C(s-t)^{-\sum_{k=1}^{n}\frac{\varrho_k+\vartheta_k}{\alpha_k}} \\
&\quad \times \int_{\mathbb{R}^{nd}} \frac{1}{\det(\mathbb{T}_{s-t})}\Big|D^{|\varrho+\vartheta|}_z p_S\Big(1,\cdot\Big)(\mathbb{T}_{s-t}^{-1}(\tilde{m}^{\tau,\xi}_{s,t}(x)-y))\Big|\mathbf{d}^{\gamma}\Big(y,\tilde{m}^{\tau,\xi}_{s,t}(x)\Big)\,dy \\
&= (s-t)^{-\sum_{k=1}^{n}\frac{\varrho_k+\vartheta_k}{\alpha_k}}\int_{\mathbb{R}^{nd}}\Big|D^{|\varrho+\vartheta|}_z p_S(1,z)\Big|\mathbf{d}^{\gamma}\Big(\mathbb{T}_{s-t}z+\tilde{m}^{\tau,\xi}_{s,t}(x),\tilde{m}^{\tau,\xi}_{s,t}(x)\Big)\,dz.
\end{aligned}
$$

To conclude, we notice that

$$
\mathbf{d}^{\gamma}\Big(\mathbb{T}_{s-t}z+\tilde{m}^{\tau,\xi}_{s,t}(x),\tilde{m}^{\tau,\xi}_{s,t}(x)\Big) \le C\sum_{i=1}^{n}|(s-t)^{\frac{1+\alpha(i-1)}{\alpha}}z_i|^{\frac{\gamma}{1+\alpha(i-1)}} = (s-t)^{\frac{\gamma}{\alpha}}\sum_{i=1}^{n}|z_i|^{\frac{\gamma}{1+\alpha(i-1)}}
$$

and use it to write that

$$\begin{aligned}
&\int_{\mathbb{R}^{nd}}\Bigl|D^\varrho_y D^\vartheta_x \tilde{p}^{\tau,\xi}(t,s,x,y)\Bigr|\mathbf{d}^\gamma\Bigl(y,\tilde{m}^{\tau,\xi}_{s,t}(x)\Bigr)\,dy \\
&\le\, C(s-t)^{\frac{\gamma}{\alpha}-\sum_{k=1}^n\frac{\varrho_k+\vartheta_k}{\alpha_k}}\sum_{i=1}^n\int_{\mathbb{R}^{nd}}\Bigl|D^{|\varrho+\vartheta|}_z p_S(1,z)\Bigr||z_i|^{\frac{\gamma}{1+\alpha(i-1)}}\,dz\,\le\, C(s-t)^{\frac{\gamma}{\alpha}-\sum_{k=1}^n\frac{\varrho_k+\vartheta_k}{\alpha_k}},
\end{aligned}$$

where in the last passage we used the smoothing effect for p_S (Equation (2.13)), recalling that for any i in $[\![1,n]\!]$, it holds that

$$\frac{\gamma}{1+\alpha(i-1)}\,\le\,\gamma\,<\,\alpha$$

and we have thus the required integrability.
To prove instead Inequality (3.7), we use a cancellation argument to write

$$\begin{aligned}
\Bigl|D^\vartheta_x\tilde{P}^{\tau,\xi}_{s,t}\phi(x)\Bigr|\,&=\,\Bigl|\int_{\mathbb{R}^{nd}}D^\vartheta_x\tilde{p}^{\tau,\xi}(t,s,x,y)\Bigl[\phi(y)-\phi(\tilde{m}^{\tau,\xi}_{s,t}(x))\Bigr]\,dy\Bigr| \\
&\le\,\int_{\mathbb{R}^{nd}}|D^\vartheta_x\tilde{p}^{\tau,\xi}(t,s,x,y)|\,|\phi(y)-\phi(\tilde{m}^{\tau,\xi}_{s,t}(x))|\,dy.
\end{aligned}$$

But since we assume ϕ to be in $C^\gamma_d(\mathbb{R}^{nd})$, we can control the last expression as

$$\begin{aligned}
\Bigl|D^\vartheta_x\tilde{P}^{\tau,\xi}_{s,t}\phi(x)\Bigr|\,&\le\,\|\phi\|_{C^\gamma_d}\int_{\mathbb{R}^{nd}}\mathbf{d}^\gamma\Bigl(y,\tilde{m}^{\tau,\xi}_{s,t}(x)\Bigr)|D^\vartheta_x\tilde{p}^{\tau,\xi}(t,s,x,y)|\,dy \\
&\le\,C\|\phi\|_{C^\gamma_d}(s-t)^{\frac{\gamma}{\alpha}-\sum_{k=1}^n\frac{\vartheta_k}{\alpha_k}},
\end{aligned}$$

where in the last passage we used Equation (3.6). □

We can define now our candidate to be the mild solution of the "frozen" IPDE. If it exists and it is smooth enough, such a candidate appears to be a representation of the solution of Equation (3.2) obtained through the Duhamel principle. For this reason, the following expression

$$\tilde{u}^{\tau,\xi}(t,x)\,:=\,\tilde{P}^{\tau,\xi}_{T,t}u_T(x)+\int_t^T\tilde{P}^{\tau,\xi}_{s,t}f(s,x)\,ds\quad\text{ for any }(t,x)\text{ in }[0,T]\times\mathbb{R}^{nd},\tag{3.10}$$

will be called the *Duhamel representation of the proxy.* As it seems, under our assumption [**A**] such a representation is robust enough to satisfy Schauder estimates similar to (2.26). Since the proof of this result is quite long, we will postpone it to Section 4.2 for clarity.

Proposition 3.3. *(Schauder Estimates for Proxy) Under [**A**], there exists a constant $C:=C(T)$ such that*

$$\|\tilde{u}^{\tau,\xi}\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\,\le\,C\Bigl[\|f\|_{L^\infty(C^\beta_{b,d})}+\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Bigr]\tag{3.11}$$

for any freezing couple (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$.

We conclude this section showing that the function $\tilde{u}^{\tau,\xi}$ is indeed a mild solution in $L^\infty(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd}))$ of the "frozen" IPDE (3.2). Moreover, the converse statement is also true. If regular enough, any solution of Equation (3.2) corresponds to the Duhamel Representation (3.10).

Proposition 3.4. *Let us assume to be under assumption [**A**]. Then,*

— *the function $\tilde{u}^{\tau,\xi}$ defined in* (3.10) *is a mild solution in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ of the "frozen" IPDE* (3.2) *for any freezing couple (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$;*

— *Fixed (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$, let $\tilde{v}^{\tau,\xi}$ be a mild solution in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ of IPDE* (3.2)*. Then,*

$$\tilde{v}^{\tau,\xi}(t,x) \;=\; \tilde{P}^{\tau,\xi}_{T,t}u_T(x)+\int_t^T\tilde{P}^{\tau,\xi}_{s,t}f(s,x)\,ds.$$

Proof. The first assertion is quite straightforward. Let us consider three sequences $\{f_m\}_{m\in\mathbb{N}}$, $\{u_{T,m}\}_{m\in\mathbb{N}}$ and $\{F_m\}_{m\in\mathbb{N}}$ of smooth and bounded coefficients such that f_m converges to f in $L^\infty\Big(0,T;C^{\beta}_{b,d}(\mathbb{R}^{nd})\Big)$, $u_{T,m}$ to u_T in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})$ and $\|F_m-F\|_H\to 0$. Denoting now by $\{\tilde{P}^{m,\tau,\xi}_{s,t}\}_{t\le s}$ the semigroup associated with the "regularized" operator

$$\mathcal{L}_\alpha+\langle Ax+F_m(t,\theta_{t,\tau}(\xi)),D_x\rangle,$$

it is not difficult to show that for any fixed m in $\mathbb{N}$, the following

$$\tilde{u}^{\tau,\xi}_m \;:=\; \tilde{P}^{m,\tau,\xi}_{T,t}u_{T,m}(x)+\int_t^T\tilde{P}^{m,\tau,\xi}_{s,t}f_m(s,x)\,ds$$

is a classical solution of the "frozen" IPDE (3.2) with regularized coefficients $f_m,u_{T,m}$ and F_m. A detailed guide of this result can be found, even if in the diffusive setting, in Lemma 3.3 in [KP10]. Using now the Schauder Estimates (3.11) for the regularized solutions $\tilde{u}^{\tau,\xi}_m$, it follows immediately that $\tilde{u}^{\tau,\xi}_m\to\tilde{u}^{\tau,\xi}$ in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ and thus, that $\tilde{u}^{\tau,\xi}$ is a mild solution of (3.2) in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$.
To prove the second statement, we start fixing a freezing couple (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$ and consider three sequences $\{f_m\}_{m\in\mathbb{N}}$, $\{u_{T,m}\}_{m\in\mathbb{N}}$ and $\{F_m\}_{m\in\mathbb{N}}$ of bounded and smooth coefficients such that $f_m\to f$ in $L^\infty\Big(0,T;C^{\beta}_{b,d}(\mathbb{R}^{nd})\Big)$, $u_{T,m}\to u_T$ in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})$ and $\|F_m-F\|_H\to 0$. They can be constructed through mollification.
Since $\tilde{v}^{\tau,\xi}$ is a mild solution of the "frozen" IPDE (3.2), we know that there exists a sequence $\{\tilde{v}^{\tau,\xi}_m\}_{m\in\mathbb{N}}$ of classical solutions of the "regularized frozen" IPDE (3.2) with coefficients $f_m,u_{T,m}$ and F_m such that $\tilde{v}^{\tau,\xi}_m\to\tilde{v}^{\tau,\xi}$ in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$. Fixed m in $\mathbb{N}$, we then denote

$$h_m(t,x) \;:=\; \tilde{v}^{\tau,\xi}_m\Big(t,x-\int_t^T e^{A(t-s)}F_m(s,\theta_{s,\tau}(\xi))\,ds\Big)$$

for any t in $[0,T]$ and any x in $\mathbb{R}^{nd}$. Direct calculations imply that

$$\begin{aligned}D_xh_m(t,x) \;&=\; D_x\tilde{v}^{\tau,\xi}_m\Big(t,x-\int_t^T e^{A(t-s)}F_m(s,\theta_{s,\tau}(\xi))\,ds\Big);\\ \mathcal{L}_\alpha h_m(t,x) \;&=\; \mathcal{L}_\alpha\tilde{v}^{\tau,\xi}_m\Big(t,x-\int_t^T e^{A(t-s)}F_m(s,\theta_{s,\tau}(\xi))\,ds\Big)\end{aligned}$$

and

$$
\begin{aligned}
\partial_t h_m(t,x) \;=\; & \partial_t \tilde{v}_m^{\tau,\xi}\Big(t, x - \int_t^T e^{A(t-s)} F_m(s,\theta_{s,\tau}(\xi))\,ds\Big) \\
& + \Big\langle F_m(t,\theta_{t,\tau}(\xi)), D_x \tilde{v}_m^{\tau,\xi}\Big(t, x - \int_t^T e^{A(t-s)} F_m(s,\theta_{s,\tau}(\xi))\,ds\Big)\Big\rangle \\
& - \Big\langle A\int_t^T e^{A(t-s)} F_m(s,\theta_{s,\tau}(\xi))\,ds, D_x \tilde{v}_m^{\tau,\xi}\Big(t, x - \int_t^T e^{A(t-s)} F_m(s,\theta_{s,\tau}(\xi))\,ds\Big)\Big\rangle.
\end{aligned}
$$

Remembering that $\tilde{v}_m^{\tau,\xi}$ is a classical solution of Equation (3.2) replacing therein f, u_T and F with coefficients $f_m, u_{T,m}$ and F_m, it follows immediately that the function h_m solves for any m in $\mathbb{N}$ the following:

$$
\begin{cases}
\partial_t h_m(t,x) + \mathcal{L}_\alpha h_m(t,x) + \langle Ax, D_x h_m(t,x)\rangle \;=\; -l_m(t,x); \\
h_m(T,x) \;=\; u_{T,m}(x)
\end{cases}
\tag{3.12}
$$

where $l_m(t,x) := f_m\Big(t, x - \int_t^T e^{A(t-s)} F_m(s,\theta_{s,\tau}(\xi))\,ds\Big)$.
Since we are going to exploit reasonings in Fourier spaces, we need however to have integrability properties on the solution h_m. For this reason, we introduce now a family $\{\rho_R\}_{R>0}$ of smooth functions such that any ρ_R is equal to 1 in $B(0,R)$ and vanishes outside $B(0,R+1)$. We then denote for any $R>0$,

$$
h_{m,R}(t,x) \;:=\; h_m(t,x)\rho_R(x).
$$

It is then straightforward that $h_{m,R}$ solves

$$
\begin{cases}
\partial_t h_{m,R}(t,x) + \mathcal{L}_\alpha h_{m,R}(t,x) + \langle Ax, D_x h_{m,R}(t,x)\rangle \;=\; -\tilde{l}_{m,R}(t,x); \\
h_{m,R}(T,x) \;=\; g_{m,R}(x),
\end{cases}
\tag{3.13}
$$

where $g_{m,R}(x) = u_{T,m}(x)\rho_R(x)$ and

$$
\begin{aligned}
\tilde{l}_{m,R}(t,x) \;=\; & \rho_R(x) l_m(t,x) + h_m(t,x)\mathcal{L}_\alpha \rho_R(x) \\
& + \int_{\mathbb{R}^d} \Big[h_m(t,x+By) - h_m(t,x)\Big]\Big[\rho_R(x+By) - \rho_R(x)\Big]\,\nu_\alpha(dy).
\end{aligned}
$$

Noticing now that $\tilde{l}_{m,R}$ is integrable with integrable Fourier transform, we can apply the Fourier transform in space to Equation (3.13) in order to write that

$$
\begin{cases}
\partial_t \widehat{h}_{m,R}(t,p) + \mathcal{F}_x\Big(\Big[\mathcal{L}_\alpha + \langle Ax, D_x\rangle\Big] h_{m,R}\Big)(t,p) \;=\; -\widehat{\tilde{l}}_{m,R}(t,p); \\
\widehat{h}_{m,R}(T,p) \;=\; \widehat{u_T}_{m,R}(p).
\end{cases}
$$

We remember in particular that the above operator $\mathcal{L}_\alpha + \langle Ax, D_x\rangle$ has an associated Lévy symbol $\Phi^{\mathrm{ou}}(p)$ and, following Section 3.3.2 in [App09], it holds that

$$
\mathcal{F}_x\Big(\Big[\mathcal{L}_\alpha + \langle Ax, D_x\rangle\Big] h_{m,R}\Big)(t,p) \;=\; \Phi^{\mathrm{ou}}(p)\widehat{h}_{m,R}(t,p).
$$

We can then use it to show that $\widehat{h}_{m,R}$ is a classical solution of the following equation:

$$
\begin{cases}
\partial_t \widehat{h}_{m,R}(t,p) + \Phi^{\mathrm{ou}}(p)\widehat{h}_{m,R}(t,p) \;=\; -\widehat{\tilde{l}}_{m,R}(t,p); \\
\widehat{h}_{m,R}(T,p) \;=\; \widehat{u_T}_{m,R}(p).
\end{cases}
$$

The above equation can be easily solved by integration in time, giving the following representation of $\widehat{h}_{m,R}(t,p)$:

$$\widehat{h}_{m,R}(t,p) \;=\; e^{(T-t)\Phi^{\mathrm{ou}}(p)}\widehat{u_T}_{m,R}(p) + \int_t^T e^{(s-t)\Phi^{\mathrm{ou}}(p)}\widehat{\tilde{l}}_{m,R}(s,p)\,ds.$$

In order to go back to $\tilde{v}_m^{\tau,\xi}$, we apply now the inverse Fourier transform to write that

$$h_{m,R}(t,x) \;=\; P^{\mathrm{ou}}_{T-t}g_{m,R}(x) + \int_t^T P^{\mathrm{ou}}_{s-t}\tilde{l}_{m,R}(s,x)\,ds,$$

remembering that $\{P^{\mathrm{ou}}_t\}_{t\ge 0}$ is the convolution Markov semigroup associated with the Ornstein-Uhlenbeck operator $\mathcal{L}_\alpha + \langle Ax, D_x\rangle$. Letting m go to ∞, it then follows immediately that $g_{m,R}\to u_{T,m}$, $h_{m,R}\to h_m$ and $\tilde{l}_{m,R}\to l_m$. A change of variable allows us to show the Duhamel representation, at least in the regularized setting:

$$\begin{aligned}\tilde{v}_m^{\tau,\xi}(t,y) =\; & P^{\mathrm{ou}}_{T-t}u_{T,m}\Big(y + \int_t^T e^{A(t-s)}F_m(s,\theta_{s,\tau}(\xi))\,ds\Big)\\ &+ \int_t^T P^{\mathrm{ou}}_{s-t}f_m\Big(s, y + \int_t^s e^{A(t-u)}F_m(u,\theta_{\tau,u}(\xi))\,du\Big)\,ds.\end{aligned}$$

Letting m goes to zero and remembering that $\tilde{v}_m^{\tau,\xi}\to\tilde{v}^{\tau,\xi}$, $f_m\to f$, $u_{T,m}\to u_T$ and $F_m\to F$ in the right functional spaces, we can conclude that $\tilde{v}^{\tau,\xi}=\tilde{u}^{\tau,\xi}$. □

3.2 Expansion along the proxy

We are going to use now the "frozen" IPDE (3.2) in order to derive appropriate quantitative controls of a solution u of Equation (1.1). Up to now, the freezing parameters (τ,ξ) were set free but they will be later chosen appropriately depending on the control we aim to establish.

The main idea is to exploit the Duhamel formula (Proposition 3.4) for the proxy to expand any solution u of the original IPDE (1.1) along the proxy. To make things more precise, let u be a mild solution in $L\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ of IPDE (1.1). Mollifying if necessary, it is possible to construct three sequences $\{f_m\}_{m\in\mathbb{N}}$, $\{u_{T,m}\}_{m\in\mathbb{N}}$ and $\{F_m\}_{m\in\mathbb{N}}$ of bounded and smooth functions with bounded derivatives such that $f_m\to f$ in $L^\infty\Big(0,T;C^{\beta}_{b,d}(\mathbb{R}^{nd})\Big)$, $u_{T,m}\to u_T$ in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})$ and $\|F_m-F\|_H\to 0$. Since u is a mild solution of (1.1), we know that there exists a smooth sequence $\{u_m\}_{m\in\mathbb{N}}$ converging to u in $L\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ and such that for any fixed m in $\mathbb{N}$, u_m solves in a classical sense the "regularized" IPDE (2.25).

Exploiting now that F_m is bounded and smooth, we can define the "regularized" flow $\theta^m_{\cdot,\tau}(\xi)$ as the *unique* flow satisfying

$$\theta^m_{t,\tau}(\xi) \;=\; \xi + \int_\tau^t \Big[A\theta^m_{s,\tau}(\xi) + F_m(s,\theta^m_{s,\tau}(\xi))\Big]\,ds, \quad t\in[\tau,T]. \tag{3.14}$$

It is then easy to notice that u_m is also a classical solution in $L\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ of

$$\partial_t u_m(t,x) + \mathcal{L}_\alpha u_m(t,x) + \langle Ax + F_m(t,\theta^m_{t,\tau}(\xi)), D_x u_m(t,x)\rangle \;=\; -\Big[f_m(t,x) + R^{\tau,\xi}_m(s,x)\Big]$$

on $(0,T)\times\mathbb{R}^{nd}$ with terminal condition $u_{T,m}$. Above, we have denoted

$$R^{\tau,\xi}_m(t,x) \,:=\, \langle F_m(t,x) - F_m(t,\theta^m_{t,\tau}(\xi)), D_x u_m(t,x)\rangle. \tag{3.15}$$

Since clearly, $R^{\tau,\xi}_m$ is in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$, we can use the Duhamel Formula (Proposition 3.4) for the proxy to write that

$$u_m(t,x) \,=\, \tilde{P}^{m,\tau,\xi}_{T,t}u_{T,m}(x) + \int_t^T \tilde{P}^{m,\tau,\xi}_{s,t}\Big[f_m(s,x)+R^{\tau,\xi}_m(s,x)\Big]\,ds, \quad (t,x)\,\in\,(0,T)\times\mathbb{R}^{nd},$$

where $\{\tilde{P}^{m,\tau,\xi}_{s,t}\}_{t\le s}$ is the semigroup associated with the operator

$$\mathcal{L}_\alpha + \langle Ax + F_m(t,\theta^m_{t,\tau}(\xi)), D_x\rangle.$$

The reasoning above is summarized in the following Duhamel-type formula that allows to expand any classical solution u_m of the "regularized" IPDE (2.25) along the "regularized frozen" proxy.

Proposition 3.5 (Duhamel-Type Formula)**.** *Let (τ,ξ) a freezing couple in $[0,T]\times\mathbb{R}^{nd}$. Under [**A**], any classical solution u_m of the "regularized" IPDE* (2.25) *can be represented as*

$$u_m(t,x) \,=\, \tilde{u}^{\tau,\xi}_m(t,x) + \int_t^T \tilde{P}^{m,\tau,\xi}_{s,t}R^{m,\tau,\xi}(s,x)\,ds, \quad (t,x)\,\in\,(0,T)\times\mathbb{R}^{nd} \tag{3.16}$$

where $R^{\tau,\xi}_m$ is as in (3.15) *and $\tilde{u}^{\tau,\xi}_m$ is defined through the Duhamel Representation* (3.10) *with the "regularized" coefficients f_m, $u_{T,m}$.*

Thanks to the above representation (Equation (3.16)), we know that, since we have already shown the suitable control for the frozen solution $u^{\tau,\xi}_m$ (namely, Proposition 3.3 with $f_m,u_{T,m}$), the main term which remains to be investigated in order to show the Schauder Estimates (Theorem 2.3) is the remainder

$$\int_t^T \tilde{P}^{m,\tau,\xi}_{s,t}R^{\tau,\xi}_m(s,x)\,ds, \tag{3.17}$$

that represents exactly the error in the expansion along the proxy.

To be precise, we could have passed to the limit in Equation (3.16) in order to obtain a similar Duhamel-type formula for a mild solution u in $L^\infty\Big([0,T];C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$. However, a problem appears when trying to give a precise meaning at the limit for the remainder contribution (3.17). We already know that the limit exists point-wise by difference, but for our approach to work, we need to establish precise quantitative controls on this term. Such estimates could be obtained through duality techniques in Besov spaces (cf. Section 5.1) but only at the expense of fixing already the freezing couple as $(\tau,\xi)=(t,x)$. The drawback of this method is that it does not allow to differentiate Equation (3.16), which is needed to estimate $D_{x_1}u$.

In order to show the suitable estimates for Expression (3.17), we will need at first an additional constraint on the behaviour of the system. In particular, we will say to be under assumption [**A'**] when assumption [**A**] is considered and if moreover,

[ST] we assume to be in a small time interval, i.e. $T \leq 1$.

Under these stronger assumptions, we will then be able to show in Section 5 below that the following control holds:

Proposition 3.6 (A Priori Estimates)**.** *Let u be a mild solution of IPDE* (1.1) *in $L^\infty\Big([0,T]; C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$. Under [**A'**], there exists a constant $C \geq 1$ such that*

$$\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \leq Cc_0^{\frac{\beta-\gamma_n}{\alpha}}\Big[\|f\|_{L^\infty(C^{\beta}_{b,d})} + \|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big] + C\Big(c_0^{\frac{\beta-\gamma_n}{\alpha}}\|F\|_H + c_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}}\Big)\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}, \quad (3.18)$$

where $c_0 \in (0,1)$ is assumed to be fixed but chosen later.

We remark already that in the above control, the constants multiplying $\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}$ have to be small if one wants to derive the expected Schauder estimates. If c_0 is small enough, then $Cc_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}}$ can be made smaller than $1/4$. Anyhow, for this chosen small c_0, the quantity $c_0^{\frac{\beta-\gamma_n}{\alpha}}$ becomes large and therefore, it needs to be balanced with $C\|F\|_H$. Namely, we can conclude if for instance, $Cc_0^{\frac{\beta-\gamma_n}{\alpha}}\|F\|_H < 1/4$ that implies in particular that $\|F\|_H$ has to be small with respect to c_0.

3.3 Conclusion of proof

In the first part of this section, we prove the Schauder estimates (Theorem 2.3) from the A Priori estimates (Proposition 3.6) through a suitable scaling procedure. Roughly speaking, the idea is to start from a general dynamics and then use the scaling procedure to make the Hölder norm $\|F\|_H$ small enough in order to make a *circular* argument work. Again, if c_0 and $\|F\|_H$ are small enough in Equation (3.18), the $L^\infty\Big(0,T; C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$-norm of u on the right-hand side can be absorbed by the left-hand one. Once the Schauder estimates (2.26) hold in the scaled dynamics, we will conclude going back to the original IPDE through the inverse scaling procedure, even if for a small final time horizon T.
The second part of the section focuses on showing how to drop the additional assumption [**A'**]. The key point here is to proceed through iteration up to an arbitrary, but finite, given time T thanks to the stability of a solution u in the space $L^\infty\Big(0,T; C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$.

Scaling argument

Under [**A**], we start considering a mild solution u of IPDE (1.1) on $[0,T]$ for some final time $T \leq 1$ to be fixed later. For a scaling parameter λ in $(0,1]$ to be chosen later, we would like to analyze IPDE (1.1) under the change of variables

$$(t,x) \mapsto (\lambda t, \mathbb{T}_\lambda x), \quad (3.19)$$

where $\mathbb{T}_\lambda := \lambda^{1/\alpha}\mathbb{M}_\lambda$. Again, the scaling is performed accordingly to the homogeneity induced by the distance $\mathbf{d}_P$ in (2.14).

To this purpose, we firstly introduce the scaled solution u_λ defined by

$$u_\lambda(t,x) := u(\lambda t, \mathbb{T}_\lambda x).$$

It then follows immediately that u_λ is a mild solution of the following Equation:

$$\lambda^{-1}\partial_t u_\lambda(t,x) + \lambda^{-1}\mathcal{L}_\alpha u_\lambda + \Big\langle A\mathbb{T}_\lambda x + F(\lambda t, \mathbb{T}_\lambda x), \mathbb{T}_\lambda^{-1} D_x u_\lambda(t,x)\Big\rangle \\ = -f(\lambda t, \mathbb{T}_\lambda x) \quad \text{on } (0,T_\lambda)\times\mathbb{R}^{nd},$$

with terminal condition $u_\lambda(T_\lambda,x) = u_T(\mathbb{T}_\lambda x)$, where $T_\lambda := T/\lambda$. Since we want the scaled dynamics to satisfy assumption ($\mathbf{A'}$), we choose now T so that $T_\lambda \le 1$. It is important to notice that this is possible since we assumed λ to be fixed, even if we have not chosen it yet. Denoting now

$$\begin{aligned} f_\lambda(t,x) &:= \lambda f(\lambda t, \mathbb{T}_\lambda x); \\ u_{T,\lambda}(x) &:= u_T(\mathbb{T}_\lambda x); \\ A_\lambda &:= \lambda \mathbb{T}_\lambda^{-1} A \mathbb{T}_\lambda; \\ F_\lambda(t,x) &:= \lambda \mathbb{T}_\lambda^{-1} F(\lambda t, \mathbb{T}_\lambda x), \end{aligned}$$

we can rewrite the scaled dynamics as:

$$\begin{cases} \partial_t u_\lambda(t,x) + \Big\langle A_\lambda x + F_\lambda(t,x), D_x u_\lambda(t,x)\Big\rangle + \mathcal{L}_\alpha u_\lambda(t,x) = -f_\lambda(t,x); \\ u_\lambda(T_\lambda,x) = u_{T,\lambda}(x). \end{cases} \tag{3.20}$$

To continue, we need the following lemma that shows how the scaling procedure reflects on the norms of the coefficients. Recalling Equation (2.24) for the definition of $\|\cdot\|_H$, a direct calculation on the norms leads to the following result:

Lemma 3.7 (Scaling Homogeneity of Norms)**.** *Under [**A**], it holds that*

$$\begin{aligned} \|F_\lambda\|_H &= \lambda^{\beta/\alpha}\|F\|_H; \\ \lambda^{\frac{\alpha+\beta}{\alpha}}\|f\|_{L^\infty(C^\beta_{b,d})} \le \|f_\lambda\|_{L^\infty(C^\beta_{b,d})} &\le \|f\|_{L^\infty(C^\beta_{b,d})}; \\ \lambda^{\frac{\alpha+\beta}{\alpha}}\|u_T\|_{C^{\alpha+\beta}_{b,d}} \le \|u_{T,\lambda}\|_{C^{\alpha+\beta}_{b,d}} &\le \|u_T\|_{C^{\alpha+\beta}_{b,d}}; \\ \lambda^{\frac{\alpha+\beta}{\alpha}}\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \le \|u_\lambda\|_{L^\infty(C^{\alpha+\beta}_{b,d})} &\le \|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}. \end{aligned} \tag{3.21}$$

Since the scaled dynamics in (3.20) satisfies assumption [**A'**], we know from Proposition 3.6 that the scaled solution u_λ satisfies the a priori Estimates (3.18):

$$\|u_\lambda\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \le Cc_0^{\frac{\beta-\gamma_n}{\alpha}}\Big[\|f_\lambda\|_{L^\infty(C^\beta_{b,d})} + \|u_{T,\lambda}\|_{C^{\alpha+\beta}_{b,d}}\Big] \\ + C\Big(c_0^{\frac{\beta-\gamma_n}{\alpha}}\|F_\lambda\|_H + c_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}}\Big)\|u_\lambda\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \tag{3.22}$$

for some constant c_0 in $(0,1]$ to be chosen later.
We would like now to exploit a circular argument in order to bring to the left-hand side

of Equation (3.22) the term involving u_λ on the right-hand one. To do that, we need to choose properly λ and c_0 in order to have

$$C\Big(c_0^{\frac{\beta-\gamma n}{\alpha}}\|F_\lambda\|_H + c_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}}\Big) \,<\, 1.$$

This is true if, for example, we choose firstly c_0 such that

$$Cc_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}} \,=\, \frac{1}{4}$$

and fixed c_0, we choose λ so that

$$Cc_0^{\frac{\beta-\gamma n}{\alpha}}\lambda^{\beta/\alpha}\|F\|_H \,=\, Cc_0^{\frac{\beta-\gamma n}{\alpha}}\|F_\lambda\|_H \,=\, \frac{1}{4}.$$

With this choice, it then follows from Equation (3.22) that

$$\|u_\lambda\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \,\le\, 2Cc_0^{\frac{\beta-\gamma n}{\alpha}}\Big[\|f_\lambda\|_{L^\infty(C^{\beta}_{b,d})} + \|u_{T,\lambda}\|_{C^{\alpha+\beta}_{b,d}}\Big].$$

We can finally use Lemma 3.7 to go back to the original dynamics and write that

$$\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \,\le\, \lambda^{-\frac{\alpha+\beta}{\alpha}}\|u_\lambda\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \,\le\, \overline{C}\Big[\|f\|_{L^\infty(C^{\beta}_{b,d})} + \|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big]$$

for some constant $\overline{C} > 0$ defined by

$$\overline{C} \,:=\, 2\lambda^{-\frac{\alpha+\beta}{\alpha}}Cc_0^{\frac{\beta-\gamma n}{\alpha}}.$$

Schauder estimates for general time

Up to this point, we have assumed to be in a small enough final time horizon (i.e. $T \le 1$) to let our procedure work. We are going now to extend the Schauder estimates (Equation (2.26)) to an arbitrary but fixed final time $T_0 > 0$. Our proof will consist essentially in a backward iterative procedure through a chain of identical differential dynamics on different, small enough, time intervals. We recall indeed that the Schauder estimates precisely provide a stability result in the chosen functional space.

Proposition 3.8. *Under [**A**], let $T_0 > T$ and u a mild solution in $L^\infty\Big(0, T_0; C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ of IPDE* (1.1) *on $[0, T_0]$ that satisfies the Schauder Estimates* (2.26) *on $[0, T]$. Then, there exists a constant $C_0 := C_0(T_0)$ such that*

$$\|u\|_{L^\infty(0,T_0;C^{\alpha+\beta}_{b,d})} \,\le\, C_0\Big[\|f\|_{L^\infty(0,T_0;C^{\beta}_{b,d})} + \|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big].$$

Proof. Fixed $N = \lceil\frac{T_0}{T}\rceil$, we are going to consider a system of N Cauchy problems:

$$\begin{cases}\partial_t u_k(t,x) + \Big\langle Ax + F(t,x), D_x u_k(t,x)\Big\rangle + \mathcal{L}_\alpha u_k(t,x) \,=\, -f(t,x);\\ u_k((1-\frac{k-1}{N})T_0, x) \,=\, u_{k-1}((1-\frac{k-1}{N})T_0, x),\end{cases}$$

on $((1-\frac{k}{N})T_0,(1-\frac{k-1}{N})T_0)\times\mathbb{R}^{nd}$, for $k=1,\dots,N$ with the notation that $u_0(T_0,x)=u_T(x)$. Reasoning iteratively, we find that any mild solution of IPDE (1.1) on $[0,T_0]$ is also a mild solution of any of the equations of the system. Moreover, since any solution u_k is defined on $[(1-\frac{k}{N})T_0,(1-\frac{k-1}{N})T_0]$ and

$$(1-\frac{k-1}{N})T_0-(1-\frac{k}{N})T_0 \;=\; \frac{k}{N}T_0-\frac{k-1}{N}T_0 \;=\; \frac{1}{N}T_0 \;\le\; T,$$

the Schauder estimates (Equation (2.26)) hold for any solution u_k with terminal condition $u_{k-1}((1-\frac{k-1}{N})T_0,\cdot)$. In particular,

$$\begin{aligned}
&\|u_k\|_{L^\infty((1-\frac{k}{N})T_0,(1-\frac{k-1}{N})T_0;C^{\alpha+\beta}_{b,d})}\\
&\quad\le C\Big[\|f\|_{L^\infty((1-\frac{k}{N})T_0,(1-\frac{k-1}{N})T_0;C^{\beta}_{b,d})}+\|u_{k-1}((1-\frac{k-1}{N})T_0,\cdot)\|_{C^{\alpha+\beta}_{b,d}}\Big]\\
&\quad\le C^2\Big[\|f\|_{L^\infty((1-\frac{k}{N})T_0,(1-\frac{k-1}{N})T_0;C^{\beta}_{b,d})}+\|f\|_{L^\infty((1-\frac{k-1}{N})T_0,(1-\frac{k-2}{N})T_0;C^{\beta}_{b,d})}\\
&\qquad\qquad+\|u_{k-2}((1-\frac{k-2}{N})T_0,\cdot)\|_{C^{\alpha+\beta}_{b,d}}\Big]\\
&\quad\le C^2\Big[\|f\|_{L^\infty((1-\frac{k}{N})T_0,(1-\frac{k-2}{N})T_0;C^{\beta}_{b,d})}+\|u_{k-2}((1-\frac{k-2}{N})T_0,\cdot)\|_{C^{\alpha+\beta}_{b,d}}\Big],
\end{aligned}$$

exploiting that u_{k-1} satisfies the Schauder estimates with source f and terminal condition $u_{k-2}((1-\frac{k-2}{N})T_0,\cdot)$. Applying the same procedure recursively, we finally find that

$$\|u_k\|_{L^\infty((1-\frac{k}{N})T_0,(1-\frac{k-1}{N})T_0;C^{\alpha+\beta}_{b,d})} \;\le\; C^k\Big[\|f\|_{L^\infty((1-\frac{k}{N})T_0,T_0;C^{\beta}_{b,d})}+\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big].$$

Hence,

$$\|u\|_{L^\infty(0,T_0;C^{\alpha+\beta}_{b,d})} \;\le\; C^N\Big[\|f\|_{L^\infty(0,T_0;C^{\beta}_{b,d})}+\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big]$$

and we have concluded the proof. □

4 Schauder estimates for the proxy

The aim of this section is to show how to properly control a solution $\tilde{u}^{\tau,\xi}$ of the "frozen" IPDE (3.2) in order to prove the Schauder estimates (Proposition 3.3) for the proxy. We recall the definition of $\tilde{u}^{\tau,\xi}$ through the Duhamel Representation (3.10). Namely, for any freezing couple (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$, it holds that

$$\tilde{u}^{\tau,\xi}(t,x) \;=\; \tilde{P}^{\tau,\xi}_{T,t}u_T(x)+\tilde{G}^{\tau,\xi}_{T,t}f(t,x) \tag{4.1}$$

where we have denoted for simplicity with $\{\tilde{G}^{\tau,\xi}_{r,v}\}_{t>v\ge 0}$ the family of Green kernels associated with the frozen density $\tilde{p}^{\tau,\xi}$. More in details, we have for any $v<r$ in $[0,T]$ that

$$\tilde{G}^{\tau,\xi}_{r,v}f(t,x) \;:=\; \int_v^r\int_{\mathbb{R}^{nd}}\tilde{p}^{\tau,\xi}(t,s,x,y)f(s,y)\,dy\,ds. \tag{4.2}$$

We can then differentiate the above equation with respect to x_1 in order to obtain an analogous Duhamel type representation for the derivative $D_{x_1}\tilde{u}^{\tau,\xi}$:

$$D_{x_1}\tilde{u}^{\tau,\xi}(t,x) \;=\; D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x)+D_{x_1}\tilde{G}^{\tau,\xi}_{T,t}f(t,x). \tag{4.3}$$

It is then clear that in order to control $\tilde{u}^{\tau,\xi}(t,x)$ in the norm $\|\cdot\|_{L^\infty(C^{\alpha+\beta}_{b,d})}$, we can analyze separately the contributions appearing from the frozen semigroup $\tilde{P}^{\tau,\xi}_{T,t}u_T(x)$ and those from the frozen Green kernel $\tilde{G}^{\tau,\xi}_{T,t}f(t,x)$.

4.1 First Besov control

We focus for the moment on the contribution in the Duhamel Representation (4.1) associated with the source u_T that is, as it will be seen, the more delicate to treat. In the non-degenerate setting (i.e. with respect to x_1), it precisely write:

$$D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x) \,=\, \int_{\mathbb{R}^{nd}} D_{x_1}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy.$$

Looking at the particular structure of $\tilde{p}^{\tau,\xi}$ (cf. Equation (3.3)), it can be seen from Lemma 2.1 that

Lemma 4.1. *Let i in $[\![1,n]\!]$. Then, there exist constants $\{C_j\}_{j\in[\![i,n]\!]}$ such that*

$$D_{x_i}\tilde{p}^{\tau,\xi}(t,s,x,y) \,=\, \sum_{j=i}^{n} C_j(s-t)^{j-i}D_{y_j}\tilde{p}^{\tau,\xi}(t,s,x,y) \tag{4.4}$$

for any $t<s$ in $[0,T]$, any x,y in $\mathbb{R}^{nd}$ and any freezing couple (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$.

We can now use Wquation (4.4) to rewrite $D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x)$ as

$$\begin{aligned} \left|D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x)\right| \,&=\, \left|\int_{\mathbb{R}^{nd}} D_{x_1}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy\right| \\ &\le\, C\sum_{j=1}^{n}(s-t)^{j-1}\left|\int_{\mathbb{R}^{nd}} D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy\right|. \end{aligned} \tag{4.5}$$

Remembering that u_T is in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})$ for $\alpha+\beta>1$ by hypothesis, we know that it is differentiable with respect to the first (non-degenerate) variable x_1. Then, the above expression can be controlled easily for $j=1$ as

$$\begin{aligned} \left|\int_{\mathbb{R}^{nd}} D_{y_1}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy\right| \,&=\, \left|\int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t,T,x,y)D_{y_1}u_T(y)\,dy\right| \\ &\le\, \|D_{y_1}u_T\|_{L^\infty} \\ &\le\, \|u_T\|_{C^{\alpha+\beta}_{b,d}}, \end{aligned}$$

using integration by parts formula. We can then focus on the degenerate components in (4.5), i.e.

$$\left|\int_{\mathbb{R}^{nd}} D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy\right| \tag{4.6}$$

for some $j>1$. Since u_T is not differentiable with respect to y_j if $j>1$, we cannot apply the same reasoning above but we will need a more subtle control. Our main idea will be to use the duality in Besov spaces to derive bounds for Expression (4.6). Namely, we introduce for a given y in $\mathbb{R}^d$,

$$y_{\smallsetminus j} \,:=\, (y_1,\dots,y_{j-1},y_{j+1},\dots,y_n) \,\in\, \mathbb{R}^{(n-1)d}.$$

With this definition at hand, we then denote for any function ϕ on $\mathbb{R}^{nd}$, the function $\phi(y_{\smallsetminus j},\cdot)$ on $\mathbb{R}^d$ with a slight abuse of notation as

$$\phi(y_{\smallsetminus j},z) \;:=\; \phi(y_1,\dots,y_{j-1},z,y_{j+1},\dots,y_n). \tag{4.7}$$

The key point now is to control the Hölder modulus of $u_T(y_{\smallsetminus j},\cdot)$ on $\mathbb{R}^d$, uniformly in $y_{\smallsetminus j}\in\mathbb{R}^{(n-1)d}$. To do so, we will need the identification $C^{\alpha_j+\beta_j}_b(\mathbb{R}^d)=B^{\alpha_j+\beta_j}_{\infty,\infty}(\mathbb{R}^d)$ with the usual notations for the Besov spaces.

We recall now some useful definitions/characterizations about Besov spaces $B^{\tilde\gamma}_{p,q}(\mathbb{R}^d)$. For a more detailed analysis of this argument, we suggest the reader to see Section 2.6.4 of Triebel [Tri92]. For $\tilde\gamma$ in $(0,1)$, q,p in $(0,+\infty]$, we define the Besov space of indexes $(\tilde\gamma,p,q)$ on $\mathbb{R}^d$ as:

$$B^{\tilde\gamma}_{p,q}(\mathbb{R}^d):=\{f\in\mathcal{S}'(\mathbb{R}^d)\colon \|f\|_{\mathcal{H}^{\tilde\gamma}_{p,q}}<+\infty\},$$

where $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz class on $\mathbb{R}^d$ and

$$\|f\|_{\mathcal{H}^{\tilde\gamma}_{p,q}} \;:=\; \|(\phi_0\hat f)^\vee\|_{L^p}+\Bigl(\int_0^1 v^{(1-\frac{\tilde\gamma}{\alpha})q}\|\partial_v p_h(v,\cdot)*f\|^q_{L^p}\frac{dv}{v}\Bigr)^{\frac1q}, \tag{4.8}$$

with ϕ_0 a function in $C^\infty_0(\mathbb{R}^d)$ such that $\phi_0(0)\neq 0$ and p_h the isotropic α-stable heat kernel on $\mathbb{R}^d$, i.e. the stable density on $\mathbb{R}^d$ whose Lévy symbol is equivalent to $|\lambda|^\alpha$.
We point out that the quantities in (4.8) are well-defined for any $q\neq+\infty$. The modifications for $q=+\infty$ are obvious and can be written passing to the limit. The previous definition of $B^{\tilde\gamma}_{p,q}(\mathbb{R}^d)$ is known as the stable thermic characterization of Besov spaces and it is particularly adapted to our framework. By a little abuse of notation, we will write $\|f\|_{B^{\tilde\gamma}_{p,q}}:=\|f\|_{\mathcal{H}^{\tilde\gamma}_{p,q}}$ when this quantity is finite.

For the heat-kernel p_h, it is possible to show an improvement of the smoothing effect (cf. Equation (2.13)), due essentially to its better decay at infinity. Namely, we are no more bounded to the condition $\gamma<\alpha$ but we can integrate up to an order γ strictly smaller than $1+\alpha$.

Lemma 4.2 (Smoothing Effect of the Isotropic Stable Heat-Kernel)**.** *Let l be in $\{1,2\}$ and γ in $[0,1+\alpha)$. Then, there exists a positive constant $C:=C(\gamma)$ such that*

$$\int_{\mathbb{R}^d}|y|^\gamma|\partial_v D^l_y p_h(v,y)|\,dy \;\le\; Ct^{\frac{\gamma-l}{\alpha}-1}. \tag{4.9}$$

A proof of the above result can be derived using the estimates of Kolokoltsov [Kol00] (see also [BJ07]).

As already indicated before, it can be seen from the Thermic Characterization (4.8) that

$$C^{\tilde\gamma}_b(\mathbb{R}^d) \;=\; B^{\tilde\gamma}_{\infty,\infty}(\mathbb{R}^d). \tag{4.10}$$

Moreover, it is well known (see for example Proposition 3.6 in [LR02]) that $B^{\tilde\gamma}_{\infty,\infty}(\mathbb{R}^d)$ and $B^{-\tilde\gamma}_{1,1}(\mathbb{R}^d)$ are in duality. Namely, it holds

$$\Bigl|\int_{\mathbb{R}^d}fg\,dx\Bigr| \;\le\; C\|f\|_{B^{\tilde\gamma}_{\infty,\infty}}\|u_T\|_{B^{-\tilde\gamma}_{1,1}}, \tag{4.11}$$

for any f in $B^{\tilde{\gamma}}_{\infty,\infty}(\mathbb{R}^d)$ and any u_T in $B^{-\tilde{\gamma}}_{1,1}(\mathbb{R}^d)$.

With these definitions and properties at hand, we can now go back at Expression (4.6) to write that

$$\begin{aligned}\left|\int_{\mathbb{R}^{nd}} D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy\right| &\le \int_{\mathbb{R}^{(n-1)d}}\left|D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)dy_j\right|dy_{\smallsetminus j}\\ &\le \int_{\mathbb{R}^{(n-1)d}}\left\|D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y_{\smallsetminus j},\cdot)\right\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}\left\|u_T(y_{\smallsetminus j},\cdot)\right\|_{B^{\alpha_j+\beta_j}_{\infty,\infty}}dy_{\smallsetminus j}\\ &\le \|u_T\|_{C^{\alpha+\beta}_{b,d}}\int_{\mathbb{R}^{(n-1)d}}\left\|D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y_{\smallsetminus j},\cdot)\right\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}dy_{\smallsetminus j}.\end{aligned}$$

In order to control the above quantities, we will then need a control on the integral of the Besov norms of the derivatives of the proxy. Since however an additional derivative with respect to x_1 will often appear, for example in Equation (4.24) below, we state the following result in a more general way.

Lemma 4.3 (First Besov Control)**.** *Let j be in $[\![2,n]\!]$ and $l\in\{0,1\}$. Under [**A**], there exists a constant $C:=C(j,l)$ such that*

$$\int_{\mathbb{R}^{(n-1)d}}\left\|D_{y_j}D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\cdot)\right\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}dy_{\smallsetminus j}\;\le\;C(s-t)^{\frac{\alpha+\beta}{\alpha}-\frac{1}{\alpha_j}-\frac{l}{\alpha}},$$

for any $t<s$ in $[0,T]$, any x in $\mathbb{R}^{nd}$ and any frozen couple (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$.

Proof. To control the Besov norm in $B^{-(\alpha_j+\beta_j)}_{1,1}(\mathbb{R}^d)$, we are going to use the Thermic Characterization (4.8) with $\tilde{\gamma}=-(\alpha_j+\beta_j)$. We start considering the second term in the characterization, i.e.

$$\int_0^1 v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\partial_v p_h(v,z-y_j)D_{y_j}D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y)\,dy_j\right|dzdv.$$

Fixed a constant $\delta_j\ge 1$ to be chosen later, we split the integral with respect to v in two components:

$$\begin{aligned}&\|D_{y_j}D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\cdot)\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}\\ &\quad=\int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\partial_v p_h(v,z-y_j)D_{y_j}D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y)\,dy_j\right|dzdv\\ &\qquad+\int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\partial_v p_h(v,z-y_j)D_{y_j}D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y)\,dy_j\right|dzdv\\ &\quad=:\Big(I_1+I_2\Big)(y_{\smallsetminus j}).\end{aligned}$$

The second component I_2 has no time-singularity and can be easily controlled by

$$I_2(y_{\smallsetminus j})\;=\;\int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}D_z\partial_v p_h(v,z-y_j)\otimes D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y)\,dy_j\right|dzdv,$$

using integration by parts formula and noticing that $D_{y_j}p_h(v,z-y_j)=-D_z p_h(v,z-y_j)$. Then,

$$I_2(y_{\smallsetminus j})\;\le\;\int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|D_z\partial_v p_h(v,z-y_j)|\,|D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y)|\,dy_j\,dzdv.$$

We can then use Fubini theorem to separate the integrals and apply the smoothing effect of the heat-kernel p_h (Lemma 4.2) to show that

$$\begin{aligned}
I_2(y_{\smallsetminus j}) \,&\le\, \int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j+\beta_j}{\alpha}} \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} |D_z\partial_v p_h(v,z-y_j)|\,dz\Big)|D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y)|\,dy_j dv \\
&\le\, C\Big(\int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j+\beta_j-1}{\alpha}-1}\,dv\Big)\Big(\int_{\mathbb{R}^d} |D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y)|\,dy_j\Big) \\
&\le\, C(s-t)^{\frac{\delta_j(\alpha_j+\beta_j-1)}{\alpha}} \int_{\mathbb{R}^d} |D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y)|\,dy_j.
\end{aligned}$$

Using the smoothing effect (Equation (3.6)) of the frozen density $\tilde{p}^{\tau,\xi}$, we have thus found that

$$\begin{aligned}
\int_{\mathbb{R}^{(n-1)d}} I_2(y_{\smallsetminus j})\,dy_{\smallsetminus j} \,&\le\, (s-t)^{\frac{\delta_j(\alpha_j+\beta_j-1)}{\alpha}} \int_{\mathbb{R}^{nd}} |D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y)|\,dy \\
&\le\, C(s-t)^{\frac{\delta_j(\alpha_j+\beta_j-1)-l}{\alpha}}.
\end{aligned} \tag{4.12}$$

On the other hand, the term I_1 needs a more delicate treatment in order to avoid time-integrability problems. We start using a cancellation argument with respect to the derivative $\partial_v p_h$ of the heat-kernel to rewrite I_1 as

$$\begin{aligned}
I_1(y_{\smallsetminus j}) \,&=\, \int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j+\beta_j}{\alpha}} \int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d} \partial_v p_h(v,z-y_j) \\
&\qquad\qquad \times \Big[D_{y_j}D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y) - D_{y_j}D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},z)\Big]\,dy_j\Big|\,dzdv \\
&=\, \int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j+\beta_j}{\alpha}} \int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d} D_z\partial_v p_h(v,z-y_j) \\
&\qquad\qquad \otimes \Big[D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y) - D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},z)\Big]\,dy_j\Big|\,dzdv,
\end{aligned}$$

where in the second passage we used again integration by parts formula to move the derivative to p_h and the equality $D_{y_j}p_h(v,z-y_j) = -D_z p_h(v,z-y_j)$. We can then apply a Taylor expansion with respect to variable y_j in order to write that

$$\begin{aligned}
I_1(y_{\smallsetminus j}) \,&=\, \int_0^{(s-t)^{\delta_i}} v^{\frac{\alpha_j+\beta_j}{\alpha}} \int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d} D_z\partial_v p_h(v,z-y_j) \\
&\qquad \times \int_0^1 D_{y_j}D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},y_j+\lambda(z-y_j))\cdot(z-y_j)\,d\mu dy_j\Big|\,dzdv \\
&\le\, \int_0^{(s-t)^{\delta_i}} v^{\frac{\alpha_j+\beta_j}{\alpha}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^1 |D_z\partial_v p_h(v,z-y_j)| \\
&\qquad \times |D_{y_j}D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},y_j+\lambda(z-y_j))|\,|z-y_j|\,d\lambda dy_j dzdv.
\end{aligned}$$

We can then use the Fubini theorem and the changes of variables $\tilde{z} = z-y_j$ (fixed y_j) and $\tilde{y}_j = y_j+\lambda\tilde{z}$ (considering $\tilde{z}$ and λ fixed) to separate the integrals so that

$$\begin{aligned}
I_1(y_{\smallsetminus j}) \,&\le\, \int_0^{(s-t)^{\delta_i}} v^{\frac{\alpha_j+\beta_j}{\alpha}} \Big(\int_{\mathbb{R}^d} |D_z\partial_v p_h(v,\tilde{z})|\,|\tilde{z}|\,dz\Big) \\
&\qquad\qquad \times \Big(\int_{\mathbb{R}^d} |D_{y_j}D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\tilde{y}_j)|\,dy_j\Big)\,dv.
\end{aligned}$$

The smoothing effect of the heat-kernel p_h (Lemma 4.2) allows now to control the first term:

$$\begin{aligned}
I_1(y_{\smallsetminus j}) &\le C\Big(\int_0^{(s-t)^{\delta_i}} v^{\frac{\alpha_j+\beta_j-1}{\alpha}}\,dv\Big)\Big(\int_{\mathbb{R}^d}|D_{y_j}D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},z+\lambda(y_j-z))|\,dy_j\Big)\\
&\le C(s-t)^{\delta_j\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d}|D_{y_j}D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},z+\lambda(y_j-z))|\,dy_j.
\end{aligned}$$

It then follows using the smoothing effect of the frozen semigroup (Lemma 3.2) that

$$\begin{aligned}
\int_{\mathbb{R}^{(n-1)d}} I_1(y_{\smallsetminus j})\,dy_{\smallsetminus j} &\le C(s-t)^{\delta_j\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^{nd}}|D_{y_j}D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},z+\lambda(y_j-z))|\,dy\\
&\le C(s-t)^{\delta_j\frac{\alpha_j+\beta_j}{\alpha}-\frac{l}{\alpha}-\frac{l}{\alpha_j}}. \qquad (4.13)
\end{aligned}$$

Going back to equations (4.12) and (4.13), we notice that we need δ_j to be such that

$$\delta_j\Big[\frac{\alpha_j+\beta_j}{\alpha}\Big] = \frac{\alpha+\beta}{\alpha} \quad\text{and}\quad \delta_j\Big[\frac{\alpha_j+\beta_j-1}{\alpha}\Big] = \frac{\alpha+\beta}{\alpha}-\frac{1}{\alpha_j}.$$

Recalling Equation (2.27) for the relative definitions, we can thus conclude choosing

$$\delta_j = (\alpha+\beta)/(\alpha_j+\beta_j) = 1+\alpha(j-1).$$

Reproducing the previous computations, we can also write for a test function in ϕ_0 in $C^\infty_0(\mathbb{R}^d)$,

$$\begin{aligned}
&\int_{\mathbb{R}^{(n-1)d}}\Big\|\Big(\phi_0\big(D_{y_j}D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\cdot)\big)\,\hat{}\,\Big)^{\vee}\Big\|_{L^1}\,dy_{\smallsetminus j}\\
&\qquad= \int_{\mathbb{R}^{(n-1)d}}\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d} D_{y_j}\hat{\phi}_0(z-y_j)\cdot D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y)\,dy_j\Big|\,dzdy_{\smallsetminus j}\\
&\qquad\le C\int_{\mathbb{R}^{nd}}|D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y)|\,dy\\
&\qquad\le C(s-t)^{-\frac{l}{\alpha}}.
\end{aligned}$$

The proof is thus concluded. □

4.2 Proof of Proposition 3.3

Thanks to the first Besov control (Lemma 4.3), we are now ready to prove the Schauder estimates for the proxy (Proposition 3.3). Such a proof will be divided in three parts: the estimates for the supremum norms of the solution and its non-degenerate gradient are stated in Lemma 4.4 while the controls of the Hölder moduli of the solution and its gradient with respect to the non-degenerate variable are given in Lemmas 4.5 and 4.6, respectively.

Lemma 4.4. *(Controls on Supremum Norm) Under [**A**], there exists a constant* $C := C(T)\ge 1$ *such that for any freezing couple* (τ,ξ) *in* $[0,T]\times\mathbb{R}^{nd}$*, any* t *in* $[0,T]$ *and any* x *in* $\mathbb{R}^{nd}$,

$$|\tilde{u}^{\tau,\xi}(t,x)|+|D_{x_1}\tilde{u}^{\tau,\xi}(t,x)| \le C\Big[\|f\|_{L^\infty(C^\beta_{b,d})}+\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big].$$

Proof. We start noticing that $\tilde{P}^{\tau,\xi}_{T,t}u_T(x)$ and $\tilde{G}^{\tau,\xi}_{T,t}f(t,x)$ can be easily bounded using the supremum norm of f and u_T, respectively.
Moreover, we can use the estimates on the frozen semigroup (Equation (3.7)) to control $D_{x_1}\tilde{G}^{\tau,\xi}_{T,t}f(t,x)$. Indeed,

$$\begin{aligned}\left|D_{x_1}\tilde{G}^{\tau,\xi}_{T,t}f(t,x)\right| &\le \int_t^T \left|D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}f(s,x)\right| ds \\ &\le C(T-t)^{\frac{\alpha+\beta-1}{\alpha}}\|f\|_{L^\infty(C^\beta_{b,d})} \\ &\le CT^{\frac{\alpha+\beta-1}{\alpha}}\|f\|_{L^\infty(C^\beta_{b,d})},\end{aligned}$$

remembering in the last inequality that $\alpha+\beta-1>0$ by hypothesis [**P**].
It remains to control $D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x)$. As done in the previous Sub-section 4.1, we start using the scaling lemma 4.1 to write that

$$\begin{aligned}\left|D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x)\right| &= \left|\int_{\mathbb{R}^{nd}} D_{x_1}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy\right| \\ &\le C\sum_{j=1}^n (T-t)^{j-1}\left|\int_{\mathbb{R}^{nd}} D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy\right| \\ &=: C\sum_{j=1}^n (T-t)^{j-1}J_j.\end{aligned}$$

Since u_T is differentiable in the first, non-degenerate variable x_1, the contribution J_1 can be easily bounded using integration by parts formula:

$$J_1 = \left|\int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t,T,x,y)D_{y_1}u_T(y)\,dy\right| \le \|D_{y_1}u_T\|_{L^\infty} \le \|u_T\|_{C^{\alpha+\beta}_{b,d}}. \tag{4.14}$$

To control the other terms J_j for $j>1$, we use instead the duality in Besov spaces (Equation (4.11)) and Identification (4.10), so that

$$\begin{aligned}J_j &\le C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\int_{\mathbb{R}^{(n-1)d}} \|D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y_{\smallsetminus j},\cdot)\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}\,dy_{\smallsetminus j} \\ &\le C\|u_T\|_{C^{\alpha+\beta}_{b,d}}(T-t)^{\frac{\alpha+\beta}{\alpha}-\frac{1}{\alpha_j}},\end{aligned} \tag{4.15}$$

where in the last inequality we applied the first Besov control (Lemma 4.3).
Looking back at Equations (4.14)-(4.15), it finally holds that

$$\begin{aligned}\left|D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x)\right| &\le C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big(1+\sum_{j=2}^n (T-t)^{j-1}(T-t)^{\frac{\alpha+\beta}{\alpha}-\frac{1}{\alpha_j}}\Big) \\ &\le C\Big(1+T^{\frac{\alpha+\beta-1}{\alpha}}\Big)\|u_T\|_{C^{\alpha+\beta}_{b,d}},\end{aligned}$$

where in the last passage we used again that $\alpha+\beta-1>0$ by hypothesis [**P**]. □

Before starting with the calculations on the Hölder modulus, we will need to distinguish two cases. Fixed (t,x,x') in $[0,T]\times\mathbb{R}^{2nd}$, we will say that the *off-diagonal regime* holds if

$T-t \le c_0\mathbf{d}^\alpha(x,x')$ for a constant c_0 to be specified but meant to be smaller than 1. This means in particular that the spatial distance is larger than the characteristic time-scale up to the prescribed constant c_0 which will be useful further on in the computations for a circular argument.
On the other hand, we will say that a *global diagonal regime* is in force if $T-t \ge c_0\mathbf{d}^\alpha(x,x')$ and the spatial points are instead closer than the typical time-scale magnitude. In particular, when a time integration is involved (for example, in the control of the frozen Green kernel), the same two regime appears even in a local base. Considering a variable s in $[t,T]$, there are again a *local off-diagonal regime* if $s-t \le c_0\mathbf{d}^\alpha(x,x')$ and a *local diagonal regime* when $s-t \ge c_0\mathbf{d}^\alpha(x,x')$. In particular, we will denote with t_0 the critical time at which a change of regime occurs in the globally diagonal regime. Namely,

$$t_0 := \Big(t + c_0\mathbf{d}^\alpha(x,x')\Big) \wedge T. \tag{4.16}$$

We highlight however that this approach was already used in [CdRHM18a] to obtain Schauder estimates for degenerate Kolmogorov equations and can be adapted in the current setting.

Moreover, it is important to notice that the norm $\|\cdot\|_{C^{\alpha+\beta}_d}$ is essentially defined as the sum of the norms $\|\cdot\|_{C^{\frac{\alpha+\beta}{1+\alpha(i-1)}}}$ with respect to the i-th variable and uniformly on the other components. Thus, there is a big difference between the case $i=1$ where $\alpha+\beta$ is in $(1,2)$ and we have to deal with a proper derivative and the other situations ($i>1$) where instead $(\alpha+\beta)/(1+\alpha(i-1)) < 1$ and the norm is calculated directly on the function. For this reason, we are going to analyze the two cases separately. Lemma 4.5 will work on the non-degenerate setting ($i=1$) while Lemma 4.6 will concern the degenerate one ($i>1$).

Lemma 4.5 (Controls on Hölder Moduli: Non-Degenerate)**.** *Let x,x' be in $\mathbb{R}^{nd}$ such that $x_j = x'_j$ for any $j \neq 1$. Under [$\boldsymbol{A}$], there exists a constant $C \ge 1$ such that for any t in $[0,T]$ and any freezing couple (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$, it holds that*

$$\Big|D_{x_1}\tilde{u}^{\tau,\xi}(t,x) - D_{x_1}\tilde{u}^{\tau,\xi}(t,x')| \le Cc_0^{\frac{\alpha+\beta-2}{\alpha}}\Big(\|u_T\|_{C^{\alpha+\beta}_{b,d}} + \|f\|_{L^\infty(C^\beta_{b,d})}\Big)\mathbf{d}^{\alpha+\beta-1}(x,x').$$

Before proving the above result, we point out the control on the Hölder modulus of $\tilde{u}^{\tau,\xi}$ with respect to the degenerate variables ($i>1$):

Lemma 4.6 (Controls on Hölder Moduli: Degenerate)**.** *Let i be in $[\![2,n]\!]$ and x,x' in $\mathbb{R}^{nd}$ such that $x_j = x'_j$ for any $j\neq i$. Under [$\boldsymbol{A}$], there exists a constant $C := C(i)$ such that for any t in $[0,T]$ and any freezing couple (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$, it holds that*

$$\Big|\tilde{u}^{\tau,\xi}(t,x) - \tilde{u}^{\tau,\xi}(t,x')| \le Cc_0^{\frac{\beta-\gamma_i}{\alpha}}\Big(\|u_T\|_{C^{\alpha+\beta}_{b,d}} + \|f\|_{L^\infty(C^\beta_{b,d})}\Big)\mathbf{d}^{\alpha+\beta}(x,x').$$

Proof of Lemma 4.5 *Controls on frozen semigroup.* Let us consider firstly the off-diagonal regime, i.e. the case $T-t \le c_0\mathbf{d}^\alpha(x,x')$. Using the scaling lemma 4.1, it holds that

$$\begin{aligned} D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x) &= \int_{\mathbb{R}^{nd}} D_{x_1}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy \\ &= \sum_{j=1}^n C_j(T-t)^{j-1}\int_{\mathbb{R}^{nd}} D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy. \end{aligned}$$

It then follows that

$$\begin{aligned} &\Big|D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x) - D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x')\Big| \\ &\qquad \le C\sum_{j=1}^n (T-t)^{j-1}\Big|\int_{\mathbb{R}^{nd}}\Big[D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y) - D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x',y)\Big]u_T(y)\,dy\Big| \\ &\qquad =: C\sum_{j=1}^n (T-t)^{j-1}I^{od}_j. \end{aligned} \tag{4.17}$$

We are going to treat separately the cases $j=1$ and $j>1$ for the *off-diagonal* contributions $\{I^{od}_j\}_{j\in[\![1,n]\!]}$. Indeed, the function u_T is differentiable only with respect to the first component y_1. In this first case, we can apply integration by parts formula to move the derivative on u_T, so that

$$I^{od}_1 = \Big|\int_{\mathbb{R}^{nd}}\Big[\tilde{p}^{\tau,\xi}(t,T,x,y) - \tilde{p}^{\tau,\xi}(t,T,x',y)\Big]D_{y_1}u_T(y)\,dy\Big|.$$

Noticing that $D_{y_1}u_T$ is in $C^{\alpha+\beta-1}_{b,d}(\mathbb{R}^{nd})$ thanks to the reverse Taylor expansion (Lemma 8.4), the last expression can be then rewritten as

$$\begin{aligned} I^{od}_1 &\le \Big|\int_{\mathbb{R}^{nd}}\tilde{p}^{\tau,\xi}(t,T,x,y)\Big[D_{y_1}u_T(y) \pm D_{y_1}u_T(\tilde{m}^{\tau,\xi}_{T,t}(x))\Big] \\ &\qquad\qquad - \tilde{p}^{\tau,\xi}(t,T,x',y)\Big[D_{y_1}u_T(y) \pm D_{y_1}u_T(\tilde{m}^{\tau,\xi}_{T,t}(x'))\Big]\,dy\Big| \\ &\le C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big\{\int_{\mathbb{R}^{nd}}\Big[\tilde{p}^{\tau,\xi}(t,T,x,y)\mathbf{d}^{\alpha+\beta-1}(y,\tilde{m}^{\tau,\xi}_{T,t}(x))\Big]\,dy \\ &\qquad\qquad + \int_{\mathbb{R}^{nd}}\Big[\tilde{p}^{\tau,\xi}(t,T,x',y)\mathbf{d}^{\alpha+\beta-1}(y,\tilde{m}^{\tau,\xi}_{T,t}(x'))\Big]\,dy \\ &\qquad\qquad\qquad + \mathbf{d}^{\alpha+\beta-1}(\tilde{m}^{\tau,\xi}_{T,t}(x),\tilde{m}^{\tau,\xi}_{T,t}(x'))\Big\}. \end{aligned} \tag{4.18}$$

Now, we use the smoothing effect of $\tilde{p}^{\tau,\xi}$ (Equation (3.6)) to control the two integrals in the last expression, so that

$$I^{od}_1 \le C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big[(T-t)^{\frac{\alpha+\beta-1}{\alpha}} + \mathbf{d}^{\alpha+\beta-1}(\tilde{m}^{\tau,\xi}_{T,t}(x),\tilde{m}^{\tau,\xi}_{T,t}(x'))\Big].$$

We can then conclude the case $j=1$ recalling that the mapping $x\to\tilde{m}^{\tau,\xi}_{T,t}(x)$ is affine (see Equation (3.4) for definition of $\tilde{m}^{\tau,\xi}_{T,t}(x)$) in order to show that

$$I^{od}_1 \le C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big[(T-t)^{\frac{\alpha+\beta-1}{\alpha}} + \mathbf{d}^{\alpha+\beta-1}(x,x')\Big]. \tag{4.19}$$

Let us consider now the case $j > 1$. Using Duality (4.11) in Besov spaces and Identification (4.10), we can write from Equation (4.17) that

$$
\begin{aligned}
I_j^{od} &\leq C\|u_T\|_{C^{\alpha+\beta}_{b,d}} \int_{\mathbb{R}^{(n-1)d}} \|D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y_{\smallsetminus j},\cdot) - D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x',y_{\smallsetminus j},\cdot)\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}\, dy_{\smallsetminus j} \\
&\leq C\|u_T\|_{C^{\alpha+\beta}_{b,d}} \int_{\mathbb{R}^{(n-1)d}} \|D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y_{\smallsetminus j},\cdot)\|_{B^{-(\alpha_j+\beta_j)}_{1,1}} \\
&\qquad\qquad\qquad\qquad + \|D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x',y_{\smallsetminus j},\cdot)\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}\, dy_{\smallsetminus j} \\
&\leq C\|u_T\|_{C^{\alpha+\beta}_{b,d}}(T-t)^{\frac{\alpha+\beta}{\alpha}-\frac{1}{\alpha_j}},
\end{aligned}
\tag{4.20}
$$

where in the last inequality we applied the first Besov control (Lemma 4.3). Going back at Equations (4.19)-(4.20), we finally conclude that

$$
\begin{aligned}
&\Big|D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x) - D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x')\Big| \\
&\qquad \leq C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big[(T-t)^{\frac{\alpha+\beta-1}{\alpha}} + \mathbf{d}^{\alpha+\beta-1}(x,x') + \sum_{j=2}^{n}(T-t)^{j-1}(T-t)^{\frac{\alpha+\beta}{\alpha}-\frac{1}{\alpha_j}}\Big] \\
&\qquad \leq C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big[(T-t)^{\frac{\alpha+\beta-1}{\alpha}} + \mathbf{d}^{\alpha+\beta-1}(x,x')\Big] \\
&\qquad \leq C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\mathbf{d}^{\alpha+\beta-1}(x,x'),
\end{aligned}
\tag{4.21}
$$

where in the last passage we used that $T-t \leq c_0\mathbf{d}^\alpha(x,x')$ for some $c_0 \leq 1$.
We focus now on the diagonal regime, i.e. when $T-t > c_0\mathbf{d}^\alpha(x,x')$. Remembering that we assumed that $x_j = x'_j$ for any j in $[\![2,n]\!]$, we start using a Taylor expansion on the density $\tilde{p}^{\tau,\xi}$ with respect to the first, non-degenerate variable x_1. Namely,

$$
\begin{aligned}
D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x) - D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x') &= \int_{\mathbb{R}^{nd}}\Big[D_{x_1}\tilde{p}^{\tau,\xi}(t,T,x,y) - D_{x_1}\tilde{p}^{\tau,\xi}(t,T,x',y)\Big]u_T(y)\, dy \\
&= \int_{\mathbb{R}^{nd}}\int_0^1 D^2_{x_1}\tilde{p}^{\tau,\xi}\Big(t,T,x'+\lambda(x-x'),y\Big)(x-x')_1u_T(y)\, d\lambda dy.
\end{aligned}
$$

Moreover, from the Scaling Lemma 4.1, it holds that

$$
D^2_{x_1}\tilde{p}^{\tau,\xi}\Big(t,T,x'+\lambda(x-x'),y\Big) = \sum_{j=1}^{n} C_j(T-t)^{j-1}D_{y_j}D_{x_1}\tilde{p}^{\tau,\xi}\Big(t,T,x'+\lambda(x-x'),y\Big)
$$

and we can use it to write

$$
\begin{aligned}
&\Big|D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x) - D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x')\Big| \\
&\quad \leq C|(x-x')_1|\sum_{j=1}^{n}(T-t)^{j-1}\Big|\int_0^1\int_{\mathbb{R}^{nd}} D_{y_j}D_{x_1}\tilde{p}^{\tau,\xi}\Big(t,T,x'+\lambda(x-x'),y\Big)u_T(y)\, dyd\lambda\Big| \\
&\quad =: C|(x-x')_1|\sum_{j=1}^{n}(T-t)^{j-1}I^d_j.
\end{aligned}
\tag{4.22}
$$

Similarly to the off-diagonal regime, we are going to treat separately the cases $j=1$ and $j>1$ for the *diagonal* contributions $\{I^d_j\}_{j\in[\![1,n]\!]}$. In the first case, we can apply integration by parts formula to show that

$$
I^d_1 = \Big|\int_0^1\int_{\mathbb{R}^{nd}} D_{x_1}\tilde{p}^{\tau,\xi}\Big(t,T,x'+\lambda(x-x'),y\Big)\otimes D_{y_1}u_T(y)\, dyd\lambda\Big|.
$$

A cancellation argument with respect to $D_{x_1}\tilde{p}^{\tau,\xi}$ then leads to

$$\begin{aligned} I^d_1 \,&=\, \Big|\int_0^1\int_{\mathbb{R}^{nd}} D_{x_1}\tilde{p}^{\tau,\xi}(t,T,x'+\lambda(x-x'),y) \\ &\qquad\qquad \otimes\Big[D_{y_1}u_T(y)-D_{y_1}u_T(\tilde{m}^{\tau,\xi}_{T,t}(x'+\lambda(x-x')))\Big]\,dyd\lambda\Big| \\ &\le\, C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\int_0^1\int_{\mathbb{R}^{nd}} |D_{x_1}\tilde{p}^{\tau,\xi}(t,T,x'+\lambda(x-x'),y)| \\ &\qquad\qquad \times\mathbf{d}^{\alpha+\beta-1}\Big(y,\tilde{m}^{\tau,\xi}_{T,t}(x'+\lambda(x-x'))\Big)\,dyd\lambda. \end{aligned}$$

Since $\alpha+\beta-1<\alpha$ by hypothesis [**P**], we can conclude using the smoothing effect of $\tilde{p}^{\tau,\xi}$ (Lemma 3.2) to show that

$$I^d_1 \,\le\, C\|u_T\|_{C^{\alpha+\beta}_{b,d}}(T-t)^{\frac{\alpha+\beta-2}{\alpha}}. \tag{4.23}$$

For the case $j>1$, we use instead the duality in Besov spaces (Equation (4.11)) and Identification (4.10) to write

$$\begin{aligned} I^d_j \,&\le\, \int_0^1\int_{\mathbb{R}^{(n-1)d}} \|D_{y_j}D_{x_1}\tilde{p}^{\xi}(t,T,x'+\lambda(x-x'),y_{\smallsetminus j},\cdot)\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}\,dy_{\smallsetminus j}d\lambda \\ &\le\, C\|u_T\|_{C^{\alpha+\beta}_{b,d}}(T-t)^{\frac{\alpha+\beta}{\alpha}-\frac{1}{\alpha_j}-\frac{1}{\alpha}}, \end{aligned} \tag{4.24}$$

where in the last passage we applied the first Besov control (Lemma 4.3). From Equations (4.22), (4.23) and (4.24), it is possible to conclude that

$$\begin{aligned} \Big|D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x)-D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x')\Big| \,&\le\, C\|u_T\|_{C^{\alpha+\beta}_{b,d}}|(x-x')_1|\sum_{j=1}^n(T-t)^{j-1}(T-t)^{\frac{\alpha+\beta-1}{\alpha}-\frac{1}{\alpha_j}} \\ &\le\, C\|u_T\|_{C^{\alpha+\beta}_{b,d}}|(x-x')_1|(T-t)^{\frac{\alpha+\beta-2}{\alpha}} \\ &\le\, Cc_0^{\frac{\alpha+\beta-2}{\alpha}}\|u_T\|_{C^{\alpha+\beta}_{b,d}}\mathbf{d}^{\alpha+\beta-1}(x,x'), \end{aligned}$$

where in the last passage we used that $|(x-x')_1|=\mathbf{d}(x,x')$ and since $\frac{\alpha+\beta-2}{\alpha}<0$, that

$$|(x-x')_1|(T-t)^{\frac{\alpha+\beta-2}{\alpha}} \,\le\, c_0^{\frac{\alpha+\beta-2}{\alpha}}\mathbf{d}^{\alpha+\beta-1}(x,x').$$

Remembering that c_0 is considered fixed and bigger then zero, the searched control follows immediately.

Controls on frozen Green kernel. We recall that, in order to preserve the previous terminology of off-diagonal/diagonal regime for the frozen semigroup, we have introduced the transition time t_0, defined in (4.16). Then, while integrating in s from t to T, we will say that the local off-diagonal regime holds for $\tilde{G}^{\tau,\xi}$ if s is in $[t,t_0]$ and that the local diagonal regime holds if s is in $[t_0,T]$. With the notations of (4.2) in mind, it seems quite natural now to decompose the derivative of the frozen Green kernel with respect to t_0, i.e.

$$D_{x_1}\tilde{G}^{\tau,\xi}_{T,t}f(t,x) \,=\, D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x)+D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x).$$

We remark however that the globally off-diagonal regime is considered in the above decomposition, too. Indeed, when $T - t \le c_0 \mathbf{d}^\alpha(x,x')$, t_0 coincides with T and the second term on the right-hand side vanishes.
We start considering the off-diagonal regime represented by

$$\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x')\Big|.$$

It holds that

$$\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x')\Big| \le \int_t^{t_0}\Big[\Big|D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}f(s,x)\Big| + \Big|D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}f(s,x')\Big|\Big]\,ds.$$

We then use the control on the frozen semigroup (Equation (3.7)) to find that

$$\begin{aligned}\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x')\Big| &\le C\|f\|_{L^\infty(C^\beta_{b,d})}\int_t^{t_0}(s-t)^{\frac{\beta-1}{\alpha}}\,ds \\ &\le C\|f\|_{L^\infty(C^\beta_{b,d})}(t_0-t)^{\frac{\beta+\alpha-1}{\alpha}}.\end{aligned}$$

Our choice of t_0 (cf. Equation (4.16)) allows then to conclude that

$$\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x')\Big| \le C\|f\|_{L^\infty(C^\beta_{b,d})}\mathbf{d}^{\alpha+\beta-1}(x,x'),$$

remembering that $c_0 \le 1$ by assumption.
We can focus now on the diagonal regime represented by

$$\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x')\Big|.$$

We start applying a Taylor expansion on the derivative of the semigroup $\tilde{P}^{\tau,\xi}f(t,x)$ so that

$$\begin{aligned}\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x')\Big| &= \Big|\int_{t_0}^T\Big[D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}f(s,x) - D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}f(s,x')\Big]\,ds\Big| \\ &= \Big|\int_{t_0}^T\int_0^1 D^2_{x_1}\tilde{P}^{\tau,\xi}_{s,t}f(s,x+\lambda(x'-x))(x'-x)_1\,d\lambda ds\Big|.\end{aligned}$$

Then, Fubini theorem and the control on the frozen semigroup (Equation (3.7)) allow us to write that

$$\begin{aligned}\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x')\Big| &\le C\|f\|_{L^\infty(C^\beta_{b,d})}|(x-x')_1|\int_{t_0}^T(s-t)^{\frac{\beta-2}{\alpha}}\,ds \\ &\le C\|f\|_{L^\infty(C^\beta_{b,d})}|(x-x')_1|\Big[\frac{\alpha}{\alpha+\beta-2}(s-t)^{\frac{\alpha+\beta-2}{\alpha}}\Big]_{t_0}^T.\end{aligned}$$

Since by hypothesis [**P**], it holds that $\alpha/(\alpha+\beta-2) < 0$, it follows that

$$\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x')\Big| \le C\|f\|_{L^\infty(C^\beta_{b,d})}|(x-x')_1|(t_0-t)^{\frac{\alpha+\beta-2}{\alpha}}.$$

Using that $|(x-x')_1| = \mathbf{d}(x,x')$ and remembering our choice of t_0 in (4.16), we can then conclude that

$$\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x')\Big| \le Cc_0^{\frac{\alpha+\beta-2}{\alpha}}\|f\|_{L^\infty(C^\beta_{b,d})}\mathbf{d}^{\alpha+\beta-1}(x,x').$$

Proof of Lemma 4.6 *Controls on frozen semigroup.* Using the change of variables $z = \tilde{m}^{\tau,\xi}_{T,t}(x) - y$, we can rewrite $\tilde{P}^{\tau,\xi}_{T,t}u_T(x)$ as

$$\begin{aligned}
\tilde{P}^{\tau,\xi}_{T,t}u_T(x) &= \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy \\
&= \int_{\mathbb{R}^{nd}} \frac{1}{\det\Big(\mathbb{M}_{T-t}\Big)} p_S(T-t,\mathbb{M}^{-1}_{T-t}\Big(\tilde{m}^{\tau,\xi}_{T,t}(x)-y\Big)u_T(y)\,dy \\
&= \int_{\mathbb{R}^{nd}} \frac{1}{\det\Big(\mathbb{M}_{T-t}\Big)} p_S(T-t,\mathbb{M}^{-1}_{T-t}z\Big)u_T(\tilde{m}^{\tau,\xi}_{T,t}(x)-z)\,dz.
\end{aligned}$$

It then follows that

$$\begin{aligned}
&\Big|\tilde{P}^{\tau,\xi}_{T,t}u_T(x) - \tilde{P}^{\tau,\xi}_{T,t}u_T(x')\Big| \\
&\quad= \Big|\int_{\mathbb{R}^{nd}} \frac{1}{\det\Big(\mathbb{M}_{T-t}\Big)} p_S\Big(T-t,\mathbb{M}^{-1}_{T-t}z\Big)\Big[u_T(\tilde{m}^{\tau,\xi}_{T,t}(x)-z) - u_T(\tilde{m}^{\tau,\xi}_{T,t}(x')-z)\Big]\,dz\Big|.
\end{aligned}$$

We now observe that the function $x \to \tilde{m}^{\tau,\xi}_{T,t}(x)$ is affine (cf. Equation (3.4)) and thus, that

$$\Big(\tilde{m}^{\tau,\xi}_{T,t}(x) - z\Big)_1 = \Big(\tilde{m}^{\tau,\xi}_{T,t}(x') - z\Big)_1,$$

since $x_1 = x'_1$. It then holds that

$$\begin{aligned}
\Big|u_T(\tilde{m}^{\tau,\xi}_{T,t}(x)-z) - u_T(\tilde{m}^{\tau,\xi}_{T,t}(x')-z)\Big| &\le C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\mathbf{d}^{\alpha+\beta}\Big(\tilde{m}^{\tau,\xi}_{T,t}(x),\tilde{m}^{\tau,\xi}_{T,t}(x')\Big) \\
&\le C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\mathbf{d}^{\alpha+\beta}(x,x').
\end{aligned}$$

Hence, we can conclude using it to write

$$\begin{aligned}
\Big|\tilde{P}^{\tau,\xi}_{T,t}u_T(x) - \tilde{P}^{\tau,\xi}_{T,t}u_T(x')\Big| &\le C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\,\mathbf{d}^{\alpha+\beta}(x,x')\int_{\mathbb{R}^{nd}} \frac{p_S\Big(T-t,\mathbb{M}^{-1}_{T-t}z\Big)}{\det\mathbb{M}_{T-t}}\,dz \\
&\le C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\,\mathbf{d}^{\alpha+\beta}(x,x').
\end{aligned}$$

Controls on frozen Green kernel. We will assume the same notations appeared in the previous lemma for the frozen Green kernel. In particular, we decompose it as

$$\tilde{G}^{\tau,\xi}_{T,t}f(t,x) = \tilde{G}^{\tau,\xi}_{t_0,t}f(t,x) + \tilde{G}^{\tau,\xi}_{T,t_0}f(t,x)$$

with t_0 defined in Equation (4.16).
We start rewriting the off-diagonal regime contribution as

$$\begin{aligned}
\Big|\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x) - \tilde{G}^{\tau,\xi}_{t_0,t}f(t,x')\Big| &= \Big|\int_t^{t_0}\int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t,s,x,y)\Big[f(s,y) \pm f(s,\tilde{m}^{\tau,\xi}_{s,t}(x))\Big] \\
&\qquad - \tilde{p}^{\tau,\xi}(t,s,x',y)\Big[f(s,y) \pm f(s,\tilde{m}^{\tau,\xi}_{s,t}(x'))\Big]\,dyds\Big| \\
&\le \Big|\int_t^{t_0}\int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t,s,x,y)\Big[f(s,y) - f(s,\tilde{m}^{\tau,\xi}_{s,t}(x))\Big] \\
&\qquad - \tilde{p}^{\tau,\xi}(t,s,x',y)\Big[f(s,y) - f(s,\tilde{m}^{\tau,\xi}_{s,t}(x'))\Big]\,dyds\Big| \\
&\qquad + \Big|\int_t^{t_0} f(s,\tilde{m}^{\tau,\xi}_{s,t}(x)) - f(s,\tilde{m}^{\tau,\xi}_{s,t}(x'))\,ds\Big|.
\end{aligned}$$

We can then use the smoothing effect for $\tilde{p}^{\tau,\xi}$ (Equation (3.6)) to show that

$$\Big|\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x)-\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x')\Big| \\ \le C\|f\|_{L^\infty(C^\beta_{b,d})}\int_t^{t_0}\Big[(s-t)^{\beta/\alpha}+\mathbf{d}^\beta(\tilde{m}^{\tau,\xi}_{s,t}(x),\tilde{m}^{\tau,\xi}_{s,t}(x'))\Big]\,ds. \quad (4.25)$$

Recalling from Equation (3.4) that $x\to\tilde{m}^{\tau,\xi}_{s,t}(x)$ is affine, it follows that

$$\begin{aligned}\Big|\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x)-\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x')\Big| &\le C\|f\|_{L^\infty(C^\beta_{b,d})}\int_t^{t_0}\Big[(s-t)^{\beta/\alpha}+\mathbf{d}^\beta(x,x')\Big]\,ds\\ &\le C\|f\|_{L^\infty(C^\beta_{b,d})}\Big[(t_0-t)\mathbf{d}^\beta(x,x')+(t_0-t)^{\frac{\beta+\alpha}{\alpha}}\Big].\end{aligned}$$

Using that $t_0-t\le c_0\mathbf{d}^\alpha(x,x')$ for some $c_0\le 1$, we can finally conclude that

$$\Big|\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x)-\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x')\Big| \le C\|f\|_{L^\infty(C^\beta_{b,d})}\mathbf{d}^{\alpha+\beta}(x,x').$$

Now, we can focus our analysis to the diagonal regime contribution:

$$\Big|\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x)-\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x')\Big|.$$

We start applying a Taylor expansion on the frozen semigroup $\tilde{P}^{\tau,\xi}_{s,t}f$ with respect to the i-th variable x_i, which is, by hypothesis, the only one for which the entries of x and x' differ. Namely,

$$\begin{aligned}\Big|\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x)-\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x')\Big| &= \Big|\int_{t_0}^T \tilde{P}^{\tau,\xi}_{s,t}f(s,x)-\tilde{P}^{\tau,\xi}_{s,t}f(s,x')\,ds\Big|\\ &= \Big|\int_{t_0}^T\int_0^1 D_{x_i}\tilde{P}^{\tau,\xi}_{s,t}f(s,x+\lambda(x'-x))\cdot(x'-x)_i\,d\lambda ds\Big|.\end{aligned}$$

The control on the frozen semigroup (Equation (3.7)) then implies that

$$\Big|\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x)-\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x')\Big| \le C\|f\|_{L^\infty(C^\beta_{b,d})}|(x-x')_i|\int_{t_0}^T(s-t)^{\frac{\beta}{\alpha}-\frac{1}{\alpha_i}}\,ds. \quad (4.26)$$

Noticing from assumption [**P**] that $\beta+\alpha-1-\alpha(i-1)<0$ for $i\ge 2$, it holds that

$$\begin{aligned}\int_{t_0}^T(s-t)^{\frac{\beta}{\alpha}-\frac{1}{\alpha_i}}\,ds &= \int_{t_0}^T(s-t)^{\frac{\beta-[1+\alpha(i-1)]}{\alpha}}\,ds\\ &\le C\Big[-(s-t)^{\frac{\beta+\alpha-1-\alpha(i-1)}{\alpha}}\Big]_{t_0}^T\\ &\le C(t_0-t)^{\frac{\beta-1-\alpha(i-2)}{\alpha}}.\end{aligned}$$

Using that $|(x-x')_i|=\mathbf{d}^{1+\alpha(i-1)}(x,x')$ and our choice of t_0 (cf. Equation (4.16)), we can then conclude from Equation (4.26) that

$$\Big|\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x)-\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x')\Big| \\ \le Cc_0^{\frac{\beta-1-\alpha(i-2)}{\alpha}}\|f\|_{L^\infty(C^\beta_{b,d})}\mathbf{d}^{\alpha+\beta}(x,x') \le Cc_0^{\frac{\beta-\gamma_i}{\alpha}}\|f\|_{L^\infty(C^\beta_{b,d})}\mathbf{d}^{\alpha+\beta}(x,x'),$$

remembering the definition of γ_i given in (2.21).

5 A priori estimates

Since the aim of this section is to prove Proposition 3.6, we will assume tacitly from this point further that assumption [**A'**] holds. Moreover, we recall here that throughout this section, we are considering the regularized framework of Section 3.2.

WARNING: For notational simplicity, we drop here the subscripts and the superscripts in m associated with the regularization. For any fixed (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$, we rewrite, with some abuse in notations, the Duhamel Expansion (3.16) as:

$$u(t,x) \,=\, \tilde{u}^{\tau,\xi}(t,x)+\int_t^T \tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)\,ds, \tag{5.27}$$

where $\tilde{u}^{\tau,\xi}$ is defined through the Duhamel Representation (3.10) and

$$R^{\tau,\xi}(t,x) \,=\, \Big\langle F(t,x)-F(t,\theta_{t,\tau}(\xi)),D_xu(t,x)\Big\rangle, \quad (t,x)\,\in\,(0,T)\times\mathbb{R}^{nd}.$$

It is however important to keep in mind that f, u_T, F are now smooth and bounded functions so that all the terms above are clearly defined. We recall however that we aim at obtaining controls in the $L^\infty(C^{\alpha+\beta}_{b,d})$-norm, uniformly with respect to the regularization parameter.
From the expansion above, we know moreover that for any (t,ξ) in $[0,T]\times\mathbb{R}^{nd}$, it holds that

$$D_{x_1}u(t,x) \,=\, D_{x_1}\tilde{u}^{\tau,\xi}(t,x)+\int_t^T D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)\,ds. \tag{5.28}$$

As seen in the previous section, these decompositions will allow us to obtain a control for u in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ analyzing separately the contributions from the Duhamel representation $\tilde{u}^{\tau,\xi}$ and those from the expansion error $R^{\tau,\xi}(t,x)$, for suitable choices of freezing parameters (τ,ξ).

5.1 Second Besov control

This sub-section focuses on the contribution associated with the remainder term $R^{m,\tau,\xi}$ appearing in the Duhamel-type Expansion (5.27). We recall that we aim at controlling it with the $L^\infty(C^{\alpha+\beta}_{b,d})$-norm of the coefficients, uniformly in the regularization parameter. Let us start decomposing it through

$$\Big|\int_t^T \tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)\,ds\Big| \,=\, \Big|\sum_{j=1}^n\int_t^T\int_{\mathbb{R}^{nd}}\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\cdot D_{y_j}u(s,y)\,dyds\Big|,$$

where we have denoted for simplicity

$$\Delta^{\tau,\xi}F_j(s,y) \,:=\, F_j(s,y)-F_j(s,\theta_{s,\tau}(\xi)), \quad j\in[\![1,n]\!]. \tag{5.29}$$

We then notice that the non-degenerate contribution in the sum (corresponding to the index $j=1$) can be treated easily, remembering that u is differentiable with respect to the first component with a bounded derivative. Indeed, using the smoothing effect for the frozen density $\tilde{p}^{\tau,\xi}$ (Equation (3.6)), it holds that

$$\begin{aligned}\Big|\int_t^T \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_1(s,y)\cdot D_{y_1}u(s,y)\,dyds\Big| \\ &\le C\|D_{y_1}u(s,y)\|_{l^\infty(L^\infty)}\|F\|_H\int_t^T\int_{\mathbb{R}^{nd}}\tilde{p}^{\tau,\xi}(t,s,x,y)\mathbf{d}^{\alpha+\beta}\Big(y,\theta_{s,\tau}(\xi)\Big)\,dyds\\ &\le C\|D_{y_1}u(s,y)\|_{l^\infty(L^\infty)}\|F\|_H\int_t^T (s-t)^{\frac{\beta}{\alpha}}\,ds\\ &\le C\|D_{y_1}u(s,y)\|_{l^\infty(L^\infty)}\|F\|_H(T-t)^{\frac{\alpha+\beta}{\alpha}}.\end{aligned}$$

In order to deal with the degenerate indexes, we will use, similarly to the previous subsection, a reasoning in Besov spaces. Since u is not differentiable with respect to y_j if $j>1$, we move the derivative to the other terms using integration by parts formula:

$$\Big|\int_t^T\int_{\mathbb{R}^{nd}} D_{y_j}\cdot\Big\{\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\Big\}u(s,y)\,dyds\Big|.$$

In order to rely again on the duality in Besov spaces (Equation (4.11)), we rewrite the above expression as

$$\begin{aligned}&\Big|\int_t^T\int_{\mathbb{R}^{nd}} D_{y_j}\cdot\Big\{\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\Big\}u(s,y)\,dyds\Big|\\ &\qquad\le \int_t^T\int_{\mathbb{R}^{(n-1)d}}\Big\|D_{y_j}\cdot\Big\{\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\cdot)\Delta^{\tau,\xi}F_j(s,y_{\smallsetminus j},\cdot)\Big\}\Big\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}\\ &\qquad\qquad\times\|u(s,y_{\smallsetminus j},\cdot)\|_{B^{\alpha_j+\beta_j}_{\infty,\infty}}\,dy_{\smallsetminus j}ds.\end{aligned}$$

Remembering the identification in Equation (4.10), it holds now that

$$\begin{aligned}&\Big|\int_t^T\int_{\mathbb{R}^{nd}} D_{y_j}\cdot\Big\{\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\Big\}u(s,y)\,dyds\Big|\\ &\le \|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\int_t^T\int_{\mathbb{R}^{(n-1)d}}\Big\|D_{y_j}\cdot\Big\{\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\cdot)\Delta^{\tau,\xi}F_j(s,y_{\smallsetminus j},\cdot)\Big\}\Big\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}\,dy_{\smallsetminus j}ds.\end{aligned}$$

It then remains to control the integral of the Besov norm above. To do that, we will need a refinement of the smoothing effect (Equation (3.6)) that involves only partial differences of variables. For a fixed i in $[\![2,n]\!]$, we start denoting by $d_{i:n}(\cdot,\cdot)$ the part of the anisotropic distance considering only the last $n-(i-1)$ variables. Namely,

$$d_{i:n}(x,x') := \sum_{j=i}^n |(x-x')_j|^{\frac{1}{1+\alpha(j-1)}}.$$

Lemma 5.1 (Partial Smoothing Effect)**.** *Let i be in $[\![2,n]\!]$, γ in $(0,1\wedge\alpha(1+\alpha(i-1)))$ and ϑ, ϱ two n-multi-indexes such that $|\vartheta+\varrho|\le 3$. Then, there exists a constant $C:=C(\vartheta,\varrho,\gamma)$ such that for any $t<s$ in $[0,T]$, any x in $\mathbb{R}^{nd}$,*

$$\int_{\mathbb{R}^{nd}}|D^\varrho_yD^\vartheta_x\tilde{p}^{\tau,\xi}(t,s,x,y)|\mathbf{d}^\gamma_{i:n}\Big(y,\theta_{s,\tau}(\xi)\Big)\,dy \le C(s-t)^{\frac{\gamma}{\alpha}-\sum_{i=k}^n\frac{\vartheta_k+\varrho_k}{\alpha_k}}, \tag{5.30}$$

taking $(\tau,\xi)=(t,x)$.

The above assumption on γ should not appear too strange. Indeed, in the partial distance $\mathbf{d}^{\gamma}_{i:n}(x,x')$, the stronger term to be integrated is at level i with intensity of order $\gamma/(1+\alpha(i-1))$. Since by the smoothing effect (Equation (3.6)) of the frozen density we know we can integrate against $\tilde{p}^{\tau,\xi}$ contributions of order up to α, the condition $\gamma<\alpha(1+\alpha(i-1))$ appears naturally.
A proof of this result can be obtained mimicking with slightly modifications the proof in Lemma 3.2.

As done above for the first Besov control (Lemma 4.3), we will however state the result considering a possibly additional derivative with respect to x_1. Namely, we would like to control the following:

$$D_{y_j}\cdot\Big\{\mathbf{d}^{\vartheta}_x\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\cdot)\otimes\Big[F_j(s,y_{\smallsetminus j},\cdot)-F_j(s,\theta_{s,\tau}(\xi))\Big]\Big\}$$

where we have denoted as in (4.7), $F_j(s,y_{\smallsetminus j},\cdot):=F_j(s,y_1,\dots,y_{j-1},\cdot,y_{j+1},\dots,y_n)$ and, with a slightly abuse of notation, by $D_{y_j}\cdot$ an extended form of the divergence over the j-th variable. In other words, this "enhanced" divergence form decreases by one the order of the input tensor.

Lemma 5.2 (Second Besov Control). *Let j be in $[\![2,n]\!]$ and ϑ a multi-index in $\mathbb{N}^n$ such that $|\vartheta|\le 2$. Under (**A**'), there exists a constant $C:=C(j,\vartheta)$ such that for any x in $\mathbb{R}^{nd}$ and any $t<s$ in $[0,T]$*

$$\int_{\mathbb{R}^{(n-1)d}}\Big\|D_{y_j}\cdot\Big\{\mathbf{d}^{\vartheta}_x\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\cdot)\otimes\Big[F_j(s,y_{\smallsetminus j},\cdot)-F_j(s,\theta_{s,\tau}(\xi))\Big]\Big\}\Big\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}dy_{\smallsetminus j}$$
$$\le\ C\|F\|_H(s-t)^{\frac{\beta}{\alpha}-\sum_{k=1}^n\frac{\vartheta_k}{\alpha_k}},$$

taking $(\tau,\xi)=(t,x)$.

Proof. To control the Besov norm in $B^{-(\alpha_j+\beta_j)}_{1,1}(\mathbb{R}^d)$, we are going to use the Thermic Characterization (4.8) with $\tilde{\gamma}=-(\alpha_j+\beta_j)$. Since the first term can be controlled as in the first Besov control (Lemma 4.3), we will focus on the second one, i.e.

$$\int_0^1 v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\partial_v p_h(v,z-y_j)D_{y_j}\cdot\Big\{D^{\vartheta}_x\tilde{p}^{\tau,\xi}(t,s,x,y)\otimes\Delta^{\tau,\xi}F_j(s,y)\Big\}dy_j\Big|dzdv,$$

where we exploited the same notations for $\Delta^{\tau,\xi}F_j$ given in (5.29).
We start applying integration by parts formula noticing that

$$D_{y_j}p_h(v,z-y_j)\ =\ -D_zp_h(v,z-y_j),$$

to write that

$$\int_0^1 v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}D_z\partial_v p_h(v,z-y_j)\cdot\Big\{D^{\vartheta}_x\tilde{p}^{\tau,\xi}(t,s,x,y)\otimes\Delta^{\tau,\xi}F_j(s,y)\Big\}dy_j\Big|dzdv.$$

Fixed a constant $\delta_j \geq 1$ to be chosen later, we then split the above integral with respect to v into two components:

$$\begin{aligned}\int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j+\beta_j}{\alpha}} \int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d} D_z\partial_v p_h(v,z-y_j)\cdot\Big\{D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\Big\}\,dy_j\Big|\,dzdv\\ +\int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j+\beta_j}{\alpha}} \int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d} D_z\partial_v p_h(v,z-y_j)\cdot\Big\{D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\Big\}\,dy_j\Big|\,dzdv\\ =: \Big(I_1+I_2\Big)(y_{\smallsetminus j}).\end{aligned}$$

The second component I_2 has no time-singularity and it can be easily controlled using Fubini theorem in the following way

$$\begin{aligned}I_2(y_{\smallsetminus j}) \;\leq\; C\|F\|_H\int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|D_z\partial_v p_h(v,z-y_j)|\,dz\Big)|D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)|\\ \times\,\mathbf{d}_{j:n}^{1+\alpha(j-2)+\beta}(y,\theta_{s,\tau}(\xi))\,dy_jdv,\end{aligned}$$

remembering that $F_j(t,\cdot)$ depends only on the last $(n-j)$ variables and it is in $C_{b,d}^{1+\alpha(j-2)+\beta}(\mathbb{R}^{nd})$ by assumption [**R**]. We can then use the smoothing effect of the heat-kernel p_h (Equation (4.9)) and Fubini theorem again, in order to write that

$$\begin{aligned}I_2(y_{\smallsetminus j}) \;&\leq\; C\|F\|_H\int_{(s-t)^{\delta_j}}^1 \frac{v^{\frac{\alpha_j+\beta_j-1}{\alpha}}}{v}\int_{\mathbb{R}^d}|D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)|\mathbf{d}_{j:n}^{1+\alpha(j-2)+\beta}(y,\theta_{s,\tau}(\xi))\,dy_jdv\\ &\leq\; C\|F\|_H\Big(\int_{(s-t)^{\delta_j}}^1 \frac{v^{\frac{\alpha_j+\beta_j-1}{\alpha}}}{v}\,dv\Big)\Big(\int_{\mathbb{R}^d}|D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)|\mathbf{d}_{j:n}^{1+\alpha(j-2)+\beta}(y,\theta_{s,\tau}(\xi))\,dy_j\Big).\end{aligned}$$

Noticing from (2.27) that $\alpha_j+\beta_j-1<0$, it holds now that

$$I_2(y_{\smallsetminus j}) \;\leq\; C\|F\|_H(s-t)^{\delta_j\frac{\alpha_j+\beta_j-1}{\alpha}}\int_{\mathbb{R}^d}|D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)|\mathbf{d}_{j:n}^{1+\alpha(j-2)+\beta}(y,\theta_{s,\tau}(\xi))\,dy_j.$$

We can finally add the integral with respect to the other components $y_{\smallsetminus j}$. In order to use now the partial smoothing effect (Equation (5.30)), we take $\tau=t$ and $\xi=x$ and notice that by assumption [**P**],

$$1+\alpha(j-2)+\beta \;<\; 1+\alpha(j-1)-(1-\alpha)\Big(1+\alpha(j-1)\Big) \;=\; \alpha(1+\alpha(j-1)). \tag{5.31}$$

It then holds that

$$\begin{aligned}&\int_{\mathbb{R}^{(n-1)d}} I_2(y_{\smallsetminus j})\,dy_{\smallsetminus j}\\ &\qquad\leq\; C\|F\|_H(s-t)^{\delta_j\frac{\alpha_j+\beta_j-1}{\alpha}}\int_{\mathbb{R}^{nd}}|D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)|\mathbf{d}_{j:n}^{1+\alpha(j-2)+\beta}(y,\theta_{s,\tau}(\xi))\,dy\\ &\qquad\leq\; C\|F\|_H(s-t)^{\delta_j\frac{\alpha_j+\beta_j-1}{\alpha}+\frac{1+\alpha(j-2)+\beta}{\alpha}-\sum_{k=1}^n\frac{\vartheta_k}{\alpha_k}}.\end{aligned} \tag{5.32}$$

To control the other term I_1, we focus at first on the inner integral with respect to y_j:

$$\int_{\mathbb{R}^d} D_z\partial_v p_h(v,z-y_j)\cdot\Big\{D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)\otimes\Delta^{\tau,\xi}F_j(s,y)\Big\}\,dy_j.$$

We start using a cancellation argument with respect to the density p_h to write that

$$\int_{\mathbb{R}^d} D_z\partial_v p_h(v,z-y_j)\cdot\Big\{D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)\otimes\Big[F_j(s,y)-F_j(s,\theta_{s,\tau}(\xi))\Big\} \\ - D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},z)\otimes\Big[F_j(s,y_{\smallsetminus j},z)-F_j(s,\theta_{s,\tau}(\xi))\Big]\Big\}\,dy_j$$

We can then divide the above integral into two components J_1+J_2 given by in

$$\begin{aligned}J_1(v,y_{\smallsetminus j},z) &:= \int_{\mathbb{R}^d} D_z\partial_v p_h(v,z-y_j)\cdot\Big\{D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)\\ &\qquad\qquad\otimes\Big[F_j(s,y)-F_j(s,y_{\smallsetminus j},z)\Big]\Big\}\,dy_j;\\ J_2(v,y_{\smallsetminus j},z) &:= \int_{\mathbb{R}^d} D_z\partial_v p_h(v,z-y_j)\cdot\Big\{\Big[D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)-D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},z)\Big]\\ &\qquad\qquad\otimes\Big[F_j(s,y_{\smallsetminus j},z)-F_j(s,\theta_{s,\tau}(\xi))\Big]\Big\}\,dy_j.\end{aligned}$$

Remembering the notation for $F_j(s,y_{\smallsetminus j},z)$ in (4.7) and that F_j is $\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}$-Hölder continuous with respect to its j-th variable by assumption [**R**], the first component J_1 can be easily controlled using Fubini theorem by

$$\begin{aligned}&\int_{\mathbb{R}^d}\Big|J_1(v,y_{\smallsetminus j},z)\Big|\,dz\\ &\quad\le C\|F\|_H\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|z-y_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}}|D_z\partial_v p_h(v,z-y_j)|\,dz\Big)|D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)|\,dy_j\\ &\quad\le C\|F\|_H v^{\frac{1}{\alpha}\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}-\frac{1}{\alpha}-1}\int_{\mathbb{R}^d}|D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)|\,dy_j,\end{aligned}$$

where in the last passage we used the smoothing effect of the heat-kernel p_h (Equation (4.9)), noticing that

$$\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)} \;=\; 1+\frac{\beta-\alpha}{1+\alpha(j-1)} \;<\; 1+\alpha,$$

since $\alpha>\beta$ by assumption [**P**]. Using now the identity

$$\frac{\alpha_j+\beta_j}{\alpha}+\frac{1}{\alpha}\Big(\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}-1\Big) \;=\; \frac{2\beta_j}{\alpha}, \tag{5.33}$$

we add the integral with respect to v and write that

$$\begin{aligned}\int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d}\Big|J_1(v,y_{\smallsetminus j},z)\Big|\,dzdv \;&\le\; C\|F\|_H\int_0^{(s-t)^{\delta_j}}\frac{v^{\frac{2\beta_j}{\alpha}}}{v}\int_{\mathbb{R}^d}|D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)|\,dy_jdv\\ &\le\; C\|F\|_H(s-t)^{\delta_j\frac{2\beta_j}{\alpha}}\int_{\mathbb{R}^d}|D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)|\,dy_j.\end{aligned}$$

Adding the integral with respect to the other components $y_{\smallsetminus j}$, we can finally conclude that

$$\begin{aligned}&\int_{\mathbb{R}^{(n-1)d}}\int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d}\Big|J_1(v,y_{\smallsetminus j},z)\Big|\,dzdv\,dy_{\smallsetminus j}\\ &\qquad\le\; C\|F\|_H(s-t)^{\delta_j\frac{2\beta_j}{\alpha}}\int_{\mathbb{R}^{nd}}|D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)|\,dy \qquad (5.34)\\ &\qquad\le\; C\|F\|_H(s-t)^{\delta_j\frac{2\beta_j}{\alpha}-\sum_{k=1}^n\frac{\vartheta_k}{\alpha_k}}.\end{aligned}$$

To control the second component J_2, we start applying a Taylor expansion on $\tilde{p}^{\tau,\xi}$ with respect to y_j:

$$\begin{aligned} J_2(v, y_{\smallsetminus j}, z) \,=\, &\int_{\mathbb{R}^d} D_z\partial_v p_h(v, z-y_j)\cdot \Big\{\Delta^{\tau,\xi}F_j(s, y_{\smallsetminus j}, z) \\ &\qquad\otimes \int_0^1 D_{y_j}D^{\vartheta}_x\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j}, y_j+\lambda(z-y_j))\cdot(z)\Big\}\,d\lambda dy_j. \end{aligned} \tag{5.35}$$

We then notice that for any fixed λ in $[0,1]$, it holds that

$$\begin{aligned} |\Delta^{\tau,\xi}F_j(s,y_{\smallsetminus j}, z)| \,&=\, |F_j(s,y_{\smallsetminus j},z)-F_j(s,\theta_{s,\tau}(\xi))| \\ &\le\, C\|F\|_H\Big\{\Big|\Big(z-\theta_{s,\tau}(\xi)\Big)_j\Big|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} + \sum_{k=j+1}^n \Big|\Big(y-\theta_{s,\tau}(\xi)\Big)_k\Big|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(k-1)}}\Big\} \\ &\le\, C\|F\|_H\Big\{\Big|\lambda(y_j-z)\Big|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} + \Big|\Big(y_j+\lambda(z-y_j)-\theta_{s,\tau}(\xi)\Big)_j\Big|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} \\ &\qquad\qquad + \sum_{k=j+1}^n \Big|\Big(y-\theta_{s,\tau}(\xi)\Big)_k\Big|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(k-1)}}\Big\} \\ &\le\, C\|F\|_H\Big\{\Big|z-y_j\Big|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} + \mathbf{d}^{1+\alpha(j-2)+\beta}_{j:n}\Big(\big(y_{\smallsetminus j}, y_j+\lambda(z-y_j)\big)\Big), \theta_{s,\tau}(\xi)\Big)\Big\}, \end{aligned}$$

where as in (4.7), we denoted

$$(y_{\smallsetminus j}, y_j+\lambda(z-y_j)) := (y_1,\dots,y_{j-1},y_1,\dots,y_{j-1},y_j+\lambda(z-y_j),y_{j+1},\dots,y_n).$$

We can thus split J_2 as

$$|J_2(v,y_{\smallsetminus j},z)| \,\le\, C\|F\|_H\int_0^1\Big(J_{2,1}+J_{2,2}\Big)(v,y_{\smallsetminus j},z,\lambda)\,d\lambda, \tag{5.36}$$

where we denoted for simplicity:

$$\begin{aligned} J_{2,1}(v,y_{\smallsetminus j},z,\lambda) \,:=\, &\int_{\mathbb{R}^d}|z-y_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}+1}|D_z\partial_v p_h(v,z-y_j)| \\ &\qquad\times|D_{y_j}D^{\vartheta}_x\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},y_j+\lambda(z-y_j))|\,dy_j \\ J_{2,2}(v,y_{\smallsetminus j},z,\lambda) \,:=\, &\int_{\mathbb{R}^d}|z-y_j|\,|D_z\partial_v p_h(v,z-y_j)| \\ &\qquad\times|D_{y_j}D^{\vartheta}_x\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},y_j+\lambda(z-y_j))| \\ &\qquad\times\mathbf{d}^{1+\alpha(j-2)+\beta}_{j:n}((y_{\smallsetminus j},y_j+\lambda(z-y_j)),\theta_{s,\tau}(\xi))\,dy_j \end{aligned}$$

Adding now the integral with respect to z, the first term $J_{2,1}$ can be rewritten as

$$\begin{aligned} \int_{\mathbb{R}^d}J_{2,1}(v,y_{\smallsetminus j},z,\lambda)\,dz \,\le\, &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|z-y_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}+1}|D_z\partial_v p_h(v,z-y_j)| \\ &\qquad\times|D_{y_j}D^{\vartheta}_x\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},y_j+\lambda(z-y_j))|\,dy_jdz. \end{aligned}$$

Fubini Theorem and the change of variables $\tilde{z}=z-y_j$ and $\tilde{y}_j=y_j+\lambda\tilde{z}$ allow then to split the integrals in the following way:

$$\begin{aligned} &\int_{\mathbb{R}^d}J_{2,1}(v,y_{\smallsetminus j},z,\lambda)\,dz \\ &\quad\le\,\Big(\int_{\mathbb{R}^d}|\tilde{z}|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}+1}|D_{\tilde{z}}\partial_v p_h(v,\tilde{z})|\,d\tilde{z}\Big)\Big(\int_{\mathbb{R}^d}|D_{\tilde{y}_j}D^{\vartheta}_x\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\tilde{y}_j)|\,d\tilde{y}_j\Big). \end{aligned}$$

Noticing now from assumption [**P**] that

$$\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}+1 \,=\, 1-\frac{\beta-\alpha}{1+\alpha(j-1)}+1 \,<\, 2-(1-\alpha) \,=\, 1+\alpha,$$

we can use the smoothing effect of the heat-kernel p_h (Equation (4.9)) to show that

$$\int_{\mathbb{R}^d} J_{2,1}(v,y_{\smallsetminus j},z,\lambda)\,dz \,\le\, \frac{v^{\frac{1+\alpha(j-2)+\beta}{\alpha(1+\alpha(j-1))}}}{v}\int_{\mathbb{R}^d}|D_{\tilde{y}_j}D^{\vartheta}_x\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\tilde{y}_j)|\,d\tilde{y}_j.$$

Remembering Equation (5.33), we can add the integral with respect to v and show that

$$\begin{aligned}\int_0^{(s-t)^{\delta_j}} & v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d} J_{2,1}(v,y_{\smallsetminus j},z,\lambda)\,dz\,dv \\ &\le\,(s-t)^{\delta_j\frac{2\beta_j+1}{\alpha}}\int_{\mathbb{R}^d}|D_{\tilde{y}_j}D^{\vartheta}_x\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\tilde{y}_j)|\,d\tilde{y}_j.\end{aligned}$$

Adding the integral with respect to $y_{\smallsetminus j}$, we can conclude with $J_{2,1}$ that

$$\begin{aligned}&\int_{\mathbb{R}^{(n-1)d}}\int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d} J_{2,1}(v,y_{\smallsetminus j},z,\lambda)\,dz\,dv\,dy_{\smallsetminus j} \\ &\le\, C(s-t)^{\delta_j\frac{2\beta_j+1}{\alpha}}\int_{\mathbb{R}^{nd}}|D_{y_j}D^{\vartheta}_x\tilde{p}^{\tau,\xi}(t,s,x,y)|\,dy \,\le\, C(s-t)^{\delta_j\frac{2\beta_j+1}{\alpha}-\frac{1}{\alpha_j}-\sum_{k=1}^n\frac{\vartheta_k}{\alpha_k}}, \end{aligned} \tag{5.37}$$

where, for simplicity, we have changed back the variable $\tilde{y}_j$ with y_j.
To control instead the term $J_{2,2}$ (cf. Equation (5.36)), we can use again Fubini theorem and the changes of variables $\tilde{z}=z-y_j$, $\tilde{y}_j=y_j+\lambda\tilde{z}$ to split the integrals and show that

$$\begin{aligned}\int_{\mathbb{R}^d} J_{2,2}(v,y_{\smallsetminus j},z,\lambda)\,dz \,&\le\, \Big(\int_{\mathbb{R}^d}|\tilde{z}|\,|D_{\tilde{z}}\partial_v p_h(v,\tilde{z})|\,d\tilde{z}\Big) \\ &\quad\times\Big(\int_{\mathbb{R}^d}|D_{\tilde{y}_j}D^{\vartheta}_x\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\tilde{y}_j)|\mathbf{d}^{1+\alpha(j-2)+\beta}_{j:n}((y_{\smallsetminus j},\tilde{y}_j),\theta_{s,\tau}(\xi))d\tilde{y}_j\Big) \\ &\le\,\frac{1}{v}\int_{\mathbb{R}^d}|D_{y_j}D^{\vartheta}_x\tilde{p}^{\tau,\xi}(t,s,x,y)|\mathbf{d}^{1+\alpha(j-2)+\beta}_{j:n}(y,\theta_{s,\tau}(\xi))dy_jdv,\end{aligned}$$

where in the second inequality we used the smoothing effect of the heat-kernel p_h (Equation (4.9)) and changed back the variable $\tilde{y}_j$ with y_j for simplicity. It then follows that

$$\begin{aligned}\int_0^{(s-t)^{\delta_j}} & v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d} J_{2,2}(v,z,y_{\smallsetminus j})\,dzdv \\ &\le\,(s-t)^{\delta_j\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d}|D_{y_j}D^{\vartheta}_x\tilde{p}^{\tau,\xi}(t,s,x,y)|\mathbf{d}^{1+\alpha(j-2)+\beta}_{j:n}(y,\theta_{s,\tau}(\xi))dy_j.\end{aligned}$$

Taking now $\tau=t$ and $\xi=x$, we conclude with $J_{2,2}$ applying the partial smoothing effect (Equation (5.30)) of $\tilde{p}^{\tau,\xi}$ under the hypothesis $1+\alpha(j-2)+\beta\le\alpha(1+\alpha(j-1))$ (see Equation (5.31)) to write that

$$\begin{aligned}\int_{\mathbb{R}^{(n-1)d}}\int_0^{(s-t)^{\delta_j}} & v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d} J_{2,2}(v,z,y_{\smallsetminus j})\,dz\,dvdy_{\smallsetminus j} \\ &\le\,(s-t)^{\delta_j\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^{nd}}|D_{y_j}D^{\vartheta}_x\tilde{p}^{\tau,\xi}(t,s,x,y)|\mathbf{d}^{1+\alpha(j-2)+\beta}_{j:n}(y,\theta_{s,\tau}(\xi))dy \\ &\le\,C(s-t)^{\delta_j\frac{\alpha_j+\beta_j}{\alpha}+\frac{1+\alpha(j-2)+\beta}{\alpha}-\frac{1}{\alpha_j}-\sum_{k=1}^n\frac{\vartheta_k}{\alpha_k}}.\end{aligned} \tag{5.38}$$

Looking back to Equations (5.32), (5.34), (5.37) and (5.38), we can finally choose the right δ_j. Since $s-t \le T-t<1$ by hypothesis [**ST**], it is enough to take δ_j such that the quantities

$$\delta_j\frac{\alpha_j+\beta_j-1}{\alpha}+\frac{1+\alpha(j-2)+\beta}{\alpha}, \qquad \delta_j\frac{2\beta_j}{\alpha}, \qquad \delta_j\frac{2\beta_j+1}{\alpha}-\frac{1}{\alpha_j}$$

and

$$\delta_j\frac{\alpha_j+\beta_j}{\alpha}+\frac{1+\alpha(j-2)+\beta}{\alpha}-\frac{1}{\alpha_j}$$

are bigger than β/α. This is true if, for example, we choose

$$\delta_j \;=\; \frac{[1+\alpha(j-2)][1+\alpha(j-1)]}{1+\alpha(j-2)-\beta}.$$

We have thus concluded th proof. □

5.2 Proof of Proposition 3.6

We have now all the tools necessary to prove the a priori estimates in Proposition 3.6. In Lemma 5.3 below, we will state the estimates for the supremum norms of the solution and its non-degenerate gradient while the controls of the Hölder moduli of the solution and its gradient with respect to the non-degenerate variable are given in Lemmas 5.7 and 5.8, respectively.
The Schauder estimates (Theorem 2.3) for a solution u in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ of IPDE (1.1) will then follows immediately.

Lemma 5.3 (Supremum Estimates)**.** *Let u be as in Equation* (5.27)*. Then, there exists a constant $C\ge 1$ such that for any t in $[0,T]$ and any x in $\mathbb{R}^{nd}$,*

$$|u(t,x)|+|D_{x_1}u(t,x)| \;\le\; C\Big[\|u_T\|_{C^{\alpha+\beta}_{b,d}}+\|f\|_{L^\infty(C^\beta_{b,d})}+\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\Big].$$

Proof. As indicated above, we can control the supremum norm of u and its gradient with respect to x_1 analyzing separately the contributions from the proxy $\tilde{u}^{\tau,\xi}$, that have already been handled in Lemma 4.4, and those from the perturbative term $R^{\tau,\xi}(s,x)$. To control the contribution $\int_t^T D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)\,ds$, we start splitting it up in the following way:

$$\begin{aligned}\int_t^T & D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)\,ds\\ &= \sum_{j=1}^n\int_t^T\int_{\mathbb{R}^{nd}} D_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y)\Big[F_j(s,y)-F_j(s,\theta_{s,\tau}(\xi))\Big]\cdot D_{y_j}u(s,y)\,dyds\\ &=: \sum_{j=1}^n I_j(t,x). \end{aligned}\tag{5.39}$$

Since by hypothesis u has a proper derivative with respect to the first variable x_1, it is possible to bound I_1 through

$$|I_1(t,x)| \;\le\; C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\int_t^T\int_{\mathbb{R}^{nd}}\Big|D_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y)\Big|\mathbf{d}^\beta(y,\theta_{s,\tau}(\xi))\,dyds.$$

We take now $(\tau,\xi)=(t,x)$ so that $\theta_{s,\tau}(\xi)=\tilde{m}^{\tau,\xi}_{s,t}$ (cf. Equation (3.5) in Lemma 3.1) and we then use the smoothing effect for the frozen density $\tilde{p}^{\tau,\xi}$ (Equation (3.6)) to show that

$$|I_1(t,x)| \,\le\, C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}(T-t)^{\frac{\beta+\alpha-1}{\alpha}}. \tag{5.40}$$

Hence, it holds that $|I_1(t,x)|\le C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}$, since $T\le 1$ and $\alpha+\beta>1$ by assumptions [**ST**] and [**P**].
The control for the terms I_j with $j>1$ can be obtained easily from the second Besov control (Lemma 5.2). For this reason, we start applying integration by parts formula to show that

$$|I_j(t,x)| \,=\, \Big|\int_t^T\int_{\mathbb{R}^{nd}} D_{y_j}\cdot\Big\{D_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\Big\}u(s,y)\,dyds\Big|,$$

where we exploited the same notations for $\Delta^{\tau,\xi}F_j$ given in (5.29). We can then use identification (4.10) and duality in Besov spaces (4.11) to write that

$$\begin{aligned}|I_j(t,x)| \,\le\, &\\ &\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\int_{\mathbb{R}^{(n-1)d}}\Big\|D_{y_j}\cdot\Big\{D_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\cdot)\Delta^{\tau,\xi}F_j(s,y_{\smallsetminus j},\cdot)\Big\}\Big\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}dy_{\smallsetminus j}.\end{aligned}$$

Taking now $(\tau,\xi)=(t,x)$, the second Besov control (Lemma 5.2) can be applied to show that

$$\begin{aligned}|I_j(t,x)| \,&\le\, C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\int_t^T(s-t)^{\frac{\beta-1}{\alpha}}\,ds\\ &\le\, C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}(T-t)^{\frac{\beta+\alpha-1}{\alpha}}.\end{aligned} \tag{5.41}$$

Since $T\le 1$ by assumption [**ST**], we can conclude that $|I_j(t,x)|\le C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}$.
The control on the pertubative term

$$\int_t^T \tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)\,ds$$

can be obtained in a similar way. Namely, Inequalities (5.40) and (5.41) hold again with $(T-t)^{\frac{\beta+\alpha-1}{\alpha}}$ replaced by $(T-t)^{\frac{\beta+\alpha}{\alpha}}$. □

As already specified in the previous sub-section, there is a big difference between the non-degenerate case $i=1$, where $\alpha+\beta$ is in $(1,2)$ and we have to deal with a proper derivative, and the other degenerate situations ($i>1$), where instead $(\alpha+\beta)/(1+\alpha(i-1))<1$ and the norm is calculated directly on the function. Again, we are going to analyze the two cases separately. Lemma 4.5 will focus on the non-degenerate setting ($i=1$) while lemma 4.6 will concerns the degenerate one ($i>1$).

Moreover, we will need to divide the proofs in two cases, depending on which regime we are considering. Since the global off-diagonal regime, i.e. when $T-t\le c_0\mathbf{d}^\alpha(x,x')$, will work essentially as the already shown Schauder estimates (Proposition 3.3) for the proxy, the proof will be quite shorter.

Instead, in the global diagonal case, such that $T-t \geq c_0\mathbf{d}^\alpha(x,x')$, when a time integration is involved (for example in the control of the frozen Green kernel or the perturbative term), two different situations appear. There are again a local off-diagonal regime if $s-t \leq c_0\mathbf{d}^\alpha(x,x')$ and a local diagonal regime when $s-t \geq c_0\mathbf{d}^\alpha(x,x')$. In order to handle these terms properly, the key tool is to be able to change the freezing points depending on which regime we are. It seems reasonable that, when the spatial points are in a local diagonal regime, the auxiliary frozen densities are considered for the same freezing parameter and conversely that, in the local off-diagonal regime, the densities are frozen along their own spatial argument. For this reason, we have postponed the relative proofs in two specific sub-sections.

Before presenting the main results of this section, we are going to state three auxiliary estimates we will need below. We refer to the Section A.2 for a precise proof of these results.
The first one concerns the sensitivity of the Hölder flow $\theta_{s,t}$ with respect to the initial point. Indeed,

Lemma 5.4 (Controls on the Flows)**.** *Let $t<s$ be two points in $[0,T]$ and x,x' two points in $\mathbb{R}^{nd}$. Then, there exists a constant $C \geq 1$ such that*

$$\mathbf{d}(\theta_{s,t}(x),\theta_{s,t}(x')) \;\leq\; C\|F\|_H\Big[\mathbf{d}(x,x')+(s-t)^{1/\alpha}\Big].$$

The second result is the following:

Lemma 5.5. *Let $t<s$ be two points in $[0,T]$ and x,x' two points in $\mathbb{R}^{nd}$ and y,y' two points in $\mathbb{R}^{nd}$ such that $y_1=y'_1$. Then, there exists a constant $C\geq 1$ such that*

$$\Big|(\tilde{m}^{t,x}_{s,t}(y)-\tilde{m}^{t,x'}_{s,t}(y'))_1\Big| \;\leq\; C\|F\|_H\Big[(s-t)\mathbf{d}^\beta(x,x')+(s-t)^{\frac{\alpha+\beta}{\alpha}}\Big].$$

Finally, the impact of the freezing point in the linearization procedure is the argument of this last Lemma. Namely,

Lemma 5.6. *Let t be in $[0,T]$ and x,x' two points in $\mathbb{R}^{nd}$. Then, there exists a constant $C\geq 1$ such that*

$$\mathbf{d}(\tilde{m}^{t,x}_{t_0,t}(x'),\tilde{m}^{t,x'}_{t_0,t}(x')) \;\leq\; Cc_0^{\frac{1}{1+\alpha(n-1)}}\|F\|_H\mathbf{d}(x,x')$$

where t_0 is the change of regime time defined in (4.16)*.*

Thanks to the above controls, we will eventually prove the following results:

Lemma 5.7 (Controls on Hölder Moduli: Non-Degenerate)**.** *Let x,x' be in $\mathbb{R}^{nd}$ such that $x_j=x'_j$ for any $j\neq 1$ and u as in Equation* (5.27)*. Then, there exists a constant $C\geq 1$ such that for any t in $[0,T]$,*

$$\begin{aligned}\Big|D_{x_1}u(t,x)-D_{x_1}u(t,x')\Big| \;\leq\; C\Big\{&c_0^{\frac{\alpha+\beta-2}{\alpha}}\Big(\|u_T\|_{C^{\alpha+\beta}}+\|f\|_{L^\infty(C^\beta)}\Big)\\&+\Big(c_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}}+c_0^{\frac{\alpha+\beta-2}{\alpha}}\|F\|_H\Big)\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\Big\}\mathbf{d}^{\alpha+\beta-1}(x,x').\end{aligned}$$

We can point out now the analogous result in the degenerate setting.

Lemma 5.8 (Controls on Hölder Moduli: Degenerate)**.** *Let i be in $[\![1,n]\!]$ and x,x' in $\mathbb{R}^{nd}$ such that $x_j = x'_j$ for any $j \neq i$ and u as in Equation* (5.27)*. Then, there exists a constant $C \geq 1$ such that for any t in $[0,T]$,*

$$\begin{aligned}\Big|u(t,x)-u(t,x')\Big| \,\le\, C\Big\{c_0^{\frac{\beta-\gamma_i}{\alpha}}\Big(\|u_T\|_{C^{\alpha+\beta}}+\|f\|_{L^\infty(C^\beta)}\Big) \\ +\Big(c_0^{\frac{\alpha+\beta}{1+\alpha(n-1)}}+c_0^{\frac{\beta-\gamma_i}{\alpha}}\|F\|_H\Big)\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\Big\}\mathbf{d}^{\alpha+\beta}(x,x').\end{aligned}$$

Off-diagonal regime

We focus here on the proof of the controls on the Hölder moduli, either in the non-degenerate setting (Lemma 5.7) and in the degenerate one (Lemma 5.8), when a off-diagonal regime is assumed. For this reason, all the statements presented in this sub-section will tacitly assume that $T-t \,\le\, c_0\mathbf{d}^\alpha(x,x')$ for some given (t,x,x') in $[0,T]\times\mathbb{R}^{2nd}$.
To show these two controls, we will need to adapt the auxiliary estimates above to the off-diagonal regime case we consider here. Namely,

$$\mathbf{d}(\tilde m^{t,x}_{T,t}(x),\tilde m^{t,x'}_{T,t}(x')) \,=\, \mathbf{d}(\theta_{T,t}(x),\theta_{T,t}(x')) \,\le\, C\|F\|_H\mathbf{d}(x,x'); \tag{5.42}$$

$$\text{if } x_1 \,=\, x'_1, \quad \Big|\Big(\tilde m^{t,x}_{T,t}(x)-\tilde m^{t,x'}_{T,t}(x')\Big)_1\Big| \,\le\, C\|F\|_H\mathbf{d}^{\alpha+\beta}(x,x') \tag{5.43}$$

They can be obtained easily from Equation (3.5) in Lemma 3.1 and the sensitivity controls (Lemmas 5.4 and 5.5, respectively), taking $s=T$ and $(y,y')=(x,x')$.

Proof of Lemma 5.7 in the off-diagonal regime. From the Duhamel-type Expansion (5.28), we can represent a mild solution u of IPDE (1.1) as

$$\begin{aligned}&|D_{x_1}u(t,x)-D_{x_1}u(t,x')| \\ &\le \Big|D_{x_1}\tilde P^{\tau,\xi}_{T,t}u_T(x)-D_{x_1}\tilde P^{\tau',\xi'}_{T,t}u_T(x')\Big|+\Big|D_{x_1}\tilde G^{\tau,\xi}_{T,t}f(t,x)-D_{x_1}\tilde G^{\tau',\xi'}_{T,t}f(t,x')\Big| \\ &\qquad+\Big|\int_t^T D_{x_1}\tilde P^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)-D_{x_1}\tilde P^{\tau',\xi'}_{s,t}R^{\tau',\xi'}(s,x')\,ds\Big|,\end{aligned}$$

for any fixed $(\tau,\xi),(\tau',\xi')$ in $[0,T]\times\mathbb{R}^{nd}$ After possible differentiations, we will choose $\tau=\tau'=t$, $\xi=x$ and $\xi'=x'$ in order to exploit the sensitivity Controls (5.43) and (5.42).

Control on the frozen semigroup. It can be handled following the analogous part in the proof of the Hölder control for the proxy (Lemma 4.5). The only difference is that we cannot control

$$\mathbf{d}(\tilde m^{\tau,\xi}_{T,t}(x),\tilde m^{\tau',\xi'}_{T,t}(x'))$$

in Equation (4.18) using the affinity of the mapping $x\to\tilde m^{\tau,\xi}_{T,t}(x)$, since the two freezing point are now different. Instead, we can take $\tau=\tau'=t$, $\xi=x$ and $\xi'=x'$ and apply the sensitivity control (5.42) to write that

$$\mathbf{d}(\tilde m^{\tau,\xi}_{T,t}(x),\tilde m^{\tau',\xi'}_{T,t}(x')) \,=\, \mathbf{d}(\theta_{T,t}(x),\theta_{T,t}(x')) \,\le\, C\|F\|_H\mathbf{d}(x,x').$$

Control on the Green kernel. It follows immediately from the proof of the Hölder control (Lemma 4.5) for the proxy, noticing that $t_0 = T$, since we are in the off-diagonal regime.

Control on the perturbative error. Since we do not exploit the difference of the spatial points (x, x') in the off-diagonal regime but instead we control the two contributions separately, we can rely on the controls on the supremum norms we have already shown in Lemma 5.3. Namely, we start writing that

$$\begin{aligned}\Big|\int_t^T D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x) - D_{x_1}\tilde{P}^{\tau',\xi'}_{s,t}R^{\tau',\xi'}(s,x')\,ds\Big| \\ \le \Big|\int_t^T D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)\,ds\Big| + \Big|\int_t^T D_{x_1}\tilde{P}^{\tau',\xi'}_{s,t}R^{\tau',\xi'}(s,x')\,ds\Big|.\end{aligned} \tag{5.44}$$

Then, we can follow the same reasonings of Lemma 5.3 concerning the remainder term (cf. Equations (5.39), (5.40) and (5.41)) to show that

$$\Big|\int_t^T D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)\,ds\Big| \le C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}(T-t)^{\frac{\alpha+\beta-1}{\alpha}}. \tag{5.45}$$

Using it in the above Equation (5.44), we can finally conclude that

$$\Big|\int_t^T D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x) - D_{x_1}\tilde{P}^{\tau',\xi'}_{s,t}R^{\tau',\xi'}(s,x')\,ds\Big| \le C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\mathbf{d}^{\alpha+\beta-1}(x,x'), \tag{5.46}$$

remembering that we assumed to be in the off-diagonal regime, i.e. $T - t \le c_0\mathbf{d}^\alpha(x, x')$ for some $c_0 \le 1$.

Proof of Lemma 5.8 in the off-diagonal regime. As done before, we are going to analyze separately the single terms appearing from the Duhamel-type Representation (5.27) of a solution u:

$$\begin{aligned}|u(t,x) - u(t,x')| \le \Big|\tilde{P}^{\tau,\xi}_{T,t}u_T(x) - \tilde{P}^{\tau',\xi'}_{T,t}u_T(x')\Big| + \Big|\tilde{G}^{\tau,\xi}_{T,t}f(t,x) - \tilde{G}^{\tau',\xi'}_{T,t}f(t,x')\Big| \\ + \Big|\int_t^T \tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x) - \tilde{P}^{\tau',\xi'}_{s,t}R^{\tau',\xi'}(s,x')\,ds\Big|,\end{aligned}$$

for some $(\tau, \xi), (\tau', \xi')$ in $[0, T] \times \mathbb{R}^{nd}$ fixed but to be chosen later as $\tau = \tau' = t$, $\xi = x$ and $\xi' = x'$.

Control on the frozen semigroup. We can essentially follow the proof of the Hölder control (Lemma 4.6) for the proxy. However, this time we cannot exploit the affinity of the mapping $x \to \tilde{m}^{\tau,\xi}_{T,t}(x)$ to control the difference

$$\Big|u_T(\tilde{m}^{\tau,\xi}_{T,t}(x) - z) - u_T(\tilde{m}^{\tau,\xi}_{T,t}(x') - z)\Big|.$$

Instead, we notice now that we can bound it as

$$\begin{aligned}&\Big|u_T(\tilde{m}^{\tau,\xi}_{T,t}(x) - z) - u_T(\tilde{m}^{\tau,\xi}_{T,t}(x') - z)\Big| \\ &\qquad \le C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big[\mathbf{d}^{\alpha+\beta}\Big(\tilde{m}^{\tau,\xi}_{T,t}(x), \tilde{m}^{\tau,\xi}_{T,t}(x')\Big) + \Big|\Big(\tilde{m}^{\tau,\xi}_{T,t}(x) - \tilde{m}^{\tau,\xi}_{T,t}(x')\Big)_1\Big|\Big],\end{aligned}$$

since u_T is differentiable and thus Lipschitz continuous, in the first non-degenerate variable.
Taking now $\tau = \tau' = t$, $\xi = x$ and $\xi' = x'$, we can use the sensitivity Controls (5.42)-(5.43) (noticing that by assumption, $x_1 = x'_1$) to write that

$$\left| u_T(\tilde{m}^{\tau,\xi}_{T,t}(x) - z) - u_T(\tilde{m}^{\tau,\xi}_{T,t}(x') - z) \right| \le C\|F\|_H \|u_T\|_{C^{\alpha+\beta}_{b,d}} \mathbf{d}^{\alpha+\beta}(x,x').$$

Control on the Green kernel. It can be obtained following the analogous part in the proof of the Hölder control (Lemma 4.6) for the proxy. Similarly to the paragraph "Control on the frozen semigroup" in the previous proof, we need to take $(\tau,\xi) = (t,x)$, $(\tau,\xi') = (t,x)$ and apply the sensitivity Control (5.42) to bound the term

$$\mathbf{d}(\tilde{m}^{\tau,\xi}_{T,t}(x), \tilde{m}^{\tau',\xi'}_{T,t}(x'))$$

appearing in Equation (4.25).

Control on the perturbative error. The proof of this estimate essentially matches the previous, analogous one in the non-degenerate setting. Namely, Equations (5.44), (5.45) and (5.46) hold again with $(T-t)^{\frac{\beta+\alpha}{\alpha}}$ instead of $(T-t)^{\frac{\beta+\alpha-1}{\alpha}}$.

Diagonal regime

Since the aim of this section is to prove Lemmas 5.7 and 5.8 when a diagonal regime is assumed, we will assume from this point further that $T - t \ge c_0 \mathbf{d}^\alpha(x,x')$ for some given (t,x,x') in $[0,T]\times\mathbb{R}^{2nd}$.
As preannounced in the introduction of this section, we need here a modification of the Duhamel-type Representation (3.16) that allows to change the freezing points along the time integration variable. Remembering the previous notations for $\tilde{G}^{\tau,\xi}_{r,v}$ and $R^{\tau,\xi}$ in (4.2) and (3.15) respectively, it holds that

Lemma 5.9 (Change of frozen point)**.** *Let (τ,ξ) be a freezing couple in $[0,T]\times\mathbb{R}^{nd}$ and $\tilde{\xi}$ another freezing point in $\mathbb{R}^{nd}$. Then, any classical solution u in $L^\infty(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd}))$ of IPDE* (1.1) *can be represented for any (t,x) in $[0,T]\times\mathbb{R}^{nd}$ as*

$$\begin{aligned} u(t,x) \;=\; & \tilde{P}^{\tau,\tilde{\xi}}_{T,t}u_T(x) + \tilde{G}^{\tau,\xi}_{t_0,t}f(t,x) + \tilde{G}^{\tau,\tilde{\xi}}_{T,t_0}f(t,x) \\ & + \int_t^{t_0} \tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)\,ds + \int_{t_0}^T \tilde{P}^{\tau,\tilde{\xi}}_{s,t}R^{\tau,\tilde{\xi}}(s,x)\,ds + \tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x) - \tilde{P}^{\tau,\tilde{\xi}}_{t_0,t}u(t_0,x), \end{aligned} \tag{5.47}$$

where t_0 is the change of regime time defined in (4.16).

Proof. Fixed t in $(0,T)$, we start considering another point r in (t,T). On $(0,r)$, it is clear that u is again a mild solution of IPDE (1.1) but with terminal condition $u(r,x)$. Then, Duhamel Expansion (3.16) can be applied with respect to the frozen couple (τ,ξ), allowing us to write that

$$u(t,x) \;=\; \tilde{P}^{\tau,\xi}_{r,t}u_T(x) + \int_t^r \tilde{P}^{\tau,\xi}_{s,t}f(s,x)\,ds + \int_t^r \tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}u(s,x)\,ds.$$

Noticing that u is independent from r, it is possible now to differentiate the above equation with respect to r in (t,T) in order to show that

$$0 \;=\; \partial_r\Big[\tilde{P}^{\tau,\xi}_{r,t}u(r,x)\Big] + \tilde{P}^{\tau,\xi}_{r,t}f(r,x) + \tilde{P}^{\tau,\xi}_{r,t}R^{\tau,\xi}(r,x). \tag{5.48}$$

We highlight now that the above expression holds for any chosen frozen couple (τ,ξ) and any fixed time r. Thus, it is possible to integrate it with respect to r for a fixed ξ between t and t_0 and for another frozen point $\tilde{\xi}$ between t_0 and T, leading to

$$\begin{aligned}0 \;=\; &\tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x) - \tilde{P}^{\tau,\xi}_{t,t}u(t,x) + \int_t^{t_0}\tilde{P}^{\tau,\xi}_{r,t}f(r,x)\,dr + \int_t^{t_0}\tilde{P}^{\tau,\xi}_{r,t}R^{\tau,\xi}(r,x)\,dr\\ &+ \tilde{P}^{\tau,\tilde{\xi}}_{T,t}u(T,x) - \tilde{P}^{\tau,\tilde{\xi}}_{t_0,t}u(t_0,x) + \int_{t_0}^{T}\tilde{P}^{\tau,\tilde{\xi}}_{r,t}f(r,x)\,dr + \int_{t_0}^{T}\tilde{P}^{\tau,\tilde{\xi}}_{r,t}R^{\tau,\tilde{\xi}}(r,x)\,dr.\end{aligned}$$

With our previous notations, the above expression can be finally rewritten as

$$\begin{aligned}0 \;=\; &\tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x) - u(t,x) + \tilde{G}^{\tau,\xi}_{t_0,t}f(t,x) + \int_t^{t_0}\tilde{P}^{\tau,\xi}_{r,t}R^{\tau,\xi}(r,x)\,dr\\ &+ \tilde{P}^{\tau,\tilde{\xi}}_{T,t}u_T(x) - \tilde{P}^{\tau,\tilde{\xi}}_{t_0,t}u(t_0,x) + \tilde{G}^{\tau,\tilde{\xi}}_{T,t_0}f(t,x) + \int_{t_0}^{T}\tilde{P}^{\tau,\tilde{\xi}}_{r,t}R^{\tau,\tilde{\xi}}(r,x)\,dr\end{aligned}$$

and we have concluded. $\square$

Similarly to the off-diagonal case, we are going to apply the auxiliary estimates associated with the proxy (Lemmas 5.5 and 5.6) in the current diagonal regime. Namely, taking $s=t_0$ and $(y,y')=(x,x)$ in Lemma 5.5, we know that there exists a constant $C\geq 1$ such that for any t in $[0,T]$ and any x,x' in $\mathbb{R}^{nd}$,

$$\text{if } x_1 \;=\; x'_1, \quad \Big|(\tilde{m}^{t,x}_{t_0,t}(x) - \tilde{m}^{t,x'}_{t_0,t}(x))_1\Big| \;\leq\; C\|F\|_H\mathbf{d}^{\alpha+\beta}(x,x'). \tag{5.49}$$

Moreover, in order to control the perturbative term when a local diagonal regime appears, i.e. when the time integration variable s is in $[t_0,T]$, we will quite often use a Taylor expansion on the frozen density. To be able to exploit the already proven controls, such that the smoothing effect for the frozen density (Equation (3.6)) or the second Besov control (Lemma 5.2), we will need the following:

$$\text{if } s-t\geq c_0\mathbf{d}^\alpha(x,x'), \quad \Big|D^\vartheta_x\tilde{p}^{\tau,\xi'}(t,s,x+\lambda(x'-x),y)\Big| \;\leq\; C\Big|D^\vartheta_x\tilde{p}^{\tau,\xi'}(t,s,x,y)\Big|, \tag{5.50}$$

for any multi-index ϑ in $\mathbb{N}^d$ such that $|\vartheta|\leq 2$ and any λ in $[0,1]$. The proof of these results can be found in Section A.2.

We are now ready to prove Lemmas 5.7 and 5.8 when a global diagonal regime is considered.

Proof of Lemma 5.7 in the diagonal regime. We start recalling that in Lemma 5.7 we assumed fixed a time t in $[0,T]$ and two spatial points x,x' in $\mathbb{R}^{nd}$ such that $x_j=x'_j$ if $j\neq 1$.

From the above Representation (5.47) and the Duhamel-type Formula (3.16), we know that

$$
\begin{aligned}
D_{x_1}u(t,x)-D_{x_1}u(t,x') &= \Big(D_{x_1}\tilde{P}^{\tau,\tilde{\xi}}_{T,t}u_T(x)-D_{x_1}\tilde{P}^{\tau',\xi'}_{T,t}u_T(x')\Big)\\
&\quad+\Big(D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x)+D_{x_1}\tilde{G}^{\tau,\tilde{\xi}}_{T,t_0}f(t,x)-D_{x_1}\tilde{G}^{\tau',\xi'}_{T,t}f(t,x')\Big)\\
+\Big(\int_t^{t_0}D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)\,ds&+\int_{t_0}^{T}D_{x_1}\tilde{P}^{\tau,\tilde{\xi}}_{s,t}R^{\tau,\tilde{\xi}}(s,x)\,ds-\int_t^{T}D_{x_1}\tilde{P}^{\tau',\xi'}_{s,t}R^{\tau',\xi'}(s,x')\,ds\Big)\\
&\quad+\Big(D_{x_1}\tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x)-D_{x_1}\tilde{P}^{\tau,\tilde{\xi}}_{t_0,t}u(t_0,x)\Big),
\end{aligned}
$$

for some freezing couples $(\tau,\xi),(\tau,\tilde{\xi}),(\tau',\xi')$ in $[0,T]\times\mathbb{R}^{nd}$ fixed but to be chosen later. To help the readability of the following, we assume from this point further $\tau=\tau'$ and $\tilde{\xi}=\xi'$.

Control on frozen semigroup. We start focusing on the control of the frozen semigroup, i.e.

$$
\Big|D_{x_1}\tilde{P}^{\tau,\xi'}_{T,t}u_T(x)-D_{x_1}\tilde{P}^{\tau,\xi'}_{T,t}u_T(x')\Big|.
$$

Since the freezing couples coincide, the control on the frozen semigroup can be obtained following the proof of the Hölder control (Lemma 4.5) for the proxy.

Control on the Green kernel. As done before, we split the analysis with respect to the change of regime time t_0. Namely, we write

$$
\begin{aligned}
&\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x)+D_{x_1}\tilde{G}^{\tau,\tilde{\xi}}_{T,t_0}f(t,x)-D_{x_1}\tilde{G}^{\tau,\xi'}_{T,t}f(t,x')\Big|\\
&\qquad\le\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x)-D_{x_1}\tilde{G}^{\tau,\xi'}_{t_0,t}f(t,x')\Big|+\Big|D_{x_1}\tilde{G}^{\tau,\tilde{\xi}}_{T,t_0}f(t,x)-D_{x_1}\tilde{G}^{\tau,\xi'}_{T,t_0}f(t,x')\Big|.
\end{aligned}
$$

While in the local off-diagonal regime, the first term in the r.h.s. of the above expression can be handled as in the global off-diagonal regime, the local diagonal regime contribution represented by

$$
\Big|D_{x_1}\tilde{G}^{\tau,\tilde{\xi}}_{T,t_0}f(t,x)-D_{x_1}\tilde{G}^{\tau,\xi'}_{T,t_0}f(t,x')\Big| = \Big|D_{x_1}\tilde{G}^{\tau,\xi'}_{T,t_0}f(t,x)-D_{x_1}\tilde{G}^{\tau,\xi'}_{T,t_0}f(t,x')\Big|
$$

since $\tilde{\xi}=\xi'$, can be controlled following again the proof of the Hölder control (Lemma 4.5) for the proxy.

Control on the discontinuity term. We can now focus on the contribution

$$
\Big|D_{x_1}\tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x)-D_{x_1}\tilde{P}^{\tau,\tilde{\xi}}_{t_0,t}u(t_0,x)\Big|,
$$

arising from the change of freezing point in the Representation (5.47).
Since at fixed time t_0, the function u shows the same spatial regularity of u_T, this control can be handled following the paragraph in the proof of the Hölder control for the proxy (Lemma 4.5) concerning the frozen semigroup in the off-diagonal regime. The only main difference is in Equation (4.18) where, this time, we need to take $(\tau,\xi,\xi')=(t,x,x')$ and exploit the sensitivity estimate (Lemma 5.6) to control the quantity

$$
\mathbf{d}(\tilde{m}^{\tau,\xi}_{t_0,t}(x),\tilde{m}^{\tau,\xi'}_{t_0,t}(x)).
$$

In the end, it is possible to show again (cf. Equation (4.21)) that

$$\left|D_{x_1}\tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x)-D_{x_1}\tilde{P}^{\tau,\tilde{\xi}}_{t_0,t}u(t_0,x)\right|\le C\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}c_0^{\frac{\alpha+\beta-1}{\alpha}}\mathbf{d}^{\alpha+\beta-1}(x,x').$$

Control on the perturbative term. We start splitting the analysis into two cases with respect to the critical time t_0 giving the change of regime. Namely, we write

$$\begin{aligned}\Big|\int_t^{t_0}D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)\,ds+\int_{t_0}^T D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x)\,ds-\int_t^T D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x')\,ds\Big|&\\
\le\Big|\int_t^{t_0}D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)-D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x')\,ds\Big|&\\
+\Big|\int_{t_0}^T D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x)-D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x')\,ds\Big|&.\end{aligned}$$

We then notice that the local off-diagonal regime represented by

$$\Big|\int_t^{t_0}D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)-D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x')\,ds\Big|$$

can be handled following the proof in the global off-diagonal regime of Lemma 5.7. We can then focus our attention on the local diagonal regime, i.e.

$$\Big|\int_{t_0}^T D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x)-D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x')\,ds\Big|.$$

Since the freezing couples coincide, we can use a Taylor expansion with respect to the first variable x_1 in order to write that

$$\begin{aligned}&\Big|\int_{t_0}^T D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x)-D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x')\,ds\Big|\\
&\qquad=\Big|\int_{t_0}^T\int_{\mathbb{R}^{nd}}\int_0^1 D^2_{x_1}\tilde{p}^{\tau,\xi'}(t,s,x+\lambda(x'-x),y)(x'-x)_1R^{\tau,\xi'}(s,y)\,dyds d\lambda\Big|.\end{aligned}$$

Noticing that we are integrating from t_0 to T, Equation (5.50) can be rewritten as

$$\begin{aligned}&\Big|\int_{t_0}^T D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x)-D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x')\,ds\Big|\\
&\qquad\le|(x'-x)_1|\sum_{j=1}^n\int_0^1\int_{t_0}^T\Big|\int_{\mathbb{R}^{nd}}D^2_{x_1}\tilde{p}^{\tau,\xi'}(t,s,x,y)\\
&\qquad\qquad\times\Big\{\Big[F_j(s,y)-F_j(s,\theta_{s,t}(\xi'))\Big]\cdot D_{y_j}u(s,y)\Big\}\,dy\Big|dsd\lambda\\
&\qquad=:|(x-x')_1|\sum_{j=1}^n\int_{t_0}^T I^d_j(s)ds.\end{aligned}\tag{5.51}$$

As done before, we are going to treat separately the cases $j=1$ and $j>1$. In the first case, the term I^d_1 can be easily controlled by

$$\begin{aligned}I^d_1(s)&\le\|D_{y_1}u\|_{L^\infty(L^\infty)}\int_{\mathbb{R}^{nd}}\Big|D^2_{x_1}\tilde{p}^{\tau,\xi'}(t,s,x,y)\Big|\,\Big|F_1(s,y)-F_1(s,\theta_{s,t}(\xi'))\Big|\,dy\\
&\le C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}(s-t)^{\frac{\beta-2}{\alpha}},\end{aligned}\tag{5.52}$$

where in the last passage we used the smoothing effect for the frozen density $\tilde{p}^{\tau,\xi}$ (Equation (3.6)).

On the other side, the case $j > 1$ can be exploited using the second Besov control (Lemma 5.2). For this reason, we start using integration by parts formula to show that

$$I_j^d(s) \; = \; \Big|\int_{\mathbb{R}^{nd}} D_{y_j} \cdot \Big\{ D^2_{x_1}\tilde{p}^{\tau,\xi'}(t,s,x,y) \otimes \Big[F_j(s,y) - F_j(s,\theta_{s,t}(\xi'))\Big]\Big\} u(s,y)\, dy\Big|.$$

Through Duality (4.11) in Besov spaces and the identification in Equation (4.10), we then write that

$$\begin{aligned} I_j^d(s) \; \le \; C\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \int_{\mathbb{R}^{(n-1)d}} \|D_{y_j} \cdot \Big\{ D^2_{x_1}\tilde{p}^{\tau,\xi'}(t,s,x,y_{\smallsetminus j},\cdot) \\ \otimes \Big[F_j(s,y_{\smallsetminus j},\cdot) - F_j(s,\theta_{s,t}(\xi'))\Big]\Big\}\|_{B^{-(\alpha_j+\beta_j)}_{1,1}} \, dy_{\smallsetminus j}. \end{aligned}$$

We can now apply the second Besov control (Lemma 5.2) to show that

$$I_j^d(s) \; \le \; C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}(s-t)^{\frac{\beta-2}{\alpha}}. \tag{5.53}$$

Going back at Equations (5.51),(5.52) and (5.53), we then notice that

$$\begin{aligned} &\Big|\int_{t_0}^T D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x) - D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x')\, ds\Big| \qquad (5.54)\\ &\qquad \le \; C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}|(x-x')_1| \int_{t_0}^T (s-t)^{\frac{\beta-2}{\alpha}} ds \\ &\qquad \le \; C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}|(x-x')_1|(t_0-t)^{\frac{\alpha+\beta-2}{\alpha}}, \end{aligned}$$

where in the last passage we used that $\frac{\alpha+\beta-2}{\alpha} < 0$ to pick the starting point t_0 in the integral.

Using that $t_0 - t = c_0\mathbf{d}^\alpha(x,x')$, we can finally write that

$$\begin{aligned} &\Big|\int_{t_0}^T D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x) - D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x')\, ds\Big| \\ &\qquad \le \; Cc_0^{\frac{\alpha+\beta-2}{\alpha}}\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\mathbf{d}^{\alpha+\beta-1}(x,x'). \end{aligned}$$

Proof of Lemma 5.8 in diagonal regime. We conclude this section showing the Hölder control in the degenerate setting when a diagonal regime is assumed. We start recalling that in Lemma 5.8, we assumed fixed a time t in $[0,T]$ and two spatial points x, x' in $\mathbb{R}^{nd}$ such that $x_j = x'_j$ if $j \neq i$ for some i in $[\![2,n]\!]$.

Representation (5.47) and Duhamel-type Expansion (3.16) allows to control the Holder modulus of a solution u analyzing separately the different terms:

$$\begin{aligned} u(t,x)-&u(t,x') \\ &= \Big(\tilde{P}^{\tau,\tilde{\xi}}_{T,t}u_T(x) - \tilde{P}^{\tau',\xi'}_{T,t}u_T(x')\Big) + \Big(\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x) + \tilde{G}^{\tau,\tilde{\xi}}_{T,t_0}f(t,x) - \tilde{G}^{\tau',\xi'}_{T,t}f(t,x')\Big) \\ &\quad + \Big(\int_t^{t_0} \tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)\, ds + \int_{t_0}^T \tilde{P}^{\tau,\tilde{\xi}}_{s,t}R^{\tau,\tilde{\xi}}(s,x)\, ds - \int_t^T \tilde{P}^{\tau',\xi'}_{s,t}R^{\tau',\xi'}(s,x')\, ds\Big) \\ &\quad + \Big(\tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x) - \tilde{P}^{\tau,\tilde{\xi}}_{t_0,t}u(t_0,x)\Big), \end{aligned}$$

for some freezing couples $(\tau,\xi),(\tau,\tilde{\xi}),(\tau,\xi')$ fixed but to be chosen later. As done before, we assume however from this point further that $\tau=\tau'$ and $\tilde{\xi}=\xi'$.

Control on the frozen semigroup. Noticing that we have taken the same freezing couples since $\tilde{\xi}=\xi'$, the control on the frozen semigroup $\left|\tilde{P}^{\tau,\xi'}_{T,t}u_T(x)-\tilde{P}^{\tau,\xi'}_{T,t}u_T(x')\right|$ can be obtained exploiting the same argument used in the proof of the Hölder control (Lemma 4.6) for the proxy.

Control on the Green kernel. The proof of this estimate essentially matches the previous, analogous one in the non-degenerate setting. Namely, we follow the proof in the global off-diagonal regime of Lemma 5.8 to control the local off-diagonal regime contribution $\left|\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x)-\tilde{G}^{\tau,\xi'}_{t_0,t}f(t,x')\right|$ while in the locally diagonal regime term

$$\left|\tilde{G}^{\tau,\xi'}_{T,t_0}f(t,x)-\tilde{G}^{\tau,\xi'}_{T,t_0}f(t,x')\right|,$$

the freezing couples coincide and we can thus exploit the same argument used in the proof of the Hölder control (Lemma 4.6) for the proxy.

Control on the discontinuity term. The proof of this result will follow essentially the one about the off-diagonal regime of the frozen semigroup with respect to the degenerate variables. It holds that

$$\begin{aligned}
\tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x) &= \int_{\mathbb{R}^{nd}}\tilde{p}^{\tau,\xi}(t,t_0,x,y)u(t_0,y)\,dy\\
&= \int_{\mathbb{R}^{nd}}\frac{1}{\det\Big(\mathbb{M}_{t_0-t}\Big)}p_S\Big(t_0-t,\mathbb{M}^{-1}_{t_0-t}(\tilde{m}^{\tau,\xi}_{t_0,t}(x)-y)\Big)u(t_0,y)\,dy\\
&= \int_{\mathbb{R}^{nd}}\frac{1}{\det\mathbb{M}_{t_0-t}}p_S(t_0-t,\mathbb{M}^{-1}_{t_0-t}z)u(t_0,\tilde{m}^{\tau,\xi}_{t_0,t}(x)-z)\,dz,
\end{aligned}$$

where in the last passage we used the change of variable $z=\tilde{m}^{\tau,\xi}_{t_0,t}(x)-y$. Since a similar argument works also for $\tilde{P}^{\xi'}_{t_0,t}u(t_0,x)$, it then follows that

$$\begin{aligned}
&\left|\tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x)-\tilde{P}^{\tau,\xi'}_{t_0,t}u(t_0,x)\right|\\
&\quad=\left|\int_{\mathbb{R}^{nd}}\frac{1}{\det\mathbb{M}_{t_0-t}}p_S\Big(t_0-t,\mathbb{M}^{-1}_{t_0-t}z\Big)\Big[u(t_0,\tilde{m}^{\tau,\xi}_{t_0,t}(x)-z)-u(t_0,\tilde{m}^{\tau,\xi'}_{t_0,t}(x)-z)\Big]\,dz\right|.
\end{aligned}$$

Remembering that $u(t_0,\cdot)$ is Lipschitz with respect to the first non-degenerate variable, we can write now that

$$\begin{aligned}
\left|\tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x)-\tilde{P}^{\tau,\xi'}_{t_0,t}u(t_0,x)\right| &\le C\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\left(\int_{\mathbb{R}^{nd}}p_S\Big(t_0-t,\mathbb{M}^{-1}_{t_0-t}z\Big)\frac{dz}{\det\mathbb{M}_{t_0-t}}\right)\\
&\quad\times\Big[\mathbf{d}^{\alpha+\beta}(\tilde{m}^{\tau,\xi}_{t_0,t}(x),\tilde{m}^{\tau,\xi'}_{t_0,t}(x))+\Big|(\tilde{m}^{\tau,\xi}_{t_0,t}(x)-\tilde{m}^{\tau,\xi'}_{t_0,t}(x))_1\Big|\Big]\\
&\le C\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\Big[\mathbf{d}^{\alpha+\beta}(\tilde{m}^{\tau,\xi}_{t_0,t}(x),\tilde{m}^{\tau,\xi'}_{t_0,t}(x))+\Big|(\tilde{m}^{\tau,\xi}_{t_0,t}(x)-\tilde{m}^{\tau,\xi'}_{t_0,t}(x))_1\Big|\Big].
\end{aligned}$$

Taking $\xi=\xi'=x$, we can then use the sensitivity controls (Lemma 5.6 and Equation (5.49)) to show that

$$\left|\tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x)-\tilde{P}^{\tau,\xi'}_{t_0,t}u(t_0,x)\right|\le C\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\|F\|_H c_0^{\frac{\alpha+\beta}{1+\alpha(n-1)}}\mathbf{d}^{\alpha+\beta}(x,x).$$

Control on the perturbative term. The proof of this estimate essentially matches the previous, analogous one in the non-degenerate setting. Namely, Inequalities (5.52), (5.53) and (5.54) hold again with $(s-t)^{\frac{\beta-2}{\alpha}}$ replaced by $(s-t)^{\frac{\beta}{\alpha}-\frac{1}{\alpha_i}}$.

Mollifying procedure

We now make the mollifying parameter m appear again using the notations introduced in Section 3.2 (see Equation (3.16)). Then, Lemmas 5.3, 5.7 and 5.8 rewrite together in the following way. There exists a constant $C>0$ such that for any m in $\mathbb{N}$,

$$\begin{aligned}\|u_m\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \le\, &Cc_0^{\frac{\beta-\gamma_n}{\alpha}}\Big[\|u_{T,m}\|_{C^{\alpha+\beta}_{b,d}}+\|f_m\|_{L^\infty(C^\beta_{b,d})}\Big]\\ &+C\Big(c_0^{\frac{\beta-\gamma_n}{\alpha}}\|F_m\|_H+c_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}}\Big)\|u_m\|_{L^\infty(C^{\alpha+\beta}_{b,d})},\end{aligned}\tag{5.1}$$

where c_0 is assumed to be fixed but chosen later. Importantly, c_0 and C does not depends on the regularizing parameter m. Thus, letting m go to ∞ and remembering Definition 2.2 of a mild solution u, the above expression immediately implies the a priori estimates (Proposition 3.6).

6 Existence result

The aim of this section is to show the well-posedness in a mild sense of the original IPDE (1.1). Recalling Definition 2.2 for a mild solution of IPDE (1.1), let us consider three sequences $\{f_m\}_{m\in\mathbb{N}}$, $\{u_{T,m}\}_{m\in\mathbb{N}}$ and $\{F_m\}_{m\in\mathbb{N}}$ of "regularized" coefficients such that

— $\{f_m\}_{m\in\mathbb{N}}$ is in $C^\infty_b((0,T)\times\mathbb{R}^{nd})$ and f_m converges to f in $L^\infty\Big(0,T;C^\beta_{b,d}(\mathbb{R}^{nd})\Big)$;

— $\{u_{T,m}\}_{m\in\mathbb{N}}$ is in $C^\infty_b(\mathbb{R}^{nd})$ and $u_{T,m}$ converges to u_T in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})$;

— $\{F_m\}_{m\in\mathbb{N}}$ is in $C^\infty_b((0,T)\times\mathbb{R}^{nd};\mathbb{R}^{nd})$ and $\|F_m-F\|_H$ converges to 0.

It can be derived through stochastic flows techniques (see e.g. [Kun04]) that there exists a solution u_m in $C^\infty_b\Big((0,T)\times\mathbb{R}^{nd}\Big)$ of the "regularized" IPDE:

$$\begin{cases}\partial_t u_m(t,x)+\mathcal{L}_\alpha u_m(t,x)+\langle Ax+F_m(t,x),D_xu_m(t,x)\rangle \,=\, -f_m(t,x) & \text{on } (0,T)\times\mathbb{R}^{nd};\\ u_m(T,x)\,=\,u_{T,m}(x) & \text{on } \mathbb{R}^{nd}.\end{cases}$$

In order to pass to the limit in m, we notice now that the arguments used above for the proof of the Schauder estimates (Equation (2.26)) can be applied to the above dynamics, too. Namely, there exists a constant $C>0$ such that

$$\|u_m\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\,\le\, C\Big[\|f_m\|_{L^\infty(C^\beta_{b,d})}+\|u_{T,m}\|_{C^{\alpha+\beta}_{b,d}}\Big]\,\le\, C\Big[\|f\|_{L^\infty(C^\beta_{b,d})}+\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big].$$

Importantly, the above estimates is uniformly in m and thus, the sequence $\{u_m\}_{m\in\mathbb{N}}$ is bounded in the space $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$. From Arzelà-Ascoli Theorem, we deduce now that there exists u in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ and a sequence $\{u_{m_k}\}_{k\in\mathbb{N}}$ of smooth and bounded functions converging to u in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ and such that u_{m_k} is solution of the "regularized" IPDE (2.25). It is then clear that u is a mild solution of the original IPDE (1.1).

From mild to weak solutions We conclude showing that any mild solution u of the IPDE (1.1) is indeed a weak solution. The proof of this result will be essentially an application of the arguments presented before, especially the second Besov control (Lemma 5.2). Let u be a mild solution of the IPDE (1.1) in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$. Recalling the definition of weak solution in (2.23), we start fixing a test function ϕ in $C^\infty_0\Big((0,T]\times\mathbb{R}^{nd}\Big)$ and passing to the "regularized" setting (see Definition 2.2), we then notice that it holds that

$$\int_0^T\int_{\mathbb{R}^{nd}}\phi(t,y)\Big(\partial_t+L^m_t\Big)u_m(t,y)\,dy \;=\; -\int_0^T\int_{\mathbb{R}^{nd}}\phi(t,y)f_m(t,y)\,dy,$$

where L^m_t is the "complete" operator defined in (2.22) but with respect to the regularized coefficients. Integration by parts formula allows now to move the operators to the test function. Indeed, remembering that $u_m(T,\cdot)=u_{T,m}(\cdot)$, it holds that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^{nd}}\Big(-\partial_t+(L^m_t)^*\Big)\phi(t,y)u_m(t,y)\,dydt+\int_{\mathbb{R}^{nd}}\phi(T,y)u_{T,m}(y)\,dy&\\ =-\int_0^T\int_{\mathbb{R}^{nd}}\phi(t,y)f_m(t,y)\,dydt,&\quad (6.2)\end{aligned}$$

where $\mathcal{L}^*_{m,\alpha}$ denotes the formal adjoint of L^m_t. We would like now to go back to the solution u, letting m go to ∞. We start rewriting the right-hand side term in the following way:

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^{nd}}\phi(t,y)f_m(t,y)\,dydt\\ &\qquad=\int_0^T\int_{\mathbb{R}^{nd}}\phi(t,y)f(t,y)\,dydt+\int_0^T\int_{\mathbb{R}^{nd}}\phi(t,y)\Big[f_m-f\Big](t,y)\,dydt.\end{aligned}$$

Exploiting that f_m converges to f in $L^\infty\Big(0,T;C^\beta_{b,d}(\mathbb{R}^{nd})\Big)$ by assumption, it is easy to see that the second contribution above goes to 0 if we let m go to ∞. A similar argument can be used to show that

$$\int_{\mathbb{R}^{nd}}\phi(T,y)u_{T,m}(y)\,dy \overset{m}{\to} \int_{\mathbb{R}^{nd}}\phi(T,y)u_T(y)\,dy.$$

On the other hand, we can decompose the first term in the left-hand side of Equation (6.2) as

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^{nd}}\Big(-\partial_t+(L^m_t)^*\Big)\phi(t,y)u_m(t,y)\,dydt\\ &\qquad=\int_0^T\int_{\mathbb{R}^{nd}}\Big(-\partial_t+L^*_t\Big)\phi(t,y)u(t,y)\,dydt+R^1_m+R^2_m,\quad (6.3)\end{aligned}$$

where we have denoted

$$\begin{aligned}R^1_m&=\int_0^T\int_{\mathbb{R}^{nd}}\Big[\mathcal{L}^*_\alpha-(L^m_t)^*\Big]\phi(t,y)u_m(t,y)\,dydt;\\ R^2_m&=\int_0^T\int_{\mathbb{R}^{nd}}\Big(-\partial_t+\mathcal{L}^*_\alpha\Big)\phi(t,y)\Big[u_m(t,y)-u(t,y)\Big]\,dydt,\end{aligned}$$

with $\mathcal{L}_\alpha^*$ as the formal adjoint of the complete operator L_t. Noticing that

$$\Big[\mathcal{L}_\alpha^* - (L_t^m)^*\Big]\phi(t,y) \,=\, D_y \cdot \Big\{\phi(t,y)[F(t,y) - F_m(t,y)]\Big\},$$

it is clear that the remainder contribution R^1_m can be essentially handled as in the introduction of Section 5.1, exploiting that $\|F - F_m\|_H \to 0$.
To control instead the second contribution R^2_m, we start decomposing it as

$$\begin{aligned} R^2_m \,&=\, -\int_0^T \int_{\mathbb{R}^{nd}} \partial_t\phi(t,y)\Big[u_m(t,y) - u(t,y)\Big]\, dydt \\ &\qquad\qquad + \sum_{j=1}^n \int_0^T \int_{\mathbb{R}^{nd}} D_{y_j}\Big[\phi F_j\Big](t,y)\Big[u_m(t,y) - u(t,y)\Big]\, dydt \\ &=:\, R^2_{0,m} + \sum_{j=1}^n R^2_{j,m}. \end{aligned}$$

We firstly observe that $|R^2_{0,m}|$ goes to 0 if we let m go to ∞, since $\|u - u_m\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \overset{m}{\to} 0$. On the other hand, integration by parts formula allows to show that

$$|R^2_{1,m}| \,=\, \Big|\int_0^T \int_{\mathbb{R}^{nd}} \Big[\phi F\Big](t,y) D_{y_j}\Big[u_m - u\Big](t,y)\, dydt\Big|$$

which again tends to 0 when m goes to ∞. To control instead the contributions $R^m_{j,m}$ for $j > 1$, the point is to use the Besov duality argument again. Namely, from Equations (4.11), (4.10) and with the notations in (4.7), it holds that

$$\begin{aligned} |R^2_{j,m}| \,&\le\, \int_0^T \int_{\mathbb{R}^{d(n-1)}} \Big\|D_{y_j}\Big[\phi F\Big](t, y_{\smallsetminus j}, \cdot)\Big\|_{B^{-(\alpha_j+\beta_j)}_{1,1}} \Big\|\Big[u_m - u\Big](t, y_{\smallsetminus j}, \cdot)\Big\|_{B^{\alpha_j+\beta_j}_{\infty,\infty}}\, dy_{\smallsetminus j}dt \\ &\le\, \int_0^T \int_{\mathbb{R}^{d(n-1)}} \Big\|D_{y_j}\Big[\phi F\Big](t, y_{\smallsetminus j}, \cdot)\Big\|_{B^{-(\alpha_j+\beta_j)}_{1,1}} \Big\|\Big[u_m - u\Big](t, y_{\smallsetminus j}, \cdot)\Big\|_{C^{\alpha_j+\beta_j}_b}\, dy_{\smallsetminus j}dt. \end{aligned}$$

Following the same arguments used in the proof of the second Besov control (Lemma 5.2), we know that there exists a constant C such that $\Big\|D_{y_j}\Big[\phi F\Big](t, y_{\smallsetminus j}, \cdot)\Big\|_{B^{\alpha_j+\beta_j}_{1,1}} \le C\psi_j(t, y_{\smallsetminus j})$, where ψ_j has compact support on $\mathbb{R}^{d(n-1)}$.
Since moreover $\|u_m - u\|$ goes to zero with m, we easily deduce that $R^2_{m,j} \overset{m}{\to} 0$ for any j in $[\![2, n]\!]$. From the above controls, we can deduce now that $R^1_m + R^2_m \overset{m}{\to} 0$. From Equation (6.3), it then follows that

$$\int_0^T \int_{\mathbb{R}^{nd}} \Big(-\partial_t + (L^m_t)^*\Big)\phi(t,y)u_m(t,y)\, dydt \,\overset{m}{\to}\, \int_0^T \int_{\mathbb{R}^{nd}} \Big(-\partial_t + L^*_t\Big)\phi(t,y)u(t,y)\, dydt$$

and the proof is concluded.

7 Extensions

As already said in the introduction, our assumption of (global) Hölder regularity on the drift $\bar{F}$, as well as the choice of considering a perturbed Ornstein-Uhlenbeck operator instead of a more general non-linear dynamics, was done to preserve, as possible, the clarity and understandability of the article. In this conclusive section, we would like to explain briefly how it possible to naturally extend it.

7.1 General drift

Here, we illustrate how the perturbative method explained above can be easily adapted to work in a more general setting. In particular, the same results (well-posedness of the IPDE (1.1) and associated Schauder estimates) can be proven to hold also for an equation of the form:

$$\begin{cases}\partial_t u(t,x) + \mathcal{L}_\alpha u(t,x) + \langle \bar{F}(t,x), D_x u(t,x)\rangle \; = \; -f(t,x), & \text{on } (0,T)\times\mathbb{R}^{nd};\\ u(T,x) \; = \; u_T(x) & \text{on } \mathbb{R}^{nd},\end{cases} \tag{7.1}$$

where $\bar{F}(t,x) = \Big(\bar{F}_1(t,x),\dots,\bar{F}_n(t,x)\Big)$ has the following structure

$$\bar{F}_i(t,x_{(i-1)\vee 1},\dots,x_n).$$

We remark in particular that if for any i in $[\![2,n]\!]$, $\bar{F}_i$ is linear with respect to x_{i-1} and independent from time, the previous analysis works since we can rewrite $\bar{F}(t,x) = Ax + F(t,x)$.

In order to deal with this more general dynamics addressed in the diffusive setting in [CdRHM18a], we will need however to add some additional constraints and to modify slightly the ones presented in assumption [**A**]. First of all, the non-degeneracy assumption [**H**] does not make sense in this new framework and it will be replaced by the following one:

[**H'**] the matrix $D_{x_{i-1}}\bar{F}_i(t,x)$ has full rank d for any i in $[\![2,n]\!]$ and any (t,x) in $[0,T]\times\mathbb{R}^{nd}$.

In particular, we will say that assumption [$\bar{\mathbf{A}}$] is in force when

[**S'**] assumption [**ND**] and [**H'**] are satisfied and the drift $\bar{F} = (\bar{F}_1,\dots,\bar{F}_n)$ is such that for any i in $[\![2,n]\!]$, $\bar{F}_i$ depends only on time and on the last $n-(i-2)\vee 0$ components, i.e. $\bar{F}_i(t,x_{i-1},\dots,x_n)$;

[**P'**] α is a number in $(0,2)$, β is in $(0,1)$ and it holds that

$$\beta<\alpha, \quad \alpha+\beta\in(1,2) \;\text{ and }\; \beta<(\alpha-1)(1+\alpha(n-1));$$

[**R'**] Recalling the notations in (2.19)-(2.20), the source f is in $L^\infty(0,T;C^\beta_{b,d}(\mathbb{R}^{nd}))$, the terminal condition u_T is in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})$ and for any i in $[\![1,n]\!]$, the drift $\bar{F}_i$ belongs to $L^\infty(0,T;C^{\gamma_i+\beta}_d(\mathbb{R}^{nd}))$ where γ_i was defined in (2.21).

To prove Schauder-type estimates for a solution of IPDE (7.1), our idea is to adapt the perturbative approach to this new dynamics. In particular, we can exploit the differentiability of $\bar{F}_i$ with respect to x_{i-1} to "linearize" it along a flow that takes into account the perturbation (cf. Section 3.1). Namely, we are interested in the following equation:

$$\partial_t\bar{u}^{\tau,\xi}(t,x) + \mathcal{L}_\alpha\bar{u}^{\tau,\xi}(t,x) + \Big\langle \bar{A}^{\tau,\xi}_t\Big(x-\bar{\theta}_{t,\tau}(\xi)\Big) + \bar{F}(t,\bar{\theta}_{t,\tau}(\xi)), D_x\bar{u}^{\tau,\xi}(t,x)\Big\rangle = -f(t,x); \tag{7.2}$$

with initial condition $\bar{u}^{\tau,\xi}(T,x) = u_T(x)$, where the time-dependent matrix $\bar{A}^{\tau,\xi}_t$ is defined through

$$\left[\bar{A}^{\tau,\xi}_t\right]_{i,j} = \begin{cases} D_{x_{i-1}}\bar{F}_i(t,\theta_{t,\tau}(\xi)), & \text{if } j=i-1; \\ 0_{d\times d}, & \text{otherwise} \end{cases}$$

and $\bar{\theta}_{t,\tau}(\xi))$ is a fixed flow satisfying the dynamics

$$\bar{\theta}_{t,\tau}(\xi) = \xi + \int_\tau^t \bar{F}(v,\bar{\theta}_{v,\tau}(\xi))\,dv. \tag{7.3}$$

A first significant difference with respect to the previous approach consists in handling a time-dependent matrix $\bar{A}^{\tau,\xi}_t$. Indeed, it is possible to modify slightly the presentation in [PZ09] (allowing time-dependency on A) in order to show that under assumption [**S'**], the two parameters semigroup $\{\bar{P}^{\tau,\xi}_{s,t}\}_{t\le s}$ associated with the proxy operator

$$\mathcal{L}_\alpha + \langle \bar{A}^{\tau,\xi}_t\Big(x-\bar{\theta}_{t,\tau}(\xi)\Big) + \bar{F}(t,\bar{\theta}_{t,\tau}(\xi)), D_x\rangle$$

admits a density $\bar{p}^{\tau,\xi}$ and that it can be written as

$$\bar{p}^{\tau,\xi}(t,s,x,y) = \frac{1}{\det \mathbb{M}_{s-t}} p_S\Big(s-t, \mathbb{M}^{-1}_{s-t}(y-\bar{m}^{\tau,\xi}_{s,t}(x))\Big).$$

Here, the notations for p_S and $\mathbb{M}_t$ remain the same of above while this time the shift $\bar{m}^{\tau,\xi}_{s,t}$ is defined through

$$\bar{m}^{\tau,\xi}_{s,t}(x) = \mathcal{R}^{\tau,\xi}_{s,t}x + \int_t^s \mathcal{R}^{\tau,\xi}_{s,v}\Big[\bar{F}(v,\bar{\theta}_{v,\tau}(\xi)) - \bar{A}^{\tau,\xi}_v\bar{\theta}_{v,\tau}(\xi)\Big]\,dv,$$

where $\mathcal{R}^{\tau,\xi}_{s,t}$ is the time-ordered resolvent of $\bar{A}^{\tau,\xi}_s$ starting at time t, i.e.

$$\begin{cases} d\mathcal{R}^{\tau,\xi}_{s,t} = \bar{A}^{\tau,\xi}_s\mathcal{R}^{\tau,\xi}_{s,t}ds, & \text{on } [t,T]; \\ \mathcal{R}^{\tau,\xi}_{t,t} = I. \end{cases}$$

We can as well refer to [HM16] for related issues (see Proposition 3.2 and Section C about the linearization, therein).

Following the same reasonings of Propositions 3.4 and 3.5, it is then possible to state a Duhamel type formula suitable for IPDE (7.1):

$$u(t,x) = \bar{P}^{\tau,\xi}_{T,t}u_T(x) + \int_t^T \bar{P}^{\tau,\xi}_{s,t}\Big[f(s,\cdot) + \bar{R}^{\tau,\xi}(s,\cdot)\Big](x)\,ds, \tag{7.4}$$

where the remainder term is given now by

$$\bar{R}^{\tau,\xi}(t,x) = \langle F(t,x) - F(t,\bar{\theta}_{t,\tau}(\xi)) - \bar{A}^{\tau,\xi}_t\Big(x-\bar{\theta}_{t,\tau}(\xi)\Big), D_xu(t,x)\rangle.$$

Looking back at the first part of the article, it is important to notice that the main steps of proof (cf. Equation (3.6), Propositions 3.3 and 3.6 and Section 3.3) does not rely on the explicit formulas for $\bar{m}^{\tau,\xi}_{s,t}(x)$ and $\bar{R}^{\tau,\xi}$ but instead, they exploit only the

Besov controls for the remainder $\bar{R}^{\tau,\xi}$ (cf. Section 5.1) and the controls on the shift $\bar{m}^{\tau,\xi}_{s,t}(x)$ (Section A.2). Hence, once we have proven the suitable controls, the proofs of the analogous results for the new IPDE (7.1) can be obtained easily modifying slightly the notations and following the same reasonings above.
For example, exploiting that

$$\bar{m}^{\tau,\xi}_{s,t}(x) \,=\, x + \int_t^s \mathcal{R}^{\tau,\xi}_{v,t}\Big(\bar{m}^{\tau,\xi}_{v,t}(x) - \theta_{v,\tau}(\xi)\Big) + F(v, \theta_{v,\tau}(\xi))\, dv,$$

we can follow the same method of proof in the above lemma 3.1 to show again that

$$\bar{m}^{\tau,\xi}_{s,t}(x) \,=\, \bar{\theta}_{s,\tau}(\xi),$$

taking $\tau = t$ and $\xi = x$.

Letting the interested reader look in the appendix for the suggestions on how to extend the controls on the shift $\bar{m}^{\tau,\xi}_{s,t}(x)$ in this more general setting, we will focus now on proving the Besov controls. First of all, we notice immediately that the proof of the first Besov control (Lemma 4.3) relies essentially only on the smoothing effect (Equation (3.6)) and thus, it can be obtained following the same reasoning above. The proof of the second Besov control (Lemma 5.2) in this framework is a bit more involved and we are going to explain it below more in details.
We start noticing that Lemma 5.2 can be reformulated for the new dynamics in the following way:

$$\begin{aligned}\int_{\mathbb{R}^{(n-1)d}} \Big\| D_{y_j} \cdot \Big\{ \mathbf{d}^{\vartheta}_x \bar{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\cdot) \otimes \bar{\Delta}^{\tau,\xi}_j(s,y_{\smallsetminus j},\cdot)\Big\}\Big\|_{B^{-(\alpha_j+\beta_j)}_{1,1}} dy_{\smallsetminus j} \\ \le\, C\|\bar{F}\|_H (s-t)^{\frac{\beta}{\alpha}-\sum_{k=1}^n \frac{\vartheta_k}{\alpha_k}},\end{aligned}$$

taking $(\tau,\xi) = (t,x)$, where we have denoted for simplicity

$$\bar{\Delta}^{\tau,\xi}_j(s,y) \,:=\, \bar{F}_j(s,y) - \bar{F}_j(s,\theta_{s,\tau}(\xi)) - D_{x_{j-1}}\bar{F}_j(s,\theta_{s,\tau}(\xi))\Big(y - \theta_{s,\tau}(\xi)\Big)_{j-1},$$

for any j in $[\![2,n]\!]$. The above control can be obtained mimicking the proof in the second Besov control (Lemma 5.2), exploiting this time that

$$|\bar{\Delta}^{\tau,\xi}_j(s,y)| \le C\|\bar{F}\|_H \mathbf{d}^{1+\alpha(j-2)+\beta}_{j-1:n}\Big(y, \bar{\theta}_{s,\tau}(\xi)\Big)$$

and the additional assumption [**P'**] in order to make the partial smoothing effect (Equation (5.30)) work in this framework, too.
The main difference in the proof is related to the control of the component $J_2(v, y_{\smallsetminus j}, z)$ appearing in Equation (5.35). Namely,

$$\begin{aligned}\int_{\mathbb{R}^d} D_z\partial_v p_h(v, z-y_j) \cdot \Big\{ \bar{\Delta}^{\tau,\xi}_j(s,y_{\smallsetminus j},z) \\ \otimes \int_0^1 D_{y_j} D^{\vartheta}_x \bar{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j}, z+\lambda(y_j - z)) \cdot (y_j - z)\Big\}\, d\lambda dy_j\end{aligned}$$

with our new notations. Indeed, the dependence of $\bar{F}$ on x_{j-1} pushes us to add a new term in the difference $|\bar{F}_j(s,y_{\smallsetminus j},z)-\bar{F}_j(s,\theta_{s,\tau}(\xi))|$ (now, $|\bar{\Delta}^{\tau,\xi}_j(s,y_{\smallsetminus j},z)|$) before splitting it up. In particular, it holds that

$$\begin{aligned}
|\bar{\Delta}^{\tau,\xi}_j(s,y_{\smallsetminus j},z)| \,&=\, \Big|\bar{F}_j(s,y_{\smallsetminus j},z)-\bar{F}_j(s,\theta_{s,\tau}(\xi))-D_{x_{j-1}}\bar{F}_j(s,\bar{\theta}_{s,\tau}(\xi))\big(y-\bar{\theta}_{s,\tau}(\xi)\big)_{j-1}\\
&\qquad\pm \bar{F}_j\big(s,y_{1:j-1},\big(\bar{\theta}_{s,\tau}(\xi)\big)_{j:n}\big)\Big|\\
&\le\, C\|\bar{F}\|_H\Big(|z-\big(\bar{\theta}_{s,\tau}(\xi)\big)_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}}+\sum_{k=j+1}^{n}|\big(y-\bar{\theta}_{s,\tau}(\xi)\big)_k|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(k-1)}}\\
&\qquad+|\big(y-\bar{\theta}_{s,\tau}(\xi)\big)_{j-1}|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-2)}}\Big)\\
&\le\, C\|\bar{F}\|_H\Big(|\lambda(z-y_j)|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}}+|z+\lambda(y_j-z)-\theta_{s,\tau}(\xi)_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}}\\
&\qquad+\sum_{k=j+1}^{n}|y-\bar{\theta}_{s,\tau}(\xi)_k|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(k-1)}}+|\big(y-\bar{\theta}_{s,\tau}(\xi)\big)_{j-1}|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-2)}}\Big)\\
&\le\, C\|\bar{F}\|_H\Big(|z-y_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}}+\mathbf{d}^{1+\alpha(j-2)+\beta}_{j+1:n}((y_{\smallsetminus j},z+\lambda(y_j-z)),\bar{\theta}_{s,\tau}(\xi))\Big).
\end{aligned}$$

The remaining part of the proof exactly matches the original method in Lemma 5.2.

Even in this more general framework, it is thus possible to obtain the following:

Theorem 7.1 (Well-posedness)**.** *Under* $[\bar{\boldsymbol{A}}]$*, there exists a unique mild solution* u *of IPDE* (7.1)*. Moreover, there exists a constant* $C:=C(T)$ *such that*

$$\|u\|_{L^\infty(C^{\alpha+\beta}_d)}\,\le\, C\Big[\|f\|_{L^\infty(C^\beta_{b,d})}+\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big].$$

7.2 Locally Hölder drift

This part is designed to give a brief explanation on how it is possible to deal with the general IPDE (7.1) when the drift $\bar{F}$ is only locally Hölder continuous in space. Namely, we assume with the notations in (2.21) that

[LR'] there exists a constant $K_0>0$ such that for any i in $[\![1,n]\!]$

$$\mathbf{d}(\bar{F}(t,x),\bar{F}(t,x'))\,\le\, K_0\mathbf{d}^{\beta+\gamma_i}(x,x'),\quad t\,\in[0,T],\ x,x'\in\mathbb{R}^{nd}\ \text{ s.t. }\ \mathbf{d}(x,x')<1.$$

In other words, it is required that $\bar{F}_i$ is in $L^\infty(0,T;C^{\beta+\gamma_i}(B(x_0,1/2)))$, uniformly in $x_0\in\mathbb{R}^{nd}$.

Under assumption $[\bar{\mathbf{A}}]$ (with condition [**R'**] replaced by [**LR'**]), it is possible to recover the Schauder-type estimates (Theorem 2.3), following the approach developed successfully in [CdRMP20a] for the non-degenerate, super-critical stable setting. Roughly speaking, in order to handle the local assumption, as well as the potentially unboundedness of the drift $\bar{F}$, we need to introduce a "localized" version of the Duhamel formulation (cf. Equation (3.16)). The key point here is to multiply a solution u by a suitable bump function $\bar{\eta}^{\tau,\xi}$ that "localizes" in space along the deterministic flow $\bar{\theta}_{t,\tau}(\xi)$ that characterizes the proxy. Namely, we fix a smooth function ρ that is equal to 1 on $B(0,1/2)$ and vanishes outside $B(0,1)$) and then define for any (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$,

$$\bar{\eta}^{\tau,\xi}(t,x)\,:=\,\rho(x-\bar{\theta}_{t,\tau}(\xi)).$$

We mention however that in the setting of [CdRMP20a], the "localization" with the cut-off function $\bar{\eta}^{\tau,\xi}$ is not simply motivated by the local Hölder continuity condition but it is also needed to give a proper meaning to the Duhamel formulation for a solution (cf. Proposition 3.5) when $\alpha < 1/2$, because of the low integrability properties of the underlying stable density. Such a problem does not however appear here since condition [**P**] forces us to consider only the case $\alpha > 1/2$.

Given a mild solution u of IPDE (7.1) and assuming $\bar{F}$ to be only locally Hölder continuous as in [**LR'**], it is possible to show, at least formally, that the function $\bar{v}^{\tau,\xi} := u\bar{\eta}^{\tau,\xi}$ solves the following equation on $(0,T) \times \mathbb{R}^{nd}$:

$$\begin{cases}\partial_t \bar{v}^{\tau,\xi}(t,x) + \langle \bar{F}(t,x), D_x\bar{v}^{\tau,\xi}(t,x)\rangle + \mathcal{L}_\alpha \bar{v}^{\tau,\xi}(t,x) \,=\, -\Big[\bar{\eta}^{\tau,\xi} f + \bar{\mathbb{S}}^{\tau,\xi}\Big](t,x); \\ \bar{v}^{\tau,\xi}(T,x) \,=\, \bar{\eta}^{\tau,\xi}(T,x)u_T(x),\end{cases} \tag{7.5}$$

where we have denoted

$$\begin{aligned}\bar{\mathbb{S}}^{\tau,\xi}(t,x) \,:=\, \int_{\mathbb{R}^d}\Big[u(t,x+By) - u(t,x)\Big]\Big[\bar{\eta}^{\tau,\xi}(t,t,x+By) - \bar{\eta}^{\tau,\xi}(t,x)\Big]\,\nu_\alpha(dy) \\ - u(t,x)\langle \bar{F}(t,x) - \bar{F}(t,\bar{\theta}_{t,\tau}(\xi)), D\rho(x-\bar{\theta}_{t,\tau}(\xi))\rangle.\end{aligned}$$

Equations as (7.5) can be essentially seen as a "local" version of the original one (7.1), depending on the freezing parameter (τ,ξ). In particular, it is important to notice that the difference

$$\bar{F}(t,x) - \bar{F}(t,\bar{\theta}_{t,\tau}(\xi))$$

appearing in the "localizing" error $\bar{\mathbb{S}}^{\tau,\xi}$ can be controlled exactly because it is multiplied by the derivative of the bump function ρ in the right point $x - \bar{\theta}_{t,\tau}(\xi)$, allowing us to exploit the *local* Hölder regularity. On the other hand, the first integral term in the r.h.s. can be seen as a commutator which involves only the non-degenerate variables and thus, that can be handled with interpolation techniques as in [CdRMP20a].

Even with the additional difficulty in controlling the remainder term, the perturbative approach explained in Section 3 can be applied, leading to show Schauder-type estimates as in Theorem 2.3 and the well-posedness of the IPDE (7.1) when assuming $\bar{F}$ to be only locally Hölder continuous.

Our procedure could be also used in order to establish Schauder-type estimates for the full Ornstein-Uhlenbeck operator as done, for example, in [Lun97] for the diffusive case. Indeed, a general operator of the form $\langle \bar{A}x, D_x\rangle + \mathcal{L}_\alpha$ can be treated, decomposing the matrix as $\bar{A} = A + U$ where A is, as before, the sub-diagonal matrix that makes the Ornstein-Ulhenbeck operator invariant by the dilation operator associated with the distance d, while U is an upper triangular matrix that could be seen as an additional *locally* Hölder term.

7.3 Diffusion coefficient

We conclude the article showing briefly how an additional diffusion term $\sigma\colon [0,T]\times\mathbb{R}^{nd} \to \mathbb{R}^d \otimes \mathbb{R}^d$ can be handled in the IPDE (7.1) with an operator $\mathcal{L}_{\alpha,t}$ of the form:

$$\mathcal{L}_{\alpha,t}\phi(t,x) \,:=\, \text{p.v.}\int_{\mathbb{R}^d}\Big[\phi(t,x+B\sigma(t,x)y) - \phi(t,x)\Big]\nu_\alpha(dy).$$

In this framework, it is quite standard (cf. [HWZ20] and [ZZ18]) to assume the Lévy measure ν_α to be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ i.e. $\nu_\alpha(dy) = f(y)dy$, for some Lipschitz function $f\colon \mathbb{R}^d \to \mathbb{R}$. In particular, since ν_α is a symmetric, α-stable, Lévy measure, it holds passing to polar coordinates $y = \rho s$ where $(\rho, s) \in [0,\infty) \times \mathbb{S}^{d-1}$ that

$$f(y) \;=\; \frac{g(s)}{\rho^{d+\alpha}}$$

for an even, Lipschitz function g on $\mathbb{S}^{d-1}$ (see also Equation (2.6)). Moreover, σ is considered uniformly elliptic and in $L^\infty(0,T;C^\beta(\mathbb{R}^n,\mathbb{R}))$.
Introducing now the "frozen" operator

$$\bar{\mathcal{L}}^{\tau,\xi}_{\alpha,t}\phi(t,x) \;=\; \text{p.v.}\int_{\mathbb{R}^d}\Big[\phi(t,x+B\sigma(t,\bar{\theta}_{t,\tau}(\xi))y)-\phi(t,x)\Big]\nu_\alpha(dy),$$

this would lead to consider for the IPDE an additional term in the Duhamel formula (cf. Equation (7.4)) that would write:

$$u(t,x) \;=\; \breve{P}^{\tau,\xi}_{T,t}u_T(x) + \int_t^T \breve{P}^{\tau,\xi}_{s,t}f(s,x) + \breve{P}^{\tau,\xi}_{s,t}\bar{R}^{\tau,\xi}(s,x) + \breve{P}^{\tau,\xi}_{s,t}\Big[\Big(\mathcal{L}_{\alpha,t}-\bar{\mathcal{L}}^{\tau,\xi}_{\alpha,t}\Big)u(s,\cdot)\Big](x)\,ds. \tag{7.6}$$

Here, $\{\breve{P}^{\tau,\xi}_{s,t}\}_{t\le s}$ denotes the two parameter semigroup associated with the proxy operator

$$\bar{L}^{\tau,\xi}_\alpha + \langle \bar{A}^{\tau,\xi}_t\Big(x-\bar{\theta}_{t,\tau}(\xi)\Big) + \bar{F}(t,\bar{\theta}_{t,\tau}(\xi)), D_x\rangle.$$

Let us focus on the last term in the integral of Equation (7.6). Looking back at the proof of the a priori estimates (Proposition 3.6), we notice in particular that we aim to establish the following control:

$$\Big|\Big(\mathcal{L}_{\alpha,t}-\bar{\mathcal{L}}^{\tau,\xi}_{\alpha,t}\Big)u(t,x)\Big| \;\le\; C\|\sigma\|_{L^\infty(C^\beta_{b,d})}\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\mathbf{d}^\beta(x,\bar{\theta}_{t,\tau}(\xi)) \tag{7.7}$$

in order to apply the same reasoning above in this new framework. To this end, we write that

$$\begin{aligned}
\Big(\mathcal{L}_{\alpha,t}-\bar{\mathcal{L}}^{\tau,\xi}_{\alpha,t}\Big)u(t,x) \;&=\; \text{p.v.}\int_{\mathbb{R}^d}\Big\{u(t,x+B\sigma(t,x)y)-u(t,x)\Big\}\,\nu_\alpha(dy)\\
&\qquad - \text{p.v.}\int_{\mathbb{R}^d}\Big\{u(t,x+B\sigma(t,\bar{\theta}_{t,\tau}(\xi))y)-u(t,x)\Big\}\,\nu_\alpha(dy)\\
&=\; \text{p.v.}\int_{\mathbb{R}^d}\Big\{u(t,x+Bz)-u(t,x)\Big\}\frac{f\Big(\sigma^{-1}(t,x)z\Big)}{\det\sigma(t,x)}\,dz\\
&\qquad - \int_{\mathbb{R}^d}\Big\{u(t,x+Bz)-u(t,x)\Big\}\frac{f\Big(\sigma^{-1}(t,\bar{\theta}_{t,\tau}(\xi))z\Big)}{\det\sigma(t,\bar{\theta}_{t,\tau}(\xi))}\,dz\\
&=\; \text{p.v.}\int_0^\infty\frac{1}{\rho^{1+\alpha}}\int_{\mathbb{S}^{d-1}}\Big\{u(t,x+B\rho s)-u(t,x)\Big\}\bar{D}^{\tau,\xi}(t,x,s)\,dsd\rho
\end{aligned}$$

where we have denoted, for notational convenience

$$\bar{D}^{\tau,\xi}(t,x,s) \;:=\; \left\{\frac{g\Big(\frac{\sigma^{-1}(t,x)s}{|\sigma^{-1}(t,x)s|}\Big)}{|\sigma^{-1}(t,x)s|^{d+\alpha}\det\sigma(t,x)} - \frac{g\Big(\frac{\sigma^{-1}(t,\bar{\theta}_{t,\tau}(\xi))s}{|\sigma^{-1}(t,\bar{\theta}_{t,\tau}(\xi))s|}\Big)}{|\sigma^{-1}(t,\bar{\theta}_{t,\tau}(\xi))s|^{d+\alpha}\det\sigma(t,\bar{\theta}_{t,\tau}(\xi))}\right\}.$$

Using now that g is Lipschitz and the assumptions on σ, we can show that

$$|\bar{D}^{\tau,\xi}(t,x,s)| \le C|\sigma(t,x)-\sigma(t,\bar{\theta}_{t,\tau}(\xi))| \le C\|\sigma\|_{L^\infty(C^\beta_{b,d})}\mathbf{d}^\beta(x,\bar{\theta}_{t,\tau}(\xi)). \tag{7.8}$$

Finally, Equation (7.7) follows from the previous controls using Taylor expansions and the symmetry condition on ν_α. Namely, considering the case $\alpha \ge 1$, which is the most delicate one for this part and precisely requires the symmetry of u_T, we write that

$$\begin{aligned}
\Big|\Big(\mathcal{L}_{\alpha,t}-\bar{\mathcal{L}}^{\tau,\xi}_{\alpha,t}\Big)u(t,x)\Big| &= \Big|\text{p.v.}\int_0^\infty \frac{1}{\rho^{1+\alpha}}\int_{\mathbb{S}^{d-1}}\Big\{u(t,x+B\rho s)-u(t,x)\Big\}\bar{D}^{\tau,\xi}(t,x,s)\,dsd\rho\Big| \\
&\le \Big|\text{p.v.}\int_{(0,1)} \frac{1}{\rho^{1+\alpha}}\int_{\mathbb{S}^{d-1}}\Big\{u(t,x+B\rho s)-u(t,x)\Big\}\bar{D}^{\tau,\xi}(t,x,s)\,dsd\rho\Big| \\
&\quad + \int_{(1,\infty)} \frac{1}{\rho^{1+\alpha}}\int_{\mathbb{S}^{d-1}}\Big|u(t,x+B\rho s)-u(t,x)\Big|\,|\bar{D}^{\tau,\xi}(t,x,s)|\,dsd\rho \\
&=: \Big[\bar{I}^{\tau,\xi}_s+\bar{I}^{\tau,\xi}_l\Big](t,x).
\end{aligned} \tag{7.9}$$

The *large jump* contribution $\bar{I}^{\tau,\xi}_l$ is easily handled from Equation (7.8). We get that

$$\begin{aligned}
\bar{I}^{\tau,\xi}_l(t,x) &\le 2C\|\sigma\|_{L^\infty(C^\beta_{b,d})}\|u\|_{L^\infty(L^\infty)}\mathbf{d}^\beta(x,\bar{\theta}_{t,\tau}(\xi)) \\
&\le 2C\|\sigma\|_{L^\infty(C^\beta_{b,d})}\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\mathbf{d}^\beta(x,\bar{\theta}_{t,\tau}(\xi)).
\end{aligned} \tag{7.10}$$

On the other hand, from the symmetry assumption on ν_α, which transfers to u_T, we can control the *small jump* contribution $\bar{I}^{\tau,\xi}_s$ through Taylor expansion and a centering argument. Indeed,

$$\begin{aligned}
&\bar{I}^{\tau,\xi}_s(t,x) \\
&= \Big|\text{p.v.}\int_{(0,1)}\frac{1}{\rho^{1+\alpha}}\int_{\mathbb{S}^{d-1}}\int_0^1\Big[D_{x_1}u(t,x+\lambda B\rho s)-D_{x_1}u(t,x)\Big]\rho s\bar{D}^{\tau,\xi}(t,x,s)\,d\lambda dsd\rho\Big| \\
&\le C\|\sigma\|_{L^\infty(C^\beta_{b,d})}\mathbf{d}^\beta(x,\bar{\theta}_{t,\tau}(\xi))\int_{(0,1)}\frac{1}{\rho^\alpha}\int_{\mathbb{S}^{d-1}}\int_0^1\Big|D_{x_1}u(t,x+\lambda B\rho s)-D_{x_1}u(t,x)\Big|\,d\lambda dsd\rho \\
&\le C\|\sigma\|_{L^\infty(C^\beta_{b,d})}\|D_{x_1}u\|_{L^\infty(C^{\alpha+\beta-1}_{b,d})}\mathbf{d}^\beta(x,\bar{\theta}_{t,\tau}(\xi))\int_{(0,1)}\frac{1}{\rho^\alpha}\rho^{\alpha+\beta-1}\,d\rho \\
&\le C\|\sigma\|_{L^\infty(C^\beta_{b,d})}\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\mathbf{d}^\beta(x,\bar{\theta}_{t,\tau}(\xi)).
\end{aligned} \tag{7.11}$$

Using Controls (7.10) and (7.11) in Equation (7.9), we obtain the expected bound (Equation (7.7)). We remark that the case $\alpha < 1$ could be handled similarly for the contribution $\bar{I}^{\tau,\xi}_l$ and even more directly for $\bar{I}^{\tau,\xi}_s$. Indeed, in that case, the centering argument is not needed since the Taylor expansion already yields an integrable singularity.

8 Appendix: proofs of complementary results

8.1 Smoothing effects for Ornstein-Ulhenbeck operator

We state and prove here some of the key properties of the Ornstein-Uhlenbeck operator. Namely, we will prove the representation (2.11) and the associated α-smoothing effect

(2.13). We highlight however that these results are only a slight modification to our purpose of those in [HMP19].

The two lemma below presents a deep connection with stochastic analysis and their proofs relies on tools that are more familiar in the probabilistic realm. For this reason, we are going to consider the stochastic counterpart of the Ornstein-Ulhenbeck operator L^{ou}. Namely, for a given starting point x in $\mathbb{R}^{nd}$, we are interested in the following dynamics

$$\begin{cases} dX_t \;=\; AX_t dt + BdZ_t, \quad \text{on } [0,T] \\ X_0 \;=\; x \end{cases} \tag{8.1}$$

where $(Z_t)_{t\geq 0}$ is an α-stable, $\mathbb{R}^{nd}$-dimensional process with Lévy measure ν_α, defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Lemma 8.1 (Representation)**.** *Under [**A**], the semigroup $\{P^{ou}_t\}_{t>0}$ generated by the Ornstein-Ulehnbeck operator L^{ou} (defined in* (2.9)*) admits for any fixed $t>0$, a density $p^{ou}(t,\cdot)$ which writes for any $t>0$ and any x,y in $\mathbb{R}^{nd}$*

$$p^{ou}(t,x,y) \;=\; \frac{1}{\det \mathbb{M}_t} p_S(t, \mathbb{M}_t^{-1}\Big(e^{At}x-y)\Big)$$

where $\mathbb{M}_t$ is the matrix defined in (2.12) *and p_S is the smooth density of an $\mathbb{R}^{nd}$-valued, symmetric and α-stable process S whose Lévy measure μ_S satisfies the non-degeneracy assumption [**ND**] on $\mathbb{R}^{nd}$.*

Proof. We start noticing that the above dynamics (8.1) can be explicitly integrated and gives

$$X_t \;=\; e^{tA}x + \int_0^t e^{(t-s)A} B\, dZ_s.$$

It is then readily derived from [PZ09] that, for any $t>0$, the random variable X_t has a density $p_X(t,x,\cdot)$ with respect to the Lebesgue measure on $\mathbb{R}^{nd}$ and it is moreover well known (see for example [Dyn65]) that p_X coincides with the density p^{ou} of the Ornstein-Ulhenbeck operator L^{ou} .
For this reason, we fix $t\geq 0$ and consider, for a given N in $\mathbb{N}$, a uniform partition $\{t_i\}_{i\in[\![0,N]\!]}$ of $[0,t]$. Then, it holds for any p in $\mathbb{R}^{nd}$,

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(i\langle p, \sum_{i=1}^N e^{(t-t_{i-1})A}B\Big(Z_{t_i}-Z_{t_{i-1}}\Big)\Big)\rangle\Big] \\ \;=\; \exp\Big(-\frac{1}{N}\sum_{i=1}^N \int_{\mathbb{S}^{d-1}} |\langle B^* e^{(t-t_{i-1})A^*}p, s\rangle|^\alpha \,\mu(ds)\Big)\end{aligned}$$

where μ is the spherical measure associated with ν_α (see Equation (2.7)). By dominated convergence theorem, we let m goes to infinity and show that

$$\mathbb{E}\Big[\exp\Big(i\langle p, \int_0^t e^{(t-s)A}B\, dZ_s\Big)\Big] \;=\; \exp\Big(-\int_0^t \int_{\mathbb{S}^{d-1}} |\langle e^{uA^*}p, Bs\rangle \,\mu(ds)du\Big).$$

Thanks to the above equation, we can rewrite the characteristic function of X_t as:

$$\begin{aligned}\psi_{X_t}(p) &= \mathbb{E}\Big[\exp\Big(i\langle p, e^{tA}x + \int_0^t e^{(t-s)A}B\,dZ_s\Big)\Big]\\ &= \exp\Big(i\langle p, e^{tA}x\rangle - \int_0^t\int_{\mathbb{S}^{d-1}} |\langle e^{uA^*}p, Bs\rangle|^\alpha\,\mu(ds)du\Big)\\ &= \exp\Big(i\langle p, e^{tA}x\rangle - t\int_0^1\int_{\mathbb{S}^{d-1}} |\langle e^{vtA^*}p, Bs\rangle|^\alpha\,\mu(ds)dv\Big)\end{aligned}$$

where in the last passage we used the change of variables $u = vt$. For the next step, we firstly notice that it holds

$$e^{tA} = \mathbb{M}_t e^A \mathbb{M}_t^{-1},$$

shown using the definition of matrix exponential and the trivial relation $\mathbb{M}_t A\mathbb{M}_t^{-1} = tA$. Exploiting the above identity, we then find that

$$\begin{aligned}\psi_{X_t}(p) &= \exp\Big(i\langle p, e^{tA}x\rangle - t\int_0^1\int_{\mathbb{S}^{d-1}} |\langle \mathbb{M}_t p, e^{vA}\mathbb{M}_t^{-1}Bs\rangle|^\alpha\,\mu(ds)dv\Big)\\ &= \exp\Big(i\langle p, e^{tA}x\rangle - t\int_0^1\int_{\mathbb{S}^{d-1}} |\langle \mathbb{M}_t p, e^{vA}Bs\rangle|^\alpha\,\mu(ds)dv\Big)\end{aligned}$$

where in the last passage we used the straightforward identity $\mathbb{M}_t^1 By = By$. We focus now only on the double integral

$$\int_0^1\int_{\mathbb{S}^{d-1}} |\langle \mathbb{M}_t p, e^{vA}Bs\rangle|^\alpha\,\mu(ds)dv.$$

If we consider the measure $m_\alpha(dv, ds) := |e^{vA}Bs|^\alpha\mu(ds)dv$ on $[0,1]\times\mathbb{S}^{d-1}$ and the normalized lift function $l\colon [0,1]\times\mathbb{S}^{d-1} \to \mathbb{S}^{nd-1}$ given by

$$l(v,s) := \frac{e^{vA}Bs}{|e^{vA}Bs|},$$

it then follows that

$$\begin{aligned}\int_0^1\int_{\mathbb{S}^{d-1}} |\langle \mathbb{M}_t p, e^{vA}Bs\rangle|^\alpha\,\mu(ds)dv &= \int_0^1\int_{\mathbb{S}^{d-1}} |\langle \mathbb{M}_t p, \frac{e^{vA}Bs}{|e^{vA}Bs|}\rangle|^\alpha\, m_\alpha(ds,dv)\\ &= \int_{\mathbb{S}^{nd-1}} |\langle \mathbb{M}_t p, \xi\rangle|^\alpha\,\mu_S(d\xi),\end{aligned}$$

where $\mu_S := \mathrm{Sym}(l_*(m_\alpha))$ is the symmetrized version of the measure m_α push-forwarded through l.
Noticing that μ_S is the Lévy measure of a symmetric α-stable process $\{S_t\}_{t\geq 0}$ satisfying assumption [**ND**] on $\mathbb{R}^{nd}$, we can finally write that

$$\psi_{X_t}(p) = \exp\Big(i\langle p, e^{tA}x\rangle - t\Phi_S(\mathbb{M}_t p)\Big)$$

where Φ_S is the Lévy symbol associated with S_t (cf. Equation (2.7)).
From Lemma A.1 in [HMP19], we know that under assumption [**ND**], the above calculations implies that

$$\int_0^1\int_{\mathbb{S}^{d-1}} \Big|(\mathbb{M}_t p)\cdot(e^{Av}Bs)\Big|^\alpha\,\mu_S(ds)dv \geq C|\mathbb{M}_t p|^\alpha$$

for some constant $C > 0$. It follows in particular that the function $p \to \psi_{X_t}(p)$ is in $L^1(\mathbb{R}^{nd})$. Thus, by inverse fourier transform and a change of variables, we can prove that

$$\begin{aligned}
\mathcal{F}^{-1}\Big[\psi_{X_t}\Big](y) &= \frac{1}{(2\pi)^{nd}}\int_{\mathbb{R}^{nd}} e^{-i\langle p,y\rangle}\exp\Big(i\langle p, e^{tA}x\rangle - t\Phi_S(\mathbb{M}_t p)\Big)\,dp \\
&= \frac{\det(\mathbb{M}_t^{-1})}{(2\pi)^{nd}}\int_{\mathbb{R}^{nd}} \exp\Big(-i\Big\langle \mathbb{M}_t^{-1}p, y - e^{tA}x\Big\rangle\Big)e^{-t\Phi(p)}\,dp \\
&= \frac{\det(\mathbb{M}_t^{-1})}{(2\pi)^{nd}}\int_{\mathbb{R}^{nd}} \exp\Big(-i\Big\langle p, \mathbb{M}_t^{-1}\Big(y - e^{tA}x\Big)\Big\rangle\Big)e^{-t\Phi(p)}\,dp \\
&= \frac{1}{\det \mathbb{M}_t} p_S(t, \mathbb{M}^{-1}(y - e^{At}x))
\end{aligned}$$

and we have concluded since p_S is symmetric. □

We can now point out the smoothing effect (Equation (2.13)) associated with the Ornstein-Uhlenbeck density p^{ou}.

Lemma 8.2 (Smoothing effect)**.** *Under [$\boldsymbol{A}$], there exists a family $\{q(t,\cdot)\colon t \in [0,T]\}$ of densities on $\mathbb{R}^{nd}$ such that*

- *for any l in $[\![0,3]\!]$, there exists a constant $C := C(l, nd)$ such that $|D^l_y p_S(t,y)| \le Cq(t,y)t^{-l/\alpha}$ for any t in $[0,T]$ and any y in $\mathbb{R}^{nd}$;*
- *(stable scaling property) $q(t,y) = t^{-nd/\alpha}q(1, t^{-1/\alpha}y)$ for any t in $[0,T]$ and any y in $\mathbb{R}^{nd}$;*
- *(stable smoothing effect) for any γ in $[0,\alpha)$, there exists a constant $c := c(\gamma, nd)$ such that*
$$\int_{\mathbb{R}^{nd}} q(t,y)\,|y|^\gamma\,dy \;\le\; ct^{\gamma/\alpha} \text{ for any } t > 0. \tag{8.2}$$

Proof. Fixed a time $t > 0$, we start applying the Ito-Lévy decomposition to S at the associated characteristic stable time scale, i.e. we choose to truncate at threshold $t^{1/\alpha}$, so that we can write $S_t = M_t + N_t$ for some M_t, N_t independent random variables corresponding to the small jumps part and the large jumps part, respectively. Namely, we denote for any $s > 0$

$$N_s := \int_0^s \int_{|x|>t^{1/\alpha}} xP(du,dx) \quad\text{and}\quad M_s := S_s - N_s$$

where P is the Poisson random measure associated with the process S. We can thus rewrite the density p_S in the following way

$$p_S(t,x) = \int_{\mathbb{R}^{nd}} p_M(t, x-y)P_{N_t}(dy)$$

where $p_M(t,\cdot)$ corresponds to the density of M_t and P_{N_t} is the law of N_t.
It is important now to notice that it is precisely our choice of the cutting threshold $t^{1/\alpha}$ that gives M and N the α-similarity property (for any fixed t)

$$N_t \overset{law}{=} t^{1/\alpha}N_1 \quad\text{and}\quad M_t \overset{law}{=} t^{1/\alpha}M_1$$

we will need below. Indeed, to show the assertion for N, we can start from the Lévy-Khintchine formula for the characteristic function of N:

$$\mathbb{E}\Big[e^{i\langle p,N_t\rangle}\Big] \;=\; \exp\Big[t\int_{\mathbb{S}^{nd-1}}\int_{t^{1/\alpha}}^{\infty}\Big(\cos(\langle p,r\xi\rangle)-1\Big)\frac{dr}{r^{1+\alpha}}\overline{\mu}_S(d\xi)\Big]$$

for any p in $\mathbb{R}^{nd}$. We then use the change of variable $rt^{-1/\alpha}=s$ to get that

$$\mathbb{E}\Big[e^{i\langle p,N_t\rangle}\Big] \;=\; \mathbb{E}\Big[e^{i\langle p,t^{1/\alpha}N_1\rangle}\Big].$$

This implies in particular our assertion on N. In a similar way, it is possible to get the analogous assertion on M.
From lemma A.2 in [HMP19] with $m=3$, we know that there exist a family $\{p_{\overline{M}}(t,\cdot)\}_{t>0}$ of densities on $\mathbb{R}^{nd}$ and a constant $C:=C(m,\alpha)$ such that

$$|D^l_y p_M(t,y)| \;\le\; Cp_{\overline{M}}(t,y)t^{-l/\alpha}$$

for any $t>0$, any x in $\mathbb{R}^{nd}$ and any $l\in\{0,1,2\}$.
Moreover, denoting $\overline{M}_t$ the random variable with density $p_{\overline{M}}(t,\cdot)$ and independent from N_t, we can easily check from $p_{\overline{M}}(t,y)=t^{-nd/\alpha}p_{\overline{M}}(1,t^{-1/\alpha}x)$ that $\overline{M}$ is α-selfsimilar

$$\overline{M}_t \overset{law}{=} t^{1/\alpha}\overline{M}_1.$$

We can finally define the family $\{q(t,\cdot)\}_{t>0}$ of densities as

$$q(t,x) \;:=\; \int_{\mathbb{R}^{nd}} p_{\overline{M}}(t,x-y)P_{N_t}(dy)$$

corresponding to the density of the random variable

$$\overline{S}_t \,;=\; \overline{M}_t+N_t$$

for any fixed $t>0$. Using Fourier transform and the already proven α-selfsimilarity of $\overline{M}$ and N, we can show now that

$$\overline{S}_t \overset{law}{=} t^{1/\alpha}\overline{S}_1$$

or equivalently, that

$$q(t,y)=t^{-nd/\alpha}q(1,t^{-1/\alpha}y)$$

for any t in $[0,T]$ and any y in $\mathbb{R}^{nd}$. Moreover,

$$\mathbb{E}[|\overline{S}_t|^\gamma] \;=\; \mathbb{E}[|\overline{M}_t+N_t|^\gamma] \;=\; Ct^{\gamma/\alpha}\Big(\mathbb{E}[|\overline{M}_1|^\gamma]+\mathbb{E}[|N_t|^\gamma]\Big) \;\le\; Ct^{\gamma/\alpha}.$$

This shows in particular that equation (8.2) holds. □

We conclude this sub-section showing Control (5.50) appearing in the proof of Proposition 3.6 for the diagonal regime. First of all, we will need the following lemma:

Lemma 8.3. *Let t in $[0,T]$, x,b in $\mathbb{R}^{nd}$ such that $|b|\le ct^{1/\alpha}$ for some constant $c>0$. Under [**A**], there exists a constant $C:=C(c)$ such that*

$$|D^l_x p_S\Big(t,x+b)| \;\le\; \tilde{C}|D^l_x p_S\Big(t,x)|$$

Proof. Looking back at the proof of the previous lemma 8.2, we know that

$$D^l_x p_S(t,x+b) \;=\; \int_{\mathbb{R}^{nd}} D^l_x p_M(t,x+b-y)P_{N_t}(dy)$$

where $p_M(t,\cdot)$ is the density of M_t and P_{N_t} is the law of N_t, corresponding to the small and big jumps in the Ito-Lévy decomposition.
From lemma $A.2$ in [HMP19] we know moreover that

$$|D^l_x p_M(t,x+b-y)| \;\le\; \frac{C}{t^{\frac{l}{\alpha}}} p_{\overline{M}}(t,x+b-y) \;\text{ where }\; p_{\overline{M}}(t,z) \;=\; \frac{C}{t^{\frac{nd}{\alpha}}}\frac{1}{\left(1+\frac{|z|}{t^{\frac{1}{\alpha}}}\right)^3}.$$

It is then enough to show that

$$\begin{aligned}
p_{\overline{M}}(t,z+b) \;&=\; \frac{C}{t^{\frac{nd}{\alpha}}}\frac{1}{\left(1+\frac{|z+b|}{t^{\frac{1}{\alpha}}}\right)^3} \;\le\; \frac{\tilde{C}}{t^{\frac{nd}{\alpha}}}\frac{1}{\left(1+c+\frac{|z+b|}{t^{\frac{1}{\alpha}}}\right)^3}\\
&\le\; \frac{C}{t^{\frac{nd}{\alpha}}}\frac{1}{\left(1+c\frac{|z|}{t^{\frac{1}{\alpha}}}-\frac{|b|}{t^{\frac{1}{\alpha}}}\right)^3} \;\le\; \frac{C}{t^{\frac{nd}{\alpha}}}\frac{1}{\left(1+\frac{|z|}{t^{\frac{1}{\alpha}}}\right)^3}\\
&\le\; Cp_{\overline{M}}(t,z).
\end{aligned}$$

to conclude the proof. □

Proof of Equation (5.50). We start looking back to the proof of Lemma 3.2 to find that

$$\begin{aligned}
&\left|D^{\vartheta}_x \tilde{p}^{\tau,\xi'}(t,s,x+\lambda(x'-x),y)\right|\\
&\qquad =\; C(s-t)^{-\sum_{k=1}^n \frac{\vartheta_k}{\alpha_k}}\frac{1}{\det \mathbb{M}_{s-t}}\left|\mathbf{d}^{|\vartheta|}_z p_S\Big(s-t,\cdot)(\mathbb{M}^{-1}_{s-t}(\tilde{m}^{\tau,\xi}_{s,t}(x)-y)\Big)\right|
\end{aligned}$$

Moreover, we notice that

$$\mathbb{M}^{-1}_{s-t}\Big(\tilde{m}^{\tau,\xi}_{s,t}(x+\lambda(x-x'))-y\Big) \;=\; \mathbb{M}^{-1}_{s-t}\Big(\tilde{m}^{\tau,\xi}_{s,t}(x)-y\Big)+\lambda\mathbb{M}^{-1}_{s-t}e^{A(s-t)}(x-x').$$

Then, Control (5.50) follows immediately from the previous lemma once we have shown that

$$\left|\lambda\mathbb{M}^{-1}_{s-t}e^{A(s-t)}(x-x')\right| \;\le\; C(s-t)^{1/\alpha}$$

for some constant $C:=C(A)$. Indeed, fixed i in $[\![1,n]\!]$, we can exploit the structure of A and $\mathbb{M}_{s-t}$ (cf. Equation (2.18) in Scaling Lemma 2.1) to write that

$$\begin{aligned}
\Big[\mathbb{M}^{-1}_{s-t}e^{A(s-t)}(x-x')\Big]_i \;&=\; \sum_{j=1}^n\sum_{k=1}^n\Big[\mathbb{M}^{-1}_{s-t}\Big]_{i,k}\Big[e^{A(s-t)}\Big]_{k,j}(x-x')_j\\
&=\; \sum_{j=i}^n (s-t)^{-(i-1)}C_j(s-t)^{i-j}(x-x')_j.
\end{aligned}$$

Since moreover we assumed to be in a local diagonal regime, i.e. $\mathbf{d}^\alpha(x,x') \le (s-t)^{1/\alpha}$, we have that

$$
\begin{aligned}
\Big|\Big[\mathbb{M}_{s-t}^{-1}e^{A(s-t)}(x-x')\Big]_i\Big| &\le C\sum_{j=i}^{n}(s-t)^{-(j-1)}|(x-x')_j| \\
&\le C\sum_{j=i}^{n}(s-t)^{-(j-1)}(s-t)^{\frac{1+\alpha(j-1)}{\alpha}} \\
&= C(s-t)^{1/\alpha}.
\end{aligned}
$$

The proof is thus concluded. □

8.2 Technical tools

In this section, we present the proof of some technical results already used in the article, for the sake of completeness.
We recall moreover that the results below can be proven also for the flow $\bar{\theta}_{s,\tau}(\xi)$ driven by a more general perturbation F under assumption $[\bar{\mathbf{A}}]$ (cf. Section 7.1), exploiting that $\bar{F}_i$ is Lipschitz continuous in the x_{i-1} variable for any i in $[\![2,n]\!]$.

We begin proving Lemma 5.4 about the sensitivity of the Hölder flows, appearing in the proof of the a priori estimates (3.18) of Proposition 3.6. For this reason, we will assume from this point further to be under assumption [**A'**].

Proof of Lemma 5.4. We start noticing that our result follows immediately using Young inequality, once we have shown that it holds

$$
\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_i\Big| \le C\Big[(s-t)^{\frac{1+\alpha(i-1)}{\alpha}} + \mathbf{d}^{1+\alpha(i-1)}(x,x')\Big] \quad \text{for any } i \text{ in } [\![1,n]\!]. \tag{8.3}
$$

Our proof will rely essentially in iterative applications of the Grönwall lemma. We notice however that under [**A**], the perturbation F_i is only Hölder continuous with respect to its i-th variable. To overcome this problem, we are going to mollify (but only with respect to the variable of interest) the function F in the following way: fixed a mollifier ρ on $\mathbb{R}^d$, i.e. a compactly supported, non-negative, smooth function such that $\|\rho\|_{L^1}=1$ and a family δ_i of positive constants to be chosen later, the mollified version of the perturbation is given by $F^\delta = (F_1, F_2^{\delta_2},\dots,F_n^{\delta_n})$ where

$$
F_i^{\delta_i}(t,z_{i:n}) := F_i *_i \rho_{\delta_i}(t,z_{i:n}) = \int_{\mathbb{R}^d} F_i(t,z_i-\omega,z_{i+1},\dots,z_n)\frac{1}{\delta_i^d}\rho(\frac{\omega}{\delta_i})\,d\omega.
$$

We remark in particular that we do not need to mollify the first component F_1 since it is regular enough, say β-Hölder continuous in the first d-dimensional variable x_1, by assumption [**R**].
Then, standard results on mollifier theory and our current assumptions on F show us

that the following controls hold

$$
\begin{aligned}
|F_i(u,z)-F_i^\delta(u,z)| &\le \|F_i\|_{L^\infty(C_d^{\gamma+\beta})}\delta_i^{\frac{\gamma_i+\beta}{1+\alpha(i-1)}}, &(8.4)\\
|F_i^\delta(u,z)-F_i^\delta(u,z')| &\le C\|F_i\|_{L^\infty(C_d^{\gamma+\beta})}\Big[\delta_i^{\frac{\gamma_i+\beta}{1+\alpha(i-1)}-1}|(z-z')_i|+\sum_{j=i+1}^{n}|(z-z')_j|^{\frac{\gamma_i+\beta}{1+\alpha(j-1)}}\Big]. &(8.5)
\end{aligned}
$$

We choose now δ_i for any i in $[\![2,n]\!]$ in order to have any contribution associated with the mollification appearing in (8.4) at a good current scale time. Namely, we would like δ_i to satisfy

$$\Big|\Big((s-t)^{\frac{1}{\alpha}}\mathbb{M}_{s-t}\Big)^{-1}\Big(F(u,z)-F^\delta(u,z)\Big)\Big| \le C(s-t)^{-1}$$

for any u in $[t,s]$ and any z in $\mathbb{R}^{nd}$. Using the mollifier controls (8.4), it is enough to ask for

$$\sum_{i=2}^{n}(s-t)^{-\frac{1}{\alpha_i}}\delta_i^{\frac{\gamma_i+\beta}{1+\alpha(i-1)}} \le C(s-t)^{-1}.$$

Recalling that $\gamma_i := 1+\alpha(i-2)$ by assumption [**R**], this is true if we fix for example,

$$\delta_i = (s-t)^{\frac{\gamma_i}{\alpha}\frac{1+\alpha(i-1)}{\gamma_i+\beta}} \quad \text{for } i \text{ in } [\![2,n]\!]. \tag{8.6}$$

After this introductive part, we start controlling the last component of the flow. By construction of $\theta_{s,t}$, we can write that

$$
\begin{aligned}
&\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_n\Big| \qquad\qquad (8.7)\\
&\quad=\Big|(x-x')_n+\int_t^s\Big\{\Big[A(\theta_{v,t}(x)-\theta_{v,t}(x'))\Big]_n+F_n(v,\theta_{v,t}(x))-F_n(v,\theta_{v,t}(x'))\Big\}\,dv\Big|\\
&\quad\le |(x-x')_n|\\
&\qquad+\int_t^s\Big\{A_{n,n-1}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-1}|+\Big|F_n(v,\theta_{v,t}(x))-F_n(v,\theta_{v,t}(x'))\Big|\Big\}\,dv
\end{aligned}
$$

where in the last passage we have exploited the sub-diagonal structure of A (cf. Equation (1.2)). If we focus only on the last term involving the difference of the drifts, It holds now that

$$
\begin{aligned}
\Big|F_n(v,\theta_{v,t}(x))-F_n(v,\theta_{v,t}(x'))\Big| &\le \Big|F_n(v,\theta_{v,t}(x))-F_n^\delta(v,\theta_{v,t}(x))\Big|\\
&\quad+\Big|F_n(v,\theta_{v,t}(x'))-F_n^\delta(v,\theta_{v,t}(x'))\Big|+\Big|F_n^\delta(v,\theta_{v,t}(x))-F_n^\delta(v,\theta_{v,t}(x'))\Big|.
\end{aligned}
$$

Using the controls (8.4), (8.5) on the mollified drifts, we then write from (8.7) and the previous equation that

$$
\begin{aligned}
\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_n\Big| &\le |(x-x')_n|+2(s-t)\delta_n^{\frac{\gamma_n+\beta}{1+\alpha(n-1)}}\\
&\quad+C\int_t^s\Big\{\Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-1}\Big|+\delta_n^{\frac{\gamma_n+\beta}{1+\alpha(n-1)}-1}\Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_n\Big|\Big\}\,dv.
\end{aligned}
$$

We now apply the Grönwall lemma to show that

$$\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_n\Big| \le C\Big[|(x-x')_n| + (s-t)\delta_n^{\frac{\gamma_n+\beta}{1+\alpha(n-1)}} + \int_t^s \Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-1}\Big|\, dv\Big].$$

From our previous choice for δ_n (cf. Equation (8.6)), we know that

$$(s-t)^{-\frac{1}{\alpha_n}}\delta_n^{\frac{\gamma_n+\beta}{1+\alpha(n-1)}} \le C(s-t)^{-1}$$

and thus, we can rewrite the last inequality as

$$\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_n\Big| \le C\Big[\Big|(x-x')_n\Big| + (s-t)^{\frac{1+\alpha(n-1)}{\alpha}} + \int_t^s \Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-1}\Big|\, dv\Big]. \quad (8.8)$$

We would like now to obtain a similar control on the $(n-1)$-th term. As already done at the beginning of the proof, we can write that

$$\begin{aligned}\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_{n-1}\Big| \le{}& \Big|(x-x')_{n-1}\Big| + C\delta_{n-1}^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-2)}}(s-t) + \int_t^s \Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-2}\Big| \\ &+ \delta_{n-1}^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-2)}-1}\Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-1}\Big| + \Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_n\Big|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}}\, dv\end{aligned}$$

We then apply the Grönwall lemma to find that

$$\begin{aligned}\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_{n-1}\Big| \le{}& C\Big[\Big|(x-x')_{n-1}\Big| + \delta_{n-1}^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-2)}}(s-t) \\ &+ \int_t^s \Big\{\Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-2}\Big| + \Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_n\Big|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}}\Big\}\, dv\Big].\end{aligned}$$

Remembering our previous choice of δ_{n-1}, it holds now that

$$\begin{aligned}\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_{n-1}\Big| \le{}& C\Big[|(x-x')_{n-1}| + (s-t)^{\frac{1+\alpha(n-2)}{\alpha}} + \int_t^s \Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-2}\Big| \\ &+ \Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_n\Big|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}}\, dv\Big]. \quad (8.9)\end{aligned}$$

We then use equation (8.8) and the Jensen inequality to write

$$\begin{aligned}&\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_{n-1}\Big| \\ &\le C\Big[|(x-x')_{n-1}| + (s-t)^{\frac{1+\alpha(n-2)}{\alpha}} + \int_t^s \Big\{\Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-2}\Big| + \Big|(x-x')_n\Big|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} \\ &\quad + (v-t)^{\frac{\gamma_{n-1}+\beta}{\alpha}} + \Big(\int_t^v \Big|(\theta_{\omega,t}(x)-\theta_{\omega,t}(x'))_{n-1}\Big|\, d\omega\Big)^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}}\Big\}\, dv\Big]. \quad (8.10)\end{aligned}$$

The idea now is to use Grönwall lemma again. To do so, we firstly move the exponent from the last integral term involving the $(n-1)$-th term using the Young inequality:

$$\begin{aligned}&\Big(\int_t^v \Big|(\theta_{\omega,t}(x)-\theta_{\omega,t}(x'))_{n-1}\Big|\, d\omega\Big)^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} \\ &\qquad\le B^{-\frac{1+\alpha(n-1)}{\gamma_{n-1}+\beta}}\int_t^v \Big|(\theta_{\omega,t}(x)-\theta_{\omega,t}(x'))_{n-1}\Big|\, d\omega + B^{\frac{1+\alpha(n-1)}{2\alpha-\beta}}\end{aligned}$$

for a quantity B to be fixed later.
Since we need homogeneity with respect to time in equation (8.9), we choose B such that

$$B^{\frac{1+\alpha(n-1)}{2\alpha-\beta}} = (v-t)^{\frac{\gamma_{n-1}+\beta}{\alpha}} \Leftrightarrow B = (v-t)^{\frac{\gamma_{n-1}+\beta}{\alpha}\frac{2\alpha-\beta}{1+\alpha(n-1)}}.$$

Plugging it into the general expression in (8.10), we find that

$$\begin{aligned}
\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_{n-1}\Big| &\le C\Big[|(x-x')_{n-1}| + (s-t)^{\frac{1+\alpha(n-2)}{\alpha}} \\
&\quad + \int_t^s \Big\{\Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-2}\Big| + \Big|(x-x')_n\Big|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} + (v-t)^{\frac{\gamma_{n-1}+\beta}{\alpha}} \\
&\qquad + (v-t)^{\frac{\beta}{\alpha}-2}\int_t^v \Big|(\theta_{\omega,t}(x)-\theta_{\omega,t}(x'))_{n-1}\Big| d\omega\Big\} dv\Big] \\
&\le C\Big[|(x-x')_{n-1}| + (s-t)^{\frac{1+\alpha(n-1)}{\alpha}} + (s-t)\Big|(x-x')_n\Big|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} + (s-t)^{\frac{\gamma_{n-1}+\beta+\alpha}{\alpha}} \\
&\quad + \int_t^s \Big\{\Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-2}\Big| + (v-t)^{\frac{\beta}{\alpha}-1}\sup_{\omega\in[t,v]}\Big|(\theta_{\omega,t}(x)-\theta_{\omega,t}(x'))_{n-1}\Big|\Big\} dv\Big].
\end{aligned}$$

Since the previous inequality is also true for any $\overline{s}$ in $[t,s]$, it follows that

$$\begin{aligned}
&\sup_{\overline{s}\in[0,s]}\Big|(\theta_{\overline{s},t}(x)-\theta_{\overline{s},t}(x'))_{n-1}\Big| \\
&\quad\le C\Big[|(x-x')_{n-1}| + (s-t)^{\frac{1+\alpha(n-2)}{\alpha}} + (s-t)\Big|(x-x')_n\Big|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} + (s-t)^{\frac{\gamma_{n-1}+\beta+\alpha}{\alpha}} \\
&\qquad + \int_t^s \Big\{\Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-2}\Big| + (v-t)^{\frac{\beta}{\alpha}-1}\sup_{\omega\in[t,v]}\Big|(\theta_{\omega,t}(x)-\theta_{\omega,t}(x'))_{n-1}\Big|\Big\} dv\Big].
\end{aligned}$$

We can finally apply the Grönwall lemma to show that for any s in $[t,T]$, there exists a constant C such that

$$\begin{aligned}
\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_{n-1}\Big| &\le C\Big[|(x-x')_{n-1}| + (s-t)^{\frac{1+\alpha(n-2)}{\alpha}} + (s-t)|(x-x')_n|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} \\
&\quad + \int_t^s \Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-2}\Big| dv\Big].
\end{aligned}$$

Moreover, thanks to the Young inequality we know that

$$(s-t)\Big|(x-x')_n\Big|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} \le C\Big\{(s-t)^{\frac{1+\alpha(n-2)}{\alpha}} + |(x-x')_n|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}\frac{1+\alpha(n-2)}{1+\alpha(n-3)}}\Big\}$$

and remembering that $\mathbf{d}(x,x') \le 1$ by hypothesis,

$$|(x-x')_n|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}\frac{1+\alpha(n-2)}{1+\alpha(n-3)}} \le |(x-x')_n|^{\frac{\gamma_{n-1}+\beta}{\gamma_{n-1}}\frac{1+\alpha(n-2)}{1+\alpha(n-1)}} \le |(x-x')_n|^{\frac{1+\alpha(n-2)}{1+\alpha(n-1)}}.$$

We then use it to write for any v in $[t,T]$,

$$\begin{aligned}
\Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-1}\Big| &\le C\Big[|(x-x')_{n-1}| + (v-t)^{\frac{1+\alpha(n-2)}{\alpha}} + |(x-x')_n|^{\frac{1+\alpha(n-2)}{1+\alpha(n-1)}} \\
&\quad + \int_t^v \Big|(\theta_{\omega,t}(x)-\theta_{\omega,t}(x'))_{n-2}\Big| d\omega\Big].
\end{aligned}$$

Going back to equation (8.8), we plug in the last bound to find that

$$\begin{aligned}\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_n\Big| \,&\le\, C\Big[|(x-x')_n|+(s-t)^{\frac{1+\alpha(n-1)}{\alpha}}+(s-t)|(x-x')_{n-1}|\\&\quad+(s-t)|(x-x')_n|^{\frac{1+\alpha(n-2)}{1+\alpha(n-1)}}+\int_t^s\int_t^v\Big|(\theta_{t,\omega}(x)-\theta_{\omega,t}(x'))_{n-2}\Big|\,d\omega dv\Big]\\&\le\, C\Big[|(x-x')_n|+(s-t)^{\frac{1+\alpha(n-1)}{\alpha}}+|(x-x')_{n-1}|^{\frac{1+\alpha(n-1)}{1+\alpha(n-2)}}\\&\quad+\int_t^s\int_t^v\Big|(\theta_{\omega,t}(x)-\theta_{\omega,t}(x'))_{n-2}\Big|\,d\omega dv\Big]\end{aligned}$$

where in the last passage we used again the Young inequality to show that

$$(s-t)|(x-x')_{n-1}|\,\le\, C(s-t)^{\frac{1+\alpha(n-1)}{\alpha}}+|(x-x')_{n-1}|^{\frac{1+\alpha(n-1)}{1+\alpha(n-2)}}$$

and

$$(s-t)|(x-x')_n|^{\frac{1+\alpha(n-2)}{1+\alpha(n-1)}}\,\le\, C(s-t)^{\frac{1+\alpha(n-1)}{\alpha}}+|(x-x')_n|.$$

This approach may be naturally iterated up to the first term of the chain, so that

$$\begin{aligned}\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_n\Big|\,&\le\, C\Big[\sum_{j=2}^n|(x-x')_j|^{\frac{1+\alpha(n-1)}{1+\alpha(j-1)}}+(s-t)^{\frac{1+\alpha(n-1)}{\alpha}}\\&\quad+\int_t^{v_n=s}dv_{n-1}\dots\int_t^{v=2}dv_1\Big|(\theta_{v_1,t}(x)-\theta_{v_1,t}(x'))_1\Big|\Big].\end{aligned}$$

In a similar manner, we can show for any i in $[\![2,n]\!]$,

$$\begin{aligned}\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_i\Big|\,&\le\, C\Big[\sum_{j=2}^n|(x-x')_j|^{\frac{1+\alpha(i-1)}{1+\alpha(j-1)}}+(s-t)^{\frac{1+\alpha(i-1)}{\alpha}}\\&\quad+\int_t^{v_i=s}dv_{i-1}\dots\int_t^{v=2}dv_1\Big|(\theta_{v_1,t}(x)-\theta_{v_1,t}(x'))_1\Big|\Big].\end{aligned}\tag{8.11}$$

Since all the non-integral terms in (8.11) are compatible with the statement of the lemma, it remains to find the proper bound for the first component of the flow. As before, let us consider $\overline{s}$ in $[t,s]$. We can write

$$|(\theta_{\overline{s},t}(x)-\theta_{\overline{s},t}(x'))_1|\,\le\,|(x-x')_1|+C\sum_{j=1}^n\int_t^{\overline{s}}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_j|^{\frac{\beta}{1+\alpha(j-1)}}\,dv$$

or, passing to the supremum on both sides,

$$\begin{aligned}\sup_{\overline{s}\in[t,s]}|(\theta_{\overline{s},t}(x)-\theta_{\overline{s},t}(x'))_1|\,&\le\,|(x-x')_1|+C\Big\{(s-t)\Big(\sup_{v\in[t,s]}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_1|\Big)^{\beta}\\&\quad+\sum_{j=2}^n\int_t^s|(\theta_{v,t}(x)-\theta_{v,t}(x'))_j|^{\frac{\beta}{1+\alpha(j-1)}}\,dv\Big\}.\end{aligned}$$

Using equation (8.11), it holds now that

$$\begin{aligned}\sup_{\overline{s}\in[t,s]}(|\theta_{\overline{s},t}(x)-\theta_{\overline{s},t}(x'))_1| \;\le\; & |(x-x')_1| + C\Big\{(s-t)\Big(\sup_{v\in[t,s]}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_1|\Big)^{\beta} \\ & + \sum_{j=2}^{n}\Big[(s-t)\Big((s-t)^{\frac{1+\alpha(j-1)}{\alpha}} + \sum_{k=2}^{n}|(x-x')_k|^{\frac{1+\alpha(j-1)}{1+\alpha(k-1)}} \\ & + (s-t)^{j-1}\sup_{v\in[t,s]}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_1|\Big)^{\frac{\beta}{1+\alpha(j-1)}}\Big]\Big\}.\end{aligned} \tag{8.12}$$

We then apply the Jensen inequality to show that

$$\begin{aligned}\sup_{\overline{s}\in[t,s]}(|\theta_{\overline{s},t}(x)-\theta_{\overline{s},t}(x'))_1| \;\le\; & |(x-x')_1| + C\Big\{(s-t)\Big[\sup_{v\in[t,s]}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_1|\Big]^{\beta} \\ & + \sum_{j=2}^{n}C(s-t)\Big[(s-t)^{\frac{\beta}{\alpha}} + \sum_{k=2}^{n}|(x-x')_k|^{\frac{\beta}{1+\alpha(k-1)}} \\ & + (s-t)^{\frac{(j-1)\beta}{1+\alpha(j-1)}}\sup_{v\in[t,s]}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_1|^{\frac{\beta}{1+\alpha(j-1)}}\Big]\Big\} \\ \le\; & C\Big\{|(x-x')_1| + (s-t)^{\frac{\alpha+\beta}{\alpha}} + (s-t)\sum_{k=2}^{n}|(x-x')_k|^{\frac{\beta}{1+\alpha(k-1)}} \\ & + \sum_{j=1}^{n}(s-t)^{1+\frac{(j-1)\beta}{1+\alpha(j-1)}}\sup_{v\in[t,s]}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_1|^{\frac{\beta}{1+\alpha(j-1)}}\Big\}.\end{aligned} \tag{8.13}$$

From Young inequality, we can deduce now that

$$(s-t)|(x-x')_k|^{\frac{\beta}{1+\alpha(k-1)}} \;\le\; C\Big((s-t)^{\frac{1}{1-\beta}} + |(x-x')_k|^{\frac{1}{1+\alpha(k-1)}}\Big)$$

and

$$\begin{aligned}&(s-t)^{1+\frac{(j-1)\beta}{1+\alpha(j-1)}}\sup_{v\in[t,s]}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_1|^{\frac{\beta}{1+\alpha(j-1)}} \\ &\qquad\le C\Big\{(s-t)^{\frac{1+(\alpha+\beta)(j-1)}{1+\alpha(j-1)-\beta}} + \sup_{v\in[t,s]}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_1|\Big\}\end{aligned}$$

Plugging these inequalities into the main one (8.13), we find that

$$\begin{aligned}\sup_{\overline{s}\in[t,s]}(|\theta_{\overline{s},t}(x)-\theta_{\overline{s},t}(x'))_1| \;\le\; & C\Big\{|(x-x')_1| + (s-t)^{\frac{\alpha+\beta}{\alpha}} + \sum_{k=2}^{n}|(x-x')_k|^{\frac{1}{1+\alpha(k-1)}} \\ & + \sum_{j=1}^{n}(s-t)^{\frac{1+(\alpha+\beta)(j-1)}{1+\alpha(j-1)-\beta}} + \sup_{v\in[t,s]}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_1|\Big\} \\ \le\; & C\Big\{(s-t)^{\frac{\alpha+\beta}{\alpha}} + (s-t)^{\frac{1}{1-\beta}} + d(x,x') \\ & + \sum_{j=1}^{n}(s-t)^{\frac{1+(\alpha+\beta)(j-1)}{1+\alpha(j-1)-\beta}} + \sup_{v\in[t,s]}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_1|\Big\}\end{aligned}$$

Remembering that $s-t\le T-t\le 1$, it finally holds that

$$|\theta_{s,t}(x)-\theta_{s,t}(x'))_1| \le C\Big((s-t)^{1/\alpha}+\mathbf{d}(x,x')\Big)$$

since by assumption [**P**],

$$\frac{\alpha+\beta}{\alpha} > \frac{1}{1-\beta} > \frac{1}{\alpha}$$

and

$$\frac{1+(\alpha+\beta)(j-1)}{1+\alpha(j-1)-\beta} = 1+\frac{\beta j}{1+\alpha j-(\alpha+\beta)} > 1+\frac{\beta j}{\alpha j} > 1+\Big(\frac{1-\alpha}{\alpha}\Big) = \frac{1}{\alpha}.$$

Plugging this control in equation (8.11), we then conclude since

$$\begin{aligned}
&\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_i\Big| \\
&\quad\le C\Big(\mathbf{d}^{1+\alpha(i-1)}(x,x')+(s-t)^{\frac{1+\alpha(i-1)}{\alpha}}+(s-t)^{i-1}\sup_{\overline{s}\in[t,s]}(|\theta_{\overline{s},t}(x)-\theta_{\overline{s},t}(x'))_1|\Big)\\
&\quad\le C\Big(\mathbf{d}^{1+\alpha(i-1)}(x,x')+(s-t)^{\frac{1+\alpha(i-1)}{\alpha}}+(s-t)^{i-1}\Big((s-t)^{1/\alpha}+\mathbf{d}(x,x')\Big)\Big)\\
&\quad\le C\Big((s-t)^{\frac{1+\alpha(i-1)}{\alpha}}+\mathbf{d}^{1+\alpha(i-1)}(x,x')\Big),
\end{aligned}$$

using again the Young inequality in the last passage. The proof is complete.

We can now prove the two results (Lemmas 5.5 and Lemma 5.6) concerning the sensitivity of the frozen shift $\tilde{m}^{\tau,\xi}_{s,t}$.

Proof of Lemma 5.5. From the integral representation of $\tilde{m}^{t,x}_{s,t}(y)$ (cf. Equation (3.4)), we can write that

$$\begin{aligned}
\Big|\Big(\tilde{m}^{t,x}_{s,t}(y)-\tilde{m}^{t,x'}_{s,t}(y')\Big)_1\Big| &\le \int_t^s\Big|F_1(v,\theta_{v,t}(x))-F_1(v,\theta_{v,t}(x'))\Big|\,dv\\
&\le C\|F\|_H\int_t^s\mathbf{d}^\beta\Big(\theta_{v,t}(x),\theta_{v,t}(x')\Big)\,dv
\end{aligned}$$

where in the second passage we used that F_1 is in $C^\beta_{b,d}(\mathbb{R}^{nd})$. Thanks to the Control on the flows (Lemma 5.4), it then holds that

$$\Big|\Big(\tilde{m}^{t,x}_{s,t}(y)-\tilde{m}^{t,x'}_{s,t}(y')\Big)_1\Big| \le C\|F\|_H(s-t)\Big[\mathbf{d}^\beta(x,x')+(s-t)^{\frac{\beta}{\alpha}}\Big]$$

and we have concluded.

Proof of Lemma 5.6. We know from Lemma 3.1 that $\tilde{m}^{t,x'}_{t_0,t}(x')=\theta_{t_0,t}(x')$. Fixed i in $[\![1,n]\!]$, we can then write that

$$\begin{aligned}
\Big(\tilde{m}^{t,x}_{t_0,t}(x')-\tilde{m}^{t,x'}_{t_0,t}(x')\Big)_i &= \Big(\tilde{m}^{t,x}_{t_0,t}(x')-\theta_{t_0,t}(x')\Big)_i\\
&= \Big(\tilde{m}^{t,x}_{t_0,t}(x')-\theta_{t_0,t}(x)\Big)_i+\Big(\theta_{t_0,t}(x)-\theta_{t_0,t}(x')\Big)_i.
\end{aligned}$$

We start focusing on the first term of the above expression. From the integral representation of $\tilde{m}^{t,x}_{t_0,t}(x')$ and $\theta_{t_0,t}(x)$, it holds that

$$\tilde{m}^{t,x}_{t_0,t}(x') - \theta_{t_0,t}(x) \,=\, x' - x + \int_t^{t_0} A\Big[\tilde{m}^{t,x}_{v,t}(x') - \theta_{v,t}(x)\Big]\, dv. \tag{8.14}$$

Remembering from (1.2) that A is sub-diagonal, it follows that

$$\Big(\tilde{m}^{t,x}_{t_0,t}(x') - \theta_{t_0,t}(x)\Big)_i \,=\, (x'-x)_i + A_{i,i-1}\int_t^{t_0} \Big(\tilde{m}^{t,x}_{v,t}(x') - \theta_{v,t}(x)\Big)_{i-1}\, dv \tag{8.15}$$

for any i in $[\![2,n]\!]$ and

$$\Big(\tilde{m}^{t,x}_{t_0,t}(x') - \theta_{t_0,t}(x)\Big)_1 \,=\, (x'-x)_1.$$

Iterating the process, we can find that

$$\Big|\Big(\tilde{m}^{t,x}_{t_0,t}(x') - \theta_{t_0,t}(x)\Big)_i\Big| \,\le\, C\sum_{k=1}^{i}\Big|(x'-x)_k\Big|(t_0-t)^{i-k}.$$

On the other side, the integral representation of $\theta_{s,\tau}(\xi)$ (Equation (3.1)) allows us to write that

$$\begin{aligned}\Big(\theta_{t_0,t}(x) - \theta_{t_0,t}(x')\Big)_i \,=\, (x-x')_i + A_{i,i-1}\int_t^{t_0}\Big\{&\Big(\theta_{t_0,t}(x) - \theta_{t_0,t}(x')\Big)_{i-1} \\ &+ F_i(v,\theta_{v,t}(x)) - F_i(v,\theta_{v,t}(x'))\Big\}\, dv\end{aligned} \tag{8.16}$$

for any i in $[\![2,n]\!]$ and

$$\Big(\theta_{t_0,t}(x) - \theta_{t_0,t}(x')\Big)_1 \,=\, (x-x')_1 + \int_t^{t_0}\Big\{F_1(v,\theta_{v,t}(x)) - F_1(v,\theta_{v,t}(x'))\Big\}\, dv. \tag{8.17}$$

Fixed i in $[\![2,n]\!]$, it then follows from (8.14) and (8.16) that

$$\begin{aligned}\Big|\Big(\tilde{m}^{t,x}_{t_0,t}(x') - \tilde{m}^{t,x'}_{t_0,t}(x')\Big)_i\Big| \,\le\, C\|F\|_H\Big(&\sum_{k=1}^{i-1}|(x'-x)_k|(t_0-t)^{i-k} \\ &+ \int_t^{t_0}\Big\{\Big|\Big(\theta_{v,t}(x) - \theta_{v,t}(x')\Big)_{i-1}\Big| + \sum_{j=i}^{n}\Big|\Big(\theta_{v,t}(x) - \theta_{v,t}(x')\Big)_j\Big|^{\frac{\gamma_i+\beta}{1+\alpha(j-1)}}\Big\}\, dv\Big).\end{aligned}$$

Also, from (8.15) and (8.17), it holds that

$$\Big|\Big(\tilde{m}^{t,x}_{t_0,t}(x') - \tilde{m}^{t,x'}_{t_0,t}(x')\Big)_1\Big| \,\le\, C\|F\|_H\int_t^{t_0}\sum_{j=1}^{n}\Big|\Big(\theta_{v,t}(x) - \theta_{v,t}(x')\Big)_j\Big|^{\frac{\beta}{1+\alpha(j-1)}}\, dv.$$

Using now Lemma 5.4, we can show that

$$\begin{aligned}\Big|\Big(\tilde{m}^{t,x}_{t_0,t}(x') - \tilde{m}^{t,x'}_{t_0,t}(x')\Big)_i\Big| \,\le\, C\|F\|_H\Big(&\sum_{k=1}^{i-1}|(x'-x)_k|(t_0-t)^{i-k} + (t_0-t)^{\frac{1+\alpha(i-2)}{\alpha}+1} \\ &+ (t_0-t)\mathbf{d}^{1+\alpha(i-2)}(x,x') + (t_0-t)^{\frac{1+\alpha(i-2)+\beta}{\alpha}+1} + (t_0-t)\mathbf{d}^{1+\alpha(i-2)+\beta}(x,x')\Big)\end{aligned}$$

for any i in $[\![2,n]\!]$ and

$$\Big|\Big(\tilde{m}^{t,x}_{t_0,t}(x')-\tilde{m}^{t,x'}_{t_0,t}(x')\Big)_1\Big| \,\le\, C\|F\|_H(t_0-t)^{\frac{\beta+\alpha}{\alpha}}+(t_0-t)\mathbf{d}^{\beta}(x,x').$$

Since $t_0-t=c_0\mathbf{d}^{\alpha}(x,x')$ by Equation (4.16), we can conclude that

$$\begin{aligned}\Big|\Big(\tilde{m}^{t,x}_{t_0,t}(x')-\tilde{m}^{t,x'}_{t_0,t}(x')\Big)_i\Big| \,&\le\, C\|F\|_H\Big\{\sum_{k=1}^{i-1}\mathbf{d}^{1+\alpha(k-1)}(x',x)c_0^{i-k}\mathbf{d}^{\alpha(i-k)}(x,x')\\&\qquad+c_0^{\frac{1+\alpha(i-1)}{\alpha}}\mathbf{d}^{1+\alpha(i-1)}(x,x')+c_0\mathbf{d}^{1+\alpha(i-1)}(x,x')\\&\qquad+c_0^{\frac{1+\alpha(i-2)+\beta}{\alpha}+1}\mathbf{d}^{1+\alpha(i-1)+\beta}(x,x')+c_0\mathbf{d}^{1+\alpha(i-1)+\beta}(x,x')\Big\}\\&\le\, C\|F\|_H\Big\{\Big(c_0+c_0^{\frac{1+\alpha(i-1)}{\alpha}}\Big)\mathbf{d}^{1+\alpha(i-1)}(x,x')\\&\qquad+\Big(c_0+c_0^{\frac{1+\alpha(i-1)+\beta}{\alpha}}\Big)\mathbf{d}^{1+\alpha(i-1)+\beta}(x,x')\Big\}\\&\le\, Cc_0\|F\|_H\mathbf{d}^{1+\alpha(i-1)}(x,x')\end{aligned}$$

for any i in $[\![2,n]\!]$ and

$$\Big|\Big(\tilde{m}^{t,x}_{t_0,t}(x')-\tilde{m}^{t,x'}_{t_0,t}(x')\Big)_1\Big| \,\le\, C\|F\|_H\Big(c_0^{\frac{\beta+\alpha}{\alpha}}+c_0\Big)\mathbf{d}^{\alpha+\beta}(x,x') \,\le\, Cc_0\|F\|_H\mathbf{d}^{\alpha+\beta}(x,x'),$$

where in the last passage we used that $c_0\le 1$ and $\mathbf{d}(x,x')\le 1$. After summing all the terms together at the right scale, we finally show that

$$\mathbf{d}(\tilde{m}^{t,x}_{t_0,t}(x'),\tilde{m}^{t,x'}_{t_0,t}(x')) \,\le\, Cc_0^{\frac{1}{1+\alpha(n-1)}}\|F\|_H\mathbf{d}(x,x')$$

thanks to convexity inequalities and $c_0\le 1$.

We conclude this section showing the reverse Taylor formula which was used in the proof of Lemma 5.7 in the diagonal regime to handle the discontinuity term:

Lemma 8.4 (Reverse Taylor expansion)**.** *Let γ be in $(1,2)$, ϕ a function in $C^{\gamma}_{b,d}(\mathbb{R}^{nd})$ and x,x' two points in $\mathbb{R}^{nd}$. Then, there exists a constant $C:=C(\gamma)$ such that*

$$|D_{x_1}\phi(x)-D_{x_1}\phi(x')| \,\le\, C\|\phi\|_{C^{\gamma}_{b,d}}\mathbf{d}^{\gamma-1}(x,x').$$

Proof. We start decomposing the left-hand side $D_{x_1}\phi(x)-D_{x_1}\phi(x')$ into $I_1+I_2+I_3$ where we denoted

$$\begin{aligned}I_1 \,&:=\, \Big(\int_0^1 D_{x_1}\phi(x)-D_{x_1}\phi(x_1+\lambda\mathbf{d}(x,x'),(x)_{2:n})\,d\lambda\Big)\\I_2 \,&:=\, -\Big(\int_0^1 D_{x_1}\phi(x')-D_{x_1}\phi(x_1+\lambda\mathbf{d}(x,x'),(x')_{2:n})\,d\lambda\Big)\\I_3 \,&:=\, -\Big(\int_0^1 D_{x_1}\phi(x_1+\lambda\mathbf{d}(x,x'),(x')_{2:n})-D_{x_1}\phi(x_1+\lambda\mathbf{d}(x,x'),(x)_{2:n})\,d\lambda\Big).\end{aligned}$$

The first two components can be treated directly using that $D_{x_1}\phi$ is in $C^{\gamma-1}(\mathbb{R}^d)$ with respect to the first non-degenerate variable. Indeed,

$$\begin{aligned}|I_1| \,&\le\, \int_0^1 |D_{x_1}\phi(x) - D_{x_1}\phi(x_1+\lambda\mathbf{d}(x,x'),(x)_{2:n})|\,d\lambda \\ &\qquad\qquad \le\, C\|\phi\|_{C^\gamma}\int_0^1 |\lambda\mathbf{d}(x,x')|^{\gamma-1}\,d\lambda \,\le\, C\|\phi\|_{C^\gamma}\mathbf{d}^{\gamma-1}(x,x')\end{aligned}$$

and

$$\begin{aligned}|I_2| \,&\le\, \int_0^1 |D_{x_1}\phi(x') - D_{x_1}\phi(x_1+\lambda\mathbf{d}(x,x'),(x')_{2:n})|\,d\lambda \\ &\le\, C\|\phi\|_{C^\gamma}\int_0^1 |(x'-x)_1+\lambda\mathbf{d}(x,x')|^{\gamma-1}\,d\lambda \\ &\le\, C\|\phi\|_{C^\gamma}\mathbf{d}^{\gamma-1}(x,x')\end{aligned}$$

where in the last expression we used Young inequality.
To control the last term, we assume for the sake of brevity to be in the scalar case, i.e. $d=1$. In the general setting, the proof below can be reproduced component-wise. The idea is to use a reverse Taylor expansion to pass from the derivative to the function itself. Namely,

$$\begin{aligned}|I_3| \,&=\, \frac{1}{\mathbf{d}(x,x')}\left|\int_0^1\Big[\partial_\lambda\phi(x_1+\lambda\mathbf{d}(x,x'),(x')_{2:n}) - \partial_\lambda\phi(x_1+\lambda\mathbf{d}(x,x'),(x)_{2:n})\Big]\,d\lambda\right| \\ &\le\, \frac{1}{\mathbf{d}(x,x')}\Big|\phi(x_1+\mathbf{d}(x,x'),(x')_{2:n}) - \phi(x_1,(x')_{2:n}) + \phi(x_1+\mathbf{d}(x,x'),(x)_{2:n}) - \phi(x)\Big| \\ &\le\, C\|\phi\|_{C^\gamma}\mathbf{d}^{\gamma-1}(x,x').\end{aligned}$$

We have thus concluded the proof. □

Chapter 3

Schauder estimates for degenerate Lévy Ornstein-Uhlenbeck operators

Abstract: We establish global Schauder estimates for integro-partial differential equations (IPDE) driven by a possibly degenerate Lévy Ornstein-Uhlenbeck operator, both in the elliptic and parabolic setting, using some suitable anisotropic Hölder spaces. The class of operators we consider is composed by a linear drift plus a Lévy operator that is comparable, in a suitable sense, with a possibly truncated stable operator. It includes for example, the relativistic, the tempered, the layered or the Lamperti stable operators. Our method does not assume neither the symmetry of the Lévy operator nor the invariance for dilations of the linear part of the operator. Thanks to our estimates, we prove in addition the well-posedness of the considered IPDE in suitable functional spaces. In the final section, we extend some of these results to more general operators involving non-linear, space-time dependent drifts.

1 Introduction

Fixed an integer N in $\mathbb{N}$, we consider the following integro-partial differential operator of Ornstein-Uhlenbeck type:

$$L^{\text{ou}} := \mathcal{L} + \langle Ax, D_x \rangle \quad \text{on } \mathbb{R}^N, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^N$, A is a matrix in $\mathbb{R}^N \otimes \mathbb{R}^N$ and $\mathcal{L}$ is a possibly degenerate, Lévy operator acting non-degenerately only on a subspace of $\mathbb{R}^N$. We are interested in showing the *well-posedness* and the associated *Schauder estimates* for elliptic and parabolic equations involving the operator L^{ou} and with coefficients in a generalized family of Hölder spaces.
We only assume that A satisfies a natural controllability assumption, the so-called Kalman rank condition (condition [**K**] below), and that the operator $\mathcal{L}$ is comparable, in a suitable sense, to a non-degenerate, truncated α-stable operator on the same subspace of $\mathbb{R}^N$, for some $\alpha < 2$ (condition [**SD**] below).
The topic of Schauder estimates for Ornstein-Uhlenbeck operators has been widely

studied in the last decades, especially in the diffusive, local setting, i.e. when $\mathcal{L} = \frac{1}{2}\mathrm{Tr}\Big(QD_x^2\Big)$ for some suitable matrix Q, and it is now quite well-understood. See e.g. [GT01].
On the other hand, a literature on the topic for the pure jump, non-local framework has been developed only in the recent years ([Bas09], [DK13], [BK15], [ROS16]), [FRRO17], [IJS18], [CdRMP20a], [Küh19], but mainly in the non-degenerate, α-stable setting, i.e. when $\mathcal{L} = \Delta_x^{\alpha/2}$ is the fractional Laplacian on $\mathbb{R}^N$ or similar. To the best of our knowledge, the only two articles dealing with the degenerate, non-local framework (if $\mathcal{L} = \Delta_x^{\alpha/2}$ acts non-degenerately only on a sub-space of $\mathbb{R}^N$) are [HPZ19], that takes into account the kinetics dynamics ($N = 2d$), and [Mar20], for the general chain. In order to use [HPZ19] or [Mar20] for our operator (1.1), we would need to impose the additional strong assumption of invariance for dilations of the matrix A.
The analysis of Ornstein-Uhlenbeck operators has been mainly developed following two different approaches. On the one hand, Da Prato and Lunardi in [DPL95] have been the first to use a priori estimates for the corresponding semi-group between suitable function spaces (See also [Lun97, Lor05, CdRHM18a, Pri18]). Such a semi-group approach only adresses the regularity in space and indeed, the associated anisotropic Hölder spaces and Schauder estimates reflect this fact. In particular, the parabolic Schauder estimates do not present a bootstrap effect with respect to the initial condition.
The second approach, introduced by Manfredini in [Man97], exploits instead the general analysis on Lie groups to construct intrinsic Hölder spaces (see [PPP16] for a definition) that takes into account the joint space-time regularity of the involved functions. For a more thorough explanation along this direction, we suggest the interested reader to see, for example, [Pas03], [DFP06] or the recent paper [IM21].

Even if the Ornstein-Uhlenbeck operator is usually exploited as a "toy model" for more general operators with space-time dependent, non-linear coefficients, we highlight that they appear naturally in various scientific contexts: for example in physics, for the analysis of anomalous diffusions phenomena or for Hamiltonian models in a turbulent regime (see e.g. [BBM01], [CPKM05] and the references therein) or in mathematical finance and econometrics (see e.g. [Bro01], [BNS01]). The interest in Schauder estimates involving this type of operator also follows from the natural application which consists in establishing the well-posedness of stochastic differential equations (SDE) driven by Lévy processes and the associated stochastic control theory. See e.g. [FM82], [CdRM20b], [HWZ20].

Under our assumptions, we have been able to consider more general Lévy operators not usually included in the literature, such as the relativistic stable process, the layered stable process or the Lamperti one (see Paragraph "Main Operators Considered" below for details). Moreover, we do not require the operator $\mathcal{L}$ to be symmetric. Here, we only mention one important example that satisfies our hypothesis, the Ornstein-Uhlenbeck operator on $\mathbb{R}^2$ driven by the relativistic fractional Laplacian $\Delta_{\mathrm{rel}}^{\alpha/2}$ and acting only on the first component:

$$x_1(D_{x_1}\phi(x) + D_{x_2}\phi(x)) + \text{p.v.}\int_{\mathbb{R}}\Big[\phi(\begin{pmatrix} x_1+z \\ x_2\end{pmatrix}) - \phi(\begin{pmatrix} x_1 \\ x_2\end{pmatrix})\Big]\frac{1+|z|^{\frac{d+\alpha-1}{2}}}{|z|^{d+\alpha}}e^{-|z|}\,dz$$
$$= \langle Ax, D_x\phi(x)\rangle + \mathcal{L}\phi(x) \quad (1.2)$$

where $x=(x_1,x_2)$ in $\mathbb{R}^2$. Such an example is included in the framework of Equation (1.1) considering $A = \begin{pmatrix} 1 & 0 \\ 1 & 0\end{pmatrix}$. This operator appears naturally as a fractional generalization of the relativistic Schrödinger operator (See [Ryz02] for more details).
We remark that example (1.2) cannot be considered in [HPZ19] or in our previous work [Mar20]. Indeed, the matrix A_0 is not "dilation-invariant" (see example 2.1 below) and thus, it cannot be rewritten in the form used in [Mar20] (see also [LP94] Proposition 2.2 for a more thorough explanation). Furthermore, operators like the relativistic fractional Laplacian cannot be treated in [HPZ19] or [Mar20] that indeed have taken into account only stable-like operators on $\mathbb{R}^N$. Another useful advantage of our technique is that we do not need anymore the symmetry of the Lévy measure ν which was, again, a key assumption in [Mar20].

More in details, given an integer $d \le N$ and a matrix B in $\mathbb{R}^N \otimes \mathbb{R}^d$ such that $\text{rank}(B) = d$, we consider a family of operators $\mathcal{L}$ that can be represented for any sufficiently regular function $\phi\colon \mathbb{R}^N \to \mathbb{R}$ as

$$\mathcal{L}\phi(x) := \frac{1}{2}\text{Tr}\Big(BQB^*D^2_x\phi(x)\Big) + \langle Bb, D_x\phi(x)\rangle$$
$$+ \int_{\mathbb{R}^d_0}\Big[\phi(x+Bz) - \phi(x) - \langle D_x\phi(x), Bz\rangle \mathbb{1}_{B(0,1)}(z)\Big]\,\nu(dz), \quad (1.3)$$

where b is a vector in $\mathbb{R}^d$, Q is a symmetric, non-negative definite matrix in $\mathbb{R}^d\otimes\mathbb{R}^d$ and ν is a Lévy measure on $\mathbb{R}^d_0 := \mathbb{R}^d \smallsetminus \{0\}$, i.e. a σ-finite measure on $\mathcal{B}(\mathbb{R}^d_0)$, the Borel σ-algebra on $\mathbb{R}^d_0$, such that $\int(1\wedge|z|^2)\,\nu(dz)$ is finite. We then suppose ν to satisfy the following *stable domination* condition:

[SD] there exists $r_0>0$, α in $(0,2)$ and a finite, non-degenerate measure μ on the unit sphere $\mathbb{S}^{d-1}$ such that

$$\nu(\mathcal{A}) \ge \int_0^{r_0}\int_{\mathbb{S}^{d-1}} \mathbb{1}_{\mathcal{A}}(r\theta)\,\mu(d\theta)\frac{dr}{r^{1+\alpha}}, \quad \mathcal{A}\in\mathcal{B}(\mathbb{R}^d_0).$$

We recall that a measure μ on $\mathbb{R}^d$ is non-degenerate if there exists a constant $\eta\ge 1$ such that

$$\eta^{-1}|p|^\alpha \le \int_{\mathbb{S}^{d-1}} |p\cdot s|^\alpha\,\mu(ds) \le \eta|p|^\alpha, \quad p\in\mathbb{R}^d, \quad (1.4)$$

where " $\cdot$ " stands for the inner product on the smaller space $\mathbb{R}^d$. Since any α-stable Lévy measure ν_α can be decomposed into a spherical part μ on $\mathbb{S}^{d-1}$ and a radial part $r^{-(1+\alpha)}dr$ (see e.g. Theorem 14.3 in [Sat13]), assumption [**SD**] roughly states that the Lévy measure of the integro-differential part of $\mathcal{L}$ is bounded from below by the Lévy measure of a possibly truncated, α-stable operator on $\mathbb{R}^d$.
It is assumed moreover that the matrices A, B satisfy the following *Kalman condition*:

[K] It holds that $N \,=\, \text{rank}\Big[B, AB, \dots, A^{N-1}B\Big]$,

where $\Big[B, AB, \dots, A^{N-1}B\Big]$ is the matrix in $\mathbb{R}^N \otimes \mathbb{R}^{dN}$ whose columns are given by $B, AB, \dots, A^{N-1}B$.
Such an assumption is equivalent, in the linear framework, to the Hörmander condition (see [Hör67]) on the commutators, ensuring the hypoellipticity of the operator $\partial_t - L^{\text{ou}}$. Moreover, condition [**K**] is well-known in control theory (see e.g. [Zab92], [PZ09]).

Mathematical outline. In the present paper, we aim at establishing global Schauder estimates for equations involving the operator L^{ou} on $\mathbb{R}^N$, both in the elliptic and parabolic settings. Namely, we consider for a fixed $\lambda > 0$ the following elliptic equation:

$$\lambda u(x) - L^{\text{ou}}u(x) \,=\, g(x), \quad x \in \mathbb{R}^N, \tag{1.5}$$

and, for a fixed time horizon $T > 0$, the following parabolic Cauchy problem:

$$\begin{cases} \partial_t u(t,x) \,=\, L^{\text{ou}}u(t,x) + f(t,x), & (t,x) \in (0,T) \times \mathbb{R}^N; \\ u(0,x) \,=\, u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.6}$$

where f, g, u_0 are given functions. Since our aim is to show optimal regularity results in Hölder spaces, we will assume for the elliptic case (Equation (1.5)) that the source g belongs to a suitable *anisotropic* Hölder space $C^\beta_{b,d}(\mathbb{R}^N)$ for some β in $(0,1)$, where the Hölder exponent depends on the "direction" considered. The space $C^\beta_{b,d}(\mathbb{R}^N)$ can be understood as composed by the bounded functions on $\mathbb{R}^N$ that are Hölder continuous with respect to a distance $\mathbf{d}$ somehow induced by the operator L^{ou}. We refer to Section 2 for a detailed exposition of such an argument but we highlight already that the above mentioned distance $\mathbf{d}$ can be seen as a generalization of the classical parabolic distance, adapted to our degenerate, non-local framework. It is precisely assumption [**K**], or equivalently the hypoellipticity of $\partial_t + L^{\text{ou}}$, that ensures the existence of such a distance $\mathbf{d}$ and gives it its anisotropic nature. Roughly speaking, it allows the smoothing effect of the Lévy operator $\mathcal{L}$ acting non-degenerately only on some components, say $B\mathbb{R}^N$, to spread in the whole space $\mathbb{R}^N$, even if with lower regularizing properties.
Concerning the parabolic problem (1.6), we assume similarly that u_0 is in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)$ and that $f(t,\cdot)$ is in $C^\beta_{b,d}(\mathbb{R}^N)$, uniformly in $t \in (0,T)$. The typical estimates we want to prove can be stated in the parabolic setting in the following way: there exists a constant C, depending only on the parameters of the model, such that any distributional solution u of the Cauchy problem (1.6) satisfies

$$\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \,\le\, C\Big[\|u_0\|_{C^{\alpha+\beta}_{b,d}} + \|f\|_{L^\infty(C^\beta_{b,d})}\Big]. \tag{$\mathscr{S}$}$$

As a by-product of the Schauder Estimates ($\mathscr{S}$), we will obtain the well-posedness of the Cauchy problem (1.6) in the space $L^\infty\Big(0,T; C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)\Big)$, once the existence of a solution is established. The additional regularity for the solution u with respect to the source f reflects the appearance of a smoothing effect associated with L^{ou} of order α, as it is expected by condition [**SD**]. It can be seen as a generalization of the "standard" parabolic bootstrap to our degenerate, non-local setting. We highlight that the parabolic

bootstrap in $(\mathscr{S})$ is precisely derived from the non-degenerate stable-like part in $\mathcal{L}$ (lowest regularizing effect in the operator).
To show our result, we will follow the semi-group approach as firstly introduced in [DPL95], which became afterwards a very robust tool to study Schauder estimates in a wide variety of frameworks ([Lun97], [Lor05], [Sai07], [Pri09], [Pri12], [DK13], [KK15], [CdRHM18a], [Küh19]). The main idea is to consider the Markov transition semi-group P_t associated with L^{ou} and then, in the elliptic case, to use the Laplace transform formula in order to represent the unique distributional solution u of Equation (1.5) as:

$$u(x) \,=\, \int_0^\infty e^{-\lambda t}\Big[P_t g\Big](x)\, dt \,=:\, \int_0^\infty e^{-\lambda t} P_t g(x)\, dt.$$

In the parabolic setting, we exploit instead the variation of constants (or Duhamel) formula in order to show a similar representation for the weak solution of the Cauchy problem (1.6):

$$u(t,x) \,=\, P_t u_0(x) + \int_0^t \Big[P_{t-s} f(s,\cdot)\Big](x)\, ds \,=:\, P_t u_0(x) + \int_0^t P_{t-s} f(s,x)\, ds.$$

In order to prove global regularity estimates for the solutions, the crucial point is to understand the action of the operator P_t on the anisotropic Hölder spaces. In particular, we will show in Corollary 4.4 the continuity of P_t as an operator from $C^\beta_{b,d}(\mathbb{R}^N)$ to $C^\gamma_{b,d}(\mathbb{R}^N)$ for $\beta < \gamma$ and, more precisely, that it holds:

$$\|P_t\phi\|_{C^\gamma_{b,d}} \,\le\, C\|\phi\|_{C^\beta_{b,d}}\Big(1 + t^{-\frac{\gamma-\beta}{\alpha}}\Big), \quad t > 0. \tag{1.7}$$

The above estimate can be obtained through interpolation techniques (see Equation (4.9)), once sharp controls in supremum norm (Theorem 4.3 below) are established for the spatial derivatives of $P_t\phi$ when $\phi \in C^\beta_{b,d}(\mathbb{R}^N)$. We think that such an estimate (1.7) and the controls in Theorem 4.3 can be of independent interest and used also beyond our scope in other contexts.
We face here two main difficulties to overcome. While in the gaussian setting, L^∞-estimates of this type have been established exploiting, for example, explicit formulas for the density of the semi-group P_t ([Lun97]), a priori controls of Bernstein type combined with interpolation methods ([Lor05] and [Sai07], when $n = 2$ in (2.2) below) or probabilistic representations of the semi-group P_t, allowing Malliavin calculus ([Pri09]), we cannot rely on these techniques in our non-local framework, mainly due to the lower integrability properties for P_t. Instead, we are going to use a *perturbative approach* which consists in considering the Lévy operator $\mathcal{L}$ as a perturbation, in a suitable sense, of an α-stable operator, at least for the associated small jumps. Indeed, we can "decompose" the operator $\mathcal{L}$ in a smoother part, $\mathcal{L}^\alpha$, whose Lévy measure is given by

$$\mu(d\theta)\frac{\mathbb{1}_{(0,r_0]}(r)}{r^{1+\alpha}}dr$$

and a remainder part. It is precisely condition [**SD**] that allows such a decomposition, since it ensures the positivity of the Lévy measure

$$d\nu - d\mu\frac{\mathbb{1}_{[0,r_0]}}{r^{1+\alpha}}dr$$

associated with the remainder term. The main difference with the previous techniques in the diffusive setting is that we will work mainly on the truncated α-stable contribution $\mathcal{L}^\alpha$, being the remainder term only bounded.
Following [SSW12], we will establish that the Hartman-Winter condition holds, ensuring the existence of a smooth density for the semi-group associated with $\mathcal{L}^\alpha$ and then, the required gradient estimates. Indeed, assumption [**SD**] roughly states that the small jump contributions of ν, the ones responsible for the creation of a density, are controlled from below by an α-stable measure, whose absolute continuity is well-known in our framework.
On the other hand, we will have to deal with the degeneracy of the operator $\mathcal{L}$, that acts non-degenerately, through the embedding matrix B, only on a subspace of dimension d. It will be managed by adapting the reasonings firstly appeared in [HM16]. Namely, we will show that the semi-group associated with the Ornstein-Uhlenbeck operator L^{ou} coincides with a non-degenerate one but "multiplied" by a time-dependent matrix that precisely takes into account the original degeneracy of the operator (see definition of matrix $\mathbb{M}_t$ in Section 2.1).

Main operators considered. We conclude this introduction showing that assumption [**SD**] applies to a large class of Lévy operators on $\mathbb{R}^d$. As already pointed out in [SSW12], it is satisfied by any Lévy measure ν that can be decomposed in polar coordinates as

$$\nu(\mathcal{A}) \,=\, \int_0^\infty \int_{\mathbb{S}^{d-1}} \mathbb{1}_{\mathcal{A}}(r\theta)Q(r,\theta)\,\mu(d\theta)\frac{dr}{r^{1+\alpha}}, \quad \mathcal{A}\in\mathcal{B}(R^d_0),$$

for a finite, non-degenerate (in the sense of Equation (1.4)), measure μ on $\mathbb{S}^{d-1}$ and a Borel function $Q\colon (0,\infty)\times\mathbb{S}^{d-1}\to\mathbb{R}$ such that there exists $r_0>0$ so that

$$Q(r,\theta) \,\ge\, c>0, \quad \text{a.e. in } [0,r_0]\times\mathbb{S}^{d-1}.$$

In particular, assumption [**SD**] holds for the following families of "stable-like" examples with $\alpha\in(0,2)$:

1. Stable operator [Sat13]:
$$Q(r,\theta) \,=\, 1;$$
2. Truncated stable operator with $r_0>0$ [KS08]:
$$Q(r,\theta) \,=\, \mathbb{1}_{(0,r_0]}(r);$$
3. Layered stable operator with β in $(0,2)$ and $r_0>0$ [HK07]:
$$Q(r,\theta) \,=\, \mathbb{1}_{(0,r_0]}(r)+\mathbb{1}_{(r_0,\infty)}(r)r^{\alpha-\beta};$$
4. Tempered stable operator [Ros07]:
$$Q(\cdot,\theta) \text{ completely monotone, } Q(0,\theta)>0 \text{ and } Q(\infty,\theta)=0 \text{ a.e. in } S^{d-1};$$
5. Relativistic stable operator [CMS90], [BMR09]:
$$Q(r,\theta) \,=\, (1+r)^{(d+\alpha-1)/2}e^{-r};$$
6. Lamperti stable operator with $f\colon S^{d-1}\to\mathbb{R}$ such that $\sup f(\theta)<1+\alpha$ [CPP10]:
$$Q(r,\theta) \,=\, e^{rf(\theta)}\Big(\frac{r}{e^r-1}\Big)^{1+\alpha}.$$

Organization of the paper. The article is organized as follows. Section 2 introduces some useful notations and then, the anisotropic distance $\mathbf{d}$ induced by the dynamics as well as Zygmund-Hölder spaces associated with such a distance. In Section 3, we are going to show some analytical properties of the semi-group P_t generated by L^{ou}, such as the existence of a smooth density and, at least for small times, some controls for its derivatives. Section 4 is then dedicated to different estimates in the L^∞-norm for P_tf and its spatial derivatives, involving the supremum or the Hölder norm of the function f. In particular, we show here the continuity of P_t as an operator between anisotropic Zygmund-Hölder spaces. In Section 5, we use the controls established in the previous parts in order to prove the elliptic Schauder estimates and show that Equation (1.5) has a unique solution. Similarly, we establish the well-posedness of the Cauchy problem (1.6) as well as the associated parabolic Schauder estimates. In the final section of the article, we briefly explain some possible extensions of the previous results to non-linear, space-time dependent operators.

2 Geometry of the dynamics

In this section, we are going to choose the right functional space "in which" to state our Schauder estimates. The idea is to construct an Hölder space $C^\beta_{b,d}(\mathbb{R}^N)$ with respect to a distance $\mathbf{d}$ that it is homogeneous to the dynamics, i.e. such that for any f in $C^\beta_{b,d}(\mathbb{R}^N)$, any distributional solution u of

$$L^{\mathrm{ou}}u(x) \,=\, \mathcal{L}u(x) + \langle Ax, Du(x)\rangle \,=\, f(x), \quad x \in \mathbb{R}^N \tag{2.1}$$

is in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)$, the expected parabolic bootstrap associated to this kind of operator. We recall in particular that the Kalman rank condition [**K**] is equivalent to the hypoellipticity (in the sense of Hörmander [Hör67]) of the operator L^{ou} that ensures the existence and smoothness of a distributional solution of Equation (2.1) for sufficiently regular f. See e.g. [Ish16] or [HPZ19] for more details.

2.1 The distance associated with the dynamics

To construct the suitable distance $\mathbf{d}$, we start noticing that the Kalman rank condition [**K**] allows us to denote

$$n \,:=\, \min\{r \in \mathbb{N} \colon N \,=\, \mathrm{rank}\Big[B, AB, \dots, A^{r-1}B\Big]\}. \tag{2.2}$$

Clearly, n is in $[\![1, N]\!]$, where $[\![\cdot,\cdot]\!]$ denotes the set of all the integers in the interval, and $n = 1$ if and only if $d = N$, i.e. if the dynamics is non-degenerate.
As done in [Lun97], the space $\mathbb{R}^N$ will be decomposed with respect to the family of linear operators $B, AB, \dots, A^{n-1}B$. We start defining the family $\{V_h \colon h \in [\![1, n]\!]\}$ of subspaces of $\mathbb{R}^N$ through

$$V_h \,:=\, \begin{cases} \mathrm{Im}(B), & \text{if } h = 1,\\ \bigoplus_{k=1}^h \mathrm{Im}(A^{k-1}B), & \text{otherwise.}\end{cases}$$

It is easy to notice that $V_h \neq V_k$ if $k \neq h$ and $V_1 \subset V_2 \subset \dots V_n = \mathbb{R}^N$. We can then construct iteratively the family $\{E_h \colon h \in [\![1,n]\!]\}$ of orthogonal projections from $\mathbb{R}^N$ as

$$E_h := \begin{cases} \text{projection on } V_1, & \text{if } h = 1; \\ \text{projection on } (V_{h-1})^\perp \cap V_h, & \text{otherwise.} \end{cases}$$

With a small abuse of notation, we will identify the projection operators E_h with the corresponding matrices in $\mathbb{R}^N \otimes \mathbb{R}^N$. It is clear that $\dim E_1(\mathbb{R}^N) = d$. Let us then denote $d_1 := d$ and

$$d_h := \dim E_h(\mathbb{R}^N), \quad \text{for } h > 1.$$

We can define now the distance $\mathbf{d}$ through the decomposition $\mathbb{R}^N = \bigoplus_{h=1}^n E_h(\mathbb{R}^N)$ as

$$\mathbf{d}(x, x') := \sum_{h=1}^n |E_h(x - x')|^{\frac{1}{1+\alpha(h-1)}}.$$

The above distance can be seen as a generalization of the usual Euclidean distance when $n = 1$ (non-degenerate dynamics) as well as an extension of the standard parabolic distance for $\alpha = 2$. It is important to highlight that it does not induce a norm since it lacks of linear homogeneity.
The anisotropic distance $\mathbf{d}$ can be understood direction-wise: we firstly fix a "direction" h in $[\![1,n]\!]$ and then calculate the standard Euclidean distance on the associated subspace $E_h(\mathbb{R}^N)$, but scaled according to the dilation of the system in that direction. We conclude summing the contributions associated with each component. The choice of such a dilation will be discussed thoroughly in the example at the end of this section.

As emphasized by the result from Lanconelli and Polidoro recalled below (cf. [LP94], Proposition 2.1), the decomposition of $\mathbb{R}^N$ with respect to the projections $\{E_h \colon h \in [\![1,n]\!]\}$ determines a particular structure of the matrices A and B. It will be often exploited in the following.

Theorem 2.1 ([LP94])**.** *Let $\{e_i \colon i \in [\![1,N]\!]\}$ be an orthonormal basis consisting of generators of $\{E_h(\mathbb{R}^N) \colon h \in [\![1,n]\!]\}$. Then, the matrices A and B have the following form:*

$$B = \begin{pmatrix} B_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \textit{and} \quad A = \begin{pmatrix} * & * & \dots & \dots & * \\ A_2 & * & \ddots & \ddots & \vdots \\ 0 & A_3 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ 0 & \dots & 0 & A_n & * \end{pmatrix} \tag{2.3}$$

where B_0 is a non-degenerate matrix in $\mathbb{R}^{d_1} \otimes \mathbb{R}^{d_1}$ and A_h are matrices in $\mathbb{R}^{d_h} \otimes \mathbb{R}^{d_{h-1}}$ with rank$(A_h) = d_h$ for any h in $[\![2,n]\!]$. Moreover, $d_1 \geq d_2 \geq \dots \geq d_n \geq 1$.

Applying a change of variables if necessary, we will assume from this point further to have fixed such a canonical basis $\{e_i \colon i \in [\![1,N]\!]\}$. For notational simplicity, we denote by I_h, $h \in [\![1,n]\!]$, the family of indexes i in $[\![1,N]\!]$ such that $\{e_i \colon i \in I_h\}$ spans $E_h(\mathbb{R}^N)$.

The particular structure of A and B given by Theorem 2.1 allows us to decompose accurately the exponential e^{tA} of the matrix A in order to make the intrinsic scale of the system appear. Further on, we will consider fixed a time-dependent matrix $\mathbb{M}_t$ on $\mathbb{R}^N \otimes \mathbb{R}^N$ given by

$$\mathbb{M}_t := \operatorname{diag}(I_{d_1\times d_1}, tI_{d_2\times d_2}, \dots, t^{n-1}I_{d_n\times d_n}), \quad t\geq 0.$$

Lemma 2.2. *There exists a time-dependent matrix $\{R_t \colon t\in[0,1]\}$ in $\mathbb{R}^N\otimes\mathbb{R}^N$ such that*

$$e^{tA}\mathbb{M}_t \;= \mathbb{M}_t R_t, \quad t\in[0,1]. \tag{2.4}$$

Moreover, there exists a constant $C>0$ such that for any t in $[0,1]$,

— *any l,h in $[\![1,n]\!]$ and any θ in $\mathbb{S}^{N-1}$, it holds that*

$$\Big|E_l e^{tA}E_h\theta\Big| \,\leq\, \begin{cases} Ct^{l-h}, & \text{if } l\geq h\\ Ct, & \text{if } l<h. \end{cases}$$

— *any θ in $\mathbb{S}^{d-1}$, it holds that*

$$\Big|R_tB\theta\Big| \,\geq\, C^{-1}.$$

Proof. By definition of the matrix exponential, we know that

$$E_le^{tA}E_h \;=\; \sum_{k=0}^{\infty}\frac{t^k}{k!}E_lA^kE_h. \tag{2.5}$$

Using now the representation of A given by Theorem 2.1, it is easy to check that $E_lA^kE_h=0$ for $k<l-h$ (when $l-h$ is non-negative). Thus, for $l\geq h$, it holds that

$$\Big|E_le^{tA}E_h\theta\Big| \;=\; \Big|\sum_{k=l-h}^{\infty}\frac{t^k}{k!}E_lA^kE_h\theta\Big| \,\leq\, Ct^{l-h},$$

where we exploited that t is in $[0,1]$ and $|\theta|=1$. Assuming instead that $l<h$, it is clear that $E_lI_{N\times N}E_h$ vanishes. We can then write that

$$\Big|E_le^{tA}E_h\theta\Big| \;=\; \Big|\sum_{k=l}^{\infty}\frac{t^k}{k!}E_lA^kE_h\theta\Big| \,\leq\, Ct,$$

using again that t is in $[0,1]$ and $|\theta|=1$.
To show the other control, we highlight that the matrix $\mathbb{M}_t$ is not invertible in $t=0$ and for this reason, we define the time-dependent matrix R_t as

$$R_t \;:=\; \begin{cases} I_{N\times N}, & \text{if } t=0;\\ \mathbb{M}_t^{-1}e^{tA}\mathbb{M}_t, & \text{if } t\in(0,1]. \end{cases}$$

We could have also defined $R_t := \Big(\tilde{R}^t_s\Big)_{|s=1}$ where $\tilde{R}^t_s$ solves the following ODE:

$$\begin{cases} \partial_s\tilde{R}^t_s \;=\; \mathbb{M}_t^{-1}tA\mathbb{M}_t\tilde{R}^t_s, & \text{on } (0,1],\\ \tilde{R}^t_0 \;=\; I_{N\times N}. \end{cases}$$

Equivalently, $\tilde{R}^t_s$ is the resolvent matrix associated with $\mathbb{M}_t^{-1}tA\mathbb{M}_t$, whose sub-diagonal entries are "macroscopic" from the structure of A and $\mathbb{M}_t$.
It follows immediately that Equation (2.4) holds. Moreover, we notice that

$$\Big|R_tB\theta\Big| \;\geq\; \Big|E_1R_tB\theta\Big| \;=\; \Big|E_1e^{tA}E_1B\theta\Big|.$$

Remembering the definition of matrix exponential (Equation (2.5) with $l = h = 1$), we use now that

$$E_1A^kE_1 \;=\; (E_1AE_1)^k \;=\; (A_{1,1})^kE_1,$$

where in the last expression the multiplication is meant block-wise, in order to conclude that

$$\Big|R_tB\theta\Big| \;\geq\; \Big|e^{tA_{1,1}}B_0\theta\Big|.$$

Using that $e^{tA_{1,1}}B_0$ is non-degenerate and continuous in time and that θ is in $\mathbb{S}^{d-1}$, it is easy to conclude. □

We conclude this sub-section with a simpler example taken from [HMP19]. We hope that it will help the reader to understand the introduction of the anisotropic distance $\mathbf{d}$.

Example 2.1. Fixed $N = 2d$, $n = 2$ and $d = d_1 = d_2$, we consider the following operator:

$$L^{\mathrm{ou}}_\alpha \;=\; \Delta^{\frac{\alpha}{2}}_{x_1} + x_1 \cdot \nabla_{x_2} \quad \text{ on } \mathbb{R}^{2d},$$

where $(x_1, x_2) \in \mathbb{R}^{2d}$ and $\Delta^{\frac{\alpha}{2}}_{x_1}$ is the fractional Laplacian with respect to x_1. In our framework, it is associated with the matrices

$$A \;:=\; \begin{pmatrix} 0 & 0 \\ I_{d\times d} & 0 \end{pmatrix} \quad \text{and} \quad B \;:=\; \begin{pmatrix} I_{d\times d} \\ 0 \end{pmatrix}.$$

The operator L^{ou}_α can be seen as a generalization of the classical Kolmogorov example (see e.g. [Kol34]) to our non-local setting.
In order to understand how the system typically behaves, we search for a dilation

$$\delta_\lambda \colon [0,\infty) \times \mathbb{R}^{2d} \to [0,\infty) \times \mathbb{R}^{2d}$$

which is invariant for the considered dynamics, i.e. a dilation that transforms solutions of the equation

$$\partial_t u(t,x) - L^{\mathrm{ou}}_\alpha u(t,x) \;=\; 0 \quad \text{ on } (0,\infty) \times \mathbb{R}^{2d}$$

into other solutions of the same equation.
Due to the structure of A and the α-stability of $\Delta^{\frac{\alpha}{2}}$, we can consider for any fixed $\lambda > 0$, the following

$$\delta_\lambda(t, x_1, x_2) := (\lambda^\alpha t, \lambda x_1, \lambda^{1+\alpha} x_2).$$

It then holds that

$$\Big(\partial_t - L^{\mathrm{ou}}_\alpha\Big)u = 0 \implies \Big(\partial_t - L^{\mathrm{ou}}_\alpha\Big)(u \circ \delta_\lambda) = 0.$$

Introducing now the complete time-space distance $\mathbf{d}_P$ on $[0,\infty) \times \mathbb{R}^{2d}$ given by

$$\mathbf{d}_P\Big((t,x),(s,x')\Big) \;:=\; |s-t|^{\frac{1}{\alpha}} + \mathbf{d}(x,x') \;=\; |s-t|^{\frac{1}{\alpha}} + |x_1 - x_1'| + |x_2 - x_2'|^{\frac{1}{1+\alpha}}, \quad (2.6)$$

we notice that it is homogeneous with respect to the dilation δ_λ, so that

$$\mathbf{d}_P\Big(\delta_\lambda(t,x);\delta_\lambda(s,x')\Big) = \lambda\mathbf{d}_P\Big((t,x);(s,x')\Big).$$

Precisely, the exponents appearing in Equation (2.6) are those which make each space-component homogeneous to the characteristic time scale $t^{1/\alpha}$. From a more probabilistic point of view, the exponents in Equation (2.6), can be related to the characteristic time scales of the iterated integrals of an α-stable process. It can be easily seen from the example, noticing that the operator L^{ou}_α corresponds to the generator of an isotropic α-stable process and its time integral.
Going back to the general setting, the appearance of this kind of phenomena is due essentially to the particular structure of the matrix A (cf. Theorem 2.1) that allows the smoothing effect of the operator $\mathcal{L}$, acting only on the first "component" given by B_0, to propagate into the system.

2.2 Anisotropic Zygmund-Hölder spaces

We are now ready to define the Zygmund-Hölder spaces $C^\gamma_{b,d}(\mathbb{R}^N)$ with respect to the distance $\mathbf{d}$. We start recalling some useful notations we will need below.
Given a function $f\colon \mathbb{R}^N \to \mathbb{R}$, we denote by $Df(x)$, $D^2f(x)$ and $D^3f(x)$ the first, second and third Fréchet derivative of f at a point x in $\mathbb{R}^N$ respectively, when they exist. For simplicity, we will identify $D^3f(x)$ as a 3-tensor so that $[D^3f(x)](u,v)$ is a vector in R^N for any u,v in $\mathbb{R}^N$. Moreover, fixed h in $[\![1,n]\!]$, we will denote by $D_{E_h}f(x)$ the gradient of f at x along the direction $E_h(\mathbb{R}^N)$. Namely,

$$D_{E_h}f(x) \;:=\; E_hDf(x).$$

A similar notation will be used for the higher derivatives, too.
Given X,Y two real Banach spaces, $\mathcal{L}(X,Y)$ will represent the family of linear continuous operators between X and Y.
In the following, c or C denote generic *positive* constants whose precise value is unimportant. They may change from line to line and they will depend only on the parameters given by the model and assumptions [**SD**], [**K**]. Namely, d,N,A,B,α,ν,r_0 and μ. Other dependencies that may occur will be explicitly specified.

Let us introduce now some function spaces we are going to use. We denote by $B_b(\mathbb{R}^N)$ the family of Borel measurable and bounded functions $f\colon \mathbb{R}^N \to \mathbb{R}$. It is a Banach space endowed with the supremum norm $\|\cdot\|_\infty$. We will consider also its closed subspace $C_b(\mathbb{R}^N)$ consisting of all the uniformly continuous functions.
Fixed some k in $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$ and β in $(0,1]$, we follow Lunardi [Lun97] denoting the Zygmund-Hölder semi-norm for a function $\phi\colon \mathbb{R}^N \to \mathbb{R}$ as

$$[\phi]_{C^{k+\beta}} \;:=\; \begin{cases} \sup_{|\vartheta|=k}\sup_{x\neq y}\frac{|D^\vartheta\phi(x)-D^\vartheta\phi(y)|}{|x-y|^\beta}, & \text{if } \beta\neq 1;\\ \sup_{|\vartheta|=k}\sup_{x\neq y}\frac{\left|D^\vartheta\phi(x)+D^\vartheta\phi(y)-2D^\vartheta\phi(\frac{x+y}{2})\right|}{|x-y|}, & \text{if } \beta = 1.\end{cases}$$

Consequently, The Zygmund-Hölder space $C^{k+\beta}_b(\mathbb{R}^N)$ is the family of functions $\phi\colon \mathbb{R}^N \to$

$\mathbb{R}$ such that ϕ and its derivatives up to order k are continuous and the norm

$$\|\phi\|_{C^{k+\beta}_b} := \sum_{i=1}^{k} \sup_{|\vartheta|=i} \|D^\vartheta \phi\|_{L^\infty} + [\phi]_{C^{k+\beta}_b} \text{ is finite.}$$

We can define now the anisotropic Zygmund-Hölder spaces associated with the distance $\mathbf{d}$. Fixed $\gamma > 0$, the space $C^\gamma_{b,d}(\mathbb{R}^N)$ is the family of functions $\phi\colon \mathbb{R}^N \to \mathbb{R}$ such that for any h in $[\![1, n]\!]$ and any x_0 in $\mathbb{R}^N$, the function

$$z \in E_h(\mathbb{R}^N) \to \phi(x_0 + z) \in \mathbb{R} \text{ belongs to } C^{\gamma/(1+\alpha(h-1))}_b\Big(E_h(\mathbb{R}^N)\Big),$$

with a norm bounded by a constant independent from x_0. It is endowed with the norm

$$\|\phi\|_{C^\gamma_{b,d}} := \sum_{h=1}^{n} \sup_{x_0 \in \mathbb{R}^N} \|\phi(x_0 + \cdot)\|_{C^{\gamma/(1+\alpha(h-1))}_b}. \tag{2.7}$$

We highlight that it is possible to recover the expected joint regularity for the partial derivatives, when they exist, as in the standard Hölder spaces. In such a case, they actually turn out to be Hölder continuous with respect to the distance $\mathbf{d}$ with order one less than the function (See Lemma 2.1 in [Lun97] for more details).

It will be convenient in the following to consider an equivalent norm in the "standard" Hölder-Zygmund spaces $C^\gamma_b(E_h(\mathbb{R}^N))$ that does not take into account the derivatives with respect to the different directions. We suggest the interested reader to see [Lun97], Equation (2.2) or [Pri09] Lemma 2.1 for further details.

Lemma 2.3. *Fixed γ in $(0, 3)$ and h in $[\![1, n]\!]$ and ϕ in $C_b(E_h(\mathbb{R}^N))$, let us introduce*

$$\Delta^3_{x_0}\phi(z) := \phi(x_0+3z)-3\phi(x_0+2z)+3\phi(x_0+z)-\phi(x_0), \quad x_0 \in \mathbb{R}^N;\ z \in E_h(\mathbb{R}^N). \tag{2.8}$$

Then, ϕ is in $C^\gamma_b(E_h(\mathbb{R}^N))$ if and only if

$$\sup_{x_0 \in \mathbb{R}^N} \sup_{z \in E_h(\mathbb{R}^N); z \neq 0} \frac{\left|\Delta^3_{x_0}\phi(z)\right|}{|z|^\gamma} < \infty.$$

We conclude this subsection with a result concerning the interpolation between the anisotropic Zygmund-Hölder spaces $C^\gamma_{b,d}(\mathbb{R}^N)$. We refer to Theorem 2.2 and Corollary 2.3 in [Lun97] for details.

Theorem 2.4. *Let r be in $(0, 1)$ and β, γ in $[0, \infty)$ such that $\beta \leq \gamma$. Then, it holds that*

$$\Big(C^\beta_{b,d}(\mathbb{R}^N), C^\gamma_{b,d}(\mathbb{R}^N)\Big)_{r,\infty} = C^{r\gamma+(1-r)\beta}_{b,d}(\mathbb{R}^N)$$

with equivalent norms, where we have denoted for simplicity: $C^0_{b,d}(\mathbb{R}^N) := C_b(\mathbb{R}^N)$.

3 Smoothing effect for truncated density

We present here some analytical properties of the semi-group generated by the operator L^{ou}. Following [SSW12] and [SW12], we will show the existence of a smooth density for such a semi-group and its anisotropic smoothing effect, at least for small times.

Throughout this section, we consider fixed a stochastic base $\Big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\Big)$ satisfying the usual assumptions (see [App09], page 72). Let us then consider the (unique in law) Lévy process $\{Z_t\}_{t\geq 0}$ on $\mathbb{R}^d$ characterized by the Lévy symbol

$$\Phi(p) \,=\, -ib\cdot p + \frac{1}{2}p\cdot Qp + \int_{\mathbb{R}^d_0}\Big(1 - e^{ip\cdot z} + ip\cdot z\mathbb{1}_{B(0,1)}(z)\Big)\,\nu(dz), \quad p\in\mathbb{R}^d.$$

It is well-known by the Lévy-Kitchine formula (see [Jac01]), that the infinitesimal generator of the process $\{BZ_t\}_{t\geq 0}$ is then given by $\mathcal{L}$ on $\mathbb{R}^N$.
Fixed x in $\mathbb{R}^N$, we denote by $\{X_t\}_{t\geq 0}$ the N-dimensional Ornstein-Uhlenbeck process driven by BZ_t, i.e. the unique (strong) solution of the following stochastic differential equation:

$$X_t \,=\, x + \int_0^t AX_s\,ds + BZ_t, \quad t\geq 0, \ \mathbb{P}\text{-almost surely.}$$

By the variation of constants method, it is easy to check that

$$X_t \,=\, e^{tA}x + \int_0^t e^{(t-s)A}B\,dZ_s, \qquad t\geq 0, \ \mathbb{P}\text{-almost surely.} \tag{3.1}$$

The *transition semi-group* associated with L^{ou} is then defined as the family $\{P_t\colon t\geq 0\}$ of linear contractions on $B_b(\mathbb{R}^N)$ given by

$$P_t\phi(x) \,=\, \mathbb{E}\Big[\phi(X_t)\Big], \quad x\in\mathbb{R}^N, \, \phi\in B_b(\mathbb{R}^N). \tag{3.2}$$

We recall that P_t is generated by L^{ou} in the sense that its infinitesimal generator $\mathcal{A}$ coincides with L^{ou} on $C^\infty_c(\mathbb{R}^N)$, the family of smooth functions with compact support.

The next result shows that the random part of X_t (see Equation (3.3)) satisfies again the non-degeneracy assumption [**SD**], even if re-scaled with respect to the anisotropic structure of the dynamics.

Proposition 3.1 (Decomposition)**.** *For any t in $(0,1]$, there exists a Lévy process $\{S^t_u\}_{u\geq 0}$ such that*

$$X_t \overset{law}{=} e^{tA}x + \mathbb{M}_t S^t_t.$$

*Moreover, $\{S^t_u\}_{u\geq 0}$ satisfies assumption [**SD**] with same α as before.*

Proof. For simplicity, we start denoting

$$\Lambda_t \,:=\, \int_0^t e^{(t-s)A}B\,dZ_s, \quad t>0, \tag{3.3}$$

so that $X_t = e^{tA}x + \Lambda_t$. To conclude, we need to construct a Lévy process $\{S^t_u\}_{u\geq 0}$ on $\mathbb{R}^N$ satisfying assumption [**SD**] and

$$\Lambda_t \overset{law}{=} \mathbb{M}_t S^t_t. \tag{3.4}$$

To show the identity in law, we are going to reason in terms of the characteristic functions. By Lemma 2.2 in [SW12], we know that Λ_t is an infinitely divisible random variable with associated Lévy symbol

$$\Phi_{\Lambda_t}(\xi) \,:=\, \int_0^t \Phi\Big((e^{sA}B)^*\xi\Big)\,ds, \quad \xi\in\mathbb{R}^N.$$

Remembering the decomposition $e^{sA}B = e^{sA}\mathbb{M}_sB = \mathbb{M}_sR_sB$ from Lemma 2.2, we can now rewrite Φ_{Λ_t} as

$$\Phi_{\Lambda_t}(\xi) \,=\, t\int_0^1 \Phi\Big((e^{stA}B)^*\xi\Big)\,ds \,=\, t\int_0^1 \Phi\Big((R_{st}B)^*\mathbb{M}_s\mathbb{M}_t\xi\Big)\,ds.$$

The above equality suggests us to define, for any fixed t in $(0,1]$, the (unique in law) Lévy process $\{S^t_u\}_{u\geq 0}$ associated with the Lévy symbol

$$\tilde{\Phi}^t(\xi) \,:=\, \int_0^1 \Phi\Big((R_{st}B)^*\mathbb{M}_s\xi\Big)\,ds, \quad \xi\in\mathbb{R}^N.$$

It is not difficult to check that $\tilde{\Phi}^t$ is indeed a Lévy symbol associated with the Lévy triplet $(\tilde{Q}^t,\tilde{b}^t,\tilde{\nu}^t)$ given by

$$\tilde{Q}^t \,=\, \int_0^1 \mathbb{M}_sR_{st}BQ(\mathbb{M}_sR_{st}B)^*\,ds; \tag{3.5}$$

$$\tilde{b}^t \,=\, \int_0^1 \mathbb{M}_sR_{st}Bb\,ds + \int_0^1\int_{\mathbb{R}^d} \mathbb{M}_sR_{st}Bz\Big[\mathbb{1}_{B(0,1)}(\mathbb{M}_sR_{st}Bz) - \mathbb{1}_{B(0,1)}(z)\Big]\,\nu(dz)ds; \tag{3.6}$$

$$\tilde{\nu}^t(\mathcal{A}) \,=\, \int_0^1 \nu\Big((\mathbb{M}_sR_{st}B)^{-1}\mathcal{A}\Big)\,ds, \quad \mathcal{A}\in\mathcal{B}(\mathbb{R}^d_0). \tag{3.7}$$

Since we have that

$$\mathbb{E}\Big[e^{i\langle\xi,\Lambda_t\rangle}\Big] \,=\, e^{-\Phi_{\Lambda_t}(\xi)} \,=\, e^{-t\tilde{\Phi}^t(\mathbb{M}_t\xi)} \,=\, \mathbb{E}\Big[e^{i\langle\xi,\mathbb{M}_tS^t_t\rangle}\Big],$$

it follows immediately that the identity (3.4) holds.
It remains to show that the family of Lévy measure $\{\tilde{\nu}^t\colon t\in(0,1]\}$ satisfies the assumption [**SD**]. Recalling that condition [**SD**] is assumed to hold for ν, we know that

$$\tilde{\nu}^t(\mathcal{A}) \,=\, \int_0^1 \nu\Big((\mathbb{M}_sR_{st}B)^{-1}\mathcal{A}\Big)\,ds \,\geq\, \int_0^1\int_0^{r_0}\int_{\mathbb{S}^{d-1}} \mathbb{1}_{\mathcal{A}}(r\mathbb{M}_sR_{st}B\theta)\mu(d\theta)\frac{dr}{r^{1+\alpha}}ds, \tag{3.8}$$

for any $\mathcal{A}$ in $\mathcal{B}(\mathbb{R}^d_0)$. Furthermore, it holds from Lemma 2.2 that

$$\inf_{s\in(0,1),\,t\in(0,1],\,\theta\in S^{d-1}}\Big|\mathbb{M}_sR_{st}B\theta\Big| \,=:\, R_0>0. \tag{3.9}$$

It allows us to define two functions $l^t\colon[0,1]\times S^{d-1}\to S^{N-1}$, $m^t\colon[0,1]\times S^{d-1}\to\mathbb{R}$, given by

$$l^t(s,\theta) \,:=\, \frac{\mathbb{M}_sR_{st}B\theta}{|\mathbb{M}_sR_{st}B\theta|} \;\text{ and }\; m^t(s,\theta) \,:=\, |\mathbb{M}_sR_{st}B\theta|.$$

Using the Fubini theorem, we can now rewrite Equation (3.8) as

$$\begin{aligned}\tilde{\nu}^t(\mathcal{A}) &\geq \int_0^1 \int_{\mathbb{S}^{d-1}} \int_0^{r_0} \mathbb{1}_{\mathcal{A}}(l^t(s,\theta)m^t(s,\theta)r)\frac{dr}{r^{1+\alpha}}\mu(d\theta)ds \\ &= \int_0^1 \int_{\mathbb{S}^{d-1}} \int_0^{m^t(s,\theta)r_0} \mathbb{1}_{\mathcal{A}}(l^t(s,\theta)r)\frac{dr}{r^{1+\alpha}}[m^t(s,\theta)]^\alpha\mu(d\theta)ds.\end{aligned}$$

Exploiting again Control (3.9), we can conclude that

$$\tilde{\nu}^t(C) \geq \int_0^1 \int_{\mathbb{S}^{d-1}} \int_0^{R_0} \mathbb{1}_C(l^t(s,\theta)r)\frac{dr}{r^{1+\alpha}}\tilde{m}^t(ds,d\theta) = \int_0^{R_0} \int_{\mathbb{S}^{N-1}} \mathbb{1}_C(\tilde{\theta}r)\tilde{\mu}^t(d\tilde{\theta})\frac{dr}{r^{1+\alpha}}, \tag{3.10}$$

where $\tilde{m}^t(ds,d\theta)$ is a measure on $[0,1]\times S^{d-1}$ given by

$$\tilde{m}^t(ds,d\theta) := [m^t(s,\theta)]^\alpha\mu(d\theta)ds$$

and $\tilde{\mu}^t := (l^t)_*\tilde{m}^t$ is the measure $\tilde{m}^t$ push-forwarded through l^t on S^{N-1}. It is easy to check that the measure $\tilde{\mu}^t$ is finite and non-degenerate in the sense of (1.4), replacing therein d by N. □

An immediate application of the above result is a first representation formula for the transition semi-group $\{P_t\colon t\geq 0\}$ associated with the Ornstein-Uhlenbeck process $\{X_t\}_{t\geq 0}$, at least for small times. Indeed, denoting by $\mathbb{P}_X$ the law of a random variable X, Equation (3.4) implies that for any ϕ in $B_b(\mathbb{R}^N)$, it holds that

$$P_t\phi(x) = \int_{\mathbb{R}^N}\phi(e^{tA}x+y)\,\mathbb{P}_{\Lambda_t}(dy) = \int_{\mathbb{R}^N}\phi(e^{tA}x+\mathbb{M}_t y)\,\mathbb{P}_{S^t_t}(dy), \quad x\in\mathbb{R}^N,\, t\in(0,1]. \tag{3.11}$$

Moreover, condition [**SD**] for $\{S^t_u\}_{u\geq 0}$ allows us to decompose it into two components: a truncated, α-stable part and a remainder one. Indeed, if we denote by ν^t_α the measure serving as lower bound to the Lévy measure $\tilde{\nu}^t$ in (3.10), i.e.

$$\nu^t_\alpha(\mathcal{A}) := \int_0^{R_0}\int_{\mathbb{S}^{N-1}}\mathbb{1}_{\mathcal{A}}(\theta r)\tilde{\mu}^t(d\theta)\frac{dr}{r^{1+\alpha}}, \quad C\in\mathcal{B}(\mathbb{R}^N_0), \tag{3.12}$$

we can consider $\{Y^t_u\}_{u\geq 0}$, the Lévy process on $\mathbb{R}^N$ associated with the Lévy triplet $(0,0,\nu^t_\alpha)$. We recall now a useful fact involving the Lévy symbol Φ^t_α of the process Y^t. The non-degeneracy of the measure $\tilde{\mu}^t$ is equivalent to the existence of a constant $C>0$ such that

$$\Phi^t_\alpha(\xi) \geq C|\xi|^\alpha, \quad \xi\in\mathbb{R}^N. \tag{3.13}$$

A proof of this result can be found, for example, in [Pri12] p.424.
In order to apply the results in [SSW12], we are going to truncate the above process at the typical time scale for an α-stable process. This is $t^{1/\alpha}$ when considering the process at time t (cf. Example 2.1). Namely, we consider the family $\{\mathbb{P}^{\mathrm{tr}}_t\}_{t\geq 0}$ of infinitely divisible probabilities whose characteristic function has the form $\widehat{\mathbb{P}^{\mathrm{tr}}_t}(\xi) := \exp[-\Phi^{\mathrm{tr}}_t(\xi)]$, where

$$\Phi^{\mathrm{tr}}_t(\xi) := \int_{|z|\leq t^{\frac{1}{\alpha}}}\left[1-e^{i\langle\xi,z\rangle}+i\langle\xi,z\rangle\right]\nu^t_\alpha(dz).$$

On the other hand, since the measure $\tilde{\nu}^t$ satisfies assumption [**SD**], we know that the remainder $\tilde{\nu}^t - \mathbb{1}_{B(0,t^{1/\alpha})}\nu^t_\alpha$ is again a Lévy measure on $\mathbb{R}^N$. Let $\{\pi_t\}_{t\geq 0}$ be the family of infinitely divisible probability associated with the following Lévy triplet:

$$(\tilde{Q}^t, \tilde{b}^t, \tilde{\nu}^t - \mathbb{1}_{B(0,t^{1/\alpha})}\nu^t_\alpha).$$

It follows immediately that $\mathbb{P}_{S^t_t} = \mathbb{P}^{\text{tr}}_t * \pi_t$ for any $t > 0$. We can now disintegrate the measure $\mathbb{P}_{S^t_t}$ in Equation (3.11) in order to obtain

$$P_t\phi(x) \,=\, \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi\Big(e^{tA}x + \mathbb{M}_t(y_1+y_2)\Big)\,\mathbb{P}^{\text{tr}}_t(dy_1)\pi_t(dy_2). \tag{3.14}$$

The next step is to use Proposition 2.3 in [SSW12] to show a smoothing effect for the family of truncated stable measures $\{\mathbb{P}^{\text{tr}}_t \colon t\geq 0\}$, at least for small times. Namely,

Proposition 3.2. *Fixed m in $\mathbb{N}_0$, there exists $T_0 := T_0(m) > 0$ such that for any t in $(0,T_0]$, the probability $\mathbb{P}^{tr}_t$ has a density $p^{tr}(t,\cdot)$ that is m-times continuously differentiable on $\mathbb{R}^N$.*
Moreover, for any ϑ in $\mathbb{N}^N$ such that $|\vartheta|\leq m$, there exists a constant $C := C(m,|\vartheta|)$ such that

$$|D^\vartheta p^{tr}(t,y)| \,\leq\, Ct^{-\frac{N+|\vartheta|}{\alpha}}\Big(1+\frac{|y|}{t^{1/\alpha}}\Big)^{|\vartheta|-m}, \quad t\in(0,T_0],\, y\in\mathbb{R}^N.$$

Proof. The result follows immediately applying Proposition 2.3 in [SSW12]. To do so, we need to show that the Lévy symbol Φ^t_α of the process $\{Y^t_u\}_{u\geq 0}$ satisfies the following assumptions:

— *Hartman-Wintner condition.* There exists $T > 0$ such that

$$\liminf_{|\xi|\to\infty} \frac{\text{Re}\Phi^t_\alpha(\xi)}{\ln(1+|\xi|)} \,=\, \infty, \quad t\in(0,T];$$

— *Controllability condition.* There exist $T > 0$ and $c > 0$ such that

$$\int_{\mathbb{R}^N} e^{-t\text{Re}\Phi^t_\alpha(\xi)}|\xi|^m \,\leq\, ct^{-\frac{m+N}{\alpha}}, \quad t\in(0,T].$$

In order to show that the above conditions hold, we fix $T\leq 1$ and we recall that the Lévy symbol Φ^t_α of Y_t, the truncated α-stable process with Lévy measure introduced in (3.12), can be written through the Lévy-Kitchine formula as

$$\Phi^t_\alpha(\xi) \,=\, \int_{\mathbb{R}^N_0}\Big(1-e^{i\langle\xi,z\rangle}+i\langle\xi,z\rangle\Big)\nu^t_\alpha(dz) \,=\, \int_0^{R_0}\int_{\mathbb{S}^{N-1}}\Big(1-\cos(\langle\xi,r\theta\rangle)\Big)\,\tilde{\mu}^t(d\theta)\frac{dr}{r^{1+\alpha}}.$$

We have seen in Equation (3.13) that the non-degeneracy of $\tilde{\mu}^t$ implies that $\Phi^t_\alpha(\xi)\geq C|\xi|^\alpha$. The Hartman-Wintner condition then follows immediately since

$$\liminf_{|\xi|\to\infty}\frac{\text{Re}\Phi^t_\alpha(\xi)}{\ln(1+|\xi|)} \,\geq\, \liminf_{|\xi|\to\infty}\frac{c|\xi|^\alpha}{\ln(1+|\xi|)} \,=\, \infty.$$

To show instead the controllability assumption, let us firstly notice that

$$e^{-t\mathrm{Re}\Phi^t_\alpha(\xi)} \;\le\; \begin{cases} 1, & \text{if } |\xi|\le R; \\ e^{-ct|\xi|^\alpha}, & \text{if } |\xi|>R, \end{cases}$$

for some $R>0$. It then follows that

$$\begin{aligned}\int_{\mathbb{R}^N} e^{-t\mathrm{Re}\Phi^t_\alpha(\xi)}|\xi|^m\,d\xi \;&=\; \int_{|\xi|\le R}|\xi|^m\,d\xi + \int_{|\xi|>R} e^{-ct|\xi|^\alpha}|\xi|^m\,d\xi \\ &\le\; C + t^{-\frac{m+N}{\alpha}}\int_{|\xi|>t^{1/\alpha}R} e^{-c|\xi|^\alpha}|\xi|^m\,d\xi \\ &\le\; C + t^{-\frac{m+N}{\alpha}}\int_{\mathbb{R}^N} e^{-c|\xi|^\alpha}|\xi|^m\,d\xi \\ &\le\; Ct^{-\frac{m+N}{\alpha}},\end{aligned}$$

where in the last step we used that $1\le t^{-\frac{m+N}{\alpha}}$. □

4 Estimates for transition semi-group

The results in the previous section (Proposition 3.2 and Equation (3.14)) allow us to represent the semi-group P_t of the Ornstein-Uhlenbeck process $\{X_t\}_{t\ge 0}$ as

$$P_t\phi(x) \;=\; \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi(\mathbb{M}_t(y_1+y_2)+e^{tA}x)p^{\mathrm{tr}}(t,y_1)\,dy_1\pi_t(dy_2), \quad x\in\mathbb{R}^N, \tag{4.1}$$

at least for small time intervals.
Here, we will focus on estimates in $\|\cdot\|_\infty$-norm of the transition semi-group $\{P_t\colon t\ge 0\}$ given in Equation (3.2) and its derivatives. The main result in this section is Corollary 4.4 that shows the continuity of P_t between anisotropic Zygmund-Hölder spaces. These controls will be fundamental in the next section to prove Schauder Estimates in the elliptic and parabolic settings.
As we will see in the following result, the derivatives of the semi-group P_t with respect to a component i in I_h induces an additional time singularity of order $\frac{1+\alpha(h-1)}{\alpha}$, corresponding to the intrinsic time scale of the considered component.

Proposition 4.1. *Let h,h',h'' be in $[\![1,n]\!]$ and ϕ in $B_b(\mathbb{R}^N)$. Then, there exists a constant $C>0$ such that for any i in I_h, any j in $I_{h'}$ and any k in $I_{h''}$, it holds that*

$$\|D_iP_t\phi\|_\infty \;\le\; C\|\phi\|_\infty\Big(1+t^{-\frac{1+\alpha(h-1)}{\alpha}}\Big), \quad t>0; \tag{4.2}$$

$$\|D^2_{i,j}P_t\phi\|_\infty \;\le\; C\|\phi\|_\infty\Big(1+t^{-\frac{2+\alpha(h+h'-2)}{\alpha}}\Big), \quad t>0; \tag{4.3}$$

$$\|D^3_{i,j,k}P_t\phi\|_\infty \;\le\; C\|\phi\|_\infty\Big(1+t^{-\frac{3+\alpha(h+h'+h''-3)}{\alpha}}\Big), \quad t>0. \tag{4.4}$$

Proof. We start fixing a time horizon $T:=1\wedge T_0(N+4)>0$, where $T_0(m)$ was defined in Proposition 3.2. Our choice of $N+4$ is motivated by the fact that we consider derivatives up to order 3.
On the interval $(0,T]$, the representation formula (4.1) holds and $P_t\phi$ is three times

differentiable for any ϕ in $B_b(\mathbb{R}^N)$. We are going to show only Estimate (4.2) since the controls for the higher derivatives can be obtained similarly.
Fixed $t \le T$, let us consider i in I_h for some h in $[\![1,n]\!]$. When $t \le T$, we recall from Equation (4.1) that, up to a change of variables, it holds that

$$\left|D_iP_t\phi(x)\right| = \left|D_i\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi(\mathbb{M}_t(y_1+y_2))p^{\mathrm{tr}}(t,y_1-\mathbb{M}_t^{-1}e^{tA}x)\,dy_1\pi_t(dy_2)\right|.$$

We can then move the derivative inside the integral and write that

$$\begin{aligned}\left|D_iP_t\phi(x)\right| &= \left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi(\mathbb{M}_t(y_1+y_2))\langle\nabla p^{\mathrm{tr}}(t,y_1-\mathbb{M}_t^{-1}e^{tA}x),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2)\right|\\ &\le |\mathbb{M}_t^{-1}e^{tA}e_i|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\phi(\mathbb{M}_t(y_1+y_2))|\,|\nabla p^{\mathrm{tr}}(t,y_1-\mathbb{M}_t^{-1}e^{tA}x))|\,dy_1\pi_t(dy_2)\\ &\le Ct^{-(h-1)}\|\phi\|_\infty\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\nabla p^{\mathrm{tr}}(t,y_1)|\,dy_1\pi_t(dy_2),\end{aligned}\tag{4.5}$$

where in the last step we exploited Lemma 2.2 to control

$$|\mathbb{M}_t^{-1}e^{tA}e_i| \le \sum_{k=1}^n|\mathbb{M}_t^{-1}E_ke^{tA}E_he_i| \le C\Big[\sum_{k=1}^{h-1}t^{k-h}t+\sum_{k=h}^nt^{-(k-1)}t^{-(h-1)}\Big] \le Ct^{-(h-1)},\tag{4.6}$$

remembering that $t \le 1$. We conclude the case $t \le T$ using the control on p^{tr} (Proposition 3.2 with $m = N+2$) to write that

$$\begin{aligned}\left|D_iP_t\phi(x)\right| &\le C\|\phi\|_\infty\pi_t(\mathbb{R}^N)t^{-(h-1)}\int_{\mathbb{R}^N}t^{-\frac{N+1}{\alpha}}\Big(1+\frac{|y_1|}{t^{1/\alpha}}\Big)^{-(N+1)}\,dy_1\\ &\le C\|\phi\|_\infty t^{-\frac{1+\alpha(h-1)}{\alpha}}\int_{\mathbb{R}^N}(1+|z|)^{-(N+1)}\,dz\\ &\le C\|\phi\|_\infty t^{-\frac{1+\alpha(h-1)}{\alpha}}.\end{aligned}\tag{4.7}$$

Above, we used the change of variables $z = t^{-1/\alpha}y_1$. When $t > T$, we can exploit the already proven controls for small times, the semi-group and the contraction properties of $\{P_t\colon t \ge 0\}$ on $B_b(\mathbb{R}^N)$ to write that

$$\|D_iP_t\phi\|_\infty = \|D_iP_T\Big(P_{t-T}\phi\Big)\|_\infty \le C_T\|P_{t-T}\phi\|_\infty \le C\|\phi\|_\infty.\tag{4.8}$$

We have thus shown Control (4.2) for any $t > 0$. □

The following interpolation inequality (see e.g. [Tri92])

$$\|\phi\|_{C_b^{r\delta_1+(1-r)\delta_2}} \le C\|\phi\|^r_{C_b^{\delta_1}}\|\phi\|^{1-r}_{C_b^{\delta_2}}\tag{4.9}$$

valid for $0 \le \delta_1 < \delta_2$, r in $(0,1)$ and ϕ in $C^{\delta_2}(\mathbb{R}^N)$, allows us to extend easily the above result.

Corollary 4.2. *Let γ be in $[0,1+\alpha)$. Then, there exists a constant $C > 0$ such that*

$$\|P_t\|_{\mathcal{L}(C_b,C^\gamma_{b,d})} \le C\Big(1+t^{-\frac{\gamma}{\alpha}}\Big),\quad t > 0.\tag{4.10}$$

Proof. Let us firstly assume that γ is in $(0,1]$. Remembering the definition of $C^\gamma_{b,d}$-norm in (2.7), we start fixing a point x_0 in $\mathbb{R}^N$ and h in $[\![2,n]\!]$. Then, the contraction property of the semi-group implies that

$$\|P_t\phi(x_0+\cdot)_{|E_h(\mathbb{R}^N)}\|_\infty \,\le\, C\|\phi\|_\infty.$$

Moreover, Control (4.2) in Proposition 4.1 ensures that

$$\|D_iP_t\phi(x_0+\cdot)_{|E_h(\mathbb{R}^N)}\|_\infty \,\le\, C\|\phi\|_\infty\Big(1+t^{-\frac{1+\alpha(h-1)}{\alpha}}\Big).$$

It follows immediately that

$$\|P_t\phi(x_0+\cdot)_{|E_h(\mathbb{R}^N)}\|_{C^1_b} \,\le\, C\|\phi\|_\infty(1+t^{-\frac{1+\alpha(h-1)}{\alpha}}).$$

We can now apply the interpolation inequality (4.9) with $\delta_1=0$, $\delta_2=1$ and $r=\gamma/(1+\alpha(h-1))$ in order to obtain that

$$\begin{aligned}\|P_t\phi(x_0+\cdot)_{|E_h(\mathbb{R}^N)}\|_{C^r_b} \,&\le\, C\|P_t\phi(x_0+\cdot)_{|E_h(\mathbb{R}^N)}\|^r_{C^1_b}\|P_t\phi(x_0+\cdot)_{|E_h(\mathbb{R}^N)}\|^{1-r}_\infty\\ &\le\, C\|\phi\|_\infty\Big(1+t^{-\frac{\gamma}{\alpha}}\Big).\end{aligned}$$

The argument is analogous for γ in $(1,3)$, considering only the case $h=0$. □

The next result allows us to extend the controls in Proposition 4.1 to functions in the anisotropic Zygmund-Hölder spaces. Roughly speaking, it states that the anisotropic γ-Hölder regularity induces a "homogeneous" gain in time of order γ/α that can be used to weaken, at least partially, the time singularities associated with the derivatives. The general argument of proof will mimic the one of Proposition 4.1 even if, this time, we will need to make the Hölder modulus of ϕ appear. It will be managed exploiting some "partial" cancellation arguments (cf. (4.17)).

Theorem 4.3. *Let h,h',h'' be in $[\![1,n]\!]$ and ϕ in $C^\gamma_{b,d}(\mathbb{R}^N)$ for some γ in $[0,1+\alpha)$. Then, there exists a constant $C>0$ such that for any i in I_h, any j in $I_{h'}$ and any k in $I_{h''}$, it holds that*

$$\|D_iP_t\phi\|_\infty \,\le\, C\|\phi\|_{C^\gamma_{b,d}}\Big(1+t^{\frac{\gamma-(1+\alpha(h-1))}{\alpha}}\Big), \quad t>0; \tag{4.11}$$

$$\|D^2_{i,j}P_t\phi\|_\infty \,\le\, C\|\phi\|_{C^\gamma_{b,d}}\Big(1+t^{\frac{\gamma-(2+\alpha(h+h'-2))}{\alpha}}\Big), \quad t>0; \tag{4.12}$$

$$\|D^3_{i,j,k}P_t\phi\|_\infty \,\le\, C\|\phi\|_{C^\gamma_{b,d}}\Big(1+t^{\frac{\gamma-(3+\alpha(h+h'+h''-3))}{\alpha}}\Big), \quad t>0. \tag{4.13}$$

Proof. Similarly to Proposition 4.1, we start fixing a time horizon

$$T \,:=\, 1\wedge T_0(N+6)>0. \tag{4.14}$$

Then, Corollary 4.2 implies the continuity of P_t on $C^\gamma_{b,d}(\mathbb{R}^N)$, for any $t\ge T/2$. Indeed,

$$\|P_t\phi\|_{C^\gamma_{b,d}} \,\le\, C\|\phi\|_\infty\Big(1+t^{-\frac{\gamma}{\alpha}}\Big) \,\le\, C_T\|\phi\|_{C^\gamma_{b,d}}. \tag{4.15}$$

The same argument shown in Equation (4.8) can now be applied to prove Control (4.11) for $t > T$. Namely,

$$\|D_iP_t\phi\|_\infty \,=\, \|D_iP_{T/2}\Big(P_{t-T/2}\phi\Big)\|_\infty \,\le\, C_T\|P_{t-T/2}\phi\|_\infty \,\le\, C\|\phi\|_\infty.$$

The same reasoning can be used for the higher derivatives, too.
When $t \le T$, let us assume $\alpha > 1$, so that $1+\alpha > 2$. The case $\alpha \le 1$ can be handled similarly taking into account one less derivative. Moreover, we notice that we need to prove Controls (4.11)-(4.13) only for γ in $(2, 1+\alpha)$ thanks to interpolation techniques. Indeed, if we want, for example, to prove Estimates (4.11) for some γ' in $(0, 2]$, we can use Theorem 2.4 to show that

$$\|D_iP_t\|_{\mathcal{L}(C^{\gamma'}_{b,d};B_b)} \,\le\, \Big(\|D_iP_t\|_{\mathcal{L}(B_b)}\Big)^{1-\gamma'/\gamma}\Big(\|D_iP_t\|_{\mathcal{L}(C^{\gamma}_{b,d},B_b)}\Big)^{\gamma'/\gamma} \,\le\, C\Big(1+t^{\frac{\gamma'-(1+\alpha(h-1))}{\alpha}}\Big),$$

once we have proven Estimate (4.11) for $\gamma > 2$.
We are only going to show Control (4.11) for $t \le T$ and γ in $(2, 1+\alpha)$. The estimates (4.12) and (4.13) involving the higher derivatives can be obtained in an analogous way. Fixed i in I_h for some h in $[\![1,n]\!]$, we start noticing from Equation (4.1) that, up to the change of variables $\tilde{y}_1 = y_1 + \mathbb{M}_t^{-1}e^{tA}x$, it holds that

$$\begin{aligned} D_iP_t\phi(x) \,&=\, D_i\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi(\mathbb{M}_t(\tilde{y}_1+y_2))p^{\mathrm{tr}}(t,\tilde{y}_1-\mathbb{M}_t^{-1}e^{tA}x)\,d\tilde{y}_1\pi_t(dy_2) \\ &=\, \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi(\mathbb{M}_t(\tilde{y}_1+y_2))D_i\Big[p^{\mathrm{tr}}(t,\tilde{y}_1-\mathbb{M}_t^{-1}e^{tA}x)\Big]\,d\tilde{y}_1\pi_t(dy_2). \end{aligned} \tag{4.16}$$

Recalling that here, D_i stands for the derivative with respect to the variable x_i, we then notice that

$$\int_{\mathbb{R}^N} D_i\Big[p^{\mathrm{tr}}(t,\tilde{y}_1-\mathbb{M}_t^{-1}e^{tA}x)\Big]\,d\tilde{y}_1 \,=\, D_i\int_{\mathbb{R}^N}\Big[p^{\mathrm{tr}}(t,\tilde{y}_1-\mathbb{M}_t^{-1}e^{tA}x)\Big]\,d\tilde{y}_1 \,=\, 0.$$

In particular, it immediately follows that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi(\mathbb{M}_ty_2+e^{tA}x)D_i\Big[p^{\mathrm{tr}}(t,\tilde{y}_1-\mathbb{M}_t^{-1}e^{tA}x)\Big]\,dy_1\pi_t(dy_2) \,=\, 0.$$

This property will allow to use a cancellation argument in Equation (4.17) below, once we split the small jumps in the non-degenerate contributions and the other ones. We thus get from (4.16) that

$$\begin{aligned} &D_iP_t\phi(x) \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big((\phi(\mathbb{M}_t(\tilde{y}_1+y_2))) - \phi(\mathbb{M}_ty_2+e^{tA}x)\Big)\,D_i\Big[p^{\mathrm{tr}}(t,\tilde{y}_1-\mathbb{M}_t^{-1}e^{tA}x)\Big]\,dy_1\pi_t(dy_2) \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Delta\phi(t,y_1,y_2,x)\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2), \end{aligned} \tag{4.17}$$

where, after the backward change of variables, we have also denoted:

$$\Delta\phi(t,y_1,y_2,x) \,:=\, \phi(\mathbb{M}_t(y_1+y_2)+e^{tA}x) - \phi(\mathbb{M}_ty_2+e^{tA}x).$$

We can then decompose the difference $\Delta\phi$ in the following way:

$$\Delta\phi(t,y_1,y_2,x) \,=\, \Lambda_0(t,y_1,y_2,x) + \Lambda_1(t,y_1,y_2,x) \tag{4.18}$$

where we denoted

$$\begin{aligned}\Lambda_0(t,y_1,y_2,x) &:= \phi(E_1y_1+\mathbb{M}_ty_2+e^{tA}x)-\phi(\mathbb{M}_ty_2+e^{tA}x);\\ \Lambda_1(t,y_1,y_2,x) &:= \phi(\mathbb{M}_ty_1+\mathbb{M}_ty_2+e^{tA}x)-\phi(E_1y_1+\mathbb{M}_ty_2+e^{tA}x).\end{aligned}$$

Noticing now that the first contribution can be easily controlled

$$|\Lambda_1(t,y_1,y_2,x)| \le \|\phi\|_{C^\gamma_{b,d}}\sum_{k=2}^n |E_k\mathbb{M}_ty_1|^{\frac{\gamma}{1+\alpha(k-1)}}, \tag{4.19}$$

we can then write that

$$\begin{aligned}&\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Lambda_1(t,y_1,y_2,x)\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2)\right|\\ &\qquad\le \|\phi\|_{C^\gamma_{b,d}}t^{-(h-1)}\sum_{k=2}^n\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\nabla p^{\mathrm{tr}}(t,y_1)||E_k\mathbb{M}_ty_1|^{\frac{\gamma}{1+\alpha(k-1)}}\,dy_1\pi(dy_2),\end{aligned}$$

where we also exploited the control in (4.6).
The above expression allows us to conclude the control for Λ_1 as in (4.7), using Proposition 3.2 with $m=N+4$ and $|\vartheta|=1$. Namely,

$$\begin{aligned}&\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Lambda_1(t,y_1,y_2,x)\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2)\right|\\ &\qquad\le C\|\phi\|_{C^\gamma_{b,d}}t^{-(h-1)}\sum_{k=2}^n\int_{\mathbb{R}^N}t^{-\frac{N+1}{\alpha}}\Big(1+\frac{|y_1|}{t^{\frac{1}{\alpha}}}\Big)^{-(N+3)}|E_k\mathbb{M}_ty_1|^{\frac{\gamma}{1+\alpha(k-1)}}\,dy_1\\ &\qquad\le C\|\phi\|_{C^\gamma_{b,d}}t^{\frac{\gamma-(1+\alpha(h-1))}{\alpha}}\sum_{k=2}^n\int_{\mathbb{R}^N}\Big(1+|z|\Big)^{-(N+3)}|z|^{\frac{\gamma}{1+\alpha(k-1)}}\,dz\\ &\qquad\le C\|\phi\|_{C^\gamma_{b,d}}t^{\frac{\gamma-(1+\alpha(h-1))}{\alpha}},\end{aligned}\tag{4.20}$$

where in the second step we used again the change of variable $z=y_1t^{-1/\alpha}$.
For the term Λ_0, we start instead applying a Taylor expansion of second order along E_1y_1:

$$\begin{aligned}\Lambda_0(t,x,y_1,y_2) &= \phi(E_1y_1+\mathbb{M}_ty_2+e^{tA}x)-\phi(\mathbb{M}_ty_2+e^{tA}x)\\ &= \langle\nabla\phi(\mathbb{M}_ty_2+e^{tA}x),E_1y_1\rangle+\int_0^1\langle D^2\phi(\mathbb{M}_ty_2+e^{tA}x+\lambda E_1y_1)R_1y_1,E_1y_1\rangle\,d\lambda\\ &=: \Lambda_2(t,x,y_1,y_2)+\Lambda_3(t,x,y_1,y_2).\end{aligned}\tag{4.21}$$

Now, we want to control the component involving the second term Λ_2:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Lambda_2(t,y_1,y_2,x)\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2).$$

To do it, we exploit an integration by part formula with respect to the derivative of p^{tr}. Indeed, reasoning component-wise if necessary, it is not difficult to check that

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Lambda_2(t,y_1,y_2,x)\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2)\qquad(4.22)\\ &\qquad= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\langle\nabla\phi(\mathbb{M}_ty_2+e^{tA}x),E_1y_1\rangle\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2)\\ &\qquad= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\langle\nabla\phi(\mathbb{M}_ty_2+e^{tA}x),E_1\mathbb{M}_t^{-1}e^{tA}e_i\rangle p^{\mathrm{tr}}(t,y_1)\,dy_1\pi_t(dy_2).\end{aligned}$$

It then follows immediately that

$$\begin{aligned}\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Lambda_2(t,y_1,y_2,x)\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2)\Big| \\ \le C\|\phi\|_{C^\gamma_{b,d}}|E_1e^{tA}e_i|\int_{\mathbb{R}^N}|p^{\mathrm{tr}}(t,y_1)|\,dy_1 \quad (4.23)\\ \le C\|\phi\|_{C^\gamma_{b,d}},\end{aligned}$$

where in the last passage we also used Lemma 2.2.
To control the contribution involving the third term Λ_3, we will need an additional cancellation argument. Let us assume for the moment that the family of truncated probabilities $\{\mathbb{P}^{\mathrm{tr}}_t\}_{t\ge0}$ has zero mean value, so that it holds that:

$$\int_{\mathbb{R}^N}E_1y_1p^{\mathrm{tr}}(t,y_1)\,dy_1 \,=\, 0_{\mathbb{R}^d}.$$

Under this additional hypothesis, it is possible to show that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\langle D^2\phi(\mathbb{M}_ty_2+e^{tA}x)E_1y_1,E_1y_1\rangle\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2) \,=\, 0. \quad (4.24)$$

Indeed, applying again an integration by parts formula with respect to the derivative on p^{tr}, we notice, reasoning as well component-wise as in (4.22), that:

$$\begin{aligned}&\int_{\mathbb{R}^N}\langle D^2\phi(\mathbb{M}_ty_2+e^{tA}x)E_1y_1,E_1y_1\rangle\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\\ &\quad= \int_{\mathbb{R}^N}\langle D^2\phi(\mathbb{M}_ty_2+e^{tA}x)E_1y_1,E_1\mathbb{M}_t^{-1}e^{tA}e_i\rangle p^{\mathrm{tr}}(t,y_1)\,dy_1\\ &\quad= \Big\langle D^2\phi(\mathbb{M}_ty_2+e^{tA}x)\Big[\int_{\mathbb{R}^N}E_1y_1p^{\mathrm{tr}}(t,y_1)\,dy_1\Big],E_1e^{tA}e_i\Big\rangle\\ &\quad= 0.\end{aligned}$$

The cancellation argument in (4.24) allows now to write that

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Lambda_3(t,y_1,y_2,x)\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2)\\ &\quad= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_0^1\Big\langle\big[D^2\phi(\mathbb{M}_ty_2+e^{tA}x+\lambda E_1y_1)-D^2\phi(\mathbb{M}_ty_2+e^{tA}x)\big]E_1y_1,E_1y_1\Big\rangle\\ &\qquad\qquad\times\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2).\end{aligned}$$

The same arguments presented in Control (4.20) can be also applied here to show that

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Lambda_3(t,y_1,y_2,x)\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2)\Big|\\ &\quad\le C\|\phi\|_{C^\gamma_{b,d}}t^{-(h-1)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|E_1y_1|^{\gamma-2}|E_1y_1|^2|\nabla p^{\mathrm{tr}}(t,y_1)|\,dy_1\pi_t(dy_2)\\ &\quad\le C\|\phi\|_{C^\gamma_{b,d}}t^{\frac{\gamma}{\alpha}-\frac{1+\alpha(h-1)}{\alpha}}. \qquad (4.25)\end{aligned}$$

Going back to Expression (4.17), we notice that the decompositions in (4.18) and (4.21) implies immediately that

$$|D_iP_t\phi(x)| \,\le\, \sum_{i=1}^3\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Lambda_i(t,y_1,y_2,x)\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2)\Big|.$$

Then, we can finally use the estimates in (4.20), (4.23) and (4.25) to conclude that

$$\Big|D_iP_t\phi(x)\Big| \,\le\, C\|\phi\|_{C^\gamma_{b,d}}\Big(1+t^{\frac{\gamma}{\alpha}-\frac{1+\alpha(i-1)}{\alpha}}\Big).$$

In the general case, when the family of probabilities $\{\mathbb{P}^{\text{tr}}_t\}_{t\ge0}$ possibly has non-zero mean value, it is then enough to follow the same reasoning above, taking into account, in the first cancellation argument, the additional error given by the mean.
Namely, we will need to consider $\phi(\mathbb{M}_ty_2+e^{tA}x-m_t)$, where m_t is the mean value associated with $\mathbb{P}_t$, in Equation (4.17). □

Next, we are going to use the controls in Theorem 4.3 to show the main result of this section. It states the continuity of the semi-group P_t between anisotropic Zygmund-Hölder spaces at a cost of additional time singularities.

Corollary 4.4. *Let β,γ be in $[0,1+\alpha)$ such that $\beta\le\gamma$. Then, there exists a constant $C>0$ such that*

$$\|P_t\|_{\mathcal{L}(C^\beta_{b,d},C^\gamma_{b,d})} \,\le\, C\Big(1+t^{\frac{\beta-\gamma}{\alpha}}\Big), \quad t>0. \tag{4.26}$$

Proof. It is enough to show the result only for $\gamma=\beta$ non-integer, thanks to interpolation techniques. Indeed, fixed $\beta<\gamma$, we can use Theorem 2.4 to show that

$$\|P_t\|_{\mathcal{L}(C^\beta_{b,d}(\mathbb{R}^N),C^\gamma_{b,d}(\mathbb{R}^N))} \,\le\, \Big(\|P_t\|_{\mathcal{L}(C_b(\mathbb{R}^N),C^\gamma_{b,d}(\mathbb{R}^N))}\Big)^{1-\frac{\beta}{\gamma}}\Big(\|P_t\|_{\mathcal{L}(C^\gamma_{b,d}(\mathbb{R}^N))}\Big)^{\frac{\beta}{\gamma}}.$$

On the other hand, if we fix γ integer, we can take γ' in $(\gamma,1+\alpha)$ non-integer such that Theorem 2.4 implies:

$$\|P_t\|_{\mathcal{L}(C^\gamma_{b,d}(\mathbb{R}^N))} \,\le\, \Big(\|P_t\|_{\mathcal{L}(C_b(\mathbb{R}^N)}\Big)^{1-\frac{\gamma}{\gamma'}}\Big(\|P_t\|_{\mathcal{L}(C^{\gamma'}_{b,d}(\mathbb{R}^N))}\Big)^{\frac{\gamma}{\gamma'}}.$$

The general result will then follows from the two above controls and Equation (4.10), once we have shown Estimate (4.26) for $\gamma=\beta$ non-integer.
Fixed again the time horizon T given in (4.14), we start noticing that Control (4.26) for $t\ge T$ has already been shown in Equation (4.15) .
To prove it when $t\le T$, we are going to exploit the equivalent norm defined in (2.8) of Lemma 2.3. For this reason, we fix h in $[\![1,n]\!]$, a point x_0 in $\mathbb{R}^N$ and $z\neq0$ in $E_h(\mathbb{R}^N)$ and we would like to show that

$$\Big|\Delta^3_{x_0}\Big(P_t\phi\Big)(z)\Big| \,\le\, C\|\phi\|_{C^\gamma_{b,d}}|z|^{\frac{\gamma}{1+\alpha(h-1)}}, \tag{4.27}$$

for some constant $C>0$ independent from x_0. Before starting with the calculations, we highlight the presence of three different "regimes" appearing below. On the one hand, we will firstly consider a *macroscopic regime* appearing for $|z|\ge1$. On the other hand, we will say that the *off-diagonal regime* holds if $t^{\frac{1+\alpha(h-1)}{\alpha}}\le|z|\le1$. It will mean in particular that the spatial distance is larger than the characteristic time-scale. Finally, a *diagonal regime* will be in force when $t^{\frac{1+\alpha(h-1)}{\alpha}}\ge|z|$ and the spatial point will be instead smaller than the typical time-scale magnitude. While for the two first regimes, we are

going to use the contraction property of the semi-group, the third regime will require to exploit the controls in Hölder norms given by Theorem 4.3.
As said above, Estimate (4.27) in the macroscopic regime (i.e. $|z| \ge 1$) follows immediately from the contraction property of P_t on $B_b(\mathbb{R}^N)$. Indeed,

$$\left|\Delta^3_{x_0}\Big(P_t\phi\Big)(z)\right| \le C\|P_t\phi\|_\infty \le C\|\phi\|_{C^\gamma_{b,d}}|z|^{\frac{\gamma}{1+\alpha(h-1)}}. \tag{4.28}$$

For $t^{\frac{1+\alpha(h-1)}{\alpha}} \le |z| \le 1$ and l in $[\![0,3]\!]$, we start noticing from Equation (4.1) that

$$\begin{aligned} P_t\phi(x_0+lz) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi\Big(\mathbb{M}_t(y_1+y_2)+e^{tA}(x_0+lz)\Big)p^{\mathrm{tr}}(t,y_1)\,dy_1\pi_t(dy_2) \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi\Big(\xi_0+le^{tA}z\Big)p^{\mathrm{tr}}(t,y_1)\,dy_1\pi_t(dy_2), \end{aligned}$$

where we have denoted for simplicity $\xi_0 = \mathbb{M}_t(y_1+y_2)+e^{tA}x_0$. We can then exploit Lemma 2.2 to write that

$$\begin{aligned} \left|\Delta^3_{x_0}\Big(P_t\phi\Big)(z)\right| &\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|\Delta^3_{\xi_0}\phi(e^{tA}z)\Big)\right|p^{\mathrm{tr}}(t,y_1)\,dy_1\pi_t(dy_2) \\ &\le \pi_t(\mathbb{R}^N)\|\phi\|_{C^\gamma_{b,d}}\sum_{k=1}^n|E_ke^{tA}z|^{\frac{\gamma}{1+\alpha(k-1)}} \\ &\le C\|\phi\|_{C^\gamma_{b,d}}\left[\sum_{k=1}^{h-1}\Big(t|z|\Big)^{\frac{\gamma}{1+\alpha(k-1)}}+\sum_{k=h}^n\Big(t^{k-h}|z|\Big)^{\frac{\gamma}{1+\alpha(k-1)}}\right] \\ &\le C\|\phi\|_{C^\gamma_{b,d}}|z|^{\frac{\gamma}{1+\alpha(h-1)}}. \end{aligned} \tag{4.29}$$

For $|z| \le t^{\frac{1+\alpha(h-1)}{\alpha}}$, we are going to apply Taylor expansion three times in order to make $D^3_{I_h}$ appear. Namely,

$$\begin{aligned} &\left|\Delta^3_{x_0}\Big(P_t\phi\Big)(z)\right| \\ &\quad= \left|\int_0^1\langle D_{I_h}P_t\phi(x_0+\lambda z)-2D_{I_h}P_t\phi(x_0+z+\lambda z)+D_{I_h}P_t\phi(x_0+2z+\lambda z),z\rangle\,d\lambda\right| \\ &\quad\le \left|\int_0^1\int_0^1\langle\Big[D^2_{I_h}P_t\phi(x_0+(\lambda+\mu)z)-D^2_{I_h}P_t\phi(x_0+z+(\lambda+\mu)z)\Big]z,z\rangle\,d\lambda d\mu\right| \\ &\quad\le \left|\int_0^1\int_0^1\int_0^1\langle\Big[D^3_{I_h}P_t\phi(x_0+(\lambda+\mu+\nu)z)\Big](z,z),z\rangle\,d\lambda d\mu d\nu\right| \\ &\quad\le C\|D^3_{I_h}P_t\phi\|_\infty|z|^3 \\ &\quad\le C\|\phi\|_{C^\gamma_{b,d}}\Big(1+t^{\frac{\gamma-3(1+\alpha(h-1))}{\alpha}}\Big)|z|^3, \end{aligned} \tag{4.30}$$

where in the last step we used Control (4.13) with $h = h' = h''$. Since $|z| \le t^{\frac{1+\alpha(h-1)}{\alpha}}$ and noticing that $\gamma-3(1+\alpha(h-1)) < 0$, it holds that

$$\Big(1+t^{\frac{\gamma-3(1+\alpha(h-1))}{\alpha}}\Big)|z|^3 \le |z|^{\frac{\gamma-3(1+\alpha(h-1))}{1+\alpha(h-1)}}|z|^3 = |z|^{\frac{\gamma}{1+\alpha(h-1)}}.$$

We can then conclude that

$$\left|\Delta^3_{x_0}\Big(P_t\phi\Big)(z)\right| \le C\|\phi\|_{C^\gamma_{b,d}}|z|^{\frac{\gamma}{1+\alpha(h-1)}}. \tag{4.31}$$

Going back to Controls (4.28), (4.29) and (4.31), we have thus proven Estimate (4.27) for any non-integer $\gamma = \beta$. □

5 Elliptic and parabolic Schauder estimates

In this section, we use the controls shown before to prove Schauder Estimates both for the elliptic and the parabolic equation driven by the Ornstein-Ulhenbeck operator L^{ou}.

Fixed $\lambda > 0$ and g in $C_b(\mathbb{R}^N)$, we say that a function $u\colon \mathbb{R}^N \to \mathbb{R}^N$ is a *distributional solution* of Elliptic Equation (1.5) if u is in $C_b(\mathbb{R}^N)$ and for any ϕ in $C^\infty_c(\mathbb{R}^N)$ (i.e. smooth functions with compact support), it holds that

$$\int_{\mathbb{R}^N} u(x)\Big[\lambda\phi(x) - \Big(L^{\text{ou}}\Big)^*\phi(x)\Big]\,dx \;=\; \int_{\mathbb{R}^N} \phi(x)g(x)\,dx, \tag{5.1}$$

where $\Big(L^{\text{ou}}\Big)^*$ denotes the formal adjoint of L^{ou} on $L^2(\mathbb{R}^N)$, i.e.

$$\Big(L^{\text{ou}}\Big)^*\phi(x) \;=\; \mathcal{L}^*\phi(x) - \langle Ax, D_x\phi(x)\rangle - \operatorname{Tr}(A)\phi(x), \quad (t,x)\in[0,T]\times\mathbb{R}^N, \tag{5.2}$$

and $\mathcal{L}^*$ is the adjoint of the operator $\mathcal{L}$ on $L^2(\mathbb{R}^N)$. It is well-known (see e.g. Section 4.2 in [App19]) that it can be represented for any ϕ in $C^\infty_c(\mathbb{R}^N)$ as

$$\begin{aligned}\mathcal{L}^*\phi(x) \;=\; &\frac{1}{2}\operatorname{Tr}\Big(BQB^*D^2\phi(x)\Big) - \langle Bb, D\phi(x)\rangle \\ &+ \int_{\mathbb{R}^d_0}\Big[\phi(x-Bz) - \phi(x) + \langle D\phi(x), Bz\rangle\mathbb{1}_{B(0,1)}(z)\Big]\,\nu(dz).\end{aligned}$$

We state now the main result for the elliptic case, ensuring the well-posedness (in a distributional sense) for Equation (1.5).

Theorem 5.1. *Fixed $\lambda > 0$, let g be in $C_b(\mathbb{R}^N)$. Then, the function $u\colon \mathbb{R}^N \to \mathbb{R}$ given by*

$$u(x) \;:=\; \int_0^\infty e^{-\lambda t}P_tg(x)\,dt, \quad x\in\mathbb{R}^N, \tag{5.3}$$

is the unique distributional solution of Equation (1.5)*.*

Proof. Existence. We are going to show that the function u given in Equation (5.3) is indeed a distributional solution of the elliptic problem (1.5). It is straightforward to notice that u is in $C_b(\mathbb{R}^N)$, thanks to the contraction property of P_t on $C_b(\mathbb{R}^N)$. Fixed ϕ in $C^\infty_c(\mathbb{R}^N)$, we then use Fubini Theorem to write that

$$\begin{aligned}\int_{\mathbb{R}^N} u(x)\Big(L^{\text{ou}}\Big)^*\phi(x)\,dx \;&=\; \lim_{\epsilon\to 0^+}\int_\epsilon^\infty\int_{\mathbb{R}^N} e^{-\lambda t}P_tg(x)\Big(L^{\text{ou}}\Big)^*\phi(x)\,dxdt \\ &=\; \lim_{\epsilon\to 0^+}\int_\epsilon^\infty\int_{\mathbb{R}^N} e^{-\lambda t}L^{\text{ou}}P_tg(x)\phi(x)\,dxdt,\end{aligned}$$

where, in the last step, we exploited that P_tg is differentiable and bounded for $t > 0$ (Proposition 4.1). Since L^{ou} is the infinitesimal generator of the semi-group $\{P_t\colon t\ge 0\}$, we know that $\partial_t(P_tg)$ exists for any $t > 0$ and $\partial_t(P_tg)(x) = L^{\text{ou}}P_tg(x)$ for any x in $\mathbb{R}^N$. Integration by parts formula allows then to conclude that

$$\begin{aligned}\int_{\mathbb{R}^N} u(x)\Big(L^{\text{ou}}\Big)^*\phi(x)\,dx \;&=\; \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N}\phi(x)\int_\epsilon^\infty e^{-\lambda t}\partial_tP_tg(x)\,dtdx \\ &=\; \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N}\Big(-e^{-\lambda\epsilon}P_\epsilon g(x) + \lambda\int_\epsilon^\infty e^{-\lambda t}P_tg(x)\,dt\Big)\,dx \\ &=\; \int_{\mathbb{R}^N} -g(x)\phi(x)\,dx + \int_{\mathbb{R}^N}\lambda u(x)\phi(x)\,dx.\end{aligned}$$

Uniqueness. It is enough to show that any distributional solution u of Equation (1.5) for $g = 0$ coincides with the zero function, i.e. $u = 0$. To do so, we fix a function ρ in $C^\infty_c(\mathbb{R}^N)$ such that $\|\rho\|_{L^1} = 1$, $0 \le |\rho| \le 1$ and we then define the *mollifier* $\rho_m := m^N\rho(mx)$ for any m in $\mathbb{N}$. Denoting now, for simplicity, $u_m := u * \rho_m$, we define the function

$$g_m(x) \,:=\, \lambda u_m(x) - L^{\text{ou}}u_m(x). \tag{5.4}$$

Using that u is in $C_b(\mathbb{R}^N)$, it is easy to notice that g_m is also in $C_b(\mathbb{R}^N)$ for any fixed m in $\mathbb{N}$. Truncating the functions if necessary, we can assume that u_m and g_m are integrable with integrable Fourier transform so that we can apply the Fourier transform in Equation (5.4):

$$\lambda\widehat{u}_m(\xi) - \mathcal{F}_x\Big(L^{\text{ou}}u_m\Big)(\xi) \,=\, \widehat{g}_m(\xi). \tag{5.5}$$

We remember in particular that the above operator L^{ou} has an associated Lévy symbol $\Phi^{\text{ou}}(\xi)$ and, following Section 3.3.2 in [App09], it holds that

$$\mathcal{F}_x\Big(L^{\text{ou}}u_m\Big)(\xi) \,=\, \Phi^{\text{ou}}(\xi)\widehat{u}_m(\xi). \tag{5.6}$$

We can then use it to show that $\widehat{u}_m$ is a classical solution of the following equation:

$$\Big[\lambda - \Phi^{\text{ou}}(\xi)\Big]\widehat{u}_m(\xi) \,=\, \widehat{g}_m(\xi).$$

The above equation can be easily solved by direct calculation as

$$\widehat{u}_m(\xi) \,=\, \int_0^\infty e^{-\lambda t}e^{t\Phi^{\text{ou}}(\xi)}\widehat{g}_m(\xi)\,ds.$$

In order to go back to u_m, we apply now the inverse Fourier transform to write that

$$u_m(x) \,=\, \int_0^\infty e^{-\lambda t}P_tg_m(x)\,dt.$$

The contraction property of the semi-group P_t then implies that $\|u_m\|_\infty \le C\|g_m\|_\infty$. In order to conclude, we need to show that

$$\lim_{m\to\infty}\|g_m\|_\infty \,=\, 0. \tag{5.7}$$

We start noticing that, since u is a solution of Equation (1.5) with $g = 0$, it holds that

$$\begin{aligned}
g_m(x) \,&=\, \int_{\mathbb{R}^N} u(y)\Big\{\lambda\rho_m(x-y) - \mathcal{L}[\rho_m(\cdot - y)](x) - \langle Ax, D_x\rho_m(x-y)\rangle\Big\}\,dy \\
&=\, \int_{\mathbb{R}^N} u(y)\Big\{\mathcal{L}^*[\rho_m(x-\cdot)](y) - \mathcal{L}[\rho_m(\cdot - y)](x) + \langle A(x-y), D_x\rho_m(x-y)\rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \mathrm{Tr}(A)\rho_m(x-y)\Big\}\,dy \\
&=:\, R^1_m(x) + R^2_m(x) + R^3_m(x),
\end{aligned}$$

where we have denoted

$$\begin{aligned}
R^1_m(x) \,&:=\, \int_{\mathbb{R}^N} u(y)\Big[\mathcal{L}^*[\rho_m(x-\cdot)](y) - \mathcal{L}[\rho_m(\cdot - y)](x)\Big]\,dy; \\
R^2_m(x) \,&:=\, \int_{\mathbb{R}^N} u(y)\langle A(x-y), D_x\rho_m(x-y)\rangle\,dy; \\
R^3_m(x) \,&:=\, \int_{\mathbb{R}^N} u(y)\mathrm{Tr}(A)\rho_m(x-y)\,dy.
\end{aligned}$$

On the one hand, it is easy to notice that $R^1_m = 0$, since $\mathcal{L}^*[\rho_m(x-\cdot)](y) = \mathcal{L}[\rho_m(\cdot-y)](x)$ for any m in $\mathbb{N}$ and any y in $\mathbb{R}^N$. Indeed, it holds that

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}\Big(BQB^*D^2_y[\rho_m(x-\cdot)](y)\Big) &- \langle Bb, D_y[\rho_m(x-\cdot)](y)\rangle \\ &= \frac{1}{2}\mathrm{Tr}\Big(BQB^*D^2_x\rho_m(x-y)\Big) + \langle Bb, D_x\rho_m(x-y)\rangle\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^d_0}\Big[\rho_m(x-y+Bz)-\rho_m(x-y)+\langle D_y[\rho_m(x-\cdot)](y), Bz\rangle\mathbb{1}_{B(0,1)}(z)\Big]\,\nu(dz) \\ &\quad= \int_{\mathbb{R}^d_0}\Big[\rho_m((x+Bz)-y)-\rho_m(x-y)-\langle D_x\rho_m(x-y), Bz\rangle\mathbb{1}_{B(0,1)}(z)\Big]\,\nu(dz).\end{aligned}$$

On the other hand, it can be checked (see e.g. [Pri09]) that $\|R^2_m + R^3_m\|_\infty \to 0$ if m goes to infinity. Indeed, we firstly notice that R^3_m converges, when m goes to infinity, to the function $u\mathrm{Tr}(A)$, uniformly in x. On the other hand, applying the change of variables $y = x - z/m$ in R^2_m, we can obtain that

$$R^2_m(x) \;=\; m\int_{\mathbb{R}^N} u(x-z/m)\langle A(z/m), D_x\rho(z)\rangle\,dy.$$

Letting m goes to infinity above, we can then conclude that R^2_m converges to the function $-u\mathrm{Tr}(A)$, uniformly in x. □

Let us deal now with the parabolic setting. Since we are working with evolution equations, the functions we consider will often depend on time, too. We denote for any $\gamma > 0$ the space $L^\infty(0,T,C^\gamma_{b,d}(\mathbb{R}^N))$ as the family of functions ϕ in $B_b\Big([0,T]\times\mathbb{R}^N\Big)$ such that $\phi(t,\cdot)$ is in $C^\gamma_{b,d}(\mathbb{R}^N)$ at any fixed t and the norm

$$\|\phi\|_{L^\infty(C^\gamma_{b,d})} \;:=\; \sup_{t\in[0,T]}\|\phi(t,\cdot)\|_{C^\gamma_{b,d}} \text{ is finite.}$$

We define now the notion of solution we are going to consider. Fixed $T > 0$, u_0 in $C_b(\mathbb{R}^N)$ and f in $L^\infty\Big(0,T;C_b(\mathbb{R}^N)\Big)$, we say that a function $u\colon [0,T]\times\mathbb{R}^N \to \mathbb{R}^N$ is a *weak solution* of the Cauchy problem (1.6) if u is in $L^\infty\Big(0,T;C_b(\mathbb{R}^N)\Big)$ and for any ϕ in $C^\infty_c([0,T)\times\mathbb{R}^N)$, it holds that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^N} u(t,x)\Big[\partial_t\phi(t,x) + \Big(L^{\mathrm{ou}}\Big)^*\phi(t,x)\Big] + f(t,x)\phi(t,x)\,dxdt \\ + \int_{\mathbb{R}^N} u_0(x)\phi(0,x)\,dx \;=\; 0,\end{aligned} \tag{5.8}$$

where $\Big(L^{\mathrm{ou}}\Big)^*$ denotes the formal adjoint of L^{ou} on $L^2(\mathbb{R}^N)$ given in Equation (5.2).

Similarly to the elliptic setting, we show firstly the weak well-posedness of the Cauchy problem (1.6).

Theorem 5.2. *Fixed $T>0$, let u_0 be a function in $C_b(\mathbb{R}^N)$ and f in $L^\infty\Big(0,T;C_b(\mathbb{R}^N)\Big)$. Then, the function $u\colon[0,T]\times\mathbb{R}^N\to\mathbb{R}$ given by*

$$u(t,x) \,:=\, P_tu_0(x)+\int_0^t P_{t-s}f(s,x)\,ds, \quad (t,x)\in[0,T]\times\mathbb{R}^N, \tag{5.9}$$

is the unique weak solution of the Cauchy problem (1.6).

Proof. Existence. We start considering a "regularized" version of the coefficients appearing in Equation (1.6). Namely, we consider a family $\{u_{0,m}\}_{m\in\mathbb{N}}$ in $C^\infty_b(\mathbb{R}^N)$ such that $u_{0,m}\to u_0$ uniformly in x and a family $\{f_m\}_{m\in\mathbb{N}}$ in $L^\infty\Big(0,T;C^\infty_b(\mathbb{R}^N)\Big)$ such that $f_m\to f$ uniformly in t and x. They can be obtained through standard mollification methods in space.
Fixed m in $\mathbb{N}$, we denote now by $u_m\colon[0,T]\times\mathbb{R}^n\to\mathbb{R}$ the function given by

$$u_m(t,x) \,:=\, P_tu_{0,m}(x)+\int_0^t P_{t-s}f_m(s,x)\,ds, \quad t\in[0,T],x\in\mathbb{R}^N.$$

On the one hand, we use again that $\partial_t(P_tu_m)(t,x)=L^{\text{ou}}P_tu_m(t,x)$ for any (t,x) in $[0,T]\times\mathbb{R}^N$ to check that u_m is indeed a *classical* solution of the "regularized" Cauchy Problem:

$$\begin{cases}\partial_tu_m(t,x) \,=\, L^{\text{ou}}u_m(t,x)+f_m(t,x), & (t,x)\in(0,T)\times\mathbb{R}^N;\\ u_m(0,x) \,=\, u_{0,m}(x), & x\in\mathbb{R}^N.\end{cases}$$

On the other hand, we exploit the linearity and the continuity of the semi-group P_t on $C_b(\mathbb{R}^N)$ to show that

$$u_m \,=\, P_tu_{0,m}(x)+\int_0^t P_{t-s}f_m(s,x)\,ds \overset{m}{\to} P_tu_0(x)+\int_0^t P_{t-s}f(s,x)\,ds \,=\, u,$$

uniformly in t and x, where u is the function given in (5.9).
We fix now a test function ϕ in $C^\infty_0\Big([0,T)\times\mathbb{R}^N\Big)$ and we then notice that

$$\int_0^T\int_{\mathbb{R}^N}\phi(t,y)\Big(\partial_t-L^{\text{ou}}\Big)u_m(t,y)\,dydt \,=\, \int_0^T\int_{\mathbb{R}^N}\phi(t,y)f_m(t,y)\,dydt.$$

An integration by parts allows now to move the operator to the test function, being careful to remember that $u_m(0,\cdot)=u_{0,m}(\cdot)$. Indeed, it holds that

$$\begin{aligned}-\int_0^T\int_{\mathbb{R}^N}&\Big(\partial_t+\Big(L^{\text{ou}}\Big)^*\Big)\phi(t,y)u_m(t,y)\,dydt\\ &= \int_{\mathbb{R}^N}\phi(0,y)u_{0,m}(y)\,dy+\int_0^T\int_{\mathbb{R}^N}\phi(t,y)f_m(t,y)\,dydt,\end{aligned} \tag{5.10}$$

where $\Big(L^{\text{ou}}\Big)^*$ denotes the formal adjoint of L^{ou} on $L^2(\mathbb{R}^N)$.
We would like now to go back to the solution u, letting m go to infinity. We start rewriting the right-hand side term of (5.10) as $R^1_m+R^2_m$, where

$$\begin{aligned}R^1_m \,&:=\, \int_{\mathbb{R}^N}\phi(0,y)u_{0,m}(y)\,dy;\\ R^2_m \,&:=\, \int_0^T\int_{\mathbb{R}^N}\phi(t,y)f_m(t,y)\,dydt.\end{aligned}$$

We can rewrite R^2_m as

$$R^2_m = \int_0^T\int_{\mathbb{R}^N}\phi(t,y)f(t,y)\,dydt + \int_0^T\int_{\mathbb{R}^N}\phi(t,y)\Big[f_m - f\Big](t,y)\,dydt.$$

Exploiting that, by assumption, f_m converges to f uniformly in t and x, it is easy to see that the second contribution above converges to 0. A similar argument can be used to show that

$$\int_{\mathbb{R}^N}\phi(0,y)u_{0,m}(y)\,dy \overset{m}{\to} \int_{\mathbb{R}^N}\phi(0,y)u_0(y)\,dy.$$

On the other hand, we can rewrite the left-hand side of Equation (5.10) as

$$\begin{aligned}-\int_0^T\int_{\mathbb{R}^N}&\Big(\partial_t + \Big(L^{\mathrm{ou}}\Big)^*\Big)\phi(t,y)u_m(t,y)\,dydt\\ &= -\int_0^T\int_{\mathbb{R}^N}\Big(\partial_t + \Big(L^{\mathrm{ou}}\Big)^*\Big)\phi(t,y)u(t,y)\,dydt + L^1_m + L^2_m + L^3_m,\end{aligned}$$

where we have denoted

$$\begin{aligned}L^1_m &:= \int_0^T\int_{\mathbb{R}^N}\Big[\frac{1}{2}\mathrm{Tr}\Big(BQB^*D^2_y\phi(t,y)\Big) + \langle Ay + Bb, D_y\phi(t,y)\rangle + \mathrm{Tr}(A)\phi(t,y)\Big]\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times[u_m - u](t,y)\,dydt;\\ L^2_m &:= \int_0^T\int_{\mathbb{R}^N}\partial_t\phi(t,y)[u - u_m](t,y)\,dydt; \qquad\qquad (5.11)\\ L^3_m &:= \int_0^T\int_{\mathbb{R}^N}\Big[\int_{\mathbb{R}^d_0}\phi(t,y-Bz) - \phi(t,y) + \langle D_y\phi(t,y), Bz\rangle\mathbb{1}_{B(0,1)}(z)\,\nu(dz)\Big]\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times[u - u_m](t,y)dydt.\end{aligned}$$

To conclude, we need to show that the remainder $L^1_m + L^2_m + L^3_m$ is negligible, if m goes to infinity. Exploiting that ϕ has a compact support and that $\|u_m - u\|_\infty \overset{m}{\to} 0$, it is easy to show that $|L^1_m + L^2_m| \overset{m}{\to} 0$.
In order to control L^3_m, we need firstly to decompose it as $L^{3,1}_m + L^{3,2}_m$, where

$$\begin{aligned}L^{3,1}_m &:= \int_0^T\int_{\mathbb{R}^N}\Big[u(t,y) - u_m(t,y)\Big]\\ &\qquad\qquad\times\Big[\int_{0<|z|<1}\phi(t,y-Bz) - \phi(t,y) + \langle D_y\phi(t,y), Bz\rangle\,\nu(dz)\Big]dydt;\\ L^{3,2}_m &:= \int_0^T\int_{\mathbb{R}^N}\Big[u(t,y) - u_m(t,y)\Big]\Big[\int_{|z|>1}\phi(t,y-Bz) - \phi(t,y)\,\nu(dz)\Big]dydt.\end{aligned}$$

The second term $L^{3,2}_m$ can be controlled easily using the Fubini Theorem. Indeed, denoting by K the support of ϕ and by λ the Lebesgue measure on $\mathbb{R}^N$, we notice that

$$\begin{aligned}|L^{3,2}_m| &\le \|u - u_m\|_\infty\int_0^T\int_{|z|>1}\int_{\mathbb{R}^N}|\phi(t,y-Bz) - \phi(t,y)|\,dy\nu(dz)dt\\ &\le CT2\lambda(K)\nu\Big(B^c(0,1)\Big)\|u - u_m\|_\infty.\end{aligned}$$

Exploiting that $\nu\Big(B^c(0,1)\Big)$ is finite since ν is a Lévy measure, we can then conclude that $|L^{3,2}_m|$ tends to zero if m goes to infinity.

The argument for $L^{3,1}_m$ is similar but we need firstly to apply a Taylor expansion twice to make a term $|z|^2$ appear in the integral and exploit that $|z|^2\nu(dz)$ is finite on $B(0,1)$.

Uniqueness. This proof will follow essentially the same arguments as for Theorem 5.1. Let u be any weak solution of Cauchy problem (1.6) with $u_0 = f = 0$. We are going to show that $u = 0$.
We start considering a mollyfing sequence $\{\rho_m\}_{m\in\mathbb{N}}$ in $C^\infty_c((0,T)\times\mathbb{R}^N)$. Denoting for simplicity $u_m(t,x) = u * \rho_m(t,x)$, we then notice that u_m is continuously differentiable in time and that $u_m(0,x) = 0$. It makes sense to define now the function

$$f_m(t,x) \,:=\, \partial_t u_m(t,x) - L^{\mathrm{ou}}u_m(t,x). \tag{5.12}$$

Moreover, we can truncate f_m and u_m if necessary, so that they are integrable with integrable Fourier transform. Then, the same reasoning in Equations (5.5), (5.6) allows us to write that

$$\begin{cases}\partial_t\widehat{u}_m(t,\xi) - \Phi^{\mathrm{ou}}(\xi)\widehat{u}_m(t,\xi) \,=\, \widehat{f}_m(t,\xi),\\ \widehat{u}_m(0,\xi) \,=\, 0.\end{cases}$$

The above equation can be easily solved integrating in time, giving the following representation:

$$\widehat{u}_m(t,\xi) \,=\, \int_0^t e^{(t-s)\Phi^{\mathrm{ou}}(\xi)}\widehat{f}_m(s,\xi)\,ds.$$

In order to go back to u_m, we apply now the inverse Fourier transform to write that

$$u_m(t,x) \,=\, \int_0^t P_{t-s}f_m(s,x)\,ds.$$

The contraction property of P_t allows us to conclude that $\|u_m\|_\infty \le C\|f_m\|_\infty$. Letting m goes to zero, we obtain the desired result. Indeed, it is possible to show that

$$\lim_{m\to\infty}\|f_m\|_\infty \,=\, 0,$$

relying on the same reasonings used in the analogous elliptic case (Theorem 5.1). □

The next two conclusive theorems provide the Schauder estimates both in the elliptic and in the parabolic setting.

Theorem 5.3 (Elliptic Schauder estimates)**.** *Fixed $\lambda > 0$ and β in $(0,1)$, let g be in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)$. Then, the distributional solution u of Equation* (1.5) *is in $C^\beta_{b,d}(\mathbb{R}^N)$ and there exists a positive constant C such that*

$$\|u\|_{C^{\alpha+\beta}_{b,d}} \,\le\, C\Big(1+\frac{1}{\lambda}\Big)\|g\|_{C^\beta_{b,d}}. \tag{5.13}$$

Proof. Thanks to Theorem 5.1, we know that the unique solution u of the elliptic equation (1.5) is given in (5.3). In order to show that such a function u satisfies Schauder estimates (5.13), we exploit again the equivalent norm defined in (2.8) of Lemma 2.3. Namely, we fix h in $[\![1,n]\!]$ and x_0 in $\mathbb{R}^N$ and we show that

$$|\Delta^3_{x_0}u(z)| \,=\, \Big|\int_0^\infty e^{-\lambda t}\Delta^3_{x_0}\Big(P_t g\Big)(z)\,dt\Big| \,\le\, C\|g\|_{C^\beta_{b,d}}|z|^{\frac{\alpha+\beta}{1+\alpha(h-1)}}, \quad z\in E_h(\mathbb{R}^N),$$

for some constant $C > 0$ independent from x_0. For $|z| \geq 1$, it can be easily obtained from the contraction property of P_t on $B_b(\mathbb{R}^N)$:

$$\Big|\int_0^\infty e^{-\lambda t}\Delta^3_{x_0}\Big(P_tg\Big)(z)\,dt\Big| \leq 3\int_0^\infty e^{-\lambda t\|P_tg\|_\infty\,dt} \leq \frac{3}{\lambda}\|g\|_\infty|z|^{\frac{\alpha+\beta}{1+\alpha(h-1)}}. \tag{5.14}$$

When $|z| \leq 1$, we start fixing a *transition time* t_0 given by

$$t_0 \;=\; |z|^{\frac{\alpha}{1+\alpha(h-1)}}. \tag{5.15}$$

Notably, t_0 represents the transition time between the diagonal and the off-diagonal regime, accordingly to the intrinsic time scales of the system. We then decompose $\Delta^3_{x_0}u(z)$ as $R_1(z) + R_2(z)$, where

$$\begin{aligned}
R_1(z) &:= \int_0^{t_0} e^{-\lambda t}\Delta^3_{x_0}\Big(P_tg\Big)(z)\,dt;\\
R_2(z) &:= \int_{t_0}^\infty e^{-\lambda t}\Delta^3_{x_0}\Big(P_tg\Big)(z)\,dt.
\end{aligned}$$

The first component R_1 is controlled easily using Corollary 4.4 for $\beta = \gamma$. Indeed,

$$|R_1(z)| \;\leq\; \int_0^{t_0} |\Delta^3_{x_0}\Big(P_tg\Big)(z)|\,dt \;\leq\; |z|^{\frac{\beta}{1+\alpha(h-1)}}\int_0^{t_0}\|P_tg\|_{C^\beta_{b,d}}\,dt \;\leq\; C\|g\|_{C^\beta_{b,d}}|z|^{\frac{\alpha+\beta}{1+\alpha(h-1)}}. \tag{5.16}$$

On the other hand, the control for R_2 can be obtained following Equation (4.30) in order to write that

$$\begin{aligned}
|R_2(z)| &\leq C\|g\|_{C^\beta_{b,d}}|z|^3\int_{t_0}^\infty e^{-\lambda t}\Big(1 + t^{\frac{\beta-3(1+\alpha(h-1))}{\alpha}}\Big)\,dt\\
&\leq C\|g\|_{C^\beta_{b,d}}|z|^3\Big(\lambda^{-1} + |z|^{\frac{\alpha+\beta-3(1+\alpha(h-1))}{1+\alpha(h-1)}}\Big)\\
&\leq C\Big(1+\frac{1}{\lambda}\Big)\|g\|_{C^\beta_{b,d}}|z|^{\frac{\alpha+\beta}{1+\alpha(h-1)}},
\end{aligned} \tag{5.17}$$

where, in the last step, we exploited that $|z| \leq 1$. □

Theorem 5.4 (Parabolic Schauder estimates). *Fixed $T > 0$ and β in $(0, 1)$, let u_0 be in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)$ and f in $L^\infty\Big(0, T; C^\beta_{b,d}(\mathbb{R}^N)\Big)$. Then, the weak solution u of Cauchy Problem* (1.6) *is in $L^\infty\Big(0, T; C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)\Big)$ and there exists a constant $C := C(T) > 0$ such that*

$$\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \;\leq\; C\Big[\|u_0\|_{C^{\alpha+\beta}_{b,d}} + \|f\|_{L^\infty(C^\beta_{b,d})}\Big]. \tag{5.18}$$

Proof. We are going to show that any function u given by Equation (5.9) satisfies the Schauder Estimates (5.18). We start splitting the function u in $u_1 + u_2$, where

$$u_1(t,x) := P_tu_0(x); \tag{5.19}$$

$$u_2(t,x) := \int_0^t P_sf(t-s,x)\,ds. \tag{5.20}$$

Corollary 4.4 allows then to control u_1 in the following way:

$$\|u_1\|_{L^\infty(C^{\alpha+\beta}_{b,d})} = \sup_{t\in[0,T]} \|P_t u_0\|_{C^{\alpha+\beta}_{b,d}} \le C\|u_0\|_{C^{\alpha+\beta}_{b,d}}.$$

In order to deal with the contribution u_2, we will follow essentially the same reasoning for the Schauder Estimates in the elliptic setting. Namely, we use again the equivalent norm defined in (2.8) of Lemma 2.3 in order to estimate

$$\|u_2\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \le C\|f\|_{L^\infty(C^{\beta}_{b,d})}.$$

Fixed h in $[\![1,n]\!]$ and x_0 in $\mathbb{R}^N$, our aim is to show that

$$|\Delta^3_{x_0} u_2(z)| = \left|\int_0^t \Delta^3_{x_0}\Big(P_{t-s}f\Big)(s,z)\,ds\right| \le C\|f\|_{L^\infty(C^{\beta}_{b,d}\beta)}|z|^{\frac{\alpha+\beta}{1+\alpha(h-1)}}, \quad z\in E_h(\mathbb{R}^N),$$

for some constant $C>0$ independent from x_0. When $|z|\ge 1$, it can be obtained easily from the contraction property of P_t on $C_b(\mathbb{R}^N)$ as in (5.14). For $|z|\le 1$, we fix again the transition time t_0 given in (5.15) and we then decompose $\Delta^3_{x_0}u_2(t,z)$ as $\tilde{R}_1(t,z)+\tilde{R}_2(t,z)$, where

$$\begin{aligned}\tilde{R}_1(t,z) &:= \int_0^{t\wedge t_0} \Delta^3_{x_0}\Big(P_s f\Big)(t-s,z)\,ds : \\ \tilde{R}_2(t,z) &:= \int_{t\wedge t_0}^{t} \Delta^3_{x_0}\Big(P_s f\Big)(t-s,z)\,ds.\end{aligned}$$

The first component R_1 can be controlled easily as in (5.16):

$$\begin{aligned}|\tilde{R}_1(t,z)| &\le \int_0^{t\wedge t_0} |\Delta^3_{x_0}\Big(P_s f\Big)(t-s,z)|\,ds \\ &\le |z|^{\frac{\beta}{1+\alpha(h-1)}}\int_0^{t\wedge t_0} \|P_s f(t-s,\cdot)\|_{C^{\beta}_{b,d}}\,ds \\ &\le C\|f\|_{L^\infty(C^{\beta}_{b,d})}|z|^{\frac{\alpha+\beta}{1+\alpha(h-1)}}.\end{aligned}$$

On the other hand, the control for R_2 is obtained following the same steps used in Equation (5.17). Namely,

$$\begin{aligned}|\tilde{R}_2(t,z)| &\le C\|f\|_{L^\infty(C^{\beta}_{b,d})}|z|^3\int_{t\wedge t_0}^{\infty}\Big(1+s^{\frac{\beta-3(1+\alpha(h-1))}{\alpha}}\Big)\,ds \\ &\le C\|f\|_{L^\infty(C^{\beta}_{b,d})}|z|^{\frac{\alpha+\beta}{1+\alpha(h-1)}}.\end{aligned}$$

□

6 Extensions to time-dependent operators

In this final section, we would like to show some possible extensions of our method in order to include more general operators with non-linear, space-time dependent coefficients. Even in this framework, we will prove the well-posedness of the parabolic Cauchy

problem and show the associated Schauder estimates.
Following [KP10], our first step is to consider a time-dependent Ornstein-Uhlenbeck operator of the following form:

$$\begin{aligned} L^{\mathrm{ou}}_t\phi(t,x) \, := \, &\frac{1}{2}\mathrm{Tr}\Big(B_tQB^*_tD^2\phi(x)\Big) + \langle A_tx, D\phi(x)\rangle \\ &+ \int_{\mathbb{R}^d_0}\Big[\phi(x+B_tz)-\phi(x)-\langle D_x\phi(x),B_tz\rangle\mathbb{1}_{B(0,1)}(z)\Big]\,\nu(dz), \end{aligned}$$

where $B_t := B\sigma_0(t)$ and A_t, $\sigma_0(t)$ are two time-dependent matrices in $\mathbb{R}^N\otimes\mathbb{R}^N$ and $\mathbb{R}^d\otimes\mathbb{R}^d$, respectively. From this point further, we assume that the matrices A_t, $\sigma_0(t)$ are measurable in time and that they satisfy the following conditions:

[**tK**] for any fixed t in $[0,T]$, it holds that $N \, = \, \mathrm{rank}\Big[B, A_tB,\dots,A^{N-1}_tB\Big]$;

[**B**] the matrix A_t is bounded in time, i.e. there exists a constant $\eta>0$ such that

$$|A_t\xi| \, \le \, \eta|\xi|, \quad \xi\in\mathbb{R}^N;$$

[**UE**] the matrix σ_0 is uniformly elliptic, i.e. it holds that

$$\eta^{-1}|\xi|^2 \, \le \, \langle\sigma_0(t)\xi,\xi\rangle \, \le \, \eta|\xi|^2, \quad (t,\xi)\in[0,T]\times\mathbb{R}^d.$$

It is important to highlight already that this new "time-dependent" version [**tK**] of the Kalman rank condition [**K**] allows us to reproduce the same reasonings of Section 2. In particular, the anisotropic distance $\mathbf{d}$ and the Zygmund-Hölder spaces $C^\beta_{b,d}(\mathbb{R}^N)$ can be constructed under these assumptions, even if only at any *fixed* time t. A priori, the number of sub-divisions of the space $\mathbb{R}^N$ may change for different times, leading to consider a time-dependent $n(t)$ in Equation (2.2) and, consequently, time-dependent anisotropic distances and Hölder spaces. We will however drop the subscript in t below since it does not add any difficulty in the arguments but it may damage the readability of the article.

Proposition 6.1. *Let u_0 be in $C_b(\mathbb{R}^N)$ and f in $L^\infty\Big(0,T;C_b(\mathbb{R}^N)\Big)$. Then, there exists a unique solution $u\colon[0,T]\times\mathbb{R}^N\to\mathbb{R}$ of the following Cauchy problem:*

$$\begin{cases} \partial_tu(t,x) \, = \, L^{ou}_tu(t,x)+f(t,x), \quad (t,x)\in(0,T)\times\mathbb{R}^N; \\ u(0,x) \, = \, u_0(x), \quad x\in\mathbb{R}^N. \end{cases} \tag{6.1}$$

Furthermore, if u_0 is in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)$ and f in $L^\infty\Big(0,T;C^\beta_{b,d}(\mathbb{R}^N)\Big)$, then the solution u is in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)\Big)$ and there exists a constant $C:=C(T,\eta)>0$ such that

$$\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \, \le \, C\Big[\|u_0\|_{C^{\alpha+\beta}_{b,d}}+\|f\|_{L^\infty(C^\beta_{b,d})}\Big]. \tag{6.2}$$

Proof. The proof of this result can be obtained mimicking the arguments already presented in the first part of the article with some slight modifications. The main difference is the introduction of the resolvent $\mathcal{R}_{s,t}$ associated with the matrix A_t in place of the matrix exponential e^{tA}. Namely, $\mathcal{R}_{t,u}$ is a time-dependent matrix in $\mathbb{R}^N \otimes \mathbb{R}^N$ that is solution of the following ODE:

$$\begin{cases} \partial_t \mathcal{R}_{t,u} \,=\, A_t \mathcal{R}_{t,u}, \quad t \in [u,T]; \\ \mathcal{R}_{u,u} \,=\, \mathrm{Id}_{N\times N}\,. \end{cases} \tag{6.3}$$

As said before, Section 2 follows exactly in the same manner as above except for Lemma 2.2 (structure of the resolvent), whose proof can be found in [HM16], Lemmas 5.1 and 5.2. The arguments in Section 3 and 4 can be applied again, even if the formulation of some objects presented there changes slightly. For example in Equation (3.1), the N-dimensional Ornstein-Uhlenbeck process $\{X_t\}_{t\geq 0}$ driven by $B_t Z_t$ should be now represented by

$$X_t \,=\, \mathcal{R}_{t,0}x + \int_0^t \mathcal{R}_{t,u} B_u \, dZ_u, \quad t \geq 0, x \in \mathbb{R}^N.$$

Finally in Section 5, the uniform ellipticity [**UE**] of $\sigma_0(t)$ and the boundedness [**B**] of A_t allow us to control the remainder terms appearing in Equation (5.11) as done above and thus, to conclude as in Theorems 5.2 and 5.4. $\square$

Once we have shown our results for the time-dependent Ornstein-Uhlenbeck operator L^{ou}_t, we add now a non-linearity to the problem, even if only dependent in time. Namely, we are interested in operators of the following form:

$$L_t\phi(t,x) \,:=\, L^{\mathrm{ou}}_t\phi(t,x) + \langle F_0(t), D_x\phi(x)\rangle - c_0(t)\phi(x), \quad (t,x) \in [0,T]\times\mathbb{R}^N, \tag{6.4}$$

where $c_0\colon [0,T] \to \mathbb{R}$ and $F_0\colon [0,T] \to \mathbb{R}^N$ are two functions. For any sufficiently regular function $\phi\colon [0,T] \to \mathbb{R}$, we are going to denote

$$\mathcal{T}\phi(t,x) \,:=\, e^{-\int_0^t c_0(s)\,ds}\phi\Big(t, x + \int_0^t F_0(s)\,ds\Big), \quad (t,x) \in [0,T]\times\mathbb{R}^N. \tag{6.5}$$

We will see in the next result that the "operator" $\mathcal{T}$ transforms solutions of the Cauchy problem associated with $\mathcal{L}^{\mathrm{ou}}_t$ to solutions of the Cauchy problem driven by L_t, even if for a modified drift $\mathcal{T}f$.

Lemma 6.2. *Fixed $T > 0$, let u_0 be in $C_b(\mathbb{R}^N)$, f in $L^\infty\Big(0,T; C_b(\mathbb{R}^N)\Big)$ and c_0, F_0 in $C_b([0,T])$. Then, a function $u\colon [0,T]\times\mathbb{R}^N \to \mathbb{R}$ is a weak solution of Cauchy Problem* (6.1) *if and only if the function $v\colon [0,T]\times\mathbb{R}^N \to \mathbb{R}$ given by $v(t,x) = \mathcal{T}u(t,x)$ is a weak solution of the following Cauchy problem:*

$$\begin{cases} \partial_t u(t,x) \,=\, L_t u(t,x) + \mathcal{T}f(t,x), \quad (t,x) \in (0,T)\times\mathbb{R}^N; \\ u(0,x) \,=\, u_0(x), \quad x \in \mathbb{R}^N. \end{cases} \tag{6.6}$$

In particular, there exists a unique weak solution of Cauchy Problem (6.6)*.*

Proof. Given a weak solution u of Cauchy problem (6.1), we are going to show that the function v given in (6.5) is indeed a weak solution of Cauchy Problem (6.6). The inverse implication can be obtained in a similar manner and we will not prove it here.

By mollification if necessary, we can take two sequences $\{c_m\}_{m\in\mathbb{N}}$ and $\{F_m\}_{m\in\mathbb{N}}$ in $C^\infty_b([0,T])$ such that $c_m \to c_0$ and $F_m \to F_0$ uniformly in t. Furthermore, we denote for simplicity

$$\tilde{c}_m(t) := \int_0^t c_m(s)\,ds; \quad \tilde{F}_m(t) := \int_0^t F_m(s)\,ds.$$

Given a test function ϕ in $C^\infty_c\Big([0,T)\times\mathbb{R}^N\Big)$, let us consider for any m in $\mathbb{N}$, the following function

$$\psi_m(t,x) := e^{-\tilde{c}_m(t)}\phi(t,x-\tilde{F}_m(t)) \quad (t,x)\in[0,T]\times\mathbb{R}^N.$$

Since $\tilde{c}_m$ and $\tilde{F}_m$ are smooth and bounded, it is easy to check that ψ_m is in $C^\infty_c\Big([0,T)\times\mathbb{R}^N\Big)$. We can then use ψ_m in Equation (5.8) (with time-dependent A_t and B_t) to show that

$$\int_0^T\int_{\mathbb{R}^N}\Big[\partial_t+\Big(L^{\text{ou}}_t\Big)^*\Big]\psi_m(t,y)u(t,y)+f(t,y)\,dydt+\int_{\mathbb{R}^N}\psi_m(0,y)u_0(y)\,dy = 0.$$

A direct calculation then show that $\psi_m(0,y)=\phi(0,y)$ and

$$\begin{aligned}\Big(L^{\text{ou}}_t\Big)^*\psi_m(t,y) &= e^{-\tilde{c}_m(t)}\Big(L^{\text{ou}}_t\Big)^*\phi(t,y-\tilde{F}_m(t));\\ \partial_t\psi_m(t,y) &= e^{-\tilde{c}_m(t)}\Big[\partial_t\phi(t,y-\tilde{F}_m(t))-\langle F_m(t),D_y\phi(t,y-\tilde{F}_m(t))\rangle\\ &\qquad\qquad -c_m(t)\phi(t,y-\tilde{F}_m(t))\Big].\end{aligned}$$

The above calculations and a change of variable then imply that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^N}\Big[\Big(\partial_t+\Big(L^{\text{ou}}_t\Big)^*\Big)\phi(t,y)-\langle F_m(t),D_y\phi(t,y)\rangle-c_m(t)\phi(t,y)\Big]\mathcal{T}_m u(t,y)\\ +\phi(t,y)\mathcal{T}_m f(t,y)\,dydt+\int_{\mathbb{R}^N}u_0(y)\phi(0,y)\,dy = 0,\end{aligned}$$

where, analogously to (6.5), we have denoted for any function $\varphi\colon[0,T]\times\mathbb{R}^N\to\mathbb{R}$,

$$\mathcal{T}_m\varphi(t,y) := e^{-\tilde{c}_m(t)}\varphi(t,y+\tilde{F}_m(t)).$$

Following similar arguments exploited in the "existence" part in the proof of Theorem 5.2, i.e. exploiting the compact support of ϕ and the uniform convergence of the coefficients, it is possible to show that the above expression converges, when m goes to infinity, to

$$\int_0^T\int_{\mathbb{R}^N}\Big[\partial_t+\Big(L_t\Big)^*\Big]\phi(t,y)v(t,y)+\mathcal{T}f(t,y)\,dxdt+\int_{\mathbb{R}^N}\phi(0,x)u_0(x)\,dx] = 0$$

and thus, that v is a weak solution of Cauchy problem (6.6). □

Thanks to the previous lemma, we are now able to show the Schauder estimates for the solution v of the Cauchy problem (6.6) and, more importantly, without changing the constant C appearing in Equation (6.2).

Proposition 6.3. *Fixed $T > 0$, $\beta \in (0,1)$, let u_0 in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)$, f in $L^\infty\Big(0,T;C^\beta_b(\mathbb{R}^N)\Big)$ and c_0, F_0 in $B_b([0,T])$. Then, the unique solution v of Cauchy Problem* (6.6) *is in $L^\infty\Big(0,T;C^{\alpha+\beta}_b(\mathbb{R}^N)\Big)$ and it holds that*

$$\|v\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \,\le\, C\Big[\|u_0\|_{C^{\alpha+\beta}_{b,d}} + \|f\|_{L^\infty(C^\beta_{b,d})}\Big], \tag{6.7}$$

where $C := C(T,\eta) > 0$ is the same constant appearing in Theorem 5.4.

Proof. We start denoting for simplicity

$$\tilde{c}_0(t) \,:=\, \int_0^t c_0(s)\,ds \quad\text{and}\quad \tilde{F}_0(t) \,:=\, \int_0^t F_0(s)\,ds.$$

By Lemma 6.2, we know that if v is a weak solution of Cauchy problem (6.6), then the function

$$u(t,x) \,:=\, e^{\tilde{c}_0(t)}v(t,x-\tilde{F}_0(t))$$

is the weak solution of Cauchy problem (6.1) with $\tilde{f}$ instead of f, where

$$\tilde{f}(t,x) \,:=\, e^{\tilde{c}_0(t)}f(t,x-\tilde{F}_0(t)), \quad (t,x) \in (0,T)\times\mathbb{R}^N.$$

Moreover, we have that $\tilde{f}$ is in $L^\infty\Big(0,T;C^\beta_b(\mathbb{R}^N)\Big)$. Considering, if necessary, a smaller time interval $[0,t]$ for some $t \le T$, it is not difficult to check from Proposition 6.1 that

$$\|e^{\tilde{c}_0(t)}v(t,\cdot-\tilde{F}_0(t))\|_{C^{\alpha+\beta}_{b,d}} \,\le\, C\Big[\|u_0\|_{C^{\alpha+\beta}_{b,d}} + \sup_{s\in[0,t]}\|e^{\tilde{c}_0(s)}f(s,\cdot-\tilde{F}_0(t))\Big].$$

Using now the invariance of the Hölder norm under translations, we can show that

$$\begin{aligned}\|v(t,\cdot)\|_{C^{\alpha+\beta}_{b,d}} \,&\le\, C\Big[e^{-\tilde{c}_0(t)}\|u_0\|_{C^{\alpha+\beta}_{b,d}} + e^{-\tilde{c}_0(t)}\sup_{s\in[0,t]}\|e^{\tilde{c}_0(s)}f(s,\cdot)\Big]\\ &\le C\Big[\|u_0\|_{C^{\alpha+\beta}_{b,d}} + \sup_{s\in[0,t]}\|f(s,\cdot)\Big],\end{aligned}$$

where in the last step we exploited that $\tilde{c}_0(t)$ is non-decreasing. Taking the supremum with respect to t on both sides of the above inequality, we obtain our result. $\square$

Remark 6.1 (About space-time dependent coefficients)**.** We briefly explain here how to extend the Schauder estimates (5.18) to a class of non-linear, space-time dependent operators, whose coefficients are only locally Hölder continuous in space and may be unbounded. Namely, we are interested in operators of the following form:

$$\begin{aligned}L_t\phi(t,x) \,:=\, &\langle F(t,x), D_x\phi(x)\rangle\\ &+\int_{\mathbb{R}^d_0}\Big[\phi(x+B\sigma(t,x)z)-\phi(x)-\langle D_x\phi(x),B\sigma(t,x)z\rangle\mathbb{1}_{B(0,1)}(z)\Big]\,\nu(dz),\end{aligned}$$

where B is as in (2.3) and $\sigma\colon [0,T]\times\mathbb{R}^N \to \mathbb{R}^d\otimes\mathbb{R}^d$, $F\colon [0,T]\times\mathbb{R}^N \to \mathbb{R}^N$ are two measurable functions such that $F(t,0)$ is locally bounded in time and σ satisfies assumption [**UE**] at any fixed (t,x) in $[0,T]\times\mathbb{R}^N$.
We would like now the operator L_t to present a similar "dynamical" behaviour as above, i.e. the transmission of the smoothing effect of the Lévy operator to the degenerate components of the system (cf. Example 2.1). For this reason, we suppose the following:

- the drift $F = (F_1, \dots, F_n)$ is such that for any i in $[\![1, n]\!]$, F_i depends only on time and on the last $n-(i-2)$ components, i.e. $F_i(t, x_{i-1}, \dots, x_n)$;
- the matrices $D_{x_{i-1}}F_i(t,x)$ have full rank d_i at any fixed (t,x) in $[0,T]\times\mathbb{R}^N$.

As said before, the functions F and σ are assumed to be only locally Hölder in space, uniformly in time. Namely, there exists a positive constant K_0 such that

$$\mathbf{d}\Big(\sigma(t,x),\sigma(t,y)\Big) \,\le\, K_0\mathbf{d}^{\beta}(x,y); \qquad \mathbf{d}\Big(F_i(t,x),F_i(t,y)\Big) \,\le\, K_0\mathbf{d}^{\beta+\gamma_i}(x,y) \tag{6.8}$$

for any i in $[\![1, n]\!]$, any t in $[0,T]$ and any x,y in $\mathbb{R}^N$ such that $\mathbf{d}(x,y)\le 1$, where

$$\gamma_i \,:=\, \begin{cases} 1+\alpha(i-2), & \text{if } i>1; \\ 0, & \text{if } i=1. \end{cases} \tag{6.9}$$

We remark in particular that the function F may be unbounded in space.
In order to recover Schauder-type estimates even in this framework, we can follow a perturbative method firstly introduced in [KP10] that allows to exploit the already proven results for time-dependent operators. Let us assume for the moment that σ and F are *globally* Hölder continuous in space, i.e. they satisfy (6.8) for any x,y in $\mathbb{R}^N$. Informally speaking, the method links the operator L_t with the space independent operator L_t defined in (6.4), by "freezing" the coefficients of L_t along a *reference path* $\theta\colon [0,T]\to\mathbb{R}^N$ given by

$$\theta_t \,:=\, x_0 + \int_{t_0}^{t} F(s,\theta_s)\,ds,$$

for some (t_0,x_0) in $[0,T]\times\mathbb{R}^N$. It is important to highlight that, since F is only Hölder continuous, we need to fix one of the possible paths satisfying the above dynamics. We point out that the deterministic flow θ_t associated with the drift F is introduced precisely to handle the possible unboundedness of F. We could then consider a proxy operator L_t whose coefficients are given by $\sigma_0(t):=\sigma(t,\theta_t)$, $F_0(t):=F(t,\theta_t)$ and

$$\Big[A_t\Big]_{i,j} \,=\, \begin{cases} D_{x_{i-1}}F_i(t,\theta_t), & \text{if } j=i-1; \\ 0, & \text{otherwise} \end{cases}$$

In particular, Theorem 6.3 assures the well-posedness and the Schauder estimates for the Cauchy problem associated with L_t.
The final step of the proof would be to expand a solution u of the Cauchy problem associated with L_t around the proxy L_t through a Duhamel-like formula and finally show that the expansion error only brings a negligible contribution so that the Schauder estimates still hold for the original problem.
The a priori estimates for the expansion error are however quite involved (and they are the main reason why we have decided to not show here the complete proof), since they rely on some non-trivial controls in appropriate Besov norms.
In order to deal with coefficients that are only locally Hölder in space, we need in addition to introduce a "localized" version of the above reasoning. It would be necessary to multiply a solution u by a suitable bump function δ that localizes in space along the deterministic flow θ_t that characterizes the proxy. Namely, to fix a smooth function ρ

that is equal to 1 on $B(0,1/2)$ and vanishes outside $B(0,1)$ and define $\delta(t,x) := \rho(x-\theta_t)$. We would then follow the above method but with respect to the "localized" solution

$$v(t,x) := \delta(t,x)u(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^N.$$

We suggest the interested reader to see [CdRHM18a] for a detailed treatise of the argument in the degenerate diffusive setting, [CdRMP20a] in the non-degenerate stable framework or [Mar20] for the precise assumptions on the coefficients.

Chapter 4

Weak well-posedness for degenerate SDEs driven by Lévy processes

Abstract: We study the effects of the propagation of a non-degenerate Lévy noise through a chain of deterministic differential equations whose coefficients are Hölder continuous and satisfy a weak Hörmander-like condition. In particular, we assume some non-degeneracy with respect to the components which transmit the noise. Moreover, we characterize, for some specific dynamics, through suitable counter-examples, the almost sharp regularity exponents that ensure the weak well-posedness for the associated SDE. As a by-product of our approach, we also derive some Krylov-ype estimates for the density of the weak solutions of the considered SDE.

1 Introduction

We investigate the effects of the propagation of a d-dimensional Lévy noise through a chain of $n \geq 2$ differential equations. Namely, we are interested in a degenerate, Lévy-driven stochastic differential equation (SDE in short) of the following form:

$$\begin{cases} dX_t^1 \;=\; [[A_t]_{1,1}X_t^1 + \cdots + [A_t]_{1,n}X_t^n + F_1(t, X_t^1, \ldots, X_t^n)]\, dt + \sigma(t, X_{t-}^1, \ldots, X_{t-}^n)dZ_t, \\ dX_t^2 \;=\; [[A_t]_{2,1}X_t^1 + \cdots + [A_t]_{2,n}X_t^n + F_2(t, X_t^2, \ldots, X_t^n)]\, dt, \\ dX_t^3 \;=\; [[A_t]_{3,2}X_t^2 + \cdots + [A_t]_{2,n}X_t^n + F_3(t, X_t^3, \ldots, X_t^n)]\, dt, \\ \vdots \\ dX_t^n \;=\; \Big[[A_t]_{n-1,n}X_t^{n-1} + [A_t]_{n,n}X_t^n + F_n(t, X_t^n)\Big]\, dt, \end{cases} \tag{1.1}$$

where for $i \in [\![1, n]\!]$ ($[\![\cdot, \cdot]\!]$ denotes the set of all the integers in the interval), X_t^i is $\mathbb{R}^{d_i}$ valued, with $d_1 = d$ and $d_i \geq 1,\ i \in [\![2, n]\!]$. Set $N = \sum_{i=1}^n d_i$. We suppose that the $F_i \colon [0, +\infty) \times \mathbb{R}^{\sum_{j=i}^n d_j} \to \mathbb{R}^d$, $\sigma \colon [0, +\infty) \times \mathbb{R}^N \to \mathbb{R}^d \otimes \mathbb{R}^d$ are Borel and respectively locally bounded and uniformly elliptic and bounded.

We also assume the entries $([A_t]_{ij})_{1\leq i\leq n,\ i-1\leq j\leq n}$ are Borel bounded and such that the blocks $[A_t]_{i,i-1}$ in $\mathbb{R}^{d_i} \otimes \mathbb{R}^{d_{i-1}}$, $2 \leq i \leq n$ have rank d_i, uniformly in time. This is a kind of non-degeneracy assumption which can be viewed as weak Hörmander-like condition. It actually precisely allows the noise to propagate into the system.

Eventually, the noise $\{Z_t\}_{t\geq 0}$ belongs to a class of d-dimensional, symmetric, Lévy processes with suitable properties. In particular, to handle non trivial diffusion coefficients, we will assume that the Lévy measure of $\{Z_t\}_{t\geq 0}$ is absolutely continuous with respect to the Lévy measure of a rotationally invariant α-stable process (with α in $(1,2]$) and its Radon-Nikodym derivative enjoys some natural properties. The class of processes $\{Z_t\}_{t\geq 0}$ we can consider, includes for example, the tempered, the layered or the relativistic α-stable processes. In the case of an additive noise, cylindrical stable processes could be handled as well.

Here, the major issue is linked with the specific degenerate framework we consider. Indeed, the noise only acts on the first component of the dynamics (1.1) and it implies, in particular, that the random perturbation on the i-th line of SDE (1.1) only comes from the previous $(i-1)$-th one, through the non-degeneracy of the matrixes $[A_t]_{i,i-1}$. Hence, the smoothing effect associated with the Lévy noise decreases along the chain, making thus more and more difficult to regularize by noise the furthest lines of Equation (1.1).
We nevertheless prove the weak well-posedness, i.e. the existence and uniqueness in law, for the above SDE (1.1) when the drift $F=(F_1,\dots,F_n)$ and σ lie in a suitable anisotropic Hölder space with multi-indices of regularity. We assume that F_1 and σ have spatial Hölder regularity $\beta^1>0$ with respect to the j-th variable. We highlight already that we could have considered different regularity indexes β^1_j for the regularity of F_1 with respect to the j-th variable. We keep only one common index for notational simplicity. We also suppose that for fixed $j\in[\![2,n]\!]$, $(F_2,\cdots,F_j)$ has Hölder regularity β^j with respect to the j-th variable, where:

$$\beta^j \;\in\; \Big(\frac{1+\alpha(j-2)}{1+\alpha(j-1)};1\Big].$$

We indeed recall that from the dynamics (1.1) the variable x_j does not appear in the chain after level j.

Furthermore, we will show through suitable counter-examples that the above threshold for the regularity exponents β^j is "almost" sharp for a perturbation of the j^{th} level of the chain. Such counter-examples are based on Peano-type dynamics adapted to our degenerate, fractional framework.

Models of the form (1.1) naturally appear in various scientific contexts: in physics, for the analysis of anomalous diffusions phenomena or for Hamiltonian models in turbulent regimes (see e.g. [BBM01], [CPKM05], [EPRB99]); in mathematical finance and econometrics, for example in the pricing of Asian options (see e.g. [JYC09], [Bro01], [BNS01]). In particular, models that consider Lévy noises, such as SDE (1.1), seem more natural and realistic for many applications since they allow the presence of jumps.

Weak well-posedness for non-degenerate stable SDEs. The topic of weak well-posedness for non-degenerate (i.e. $d = N$) SDEs of the form:

$$X_t = x + \int_0^t F(X_s)ds + Z_t, \quad t \geq 0, \tag{1.2}$$

where $\{Z_t\}_{t\geq 0}$ is a symmetric α-stable process on $\mathbb{R}^N$, has been widely studied in the last decades, especially in the diffusive, local setting, i.e. when $\alpha = 2$ and $\{Z_t\}_{t\geq 0}$ is a Brownian Motion, and it is now quite well-understood. We can first refer to the seminal work [SV79] where the Authors considered additionally a multiplicative noise with bounded drift and non-degenerate, continuous in space diffusion coefficient. We recall moreover that in the framework of (1.2) with bounded drift, strong uniqueness also holds (cf. [Ver81]).
SDEs like (1.2) with a proper α-stable process ($\alpha < 2$) were firstly investigated in [TTW74] where the weak well-posedness was obtained for the one-dimensional case when the drift F is bounded, continuous and the Lévy exponent Φ of $\{Z_t\}_{t\geq 0}$ satisfies $\Re\Phi(\xi)^{-1} = 0(1/|\xi|)$ if $|\xi| \to \infty$. The multidimensional case ($d > 1$) can be similarly obtained following [Kom83] if the drift is bounded, continuous and the law of $\{Z_t\}_{t\geq 0}$ admits a density with respect to the Lebesgue measure on $\mathbb{R}^d$. Equations as (1.2) with drift in some suitable L^p-spaces and non-degenerate noise were also considered in [Jin18] (see also the references therein). We can eventually quote the recent work by Krylov who obtained even strong uniqueness for Brownian SDEs with drifts in critical L^p-spaces, see [Kry21].

In recent years, SDEs driven by singular (distributional) drift have gained a lot of interest, especially in the Brownian setting, where they arise as a model for diffusions in random media (see e.g. [FRW03],[FRW04],[FIR17], [DD16], [CC18]).
In the non-local α-stable framework, a first work to appear was [ABM20] where the authors considered the one-dimensional case with a time-homogeneous drift of (negative) Hölder regularity strictly greater than $(1-\alpha)/2$. We remark that in the one-dimensional framework, the regularity thresholds on the drift are the same for the strong and the weak well-posedness, since it is possible to exploit local time arguments (see also [BC01] in the diffusive setting). On the same side, the almost simultaneous works [LZ19] and [CdRM20a] take into account time-homogeneous and time-inhomogeneous, respectively, distributional drift in general Besov spaces with suitable conditions on the parameters. These results rely on Young integrals in order to give a meaningful sense to the dynamics. Beyond the Young regime, we instead refer to [KP20] where techniques such as paracontrolled products (which have also been popular in the recent developments in the SPDE theory) are exploited to analyze the martingale problem associated with a time-inhomogeneous drift of regularity index strictly greater than $(2-2\alpha)/3$.

Moreover, we would like to remark that the above works concerned the so-called *sub-critical* case, i.e. when $\alpha > 1$. Indeed, SDEs like (1.2) are much more difficult to handle if $\alpha \leq 1$ since in this case, the noise does not dominate the system for small time scales. Two recent works along this path are [Zha19] and [CdRMP20b] where the authors consider $\alpha < 1$, $(1-\alpha)$-Hölder drift F and $\alpha = 1$, continuous drift, respectively. We also mention that for Hölder drifts, the well-posedness of the associated martingale problem can be obtained following [MP14] if F is bounded or through the Schauder estimates given in [CdRMP20a] when F is unbounded.

Regularization by noise in a degenerate setting. All the above results present a common phenomenon that, following the terminology in [Fla11], is usually called *regularization by noise*. This occurs when a deterministic ODE is ill-posed (for example if the drift is less than Lipschitz) but its stochastic counterpart (SDE) is well posed in a strong or a weak sense.
To obtain such phenomenon, the noise plays a fundamental role. A usual assumption is that the noise should act on every line of the dynamics, regularizing the coefficients. It is then clear that in our degenerate framework, when the noise acts only on the first component of the chain (1.1), the situation is even more delicate. In order to obtain some kind of regularization effect in this case, we need that the noise propagates through the system, reaching all the lines of Equation (1.1). A typical assumption ensuring such type of behaviour is the so-called Hörmander condition for hypoellipticity (cf. [Hör67]). From the structure of the equation (1.1) at hand, we will consider a *weak* type Hörmander condition, i.e. up to some regularization of the diffusion coefficient, the drift is needed to span the space through Lie bracketing.

In the Hamiltonian setting $n = 2$, when $\{Z_t\}_{t\geq 0}$ is a Brownian Motion and for a more general, non-linear, drift than in (1.1) which still satisfies a weak Hörmander type condition, Chaudru de Raynal showed in [CdR18] that the associated SDE is weakly well-posed as soon as the drift is Hölder continuous in the degenerate variable with regularity index strictly greater than 1/3. It was also established through an appropriate counter-example, that the 1/3-threshold is (almost) sharp for the second component of the drift. Such a result has been then extended in [CdRM20b] in order to consider the more general case of n oscillators. Therein, the regularity thresholds that ensure weak uniqueness also depend on the variable and the level of the chain. This seems intuitively clear, the further the variable in the oscillator chain, the larger its typical time scale, the weaker the regularity needed to regularize components which are above that variable in the chain. Also, some corresponding Krylov type estimates, giving existence and integrability properties of the density of the SDE are derived. We can mention as well the recent work by Gerencsér [Ger20] who obtain similar regularization properties for the iterated time integrals of a fractional Brownian motion.

In the jump case, the situation is much more delicate. Within the proper regularization by noise framework (when the coefficients are less than Lipschitz continuous), we cite [HM16] where the Authors showed the weak well-posedness for (1.1) with $F = 0$ and a Hölder continuous diffusion coefficient, under some constraints on the dimensions d,n. In that framework, the Authors obtained as well same point-wise density estimates. The driving noises considered were stable and tempered stable processes.
Finally, we mention that it is possible to derive the weak well-posedness of dynamics (1.1) via the martingale formulation, exploiting the Schauder estimates given in [HWZ20] for the kinetic model ($n = 2$). In that work, the Authors actually characterized conditions for strong uniqueness, using Littlewood-Paley decomposition techniques.

We will here proceed through a perturbative approach. Namely, we will expand the formal generator associated with (1.1) around the one of a well understood process, with possibly time inhomogeneous coefficients which are anyhow frozen in space. We will call such a process a *proxy*. The most natural candidate to be a *proxy* for (1.1) is a degenerate Ornstein-Uhlenbeck process. In the case of time homogeneous coefficients, Priola and Zabczyk established in [PZ09] existence of the density for such processes under the same previously indicated non-degeneracy conditions on the matrix A (which turn out to be equivalent in that setting to the well known Kalman condition).

Intrinsic difficulties associated with large jumps. When Z is a strict stable process, the density of the corresponding degenerate Ornstein-Uhlenbeck process can somehow be related to the one of a multi-scale stable process which has however a very singular associated *spectral measure* (spherical part of the α-stable Lévy measure) on $\mathbb{S}^{N-1}$, see e.g. [HM16], [HMP19] and Proposition 2.10 below. From Watanabe [Wat07], it is known that the tails of stable densities are highly related to the nature of this spectral measure. Specifically, the concentration properties worsen when the measure becomes singular. This renders delicate the characterization of the smoothing properties for the proxy, especially when it depends on parameters and that one would like to obtain estimates which are uniform w.r.t. those parameters (see Proposition 2.11 and Section 2.2 below).

Even for smooth coefficients, the stable like jump setting is much more delicate to establish the existence of the density for (degenerate) SDEs. For multiplicative noises, we cannot indeed rely on the flow techniques considered in [BGJ87] or [Kun19] in the non-degenerate case, and Léandre in the degenerate one, see [Léa85],[Léa88]. Still for smooth coefficients, we can refer to the work of Zhang [Zha14] who obtained existence and smoothness results for the density of equations of type (1.1) in arbitrary dimension, for a possibly more general non linear drift, still satisfying a weak Hörmander type condition when the driving process is a rotationally invariant stable process. The strategy therein is based on the *subordinated* Malliavin calculus, which consists in applying the *usual* Malliavin calculus techniques on a Brownian motion observed along the path of an independent α-stable subordinator. In whole generality a *complete* version of the Hörmander theorem in the jump case seems to lack. We can refer to the work by Cass [Cas09] who gets smoothness of the density in the weak Hörmander framework under technical restrictions.

1.1 Complete model and assumptions

Let us now specify the assumptions on equation (1.1) that we rewrite in the shortened form:

$$dX_t \,=\, G(t,X_t)dt + B\sigma(t,X_{t-})dZ_t, \quad t \ge 0, \tag{1.3}$$

where B is the embedding from $\mathbb{R}^d$ to $\mathbb{R}^N$ given in matricial form as

$$B := \Big(I_{d\times d}, \quad 0_{d\times(N-d)},\Big)^t$$

and $G(t,x) = A_t x + F(t,x)$ with:

$$A_t := \begin{pmatrix} [A_t]_{1,1} & \dots & \dots & \dots & [A_t]_{1,n} \\ [A_t]_{2,1} & [A_t]_{2,2} & \dots & \dots & [A_t]_{2,n} \\ 0 & [A_t]_{3,2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & [A_t]_{n,n-1} & [A_t]_{n,n} \end{pmatrix}. \tag{1.4}$$

A classical assumption in this degenerate framework (cf. [SV79], [Kry04], [CdRM20b]) is the *uniform ellipticity* of the underlying non-degenerate component of the diffusion matrix at any fixed space-time point. Namely,

[UE] There exists a constant $\eta > 1$ such that for any $t \geq 0$ and any x in $\mathbb{R}^N$, it holds that

$$\eta^{-1}|\xi|^2 \leq \sigma(t,x)\xi \cdot \xi \leq \eta|\xi|^2, \quad \xi \in \mathbb{R}^d,$$

where "$\cdot$" stands for the inner product on the smaller space $\mathbb{R}^d$.

We will suppose that the drift $G(t,x) = A_t x + F(t,x)$ has a particular "upper diagonal" structure and its sub-diagonal elements are linear and non-degenerate, i.e.

[H]
- $F = (F_1, \dots, F_n)\colon [0,\infty) \times \mathbb{R}^N \to \mathbb{R}^N$ is such that F_i depends only on time and on the last $n-(i-1)$ components, i.e. $F_i(t, x_i, \dots, x_n)$, for any i in $[\![1, n]\!]$;
- $A\colon [0,\infty) \to \mathbb{R}^N \otimes \mathbb{R}^N$ is bounded and the blocks $[A_t]_{i,j} \in \mathbb{R}^{d_i} \otimes \mathbb{R}^{d_j}$ in (1.4) are such that

$$[A_t]_{i,j} = \begin{cases} \text{is non-singular (i.e. it has rank } d_i\text{) uniformly in } t, \text{ if } j = i-1; \\ 0, \text{ if } j < i-1. \end{cases}$$

Clearly, n is in $[\![1, N]\!]$ and $n = 1$ if and only if $d = N$, i.e. if the dynamics is non-degenerate.

In the linear framework ($F = 0$) and for constant diffusion coefficients ($\sigma(t,x) = \sigma$), this last assumption can be seen as a Hörmander-type condition, ensuring the hypoellipticity of the infinitesimal generator associated with the process $\{X_t\}_{t\geq 0}$, which is in this setting equivalent to the Kalman condition, see e.g. [PZ09]. We highlight however that in our framework, the "classic" Hörmander assumption (cf. [Hör67]) cannot be considered, due to the low regularity of the coefficients we will consider in (1.3) (see Theorem 2.6). This prevents us from explicitly calculating the commutators.

In Equation (1.3) above, $\{Z_t\}_{t\geq 0}$ is a d-dimensional, symmetric and adapted Lévy process with respect to some stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. We recall that a d-valued Lévy process is a stochastically continuous process on $\mathbb{R}^d$ starting from zero and such that its increments are independent and stationary. Moreover, it is well-known (see e.g. [Sat13]) that any Lévy process admits a càdlàg modification, i.e. a right continuous modification having left limits $\mathbb{P}$-almost surely. We will always assume to have chosen such a version.

A fundamental tool in the analysis of Lévy processes is given by the Lévy-Kitchine formula (see for instance [Jac01]) that allows us to represent the Lévy symbol $\Phi(\xi)$ of $\{Z_t\}_{t\geq 0}$, given by

$$\mathbb{E}[e^{i\xi\cdot Z_t}] = e^{t\Phi(\xi)}, \quad \xi \in \mathbb{R}^d$$

in terms of the generating triplet (b, Σ, ν) as:

$$\Phi(\xi) \,=\, ib\cdot\xi - \frac{1}{2}\Sigma\xi\cdot\xi + \int_{\mathbb{R}^d_0}\Bigl(e^{i\xi\cdot z} - 1 - i\xi\cdot z\mathbb{1}_{B(0,1)}(z)\Bigr)\,\nu(dy), \quad \xi\in\mathbb{R}^d,$$

where b is a vector in $\mathbb{R}^d$, Σ is a symmetric, non-negative definite matrix in $\mathbb{R}^d\otimes\mathbb{R}^d$ and ν is a Lévy measure on $\mathbb{R}^d_0 := \mathbb{R}^d\smallsetminus\{0\}$, i.e. a σ-finite measure on $\mathcal{B}(\mathbb{R}^d_0)$, the Borel σ-algebra on $\mathbb{R}^d_0$, such that $\int(1\wedge|z|^2)\,\nu(dz)$ is finite. In particular, any Lévy process is completely determined by its generating triplet (b, Σ, ν).

Importantly, we point out already that a change on the truncation set $B(0,1)$ for the Lévy-Kitchine formula does not affect the formulation of the Lévy symbol Φ, since we assumed ν to be symmetric. Namely, given a threshold $c > 0$, the Lévy symbol $\Phi(\xi)$ of $\{Z_t\}_{t\geq 0}$ could be also represented as

$$\Phi(\xi) \,=\, ib\cdot\xi - \frac{1}{2}\Sigma\xi\cdot\xi + \int_{\mathbb{R}^d_0}\Bigl(e^{i\xi\cdot z} - 1 - i\xi\cdot z\mathbb{1}_{B(0,c)}(z)\Bigr)\,\nu(dz), \quad \xi\in\mathbb{R}^d, \tag{1.5}$$

where b, Σ and ν are as above. Here, we only consider pure jump processes, i.e. $\Sigma = 0$. Indeed, the more general case, where a Gaussian component is considered, can be obtained from already existing results (cf. [CdRM20b]).

We will suppose moreover that, additionally to the symmetry, the Lévy measure ν of $\{Z_t\}_{t\geq 0}$ satisfies the following *non-degeneracy condition*:

[**ND**] there exist a Borel function $Q\colon \mathbb{R}^d\to[0,\infty)$ such that

- ess-sup$\{Q(z)\colon z\in\mathbb{R}^d\} < +\infty$;
- there exist $r_0 > 0$ and $c > 0$ such that $Q(z)\geq c$ and Lipschitz continuous in $B(0, r_0)$;
- there exists $\alpha\in(1,2)$ and a finite, non-degenerate measure μ on $\mathbb{S}^{d-1}$ such that

$$\nu(\mathcal{A}) \,=\, \int_0^\infty\int_{\mathbb{S}^{d-1}}\mathbb{1}_{\mathcal{A}}(rs)Q(rs)\,\mu(ds)\frac{dr}{r^{1+\alpha}}, \quad \mathcal{A}\in\mathcal{B}(\mathbb{R}^d_0),$$

where $\mathcal{B}(\mathbb{R}^d_0)$ stands for the Borelian σ-field on $\mathbb{R}^d_0$. We recall moreover that a spherical measure μ on $\mathbb{S}^{d-1}$ is non-degenerate (in the sense of Kolokoltsov [Kol00]) if there exists a constant $\tilde{\eta}\geq 1$ such that

$$\tilde{\eta}^{-1}|\xi|^\alpha \,\leq\, \int_{\mathbb{S}^{d-1}}|\xi\cdot s|^\alpha\,\mu(ds) \,\leq\, \tilde{\eta}|\xi|^\alpha, \quad \xi\in\mathbb{R}^d. \tag{1.6}$$

Since any α-stable Lévy measure can be decomposed into a spherical part μ on $\mathbb{S}^{d-1}$ and a radial part $r^{-(1+\alpha)}dr$ (see e.g. Theorem 14.3 in [Sat13]), assumption [**ND**] roughly states that the Lévy measure of $\{Z_t\}_{t\geq 0}$ is absolutely continuous with respect to the non-degenerate (in the sense of (1.6)), Lévy measure of a α-stable process and that their Radon-Nikodym derivative is given by the function Q. From this point further, we will denote such a Lévy measure by $\nu_\alpha(dz) := \mu(ds)r^{-(1+\alpha)}dr$ with $z = rs$.

In order to deal with a possibly multiplicative noise, i.e. in the presence of a space-inhomogeneous diffusion coefficient σ in Equation (1.3), we will need the following:

[**AC**] If $x\to\sigma(t,x)$ is non-constant for some $t\geq 0$, then the measure ν_α is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^N$ with Lipschitz Radon-Nykodim derivative.

Assumptions [**ND**] and [**AC**] together imply in particular that in the multiplicative case, the Lévy measure ν of $\{Z_t\}_{t\geq 0}$ can be decomposed as

$$\nu(dz) \,=\, Q(z)\frac{g(\frac{z}{|z|})}{|z|^{d+\alpha}}dz, \tag{1.7}$$

for some Lipschitz function $g\colon \mathbb{S}^{d-1}\to\mathbb{R}$.
At this point, we would like to remark that no regularity is assumed for the Lévy measure ν of $\{Z_t\}_{t\geq 0}$ in the additive framework (or more generally, for a space-independent σ). In particular, the measure μ in condition [**ND**] may not be absolutely continuous with respect to the Lebesgue measure on $\mathbb{S}^{d-1}$. Indeed, our model can also include very singular (with respect to the Lebesgue measure) examples such as the cylindrical α-stable process associated with $\mu=\sum_{i=1}^d\frac{1}{2}(\delta_{e_i}+\delta_{-e_i})$. See e.g. [BC06] for more details.
From this point further, we always assume that the above hypotheses on the coefficients are satisfied.

We would like to conclude the introduction with some comments concerning our assumptions with particular reference with our previous works.
In [Mar21], the Schauder estimates, an important analytical first step for proving the well-posedness of SDEs, has been showed for degenerate Ornstein-Uhlenbeck operators driven by a more general class of Lévy noises. It also includes, for example, the asymmetric version of the stable-like noises we consider in this work. We start highlighting that a similar family of noises could not have been introduced here, as in [Mar20], due to the non-linear structure of our problem and, especially, our technique of proof through a perturbative approach. Indeed, it requires more delicate regularizing properties for the involved operators and, in particular, a compatibility between some proxy and the original operator, seen as a perturbation of the first one.
Here, we have followed a backward perturbative approach as firstly introduced by McKean-Singer in [MS67]. This terminology comes from the fact that the underlying proxy process will be associated with a backward in time flow. This method appears more natural for proving weak uniqueness in a degenerate L^p-L^q framework (cf. [CdRM20b] in the diffusive case). Roughly speaking, it only requires controls on the gradients (in a weak sense) for the solutions of the associated PDE in order to apply the inversion technique on the infinitesimal generator. However, we are confident that the Schauder estimates presented in [Mar20] could be extended to the class of noises we consider here. Relying on them, we could have then proven the uniqueness in law for dynamics such as (1.3). This method appears really involved and long since it structurally requires to establish pointwise estimates for the first order derivatives with respect to the degenerate components of the dynamics. Another useful advantage of the backward perturbative approach is that it allows us to show Krylov-type estimates on the solution process X_t of SDE (1.3). These kind of controls seems of independent interest and new for random dynamics involving degenerate stable-like noises.

The drawback of our approach is that it leads to a specific structure in Equation (1.3), given by assumption [**H**]. Namely, we cannot consider drift of the form $F_i(x)=F_i(x_{i-1},\cdots,x_n)$ with non-linear dependence w.r.t. x_{i-1}, variable which transmits the noise. This case is often investigated for Brownian noises (see e.g. [DM10], [CdRM20b]).

This feature is specifically linked to the structure of the joint law of a stable process and its iterated integrals which generate a multi-scale stable process with highly singular associated spectral measure, see e.g. Proposition 2.10 and Remark 2.2 below or [HM16]. Similar issues constrain us to assume in the multiplicative noise case that the driving process has an absolutely continuous spectral measure with respect to the Lebesgue measure on $\mathbb{S}^{d-1}$. This precisely allows us to get estimates which will be uniform with respect to the parameters for the considered class of proxys.

Main driving processes considered. Here, we highlight that assumption [**ND**] applies to a large class of Lévy processes on $\mathbb{R}^d$. As already pointed out in [SSW12], it holds for the following families of stable-like examples with $\alpha \in (0,2)$:

1. Stable process [Sat13]:
$$Q(z) \;=\; 1;$$
2. Truncated stable process with $r_0 > 0$ [KS08]:
$$Q(z) \;=\; \mathbb{1}_{(0,r_0]}(|z|);$$
3. Layered stable process with $\beta > \alpha$ and $r_0 > 0$ [HK07]:
$$Q(z) \;=\; \mathbb{1}_{(0,r_0]}(|z|) + \mathbb{1}_{(r_0,\infty)}(|z|)|z|^{\alpha-\beta};$$
4. Tempered stable process [Ros07] with $Q(z) = Q(rs)$ such that for all s in $\mathbb{S}^{d-1}$,
$$r \to Q(rs) \text{ is completely monotone, } Q(0) > 0 \text{ and } \lim_{r\to+\infty} Q(rs) = 0.$$
5. Relativistic stable process [CMS90], [BMR09]:
$$Q(z) \;=\; (1+|z|)^{(d+\alpha-1)/2}e^{-|z|};$$
6. Lamperti process with $f\colon \mathbb{S}^{d-1} \to \mathbb{R}$ even such that $\sup f(s) < 1+\alpha$ [CPP10]:
$$Q(z) \;=\; \exp\Big(|z|f(\frac{z}{|z|})\Big)\Big(\frac{|z|}{e^{|z|}-1}\Big)^{1+\alpha}, \quad z \in \mathbb{R}^d_0.$$

Organization of the paper. The article is organised as follows. In Section 2, we introduce some useful notations and we present the associated martingale problem. In particular, we state there our main results. Section 3 contains all the associated analytical tools that allow to derive our results. Namely, we follow there a perturbative approach, considering a suitable linearization of our dynamics (1.3) around a Cauchy-Peano flow which takes into account the deterministic part of our model (corresponding to (1.3) with $\sigma = 0$). Section 4 is then dedicated to prove the well-posedness of the associated martingale problem, exploiting the analytical results given in Section 3. In Section 5, we finally construct an “ad hoc” Peano counter-example to the uniqueness in law for SDE (1.3).

2 Basic notations and main results

We start recalling some useful notations we will need below. In the following, C will denote a generic *positive* constant. It may change from line to line and it will depend only on the parameters appearing in the previously stated assumptions, as for instance: $d, N, \alpha, \eta, b, g, r_0, \mu$. We will explicitly specify any other dependence that may occur. Given a function $f\colon \mathbb{R}^N \to \mathbb{R}$, we denote by $Df(x)$, and $D^2f(x)$ the first and second Fréchet derivative of f at a point x in $\mathbb{R}^N$ respectively, when they exist. We denote by $B_b(\mathbb{R}^N)$ the family of all the Borel and bounded functions $f\colon \mathbb{R}^N \to \mathbb{R}$. It is a Banach space endowed with the supremum norm $\|\cdot\|_\infty$. We also consider its closed subspace $C_b(\mathbb{R}^N)$ consisting of all the continuous functions. Moreover, $C^\infty_c(\mathbb{R}^N) \subseteq C_b(\mathbb{R}^N)$ denotes the space of smooth functions with compact support.

We now recall two correlated definitions of solution associated with SDE (1.3). Let us consider fixed μ in $\mathcal{P}(\mathbb{R}^N)$, the family of the probability measures on $\mathbb{R}^N$ and an initial time $t \geq 0$.

Definition 2.1. A weak solution of SDE (1.3) with starting condition (t, μ) is a N-dimensional, càdlàg, adapted process $\{X_s\}_{s\geq 0}$ on some stocastic base $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\geq 0}, \mathbb{P})$ such that

- — the law of X_t is μ;
- — there exists a d-dimensional, adapted Lévy process $\{Z_s\}_{s\geq t}$ satisfying [**ND**] and [**AC**] such that

$$X_s \;=\; X_t + \int_t^s G(u, X_u)\, du + \int_t^s \sigma(u, X_{u-})B\, dZ_u, \quad s \geq t, \; \mathbb{P}\text{-a.s.} \tag{2.1}$$

To state our second definition, we need to consider the infinitesimal generator $\partial_s + L_s$ (formally) associated with the solutions of SDE (1.3). Noticing that the term involving the constant drift b can be absorbed in the expression for G without loss of generality, the operator L_s can be represented for any ϕ in $C^\infty_c(\mathbb{R}^N)$ as

$$\begin{aligned} L_s\phi(s,x) \;&:=\; \langle G(s,x), D_x\phi(x)\rangle + \mathcal{L}_s\phi(s,x) \\ &:=\; \langle G(s,x), D_x\phi(x)\rangle + \int_{\mathbb{R}^d_0}\Bigl[\phi(x + B(s,x)z) - \phi(x)\Bigr]\,\nu(dz), \end{aligned} \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on the bigger space $\mathbb{R}^N$ and, for brevity, $B(s,x) := B\sigma(s,x)$. As done in [Pri15b], we introduce the following definition:

Definition 2.2. A solution of the martingale problem for $\partial_s + L_s$ with initial condition (t, μ) is an N-dimensional, càdlàg process $\{X_s\}_{s\geq t}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

- — the law of X_t is μ;
- — for any ϕ in $\mathrm{dom}\Bigl(\partial_s + L_s\Bigr)$, the process

$$\Bigl\{\phi(s, X_s) - \phi(t, X_t) - \int_t^s \Bigl(\partial_u + L_u\Bigr)\phi(u, X_u)\, du\Bigr\}_{s\geq t}$$

  is a $\mathbb{P}$-martingale with respect to the natural filtration $\{\mathcal{F}^X_s\}_{s\geq 0}$ of the process $\{X_s\}_{s\geq 0}$.

We can now recall some known results that enlighten the link between the two definitions presented above. For a more thorough analysis on the topic of martingale problems in a rather abstract and general framework, we refer to Chapter 4 in [EK86].
Given a solution $\{X_s\}_{s\geq 0}$ of SDE (1.3), an application of the Itô formula immediately shows that the process $\{X_s\}_{s\geq 0}$ is a solution of the martingale problem for $\partial_s + L_s$ with initial condition (t, μ), too.
On the other hand, if there exists a solution $\{X_s\}_{s\geq 0}$ of the martingale problem for $\partial_t + L_t$ with initial condition (t, μ), it is possible to construct an "enhanced" filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}\}_{s\geq 0}, \tilde{\mathbb{P}})$ on which there exists a solution $\{\tilde{X}_s\}_{s\geq 0}$ of the SDE (1.3). Moreover, the two processes $\{X_s\}_{s\geq t}$ and $\{\tilde{X}_s\}_{s\geq t}$ have the same law (See, for more details, [Kur11]). Thus, it holds that:

Proposition 2.3. *Let μ be in $\mathcal{P}(\mathbb{R}^N)$ and $t \geq 0$. The existence of a weak solution for SDE* (1.3) *with initial condition (t, μ) is equivalent to the existence of a solution to the martingale problem for $\partial_s + L_s$ with initial condition (t, μ).*

We can now move on the notion of uniqueness associated with our problem.

Definition 2.4. We say that weak uniqueness holds for the SDE (1.3) with initial condition (t, μ) if any two solutions $\{X_s\}_{s\geq 0}$, $\{Y_s\}_{s\geq 0}$ of SDE (1.3) with initial condition (t, μ) have same finite dimensional distributions. In particular, we say that SDE (1.3) is weakly well-posed if for any μ in $\mathcal{P}(\mathbb{R}^N)$ and any $t \geq 0$, there exists a unique weak solution of SDE (1.3) with initial condition (t, μ).

Since the definition above takes into account only the law of the solutions $\{X_s\}_{s\geq t}$, $\{Y_s\}_{s\geq t}$, they may, in general, have been defined on different stochastic bases or with respect to two different underlying Lévy processes. The definition of uniqueness for a solution of the martingale problem for $\partial_s + L_s$ can be stated similarly.

It is not difficult to check that the uniqueness of the martingale problem for $\partial_s + L_s$ with initial condition (t, μ) implies the weak uniqueness of the SDE (1.3). Furthermore, it has been shown in [Kur11], Corollary 2.5 that the converse is also true.

Proposition 2.5. *Let μ be in $\mathcal{P}(\mathbb{R}^N)$ and $t \geq 0$. Then, weak uniqueness holds for SDE* (1.3) *with initial condition (t, μ) if and only if uniqueness holds for the martingale problem for $\partial_s + L_s$ with initial condition (t, μ).*

Thanks to Propositions 2.3 and 2.5, we can conclude that the two approaches, i.e. the martingale formulation and the dynamics given in (1.3), are equivalent in specifying a Lévy diffusion process on $\mathbb{R}^N$. We recall however that a third, yet equivalent, method is given by the forward Fokker-Plank equation governing the law of the process. We will not explicitly define it since we will not exploit it afterwards (see, for more details, e.g. [Fig08], [LBL08]).
From now on, we write x in $\mathbb{R}^N$ as $x = (x_1, \dots, x_n)$ where $x_i = (x_i^1, \dots, x_i^{d_i})$ is in $\mathbb{R}^{d_i}$ for any i in $[\![1, n]\!]$.
We can now state our main theorem.

Theorem 2.6. *For any j in $[\![1, n]\!]$, let β^j be an index in $(0, 1]$ such that*

— *$x_j \to \sigma(t, x_1, \dots, x_j, \dots, x_n)$ is β^1-Hölder continuous, uniformly in t and in x_i for*

$i \neq j$;

- *$x_j \to F_1(t, x_1, \dots, x_j, \dots, x_n)$ is β^1-Hölder continuous, uniformly in t and in x_i for $i \neq j$;*
- *$x_j \to F_i(t, x_i, \dots, x_j, \dots, x_n)$ is β^j-Hölder continuous, uniformly in t and in x_k, for $k \neq j$ and $2 \leq i \leq j$.*

Additionally, we suppose that there exists $K \geq 1$ such that $|F_i(t, 0)| \leq K$ for any i in $[\![1, n]\!]$ and any $t \geq 0$. Then, the SDE (1.3) *is weakly well-posed if*

$$\beta^j > \frac{1 + \alpha(j-2)}{1 + \alpha(j-1)}, \; j \geq 2. \tag{2.3}$$

Theorem 2.6 above will follow from Propositions 2.3 and 2.5, once we have shown that under the same assumptions, there exists a unique weak solution to the martingale problem for $(\partial_s + L_s, \delta_x)$ at any x in $\mathbb{R}^N$.
As a by-product of our method of proof, we have been able to show a Krylov-type estimates for the solutions of SDE (1.3). For notational convenience, we will say that two real numbers $p > 1$, $q > 1$ satisfy Condition $(\mathscr{C})$ when the following inequality holds:

$$\Big(\frac{1-\alpha}{\alpha} N + \sum_{i=1}^{n} i d_i\Big) \frac{1}{q} + \frac{1}{p} \; < \; 1. \tag{$\mathscr{C}$}$$

Roughly speaking, such a threshold guarantees the necessary integrability in time with respect to the associated intrinsic scale of the system when considering the $L^p_t - L^q_x$ theory (see Equation (2.43) for more details). Furthermore, when considering the homogeneous case, i.e. when all the components of the system has the same dimension ($d_i = d$ and $N = nd$), condition $(\mathscr{C})$ can be rewritten in the following, clearer, way:

$$\left(\frac{2 + \alpha(n-1)}{\alpha}\right) \frac{nd}{q} + \frac{2}{p} \; < \; 2.$$

In particular, taking $\alpha = 2$ above, we find the same threshold appearing in [CdRM20b] for the diffusive setting. We highlight moreover that our thresholds can be seen as a natural extension of the ones appearing in [KR05] in the non-degenerate, Brownian setting.

Corollary 2.7. *Under the same assumptions of Theorem 2.6, let $T > 0$ and $p > 1$, $q > 1$ such that Condition $(\mathscr{C})$ holds. Then, there exists a constant $C := C(T, p, q)$ such that for any f in $L^p\big(0, T; L^q(\mathbb{R}^N)\big)$, it holds*

$$\left|\mathbb{E}^{\mathbb{P}_{t,x}}\Big[\int_t^T f(s, X_s)\, ds\Big]\right| \; \leq \; C \|f\|_{L^p_t L^q_x}, \quad (t, x) \in [0, T] \times \mathbb{R}^N, \tag{2.4}$$

where $\{X_s\}_{s \geq 0}$ is the canonical process associated with $\mathbb{P}_{t,x}[\cdot] := \mathbb{E}[\cdot | X_t = x]$ which is also the unique weak solution of SDE (1.3) *with initial condition (t, x). In particular, the random variable X_s admits a density $p(t, s, x, \cdot)$ for any $t < s$ and any x in $\mathbb{R}^N$.*

Additionally, we have been able to show the following non uniqueness result.

Theorem 2.8. *Let us consider SDE* (1.3) *with* $\sigma = 1$ *and assume that*

— $x_j \to F_i(t, x_i, \ldots, x_j, \ldots, x_n)$ *is* β_i^j*-Hölder continuous, uniformly in* t *and in* x_k*, for* $k \neq j$*.*

Then, for given i*,* j *in* $[\![2, n]\!]$ *with* $j \geq i$ *there exists* $F_i(t, x_i, \ldots, x_j, \ldots, x_n) = F_i(t, x_j)$ *with*

$$\beta_i^j < \frac{1 + \alpha(i-2)}{1 + \alpha(j-1)},$$

for which weak uniqueness fails for the SDE (1.3)*.*

The above result will be proven in Section 5, showing a suitable, explicit Peano-type counter-example.

Remark 2.1. As opposed to the Gaussian driven case, we did not succeed to obtain regularity indexes which are *sharp* at any level of the chain (cf. [CdRM20b]). However, we point out that for diagonal systems of the form:

$$\begin{cases} dX_t^1 &= F_1(t, X_t^1, \ldots, X_t^n)dt + \sigma(t, X_{t-}^1, \ldots, X_{t-}^n)dZ_t, \\ dX_t^2 &= [A_t^2 X_t^1 + F_2(t, X_t^2)]\, dt, \\ dX_t^3 &= [A_t^3 X_t^2 + F_3(t, X_t^3)]\, dt, \\ \vdots & \\ dX_t^n &= \left[A_t^n X_t^{n-1} + F_n(t, X_t^n)\right] dt, \end{cases} \tag{2.5}$$

i.e. the degenerate components are perturbed by a function which only depends of the current level on the chain, we have that the previous thresholds are *almost* sharp. Indeed, in this case, we are led to consider $\beta^j > \frac{1+\alpha(j-2)}{1+\alpha(j-1)}$ which gives the well-posedness from the conditions in Theorem 2.6 while Theorem 2.8 shows that uniqueness fails as soon as $\beta_j^j < \frac{1+\alpha(j-2)}{1+\alpha(j-1)}$. For this diagonal system, Theorems 2.6 and 2.8 together then provide an "almost" complete understanding of the weak well-posedness for degenerate SDEs of type (2.5) with Hölder coefficients. Indeed, the problem for the critical exponents

$$\overline{\beta}_j^j = \frac{1 + \alpha(j-2)}{1 + \alpha(j-1)}, \quad j \in [\![1, n]\!],$$

remains to be investigated and, up to our best knowledge, there are no general available results even in the diffusive case. We can only mention [Zha18] in the kinetic case.

We present in this section the analytical tools we will need to show the well-posedness of the associated martingale problem. In particular, they will be fundamental in the derivation of our main Theorem 2.6, thanks to Propositions 2.3 and 2.5. For this reason, we will assume in this section to be under the same conditions of Theorem 2.6. Moreover, we will suppose that the final time horizon T is small enough for our scopes. Indeed, we could always exploit the Markov property of the involved processes and standard chaining in time arguments to extend the results to arbitrary (but finite) time intervals.

2.1 The "frozen" dynamics

The crucial element in our approach consists in choosing wisely a suitable proxy operator with well-known properties and controls, along which we can expand the infinitesimal

generator L_s, with an additional negligible error.
In order to deal with potentially unbounded perturbations F, it is natural to use a proxy involving a non-zero first order term associated with a flow associated with $G(t,x) := Ax+F(t,x)$, the transport part of SDE (1.3) (see e.g. [KP10] or [CdRMP20a]). Remembering that we assume F to be Hölder continuous, we know from the classical Peano-Lipschitz Theorem that there exists a solution of

$$\begin{cases} d\theta_{t,\tau}(\xi) \,=\, \Big[A_t\theta_{t,\tau}(\xi) + F(t,\theta_{t,\tau}(\xi))\Big]\,dt \quad \text{ on } [0,\tau]; \\ \theta_{\tau,\tau}(\xi) \,=\, \xi, \end{cases} \tag{2.1}$$

even if it may be not unique. For this reason, we are going to choose one particular flow, denoted by $\theta_{t,\tau}(\xi)$, and consider it fixed throughout the work. As it will be shown below in Lemma 2.13, it is always possible to take a measurable version of such a flow.
More precisely, given a freezing couple (τ,ξ) in $(0,T]\times\mathbb{R}^N$, the backward flow will be defined on $[0,\tau]$ as

$$\theta_{t,\tau}(\xi) \,=\, \xi - \int_t^\tau \Big[A_u\theta_{u,\tau}(\xi) + F(u,\theta_{u,\tau}(\xi))\Big]\,du.$$

Fixed the reference flow, the next step is to consider the stochastic dynamics linearized along the backward flow $\theta_{t,\tau}(\xi)$. Namely, for any fixed starting point (t,x) in $[0,\tau]\times\mathbb{R}^N$, we consider $\{\tilde{X}^{\tau,\xi,t,x}_s\}_{s\in[t,T]}$ solving the following SDE:

$$\begin{cases} d\tilde{X}^{\tau,\xi,t,x}_u \,=\, \Big[A_u\tilde{X}^{\tau,\xi,t,x}_u + \tilde{F}^{\tau,\xi}_u\Big]\,du + B\tilde{\sigma}^{\tau,\xi}_u\,dZ_u, \quad u\in[t,T], \\ \tilde{X}^{\tau,\xi,t,x}_t \,=\, x, \end{cases} \tag{2.2}$$

where $\tilde{\sigma}^{\tau,\xi}_s := \sigma(s,\theta_{s,\tau}(\xi))$ and $\tilde{F}^{\tau,\xi}_s := F(s,\theta_{s,\tau}(\xi))$.
In order to obtain an integral representation of the process $\{\tilde{X}^{\tau,\xi,t,x}_s\}_{s\in[t,T]}$, we now introduce the time-ordered resolvent $\mathcal{R}_{s,t}$ of the matrix A_s starting at time t. Namely, $\mathcal{R}_{s,t}$ is a time-dependent matrix in $\mathbb{R}^N\otimes\mathbb{R}^N$ that is solution of the following ODE:

$$\begin{cases} \partial_s\mathcal{R}_{s,t} \,=\, A_s\mathcal{R}_{s,t}, \quad s\in[0,T]; \\ \mathcal{R}_{t,t} \,=\, \mathrm{Id}_{N\times N}\,. \end{cases}$$

By the variation of constants method, it is now easy to check that the solution $\tilde{X}^{\tau,\xi,t,x}_s$ of SDE (2.2) satisfies that

$$\tilde{X}^{\tau,\xi,t,x}_s \,=\, \tilde{m}^{\tau,\xi}_{s,t}(x) + \int_t^s \mathcal{R}_{s,u}B\tilde{\sigma}^{\tau,\xi}_u dZ_u, \tag{2.3}$$

where the "frozen shift" $\tilde{m}^{\tau,\xi}_{s,t}(x)$ is given by:

$$\tilde{m}^{\tau,\xi}_{s,t}(x) \,=\, \mathcal{R}_{s,t}x + \int_t^s \mathcal{R}_{s,u}\tilde{F}^{\tau,\xi}_u\,du. \tag{2.4}$$

We point out already two important properties of the shift $\tilde{m}^{\tau,\xi}_{s,t}(x)$.

Lemma 2.9. *Let s in $[0,T]$ and x, y two points in $\mathbb{R}^N$. Then, for any $t < s$, it holds that*

$$\begin{aligned} \tilde{m}^{t,x}_{s,t}(x) &= \theta_{s,t}(x) && (2.5)\\ y - \tilde{m}^{s,y}_{s,t}(x) &= \theta_{t,s}(y) - x && (2.6) \end{aligned}$$

Proof. We start noticing that by construction in (2.4), $\tilde{m}^{\tau,\xi}_{s,t}(x)$ satisfies

$$\partial_s \tilde{m}^{\tau,\xi}_{s,t}(x) = A_s \tilde{m}^{\tau,\xi}_{s,t}(x) + F(s, \theta_{s,\tau}(\xi)), \tag{2.7}$$

for any freezing parameters (τ, ξ). Choosing $\tau = t$, $\xi = x$ above, it then holds that

$$\partial_s \left[\tilde{m}^{s,x}_{s,t}(x) - \theta_{s,t}(x)\right] = A_s \left[\tilde{m}^{t,x}_{s,t}(x) - \theta_{s,t}(x)\right].$$

Since, $\tilde{m}^{t,x}_{t,t}(x) = \theta_{t,t}(x) = x$, Equation (2.5) then follows immediately applying the Grönwall lemma.
The second identity in (2.6) follows in a similar manner. □

We are now interested in investigating the analytical properties of the "frozen" solution process $\tilde{X}^{\tau,\xi,t,x}_s$. In particular, we will show in the next results the existence of a density for such a process and its anisotropic regularizing effect, at least for small times. Further on, we will consider fixed a time-dependent matrix $\mathbb{M}_t$ on $\mathbb{R}^N \otimes \mathbb{R}^N$ given by

$$\mathbb{M}_t := \operatorname{diag}(I_{d_1\times d_1}, tI_{d_2\times d_2}, \dots, t^{n-1} I_{d_n\times d_n}), \quad t \geq 0, \tag{2.8}$$

which reflects the multi-scale nature of the underlying dynamics in (2.2).

Proposition 2.10 (Decomposition)**.** *Let the freezing couple (τ, ξ) be in $[0,T] \times \mathbb{R}^N$, $t < s$ in $[0,T]$ and x in $\mathbb{R}^N$. Then, there exists a Lévy process $\{\tilde{S}^{\tau,\xi,t,s}_u\}_{u\geq 0}$ such that*

$$\tilde{X}^{\tau,\xi,t,x}_s \overset{(law)}{=} \tilde{m}^{\tau,\xi}_{s,t}(x) + \mathbb{M}_{s-t} \tilde{S}^{\tau,\xi,t,s}_{s-t}. \tag{2.9}$$

In particular, the random variable $\tilde{X}^{\tau,\xi,t,x}_s$ admits a continuous density $\tilde{p}^{\tau,\xi}(t,s,x,\cdot)$ given by

$$\begin{aligned} \tilde{p}^{\tau,\xi}(t,s,x,y) &= \frac{1}{\det \mathbb{M}_{s-t}} p_{\tilde{S}^{\tau,\xi,t,s}}\left(t-s, \mathbb{M}^{-1}_{s-t}(y - \tilde{m}^{\tau,\xi}_{s,t}(x))\right) \qquad (2.10)\\ &:= \frac{\det \mathbb{M}^{-1}_{s-t}}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-i\langle \mathbb{M}^{-1}_{s-t}(y - \tilde{m}^{\tau,\xi}_{s,t}(x)), z\rangle} \exp\left((s-t)\int_{\mathbb{R}^N} [\cos(\langle z, p\rangle) - 1]\, \nu_{\tilde{S}^{\tau,\xi,t,s}}(dp)\right) dz, \end{aligned}$$

where $\nu_{\tilde{S}^{\tau,\xi,t,s}}$ and $p_{\tilde{S}^{\tau,\xi,t,s}}(u,\cdot)$ are the Lévy measure and the density associated with the process $\{\tilde{S}^{\tau,\xi,t,s}_u\}_{u\geq 0}$, respectively.

Proof. For simplicity, we start denoting

$$\tilde{\Lambda}^{\tau,\xi,t,s} := \int_t^s \mathcal{R}_{s,u} B \tilde{\sigma}^{\tau,\xi}_u dZ_u, \quad s \geq t,$$

so that we have from Equation (2.3) that $\tilde{X}^{\tau,\xi,t,x}_s = \tilde{m}^{\tau,\xi}_{s,t}(x) + \tilde{\Lambda}^{\tau,\xi,t,s}$. To conclude, we need to construct a Lévy process $\{\tilde{S}^{\tau,\xi,t,s}_u\}_{u\geq 0}$ on $\mathbb{R}^N$ such that

$$\tilde{\Lambda}^{\tau,\xi,t,s} \overset{\text{(law)}}{=} \mathbb{M}_{s-t}\tilde{S}^{\tau,\xi,t,s}_{s-t}. \tag{2.11}$$

To show the identity in law, we are going to reason in terms of the characteristic functions. We start recalling that the Lévy process $\{Z_t\}_{t\geq 0}$ on $\mathbb{R}^d$ is characterized by the Lévy symbol

$$\Phi(p) \,=\, \int_{\mathbb{R}^d_0} [\cos(p\cdot q)-1]\, Q(q)\,\nu_\alpha(dq), \quad p\in\mathbb{R}^d,$$

where $\nu_\alpha(dq)=\mu(d\theta)\frac{dr}{r^{1+\alpha}}$ is the Lévy measure of an α-stable process. It is well-known (see e.g. Lemma 2.2 in [SW12]) that at any fixed $t\leq s$ in $[0,1]$, $\tilde{\Lambda}^{\tau,\xi,t,s}$ is an infinitely divisible random variable with associated Lévy symbol

$$\Phi_{\tilde{\Lambda}^{\tau,\xi,t,s}}(z) \,:=\, \int_t^s \Phi\Big((\mathcal{R}_{s,u}B\tilde{\sigma}^{\tau,\xi}_u)^*z\Big)\,du, \quad z\in\mathbb{R}^N,$$

where, we recall, we have denoted $\tilde{\sigma}^{\tau,\xi}_u=\sigma(u,\theta_{u,\tau}(\xi))$.
Setting $v:=(u-t)/(s-t)$ and noticing that $u=u(v):=t+v(s-t)$, we can now rewrite the Lévy symbol of $\tilde{\Lambda}^{\tau,\xi,t,s}$ as

$$\Phi_{\tilde{\Lambda}^{\tau,\xi,t,s}}(z) \,:=\, (s-t)\int_0^1 \Phi\Big((\mathcal{R}_{s,u(v)}B\tilde{\sigma}^{\tau,\xi}_{u(v)})^*z\Big)\,dv. \tag{2.12}$$

From the analysis performed in [HM16], Lemmas 5.1 and 5.2 (see also [DM10] Proposition 3.7), we then know that we can decompose the first column of the resolvent $\mathcal{R}_{s,u(v)}$ in the following way:

$$\mathcal{R}_{s,u(v)}B \,=\, \mathbb{M}_{s-t}\widehat{\mathcal{R}}_vB,$$

where $\{\widehat{\mathcal{R}}_v\colon v\in[0,T]\}$ are non-degenerate and bounded matrixes in $\mathbb{R}^N\otimes\mathbb{R}^N$ and the multi-scale matrix $\mathbb{M}_t$ is given in (2.8). We can now rewrite the Lévy symbol of $\tilde{\Lambda}^{\tau,\xi,t,s}$ as

$$\Phi_{\tilde{\Lambda}^{\tau,\xi,t,s}}(z) \,=\, (s-t)\int_0^1 \Phi\Big((\widehat{\mathcal{R}}_vB\tilde{\sigma}^{\tau,\xi}_{u(v)})^*\mathbb{M}_{s-t}z\Big)\,dv, \quad z\in\mathbb{R}^N.$$

The above equality suggests us to define, for any fixed $t\leq s$ in $(0,1]$, the (unique in law) Lévy process $\{\tilde{S}^{\tau,\xi,t,s}_u\}_{u\geq 0}$ associated with the Lévy symbol

$$\begin{aligned}\Phi_{\tilde{S}^{\tau,\xi,t,s}}(z) \,&:=\, \int_0^1 \Phi\Big((\widehat{\mathcal{R}}_vB\tilde{\sigma}^{\tau,\xi}_{u(v)})^*z\Big)\,dv\\ &=\int_0^1\int_{\mathbb{R}^d}\Big[\cos\Big(\langle z,\widehat{\mathcal{R}}_vB\tilde{\sigma}^{\tau,\xi}_{u(v)}p\rangle\Big)-1\Big]\,\nu(dp)dv.\end{aligned} \tag{2.13}$$

Since we have that

$$\mathbb{E}\Big[e^{i\langle z,\tilde{\Lambda}^{\tau,\xi,t,s}\rangle}\Big] \,=\, e^{\Phi_{\tilde{\Lambda}^{\tau,\xi,t,s}}(z)} \,=\, e^{(s-t)\Phi_{\tilde{S}^{\tau,\xi,t,s}}(\mathbb{M}_tz)} \,=\, \mathbb{E}\Big[e^{i\langle z,\mathbb{M}_t\tilde{S}^{\tau,\xi,t,s}_{s-t}\rangle}\Big], \tag{2.14}$$

it follows immediately that Equation (2.11) holds.
To show the existence of a density for $\tilde{X}^{\tau,\xi,t,x}_s$, we want to exploit the Fourier inversion

formula in (2.14). To do it, we firstly need to prove that $\exp(\Phi_{\tilde{\Lambda}^{\tau,\xi,t,s}}(z))$ is integrable. From (2.13), we notice that

$$\begin{aligned}\Phi_{\tilde{S}^{\tau,\xi,t,s}}(z) &= \int_0^1\int_{\mathbb{R}^d}\left[\cos\left(\langle z, \widehat{\mathfrak{R}}_v B\tilde{\sigma}^{\tau,\xi}_{u(v)}p\rangle\right)-1\right]\nu(dp)dv \\ &= \int_0^1\int_{\mathbb{R}^d}\left[\cos\left(\langle z, \widehat{\mathfrak{R}}_v B\tilde{\sigma}^{\tau,\xi}_{u(v)}p\rangle\right)-1\right]Q(p)\,\nu_\alpha(dp)dv,\end{aligned}$$

where in the last step we used hypothesis [**ND**]. Exploiting now that the quantities above are non-positive and $Q(p) \geq c > 0$ for p in $B(0, r_0)$, we write that

$$\begin{aligned}\Phi_{\tilde{S}^{\tau,\xi,t,s}}(z) &\leq C\int_0^1\int_{B(0,r_0)}\left[\cos\left(\langle z, \widehat{\mathfrak{R}}_v B\tilde{\sigma}^{\tau,\xi}_{u(v)}p\rangle\right)-1\right]\nu_\alpha(dp)dv \\ &= C\Big\{-\int_0^1\left|(\widehat{\mathfrak{R}}_v B\tilde{\sigma}^{\tau,\xi}_{u(v)})^*z\right|^\alpha dv+\int_0^1\int_{B^c(0,r_0)}\left[1-\cos\left(\langle z, \widehat{\mathfrak{R}}_v B\tilde{\sigma}^{\tau,\xi}_{u(v)}p\rangle\right)\right]\nu_\alpha(dp)dv\Big\} \\ &\leq C\Big\{-\int_0^1\left|(\widehat{\mathfrak{R}}_v B\tilde{\sigma}^{\tau,\xi}_{u(v)})^*z\right|^\alpha dv+1\Big\}.\end{aligned}$$

To conclude, we recall that Lemma 5.4 in [HM16] states that

$$\int_0^1\left|(\widehat{\mathfrak{R}}_v B\tilde{\sigma}^{\tau,\xi}_{u(v)})^*z\right|^\alpha dv \;\geq\; C|z|^\alpha,$$

for some positive constant C independent from t, s, τ, ξ. It then follows in particular that

$$\Phi_{\tilde{S}^{\tau,\xi,t,s}}(z) \;\leq\; C\left[1-|z|^\alpha\right], \quad z\in\mathbb{R}^N. \tag{2.15}$$

Since $\exp(\Phi_{\tilde{\Lambda}^{\tau,\xi,t,s}}(z))$ is integrable, it implies that there exists a density $p_{\tilde{\Lambda}^{\tau,\xi,t,s}}(t,s,\cdot)$ of the random variable $\tilde{\Lambda}^{\tau,\xi,t,s}$. We can now apply the Fourier inversion formula in Equation (2.14) showing that $p_{\tilde{\Lambda}^{\tau,\xi,t,s}}(t,s,\cdot)$ is given by

$$p^{\tau,\xi}_{\tilde{\Lambda}}(t,s,y) \;:=\; \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}e^{-i\langle y,z\rangle}\exp\left((s-t)\Phi_{\tilde{S}^{\tau,\xi,t,s}}(z)\right)dz. \tag{2.16}$$

From Decomposition (2.9) and Equation (2.16), the representation for $\tilde{p}^{\tau,\xi}(t,s,x,\cdot)$ follows immediately. □

Once we have proven the existence of a density $\tilde{p}^{\tau,\xi}(t,s,x,\cdot)$ for the "frozen" stochastic dynamics $\tilde{X}^{\tau,\xi,t,x}_s$, we move now on determining its associated smoothing effects. In particular, we show in the following proposition that the derivatives of the "frozen" density are controlled by another density at the price of an additional time singularity of the order corresponding to the intrinsic time scale of the considered component in the stable regime. Importantly, such a control holds uniformly in the freezing parameters (τ,ξ).
Let us introduce for simplicity the following time-dependent scale matrix:

$$\mathbb{T}_t \;:=\; t^{\frac{1}{\alpha}}\mathbb{M}_t, \quad t\geq 0. \tag{2.17}$$

Proposition 2.11. *There exists a family $\{\overline{p}(u,\cdot)\colon u\geq 0\}$ of densities on $\mathbb{R}^N$ and a positive constant $C := C(N,\alpha)$ such that*

— *for any* $u \ge 0$ *and any* z *in* $\mathbb{R}^N$, $\overline{p}(u,z) = u^{-N/\alpha}\overline{p}(1,u^{-1/\alpha}z)$*; (stable scaling property)*

— *for any* γ *in* $[0,\alpha)$,

$$\int_{\mathbb{R}^N} \overline{p}(u,z)\,|z|^\gamma\,dz \;\le\; Cu^{\gamma/\alpha}, \quad u>0; \tag{2.18}$$

— *for any* k *in* $[\![0,2]\!]$*, any* i *in* $[\![1,n]\!]$*, any* $t<s$ *in* $[0,T]$ *and any* x,y *in* $\mathbb{R}^N$,

$$|D^k_{x_i}\tilde{p}^{\tau,\xi}(t,s,x,y)| \le C\frac{(s-t)^{-k\frac{1+\alpha(i-1)}{\alpha}}}{\det \mathbb{T}_{s-t}}\overline{p}\left(1,\mathbb{T}^{-1}_{s-t}(y-\tilde{m}^{\tau,\xi}_{s,t}(x))\right). \tag{2.19}$$

where we denoted, coherently with the notations introduced before Theorem 2.6, $D_{x_i} = \left(D_{x_i^1},\dots,D_{x_i^{d_i}}\right)$.

Remark 2.2 (About the freezing parameters)**.** We carefully point out that since we will later on choose as parameters $(\tau,\xi)=(s,y)$, it is particularly important that we manage to obtain an upper bound by a density which is independent from those parameters, since they will be as well the integration variables (see Section 2.2 below). This is precisely why we actually impose the specific semi-linear drift structure in SDE (1.3) (cf. assumption **[H]**), as opposed to the more general one that can be handled in the Gaussian case [CdRM20b]. This is a framework which naturally gives the independence of the large jumps of the *proxy* process $\tilde{X}^{\tau,\xi,t,x}_s$ as used in (2.24) below. The more general case for the first order dynamics considered in [CdRM20b] would actually lead to linearize around a matrix which would depend on the freezing parameters. For such models, we did not succeed in proving that the corresponding densities can be bounded independently of the parameters (see also the proof of Lemma 5.2 below for a similar issue regarding the diffusion coefficient).

Proof. Fixed the freezing parameters (τ,ξ) in $[0,T]\times\mathbb{R}^N$, and the times $t<s$ in $[0,T]$, we start applying the Itô-Lévy decomposition to the process $\{\tilde{S}^{\tau,\xi,t,s}_u\}_{u\ge0}$ introduced in Proposition 2.10 at the associated characteristic stable time, i.e. we choose to truncate at threshold $u^{1/\alpha}$. Thus, we can write

$$\tilde{S}^{\tau,\xi,t,s}_u \;=\; \tilde{M}^{\tau,\xi,t,s}_u + \tilde{N}^{\tau,\xi,t,s}_u \tag{2.20}$$

for some $\tilde{M}^{\tau,\xi,t,s}_u, \tilde{N}^{\tau,\xi,t,s}_u$ independent random variables corresponding to the small jumps part and the large jumps part, respectively. Namely, we denote for any $v>0$,

$$\tilde{N}^{\tau,\xi,t,s}_v \;:=\; \int_0^v\int_{|z|>u^{1/\alpha}} zP_{\tilde{S}^{\tau,\xi,t,s}}(dr,dz) \;\text{ and }\; \tilde{M}^{\tau,\xi,t,s}_v :=\; \tilde{S}^{\tau,\xi,t,s}_v - \tilde{N}^{\tau,\xi,t,s}_v, \tag{2.21}$$

where $P_{\tilde{S}^{\tau,\xi,t,s}}(dr,dz)$ is the Poisson random measure associated with the process $\tilde{S}^{\tau,\xi,t,s}$. It can be shown, similarly to Proposition 2.10, that the process $\{\tilde{M}^{\tau,\xi,t,s}_u\}_{u\ge0}$ admits a density $p_{\tilde{M}^{\tau,\xi,t,s}}(u,\cdot)$. Indeed, it is well-known that the small jump part leads to a density which is in the Schwartz class $\mathcal{S}(\mathbb{R}^N)$ (see Lemma 5.1 below). We can then rewrite the density $p_{\tilde{S}^{\tau,\xi,t,s}}$ of $\tilde{S}^{\tau,\xi,t,s}$ in the following way:

$$p_{\tilde{S}^{\tau,\xi,t,s}}(u,z) \;=\; \int_{\mathbb{R}^N} p_{\tilde{M}^{\tau,\xi,t,s}}(u,z-y)P_{\tilde{N}^{\tau,\xi,t,s}_u}(dy) \tag{2.22}$$

where $P_{\tilde{N}_u^{\tau,\xi,t,s}}$ is the law of $\tilde{N}_u^{\tau,\xi,t,s}$.
We need now to control the modulus of the density $p_{\tilde{S}^{\tau,\xi,t,s}}$ with another density, independently from the parameters τ, ξ. From Lemma 5.1 in the Appendix (see also Lemma B.2 in [HM16]) with $m = N+1$, we know that there exists a positive constant C, independent from τ, ξ such that

$$\left|D_z^k p_{\tilde{M}^{\tau,\xi,t,s}}(u,z)\right| \le Cu^{-(N+k)/\alpha}\left(\frac{u^{1/\alpha}}{u^{1/\alpha}+|z|}\right)^{N+2} =: Cu^{-k/\alpha}p_{\overline{M}}(u,z), \tag{2.23}$$

for any k in $[\![0,2]\!]$, any $u>0$, and any z in $\mathbb{R}^N$.
Moreover, denoting by $\overline{M}_u$ the random variable with density $p_{\overline{M}}(u,\cdot)$ that is independent from $\tilde{N}_u^{\tau,\xi,t,s}$, we can easily check that $p_{\overline{M}}(u,z) = u^{-N/\alpha}p_{\overline{M}}(1,u^{-1/\alpha}z)$ and thus, that $\overline{M}$ is α-selfsimilar:

$$\overline{M}_u \overset{\text{law}}{=} u^{1/\alpha}\overline{M}_1.$$

On the other hand, Lemma 5.2 in the Appendix (see also Lemma A.2 in [FKM21]) ensures the existence of a family $\{\overline{P}_u\}_{u\ge 0}$ of probability measures such that

$$P_{\tilde{N}_u^{\tau,\xi,t,s}}(\mathcal{A}) \le C\overline{P}_u(\mathcal{A}), \quad \mathcal{A}\in\mathcal{B}(\mathbb{R}^N), \tag{2.24}$$

for some positive constant C independent from the parameters τ,ξ,t,s.
For any fixed $u\ge 0$, let us now denote by $\overline{N}_u$ the random variable with law $\overline{P}_u$ that is independent from $\overline{M}_u$. Thanks to the representation of the measure $\overline{P}_u$ in (5.7), it is then immediate to check that

$$\overline{N}_u \overset{\text{(law)}}{=} u^{1/\alpha}\overline{N}_1.$$

We can finally define the family $\{\overline{p}(u,\cdot)\}_{u\ge 0}$ of densities as

$$\overline{p}(u,z) := \int_{\mathbb{R}^N} p_{\overline{M}}(u,z-w)\overline{P}_u(dw), \tag{2.25}$$

which corresponds to the density of the following random variable:

$$\overline{S}_u := \overline{M}_u + \overline{N}_u$$

for any fixed $u\ge 0$. Using Fourier transform and the already proven α-selfsimilarity of $\overline{M}$ and $\overline{N}$, we now show that

$$\overline{S}_u \overset{\text{(law)}}{=} u^{1/\alpha}\overline{S}_1,$$

or equivalently, that

$$\overline{p}(u,z) = u^{-N/\alpha}\overline{p}(1,u^{-1/\alpha}z)$$

for any $u\ge 0$ and any z in $\mathbb{R}^N$. Moreover,

$$\mathbb{E}[|\overline{S}_u|^\gamma] = \mathbb{E}[|\overline{M}_u+\overline{N}_u|^\gamma] = Cu^{\gamma/\alpha}\Big(\mathbb{E}[|\overline{M}_1|^\gamma]+\mathbb{E}[|\overline{N}_1|^\gamma]\Big) \le Cu^{\gamma/\alpha},$$

for any $\gamma<\alpha$. In particular, Equation (2.18) holds. We emphasize that the integrability constraints precisely come from the Poisson measure $\overline{P}_u$ which behaves as the one associated with the large jumps of an α-stable density.

Equation (2.19) now follows easily from the previous arguments. From Equation (2.22), we start noticing that Controls (2.23), (2.24) and (2.25) imply that for any k in $[\![0,2]\!]$,

$$\left|D^k_z p_{\tilde{S}^{\tau,\xi,t,s}}(u,z)\right| \le Ct^{-k/\alpha}\overline{p}(u,z), \quad u\ge 0,\, z\in\mathbb{R}^N,$$

for some constant $C>0$, independent from the parameters τ,ξ,t,s. Recalling the decomposition in (2.9), Equation (2.19) for $k=0$ already follows.
To show instead the case $k=1$, we can write that

$$\begin{aligned}\left|D_{x_i}\tilde{p}^{\tau,\xi}(t,s,x,y)\right| &= \left|\frac{1}{\det(\mathbb{M}_{s-t})}D_{x_i}\left[p_{\tilde{S}^{\tau,\xi,t,s}}\Big(s-t,\mathbb{M}^{-1}_{s-t}(y-\tilde{m}^{\tau,\xi}_{s,t}(x))\Big)\right]\right| \\ &= \left|\frac{1}{\det(\mathbb{M}_{s-t})}\langle D_z p_{\tilde{S}^{\tau,\xi,t,s}}\,(s-t,\cdot)\,(\mathbb{M}^{-1}_{s-t}(y-\tilde{m}^{\tau,\xi}_{s,t}(x))),D_{x_i}\mathbb{M}^{-1}_{s-t}\tilde{m}^{\tau,\xi}_{s,t}(x)\rangle\right| \\ &= \frac{(s-t)^{-1/\alpha}}{\det(\mathbb{T}_{s-t})}\overline{p}\left(1,\mathbb{T}^{-1}_{s-t}(y-\tilde{m}^{\tau,\xi}_{s,t}(x))\right)\left|D_{x_i}\mathbb{M}^{-1}_{s-t}\tilde{m}^{\tau,\xi}_{s,t}(x)\right|,\end{aligned}$$

where in the last step we exploited the α-scaling property of $\overline{p}$. From Equation (2.7), we now notice that the function $x\to\tilde{m}^{\tau,\xi}_{s,t}(x)$ is affine, so that

$$\left|D_{x_i}\mathbb{M}^{-1}_{s-t}\tilde{m}^{\tau,\xi}_{s,t}(x)\right| \le C(s-t)^{-(i-1)}.$$

Hence, it follows that

$$|D_{x_i}\tilde{p}^{\tau,\xi}(t,s,x,y)| \le C\frac{(s-t)^{-\frac{1+\alpha(i-1)}{\alpha}}}{\det(\mathbb{T}_{s-t})}\overline{p}\left(1,\mathbb{T}^{-1}_{s-t}(y-\tilde{m}^{\tau,\xi}_{s,t}(x))\right).$$

The other case ($k=2$) can be derived in an analogous way. □

We conclude this section with a useful control on the powers of the density $\overline{p}(u,z)$.

Corollary 2.12. *Let $q\ge 1$. Then, there exists a positive constant $C:=C(q)$ such that*

$$[\overline{p}(u,z)]^q \le u^{(1-q)\frac{N}{\alpha}}C\overline{p}(u,z), \quad (u,z)\in(0,T]\times\mathbb{R}^N. \tag{2.26}$$

Proof. We start noticing that we can assume without loss of generality that $u=1$, thanks to the scaling property of $\overline{p}(u,z)$ in Proposition 2.11. Moreover, we know that there exists a constant K such that $\overline{p}(1,z)\le 1$ for any z in $B^c(0,K)$, since $\overline{p}(1,\cdot)$ is a density. It then clearly follows that

$$[\overline{p}(1,z)]^q \le \overline{p}(1,z), \quad z\in B^c(0,K).$$

On the other hand, we recall that $\overline{p}(1,\cdot)$ is continuous. For any z in $B(0,K)$, it then holds that

$$[\overline{p}(1,z)]^q = \overline{p}(1,z)[\overline{p}(1,z)]^{q-1} \le C\overline{p}(1,z),$$

where C is the maximum of $[\overline{p}(1,\cdot)]^q$ on $B(0,K)$. □

2.2 Regularity of the density along the terminal condition

We briefly explain here how we want to prove the well-posedness of the martingale formulation associated with $\partial_s + L_s$ at some starting point (t, x). We will mainly focus on the problem of uniqueness since the existence of a solution can be easily handled from already known results. Indeed, we recall that under the assumptions we consider, the main part of the operator L_s is of order $\alpha > 1$ while the perturbation is sub-linear. Thus, the existence of a solution can be obtained, for example, from Theorem 2.2 in [Str75].
In particular, uniqueness for the martingale problem will follow once the Krylov-like estimates (2.4) have been shown.
Starting from a solution $\{X^{t,x}_s\}_{s\in[0,T]}$ of the martingale problem with starting point (t, x), the idea is to exploit the properties of the frozen dynamics $\{\tilde{X}^{\tau,\xi,t,x}_s\}_{s\in[0,T]}$ in (2.3). For this reason, let us denote by $\tilde{L}^{\tau,\xi}_s$ its infinitesimal generator and define for f in $C^{1,2}_c([0, T) \times \mathbb{R}^N)$ the associated Green kernel:

$$\tilde{G}^{\tau,\xi}(t, x) = \int_t^T ds \int_{\mathbb{R}^N} \tilde{p}^{\tau,\xi}(t, s, x, y)f(s, y)dy.$$

Standard results now give that

$$\left(\partial_t + \tilde{L}^{\tau,\xi}_t\right) \tilde{G}^{\tau,\xi} f(t, x) \;=\; -f(t, x), \quad (t, x) \in [0, T) \times \mathbb{R}^N, \tag{2.27}$$

for any (fixed) freezing parameters (τ, ξ).
The first step of our method then consists in applying the Itô formula on the function $\tilde{G}^{\tau,\xi} f$, which is indeed smooth enough, and the solution process $\{X^{t,x}_s\}_{s\in[0,T]}$:

$$\tilde{G}^{\tau,\xi} f(t, x) + \mathbb{E}\left[\int_t^T (\partial_s + L_s)\tilde{G}^{\tau,\xi} f(s, X^{t,x}_s)\, ds\right] \;=\; 0.$$

Exploiting (2.27), we can then write

$$\tilde{G}^{\tau,\xi} f(t, x) - \mathbb{E}\Big[\int_t^T f(s, X^{t,x}_s)\, ds\Big] + \mathbb{E}\Big[\int_t^T \Big(L_s - \tilde{L}^{\tau,\xi}_s\Big)\tilde{G}^{\tau,\xi} f(s, X^{t,x}_s)\, ds\Big] \;=\; 0$$

or, equivalently,

$$\mathbb{E}\Big[\int_t^T f(s, X^{t,x}_s)\, ds\Big] \;=\; \tilde{G}^{\tau,\xi} f(t, x) + \mathbb{E}\Big[\int_t^T \Big(L_s - \tilde{L}^{\tau,\xi}_s\Big)\tilde{G}^{\tau,\xi} f(s, X^{t,x}_s)\, ds\Big].$$

While an estimate of the frozen Green kernel $\tilde{G}^{\tau,\xi} f$ can be obtained from Proposition 2.11, the main difficulty of our approach will be to control, uniformly in (t, x), the following quantity:

$$\int_t^T \int_{\mathbb{R}^N} \Big(L_s - \tilde{L}^{\tau,\xi}_s\Big)\tilde{G}^{\tau,\xi} f(s, x)\, ds.$$

Focusing for example only on the component associated with the deterministic drift F, i.e.

$$\int_t^T \int_{\mathbb{R}^N} \langle F(t, x) - F(t, \theta_{t,\tau}(\xi)), D_x\tilde{p}^{\tau,\xi}(t, s, x, y)\rangle f(s, y)\, dyds,$$

it is clear that we need some kind of compatibility between the arguments of the drift F and those of the frozen density $\tilde{p}^{\tau,\xi}(t,s,x,\cdot)$, in order to exploit the associated smoothing effect (Proposition 2.11). Namely, we need to compare the quantities $(x-\theta_{t,\tau}(\xi))$ and $(y-\tilde{m}^{\tau,\xi}_{s,t}(x))$.
Noticing that for $\tau=s$ and $\xi=y$, $(y-\tilde{m}^{\tau,\xi}_{s,t}(x))=\theta_{t,s}(y)-x$, it follows from Proposition 2.11 that this choice of freezing parameters gives the natural compatibility between the difference of the generators and the upper-bounds of the derivatives of the corresponding proxy.

The above reasoning requires however a more thorough analysis on the properties of the "density" $\tilde{p}^{s,y}(t,s,x,\cdot)$ frozen along the terminal condition (τ,ξ). Indeed, the freezing parameter y appears also as the integration variable. In other words, with this approach, the freezing parameter cannot be fixed once for all. The present section is precisely dedicated to the handling of such a choice. This will lead us to introduce a *pseudo* Green kernel, see (2.41) below, from which will then derive uniqueness to the martingale problem following the Stroock and Varadhan approach (see Chapter 7 in [SV79]), through appropriate inversion in $L^q_t-L^p_x$ spaces, proving that the remainder has a small corresponding norm.

We start with a lemma showing the existence of at least one version of the flow $\theta_{t,s}(y)$ which is measurable in s and y. This result will be fundamental to make licit any integration of this flow along the terminal condition y.

Lemma 2.13. *There exists a measurable mapping $\theta\colon[0,T]^2\times\mathbb{R}^N\to\mathbb{R}^N$ such that*

$$\theta(t,s,z)\;:=\;\theta_{t,s}(z)\;=z+\int_t^s\Big[A_u\theta_{u,s}(z)+F(u,\theta_{u,s}(z))\Big]\,du. \tag{2.28}$$

Proof. The result can be obtained from [Zub12] and a standard compactness argument. □

From this point further, we assume without loss of generality to have chosen such a measurable version $\theta_{t,s}(x)$ of the reference flow.

The next Lemma 2.14 (Approximate Lipschitz condition of the flows) will be a key technical tool for our method. Roughly speaking, it says that a kind of equivalence between the rescaled forward and backward flows appears even in our framework (where the drift F is not regular enough), up to an additional constant contribution, for any two measurable flows satisfying Equation (2.1). We only remark that similar results has been thoroughly exploited in [DM10, Men11, Men18] when considering Lipschitz drifts or [CdRM20b] in the degenerate diffusive setting with Hölder coefficients.
This *approximated* Lipschitz property will be fundamental later on in the proof of Lemma 2.18 (Dirac Convergence of frozen density) below. It will be proved in Appendix A.1, adapting the lines of [CdRM20b].

Lemma 2.14. *Let $\theta\colon[0,T]^2\times\mathbb{R}^N\to\mathbb{R}^N$, $\check{\theta}\colon[0,T]^2\times\mathbb{R}^N\to\mathbb{R}^N$ be two measurable flows satisfying Equation* (2.28)*. Then, there exist two positive constants $(C,C'):=(C,C')(T)$ such that for any $t<s$ in $[0,T]$ and any x,y in $\mathbb{R}^N$,*

$$C^{-1}|\mathbb{T}^{-1}_{s-t}(\check{\theta}_{s,t}(x)-y)|-C'\;\le\;|\mathbb{T}^{-1}_{s-t}(x-\theta_{t,s}(y))|\;\le\;C\left[|\mathbb{T}^{-1}_{s-t}(\check{\theta}_{s,t}(x)-y)|+1\right]. \tag{2.29}$$

From the above lemma, we also derive the following important estimate for the rescaled difference between the forward flow $\theta_{s,t}(x)$ and the linearized forward dynamics $\tilde{m}^{s,y}_{s,t}(x)$ (defined in (2.4)) where the linearization is considered along any backward flow.

Corollary 2.15. *Let $\theta\colon [0,T]^2\times\mathbb{R}^N\to\mathbb{R}^N$ be a measurable flow satisfying Equation (2.28). Then, there exist a positive constant $C:=C(T)$ and ζ in $(0,1)$ such that for any $t<s$ in $[0,T]$ and any x,y in $\mathbb{R}^N$,*

$$|\mathbb{T}^{-1}_{s-t}(\theta_{s,t}(x)-\tilde{m}^{s,y}_{s,t}(x))| \,\le\, C(s-t)^{\frac{1}{\alpha}\wedge\zeta}\left(1+|\mathbb{T}^{-1}_{s-t}(\theta_{s,t}(x)-y)|\right). \tag{2.30}$$

Proof. We start exploiting the differential dynamics given in Equation (2.7) to write that

$$\begin{aligned}\mathbb{T}^{-1}_{s-t}\left(\theta_{s,t}(x)-\tilde{m}^{s,y}_{s,t}(x)\right) \,&=\, \mathbb{T}^{-1}_{s-t}\int_t^s\Big\{\Big[F(u,\theta_{u,t}(x))-F(u,\theta_{u,s}(y))\Big]\\&\qquad\qquad+\Big[A_u(\theta_{u,t}(x)-\tilde{m}^{s,y}_{u,t}(x))\Big]\Big\}\,du\\&:=\,(\mathcal{I}^1_{s,t}+\mathcal{I}^2_{s,t})(x,y).\end{aligned} \tag{2.31}$$

We start dealing with $\mathcal{I}^1_{s,t}(x,y)$. The key idea is to use the sub-linearity of F and the appropriate Hölder exponents. Namely, using the Young inequality, we derive that

$$\begin{aligned}|\mathcal{I}^1_{s,t}(x,y)| \,&\le\, C\sum_{i=1}^n(s-t)^{-\frac{1+\alpha(i-1)}{\alpha}}\sum_{j=i}^n\int_t^s|(\theta_{u,t}(x)-\theta_{u,s}(y))_j|^{\beta^j}\,du\\ &\le C\Big\{(s-t)^{-\frac{1}{\alpha}}\int_t^s\left[|\theta_{u,t}(x)-\theta_{u,s}(y)|+1\right]du\\ &\quad+\sum_{i=2}^n(s-t)^{-\frac{1+\alpha(i-1)}{\alpha}}\sum_{j=i}^n\int_t^s\Big[(s-t)^{-\gamma^j}|((\theta_{u,t}(x)-\theta_{u,s}(y))_j)|+(s-t)^{\gamma^j\frac{\beta^j}{1-\beta^j}}\Big]\,du\Big\},\end{aligned}$$

for some parameters $\gamma^j>0$ to be specified below. Denoting now for simplicity,

$$\Gamma_j \,:=\, -\frac{1+\alpha(i-1)}{\alpha}+\gamma^j\frac{\beta^j}{1-\beta^j},$$

we get that

$$\begin{aligned}|\mathcal{I}^1_{s,t}(x,y)| \,&\le\, C\Big\{(s-t)^{\frac{\alpha-1}{\alpha}}+\int_t^s|\mathbb{T}^{-1}_{s-t}(\theta_{u,t}(x)-\theta_{u,s}(y))|\,du\\&\quad+\sum_{i=2}^n\sum_{j=i}^n\int_t^s\Big[(s-t)^{-i+j-\gamma^j}\Big(\frac{|((\theta_{u,t}(x)-\theta_{u,s}(y))_j)|}{(s-t)^{\frac{1+\alpha(j-1)}{\alpha}}}\Big)+(s-t)^{-\Gamma_j}\Big]\,du\Big\}\\ &\le\, C\Big\{(s-t)^{\frac{\alpha-1}{\alpha}}+\int_t^s|\mathbb{T}_(s-t)^{-1}(\theta_{u,t}(x)-\theta_{u,s}(y))|\,du\\&\quad+\sum_{i=2}^n\sum_{j=i}^n\int_t^s\Big[(s-t)^{-i+j-\gamma^j}|\mathbb{T}^{-1}_{s-t}(\theta_{u,t}(x)-\theta_{u,s}(y))|+(s-t)^{\Gamma_j}\Big]\,du\Big\}.\end{aligned}$$

We now use Lemma 2.14 (Approximate Lipschitz condition of the flows) to derive that

$$|\mathbb{T}^{-1}_{s-t}(\theta_{u,t}(x)-\theta_{u,s}(y))| \,\le\, C(|\mathbb{T}^{-1}_{s-t}(\theta_{s,t}(x)-y)|+1).$$

We emphasize here that in our current framework we should *a priori* write $\theta_{s,u}(\theta_{u,t}(x))$ in the above equation since we do not have the flow property. Anyhow, since Lemma 2.14 (Approximate Lipschitz condition of the flows) is valid for any flow starting from $\theta_{u,t}(x)$ at time u associated with the ODE (see Equation (2.29)) we can proceed along the previous one, i.e. $(\theta_{v,t}(x))_{v\in[u,s]}$. The previous reasoning yields that

$$\begin{aligned} |\mathcal{I}^1_{s,t}(x,y)| \,\le\, C\Bigg\{ &(s-t)^{\frac{\alpha-1}{\alpha}} + (s-t)\Big[|\mathbb{T}^{-1}_{s-t}(\theta_{s,t}(x)-y)|+1\Big] \\ &\times\left[1+\sum_{i=2}^n\sum_{j=i}^n\Big((s-t)^{-i+j-\gamma^j}+(s-t)^{-\frac{1+\alpha(i-1)}{\alpha}+\gamma^j\frac{\beta^j}{1-\beta^j}}\Big)\right]\Bigg\}. \end{aligned} \tag{2.32}$$

We now choose for j in $[\![i,n]\!]$,

$$-i+j-\gamma^j \,=\, -\frac{1+\alpha(i-1)}{\alpha}+\gamma^j\frac{\beta^j}{1-\beta^j} \Leftrightarrow \gamma^j \,=\, \Big(j-\frac{\alpha-1}{\alpha}\Big)\Big(1-\beta^j\Big),$$

to balance the two previous contributions associated with the indexes i,j.
To obtain a global smoothing effect with respect to $s-t$ in (2.32) we need to impose:

$$-\,i+j-\gamma^j \,>\, -1 \Leftrightarrow \beta^j \,>\, \frac{1+\alpha(i-2)}{1+\alpha(j-1)}, \quad \forall i\le j. \tag{2.33}$$

Hence, under our assumptions, we have that there exists ζ in $(0,1)$ depending on β^j for any $j\in[\![i,n]\!]$ such that

$$|\mathcal{I}^1_{s,t}(x,y)| \,\le\, C(s-t)^{\zeta}\Big[1+|\mathbb{T}^{-1}_{s-t}(\theta_{s,t}(x)-y)|\Big]. \tag{2.34}$$

Recalling from the structure of A that

$$|\mathbb{T}^{-1}_{s-t}A_u\mathbb{T}_{s-t}| \le\, C(s-t)^{-1},$$

Control (2.30) now follows from (2.31), (2.34) and the Grönwall lemma. □

Thanks to the Approximate Lipschitz property of the flow presented in Lemma 2.14 above and Corollary 2.15, we can now adapt the controls on the derivatives of the frozen density (Proposition 2.11) to the "density" $\tilde{p}^{s,y}(t,s,x,y)$. Indeed, we recall again that the function $\tilde{p}^{s,y}(t,s,x,y)$ is not a proper density in y since the integration variable y stands also as freezing parameter. This is one of the main difficulties of the approach.

The following result is the key to our analysis since it precisely quantifies the smoothing effect in time of the proxy we chose.

Corollary 2.16. *There exists a positive constant $C:=C(N,\alpha)$ such that for any γ in $[0,\alpha)$, any $t<s$ in $[0,T]$ and any x,y in $\mathbb{R}^N$,*

$$\int_{\mathbb{R}^N}\frac{|\mathbb{T}^{-1}_{s-t}(\theta_{t,s}(y)-x)|^{\gamma}}{\det\mathbb{T}_{s-t}}\overline{p}(1,\mathbb{T}^{-1}_{s-t}(\theta_{t,s}(y)-x))\,dy \,\le\, C. \tag{2.35}$$

Moreover, if $K > 0$ is large enough, it holds that

$$\int_{\mathbb{R}^N} \mathbb{1}_{|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|\geq K} \frac{1}{\det \mathbb{T}_{s-t}} \overline{p}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x))\, dy \\ \leq C \int_{\mathbb{R}^N} \mathbb{1}_{|z|\geq \frac{K}{2}} \check{p}(1, z) dz, \quad (2.36)$$

where $\check{p}$ enjoys the same integrability properties as $\overline{p}$ (stated in Proposition 2.11).

The strengthened assumptions concerning the integrability thresholds in Theorem 2.6 with respect to the natural ones appearing in (2.33) might seem awkward at first sight. It is actually the specific current framework, which involves as a proxy a stochastic integral with respect to a stable-like jump process and its associated iterated integrals that leads to additional constraints on the regularity indexes needed for our method to work.

The natural approach to get rid of the flow involving the integration variable in (2.35) would have been to use the approximate Lipschitz property of the flow established in Lemma 2.14. This indeed readily yields that:

$$|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x)|^\gamma \leq C(1 + |\mathbb{T}_{s-t}^{-1}(y - \theta_{s,t}(x))|^\gamma).$$

The main difficulty is that we do not actually succeed in establishing in whole generality that:

$$\overline{p}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x)) \leq C\check{p}(1, \mathbb{T}_{s-t}^{-1}(y - \theta_{s,t}(x)), \quad (2.37)$$

for a density $\check{p}$ which shares the same integrability properties as $\bar{p}$.

Equation (2.37) is absolutely direct in the diffusive setting from the explicit form of the Gaussian density and it has been thoroughly used in [CdRM20b] to derive sharp thresholds for weak uniqueness. It is clear that the above control has to be considered point-wise and one of the huge difficulties with stable type processes consists in describing precisely their tail behavior which is actually very much related to the geometry of their corresponding spectral measure on the sphere. We refer to the seminal work of Watanabe [Wat07] for a precise description of the tails in terms of the dimension of the support of the spectral measure, in the stable case, and to the extension by Sztonyk [Szt10a] for the tempered stable case. The delicate point comes of course from the behavior of the Poisson measure (large jumps) as illustrated in the following computation. From (2.23) and (2.25), we write that

$$\begin{aligned}\overline{p}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x)) &= \int_{\mathbb{R}^N} p_{\bar{M}}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x) - w)\, \overline{P}_1(dw) \\ &\leq C \int_{\mathbb{R}^N} \frac{1}{(C + |\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x) - w|)^M}\, \overline{P}_1(dw).\end{aligned}$$

- Let us first emphasize that, when $|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x)| \leq K$ (*diagonal type regime*) for some fixed K, then Control (2.37) holds. Indeed, since from Corollary 2.15,

$$|\mathbb{T}_{s-t}^{-1}(\tilde{m}_{s,t}^{t,y}(x) - \theta_{s,t}(x))| \leq \tilde{C}(s-t)^{\frac{1}{\alpha}\wedge\zeta} \left(1 + |\mathbb{T}_{s-t}^{-1}(\theta_{s,t}(x) - y)|\right),$$

we would get, recalling from Lemma 2.9, Equation (2.6) that $\theta_{t,s}(y) - x = y - \tilde{m}_{s,t}^{s,y}(x)$, that

$$\begin{aligned}
\overline{p}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x)) &\leq C \int_{\mathbb{R}^N} \frac{1}{(C + |\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x) - w|)^M} \overline{P}_1(dw) \\
&\leq C \int_{\mathbb{R}^N} \frac{1}{([C + |\mathbb{T}_{s-t}^{-1}(y - \theta_{s,t}(x)) - w| - (s-t)^{\frac{1}{\alpha}\wedge\zeta} |\mathbb{T}_{s-t}^{-1}(\theta_{s,t}(x) - y)|] \vee 1)^M} \overline{P}_1(dw) \\
&\leq C \int_{\mathbb{R}^N} \frac{1}{([\check{C} + |\mathbb{T}_{s-t}^{-1}(y - \theta_{s,t}(x)) - w|)^M} \overline{P}_1(dw) \\
&=: \check{p}(1, \mathbb{T}_{s-t}^{-1}(y - \theta_{s,t}(x))),
\end{aligned}$$

and $\check{p}$ plainly satisfies the required integrability conditions. These computations actually emphasize that (2.37) holds, up to a modification of $\check{C}$ above, up to the threshold

$$|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x)| \leq c_0 (s-t)^{-(\frac{1}{\alpha}\wedge\zeta)},$$

for some $c_0 > 0$ small enough with respect to C. It would therefore remain to investigate the complementary *very off-diagonal regime*.

- Let us now concentrate on the *off-diagonal regime* $|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x)| > K$. In that case, we write:

$$\begin{aligned}
\overline{p}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x)) &\leq C \int_{\mathbb{R}^N} \frac{1}{(C + |\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x) - w|)^M} \overline{P}_1(dw) \\
&\leq C \int_0^1 \overline{P}_1(\{w \in \mathbb{R}^N : (1 + |\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x) - w|)^{-M}| > u\}) du \\
&\leq C \int_0^1 \overline{P}_1(B(\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x), u^{-1/M}) du. \qquad (2.38)
\end{aligned}$$

It now follows from the proof of Proposition 2.10 that the support of the spectral measure on $\mathbb{S}^{N-1}$ associated with $\{\tilde{S}_u^{\tau,\xi,t,s}\}_{u\geq 0}$ has dimension d. The related concentration properties also transmit to $\bar{N}_1$ (see the proofs of Proposition 2.11 and Lemma 5.2). Thus, we get from [Wat07], [Szt10a] (see respectively Lemma 3.1 and Corollary 6 in those references) that there exists a constant $C > 0$ such that for all z $\mathbb{R}^N$ and $r > 0$:

$$\overline{P}_1(B(z, r)) \leq C r^{d+1} (1 + r^\alpha) |z|^{-(d+1+\alpha)}. \qquad (2.39)$$

In other words, the global bound is given by the worst decay deriving from the dimension of the support of the spectral measure. In the current case $|z| \geq K$, this bound is clearly of interest for *large* values of z. Hence, from (2.38) and (2.39), it holds that

$$\begin{aligned}
\overline{p}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x)) &\leq C \int_0^1 u^{-(d+1)/M} (1 + u^{-\alpha/M}) \, du |\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x)|^{-(d+1+\alpha)} \\
&\leq C (1 + |\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x)|)^{-(d+1+\alpha)} \int_0^1 [u^{-(d+1)/M} + u^{-(d+1+\alpha)/M} du],
\end{aligned}$$

Choosing $M > d + 1 + \alpha$ then gives that there exists $C \geq 1$ such that

$$\overline{p}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x)) \leq C (1 + |\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x)|)^{-(d+1+\alpha)}.$$

We thus get from Lemma 2.14, up to a modification of C, that:

$$\bar{p}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x)) \le C(1 + |\mathbb{T}_{s-t}^{-1}(y - \theta_{s,t}(x))|)^{-(d+1+\alpha)}. \tag{2.40}$$

This actually leads to strong dimension constraints for this bound to be integrable. This phenomenon already appeared e.g. in [HM16] and induced therein to consider $d = 1, n = 3$ at most to address the well posedness of the martingale problem associated with a linear drift and a multiplicative isotropic stable noise. Those thresholds and dimension constraints remain with this approach.
Actually, from the threshold appearing in (2.3), we would like to consider the left-hand side of (2.35) with $\gamma > \frac{1+\alpha}{1+2\alpha}$ corresponding to $j = 3 = n$ therein. From Control (2.40), this would require $-\frac{1+\alpha}{1+2\alpha} + (d + 1 + \alpha) > 3, d = 1 \iff \alpha^2 - \alpha - 1 > 0$, which in our framework imposes that $\alpha \in (\frac{1+\sqrt{5}}{2}, 2)$.

Another possibility would have been, in the tempered case, to keep track of the tempering function, instead of bounding $\tilde{p}^{\tau,\xi}$ by a self-similar density $\bar{p}$, in order to benefit from the tempering at infinity to compensate the bad concentration rate in (2.40). However, see [HM16] and [Szt10a], we would have obtained bounds of the form

$$\tilde{p}^{\tau,\xi}(t, s, x, y) \le C(1 + |\mathbb{T}_{s-t}^{-1}(y - \theta_{s,t}(x))|)^{-(d+1+\alpha)} Q\left(|\mathbb{M}_{s-t}^{-1}(y - \theta_{s,t}(x))|\right).$$

Such a bound will give space integrability but deteriorates as well the time-integrability. This difficulty would occur even in the truncated case, thoroughly studied in the non-degenerate case by Chen *et al.* [CKK08]. Thus, we will develop here another approach.

Namely, we would like to change variable to $\bar{y} := \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x)$ in the left-hand side of Equation (2.35). Of course, this is not bluntly possible since the coefficients at hand are not smooth enough. The point is then to introduce a flow $\theta_{t,s}^{\delta}(y)$ associated with mollified coefficients (for which the difference with respect to the initial flow will be controlled similarly to what is done to establish the approximate Lipschitz property of the flows in Lemma 2.14) and then, to control $\det(\nabla\theta_{t,s}^{\delta}(y))$ (see Lemma 5.3 below). Since we do not have here a summation with respect to the single rescaled components as in the previous Lemma 2.14 above or as in Corollary 2.15, this will conduct to reinforce our assumptions and suppose that $(F_i)_{i\in[\![2,n]\!]}$ has the same regularity with respect to the variable x_j, $j \in [\![2, n]\!]$, whatever the level of the chain. This is precisely what leads to consider the condition

— $x_j \to F_i(t, x_i, \dots, x_j, \dots, x_n)$ is β^j-Hölder continuous, uniformly in t and in x_k for $k \ne j$, with
$$\beta^j > \frac{1 + \alpha(j-2)}{1 + \alpha(j-1)}.$$

For the sake of clarity the proof of Corollary 2.16 is postponed to the Appendix.

Let us introduce now some useful tools for the study of the martingale problem for $\partial_s + L_s$. The first step is to consider a suitable Green-type kernel associated with the frozen density $\tilde{p}^{s,y}$ and establish which Cauchy-like problem it solves. Namely, we define for any function $f\colon [0, T] \times \mathbb{R}^N \to \mathbb{R}$ regular enough, the *pseudo* Green kernel $\tilde{G}_\epsilon$ given by:

$$\tilde{G}_\epsilon f(t, x) := \int_{(t+\epsilon)\wedge T}^{T} \int_{\mathbb{R}^N} \tilde{p}^{s,y}(t, s, x, y) f(s, y)\, dyds, \quad (t, x) \in [0, T) \times \mathbb{R}^N, \tag{2.41}$$

where ϵ is meant to be small.
We only remark that the above Green kernel $\tilde{G}_\epsilon$ is well-defined, since the frozen density $\tilde{p}^{s,y}(t,s,x,y)$ is measurable in (s,y) thanks to Lemma 2.13 (measurability of the flow in these parameters).

Proposition 2.17. *Let p, q in $(1,+\infty)$ such that the integrability Condition $(\mathscr{C})$ holds. Then, there exists a positive constant $C := C(T,p,q)$ such that for any f in $L^p\Big(0,T;L^q(\mathbb{R}^N)\Big)$,*

$$\|\tilde{G}_\epsilon f\|_\infty \,\le\, C\|f\|_{L^p_tL^q_x}.$$

Moreover, it holds that $\lim_{T\to 0} C(T,p,q) = 0$.

Proof. We start using the Hölder inequality in order to split the component with f and the part with the density $\tilde{p}(t,s,x,y)$:

$$\begin{aligned}|\tilde{G}_\epsilon f(t,x)| \,&\le\, C\|f\|_{L^p_tL^q_x}\left(\int_{(t+\epsilon)\wedge T}^T\left(\int_{\mathbb{R}^N}|\tilde{p}^{s,y}(t,s,x,y)|^{q'}\,dy\right)^{\frac{p'}{q'}}ds\right)^{\frac{1}{p'}}\\ &=:\, C\|f\|_{L^p_tL^q_x}|I_\epsilon(t,x)|,\end{aligned}$$

where we have denoted by p', q' the conjugate of p and q, respectively.
In order to control the remainder term $I_\epsilon(t,x)$, we now apply (2.19) from Proposition 2.11 with $k=0$ and $(\tau,\xi)=(s,y)$ to write that

$$|I_\epsilon(t,x)|^{p'} \,\le\, C\int_{(t+\epsilon)\wedge T}^T\left(\int_{\mathbb{R}^N}\left(\frac{1}{\det\mathbb{T}_{s-t}}\overline{p}\left(1,\mathbb{T}^{-1}_{s-t}(y-\tilde{m}^{s,y}_{s,t}(x))\right)\right)^{q'}dy\right)^{\frac{p'}{q'}}ds,$$

where we recall that $\mathbb{T}_t = t^{1/\alpha}\mathbb{M}_t$ (see (2.17) and (2.8)).
From Corollaries 2.12 and 2.16, we then write that

$$\begin{aligned}|I_\epsilon(t,x)|^{p'} \,&\le\, C\int_{(t+\epsilon)\wedge T}^T\left(\int_{\mathbb{R}^N}\frac{1}{(\det\mathbb{T}_{s-t})^{q'}}\overline{p}\left(1,\mathbb{T}^{-1}_{s-t}(y-\tilde{m}^{s,y}_{s,t}(x))\right)dy\right)^{\frac{p'}{q'}}ds\\ &\le\, C\int_{(t+\epsilon)\wedge T}^T(\det\mathbb{T}_{s-t})^{\frac{p'}{q'}-p'}ds \,=\, C\int_{(t+\epsilon)\wedge T}^T\frac{1}{(\det\mathbb{T}_{s-t})^{\frac{p'}{q}}}ds.\end{aligned}$$

Since by definition of matrix $\mathbb{T}_t$, it holds that

$$\det\mathbb{T}_{s-t} \,=\, (s-t)^{\sum_{i=1}^n d_i\frac{1+\alpha(i-1)}{\alpha}}, \tag{2.42}$$

we can conclude that under the integrability assumption $(\mathscr{C})$, we have that

$$\Big(\sum_{i=1}^n d_i\frac{1+\alpha(i-1)}{\alpha}\Big)\frac{p'}{q} \,<\, 1 \,\Leftrightarrow\, \Big(\sum_{i=1}^n d_i\frac{1+\alpha(i-1)}{\alpha}\Big)\frac{1}{q}+\frac{1}{p} \,<\, 1. \tag{2.43}$$

The proof is complete. □

Now, we understand which Cauchy-like problem is solved by the "density" $\tilde{p}^{s,y}(t,s,x,y)$ frozen at the terminal point (s,y). We start denoting by $\tilde{L}^{s,y}_t$ the infinitesimal generator of the proxy process $\{\tilde{X}^{s,y,t,x}_s\}_{s\in[t,T]}$. For any smooth function $\phi\colon \mathbb{R}^N \to \mathbb{R}$, it writes:

$$\begin{aligned}\tilde{L}^{s,y}_t\phi(x) \,&:=\, \langle A_tx + \tilde{F}^{s,y}_t, D_x\phi(x)\rangle + \tilde{\mathcal{L}}^{s,y}_t \\ &:= \langle A_tx + \tilde{F}^{s,y}_t, D_x\phi(x)\rangle + \int_{\mathbb{R}^d_0}\Big[\phi(x+B\tilde{\sigma}^{s,y}_t w) - \phi(x)\Big]\,\nu(dw),\end{aligned} \tag{2.44}$$

where, we recall, $\tilde{F}^{s,y}_t := F(t,\theta_{t,s}(y))$ and $\tilde{\sigma}^{s,y}_t := \sigma(t,\theta_{t,s}(y))$.
By direct calculation, it is not difficult to check now that for any (s,x,y) in $[0,T]\times\mathbb{R}^{2N}$ it holds that

$$\Big(\partial_t + \tilde{L}^{s,y}_t\Big)\,\tilde{p}^{s,y}(t,s,x,z) \,=\, 0, \quad (t,z)\in[0,s)\times\mathbb{R}^N. \tag{2.45}$$

However, we carefully point out that some attention is requested to establish the following lemma, which is crucial to derive which Cauchy-type problem the function $\tilde{G}f := \lim_{\epsilon\to 0}\tilde{G}_\epsilon f$ actually solves. In particular, it is important to highlight that Lemma 2.18 (Dirac Convergence of frozen density) below cannot be obtained directly from the convergence in law of the frozen process $\tilde{X}^{s,y,t,x}_s$ towards the Dirac mass (cf. Equation (2.45)). Indeed, the integration variable y also appears as a freezing parameter which makes the argument more complicated.
The proofs of the following two lemmas is quite involved and technical. For this reason, we decided to postpone them to the Appendix, Section 5.2.

Lemma 2.18. *Let (t,x) be in $[0,T)\times\mathbb{R}^N$ and $f\colon\mathbb{R}^N\to\mathbb{R}$ a bounded continuous function. Then,*

$$\lim_{\epsilon\to 0}\Big|\int_{\mathbb{R}^N} f(y)\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)\,dy - f(x)\Big| \,=\, 0.$$

Moreover, the above limit is uniform with respect to t in $[0,T]$.

A similar result involving the $L^p_tL^q_x$-norm can also be obtained. For notational simplicity, let us set

$$I_\epsilon f(t,x) \,:=\, \int_{\mathbb{R}^N} f(t+\epsilon,y)\mathbb{1}_{[0,T-\epsilon]}(t)\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)\,dy \tag{2.46}$$

for any sufficiently regular function $f\colon[0,T]\times\mathbb{R}^N\to\mathbb{R}$.

Lemma 2.19. *Let $p>1$, $q>1$ and f in $C^{1,2}_c([0,T)\times\mathbb{R}^N)$. Then,*

$$\lim_{\epsilon\to 0}\|I_\epsilon f - f\|_{L^p_tL^q_x} \,=\, 0.$$

We want now to understand which Cauchy-like problem is solved by our frozen Green kernel $\tilde{G}_\epsilon f(t,x)$. For this reason, we introduce for any function f in $C^{1,2}_0([0,T)\times\mathbb{R}^N,\mathbb{R})$ the following quantity:

$$\tilde{M}_\epsilon f(t,x) \,:=\, \int_{t+\epsilon}^T\int_{\mathbb{R}^N}\tilde{L}^{s,y}_t\tilde{p}^{s,y}(t,s,x,y)f(s,y)\,dyds, \quad (t,x)\in[0,T)\times\mathbb{R}^N, \tag{2.47}$$

for some fixed $\epsilon>0$ that is assumed to be small enough. Then, we can derive from Equation (2.45) and Proposition 2.11 that the following equality holds:

$$\partial_t\tilde{G}_\epsilon f(t,x) + \tilde{M}_\epsilon f(t,x) \,=\, -I_\epsilon f(t,x), \quad (t,x)\in[0,T)\times\mathbb{R}^N, \tag{2.48}$$

where we used the same notation in (2.46) for $I_\epsilon f$. We point out that the localization with respect to ϵ is precisely needed to exploit directly (2.45) and thus, to derive (2.48) for any fixed $\epsilon > 0$, by usual dominated convergence arguments. In particular, we point out that in the limit case ($\epsilon \to 0$), the smoothness on f is not a sufficient condition to derive the smoothness of $\tilde{G}f$. This is again due to the dependence of the proxy upon the integration variable.

3 Well-posedness of the martingale problem

This section is devoted to the proof of the well-posedness of the martingale problem for $\partial_s + L_s$ with initial condition (t, x), under the assumptions of Theorem 2.6.

Since by definition the paths of any solution $\{X_t\}_{t\geq 0}$ of the martingale problem for $\partial_s + L_s$ are càdlàg, it will be convenient afterwards to give an alternative definition. We denote by $\mathcal{D}[0,\infty)$ the family of all the càdlàg paths from $[0,\infty)$ to $\mathbb{R}^N$, equipped with the "standard" Skorokhod topology. For further details, we suggest the interested reader to see [Bas11], [EK86] or [JS03].
Fixed a starting point (t, x) in $[0,\infty) \times \mathbb{R}^N$, we will say that a a probability measure $\mathbb{P}$ on $\mathcal{D}[0,\infty)$ is a solution of the martingale problem for $\partial_t + L_t$ starting at (t, x) if the coordinate process $\{y_t\}_{t\geq 0}$ on $\mathcal{D}[0,\infty)$, defined by

$$y_t(\omega) \,=\, \omega(t), \quad \omega \in \mathcal{D}[0,\infty)$$

is a solution (in the previous sense) of the martingale problem for $\partial_s + L_s$.
Similarly, we will say that uniqueness holds for the martingale problem for $\partial_s + L_s$ with starting point (t, x) if

$$\mathbb{P} \circ y^{-1} \,=\, \tilde{\mathbb{P}} \circ y^{-1},$$

for any two solutions $\mathbb{P}$, $\tilde{\mathbb{P}}$ of the martingale problem for $\partial_s + L_s$ starting at (t, x).

The existence of a solution $\mathbb{P}$ of the martingale problem for $\partial_s + L_s$ can be obtained adapting the proof of Theorem 2.2 in [Str75] exploiting the sublinear structure of the drift F and localization arguments in order to deal with possibly unbounded coefficients.

Proposition 3.1 (Existence)**.** *Under the assumptions of Theorem 2.6, let (t, x) be in $[0,\infty) \times \mathbb{R}^N$. Then, there exists a solution $\mathbb{P}$ of the martingale problem for $\partial_s + L_s$ starting at (t, x).*

We move to the question of uniqueness for the martingale problem associated with $\partial_s + L_s$. As shown already in the introduction of Section 3, the analytical properties on the frozen process $(\tilde{X}^{s,y,t,x}_u)_{u\in[t,T]}$ we presented there will be the crucial tools for the reasoning in the following section.
We will start proving directly that the Krylov-type estimates (2.4) holds but first for p, q big enough (but finite). It will imply in particular the existence of a density for the canonical process associated with any solution of the martingale problem. As a consequence, the weak well-posedness of SDE (1.3) under our assumptions can be shown to hold.
Only in a second moment, we will then show that the Krylov estimates holds for *any* p,

q satisfying condition ($\mathscr{C}$) through a regularization technique. Namely, we regularize the driving noise Z_t by introducing an additional isotropic α-stable process depending from a regularizing parameter. Following the previous arguments for the regularized dynamics, we will then prove that the solution process satisfies again the Krylov-type estimates for any p, q in the considered range, *uniformly* with respect to the regularizing parameter.
Letting the regularizing parameter go to zero, we will then conclude the proof of Corollary 2.7.

3.1 Uniqueness of the martingale problem

The first step in proving the uniqueness of the Martingale problem for $\partial_s + L_s$ is to show that any solution to the martingale problem satisfies the Krylov-like estimates in Equation (2.4). To do so, we prove that the difference operator between the genuine generator L_t and a suitable associated perturbation (associated with the frozen generator $\tilde{L}^{s,y}_t$ given in (2.44)) has small $L^p_tL^q_x$-norm when considering a sufficiently small final horizon T. Namely, we introduce the following remainder:

$$\tilde{R}_\epsilon f(t,x) \,:=\, (L_t\tilde{G}_\epsilon f-\tilde{M}_\epsilon f)(t,x) \,=\, \int_{t+\epsilon}^T\int_{\mathbb{R}^N}(L_t-\tilde{L}^{s,y}_t)\tilde{p}^{s,y}(t,s,x,y)f(s,y)\,dyds, \tag{3.1}$$

for some ϵ to be small enough. We recall that $\tilde{G}_\epsilon f$, $\tilde{M}_\epsilon f$ and $\tilde{p}^{s,y}(t,s,x,y)$ were defined in (2.41), (2.47) and (2.10), respectively.
We firstly present a point-wise control for the remainder term $\tilde{R}_\epsilon f$. Importantly, the constant C below does not depend on ϵ, allowing to pass to the limit in Equation (3.1). This will be discussed at the end of the present section.

Proposition 3.2. *There exist $q_0>1$, $p_0>1$ and $C:=C(T,p_0,q_0)$ such that for any $q\ge q_0$, $p\ge p_0$ and any f in $L^p([0,T];L^q(\mathbb{R}^N))$, it holds that*

$$\|\tilde{R}_\epsilon f\|_\infty \,\le\, C\|f\|_{L^p_tL^q_x}. \tag{3.2}$$

Proof. We start recalling from (2.2)-(2.44) (exploiting also the change of truncation in (1.5))) that we can decompose $\tilde{R}_\epsilon f$ in the following way:

$$\begin{aligned}\tilde{R}_\epsilon f(t,x) \,&=\, \int_{t+\epsilon}^T\int_{\mathbb{R}^N}(\mathcal{L}_t-\tilde{\mathcal{L}}^{s,y}_t)\tilde{p}^{s,y}(t,s,x,y)f(s,y)\,dyds \\ &\qquad\qquad + \int_{t+\epsilon}^T\int_{\mathbb{R}^N}\langle F(t,x)-\tilde{F}^{s,y}_t, D_x\tilde{p}^{s,y}(t,s,x,y)\rangle f(s,y)\,dyds \\ &=: \, \tilde{R}^0_\epsilon f(t,x)+\tilde{R}^1_\epsilon f(t,x)\end{aligned} \tag{3.3}$$

where the operators $\mathcal{L}_t$ and $\mathcal{L}^{\tilde{s},y}_t$ have been defined in (2.2) and (2.44), respectively.
Since by assumptions, $x_j\to F_i(t,x)$ is β^j-Hölder continuous, we can control the second term $\tilde{R}^1_\epsilon f$, associated with the difference of the drifts, using Proposition 2.11 with

$(\tau, \xi) = (s, y)$:

$$
\begin{aligned}
\left|\langle F(t,x) - \tilde{F}_t^{s,y}, D_x\tilde{p}^{s,y}(t,s,x,y)\rangle\right| &\le \sum_{i=1}^{n} |F_i(t,x) - F_i(t,\theta_{t,s}(y))|\; |D_{x_i}\tilde{p}^{s,y}(t,s,x,y)| \\
&\le C\sum_{i=1}^{n}(s-t)^{-\frac{1+\alpha(i-1)}{\alpha}}\frac{\overline{p}(1,\mathbb{T}_{s-t}^{-1}(y-\tilde{m}_{s,t}^{s,y}(x)))}{\det \mathbb{T}_{s-t}}\sum_{j=i}^{n}|(x-\theta_{t,s}(y))_j|^{\beta^j} \\
&\le C\sum_{i=1}^{n}\sum_{j=i}^{n}(s-t)^{\zeta_i^j}\left|\mathbb{T}_{s-t}^{-1}(x-\theta_{t,s}(y))\right|^{\beta^j}\frac{\overline{p}(1,\mathbb{T}_{s-t}^{-1}(y-\tilde{m}_{s,t}^{s,y}(x)))}{\det \mathbb{T}_{s-t}},
\end{aligned}
$$

with the following notation at hand:

$$
\zeta_i^j \;:=\; -\frac{1+\alpha(i-1)}{\alpha} + \beta^j\frac{1+\alpha(j-1)}{\alpha}.
$$

Then, we write with the notations of (3.3) that

$$
\begin{aligned}
\left|\tilde{R}_\epsilon^1 f(t,x)\right| &\le C\sum_{i=1}^{n}\sum_{j=i}^{n}\int_t^T\int_{\mathbb{R}^N}|f(s,y)|\frac{\overline{p}(1,\mathbb{T}_{s-t}^{-1}(y-\tilde{m}_{s,t}^{s,y}(x)))}{\det \mathbb{T}_{s-t}}\frac{\left|\mathbb{T}_{s-t}^{-1}(x-\theta_{t,s}(y))\right|^{\beta^j}}{(s-t)^{-\zeta_i^j}}\,dyds \\
&=: C\sum_{i=1}^{n}\sum_{j=i}^{n}\int_t^T\int_{\mathbb{R}^N}|f(s,y)|\mathcal{I}_{ij}(t,s,x,y)\,dyds. \qquad (3.4)
\end{aligned}
$$

Then, from the Hölder inequality,

$$
\left|\tilde{R}_\epsilon^1 f(t,x)\right| \le C\|f\|_{L_t^pL_x^q}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\int_t^T\left(\int_{\mathbb{R}^N}[\mathcal{I}_{ij}(t,s,x,y)]^{q'}\,dy\right)^{\frac{p'}{q'}}ds\right)^{\frac{1}{p'}}, \qquad (3.5)
$$

where q' and p' are the conjugate exponents of q and p, respectively.
Now, the integrals with respect to y can be easily controlled by Corollary 2.12. Indeed,

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\left[\mathcal{I}_{ij}(t,s,x,y)\right]^{q'}dy \qquad (3.6)\\
&\qquad \le C\left(\frac{(s-t)^{\zeta_i^j}}{\det \mathbb{T}_{s-t}}\right)^{q'}\int_{\mathbb{R}^N}\left|\mathbb{T}_{s-t}^{-1}(x-\theta_{t,s}(y))\right|^{\beta^j q'}\overline{p}(1,\mathbb{T}_{s-t}^{-1}(y-\tilde{m}_{s,t}^{s,y}(x)))\,dy.
\end{aligned}
$$

Choosing $q_0 > 1$ big enough so that $\beta^j q' < \alpha$ for any j in $[\![1,n]\!]$ and any $q \ge q_0$, we can use Corollary 2.16 to show that

$$
\int_{\mathbb{R}^N}[\mathcal{I}_{ij}(t,s,x,y)]^{q'}\,dy \;\le\; C(s-t)^{\zeta_i^j q'}\left(\det \mathbb{T}_{s-t}\right)^{1-q'}. \qquad (3.7)
$$

Going back to Equation (3.5), we can thus write that

$$
\left|\tilde{R}_\epsilon^1 f(t,x)\right| \le C\|f\|_{L_t^pL_x^q}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\int_t^T(s-t)^{\zeta_i^j p'}\left(\det \mathbb{T}_{s-t}\right)^{\frac{p'}{q'}-p'}ds\right)^{\frac{1}{p'}}.
$$

Noticing now that for any $i \le j$ in $[\![1,n]\!]$

$$\zeta_i^j > -1 \Leftrightarrow -\frac{1+\alpha(i-1)}{\alpha} + \beta^j \frac{1+\alpha(j-1)}{\alpha} > -1 \Leftrightarrow \beta^j > \frac{1+\alpha(i-2)}{1+\alpha(j-1)}, \tag{3.8}$$

we can choose $q_0 > 1$, $p_0 > 1$ large enough so that p', q' are sufficiently close to 1 in order to conclude that

$$\left|\tilde{R}^1_\epsilon f(t,x)\right| \le C\|f\|_{L^p_t L^q_x}. \tag{3.9}$$

We can now focus on the control for the first remainder term $\tilde{R}^0_\epsilon f$. Since clearly $\tilde{R}^0_\epsilon f = 0$ if $\sigma(t,x)$ is constant in space, we can assume without loss of generality that ν is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ (cf. assumption [**AC**]). In particular, we know that it can be decomposed as in (1.7):

$$\nu(dz) = Q(z)\frac{g(\frac{z}{|z|})}{|z|^{d+\alpha}}dz.$$

Given now a smooth enough function $\phi\colon \mathbb{R}^N \to \mathbb{R}$, we start noticing that

$$\begin{aligned}
\mathcal{L}_t\phi(x) &= \int_{\mathbb{R}^d_0} [\phi(x+B\sigma(t,x)z) - \phi(x)]\ \nu(dz) \\
&= \int_{\mathbb{R}^d_0} [\phi(x+B\sigma(t,x)z) - \phi(x)]\ Q(z) g\left(\frac{z}{|z|}\right)\frac{dz}{|z|^{d+\alpha}} \\
&= \int_{\mathbb{R}^d_0} [\phi(x+B\tilde{z}) - \phi(x)]\, Q(\sigma^{-1}(t,x)\tilde{z})\frac{g\left(\frac{\sigma^{-1}(t,x)\tilde{z}}{|\sigma^{-1}(t,x)\tilde{z}|}\right)}{\det\sigma(t,x)}\frac{d\tilde{z}}{|\sigma^{-1}(t,x)\tilde{z}|^{d+\alpha}},
\end{aligned}$$

where we assumed, without loss of generality from [**UE**], that $\det\sigma(t,x) > 0$. A similar representation holds for $\tilde{\mathcal{L}}^{s,y}_t\phi(x)$, too. Now, let us introduce for any z in $\mathbb{R}^d$, the following quantity:

$$\begin{aligned}
&\tilde{H}^{s,y}_{t,x}(z) \\
&:= Q(\sigma^{-1}(t,x)z)\frac{g\left(\frac{\sigma^{-1}(t,x)z}{|\sigma^{-1}(t,x)z|}\right)}{\det\sigma(t,x)|\sigma^{-1}(t,x)\frac{z}{|z|}|^{d+\alpha}} - Q((\tilde{\sigma}^{s,y}_t)^{-1}z)\frac{g\left(\frac{(\tilde{\sigma}^{s,y}_t)^{-1}z}{|(\tilde{\sigma}^{s,y}_t)^{-1}z|}\right)}{\det\tilde{\sigma}^{s,y}_t|(\tilde{\sigma}^{s,y}_t)^{-1}\frac{z}{|z|}|^{d+\alpha}},
\end{aligned}$$

where we have normalized z above in order to make the usual isotropic stable Lévy measure appear.
Fixed $\eta > 0$, local to this section, meant to be small and to be chosen later (and not to be confused with the ellipticity constant in assumption [**UE**]), we then define

$$\alpha_\eta = \alpha/(1-\eta), \tag{3.10}$$

and we decompose the integral in the difference of the generators in the following way:

$$\begin{aligned}
\left(\mathcal{L}_t - \tilde{\mathcal{L}}^{s,y}_t\right)\phi(x) &= \int_{\mathbb{R}^d_0} [\phi(x+Bz) - \phi(x)]\, \tilde{H}^{s,y}_{t,x}(z)\, s\frac{dz}{|z|^{d+\alpha}} \\
&= \int_{\Delta_\eta} [\phi(x+Bz) - \phi(x) - \langle D_x\phi(x), Bz\rangle]\, \tilde{H}^{s,y}_{t,x}(z)\, \frac{dz}{|z|^{d+\alpha}} \\
&\qquad + \int_{\Delta^c_\eta} [\phi(x+Bz) - \phi(x)]\, \tilde{H}^{s,y}_{t,x}(z)\, \frac{dz}{|z|^{d+\alpha}} \\
&=: \sum_{i=1}^2 [\Delta_i\phi(t,s,\cdot,y)](x),
\end{aligned}$$

where we have denoted, for simplicity,

$$\begin{aligned}\Delta_\eta &:= B(0,(s-t)^{\frac{1}{\alpha_\eta}});\\ \Delta_\eta^c &:= B^c(0,(s-t)^{\frac{1}{\alpha_\eta}}).\end{aligned}$$

We highlight in particular that it is precisely the symmetry of ν that ensures that the function $\tilde{H}^{s,y}_{t,x}$ is even and that allow us to introduce the odd first order term $\langle D_x\phi(x), Bz\rangle$ in the first integral above on the simmetric space Δ_η.
Noticing from Proposition 2.10 that the frozen "density" $\tilde{p}^{s,y}$ is regular enough in x, we can now replace ϕ in the above decomposition with $\tilde{p}^{s,y}(t,s,\cdot,y)$. Going back to $\tilde{R}^0_\epsilon f$ given in (3.3), we start rewriting it as

$$\begin{aligned}|\tilde{R}^0_\epsilon f(t,x)| &\le C\sum_{i=1}^2\int_t^T\int_{\mathbb{R}^N}|f(s,y)|\,|[\Delta_i\tilde{p}^{s,y}(t,s,\cdot,y)]\,(x)|\,dyds\\ &=: \sum_{i=1}^2\int_t^T\int_{\mathbb{R}^N}|f(s,y)|\mathcal{J}_{0i}(t,s,x,y)\,dyds.\end{aligned}\tag{3.11}$$

As before, we can then apply Hölder inequality to show that

$$\left|\tilde{R}^0_\epsilon f(t,x)\right| \le C\|f\|_{L^p_tL^q_x}\sum_{i=1}^2\left(\int_t^T\left(\int_{\mathbb{R}^N}[\mathcal{J}_{0i}(t,s,x,y)]^{q'}\,dy\right)^{\frac{p'}{q'}}ds\right)^{\frac{1}{p'}},\tag{3.12}$$

where q' and p' are again the conjugate exponents of q and p, respectively.
To control the second term involving $\mathcal{J}_{02}$, we start noticing that

$$|\tilde{H}^{s,y}_{t,x}(z)| \le C\tag{3.13}$$

for some constant C independent from the parameters, thanks to assumption [**UE**] for σ and the boundedness of g and Q.
Then, we can use Control (3.13), Corollary 2.12 and the Hölder inequality to write that

$$\begin{aligned}|\mathcal{J}_{02}(t,s,x,y)|^{q'} &\le C\Big[\int_{\Delta_\eta^c}|\tilde{p}^{s,y}(t,s,x+Bz,y)-\tilde{p}^{s,y}(t,s,x,y)|\frac{dz}{|z|^{d+\alpha}}\Big]^{q'}\\ &\le C\left(\int_{\Delta_\eta^c}\frac{dz}{|z|^{d+\alpha}}\right)^{\frac{q'}{q}}\int_{\Delta_\eta^c}|\tilde{p}^{s,y}(t,s,x+Bz,y)-\tilde{p}^{s,y}(t,s,x,y)|^{q'}\frac{dz}{|z|^{d+\alpha}}\\ &\le C\frac{(s-t)^{(\eta-1)\frac{q'}{q}}}{(\det\mathbb{T}_{s-t})^{q'}}\int_{\Delta_\eta^c}\left[\overline{p}(1,\mathbb{T}^{-1}_{s-t}(y-\theta_{s,t}(x+Bz)))+\overline{p}(1,\mathbb{T}^{-1}_{s-t}(y-\theta_{s,t}(x)))\right]\frac{dz}{|z|^{d+\alpha}},\end{aligned}$$

recalling from (3.10) that $\alpha_\eta=\alpha/(1-\eta)$ for the last inequality. The Fubini theorem and the change of variables $\tilde{y}=y-\theta_{s,t}(x+Bz)$ now show that

$$\begin{aligned}\int_{\mathbb{R}^N}|\mathcal{J}_{02}(t,s,x,y)|^{q'}\,dy &\le 2C\frac{(s-t)^{(\eta-1)\frac{q'}{q}}}{(\det\mathbb{T}_{s-t})^{q'-1}}\int_{B^c(0,(s-t)^{\frac{1}{\alpha_\eta}})}\int_{\mathbb{R}^N}\overline{p}(1,\tilde{y})\,d\tilde{y}\frac{dz}{|z|^{d+\alpha}}\\ &\le C(\det\mathbb{T}_{s-t})^{1-q'}(s-t)^{(\eta-1)\frac{q'}{q}}\int_{\Delta_\eta^c}\frac{dz}{|z|^{d+\alpha}}\\ &\le C(\det\mathbb{T}_{s-t})^{1-q'}(s-t)^{q'(\eta-1)}.\end{aligned}\tag{3.14}$$

Going back to Equation (3.12), we can then conclude that

$$\begin{aligned}\int_t^T \left(\int_{\mathbb{R}^N} |\mathcal{I}_{02}(t,s,x,y)|^{q'}\,dy\right)^{\frac{p'}{q'}} ds \,&\le\, C\int_t^T (\det \mathbb{T}_{s-t})^{-\frac{p'}{q}}(s-t)^{p'(\eta-1)}\,ds\\ &\le\, C\int_t^T (s-t)^{-p'(1-\eta+\frac{1}{q}\sum_{i=1}^n d_i\frac{1+\alpha(i-1)}{\alpha})}\,ds,\end{aligned}$$

where in the last step we also exploited (2.42).
Assuming now that $\eta<1$ and p, q are big enough so that

$$p'(1-\eta+\frac{1}{q}\sum_{i=1}^n d_i\frac{1+\alpha(i-1)}{\alpha}) \,<\, 1,$$

we immediately obtain that

$$\left(\int_t^T \left(\int_{\mathbb{R}^N} |\mathcal{I}_{02}(t,s,x,y)|^{q'}\,dy\right)^{\frac{p'}{q'}} ds\right)^{\frac{1}{p'}} \,\le\, C_T. \tag{3.15}$$

We can now focus on the integral with respect to y of the first term $\mathcal{I}_{01}$ in Equation (3.12). Using the Lipschitz continuity of Q in a neighborhood of zero and the Hölder regularity of the diffusion matrix σ, it is not difficult to check that

$$|\tilde{H}^{s,y}_{t,x}(z)| \,\le\, C\sum_{j=1}^n |(x-\theta_{t,s}(y))_j|^{\beta^1}.$$

Thanks to the above estimate, we exploit a Taylor expansion on the density $\tilde{p}^{s,y}$ and Proposition 2.11 with $k=2$ and $(\tau,\xi)=(s,y)$ to show that

$$\begin{aligned}\Theta(t,s,x,y,z) \,&:=\, \left|\left[\tilde{p}^{s,y}(t,s,x+Bz,y)-\tilde{p}^{s,y}(t,s,x,y)-\langle D_x\tilde{p}^{s,y}(t,s,x,y),Bz\rangle\right]\tilde{H}^{s,y}_{t,x}(z)\right|\\ &\le\, \sum_{j=1}^n\int_0^1 |(x-\theta_{t,s}(y))_j|^{\beta^1}\;|D^2_{x_1}\tilde{p}^{s,y}(t,s,x+\lambda Bz,y)||z|^2\,d\lambda\\ &\le\, \frac{C}{\det\mathbb{T}_{s-t}}\int_0^1 |z|^2\frac{\overline{p}(1,\mathbb{T}^{-1}_{s-t}(y-\tilde{m}^{s,y}_{s,t}(x+\lambda Bz)))}{(s-t)^{\frac{2}{\alpha}}}\\ &\qquad\qquad\times\left[\sum_{j=1}^n |(x+\lambda Bz-\theta_{t,s}(y))_j|^{\beta^1}+|\lambda Bz|^{\beta^1}\right]d\lambda\\ &\le\, \frac{C}{\det\mathbb{T}_{s-t}}\int_0^1 |z|^2\overline{p}(1,\mathbb{T}^{-1}_{s-t}(y-\tilde{m}^{s,y}_{s,t}(x+\lambda Bz))) \qquad (3.16)\\ &\qquad\qquad\times\left[\sum_{j=1}^n\frac{\left|\mathbb{T}^{-1}_{s-t}(x+\lambda Bz-\theta_{t,s}(y))\right|^{\beta^1}}{(s-t)^{\zeta^j_0}}+\frac{|z_1|^{\beta^1}}{(s-t)^{\frac{2}{\alpha}}}\right]d\lambda,\end{aligned}$$

where, similarly to above, we have denoted:

$$\zeta^j_0 \,:=\, \frac{2}{\alpha}-\beta^1\frac{1+\alpha(j-1)}{\alpha}. \tag{3.17}$$

It then follows from the Hölder inequality and Corollary 2.12 that

$$
\begin{aligned}
|\mathcal{I}_{01}(t,s,x,y)|^{q'} &\le \left[\int_{\Delta_\eta} \Theta(t,s,x,y,z)\,\frac{dz}{|z|^{d+\alpha}}\right]^{q'} \\
&\le \frac{C}{(\det \mathbb{T}_{s-t})^{q'}}\left(\int_{\Delta_\eta} 1\,dz\right)^{\frac{q'}{q}} \int_0^1\int_{\Delta_\eta} \overline{p}(1,\mathbb{T}^{-1}_{s-t}(y-\tilde{m}^{s,y}_{s,t}(x+\lambda Bz))) \\
&\qquad \times\left[\sum_{j=1}^n \frac{\left|\mathbb{T}^{-1}_{s-t}(x+\lambda Bz-\theta_{t,s}(y))\right|^{\beta^1}}{(s-t)^{\zeta^j_0}}+\frac{|z_1|^{\beta^1}}{(s-t)^{\frac{2}{\alpha}}}\right]^{q'}\frac{dz}{|z|^{q'(d+\alpha-2)}}d\lambda.
\end{aligned}
$$

If we now add the integral with respect to y, Fubini Theorem readily implies that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}|\mathcal{I}_{01}(t,s,x,y)|^{q'}\,dy \\
&\quad\le C\frac{(s-t)^{\frac{d}{\alpha_\eta}(q'-1)}}{(\det \mathbb{T}_{s-t})^{q'}}\int_0^1\int_{\Delta_\eta}\int_{\mathbb{R}^N} \overline{p}(1,\mathbb{T}^{-1}_{s-t}(y-\tilde{m}^{s,y}_{s,t}(x+\lambda Bz))) \\
&\qquad\times\left[\sum_{j=1}^n \frac{\left|\mathbb{T}^{-1}_{s-t}(x+\lambda Bz-\theta_{t,s}(y))\right|^{\beta^1 q'}}{(s-t)^{\zeta^j_0 q'}}+\frac{|z_1|^{\beta^1 q'}}{(s-t)^{\frac{2}{\alpha}q'}}\right]dy\frac{dz}{|z|^{q'(d+\alpha-2)}}d\lambda.
\end{aligned}
$$

If we assume to have taken q' close enough to 1 so that $\beta^1 q'<\alpha$, we can use Corollary 2.16 to show that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}|\mathcal{I}_{01}(t,s,x,y)|^{q'}\,dy \\
&\qquad\le C\frac{(s-t)^{\frac{d}{\alpha_\eta}(q'-1)}}{(\det \mathbb{T}_{s-t})^{q'-1}}\int_{B(0,(s-t)^{\frac{1}{\alpha_\eta}})}\left[\sum_{j=1}^n\frac{1}{(s-t)^{q'\zeta^j_0}}+\frac{|z_1|^{q'\beta^1}}{(s-t)^{q'\frac{2}{\alpha}}}\right]\frac{dz}{|z|^{q'(d+\alpha-2)}} \\
&\qquad\le C\frac{(s-t)^{\frac{d}{\alpha_\eta}(q'-1)}}{(\det \mathbb{T}_{s-t})^{q'-1}}\int_0^{(s-t)^{\frac{1}{\alpha_\eta}}}\left[\sum_{j=1}^n\frac{r^{d-1-(d+\alpha-2)q'}}{(s-t)^{q'\zeta^j_0}}+\frac{r^{d-1-(d+\alpha-2-\beta^1)q'}}{(s-t)^{\frac{2}{\alpha}q'}}\right]dr.
\end{aligned}
$$

Similarly, if q is big enough (so that q' is close to 1), it holds that

$$
d-1-q'(d+\alpha-2)>-1 \Leftrightarrow q'<\frac{d}{d+\alpha-2}
$$

and we can integrate with respect to r:

$$
\begin{aligned}
\int_{\mathbb{R}^N}|\mathcal{I}_{01}(t,s,x,y)|^{q'}\,dy &\le C\frac{(s-t)^{\frac{d}{\alpha_\eta}(q'-1)}}{(\det \mathbb{T}_{s-t})^{q'-1}}\left[\sum_{j=1}^n\frac{r^{d-q'(d+\alpha-2)}}{(s-t)^{q'\zeta^j_0}}+\frac{r^{d-q'(d+\alpha-2-\beta^1)}}{(s-t)^{q'\frac{2}{\alpha}}}\right]\Bigg|_0^{(s-t)^{\frac{1}{\alpha_\eta}}} \\
&\le C(\det \mathbb{T}_{s-t})^{1-q'}(s-t)^{\frac{q'}{\alpha_\eta}(2-\alpha)}\left[\sum_{j=1}^n(s-t)^{-q'\zeta^j_0}+(s-t)^{q'(\frac{\beta^1}{\alpha_\eta}-\frac{2}{\alpha})}\right]. \quad (3.18)
\end{aligned}
$$

Hence, it follows from Equation (2.42) that

$$
\begin{aligned}
&\int_t^T \left(\int_{\mathbb{R}^N} |\mathfrak{I}_{01}(t,s,x,y)|^{q'}\,dy\right)^{\frac{p'}{q'}} ds \\
&\qquad \le C \int_t^T (\det \mathbb{T}_{s-t})^{-\frac{p'}{q}} (s-t)^{\frac{p'}{\alpha_\eta}(2-\alpha)} \left[\sum_{j=1}^n (s-t)^{-p'\zeta_0^j} + (s-t)^{p'(\frac{\beta^1}{\alpha_\eta}-\frac{2}{\alpha})}\right] ds \\
&\qquad \le C \int_t^T (s-t)^{p'\left(\frac{(2-\alpha)}{\alpha_\eta}-\frac{1}{q}\sum_{i=1}^n d_i \frac{1+\alpha(i-1)}{\alpha}\right)} \left[\sum_{j=1}^n (s-t)^{-p'\zeta_0^j} + (s-t)^{p'(\frac{\beta^1}{\alpha_\eta}-\frac{2}{\alpha})}\right] ds
\end{aligned}
$$

To conclude, we need to show that the two terms above are integrable with respect to s. Namely,

$$
\begin{aligned}
p'\left(\frac{(2-\alpha)}{\alpha_\eta} - \frac{1}{q}\sum_{i=1}^n d_i \frac{1+\alpha(i-1)}{\alpha} - \zeta_0^j\right) &> -1, \quad \forall j \in [\![1,n]\!]; \\
p'\left(\frac{(2-\alpha)}{\alpha_\eta} - \frac{1}{q}\sum_{i=1}^n d_i \frac{1+\alpha(i-1)}{\alpha} + \frac{\beta^1}{\alpha_\eta} - \frac{2}{\alpha}\right) &> -1.
\end{aligned}
$$

Recalling again that we can choose p, q big enough as we want, so that Equation (2.43) holds, it is now sufficient to take η in $(0,1)$ in order to have:

$$
\frac{(2-\alpha)}{\alpha_\eta} - \zeta_0^j = \frac{(2-\alpha)}{\alpha_\eta} - \frac{2}{\alpha} + \beta^1 \frac{1+\alpha(j-1)}{\alpha} > -1, \quad \forall j \in [\![1,n]\!]; \tag{3.19}
$$

$$
\frac{(2-\alpha)}{\alpha_\eta} + \frac{\beta^1}{\alpha_\eta} - \frac{2}{\alpha} > -1. \tag{3.20}
$$

By direct calculations, recalling from (3.10) that $\alpha_\eta = \alpha/(1-\eta)$, we now notice that Conditions (3.19)-(3.20) can be rewritten as follows

$$
\begin{aligned}
\eta &< \frac{\beta^1(1+\alpha(j-1)}{2-\alpha}, \quad \forall j \in [\![1,n]\!]; \\
\eta &< \frac{\beta^1}{2+\beta^1-\alpha}.
\end{aligned}
$$

Choosing $\epsilon > 0$ so that the above conditions holds, we have that

$$
\left(\int_t^T \left(\int_{\mathbb{R}^N} |\mathfrak{I}_{01}(t,s,x,y)|^{q'}\,dy\right)^{\frac{p'}{q'}} ds\right)^{\frac{1}{p'}} \le C_T. \tag{3.21}
$$

Going back to Equation (3.12), we use Controls (3.15)-(3.21) to write that

$$
\left|\tilde{R}^0_\epsilon f(t,x)\right| \le C\|f\|_{L^p_t L^q_x}. \tag{3.22}
$$

Exploiting Controls (3.9) and (3.22) in Equation (3.3), we have concluded our proof. □

A similar control in $L^p_t L^q_x$-norms can be obtained. In particular, we point out that Equation (3.23) below implies that the operator $I - \tilde{R}_\epsilon$ is invertible in $L^p\left(0,T; L^q(\mathbb{R}^N)\right)$, provided T is small enough. From Lemma 2.19, the same holds for $I_\epsilon - \tilde{R}_\epsilon$.

Proposition 3.3. *Let $q>1$, $p>1$ be such that Condition ($\mathscr{C}$) holds. Then, there exists $C:=C(T,p,q)>0$ such that for any f in $L^p\left(0,T;L^q(\mathbb{R}^N)\right)$,*

$$\|\tilde{R}_\epsilon f\|_{L^p_tL^q_x} \,\le\, C\|f\|_{L^p_tL^q_x}. \tag{3.23}$$

In particular, it holds that $\lim_{T\to 0} C(T,p,q)=0$.

Proof. We are going to keep the same notations used in the previous proof. In particular, we recall the following decomposition

$$\tilde{R}_\epsilon f(t,x) \,=\, \tilde{R}^0_\epsilon f(t,x)+\tilde{R}^1_\epsilon f(t,x),$$

given in Equation (3.3).
In order to control the second term $\tilde{R}^1_\epsilon f$ in $L^p_tL^q_x$-norm, we start from Equation (3.4) to write that

$$\|\tilde{R}^1_\epsilon f(t,\cdot)\|_{L^q_x} \,\le\, C\sum_{i=1}^{n}\sum_{j=i}^{n}\int_t^T\left\|\int_{\mathbb{R}^N}|f(s,y)|\mathfrak{I}_{ij}(t,s,\cdot,y)\,dy\right\|_{L^q_x}ds.$$

The Young inequality now implies that

$$\begin{aligned}\left\|\int_{\mathbb{R}^N}|f(s,y)|\mathfrak{I}_{ij}(t,s,\cdot,y)\,dy\right\|^q_{L^q_x} \,&=\, \int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N}|f(s,y)|\mathfrak{I}_{ij}(t,s,x,y)\,dy\right|^q dx\\
&\le\, \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^N}|f(s,y)|^q\mathfrak{I}_{ij}(t,s,x,y)\,dy\right)\|\mathfrak{I}_{ij}(t,s,x,\cdot)\|^{q/q'}_{L^1}dx\\
&\le\, C(s-t)^{\zeta^j_i q/q'}\int_{\mathbb{R}^N}|f(s,y)|^q\left(\int_{\mathbb{R}^N}\mathfrak{I}_{ij}(t,s,x,y)\,dx\right)dy\end{aligned}$$

using Control (3.7) and the Fubini Theorem for the last inequality. From (3.4), (3.6) and the correspondence (2.6) which gives $y-\tilde{m}^{s,y}_{s,t}(x)=\theta_{t,s}(y)-x$ it is plain to derive that:

$$\int_{\mathbb{R}^N} dx\mathfrak{I}_{ij}(t,s,x,y)\le C(s-t)^{\zeta^j_i}.$$

Thus,

$$\|\tilde{R}^1_\epsilon f(t,\cdot)\|_{L^q_x} \,\le\, C\sum_{i=1}^{n}\sum_{j=i}^{n}\int_t^T(s-t)^{\zeta^j_i}\|f(t,\cdot)\|_{L^q_x}\,ds,$$

where, in the last step, we exploited Equation (3.7) with $q'=1$, recalling that $\beta^j<1<\alpha$.
We can then use the above control to write that

$$\begin{aligned}\|\tilde{R}^1_\epsilon f\|^p_{L^p_tL^q_x} \,&\le\, \sum_{i=1}^{n}\sum_{j=i}^{n}\int_0^T\|\tilde{R}^1_\epsilon f(t,\cdot)\|^p_{L^q_x}\,dt\\
&\le\, C\sum_{i=1}^{n}\sum_{j=i}^{n}\int_0^T\|f(t,\cdot)\|^p_{L^q_x}\left(\int_t^T(s-t)^{\zeta^j_i}\,ds\right)^p dt\\
&\le\, C_T\sum_{i=1}^{n}\sum_{j=i}^{n}\int_0^T\|f(t,\cdot)\|^p_{L^q_x}dt\le C_T\|f\|^p_{L^p_tL^q_x},\end{aligned}$$

where $C_T:=C(p',q',T)$ denotes a positive constant that tends to zero if T goes to zero (recall indeed from (3.8) that $\zeta^j_i>-1$) .

The control for $\tilde{R}^0_\epsilon f$ can be obtained following the same arguments above, exploiting Equation (3.11) instead of (3.4) and Equations (3.14)-(3.18) with $q' = 1$ for the controls of $\|\mathcal{I}_{0j}(t, s, x, \cdot)\|_{L^1_x}$. $\square$

Let us fix now a function f in $C^{1,2}_c([0, T) \times \mathbb{R}^N)$. The first step of our method consists in applying the Itô formula on the Green kernel $\tilde{G}_\epsilon f$ and the process $\{X^{t,x}_s\}_{s\in[t,T]}$, solution of the martingale problem with starting point (t, x):

$$\mathbb{E}\left[\tilde{G}_\epsilon f(t, x) + \int_t^T (\partial_s + L_s)\tilde{G}_\epsilon f(s, X^{t,x}_s) ds\right] = 0.$$

We then exploit Equation (2.48) to write that

$$\tilde{G}_\epsilon f(t, x) - \mathbb{E}\left[\int_t^T I_\epsilon f(s, X^{t,x}_s)\, ds\right] + \mathbb{E}\left[\int_t^T \left[L_s\tilde{G}_\epsilon f - \tilde{M}_\epsilon f\right](s, X^{t,x}_s)\, ds\right] = 0.$$

Thus, it holds that

$$\mathbb{E}\Big[\int_t^T I_\epsilon f(s, X^{t,x}_s)\, ds\Big] = \tilde{G}_\epsilon f(t, x) + \mathbb{E}\Big[\int_t^T \tilde{R}_\epsilon f(s, X^{t,x}_s)\, ds\Big]. \tag{3.24}$$

Thanks to Proposition 2.17, we know that there exists $C(T) := C(T) \xrightarrow[T\to 0]{} 0$ such that

$$\|\tilde{G}_\epsilon f\|_\infty \le C\|f\|_{L^p_t L^q_x}. \tag{3.25}$$

Let us assume for now that p, q are large enough so that the control (3.2) of Lemma 3.2 (pointwise control of the remainder) holds. From Equations (3.24), (3.25) and (3.2), we readily get that

$$\left|\mathbb{E}\left[\int_t^T I_\epsilon f(X^{t,x}_s)\, ds\right]\right| \le C\|f\|_{L^p_t L^q_x}.$$

Letting ϵ go to zero, we thus derive that any solution $\{X^{t,x}_s\}_{s\in[t,T]}$ of the martingale problem for $\partial_s + L_s$ with initial condition (t, x) satisfies

$$\left|\mathbb{E}\left[\int_t^T f(s, X^{t,x}_s)\, ds\right]\right| \le C\|f\|_{L^p_t L^q_x},$$

for any f in f in $C^{1,2}_c([0, T) \times \mathbb{R}^N)$. Above, we have exploited Lemma 2.18 for the integral in space and the bounded convergence Theorem for that in time.
To show the result for a general f in $L^p\Big(0, T; L^q(\mathbb{R}^N)\Big)$, we now use a density argument and the Fatou Lemma. Indeed, let $\{f_n\}_{n\in\mathbb{N}}$ a sequence of functions in $C^{1,2}_c([0, T) \times \mathbb{R}^N)$ such that $\|f_n - f\|_{L^p_t L^q_x} \to 0$. We then have that:

$$\begin{aligned}\left|\mathbb{E}\left[\int_t^T f(s, X^{t,x}_s)\, ds\right]\right| &\le \left|\mathbb{E}\left[\int_t^T \liminf_n f_n(s, X^{t,x}_s)\, ds\right]\right| \\ &\le \liminf_n \left|\mathbb{E}\left[\int_t^T f_n(s, X^{t,x}_s)\, ds\right]\right| \\ &\le C \liminf_n \|f_n\|_{L^p_t L^q_x} \\ &= C\|f\|_{L^p_t L^q_x}.\end{aligned} \tag{3.26}$$

This is precisely the Estimate (2.4) in Corollary 2.7, provided that p, q are large enough.

Thanks to Estimates (3.26), we then know that the process $\{X^{t,x}_s\}_{s\in[t,T]}$ has a density we will denote by $p(t, s, x, y)$. From Equation (3.24) it now follows that

$$\begin{aligned}
\tilde{G}_\epsilon f(t,x) &= \mathbb{E}\left[\int_t^T I_\epsilon f(s, X^{t,x}_s)\, ds\right] - \mathbb{E}\left[\int_t^T \tilde{R}_\epsilon f(s, X^{t,x}_s)\, ds\right] \\
&= \int_t^T \int_{\mathbb{R}^N} I_\epsilon f(s,y) p(t,s,x,y)\, dyds - \int_t^T \int_{\mathbb{R}^N} \tilde{R}_\epsilon f(s,y) p(t,s,x,y)\, dyds \\
&= \int_t^T \int_{\mathbb{R}^N} (I_\epsilon - \tilde{R}_\epsilon) f(s,y) p(t,s,x,y)\, dyds. \qquad (3.27)
\end{aligned}$$

Then, Proposition 2.17, Lemma 2.19 (with an additional approximation argument) and Control (3.23) imply that both sides of the above control are bounded in the $L^p_tL^q_x$-norm, uniformly in $\epsilon > 0$. Thus, we can conclude that Equation (3.27) holds for any f in $L^p\Big(0, T; L^q(\mathbb{R}^N)\Big)$. We then conclude from Lemma 3.2 (pointwise control of the remainder) that letting ϵ go to zero, it holds that

$$\mathbb{E}\left[\int_t^T f(s, X^{t,x}_s)\, ds\right] = \tilde{G} \circ (I - \tilde{R})^{-1} f(t,x),$$

which gives uniqueness if the final time T is small enough. Global well-posedness is again derived from a chaining argument in time.

To complete the proof of Corollary 2.7, it remains to derive the Krylov estimates (2.4) under Condition $(\mathscr{C})$ and not only for p, q large enough.
Fixed a parameter $\delta > 0$ meant to be small, we consider a "mollified" version of the solution process $X^{t,x}_s$, given by

$$\overline{X}^{t,x,\delta}_s := X^{t,x}_s + \delta \mathbb{M}_{s-t} \overline{Z}_{s-t}, \qquad (3.28)$$

where $\{\overline{Z}_s\}_{s\ge 0}$ is an isotropic α-stable process on $\mathbb{R}^N$.
Let us denote now by $p^\delta(t, s, x, \cdot)$ the density associated with the random variable $\overline{X}^{t,s,\delta}_s$. We notice that Equation (3.28) implies in particular that

$$p^\delta(t,s,x,y) = \Big[p(t,s,x,\cdot) * q^\delta(s-t,\cdot)\Big](y),$$

where $q^\delta(t, \cdot)$ is the density of the process $\delta\mathbb{M}_t\overline{Z}_t$ and thus, under the integrability condition $(\mathscr{C})$ and thanks to the Young inequality, the quantity $\|p^\delta\|_{L^{p'}_t L^{q'}_x}$, where p', q' are the conjugate exponents of p, q, respectively, is finite (possibly explosive with δ). The point is now to reproduce the previous perturbative analysis in order to prove that the controls on $\|p^\delta\|_{L^{p'}_t L^{q'}_x}$ actually do no depend on δ.

For this reason, we introduce the mollified "frozen" process $\tilde{X}^{s,y,t,x,\delta}_s$ along the flow $\theta_{t,s}(y)$ as

$$\tilde{X}^{s,y,t,x,\delta}_s := \tilde{X}^{s,y,t,x}_s + \delta\mathbb{M}_{s-t}\overline{Z}_{s-t}. \qquad (3.29)$$

Following the same arguments presented in Propositions 2.10 and 2.11, it is now possible to show that the process $\tilde{X}^{s,y,t,x,\delta}_s$ admits a density $\tilde{p}^{s,y,\delta}(t, s, x, y)$ and that it enjoys a multi-scale bound similar to (2.35). Namely,

Proposition 3.4. *There exists a positive constant $C := C(N, \alpha)$ such that for any k in $[\![0, 2]\!]$, any i in $[\![1, n]\!]$, any $t < s$ in $[0, T]$ and any x, y in $\mathbb{R}^N$,*

$$|D^k_{x_i}\tilde{p}^{s,y,\delta}(t, s, x, y)| \le C\frac{((s-t)(1+\delta))^{-k\frac{1+\alpha(i-1)}{\alpha}}}{\det \mathbb{T}_{(s-t)(1+\delta)}}\overline{p}\left(1, \mathbb{T}^{-1}_{(s-t)(1+\delta)}(y - \theta_{s,t}(x))\right). \quad (3.30)$$

A sketch of proof for the above Proposition has been briefly presented in the Appendix section. Importantly, we highlight that the constant C appearing in (3.30) is independent from the "smoothing" parameter δ.
Then, the same arguments leading to (3.24) can be applied here to show that

$$\mathbb{E}\Big[\int_t^T I_\epsilon f(s, \overline{X}^{t,x,\delta}_s)\, ds\Big] = \tilde{G}^\delta_\epsilon f(t, x) + \mathbb{E}\Big[\int_t^T \int_{\mathbb{R}^N}\Big(L^\delta_t - \tilde{L}^{s,y,\delta}_t\Big)\tilde{G}^\delta_\epsilon f(s, \overline{X}^{t,x,\delta}_s)\, ds\Big], \quad (3.31)$$

where $\tilde{G}^\delta_\epsilon$ and $\tilde{\mathcal{L}}^{s,y,\delta}$ are the frozen Green kernel and the frozen infinitesimal generator associated with the process $\tilde{X}^{s,y,t,x,\delta}_s$, respectively (cf. Equations (2.41) and (2.44)). In particular, we point out that the pointwise bound (3.25) on the Green kernel and the controls of Proposition 3.2 (pointwise control of the remainder) are uniform with respect to the additional parameter δ, thanks to Proposition 3.4.
From Equation (3.31) and Proposition 3.3 ($L^p_tL^q_x$ control of the remainder) we can then deduce that

$$\left|\int_t^T \int_{\mathbb{R}^N} I_\epsilon f(s, y)p^\delta(t, s, x, y)\, dyds\right| \le C_T\left(1 + \|p^\delta\|_{L^{p'}_t L^{q'}_x}\right)\|f\|_{L^p_tL^q_x}.$$

From the Riesz representation theorem and the above inequality, we then deduce that $\|p^\delta\|_{L^{p'}_t L^{q'}_x} \le C_T$, for T small enough and uniformly in δ. Hence,

$$\left|\int_t^T \int_{\mathbb{R}^N} I_\epsilon f(s, y)p^\delta(t, s, x, y)\, dyds\right| = \left|\int_t^T \mathbb{E}[I_\epsilon f(s, X^{t,x}_s + \delta\overline{Z}_s)]\, ds\right| \le C_T\|I_\epsilon f\|_{L^p_tL^q_x}.$$

The Krylov-type estimate (2.4) can be then derived exploiting the dominated convergence theorem and Lemma 2.18 (Dirac Convergence of frozen density), letting firstly ϵ and then δ go to zero. We have thus concluded the proof of Corollary 2.7.

4 A counter-example to uniqueness

In this section, we present a counter-example to the uniqueness in law for the equation (1.3) when the Hölder regularity in space of the coefficients is low enough. In particular, we show here the almost sharpness of the thresholds appearing in Theorem 2.6 for diagonal perturbations, proving also Theorem 2.8. In order to test the threshold associated with the critical Hölder exponent for the i-th component of the drift F with respect to the variables x_j, we adapt the *ad hoc* Peano example constructed in [CdRM20b] to our Lévy framework.
Let us briefly recall it. It is well-known that the following deterministic equation

$$\begin{cases} dy_t = \mathrm{sgn}(y_t)|y_t|^\beta dt, & t \ge 0, \\ y_0 = 0, \end{cases} \quad (4.1)$$

for some β in $(0,1)$, is ill-posed since it admits an infinite number of solutions of the form

$$y_t = \pm c(t-t_0)^{1/(1-\beta)}\mathbb{1}_{[t_0,\infty)}(t), \quad \text{for some } t_0 \text{ in } [0,+\infty).$$

Nevertheless, Bafico and Baldi in [BB81] proved that the associated SDE, obtained by adding a Brownian Motion $\{W_t\}_{t\geq 0}$ to the dynamics:

$$\begin{cases} dX_t = \operatorname{sgn}(X_t)|X_t|^\beta dt + \epsilon dW_t, & t \geq 0 \\ X_0 = 0, \end{cases}$$

is well-posed for any $\epsilon > 0$ in a strong (probabilistic) sense. Furthermore, they showed that, letting ϵ goes to zero, the limit law concentrates around the two extremal solutions $\pm ct^{1/(1-\beta)}$ of the deterministic equation (4.1), thus providing a selection "criterion" between the infinite deterministic solutions.
In a subsequent article [DF14], Delarue and Flandoli highlighted the hidden dynamical mechanism behind this counter-intuitive behaviour. Heuristically, this *regularization by noise* happens since, at least in a small time interval, the mean fluctuations of the Brownian noise are stronger than the irregularity of the deterministic drift. Indeed, they showed that before some transition time t_ϵ, the dominating noise pushes the solution to leave the drift singularity at 0, while afterwards, the deterministic part of the system prevails, constraining the (stochastic) solution to fluctuate around one of the extremal deterministic solutions, given by $\pm ct^{1/(1-\beta)}$.
More quantitatively, we can compare the fluctuations of the noise, say of order $\gamma > 0$ with the fluctuations of the deterministic extremal solutions, giving that

$$t^\gamma > t^{1/(1-\beta)}.$$

Since it should happen in small times, we then obtain that

$$\beta > 1 - \frac{1}{\gamma},$$

should be the heuristic relation that guarantees the noise dominates in short time. Clearly, the above inequality holds for any β in $(0,1)$ in the Brownian case ($\gamma = 1/2$), which would actually give $\beta > -1$. We can refer to [DD16] which is the closest work to this threshold since the authors manage to reach $-2/3^+$.

In view the above arguments, we fix $n = N$, $d_i = d = 1$ and i, j in $[\![1,n]\!]$ such that $j \geq i$ and we consider the drift

$$Ax + e_i\operatorname{sgn}(x_j)|x_j|^\beta$$

where $\{e_i \colon i \in [\![1,N]\!]\}$ is the canonical orthonormal basis for $\mathbb{R}^N$, A is the matrix in $\mathbb{R}^N \otimes \mathbb{R}^N$ given by

$$A := \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

We will assume moreover that β is in $(0,1)$ such that

$$\beta \,<\, \frac{1+\alpha(i-2)}{1+\alpha(j-1)},$$

so that we are clearly outside the framework given by condition $(\mathscr{C})$.
Our aim is to prove that uniqueness in law fails for the following equation:

$$\begin{cases} dX_t \,=\, \Big[Ax + e_i\mathrm{sgn}(X^j_t)|X^j_t|^\beta\Big]dt + BdZ_t, \quad t \geq 0, \\ X_0 \,=\, 0, \end{cases} \tag{4.2}$$

where $\{Z_t\}_{t\geq 0}$ is a symmetric, d-dimensional α-stable process such that $\mathbb{E}[|Z_1|]$ is finite. In particular, we are interested on the i-th component of the above Equation (4.2) that can be rewritten in integral form as:

$$X^j_t \,=\, \int_0^t \mathrm{sgn}\Big(I^{j-i}_t(X^j)\Big)\Big|I^{j-i}_t(X^j)\Big|^\beta dt + I^{i-1}_t(Z), \quad t \geq 0, \tag{4.3}$$

where we have denoted by $I^k_t(y)$ the k-th *iterated integral* of a càdlàg path $y\colon [0,\infty) \to \mathbb{R}$ at a time t. Namely,

$$I^k_t(y) \,:=\, \int_0^{t_k=t} \dots \int_0^{t_2} y_{t_0}\, dt_0 \dots dt_{k-1}, \quad t \geq 0. \tag{4.4}$$

In order to improve the readability of the next part, we are going to present our reasoning in a slightly more general way. It is not difficult to check that Equation (4.3) satisfies the assumptions of the following proposition.

Proposition 4.1. *Let k be in $\mathbb{N}$, β in $(0,1)$, x in $\mathbb{R}$ and $\{\mathcal{Z}_t\}_{t\geq 0}$ a continuous process on $\mathbb{R}$ such that*

- $\mathbb{E}\Big[\sup_{s\in[0,1]}|\mathcal{Z}_s|\Big] < \infty$*;*
- *it is symmetric and γ-self-similar in law for some $\gamma > 0$. Namely,*

$$\Big(\mathcal{Z}_t\Big)_{t\geq 0} \overset{(\text{law})}{=} \Big(-\mathcal{Z}_t\Big)_{t\geq 0} \text{ and } \forall \rho > 0,\ \Big(\mathcal{Z}_{\rho t}\Big)_{t\geq 0} \overset{(\text{law})}{=} \Big(\mathcal{Z}_t\rho^\gamma\Big)_{t\geq 0}.$$

Then, uniqueness in law fails for the following SDE:

$$\begin{cases} dX_t \,=\, \mathrm{sgn}\Big(I^k_t(X)\Big)\Big|I^k_t(X)\Big|^\beta dt + d\mathcal{Z}_t, \quad t \geq 0 \\ X_0 \,=\, x, \end{cases} \tag{4.5}$$

if $x = 0$ and $\beta < \frac{\gamma-1}{\gamma+k}$.

Since we can clearly apply Proposition 4.1 to Equation (4.3) taking $\gamma = i - 1 + \frac{1}{\alpha}$, $k = j - i$, it implies that SDE (4.2) lacks of uniqueness in law if

$$\beta \,<\, \frac{\gamma-1}{\gamma+k} \,=\, \frac{1+\alpha(i-2)}{1+\alpha(j-1)}.$$

Hence, to complete the proof of Theorem 2.8, it suffices to establish Proposition 4.1.

Before proving Proposition 4.1, we need however an auxiliary result. It roughly states that any solution of SDE (4.5) starting outside zero cannot immediately reach the extremal solutions of the associated deterministic Peano example. Importantly, the constant ρ appearing below does not depend on the starting point x.

Lemma 4.2. *Fixed $x > 0$ and $\beta < \frac{\gamma-1}{\gamma+k}$, let $\{X_t\}_{t\geq 0}$ be a solution of Equation* (4.5) *starting from x. Then, there exist two positive constants $\rho := \rho(k,\beta,\gamma,\mathbb{E}[\sup_{s\in[0,1]}|\mathcal{Z}_s|])$ and $c_0 := c_0(k,\beta)$ such that*

$$\mathbb{P}\Big(\tau(X) \geq \rho\Big) \,\geq\, 3/4, \tag{4.6}$$

where $\tau(X)$ is the stopping time on Ω given by

$$\tau(X) \,=\, \inf\{t\geq 0\colon X_t \,\leq\, c_0 t^{\frac{k\beta+1}{1-\beta}}\}. \tag{4.7}$$

Proof. We start noticing that the process $\{X_t\}_{t\geq 0}$ is continuous in 0, since it is càdlàg. Fixed $c_0 > 0$ to be chosen later, it implies that $\tau(X) > 0$, almost surely. In particular, it makes sense to consider the random interval $(0,\tau(X)]$.

Fixed t in $(0,\tau(X)]$, it holds, by definition of $\tau(X)$, that $X_t > c_0 t^{\frac{k\beta+1}{1-\beta}}$. It follows then that

$$\int_0^t \Big|I^k_s(X)\Big|^\beta ds \,>\, \tilde{C}c_0^\beta t^{\frac{k\beta+1}{1-\beta}} \quad\text{where}\quad \tilde{C} \,:=\, \Big(\prod_{i=1}^k \frac{k\beta+1}{1-\beta}+(i-1)\Big)^{-\beta}.$$

Since $x > 0$ by assumption and $X > 0$ on $(0,\tau(X)]$, we can now show that

$$X_t \,=\, x+\int_0^t \operatorname{sgn}\Big(I^k_s(X)\Big)\Big|I^k_s(X)\Big|^\beta ds+\mathcal{Z}_t \,>\, \tilde{C}c_0^\beta t^{\frac{k\beta+1}{1-\beta}}+\mathcal{Z}_t.$$

The next step is to write $\tilde{C}c_0^\beta = c_0+\hat{C}$ for some constant $\hat{C} > 0$. To do so, we need to choose carefully c_0. In particular, the condition above is equivalent to the following

$$\hat{C} = \tilde{C}c_0^\beta - c_0 \,>\, 0 \Leftrightarrow c_0 \,<\, \tilde{C}^{\frac{1}{1-\beta}}.$$

Fixed $c_0 = \tilde{C}^{\frac{1}{1-\beta}}/2$, it then holds that

$$X_t \,>\, c_0 t^{\frac{k\beta+1}{1-\beta}}+\hat{C}t^{\frac{k\beta+1}{1-\beta}}+\mathcal{Z}_t$$

for any t in $(0,\tau(X)]$. Fixed $\rho > 0$ to be chosen later, we can now define the event $\mathcal{A}$ in Ω as

$$\mathcal{A} \,:=\, \{\omega\in\Omega\colon \hat{C}t^{\frac{k\beta+1}{1-\beta}}+\mathcal{Z}_t > 0,\ \forall\, t\in(0,\rho]\}.$$

On $\mathcal{A}$ and for any t in $(0,\tau(X)]$, it then holds that

$$X_t \,>\, c_0 t^{\frac{k\beta+1}{1-\beta}}.$$

In particular, we have that $\tau(X) \geq \rho$ on $\mathcal{A}$ and thus, $\mathcal{A}\subseteq\{\tau(X)\geq\rho\}$ on Ω. It immediately implies that

$$\mathbb{P}\Big(\tau(X)\geq\rho\Big) \,\geq\, \mathbb{P}(\mathcal{A}).$$

It remains to choose $\rho > 0$ such that $\mathbb{P}(\mathcal{A}) \geq 3/4$. Write:

$$\begin{aligned}\mathbb{P}(\mathcal{A}) =&\mathbb{P}[\forall t\in(0,\rho],\ \hat{C}t^{\frac{k\beta+1}{1-\beta}}+\mathcal{Z}_t > 0] = \mathbb{P}[\forall t\in(0,1],\ \hat{C}(\rho t)^{\frac{k\beta+1}{1-\beta}}+\mathcal{Z}_{\rho t} > 0]\\ =&\mathbb{P}[\forall t\in(0,1],\ \hat{C}(\rho t)^{\frac{k\beta+1}{1-\beta}}+\rho^\gamma\mathcal{Z}_t > 0] = \mathbb{P}[\forall t\in(0,1],\ \hat{C}\rho^{\frac{k\beta+1}{1-\beta}-\gamma}+t^{-\frac{k\beta+1}{1-\beta}}\mathcal{Z}_t > 0],\end{aligned}$$

from the self-similarity assumption on $\mathcal{Z}$. Since by assumption $\beta < \frac{\gamma-1}{\gamma+k} \iff \frac{k\beta+1}{1-\beta} - \gamma < 0$, the statement will follow taking ρ small enough as soon as we prove the process $\mathcal{R}_t := t^{-\frac{k\beta+1}{1-\beta}} \mathcal{Z}_t,\ t \in (0,1]$, which is continuous on the open set $(0,1]$, can be extended by continuity in 0 with $\mathcal{R}_0 = 0$. Observe that $\mathbb{E}[|\mathcal{R}_t|] = t^{\gamma-\frac{k\beta+1}{1-\beta}}\mathbb{E}[|\mathcal{Z}_1|] \underset{t\to 0}{\longrightarrow} 0$. Setting $\delta := \gamma - \frac{k\beta+1}{1-\beta} > 0$ and introducing $t_n := n^{-1/\delta(1+\eta)}, \eta > 0$, we get that for all $\varepsilon > 0$,

$$\mathbb{P}[|\mathcal{R}_{t_n}| \geq \varepsilon] \leq \varepsilon^{-1}\mathbb{E}[|\mathcal{R}_{t_n}|] = \varepsilon^{-1}t_n^{\delta}\mathbb{E}[|\mathcal{Z}_1|] = \varepsilon^{-1}n^{-(1+\eta)}\mathbb{E}[|\mathcal{Z}_1|].$$

We thus get from the Borel-Cantelli lemma that $\mathcal{R}_{t_n} \underset{n,\ a.s.}{\longrightarrow} 0$. Namely, we have almost sure convergence along the subsequence t_n going to zero with n. It now remains to prove that the process $\mathcal{R}_t$ does not fluctuate much between two successive times t_n and t_{n+1}. Write for $t \in [t_{n+1}, t_n]$:

$$\begin{aligned} |R_t| := |t^{-\frac{k\beta+1}{1-\beta}}\mathcal{Z}_t| \leq & t_{n+1}^{-\frac{k\beta+1}{1-\beta}}\Big(|\mathcal{Z}_{t_{n+1}}| + \sup_{s\in[t_{n+1},t_n]}|\mathcal{Z}_s - \mathcal{Z}_{t_{n+1}}|\Big) \\ \leq & t_{n+1}^{-\frac{k\beta+1}{1-\beta}}\Big(2|\mathcal{Z}_{t_{n+1}}| + \sup_{s\in[0,t_n]}|\mathcal{Z}_s|\Big). \end{aligned} \tag{4.8}$$

The first term of the above left hand side tends almost surely to zero with n. Observe as well that, from the scaling properties of $\mathcal{Z}$, for any $\varepsilon > 0$:

$$\begin{aligned} \mathbb{P}[t_{n+1}^{-\frac{k\beta+1}{1-\beta}} \sup_{s\in[0,t_n]}|\mathcal{Z}_s| \geq \varepsilon] = \mathbb{P}[t_{n+1}^{-\frac{k\beta+1}{1-\beta}} t_n^{\gamma} \sup_{s\in[0,1]}|\mathcal{Z}_s| \geq \varepsilon] &\leq \varepsilon^{-1}t_n^{\delta}(\frac{t_n}{t_{n+1}})^{\frac{k\beta+1}{1-\beta}}\mathbb{E}[\sup_{s\in[0,1]}|\mathcal{Z}_s|] \\ &\leq C\varepsilon^{-1}n^{-(1+\eta)}\mathbb{E}[\sup_{s\in[0,1]}|\mathcal{Z}_s|], \end{aligned}$$

which again gives from the Borel-Cantelli lemma the a.s. convergence with n of the second term in the r.h.s of (4.8). We eventually derive that $\mathcal{R}_t \underset{t\to 0,a.s.}{\longrightarrow} 0$. Again, the key point is that we normalize the process $\mathcal{Z}$ at a rate, $t^{\frac{k\beta+1}{1-\beta}}$, which is lower than its own characteristic time scale, t^{γ}. This is precisely what leaves some margin to establish continuity.

□

Exploiting the lower bound for the random time $\tau(X)$ given in Lemma 4.2, we are now ready to show uniqueness in law fails for SDE (4.5) when $x = 0$ and $\beta < \frac{\gamma-1}{\gamma+k}$.

Proof of Proposition 4.1. By contradiction, we start assuming that uniqueness in law holds for SDE (4.5) starting at $x = 0$. Fixed any solution $\{X_t\}_{t\geq 0}$ of Equation (4.5) starting at zero, it follows by symmetry that $\{-X_t\}_{t\geq 0}$ is also a solution of the same dynamics. Since by hypothesis, $-\mathcal{Z}_t \overset{(\text{law})}{=} \mathcal{Z}_t$, uniqueness in law for SDE (4.5) implies that the laws of X and $-X$ are identical.
Assuming for the moment that Lemma 4.2 is applicable for $x = 0$, we easily find a contradiction. Indeed, it follows from Lemma 4.2 that

$$\mathbb{P}\Big(\tau(X) \geq \rho\Big) \geq 3/4$$

but on the same time, thanks to the uniqueness in law, we have that

$$\mathbb{P}^0\Big(\tau(-X) \geq \rho\Big) \;\geq\; 3/4,$$

which is clearly impossible. To show the validity of Lemma 4.2 in $x = 0$, we consider a a sequence $\{\{X^n_t\}_{t\geq 0} \colon n \in \mathbb{N}\}$ of solutions of SDE (4.5) starting at $1/n$. It is then easy to check that such a sequence satisfies the Aldous criterion:

$$\mathbb{E}[|X^n_t - X^n_0|^p] \;\leq\; ct^{p\gamma}, \quad t \geq 0$$

for some $p > 0$ and $c > 0$ independent from t and n. It follows (Proposition 34.8 in [Bas11]) that the sequence $\{\mathbb{P}^n\}_{n\in\mathbb{N}}$ of the laws of $\{X^n_t\}_{t\geq 0}$ is tight. Prohorov Theorem (cf. Theorem 30.4 in [Bas11]) ensures now the existence of a converging sub-sequence $\{\mathbb{P}^{n_k}\}_{k\in\mathbb{N}}$. The uniqueness in law then implies that the sequence $\{\mathbb{P}^{n_k}\}_{k\in\mathbb{N}}$ converges, as expected, to $\mathbb{P}^0$ the law of the solution starting at 0. Noticing that inequality (4.6) holds for any solution $\{X^n_t\}_{t\geq 0}$ and moreover, the constant ρ is independent from the starting points $1/n$, we find that

$$\mathbb{P}\Big(\tau(X) \geq \rho\Big) \;\geq\; 3/4.$$

The proof of Proposition 4.1 is thus concluded.

5 Appendix: proofs of complementary results

5.1 Controls on the density of the proxy process

We present here two useful lemmas needed to complete the proof of Proposition 2.11. We will analyze the behavior of the laws of the independent random variables $\tilde{M}^{\tau,\xi,t,s}$ and $\tilde{N}^{\tau,\xi,t,s}$ obtained in (2.20) by truncation of the process $\tilde{S}^{\tau,\xi,t,s}$ at the associated stable time scale $u^{1/\alpha}$.

Lemma 5.1. *Let m be in $\mathbb{N}$. Then, there exists a positive constant $C := C(m,T)$ such that for any k in $[\![0,m]\!]$,*

$$\left|D^k_z p_{\tilde{M}^{\tau,\xi,t,s}}(u,z)\right| \;\leq\; Cu^{-(N+k)/\alpha}\left(1+\frac{|z|}{u^{1/\alpha}}\right)^{-m} \;=:\; Cu^{-k/\alpha}p_{\overline{M}}(u,z),$$

for any $u > 0$, any z in $\mathbb{R}^N$, any $t \leq s$ in $[0,T]$ and any (τ,ξ) in $[0,T]\times\mathbb{R}^N$.

Proof. Similarly to the proof of Proposition 2.10 (see in particular Equation (2.16)), we start writing

$$p_{\tilde{M}^{\tau,\xi,t,s}}(u,z) \;=\; \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{-i\langle z,y\rangle}\exp\left(u\int_{|p|\leq u^{1/\alpha}}\left[\cos(\langle y,p\rangle)-1\right]\,\nu_{\tilde{S}^{\tau,\xi,t,s}}(dp)\right)dy,$$

where, we recall, $\nu^{\tau,\xi,t,s}_{\tilde{S}}$ is the Lévy measure associated with the process $\{\tilde{S}^{\tau,\xi,t,s}_u\}_{u\geq 0}$ in Proposition 2.10. Setting $u^{1/\alpha}y = \tilde{y}$ then yields

$$\begin{aligned} p_{\tilde{M}^{\tau,\xi,t,s}}(u,z) \;&=\; \frac{u^{-N/\alpha}}{(2\pi)^N}\int_{\mathbb{R}^N} e^{-i\langle z,\frac{\tilde{y}}{u^{1/\alpha}}\rangle}\exp\left(u\int_{|p|\leq u^{1/\alpha}}\left[\cos(\langle\tilde{y},\frac{p}{u^{1/\alpha}}\rangle)-1\right]\nu_{\tilde{S}^{\tau,\xi,t,s}}(dp)\right)d\tilde{y} \\ &=:\; \frac{u^{-N/\alpha}}{(2\pi)^N}\int_{\mathbb{R}^N} e^{-i\langle\frac{z}{u^{1/\alpha}},\tilde{y}\rangle}\,\hat{f}^{\tau,\xi,t,s}_u(\tilde{y})\,d\tilde{y} \end{aligned} \tag{5.1}$$

Since the Lévy measure $\nu_{\tilde S}^{\tau,\xi,t,s}$ in the expression above has finite support, Theorem 3.7.13 in Jacob [Jac01] implies that $\hat f_u^{\tau,\xi,t,s}$ is infinitely differentiable in $\tilde y$. We can thus calculate

$$\begin{aligned}|\partial_{\tilde y}\hat f_u^{\tau,\xi,t,s}(\tilde y)| \,\le\, & u\int_{|p|\le u^{1/\alpha}}\frac{|p|}{u^{1/\alpha}}\left|\sin\left(\left\langle \tilde y,\frac{p}{u^{1/\alpha}}\right\rangle\right)\right|\,\nu_{\tilde S^{\tau,\xi,t,s}}(dp)\\ &\times\exp\left(u\int_{|p|\le u^{1/\alpha}}\left[\cos\left(\left\langle\frac{\tilde y}{u^{1/\alpha}},p\right\rangle\right)-1\right]\,\nu_{\tilde S^{\tau,\xi,t,s}}(dp)\right).\end{aligned}$$

Recalling that $\alpha>1$, we can now write that

$$\begin{aligned}u\int_{|p|\le u^{1/\alpha}}\frac{|p|}{u^{1/\alpha}}\left|\sin\left(\left\langle \tilde y,\frac{p}{u^{1/\alpha}}\right\rangle\right)\right|\,\nu_{\tilde S^{\tau,\xi,t,s}}(dp) \,&\le\, Cu\int_{r\le u^{1/\alpha}}\frac{r}{u^{1/\alpha}}\frac{|\tilde y|r}{u^{1/\alpha}}\frac{dr}{r^{1+\alpha}}\\ &\le\, Cu\int_{r\le u^{1/\alpha}}|\tilde y|\frac{r^{1-\alpha}}{u^{2/\alpha}}\,dr\\ &\le\, C(1+|\tilde y|).\end{aligned}$$

It then follows that

$$\begin{aligned}&|\partial_{\tilde y}\hat f_u^{\tau,\xi,t,s}(\tilde y)|\\ &\quad\le\, C(1+|\tilde y|)\exp\left(u\int_{\mathbb R^N}\left[\cos\left(\left\langle\frac{\tilde y}{u^{1/\alpha}},p\right\rangle\right)-1\right]\,\nu_{\tilde S^{\tau,\xi,t,s}}(dp)\right)e^{2u\nu_{\tilde S^{\tau,\xi,t,s}}(B^c(0,u^{1/\alpha}))}\\ &\quad\le\, C(1+|\tilde y|)\exp(-C^{-1}|\tilde y|^\alpha),\end{aligned}$$

where in second inequality we exploited Control (2.15) and

$$\nu_{\tilde S^{\tau,\xi,t,s}}(B^c(0,u^{1/\alpha}))\le C/u. \tag{5.2}$$

Iterating the above reasoning, we can then show that for any l in $\mathbb N$,

$$|\partial^l_{\tilde y}\hat f_u^{\tau,\xi,t,s}(\tilde y)|\,\le\, C_l(1+|\tilde y|^l)\exp(-C^{-1}|\tilde y|^\alpha),$$

for some positive constant $C:=C(l)$. It implies in particular that $\hat f_u^{\tau,\xi,t,s}(\tilde y)$ is a Schwartz test function. Denoting by $f_u^{\tau,\xi,t,s}$ its inverse Fourier transform, we thus have that for any m in $\mathbb N$, there exists a positive constant $C:=C(m)$ such that

$$|f_u^{\tau,\xi,t,s}(y)|\le C_m(1+|y|)^{-m},\quad y\in\mathbb R^N.$$

The result for $k=0$ now follows immediately noticing that

$$p_{\overline M}(t-s,y)\,=\,(t-s)^{-\frac{d}{\alpha}}f_{s,t}(y/(t-s)^{\frac1\alpha}).$$

The controls on the derivatives can be derived analogously. □

We can now show a similar control on the law of the process $\tilde N^{\tau,\xi,t,s}$.

Lemma 5.2. *There exists a family $\{\overline P_u\}_{u\ge0}$ of Poisson measures and a positive constant $C:=C(T,N)$ such that for any $\mathcal A$ in $\mathcal B(\mathbb R^N)$ and $\tilde N^{\tau,\xi,t,s}$ as in (2.21),*

$$P_{\tilde N_u^{\tau,\xi,t,s}}(\mathcal A)\,\le\, C\overline P_u(\mathcal A). \tag{5.3}$$

Proof. For notational simplicity, we start introducing the truncated Lévy measure associated with the big jumps of the process $\{\tilde{S}^{\tau,\xi,t,s}_u\}_{u\geq 0}$:

$$\nu^{\tau,\xi,t,s}_{\mathrm{tr}}(dp) \,=\, \mathbb{1}_{|p|\geq u^{1/\alpha}}(p)\nu^{\tau,\xi,t,s}_{\tilde{S}}(dp).$$

It follows immediately that $\nu^{\tau,\xi,t,s}_{\mathrm{tr}}$ is a finite measure (see (5.2) above). With this notation at hand, we can write:

$$\begin{aligned}\widehat{P_{\tilde{N}^{\tau,\xi,t,s}_u}}(y) \,&=\, \exp\left(u\int_{|p|>u^{\frac{1}{\alpha}}}\left[\cos(\langle y,p\rangle)-1\right]\,\nu^{\tau,\xi,t,s}_{\tilde{S}}(dp)\right)\\ &=\, \exp\left(u\widehat{\nu^{\tau,\xi,t,s}_{\mathrm{tr}}}(y)-u\nu^{\tau,\xi,t,s}_{\mathrm{tr}}(\mathbb{R}^N)\right),\end{aligned}$$

where $\widehat{\nu}$ denotes the Fourier-Stieltjes transform of the considered measure ν. Let us introduce then the following measure:

$$\zeta^{\tau,\xi,t,s} \,:=\, u\nu^{\tau,\xi,t,s}_{\mathrm{tr}}.$$

Expanding the previous exponential and by termwise Fourier inversion, we now find that

$$P_{\tilde{N}^{\tau,\xi,t,s}_u} \,=\, \exp\left(\zeta^{\tau,\xi,t,s}-u\nu^{\tau,\xi,t,s}_{\mathrm{tr}}(\mathbb{R}^N)\right) \,=\, \exp\left(-u\nu^{\tau,\xi,t,s}_{\mathrm{tr}}(\mathbb{R}^N)\right)\sum_{n\in\mathbb{N}}\frac{\left(\zeta^{\tau,\xi,t,s}\right)^{\star n}}{n!}, \tag{5.4}$$

where, for a finite measure ρ on $\mathbb{R}^N$, $(\rho)^{\star n} := \rho\star\cdots\star\rho$ denotes its n^{th} fold convolution. For now, let us assume that $\sigma(t,x)$ is non-constant in space, so that

$$B\tilde{\sigma}^{\tau,\xi}_{u(v)} \,=\, B\sigma\left(u(v),\theta_{u(v),\tau}(\xi)\right)$$

appearing in the definition of $\nu^{\tau,\xi,t,s}_{\tilde{S}}$, truly depends on the parameters τ,ξ. Assumption [**AC**] then ensures the existence of a bounded function $g\colon\mathbb{S}^{d-1}\to\mathbb{R}$ such that

$$\nu(dp) \,=\, Q(p)\frac{g(\frac{p}{|p|})}{|p|^{d+\alpha}}dp.$$

From Equation (5.4), it is clear that we need to control the measure $\zeta^{\tau,\xi,t,s}$, uniformly in the parameters τ,ξ,t,s. Namely, for any $\mathcal{A}$ in $\mathcal{B}(\mathbb{R}^N)$, we write from (2.13) that

$$\begin{aligned}\zeta^{\tau,\xi,t,s}(\mathcal{A}) \,&=\, u\int_{|p|>u^{\frac{1}{\alpha}}}\mathbb{1}_{\mathcal{A}}(p)\,\nu^{\tau,\xi,t,s}_{\tilde{S}}(dp) \,=\, u\int_0^1\int_{|\widehat{\mathcal{R}}_vB\tilde{\sigma}^{\tau,\xi}_{u(v)}p|>u^{\frac{1}{\alpha}}}\mathbb{1}_{\mathcal{A}}(\widehat{\mathcal{R}}_vB\tilde{\sigma}^{\tau,\xi}_{u(v)}p)\,\nu(dp)dv\\ &= u\int_0^1\int_{|\widehat{\mathcal{R}}_vB\tilde{\sigma}^{\tau,\xi}_{u(v)}p|>u^{\frac{1}{\alpha}}}\mathbb{1}_{\mathcal{A}}(\widehat{\mathcal{R}}_vB\tilde{\sigma}^{\tau,\xi}_{u(v)}p)\frac{g(\frac{p}{|p|})}{|p|^{d+\alpha}}Q(p)\,dpdv\\ &\leq\, u\int_0^1\int_{|\widehat{\mathcal{R}}_vB\tilde{\sigma}^{\tau,\xi}_{u(v)}p|>u^{\frac{1}{\alpha}}}\mathbb{1}_{\mathcal{A}}(\widehat{\mathcal{R}}_vB\tilde{\sigma}^{\tau,\xi}_{u(v)}p)\frac{dp}{|p|^{d+\alpha}}dv\end{aligned}$$

We can then exploit assumption [**UE**] on σ to conclude that

$$\begin{aligned}\zeta^{\tau,\xi,t,s}(\mathcal{A}) \,&\leq\, u\int_0^1\int_{|\widehat{\mathcal{R}}_vBq|>u^{\frac{1}{\alpha}}}\mathbb{1}_{\mathcal{A}}(\widehat{\mathcal{R}}_vBq)\frac{1}{\det(\tilde{\sigma}^{\tau,\xi}_{u(v)})}\frac{dq}{|(\tilde{\sigma}^{\tau,\xi}_{u(v)})^{-1}q|^{d+\alpha}}dv\\ &\leq\, Cu\int_0^1\int_{|\widehat{\mathcal{R}}_vBq|>u^{\frac{1}{\alpha}}}\mathbb{1}_{\mathcal{A}}(\widehat{\mathcal{R}}_vBq)\frac{dq}{|q|^{d+\alpha}}dv.\end{aligned}$$

Denoting now by $\Lambda_{\mathrm{tr}} := c\mathbb{1}_{p>u^{1/\alpha}}\frac{dp}{p^{d+\alpha}}$ the truncated Lévy measure of the isotropic α-stable process and by $\overline{\nu}_{\mathrm{tr}}$ the following push-forward measure

$$\overline{\nu}_{\mathrm{tr}}(\mathcal{A}) := \int_0^1 \Lambda_{\mathrm{tr}}\left((\widehat{\mathcal{R}}_v B)^{-1}\mathcal{A}\right) dv, \quad \mathcal{A} \in \mathcal{B}(\mathbb{R}^N)$$

we derive that there exists a constant C such that for any (τ,ξ) in $[0,T]\times\mathbb{R}^N$, $t\le s$ in $[0,T]$,

$$\zeta^{\tau,\xi,t,s}(\mathcal{A}) \le Cu\int_0^1 \Lambda_{\mathrm{tr}}\left((\widehat{\mathcal{R}}_v B)^{-1}\mathcal{A}\right) dv = u\overline{\nu}_{\mathrm{tr}}(\mathcal{A}) =: \overline{\zeta}(\mathcal{A}). \tag{5.5}$$

Equation (5.3) now follows from the above control, (5.2) and (5.4), denoting

$$\overline{P}_u := \exp\left(-u\overline{\nu}_{\mathrm{tr}}(\mathbb{R}^N)\right)\sum_{n\in\mathbb{N}}\frac{(\overline{\zeta})^{\star n}}{n!},$$

up to a modification of the constant C in (5.5). Following backwards the same reasoning presented at the beginning of the proof, we then notice that

$$\begin{aligned}
\widehat{\overline{P}_u}(y) &= \exp\left(u\int_0^1\int_{\mathbb{R}^N}\left[\cos(\langle y,p\rangle)-1\right]\overline{\nu}_{\mathrm{tr}}(dp)dv\right)\\
&= \exp\left(u\int_0^1\int_{\mathbb{R}^d}\mathbb{1}_{\{|\widehat{\mathcal{R}}_v Bp|>u^{\frac{1}{\alpha}}\}}\left[\cos(\langle y,\widehat{\mathcal{R}}_v Bp\rangle)-1\right]\Lambda(dp)dv\right)\\
&= \exp\left(u\int_0^1\int_0^\infty\int_{\mathbb{S}^{d-1}}\mathbb{1}_{\{|\widehat{\mathcal{R}}_v B\theta r|>u^{\frac{1}{\alpha}}\}}\left[\cos(\langle y,\widehat{\mathcal{R}}_v B\theta r\rangle)-1\right]\mu_{\mathrm{leb}}(d\theta)\frac{dr}{r^{1+\alpha}}dv\right),
\end{aligned}$$

where we used the spherical decomposition for the Lévy measure Λ of an isotropic α-stable process:

$$\Lambda(dp) := \frac{dp}{p^{d+\alpha}} = C\mu_{\mathrm{leb}}(d\theta)\frac{dr}{r^{1+\alpha}}, \tag{5.6}$$

with $p=r\theta$ and μ_{leb} Lebesgue measure on the sphere $\mathbb{S}^{d-1}$.
We exploit now the non-degeneracy of $\widehat{\mathcal{R}}_v$ to to define two functions $k\colon[0,1]\times\mathbb{S}^{d-1}\to\mathbb{R}$ and $l\colon[0,1]\times\mathbb{S}^{d-1}\to\mathbb{S}^{N-1}$, given by

$$k(v,\theta) := |\widehat{\mathcal{R}}_v B\theta| \quad\text{and}\quad l(v,\theta) := \frac{\widehat{\mathcal{R}}_v B\theta}{|\widehat{\mathcal{R}}_v B\theta|}.$$

Using the Fubini theorem, we can now write that

$$\begin{aligned}
&\widehat{\overline{P}_u}(y)\\
&= \exp\left(u\int_0^1\int_0^\infty\int_{\mathbb{S}^{d-1}}\mathbb{1}_{\{|l(v,\theta)k(v,\theta)r|>u^{\frac{1}{\alpha}}\}}\left[\cos\left(\langle z,l(v,\theta)k(v,\theta)r\rangle\right)-1\right]\mu_{\mathrm{leb}}(d\theta)\frac{dr}{r^{1+\alpha}}dv\right)\\
&= \exp\left(u\int_0^1\int_0^\infty\int_{\mathbb{S}^{d-1}}\mathbb{1}_{\{|l(v,\theta)\tilde{r}|>u^{\frac{1}{\alpha}}\}}\left[\cos\left(\langle z,l(v,\theta)\tilde{r}\rangle\right)-1\right][k(v,\theta)]^\alpha\mu_{\mathrm{leb}}(d\theta)\frac{d\tilde{r}}{\tilde{r}^{1+\alpha}}dv\right).
\end{aligned}$$

Denoting now by $\tilde{k}(dv,d\theta)$ the measure on $[0,1]\times\mathbb{S}^{d-1}$ given by

$$\tilde{k}(dv,d\theta) := [k(v,\theta)]^\alpha\mu_{\mathrm{leb}}(d\theta)dv$$

and by $\tilde{\mu}_{\text{sym}} := \text{Sym}(l)_*\tilde{k}$ the symmetrization of the measure $\tilde{k}(dv, d\theta)$ push-forwarded through l on $\mathbb{S}^{N-1}$, we can finally conclude that

$$\begin{aligned}
\widehat{\overline{P}_u}(y) &= \exp\left(u\int_0^\infty \int_{[0,1]\times\mathbb{S}^{d-1}} \mathbb{1}_{\{|l(v,\theta)\tilde{r}|>u^{\frac{1}{\alpha}}\}} \left[\cos\left(\langle z, l(v,\theta)\tilde{r}\rangle\right)-1\right] \tilde{k}(dv,d\theta)\frac{d\tilde{r}}{\tilde{r}^{1+\alpha}}\right) \\
&= \exp\left(u\int_{|u|^{\frac{1}{\alpha}}}^\infty \int_{\mathbb{S}^{N-1}} \left[\cos\left(\langle z, \tilde{\theta}\tilde{r}\rangle\right)-1\right] \tilde{\mu}_{\text{sym}}(d\tilde{\theta})\frac{d\tilde{r}}{\tilde{r}^{1+\alpha}}\right). \qquad (5.7)
\end{aligned}$$

It is easy to check now that the measure $\tilde{\mu}_{\text{sym}}$ is finite and non-degenerate in the sense of (1.6). This concludes the proof of our result under the additional assumption that ν is absolutely continuous with respect to the Lebesgue measure.
If this is not the case, assumption [**AC**] implies immediately that $\sigma(t,x) =: \sigma_t$ does not depends on x. Thus, the "frozen" diffusion $\tilde{\sigma}_t^{\tau,\xi}$ does not depends on the parameters τ, ξ as well. The same arguments above then allow to conclude in a similar manner. $\square$

Sketch of proof for Proposition 3.4 We briefly present here the proof of Proposition 3.4 concerning the existence and the associated controls for the density of the mollified frozen process $\tilde{X}_s^{\tau,\xi,t,x,\delta}$.
We start noticing that the reasoning in the proof of Proposition 2.10 can be similarly applied. Indeed, from the definition in (3.29), it follows immediately that

$$\tilde{X}_s^{\tau,\xi,t,x,\delta} = \tilde{m}_{s,t}^{\tau,\xi}(x) + \mathbb{M}_{s-t}\left(\tilde{S}_{s-t}^{\tau,\xi,t,s} + \delta\overline{Z}_{s-t}\right),$$

and thus, that there exists a density $\tilde{p}^{\tau,\xi,\delta}(t,s,x,y)$ associated with the frozen process $\tilde{X}_s^{\tau,\xi,t,x,\delta}$. Moreover, the representation in (2.10) holds again if we change there the Lévy measure $\nu_{\tilde{S}}^{\tau,\xi,t,s}$ with the one associated with the following Lévy symbol:

$$\Phi_{\tilde{S}^{\tau,\xi,t,s,\delta}}(z) := \Phi_{\tilde{S}^{\tau,\xi,t,s}}(z) + c_\alpha\delta|z|^\alpha = \int_0^1 \Phi\Big((\widehat{\mathcal{R}}_v B\tilde{\sigma}_{u(v)}^{\tau,\xi})^*z\Big)\,dv + c_\alpha\delta|z|^\alpha.$$

Namely, it holds that

$$\begin{aligned}
\tilde{p}^{\tau,\xi,\delta}(t,s,x,y) = \frac{\det\mathbb{M}_{s-t}^{-1}}{(2\pi)^N}\int_{\mathbb{R}^N} & e^{-i\langle\mathbb{M}_{s-t}^{-1}(y-\tilde{m}_{s,t}^{\tau,\xi}(x)),z\rangle} \\
&\times \exp\left((s-t)\int_{\mathbb{R}^N}\left[\cos(\langle z,p\rangle)-1\right]\nu_{\tilde{S}^{\tau,\xi,t,s,\delta}}(dp)\right)\,dz,
\end{aligned}$$

where the Lévy measure $\nu_{\tilde{S}^{\tau,\xi,t,s,\delta}}$ is given by

$$\nu_{\tilde{S}^{\tau,\xi,t,s,\delta}}(\mathcal{A}) = \nu_{\tilde{S}^{\tau,\xi,t,s}}(A) + \delta^\alpha\nu_{\overline{Z}}(\mathcal{A}), \quad \mathcal{A}\in\mathcal{B}(\mathbb{R}^N), \qquad (5.8)$$

with $\nu_{\overline{Z}}$ Lévy measure of the isotropic α-stable process $\overline{Z}_t$. In particular, the Lévy symbol $\Phi_{\tilde{S}^\delta}$ satisfies Control (2.15) for a constant C independent from δ.
We can now move to show the controls on the derivatives of the mollified frozen density. It is not difficult to check that the arguments presented in the proofs of Proposition 2.11, Lemmas 5.1 and 5.2 can be applied again if we substitute there the Lévy measure $\nu_{\tilde{S}^{\tau,\xi,t,s}}$ with the mollified one $\nu_{\tilde{S}^{\tau,\xi,t,s,\delta}}$. Indeed, taking into account the decomposition in (5.8), we notice that the Lévy measure $\nu_{\tilde{S}^{\tau,\xi,t,s,\delta}}$ only considers an additional term

$(\delta\nu_{\overline{Z}})$ that has the same α-scaling nature considered before (but is however much less singular).
To show instead that the estimates (3.30) are indeed uniform in the parameter δ, it is sufficient to notice from (5.8) that we have that

$$\nu_{\tilde{S}^{\tau,\xi,t,s,\delta}}(\mathcal{A}) \,\le\, \nu_{\tilde{S}^{\tau,\xi,t,s}}(\mathcal{A}) + \nu_{\overline{Z}}(\mathcal{A}), \quad \mathcal{A} \in \mathcal{B}(\mathbb{R}^N).$$

To conclude the proof of Proposition 3.4, it is then enough to take $\xi = y$, $\tau = s$ and to follow the same arguments introduced in the proof of Corollary 2.16.

5.2 Proof of the technical lemmas

Proof of Lemma 2.14 (Approximate Lipschitz condition of the flows) We start considering two measurable flows $\theta, \check{\theta}$ satisfying dynamics (2.28). Recalling the decomposition $G(t,x) = A_t x + F(t,x)$, it follows immediately that:

$$\begin{aligned}\mathbb{T}^{-1}_{s-t}(x-\theta_{t,s}(y)) \,&=\, \mathbb{T}^{-1}_{s-t}\Big[\check{\theta}_{s,t}(x) - y - \int_t^s \Big(G(u,\check{\theta}_{u,t}(x)) - G(u,\theta_{u,s}(y))\Big)\,du\Big] \\ &=\, \mathbb{T}^{-1}_{s-t}\Big(\check{\theta}_{s,t}(x) - y\Big) + \mathcal{I}_{s,t}(x,y),\end{aligned} \tag{5.9}$$

where in the last step, we denoted

$$\mathcal{I}_{s,t}(x,y) \,=\, \mathbb{T}^{-1}_{s-t}\int_t^s \Big[A_u\Big(\theta_{u,s}(y) - \check{\theta}_{u,t}(x)\Big) + \Big(F(u,\theta_{u,s}(y)) - F(u,\check{\theta}_{u,t}(x))\Big)\Big]\,du.$$

To conclude, we need to show the following bound for $\mathcal{I}_{s,t}(x,y)$:

$$|\mathcal{I}_{s,t}(x,y)| \,\le\, C\left[1 + (s-t)^{-1}\int_t^s |\mathbb{T}^{-1}_{s-t}(\check{\theta}_{u,t}(x) - \theta_{u,s}(y))|\,du\right]. \tag{5.10}$$

Indeed, Control (5.10) together with (5.9) and the Gronwall lemma imply the right-hand side of Control (2.29). The left-hand side one can be obtained analogously and we will not show it here.
We start decomposing $\mathcal{I}_{s,t}$ into $\mathcal{I}^1_{s,t} + \mathcal{I}^2_{s,t}$, where we denote

$$\begin{aligned}\mathcal{I}^1_{s,t}(x,y) \,&:=\, \mathbb{T}^{-1}_{s-t}\int_t^s A_u\Big(\theta_{u,s}(y) - \check{\theta}_{u,t}(x)\Big)\,du; \\ \mathcal{I}^2_{s,t}(x,y) \,&:=\, \mathbb{T}^{-1}_{s-t}\int_t^s \Big[F(u,\theta_{u,s}(y)) - F(u,\check{\theta}_{u,t}(x))\Big]\,du.\end{aligned}$$

The first remainder $\mathcal{I}^1_{s,t}$ can be controlled easily, exploiting the linearity of $z \to A_u z$. Indeed, for any z,z' in $\mathbb{R}^N$ and any u in $[s,t]$, we have that

$$\begin{aligned}\Big|\mathbb{T}^{-1}_{s-t}A_u(z-z')\Big| \,&\le\, \sum_{i=1}^n \sum_{j=(i-1)\vee 1}^n (s-t)^{-\frac{1+\alpha(i-1)}{\alpha}}|A^{i,j}_u|\,|(z-z')_j| \\ &\le\, C(s-t)^{-1}|\mathbb{T}^{-1}_{s-t}(z-z')|.\end{aligned} \tag{5.11}$$

To control instead the second term $\mathcal{I}^2_{s,t}$, we will need to thoroughly exploit an appropriate smoothing method, due to the low regularity in space of the drift F. To overcome this problem, we are going to mollify the function F in the following way. We start fixing a family $\{\rho_i\colon i\in [\![1,n]\!]\}$ of mollifiers on $\mathbb{R}^{D_i}$ where $D_i = N-\sum_{j=1}^{i-1} d_j$, i.e. for any i in $[\![1,n]\!]$, ρ_i is a compactly supported, non-negative, smooth function on $\mathbb{R}^{D_i}$ such that $\|\rho_i\|_{L^1}=1$, and a family $\{\delta_{ij}\colon i\le j\}$ of positive constants to be chosen later. Then, the mollified version of the drift is defined by $F^\delta := (F_1, F_2^\delta,\dots,F_n^\delta)$ where

$$
\begin{aligned}
F_i^\delta(t,z) \;&:=\; F_i *_x \rho_i^\delta(t,z)\\
&:=\; \int_{\mathbb{R}^{D_i}} F_i(t,z_i-\omega_i,\dots,z_n-\omega_n)\frac{1}{\prod_{j=i}^n \delta_{ij}^{d_i}}\rho_i(\frac{\omega_i}{\delta_{ii}},\dots,\frac{\omega_n}{\delta_{in}})\,d\omega.
\end{aligned}
\tag{5.12}
$$

Roughly speaking, we have mollified any component F_i by convolution in space with a mollifier with multi-scaled dilations. Then, standard results on mollifier theory and our current assumptions on F show us that the following controls hold

$$
|F_i(u,z)-F_i^\delta(u,z)| \;\le\; C\sum_{j=i}^n \delta_{ij}^{\beta^j}, \tag{5.13}
$$

$$
|F_i^\delta(u,z)-F_i^\delta(u,z')| \;\le\; C\sum_{j=i}^n \delta_{ij}^{\beta^j-1}|(z-z')_j|. \tag{5.14}
$$

We can now pick δ_{ij} for any $i\le j$ in $[\![2,n]\!]$ in order to have any contribution associated with the mollification appearing in (5.13) at a good current scale time. Namely, we would like δ_{ij} to satisfy

$$
\left|\mathbb{T}_{s-t}^{-1}\left(F(u,z)-F^\delta(u,z)\right)\right| \;\le\; C(s-t)^{-1}, \tag{5.15}
$$

for any u in $[t,s]$ and any z in $\mathbb{R}^N$. Using the mollifier controls (5.13), it is enough to ask for

$$
\sum_{i=2}^n (s-t)^{-\frac{1+\alpha(i-1)}{\alpha}}\sum_{j=i}^n \delta_{ij}^{\beta^j} \;\le\; C(s-t)^{-1}. \tag{5.16}
$$

This is true if we fix for example,

$$
\delta_{ij} \;=\; (s-t)^{\frac{1+\alpha(i-2)}{\alpha\beta^j}} \quad \text{for } i\le j \text{ in } [\![2,n]\!]. \tag{5.17}
$$

Next, we would like to show that, for our choice of the regularization parameter δ_{ij}, the mollified drift F^δ satisfies an *approximate* Lipschitz condition with a constant that, once the drift is integrated, does not yield any additional singularity. Namely, we want to derive the following control:

$$
\left|\mathbb{T}_{s-t}^{-1}\left(F^\delta(u,z)-F^\delta(u,z')\right)\right| \;\le\; C\left[(s-t)^{-\frac{1}{\alpha}}+(s-t)^{-1}|\mathbb{T}_{s-t}^{-1}(z-z')|\right]. \tag{5.18}
$$

To show it, we start noticing that F_1 is Hölder continuous with Hölder index $\beta^1>0$. By Young inequality, it then yields that there exists a positive constant C possibly depending on β^1 such that $|z|^{\beta^1}\le C(1+|z|)$ for any z in $\mathbb{R}^N$. It then follows from

Equation (5.14) that

$$
\begin{aligned}
&|\mathbb{T}_{s-t}^{-1}\Big(F^\delta(u,z) - F^\delta(u,z')\Big)| \\
&\qquad\qquad \le C\Big[(s-t)^{-\frac{1}{\alpha}}(1+|(z-z')|) + \sum_{i=2}^n\sum_{j=i}^n (s-t)^{-\frac{1+\alpha(i-1)}{\alpha}}\delta_{ij}^{\beta^j-1}|(z-z')_j|\Big] \\
&\qquad\qquad \le C\Big[(s-t)^{-\frac{1}{\alpha}} + |\mathbb{T}_{s-t}^{-1}(z-z')|\Big(1+\sum_{i=2}^n\sum_{j=i}^n \frac{(s-t)^{j-i}}{\delta_{ij}^{1-\beta^j}}\Big)\Big].
\end{aligned}
$$

Hence, Control (5.18) follows from the fact that, from our previous choice of δ_{ij}, one gets

$$
\frac{(s-t)^{j-i}}{\delta_{ij}^{1-\beta^j}} \;=\; (s-t)^{(j-i)-\frac{1+\alpha(i-2)}{\alpha\beta^j}(1-\beta^j)} \;\le\; C(s-t)^{-1}, \tag{5.19}
$$

recalling that we assumed $s-t$ to be small enough and since from the assumption (2.3) on the indexes of Hölder continuity β^j for F:

$$
\beta^j > \frac{1+\alpha(i-2)}{1+\alpha(j-1)} \Leftrightarrow (j-i) - \frac{1+\alpha(i-2)}{\alpha\beta^j}(1-\beta^j) \;>\; -1.
$$

We recall that the above inequality should precisely give the natural threshold, namely an exponent β_i^j satisfying this condition. The current choice for β^j is sufficient to ensure this bound holds for any $i \le j$ and is *sharp* for $i=j$. We can finally show the bound for the second remainder $\mathcal{I}_{s,t}^2(x,y)$ as given in (5.10). It holds that:

$$
\begin{aligned}
|\mathcal{I}_{s,t}^2(x,y)| \;&\le\; \int_t^s \Big|\mathbb{T}_{s-t}^{-1}\Big(F(u,\theta_{u,s}(y)) - F(u,\check{\theta}_{u,t}(x))\Big)\Big|\,du \\
&\le\; \int_t^s \Big|\mathbb{T}_{s-t}^{-1}(F(u,\check{\theta}_{u,t}(x)) - F^\delta(u,\check{\theta}_{u,t}(x)))\Big|\,du \\
&\qquad\qquad + \int_t^s \Big|\mathbb{T}_{s-t}^{-1}(F^\delta(u,\check{\theta}_{u,t}(x)) - F^\delta(u,\theta_{u,s}(y)))\Big|\,du \\
&\qquad\qquad\qquad + \int_t^s \Big|\mathbb{T}_{s-t}^{-1}\Big(F^\delta(u,\theta_{u,s}(y)) - F(u,\theta_{u,s}(y))\Big)\Big|\,du \\
&=:\; \mathcal{I}_{s,t}^{21}(x,y) + \mathcal{I}_{s,t}^{22}(x,y) + \mathcal{I}_{s,t}^{23}(x,y).
\end{aligned}
$$

From Control (5.13) with our choice of δ_{ij}, we easily obtain from Control (5.15) that there exists a positive constant $C := C(T)$ such that

$$
|\mathcal{I}_{s,t}^{21}(x,y)| + |\mathcal{I}_{s,t}^{23}(x,y)| \;\le\; C, \tag{5.20}
$$

for any $t \le s$ in $[0,T]$ and x,y in $\mathbb{R}^N$. On the other hand, we exploit (5.18) to derive that

$$
|\mathcal{I}_{s,t}^{22}(x,y)| \;\le\; C\left[1+\int_t^s (s-t)^{-1}|\mathbb{T}_{s-t}^{-1}(\check{\theta}_{u,t}(x) - \theta_{u,s}(y))|\,du\right]
$$

for any $t \le s$ in $[0,T]$ and x,y in $\mathbb{R}^N$. To conclude, we finally derive (5.10) from the last inequality together with Controls (5.11)-(5.20).

Proof of Lemma 2.18 (Dirac convergence of frozen density). Fixed (t,x) in $[0,T]\times\mathbb{R}^N$ and a bounded, continuous function $f\colon\mathbb{R}^N\to\mathbb{R}$, we want to show that the following limit

$$\lim_{\epsilon\to 0}\left|\int_{\mathbb{R}^N} f(y)\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)\,dy - f(x)\right| \,=\, 0$$

holds, uniformly in $t\in[0,T]$.
We start rewriting the argument of the limit in the following way:

$$\begin{aligned}\int_{\mathbb{R}^N} f(y)\tilde{p}^{t+\epsilon,y}&(t,t+\epsilon,x,y)\,dy - f(x) \qquad (5.21)\\ &= \int_{\mathbb{R}^N} f(y)\left[\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)-\tilde{p}^{t,x}(t,t+\epsilon,x,y)\right]\,dy\\ &\qquad + \int_{\mathbb{R}^N} f(y)\tilde{p}^{t,x}(t,t+\epsilon,x,y)\,dy - f(x).\end{aligned}$$

By Proposition 2.10, we know that the second term in (5.21) tends to zero, uniformly in t in $[0,T]$ (scaling property of the upper bound for the density), when ϵ goes to zero. We can then focus on the first one. We start splitting the space $\mathbb{R}^N$ in the diagonal/off-diagonal regime associated with our anisotropic dynamics. Namely, we fix $\beta>0$ to be chosen later and we consider the following subsets:

$$\begin{aligned}D_1 &:= \{y\in\mathbb{R}^N\colon \left|\mathbb{T}_\epsilon^{-1}(y-\theta_{t+\epsilon,t}(x))\right| \,\le\, \epsilon^{-\beta}\};\\ D_2 &:= \{y\in\mathbb{R}^N\colon \left|\mathbb{T}_\epsilon^{-1}(y-\theta_{t+\epsilon,t}(x))\right| \,>\, \epsilon^{-\beta}\},\end{aligned}$$

where $\mathbb{T}_\epsilon$ was defined in (2.17). We can then decompose the first term in (5.21) in the following way:

$$\begin{aligned}\Big|\int_{\mathbb{R}^N} f(y)&\Big[\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)-\tilde{p}^{t,x}(t,t+\epsilon,x,y)\Big]\,dy\Big|\\ &\le \|f\|_\infty\int_{D_1}\left|\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)-\tilde{p}^{t,x}(t,t+\epsilon,x,y)\right|dy\\ &\qquad + \|f\|_\infty\int_{D_2}\left|\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)-\tilde{p}^{t,x}(t,t+\epsilon,x,y)\right|dy\\ &=: \|f\|_\infty\left(\mathcal{D}_1+\mathcal{D}_2\right)(t,t+\epsilon,x). \qquad (5.22)\end{aligned}$$

We will follow different approaches to control the two terms $\mathcal{D}_1$, $\mathcal{D}_2$. In the off-diagonal regime D_2, the idea is to exploit tail estimates of the single densities while in the diagonal one D_1, a more thorough sensibility analysis between the spectral measures and the Fourier transform is needed. Let us consider first the off-diagonal term $\mathcal{D}_2$. We can write that

$$\begin{aligned}\mathcal{D}_2(t,t+\epsilon,x,y) &\le \int_{D_2}\left|\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)\right| + \left|\tilde{p}^{t,x}(t,t+\epsilon,x,y)\right|\,dy\\ &\le \int_{D_2}\frac{1}{\det\mathbb{T}_\epsilon}\Big(\bar{p}(1,\mathbb{T}_\epsilon^{-1}(x-\theta_{t,t+\epsilon}(y)))+\bar{p}(1,\mathbb{T}_\epsilon^{-1}(\theta_{t+\epsilon,t}(x)-y))\Big)dy\end{aligned}$$

using Proposition 2.11 together with Lemma 2.9 for the last inequality. From Lemma 2.14 (to use the *approximate* Lipschitz property of the flows) and introducing

$$\bar{D}_2 \,:=\, \{y\in\mathbb{R}^N\colon \left|\mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y)-x)\right| \,>\, \frac{1}{2}\epsilon^{-\beta}\},$$

we thus deduce that for ϵ small enough we get:

$$\mathcal{D}_2(t,t+\epsilon,x,y) \le \int_{\bar{D}_2} \frac{1}{\det \mathbb{T}_\epsilon} \bar{p}(1,\mathbb{T}_\epsilon^{-1}(x-\theta_{t,t+\epsilon}(y)))\,dy \\ + \int_{D_2} \frac{1}{\det \mathbb{T}_\epsilon} \bar{p}(1,\mathbb{T}_\epsilon^{-1}(\theta_{t+\epsilon,t}(x)-y))\,dy.$$

Using now Equation (2.36) from Corollary 2.16 for the first integral and the direct change of variable $z = \mathbb{T}_\epsilon^{-1}(y-\theta_{t+\epsilon,t}(x))$ for the second, we can conclude that

$$\mathcal{D}_2(t,t+\epsilon,x) \le C \int_{\mathbb{R}^N} \mathbb{1}_{B^c(0,\frac{1}{2}\epsilon^{-\beta})}(z)(\check{p}+\bar{p})(1,z)\,dz,$$

where $\check{p}$ is a density enjoying the same integrability properties as $\bar{p}$.
By dominated convergence theorem, it is easy to notice that $\mathcal{D}_2(t,t+\epsilon,x)$ tends to zero if ϵ goes to zero, uniformly in the time variable t in $[0,T]$.
We can now focus on the diagonal term $\mathcal{D}_1$ appearing in (5.22). We start recalling from Equation (2.16) that the density $\tilde{p}^\omega(t,s,x,y)$ (for $\omega \in \{(t,x),(t+\epsilon,y)\}$) can be written as

$$\tilde{p}^\omega(t,t+\epsilon,x,y) = \frac{\det \mathbb{M}_\epsilon^{-1}}{(2\pi)^N} \int_{\mathbb{R}^N} e^{\mathcal{F}_{t,t+\epsilon}(z,\omega)} \exp\Big(-i\langle \mathbb{M}_\epsilon^{-1}(y-\tilde{m}^\omega_{t+\epsilon,t}(x)),z\rangle\Big)\,dz,$$

where we have denoted:

$$\mathcal{F}_\epsilon(t,z,\omega) := \epsilon \int_0^1 \int_{\mathbb{R}^d} \Big[\cos\Big(\langle z, \widehat{\mathcal{R}}_v B \tilde{\sigma}^\omega_{u(v)} p\rangle\Big) - 1\Big]\,\nu(dp)dv,$$

with $u(v) = t+\epsilon v$ (cf. notations in (2.12) of Proposition 2.10) and $\Phi(p)$ the Lévy symbol of the process $\{Z_t\}_{t\ge 0}$. We can now consider the two following terms

$$\mathcal{P}_1(t,t+\epsilon,x,y) := \frac{\det \mathbb{M}_\epsilon^{-1}}{(2\pi)^N} \int_{\mathbb{R}^N} \Big[e^{\mathcal{F}_\epsilon(t,z,t,x)} - e^{\mathcal{F}_\epsilon(t,z,t+\epsilon,y)}\Big] e^{-i\langle \mathbb{M}_\epsilon^{-1}(y-\tilde{m}^{t,x}_{t+\epsilon,t}(x)),z\rangle}\,dz$$

$$\mathcal{P}_2(t,t+\epsilon,x,y) \\ := \frac{\det \mathbb{M}_\epsilon^{-1}}{(2\pi)^N} \int_{\mathbb{R}^N} e^{\mathcal{F}_\epsilon(t,z,t+\epsilon,y)} \Big[e^{-i\langle \mathbb{M}_\epsilon^{-1}(y-\tilde{m}^{t,x}_{t+\epsilon,t}(x)),z\rangle} - e^{-i\langle \mathbb{M}_\epsilon^{-1}(y-\tilde{m}^{t+\epsilon,y}_{t+\epsilon,t}(x)),z\rangle}\Big]\,dz$$

and decompose $\mathcal{D}_1$ as follows:

$$\mathcal{D}_1 = \int_{D_1} |\mathcal{P}_1(t,t+\epsilon,x,y)| + |\mathcal{P}_2(t,t+\epsilon,x,y)|\,dy.$$

To control the first term $\mathcal{P}_1$, we can exploit a Taylor expansion. Indeed,

$$|\mathcal{P}_1(t,t+\epsilon,x,y)| \\ \le \frac{C}{\det \mathbb{M}_\epsilon} \int_{\mathbb{R}^N} \int_0^1 |\mathcal{F}_\epsilon(t,z,t+\epsilon,y) - \mathcal{F}_\epsilon(t,z,t,x)|\, e^{\lambda \mathcal{F}_\epsilon(t,z,t+\epsilon,y)+(1-\lambda)\mathcal{F}_\epsilon(t,z,t,x)}\,d\lambda dz.$$

We then notice from (2.15) that

$$\mathcal{F}_\epsilon(t,z,\omega) \le C\epsilon[1-|z|^\alpha],$$

and thus, we obtain that

$$e^{\lambda \mathcal{F}_\epsilon(t,z,t+\epsilon,y)+(1-\lambda)\mathcal{F}_\epsilon(t,z,t,x)} \,\le\, e^{C\epsilon(1-|z|^\alpha)},$$

for some constant C independent from λ in $[0,1]$. From our non-degenerate structure, any linear combination of the symbols remains homogeneous to a non-degenerate symbol. Thus, we have that

$$|\mathcal{P}_1(t,t+\epsilon,x,y)| \,\le\, \frac{C}{\det \mathbb{M}_\epsilon}\int_{\mathbb{R}^N} |\mathcal{F}_\epsilon(t,z,t+\epsilon,y)-\mathcal{F}_\epsilon(t,z,t,x)|\, e^{C\epsilon(1-|z|^\alpha)}\,dz. \tag{5.23}$$

On the other hand, we can decompose the difference in absolute value in the following way:

$$\begin{aligned}|\mathcal{F}_\epsilon(t,z,t+\epsilon,y)&-\mathcal{F}_\epsilon(t,z,t,x)|\\
&\le\, \epsilon\int_0^1\Big|\int_{\mathbb{R}^d}\Big[\cos\Big(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t+\epsilon,y}_{u(v)}p\rangle\Big)-\cos\Big(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t,x}_{u(v)}p\rangle\Big)\Big]\,\nu(dp)\Big|dv\\
&\le\, \epsilon\int_0^1\Big|\Big(\Delta^{t,\epsilon,x,y}_s+\Delta^{t,\epsilon,x,y}_l\Big)(v,z)\Big|\,dv,\end{aligned} \tag{5.24}$$

where we denoted

$$\begin{aligned}\Delta^{t,\epsilon,x,y}_s(v,z) \,&=\, \int_{B(0,r_0)}\Big[\cos\Big(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t+\epsilon,y}_{u(v)}p\rangle\Big)-\cos\Big(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t,x}_{u(v)}p\rangle\Big)\Big]\,Q(p)\,\nu_\alpha(dy);\\
\Delta^{t,\epsilon,x,y}_l(v,z) \,&=\, \int_{B^c(0,r_0)}\Big[\cos\Big(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t+\epsilon,y}_{u(v)}p\rangle\Big)-\cos\Big(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t,x}_{u(v)}p\rangle\Big)\Big]\,Q(p)\,\nu_\alpha(dp),\end{aligned}$$

with r_0 defined in assumption [**ND**]. The term $\Delta^{t,\epsilon,x,y}_l$ involving the large jumps can be easily controlled using that $\sup_{p\in\mathbb{R}^d} Q(p)<\infty$:

$$\begin{aligned}|\Delta^{t,\epsilon,x,y}_l(v,z)| \,&\le\, \int_{B^c(0,r_0)}\Big|\cos\Big(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t+\epsilon,y}_{u(v)}p\rangle\Big)-\cos\Big(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t,x}_{u(v)}p\rangle\Big)\Big|\,\nu_\alpha(dp)\\
&\le\, C.\end{aligned} \tag{5.25}$$

To bound the term $\Delta^{t,\epsilon,x,y}_s$ associated with the small jumps, we want to exploit instead that Q is Lipschitz continuous on $B(0,r_0)$. For this reason, we write that

$$\begin{aligned}&|\Delta^{t,\epsilon,x,y}_s(v,z)|\\
&\quad\le\,\Big|\int_{B(0,r_0)}\Big[\cos\Big(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t+\epsilon,y}_{u(v)}p\rangle\Big)-\cos\Big(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t,x}_{u(v)}p\rangle\Big)\Big]\,[Q(p)-Q(0)]\,\nu_\alpha(dp)\Big|\\
&\qquad\qquad+\Big|\int_{B(0,r_0)}\Big[\cos\Big(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t+\epsilon,y}_{u(v)}p\rangle\Big)-\cos\Big(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t,x}_{u(v)}p\rangle\Big)\Big]\,Q(0)\,\nu_\alpha(dp)\Big|\\
&\quad=:\,\Big(\Delta^{t,\epsilon,x,y}_{s,1}+\Delta^{t,\epsilon,x,y}_{s,2}\Big)(v,z).\end{aligned} \tag{5.26}$$

Since Q and the cosine function are Lipschitz continuous in a neighborhood of 0, we have that

$$\begin{aligned}\Delta^{t,\epsilon,x,y}_{s,1}(v,z) \,&\le\, C\int_{B(0,r_0)}|p||z|\,\Big|\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t+\epsilon,y}_{u(v)}p-\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t,x}_{u(v)}p\Big|\,\nu_\alpha(dp)\\
&\le\, C\int_{B(0,r_0)}|p||z|\,\Big|\sigma(u(v),\theta_{u(v),t+\epsilon}(y))p-\sigma(u(v),\theta_{u(v),t}(x))p\Big|\,\nu_\alpha(dp)\\
&\le\, C|z|\int_{B(0,r_0)}|p|^2\,\nu_\alpha(dp)\,\le\, C|z|,\end{aligned} \tag{5.27}$$

where in the last step, we used that the diffusion coefficient σ is bounded (cf. assumption [**UE**]).
The control of the other term $\Delta_{s,2}^{t,\epsilon,x,y}$ now follows from the classical characterization of the Lévy symbol of a non-degenerate α-stable process (see e.g. [Sat13]). Indeed,

$$\begin{aligned}
\Delta_{s,2}^{t,\epsilon,x,y}(v,z) \;&=\; \left|\int_{\mathbb{R}^d}\left[\cos\left(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t+\epsilon,y}_{u(v)}p\rangle\right)-1\right]-\left[\cos\left(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t,x}_{u(v)}p\rangle\right)-1\right]\,\nu(dp)\right| \\
&\le\; C\int_{\mathbb{S}^{d-1}}\left|\left|\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t+\epsilon,y}_{u(v)}s\rangle\right|^{\alpha}-\left|\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t,x}_{u(v)}s\rangle\right|^{\alpha}\right|\,\mu(ds).
\end{aligned}$$

We now exploit the β^1-Hölder regularity in space of the diffusion coefficient σ to show that

$$\begin{aligned}
\Delta_{s,2}^{t,\epsilon,x,y}(v,z) \;&\le\; C|z|^{\alpha}\left|\theta_{u(v),t+\epsilon}(y)-\theta_{u(v),t}(x)\right|^{\beta^1(\alpha\wedge 1)} \\
&\le\; C|z|^{\alpha}\left[|y-\theta_{t+\epsilon,t}(x)|^{\beta^1}+\epsilon^{\beta^1}\right],
\end{aligned} \tag{5.28}$$

where in the last step we used that $\alpha>1$ and the approximate Lipschitz property of the flow (cf. Lemma 2.14 up to a normalization, see also Lemma 1.1 in [MPZ21]).
We can now use Controls (5.27)-(5.28) in Equation (5.26) to show that

$$|\Delta_{s}^{t,\epsilon,x,y}(v,z)| \;\le\; C\left(|z|+|z|^{\alpha}+|y-\theta_{t+\epsilon,t}(x)|^{\beta^1}|z|^{\alpha}\right). \tag{5.29}$$

Similarly, Controls (5.29)-(5.25) with Equation (5.24) allow us to conclude that

$$|\mathcal{F}_\epsilon(t,z,t+\epsilon,y)-\mathcal{F}_\epsilon(t,z,t,x)| \;\le\; C\epsilon\left(1+|z|+\epsilon^{\beta^1}|z|^{\alpha}+|y-\theta_{t+\epsilon,t}(x)|^{\beta^1}|z|^{\alpha}\right). \tag{5.30}$$

We can now go back to Equation (5.23). Changing variable and integrating over z, we find that

$$\begin{aligned}
|\mathcal{P}_1(t,t+\epsilon,x,y)| \;&\le\; \frac{C\epsilon}{\det\mathbb{M}_\epsilon}\int_{\mathbb{R}^N}\left(1+|z|+\epsilon^{\beta^1}|z|^{\alpha}+|y-\theta_{t+\epsilon,t}(x)|^{\beta^1}|z|^{\alpha}\right)e^{C\epsilon(1-|z|^{\alpha})}\,dz \\
&\le\; \frac{C}{\det\mathbb{T}_\epsilon}\int_{\mathbb{R}^N}\left(\epsilon+\epsilon^{\frac{\alpha-1}{\alpha}}|\tilde{z}|+\epsilon^{\beta^1}|\tilde{z}|^{\alpha}+|y-\theta_{t+\epsilon,t}(x)|^{\beta^1}|\tilde{z}|^{\alpha}\right)e^{C(1-|\tilde{z}|^{\alpha})}\,dz \\
&\le\; \frac{C}{\det\mathbb{T}_\epsilon}\left(\epsilon^{(1-\frac{1}{\alpha})\wedge\beta^1}+|y-\theta_{t+\epsilon,t}(x)|^{\beta^1}\right)
\end{aligned}$$

where we recall that $\mathbb{T}_t=t^{1/\alpha}\mathbb{M}_t$.
To conclude, we apply the change of variable $\tilde{y}=y-\theta_{t+\epsilon,t}(x)$:

$$\begin{aligned}
\int_{D_1}|\mathcal{P}_1(t,t+\epsilon,x,y)|\,dy \;&\le\; \frac{C}{\det\mathbb{T}_\epsilon}\int_{D_1}\left[|y-\theta_{t+\epsilon,t}(x)|^{\beta^1}+\epsilon^{(1-\frac{1}{\alpha})\wedge\beta^1}\right]\,dy \\
&=\; C\int_{|\tilde{y}|\le\epsilon^{-\beta}}\left[|\mathbb{T}_\epsilon\tilde{y}|^{\beta^1}+\epsilon^{(1-\frac{1}{\alpha})\wedge\beta^1}\right]\,d\tilde{y} \\
&\le\; C[\epsilon^{\beta^1/\alpha-\beta(N+\beta^1)}+\epsilon^{(1-\frac{1}{\alpha})\wedge\beta^1-\beta N}].
\end{aligned}$$

The above control then tends to zero letting ϵ go to zero, if we choose β such that

$$0\;<\;\beta\;<\;\frac{\beta^1}{\alpha(N+\beta^1)}\wedge\frac{(1-\frac{1}{\alpha})\wedge\beta^1}{N}.$$

To control the second term $\mathcal{P}_2$, we use again Control (2.15) and a Taylor expansion to write, similarly to above, that

$$\begin{aligned}
&|\mathcal{P}_2(t,t+\epsilon,x,y)| \\
&\qquad \le \frac{C}{\det \mathbb{M}_\epsilon}\int_{\mathbb{R}^N} e^{C\epsilon(1-|z|^\alpha)}\left|\langle \mathbb{M}_\epsilon^{-1}(y-\tilde{m}^{t,x}_{t+\epsilon,t}(x)),z\rangle - \langle \mathbb{M}_\epsilon^{-1}(y-\tilde{m}^{t+\epsilon,y}_{t+\epsilon,t}(x)),z\rangle\right| dz \\
&\qquad \le \frac{C}{\det \mathbb{T}_\epsilon}\left|\mathbb{T}_\epsilon^{-1}\left(\theta_{t+\epsilon,t}(x)-\tilde{m}^{t+\epsilon,y}_{t+\epsilon,t}(x)\right)\right|, \qquad (5.31)
\end{aligned}$$

where in the last passage we used Lemma 2.9. To bound the above right-hand side, we now exploit Corollary 2.15 to show that

$$|\mathcal{P}_2(t,t+\epsilon,x,y)| \le C\epsilon^{\frac{1}{\alpha}\wedge\zeta}\frac{1}{\det \mathbb{T}_\epsilon}\left(1+|\mathbb{T}_\epsilon^{-1}(\theta_{t+\epsilon,t}(x)-y)|\right).$$

Similarly to above, we can then apply a change of variables:

$$\int_{D_1}|\mathcal{P}_2(t,t+\epsilon,x,y)|\,dy \le C\epsilon^{\frac{1}{\alpha}\wedge\zeta}\int_{|z|\le\epsilon^{-\beta}}(1+|z|)\,dz.$$

We can then notice again that the above control tends to zero letting ϵ goes to zero, if we choose β small enough.

Proof of Lemma 2.19. As in the previous Lemma 2.18, we want to show the following limit:

$$\lim_{\epsilon\to 0}\|I_\epsilon f - f\|_{L^p_tL^q_x} = 0,$$

for some $p\in(1,+\infty)$, $q\in(1,+\infty)$ and f in $C^{1,2}_c([0,T)\times\mathbb{R}^N)$. We start writing that

$$\|I_\epsilon f - f\|^p_{L^p_tL^q_x} = \int_0^T \|I_\epsilon f(t,\cdot)-f(t,\cdot)\|^p_{L^q}\,dt.$$

We then notice that, up to a middle point-type argument, the indicator function in the definition (2.46) of $I_\epsilon f$ can be easily controlled. We can now write that

$$\begin{aligned}
\|I_\epsilon f(t,\cdot)-f(t,\cdot)\|^p_{L^p} &= \int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N} f(t+\epsilon,y)\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)\,dy - f(t,x)\right|^p dx \\
&\le C\Big(\int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N} f(t+\epsilon,y)\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)\,dy - f(t+\epsilon,\theta_{t+\epsilon,t}(x))\right|^p dx \\
&\qquad\qquad + \int_{\mathbb{R}^N}|f(t+\epsilon,\theta_{t+\epsilon,t}(x))-f(t,x)|^p\,dx\Big) \\
&=: C\left(\mathcal{I}+\mathcal{I}'\right)(\epsilon,t). \qquad (5.32)
\end{aligned}$$

Since f is smooth and with compact support in time and space, it follows immediately that $\mathcal{I}'(\epsilon,t)$ tends to zero if ϵ goes to zero, thanks to the bounded convergence Theorem. We can then focus on the first term $\mathcal{I}(\epsilon,t)$. We start splitting it in the following way:

$$\begin{aligned}
\mathcal{I}(\epsilon,t) &\le C\Big(\int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N}\left[f(t+\epsilon,y)-f(t+\epsilon,\theta_{t+\epsilon,t}(x))\right]\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)\,dy\right|^p dx \\
&\quad + \int_{\mathbb{R}^N}\left|f(t+\epsilon,\theta_{t+\epsilon,t}(x))\int_{\mathbb{R}^N}\left[\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)-\tilde{p}^{t,x}(t,t+\epsilon,x,y)\right]dy\right|^p dx\Big) \\
&=: C\left(\mathcal{I}_1+\mathcal{I}_2\right)(\epsilon,t),
\end{aligned}$$

where we used that $\tilde{p}^{t,x}(t,s,x,y)$ is indeed a *true* density with respect to y. The second term $\mathcal{I}_2(\epsilon,t)$ already appeared in the proof of Lemma 2.18 (Dirac Convergence of frozen density) (cf. term $\mathcal{D}_2$ in (5.22)) and a similar analysis readily gives that $\mathcal{I}_2(\epsilon,t) \xrightarrow{\epsilon\to 0} 0$. To control instead the first term $\mathcal{I}_1(\epsilon,t)$, we decompose the whole space $\mathbb{R}^N$ into $\Delta_1 \cup \Delta_2$ given by

$$\begin{aligned}
\Delta_1 &:= \{x \in \mathbb{R}^N \colon |\theta_{t+\epsilon,t}(x) - \mathrm{supp}[f(t+\epsilon,\cdot)]| \le 1\};\\
\Delta_2 &:= \{x \in \mathbb{R}^N \colon |\theta_{t+\epsilon,t}(x) - \mathrm{supp}[f(t+\epsilon,\cdot)]| > 1\}.
\end{aligned}$$

Using Proposition 2.11 with $(\tau,\xi) = (t+\epsilon,y)$, we write that

$$\begin{aligned}
\mathcal{I}_1(\epsilon,t) &\le \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |f(t+\epsilon,y) - f(t+\epsilon,\theta_{t+\epsilon,t}(x))| \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(y - \tilde{m}^{t+\epsilon,y}_{t+\epsilon,t}(x))\right)}{\det \mathbb{T}_\epsilon} \, dy \right)^p dx \\
&\le \int_{\Delta_1} \left(\int_{\mathbb{R}^N} |f(t+\epsilon,y) - f(t+\epsilon,\theta_{t+\epsilon,t}(x))| \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(y - \tilde{m}^{t+\epsilon,y}_{t+\epsilon,t}(x))\right)}{\det \mathbb{T}_\epsilon} \, dy \right)^p dx \\
&\quad + \int_{\Delta_2} \left(\int_{\mathbb{R}^N} |f(t+\epsilon,y) - f(t+\epsilon,\theta_{t+\epsilon,t}(x))| \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(y - \tilde{m}^{t+\epsilon,y}_{t+\epsilon,t}(x))\right)}{\det \mathbb{T}_\epsilon} \, dy \right)^p dx \\
&=: \left(\mathcal{I}_{11} + \mathcal{I}_{12}\right)(\epsilon,t).
\end{aligned}$$

To control $\mathcal{I}_{11}$, we start noticing that f is Hölder continuous with a Hölder exponent $\gamma < \alpha$ in $(0,1]$, since it has a compact support. Moreover, Δ_1 is a bounded set (uniformly in ϵ). Then, from Lemma 2.9 (cf. Equation (2.6)), Lemma 2.14 and Corollary 2.16,

$$\begin{aligned}
\mathcal{I}_{11}(\epsilon,t) &\le C \int_{\Delta_1} \left(\int_{\mathbb{R}^N} |y - \theta_{t+\epsilon,t}(x)|^\gamma \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(y - \tilde{m}^{t+\epsilon,y}_{t+\epsilon,t}(x))\right)}{\det \mathbb{T}_\epsilon} \, dy \right)^p dx \\
&\le C\epsilon^{p\gamma/\alpha} \int_{\Delta_1} \left(\int_{\mathbb{R}^N} |\mathbb{T}_\epsilon^{-1}(y - \theta_{t+\epsilon,t}(x))|^\gamma \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y) - x)\right)}{\det \mathbb{T}_\epsilon} \, dy \right)^p dx \\
&\le C\epsilon^{p\gamma/\alpha} \int_{\Delta_1} \left(\int_{\mathbb{R}^N} \left[|\mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y) - x)|^\gamma + 1\right] \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y) - x)\right)}{\det \mathbb{T}_\epsilon} \, dy \right)^p dx \\
&\le C\epsilon^{p\gamma/\alpha}.
\end{aligned}$$

To control instead $\mathcal{I}_{12}$ we firstly notice that if x is in Δ_2, then, $\theta_{t+\epsilon,t}(x)$ is not in the support of f. Thus,

$$\begin{aligned}
\mathcal{I}_{12}(\epsilon,t) &= \int_{\Delta_2} \left(\int_{\mathbb{R}^N} |f(t+\epsilon,y) - f(t+\epsilon,\theta_{t+\epsilon,t}(x))| \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(y - \tilde{m}^{t+\epsilon,y}_{t+\epsilon,t}(x))\right)}{\det \mathbb{T}_\epsilon} \, dy \right)^p dx \\
&\le \int_{\Delta_2} \left(\int_{\mathrm{supp} f} |f(t+\epsilon,y)| \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y) - x)\right)}{\det \mathbb{T}_\epsilon} \, dy \right)^p dx \\
&\le \|f\|_\infty^p \int_{\Delta_2} \left(\int_{\mathrm{supp} f} \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y) - x)\right)}{\det \mathbb{T}_\epsilon} \, dy \right)^{p-1+1} dx \\
&\le C \int_{\mathrm{supp} f} \int_{\Delta_2} \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y) - x)\right)}{\det \mathbb{T}_\epsilon} \, dx dy,
\end{aligned}$$

where in the last step we used that, from Corollary 2.16

$$\left(\int_{\mathrm{supp} f}\frac{\overline{p}\left(1,\mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y)-x)\right)}{\det \mathbb{T}_\epsilon}\,dy\right)^{p-1} \le \left(\int_{\mathbb{R}^N}\frac{\overline{p}\left(1,\mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y)-x)\right)}{\det \mathbb{T}_\epsilon}\,dy\right)^{p-1} \le C_p.$$

We notice now that for any y in $\mathrm{supp} f$ and any x in Δ_2, we have that $|y-\theta_{t+\epsilon,t}(x))|\ge 1$. Exploiting Corollary 2.16 and Lemma 2.14, we write that

$$\begin{aligned}
\mathcal{I}_{12}(\epsilon,t) &\le \int_{\mathrm{supp} f}\int_{\Delta_2}|y-\theta_{t+\epsilon,t}(x)|\frac{\overline{p}\left(1,\mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y)-x)\right)}{\det \mathbb{T}_\epsilon}\,dxdy\\
&\le C\epsilon^{\frac{1}{\alpha}}\int_{\mathrm{supp} f}\int_{\Delta_2}|\mathbb{T}_\epsilon^{-1}(y-\theta_{t+\epsilon,t}(x))|\frac{\overline{p}\left(1,\mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y)-x)\right)}{\det \mathbb{T}_\epsilon}\,dxdy\\
&\le C\epsilon^{\frac{1}{\alpha}}\int_{\mathrm{supp} f}\int_{\mathbb{R}^N}\left[|\mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y)-x)|+1\right]\frac{\overline{p}\left(1,\mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y)-x)\right)}{\det \mathbb{T}_\epsilon}\,dxdy\\
&\le C\epsilon^{\frac{1}{\alpha}}\int_{\mathrm{supp} f}\int_{\mathbb{R}^N}\left[|z|+1\right]\overline{p}\left(1,z\right)\,dzdy\\
&\le C\epsilon^{\frac{1}{\alpha}}.
\end{aligned}$$

Knowing the convergence of $\mathcal{I}(\epsilon,t)$ and $\mathcal{I}'(\epsilon,t)$ to zero, we can finally conclude the proof using the dominated convergence theorem in (5.32).

5.3 Controls associated with the change of variable

Proof of Corollary 2.16

We first concentrate on the proof of Control (2.35). We start exploiting the decomposition of $\overline{p}(t,z)$ in terms of small and large jumps, as in (2.25), to rewrite the left-hand side of Equation (2.35) in the following way:

$$\begin{aligned}
I(s,t,x) &:= \int_{\mathbb{R}^N}\frac{|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|^\gamma}{\det \mathbb{T}_{s-t}}\overline{p}(1,\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x))\,dy\\
&= \int_{\mathbb{R}^N}\frac{|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|^\gamma}{\det \mathbb{T}_{s-t}}\int_{\mathbb{R}^N}p_{\overline{M}}(1,\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)-w)\overline{P}_1(dw)dy,
\end{aligned}$$

Then, the Fubini Theorem and the definition of $p_{\overline{M}}$ in (2.23) immediately imply that

$$\begin{aligned}
I(s,t,x) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|^\gamma}{\det \mathbb{T}_{s-t}}p_{\overline{M}}(1,\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)-w)\,dy\overline{P}_1(dw)\\
&\le C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\det \mathbb{T}_{s-t}^{-1}\frac{\left[|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y))-x-w|^\gamma+|w|^\gamma\right]}{\left[1+|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)-w|\right]^{N+2}}\,dy\overline{P}_1(dw).
\end{aligned}$$

To conclude, it is now enough to show that for any $M>N+1$, there exists $C:=C(M)$ such that

$$\int_{\mathbb{R}^N}\frac{\det \mathbb{T}_{s-t}^{-1}}{\left[1+|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)-w|\right]^M}\,dy \le C. \tag{5.33}$$

Indeed, it would follow from Control (5.33) that

$$
\begin{aligned}
I(t,s,x) \;&\le\; C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\det \mathbb{T}^{-1}_{s-t}}{\Big[1+|\mathbb{T}^{-1}_{s-t}(\theta_{t,s}(y)-x)-w|\Big]^{N+2-\gamma}}\,dy\overline{P}_1(dw)\\
&\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}[1+|w|^\gamma]\,\frac{\det \mathbb{T}^{-1}_{s-t}}{\Big[1+|\mathbb{T}^{-1}_{s-t}(\theta_{t,s}(y)-x)-w|\Big]^{N+2}}\,dy\overline{P}_1(dw)\\
&\le\; C\int_{\mathbb{R}^N}[1+|w|^\gamma]\,\overline{P}_1(dw)\;\le\; C.
\end{aligned}
$$

In order to show Control (5.33), we start noticing that it would be enough to apply the change of variable $\tilde{y}=\mathbb{T}^{-1}_{s-t}(x-\theta_{t,s}(y))-w$ and then, to control the Jacobian matrix of the transformation. Unfortunately, our coefficients are not smooth enough in order to follow this kind of reasoning. Indeed, the drift F is only Hölder continuous.
As done already in in the proof of Lemma 2.14, we firstly need to regularize F through a multi-scale mollification procedure. Namely, we reintroduce the mollified drift $F^\delta := (F^\delta_1,\dots,F^\delta_n)$ similarly to what we did in Equation (5.12). However we modify a bit the mollifying parameters and set

$$
\delta_{ij} \;=\; \bar{C}(s-t)^{\frac{1+\alpha(j-2)}{\alpha\beta j}} \quad \text{for } 2\le i\le j\le n, \tag{5.34}
$$

for a constant $\bar{C}$ meant to be large enough. We also mollify the first component F_1 at a macro scale, i.e. $\delta_{1j}=C_1$, with C_1 large enough as well.

In particular, this choice of parameters gives that the controls (5.11), (5.15) and (5.18) hold again.
We can now define the mollified flow $\theta^\delta_{t,s}(y)$ associated with the drift F^δ given by

$$
\theta^\delta_{t,s}(y) \;=\; y-\int_t^s\Big[A_u\theta^\delta_{u,s}+F^\delta(u,\theta^\delta_{u,s}(y))\Big]\,du. \tag{5.35}
$$

Denoting now, for brevity,

$$
\Delta^\delta\theta_{u,s}(y) \;:=\; \theta_{u,s}(y)-\theta^\delta_{u,s}(y),
$$

it is not difficult to check from the Grönwall Lemma and Controls (5.11), (5.15) and (5.18) that

$$
\begin{aligned}
|\mathbb{T}^{-1}_{s-t}(\theta_{t,s}(y)-\theta^\delta_{t,s}(y))| \;&\le\; \left|\int_t^s \mathbb{T}^{-1}_{s-t}\Big[A_u(\Delta^\delta\theta_{u,s}(y))+F(u,\theta_{u,s}(y))-F^\delta(u,\theta^\delta_{u,s}(y))\Big]\,du\right|\\
&\le\; \int_t^s\Big|\mathbb{T}^{-1}_{s-t}A_u(\Delta^\delta\theta_{u,s}(y))\Big|\,du+\int_t^s\Big|\mathbb{T}^{-1}_{s-t}\Big(F(u,\theta_{u,s}(y))-F^\delta(u,\theta_{u,s}(y))\Big)\Big|\,du\\
&\qquad+\int_t^s\Big|\mathbb{T}^{-1}_{s-t}\Big(F^\delta(u,\theta_{u,s}(y))-F^\delta(u,\theta^\delta_{u,s}(y))\Big)\Big|\,du\\
&\le C_0,
\end{aligned} \tag{5.36}
$$

for some positive constant C_0.
Exploiting now Control (5.36), we firstly notice that for $C\ge 2C_0$,

$$
\begin{aligned}
C+|\mathbb{T}^{-1}_{s-t}(x-\theta_{t,s}(y))-w| \;&\ge\; C+|\mathbb{T}^{-1}_{s-t}(x-\theta^\delta_{t,s}(y))-w|-|\mathbb{T}^{-1}_{s-t}(\theta_{t,s}(y)-\theta^\delta_{t,s}(y))|\\
&\ge\; C_0+|\mathbb{T}^{-1}_{s-t}(x-\theta^\delta_{s,t}(y))-w|
\end{aligned}
$$

and we then use it to write that

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{\det \mathbb{T}_{s-t}^{-1}}{\Big(1+|\mathbb{T}_{s-t}^{-1}(x-\theta_{t,s}(y))-w|\Big)^M}\,dy &\le C\int_{\mathbb{R}^N} \frac{\det \mathbb{T}_{s-t}^{-1}}{\Big(1+|\mathbb{T}_{s-t}^{-1}(x-\theta^\delta_{t,s}(y))-w|\Big)^M}\,dy \\ &= C\int_{\mathbb{R}^N} \frac{1}{(1+|\tilde y|)^M}\frac{1}{\det J^\delta_{t,s}(\tilde y)}\,dy \end{aligned} \tag{5.37}$$

where in the last step we used the change of variables $\tilde y = \mathbb{T}_{s-t}^{-1}(x-\theta^\delta_{t,s}(y))-w$ and denoted by $J^\delta_{t,s}(\tilde y)$ the Jacobian matrix of $y \to \theta^\delta_{t,s}(y)$.
It is now clear that the last term in (5.37) is indeed controlled by a constant C, if we show the existence of a positive constant c, independent from y in $\mathbb{R}^N$, $t<s$ in $[0,T]$ and δ , such that

$$|\det J^\delta_{t,s}(y)| \ \ge\ c>0. \tag{5.38}$$

This is precisely the result provided by Lemma 5.3 below. From the previous computations it is clear that (2.35) holds.

Let us now turn to the proof of Control (2.36). Following the previous approach, we can write

$$\begin{aligned}&\int_{\{|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|\ge K\}} \frac{1}{\det \mathbb{T}_{s-t}}\overline{p}(1,\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x))\,dy \\ &\le C\int_{\{|\mathbb{T}_{s-t}^{-1}(\theta^\delta_{t,s}(y)-x)|\ge K-|\mathbb{T}_{s-t}^{-1}(\Delta^\delta\theta_{u,s}(y))|\}}\int_{\mathbb{R}^N} \frac{\det \mathbb{T}_{s-t}^{-1}}{\Big(1+|\mathbb{T}_{s-t}^{-1}(x-\theta^\delta_{s,t}(y))-w|\Big)^M}\overline{P}_1(dw)dy \\ &\le C\int_{\{|\mathbb{T}_{s-t}^{-1}(\theta^\delta_{t,s}(y)-x)|\ge K-C_0\}}\int_{\mathbb{R}^N} \frac{\det \mathbb{T}_{s-t}^{-1}}{\Big(1+|\mathbb{T}_{s-t}^{-1}(x-\theta^\delta_{s,t}(y))-w|\Big)^M}\overline{P}_1(dw)dy,\end{aligned}$$

exploiting also (5.36) for the last inequality. Using now the Fubini Theorem and the change of variables $z=\mathbb{T}_{s-t}^{-1}(x-\theta^\delta_{s,t}(y))$, we derive from (5.38) that

$$\begin{aligned}\int_{\{|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|\ge K\}} \frac{1}{\det \mathbb{T}_{s-t}}\overline{p}(1,\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x))\,dy &\\ \le C\int_{\mathbb{R}^N}\int_{\{|z|\ge K-C_0\}}\frac{1}{(1+|z-w|)^M}\,dz\overline{P}_1(dw)& \\ =: C\int_{\{|z|\ge \frac{K}{2}|\}}\check p(1,z)\,dz,&\end{aligned}$$

where $\check p$ is a density satisfying the same integrability properties as $\bar p$ assuming as well K large enough. Thus (2.36) holds and the proof of Corollary 2.16 is now complete.

Jacobian of the mollified system

This is a technical part dedicated to the proof of control (5.38) appearing in the proof of key Corollary 2.16 which precisely gives the expected smoothing effect of the frozen density where the freezing parameters also correspond to the integration variable.

Lemma 5.3 (Control of the determinant for the change of variable)**.** *Let $\theta^\delta_{t,s}(y)$ denote the mollified flow associated with the drift F^δ where the mollifying parameter δ has the form* (5.34)*. Its dynamics writes:*

$$\theta^\delta_{t,s}(y) \;=\; y - \int_t^s \Big[A_u\theta^\delta_{u,s} + F^\delta(u,\theta^\delta_{u,s}(y))\Big]\, du.$$

Then, there exists a constants $c_0 := c_0(T) > 0$ s.t., denoting for $0 \le t \le s \le T$ by $J^\delta_{t,s}(y)$ the Jacobian matrix associated with the mapping $y \mapsto \theta^\delta_{s,t}(y)$

$$\det(J^\delta_{t,s}(y)) \ge c_0.$$

Importantly, c_0 does not depend on δ.

Proof. Let us first mention that the even though the coefficients F^δ are smooth, the above control is not direct because there is a subtle balance between the mollifying, matrix valued, parameter δ and the length of the considered time interval $[t,s]$. We recall indeed that the entries δ_{ij} given in (5.34) do depend on $s-t$.

We also recall that, similarly to (5.14), it holds that

$$|D_{x_j}F^\delta_1(t,z)| \;\le\; C(\delta_{1j})^{\beta^1-1},\ \forall 2\le i\le j\le n,\ |D_{x_j}F^\delta_i(t,z)| \;\le\; C(\delta_{ij})^{\beta^j-1}. \tag{5.39}$$

To prove the statement, we have thus to justify, somehow similarly to the control for the flows of Lemma 2.14, that the explosive behavior of the Lipschitz moduli is indeed well balanced by the time-integration.

Let us now start from the dynamics of $J^\delta(y)$ which writes:

$$\begin{aligned} J^\delta_{t,s}(y) =\; D_y\theta^\delta_{t,s}(y) &= \mathbb{I} - \int_t^s \Big[\Big(A_u + D_zF^\delta(u,z)|_{z=\theta^\delta_{u,s}(y)}\Big)\, D_y\theta^\delta_{u,s}(y)\Big]\, du \\ &= \mathbb{I} - \int_t^s \Big[\Big(A_u + D_zF^\delta(u,z)|_{z=\theta^\delta_{u,s}(y)}\Big)\, J^\delta_{u,s}(y)\Big]\, du. \end{aligned}$$

The above equation can be partially integrated using the resolvent $(R_{u,s})_{u\in[t,s]}$ associated with A, i.e. the $\mathbb{R}^N\otimes\mathbb{R}^N$ valued function satisfying

$$\frac{d}{du}R_{u,s} = A_uR_{u,s},\ \ R_{s,s} = I_{nd\times nd}. \tag{5.40}$$

This yields:

$$J^\delta_{t,s}(y) = R_{t,s} - \int_t^s R_{t,u}D_zF^\delta(u,z)|_{z=\theta^\delta_{u,s}(y)}J^\delta_{u,s}(y)du. \tag{5.41}$$

We actually have the following important structure property of the resolvent $(R_{u,s})_{u\in[t,s]}$. There exists a non-degenerate family of matrices $(\hat R^{t,s}_{\frac{u-t}{s-t}})_{u\in[t,s]}$, which is bounded uniformly on $u\in[t,s]$ with constants depending on T s.t.

$$R_{u,s} = \mathbb{T}_{s-t}\hat R^{t,s}_{\frac{u-t}{s-t}}(\mathbb{T}_{s-t})^{-1}. \tag{5.42}$$

Indeed, setting for all $v \in [0,1]$, $\hat{R}_v^{t,s} := (\mathbb{T}_{s-t})^{-1} R_{t+v(s-t),s} \mathbb{T}_{s-t}$ and differentiating yields:

$$\begin{aligned}
\partial_v \hat{R}_v^{t,s} &= (s-t)(\mathbb{T}_{s-t})^{-1} A_{t+v(s-t)} R_{t+v(s-t),s} \mathbb{T}_{s-t} \\
&= \Big((s-t)(\mathbb{T}_{s-t})^{-1} A_{t+v(s-t)} \mathbb{T}_{s-t} \Big) \hat{R}_v^{t,s} := A_v^{t,s} \hat{R}_v^{t,s}.
\end{aligned}$$

The identity (5.42) then actually follows from the structure of the matrix A_t (see assumption **[H]** and (1.4)) which ensures that $(A_v^{t,s})_{v\in[0,1]}$ has bounded entries.

As a by-product of (5.42), we derive that there exists $C \geq 1$ s.t. for all $(i,j) \in [\![1,n]\!]$,

$$|(R_{t,u})_{ij}| \leq C(\mathbb{1}_{j\geq i} + (s-t)^{i-j}\mathbb{1}_{i>j}). \tag{5.43}$$

From (5.41) we thus derive

$$\begin{aligned}
|J^\delta_{t,s}(y)| \leq & C + \int_t^s \sum_{i,j,k=1}^n \Big| R_{t,u} D_z F^\delta(u,z)|_{z=\theta^\delta_{u,s}(y)} \Big|_{ik} |J^\delta_{u,s}(y)|_{kj} du \\
\leq & C + \int_t^s \sum_{i,j,k=1}^n \Big| R_{t,u} D_z F^\delta(u,z)|_{z=\theta^\delta_{u,s}(y)} \Big|_{ik} |J^\delta_{u,s}(y)|_{kj} du.
\end{aligned} \tag{5.44}$$

Remember now that $D_z F^\delta(u,z)$ is upper triangular. Then for fixed $(j,k) \in [\![1,n]\!]^2$, using (5.43),

$$\begin{aligned}
\Big| R_{t,u} D_z F^\delta(u,z)|_{z=\theta^\delta_{u,s}(y)} \Big|_{ik} \leq & \sum_{\ell=1}^k |R_{t,u}|_{i\ell} |DF^\delta_{\ell k}|_\infty \\
\leq & C \sum_{\ell=1}^k (\mathbb{1}_{\ell\geq i} + (t-s)^{i-l}\mathbb{1}_{\ell<i}) |D_k F^\delta_\ell|_\infty.
\end{aligned} \tag{5.45}$$

It is now clearly seen that, for a fixed line index i and $\ell \geq i$, there is no time regularity, contrarily to what happened with the control of the renormalized flows. Recall that if we had chosen F^δ as in the proof of Lemma 2.14 then, for $\ell \geq 2$ (recall that we regularize at macro scale C_1 for F_1^δ) $|D_k F_\ell^\delta|_\infty \leq C(\delta_{\ell k})^{-1+\beta_\ell^k}$. It is then clear that the $\Big(\delta_{lk}^{-1+\beta_\ell^k}\Big)_{\ell\in[\![2,k]\!]}$ must have the same order, which precisely prevents from the choice in (5.17) which allows to consider minimal Hölder regularity exponents distinguishing the regularity with respect to the k^{th} variable in function of the level ℓ of the chain. We are here led to consider $\beta_\ell^k = \beta_k^k = \beta^k$ (condition (5.34)), imposing the strongest integrability threshold, associated with the diagonal perturbation at level k all along the previous levels (up to the second one), which in principle lead to less singularity when the corresponding gradients are considered.

Such a phenomenon naturally appears when investigating the strong uniqueness of the SDE because of the Zvonkin approach, see e.g. [HWZ20] for the Kinetic case deriving from our framework or [CdR17] for the kinetic Brownian case. It was also the case, still for the Brownian kinetic case, in [CdR18] where the parametrix approach freezing the initial coefficients was considered. The author had to impose the same regularity for the drift, in the degenerate variable, on the whole F. Hence, adapting the work [Mar20] to derive pointwise bound of the gradients, which could have been another approach would have led to the same constraints. Here, we have slightly more freedom since we manage to have arbitrary smoothness indexes for the non-degenerate component of the drift.

We thus derive from (5.44) and for $\bar{C}, C_1$ large enough there exists $c_0 > 0$ such that

$$\left[\sum_{k=2}^{n}\sum_{\ell=2}^{k}(\delta_{\ell k})^{-1+\beta^k} + \sum_{k=1}^{n}(\delta_{1k})^{-1+\beta_1^k}\right](s-t) \leq c_0$$

meant to be small that, under the current assumptions, there exists $C \geq 1$ s.t.

$$|J^\delta_{t,s}(y)| \leq C\exp(c_0),$$

and similarly, $\forall u \in [t,s]$,

$$|J^\delta_{u,s}(y)| \leq C\exp(c_0). \tag{5.46}$$

Rewriting:

$$J^\delta_{t,s}(y) = R_{t,s}\Big(I - \int_t^s R_{s,u}D_zF^\delta(u,z)|_{z=\theta^\delta_{u,s}(y)}J^\delta_{u,s}(y)du\Big),$$

we derive from (5.43), (5.46) that the matrix $\Big(I - \int_t^s R_{s,u}D_zF^\delta(u,z)|_{z=\theta^\delta_{u,s}(y)}J^\delta_{u,s}(y)du\Big)$ has diagonal dominant and this gives, from the non degeneracy of R, the statement concerning the determinant. □

Chapter 5

About the sharp constants in Sobolev and Schauder estimates for degenerate Kolmogorov operators

Abstract: We consider a possibly degenerate Kolmogorov-Ornstein-Uhlenbeck operator of the form $L = \mathrm{Tr}(BD^2) + \langle Az, D\rangle$, where A, B are $N \times N$ matrices, $z \in \mathbb{R}^N$, $N \geq 1$, which satisfy the Kalman condition which is equivalent to the hypoellipticity condition. We prove the following stability result: the Schauder and Sobolev estimates associated with the corresponding parabolic Cauchy problem remain valid, with the same constant, for the parabolic Cauchy problem associated with a second order perturbation of L, namely for $L + \mathrm{Tr}(S(t)D^2)$ where $S(t)$ is a *non-negative definite* $N \times N$ matrix depending continuously on $t \in [0, T]$. Our approach relies on the perturbative technique based on the Poisson process introduced in [KP17].

1 Introduction

Let us first consider the following parabolic Cauchy problem:

$$\begin{cases} \partial_t u(t,x,y) \;=\; \Delta_x u(t,x,y) + x \cdot \nabla_y u(t,x,y) + f(t,x,y), \\ u(0,x,y) \;=\; 0, \end{cases} \tag{1.1}$$

where (t, x, y) is in $(0, +\infty) \times \mathbb{R}^{2d}$ for some integer $d \geq 1$. The underlying differential operator

$$L^{\mathrm{K}} \;=\; \Delta_x + x \cdot \nabla_y = \sum_{i=1}^{d} \partial^2_{x_i x_i} \;+\; \sum_{i=1}^{d} x_i \partial_{y_i}$$

is the so-called Kolmogorov operator whose fundamental solution was derived in the seminal paper [Kol34]. This particular operator was also mentioned by Hörmander as the starting point for his theory of hypoelliptic operators [Hör67].

Let us write $z = (x, y) \in \mathbb{R}^{2d}$ and by ∂_{z_j} and $\partial^2_{z_i z_j}$ we denote respectively the first and the second partial derivatives with $i, j = 1, \ldots, 2d$.

We are interested in studying the influence of a second order perturbation on Equation (1.1). Precisely, for a time-dependent matrix $\{S(t)\colon t \geq 0\}$ in $\mathbb{R}^N \otimes \mathbb{R}^N$ such that $t \mapsto S(t)$ is continuous and $S(t)$ is *symmetric* and *non-negative definite* for any fixed t, we consider the *perturbed* Cauchy problem:

$$\begin{cases} \partial_t u_S(t,z) \;=\; L^{\mathrm{K}} u_S(t,z) + \sum_{i,j=1}^{2d} S_{ij}(t)\, \partial^2_{z_i z_j} u_S(t,z) + f(t,z) \\ \qquad\qquad\;\; =: \; L^{\mathrm{K},S}\, u_S(t,z) + f(t,z), \\ u_S(0,z) = 0, \quad z \in \mathbb{R}^{2d}. \end{cases} \tag{1.2}$$

In particular, we will show that Sobolev (and Schauder) estimates which hold for solutions u of the Cauchy Problem (1.1) are also true, with the same constants, for solutions u_S to (1.2). Clearly, the operator $L^{\mathrm{K},S}$ can be seen as a perturbation of L^{K} involving second order partial derivatives with continuous time-dependent coefficients.

For now, let us explain our main results in a special form for Equation (1.1) in the case of L^p-estimates (or Sobolev estimates). For a statement of our results in the whole generality, we instead refer to Section 2.
For a fixed final time $T > 0$ and a source f in $C^\infty_0((0,T) \times \mathbb{R}^{2d})$, it is known from the work of Bramanti *et al.* [BCM96], Theorem 3.1, that Equation (1.1) admits a unique classical bounded solution u which satisfies for p in $(1,+\infty)$ the following estimates:

$$\|\Delta_x u\|_{L^p((0,T)\times\mathbb{R}^{2d})} \leq C_p \,\|f\|_{L^p((0,T)\times\mathbb{R}^{2d})} = C_p \|\partial_t u - L^{\mathrm{K}} u\|_{L^p((0,T)\times\mathbb{R}^{2d})}. \tag{1.3}$$

Note that in this case $C_p = C_p(T,d) > 0$. We will actually manage to prove that the unique classical bounded solution u_S to (1.2) satisfies the estimate

$$\|\Delta_x u_S\|_{L^p((0,T)\times\mathbb{R}^{2d})} \leq C_p \,\|f\|_{L^p((0,T)\times\mathbb{R}^{2d})} = C_p \|\partial_t u - L^{\mathrm{K},S} u\|_{L^p((0,T)\times\mathbb{R}^{2d})}, \tag{1.4}$$

with the **same** previous constant C_p as in (1.3). This result seems to be new even in dimension $N = 2$ and even if we only consider $S(t) = S$, $t \in [0,T]$, where S is a 2×2 *symmetric non-negative definite* matrix.

For a uniformly elliptic second order perturbation $S(t) = S$, $t \in [0,T]$, where S is *positive definite*, we could also have appealed to [BCLP10] to derive estimates like in (1.4). For related estimates in the uniformly elliptic case, see also Section 4 in Metafune *et al.* [MPRS02]. However, note that from [BCLP10] and [MPRS02] we could only deduce that the constant C_p depends on the ellipticity constant of the perturbation (this is the first eigenvalue λ_S of S if $0 < \lambda_S \leq 1$) and on the maximum eigenvalue of S (on this respect, see also [Kry02] and [Pri15a]).

The remarkable point in (1.4) is that the L^p-estimates are stable under second order perturbations, which can be possibly degenerate. Namely, the fact that $S(t)$ might be degenerate for some t in $(0,T)$, or even in some non-empty sub-intervals of $(0,T)$, does not affect the estimates in (1.4).

To prove (1.4), we combine the results of [BCM96] with a probabilistic *perturbative* approach based on the Poisson process inspired by [KP17]. There, it was established in particular that the L^p-estimates for non-degenerate parabolic heat equations with space homogeneous coefficients are valid with constants that are independent of the dimension.

Remark 1.1. Importantly, the approach of [KP17] turns out to be sufficiently robust to handle the estimates in the degenerate directions as well. We recall that the associated maximal L^p-regularity was studied e.g. in [Bou02], [CZ19] or [HMP19]. Fixed p in $(1,+\infty)$, there exists $\tilde{C}_p > 0$ such that for f in $C_0^\infty((0,T)\times\mathbb{R}^{2d})$ the unique classical bounded solution u of (1.1) verifies

$$\|(\Delta_y)^{\frac{1}{3}}u\|_{L^p((0,T)\times\mathbb{R}^{2d})} \le \tilde{C}_p\, \|f\|_{L^p((0,T)\times\mathbb{R}^{2d})} = \tilde{C}_p\|\partial_t u - L^{\mathrm{K}}u\|_{L^p((0,T)\times\mathbb{R}^{2d})}, \tag{1.5}$$

where $(\Delta_y)^{\frac{1}{3}}$ denotes the fractional Laplacian with respect to the degenerate variables y in $\mathbb{R}^d$. It turns out that this estimate is also stable for the previously described second order perturbation. Namely, for u_S solving (1.2),

$$\|(\Delta_y)^{\frac{1}{3}}u_S\|_{L^p((0,T)\times\mathbb{R}^{2d})} \le \tilde{C}_p\, \|f\|_{L^p((0,T)\times\mathbb{R}^{2d})} = \tilde{C}_p\|\partial_t u - L^{\mathrm{K},S}u\|_{L^p((0,T)\times\mathbb{R}^{2d})}, \tag{1.6}$$

where again $\tilde{C}_p$ is the same as in (1.5). □

Remark 1.2. The same type of stability results will also hold for the corresponding global Schauder estimates, first established in the framework of anisotropic Hölder spaces for the solution of (1.1) by Lunardi [Lun97] (see also [Mar20, Mar21] and the references therein). We refer to estimate (4.70).

We point out that our results in Section 3 could be possibly obtained by using the general theorems of Section 4 in [KP17]. This section in [KP17] introduces a more general probabilistic approach and provides quite unexpected regularity results. However checking in our case all the assumptions given in that section is quite involved. On the other hand, we provide self-contained proofs inspired by Sections 2 and 3 of [KP17].

It remains a challenging open problem to have a purely analytic proof of the above regularity results.

Lp-estimates for degenerate Ornstein-Uhlenbeck operators. Let us now describe the more general framework we are going to consider here. Fixed $\mathbb{R}^N = \mathbb{R}^d\times\mathbb{R}^{d'}$ where d, d' are two *non-negative* integers such that $d + d' = N$ and $d \ge 1$. Let us introduce the non-negative, symmetric matrix B in $\mathbb{R}^N\otimes\mathbb{R}^N$ given by

$$B = \begin{pmatrix} B_0 & 0 \\ 0 & 0 \end{pmatrix},$$

where B_0 is a symmetric, positive definite matrix in $\mathbb{R}^d\otimes\mathbb{R}^d$ such that

$$\nu\sum_{i=1}^{d}\xi_i^2 \le \sum_{i,j=1}^{d}(B_0)_{ij}\xi_i\xi_j \le \frac{1}{\nu}\sum_{i=1}^{d}\xi_i^2,$$

for all $\xi \in \mathbb{R}^{d_0}$ and some $\nu > 0$.

We will use, as underlying *proxy* operators, the family of degenerate Ornstein-Uhlenbeck generators of the form

$$L^{\mathrm{ou}}f(z) = \mathrm{Tr}(BD^2f(z)) + \langle Az, Df(z)\rangle, \quad z\in\mathbb{R}^N, \tag{1.7}$$

for a matrix A in $\mathbb{R}^N\otimes\mathbb{R}^N$, where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^N$.

Moreover, we assume the Kalman condition:

[K] There exists a non-negative integer n, such that

$$\mathrm{Rank}[B, AB, \cdots, A^{n-1}B] = N, \tag{1.8}$$

where $[B, AB, ..., A^{n-1}B]$ is the $\mathbb{R}^N \otimes \mathbb{R}^{Nn}$ matrix whose blocks are $B, AB, \cdots A^{n-1}B$.

From the non-degeneracy of B_0, the above condition amounts to say that the vectors

$$\{e_1, \dots, e_d, Ae_1, \dots, Ae_d, \dots, A^{n-1}e_1, \dots, A^{n-1}e_d\} \quad \text{generate} \ \ \mathbb{R}^N, \tag{1.9}$$

where $\{e_i \colon i \in \{1, \cdots, d\}\}$ are the first d vectors of the canonical basis for $\mathbb{R}^N$.

Assumption **[K]** (which also often appears in control theory; see e.g. [Zab92]) is equivalent to the Hörmander condition on the commutators (cf. [Hör67]) ensuring the hypoellipticity of the operator $\partial_t - L^{\mathrm{ou}}$. In particular, it implies the existence and the smoothness of a distributional solution for the following equation:

$$\begin{cases} \partial_t u(t,z) \; = \; L^{\mathrm{ou}} u(t,z) + f(t,z), & \text{on } (0,T) \times \mathbb{R}^N; \\ u(0,z) \; = \; 0, & \text{on } \mathbb{R}^N, \end{cases} \tag{1.10}$$

where f is a function in $C^\infty_0((0,T) \times \mathbb{R}^N)$.

Similarly to [KP17], we will prove below the existence and the uniqueness of bounded regular solutions to (1.10) assuming that the source f belongs to $B_b\left(0,T; C^\infty_0(\mathbb{R}^N)\right)$, which contains $C^\infty_0((0,T) \times \mathbb{R}^N)$, and that can be roughly described as the family of functions which are bounded measurable in time and compactly supported in space, uniformly in time (see Section 1.2 for a precise definition). Equation (1.10) will be understood in an integral form (cf. Equation (1.25)).

By Theorem 3 in [BCLP10] and exploiting some explicit properties of the underlying heat kernel (see Section 2.3 below), it can be derived that for any fixed p in $(1, +\infty)$, there exists $C_p = C_p(B_0, d, d', T)$ such that

$$\|D^2_x u\|_{L^p((0,T)\times \mathbb{R}^N)} \; \le \; C_p \|\partial_t u - L^{\mathrm{ou}} u\|_{L^p((0,T)\times \mathbb{R}^N)} \; = \; C_p \|f\|_{L^p((0,T)\times \mathbb{R}^N)}, \tag{1.11}$$

where for any (t,z) in $[0,T] \times \mathbb{R}^N$, $D^2_x u(t,z)$ stands for the Hessian matrix in $\mathbb{R}^d \otimes \mathbb{R}^d$ with respect to the variable x. We set

$$B_I \; := \begin{pmatrix} I_{d,d} & 0_{d,d'} \\ 0_{d',d} & 0_{d',d'} \end{pmatrix}$$

and note, in particular, that (1.11) can be rewritten in the following, equivalent way:

$$\begin{aligned} \|B_I D^2 u \, B_I\|_{L^p((0,T)\times \mathbb{R}^N)} \; &= \; \|D^2_x u \,\|_{L^p((0,T)\times \mathbb{R}^N)} \\ &\le \; C_p \|\partial_t u - L^{\mathrm{ou}} u\|_{L^p((0,T)\times \mathbb{R}^N} \\ &= \; C_p \|f\|_{L^p((0,T)\times \mathbb{R}^N)}, \end{aligned} \tag{1.12}$$

where $D^2 u = D^2_z u$ represents instead the full Hessian matrix in $\mathbb{R}^N \otimes \mathbb{R}^N$ with respect to z.

Fixed a continuous mapping $t \mapsto S(t)$ such that $S(t)$ is a *symmetric* and *non-negative definite* matrix in $\mathbb{R}^N \otimes \mathbb{R}^N$ for any $t \in [0,T]$, we consider again the following perturbation of L^{ou} :

$$\begin{aligned} L_t^{\text{ou},S}\phi(z) \; &:= \; \text{Tr}(BD^2\phi(z)) + \text{Tr}(S(t)D^2\phi(z)) + \langle Az, D\phi(z)\rangle \\ &= \; L^{\text{ou}}\phi(z) + \text{Tr}(S(t)D^2\phi(z)), \end{aligned} \tag{1.13}$$

where z is in $\mathbb{R}^N$. For the solution u_s of the related Cauchy problem

$$\begin{cases} \partial_t u_S(t,z) \; = \; L_t^{\text{ou},S} u_S(t,z) + f(t,z), & \text{on } (0,T)\times\mathbb{R}^N; \\ u_S(0,z) \; = \; 0, & \text{on } \mathbb{R}^N, \end{cases} \tag{1.14}$$

we will prove the following main theorem:

Theorem 1.1. *Let us consider* (1.14) *with* $f \in B_b\Big(0,T; C_0^\infty(\mathbb{R}^N)\Big)$. *Then, there exists a unique solution* u_S *of Cauchy Problem* (1.14) *which verifies, with the same constant* C_p *as in* (1.12),

$$\begin{aligned} \|D_x^2 u_S\|_{L^p((0,T)\times\mathbb{R}^N)} \; &= \; \|B_I D^2 u_S B_I\|_{L^p((0,T)\times\mathbb{R}^N)} \\ &\leq \; C_p \|\partial_t u_S - L_t^{\text{ou},S} u_S\|_{L^p((0,T)\times\mathbb{R}^N)} \; = \; C_p\|f\|_{L^p((0,T)\times\mathbb{R}^N)}. \end{aligned} \tag{1.15}$$

We point out that for time-homogeneous non-negative definite matrices S, the corresponding elliptic L^p-estimates as in formula (5) of [BCLP10] (replacing $\mathcal{A}$ in [BCLP10] with $L^{\text{ou},S} := \text{Tr}(BD^2\cdot) + \text{Tr}(SD^2\cdot) + \langle Az, D\cdot\rangle$) *with a constant independent of* S, could also be derived from (1.15) using an argument given in [BCLP10].

For more information on the OU operator L^{ou} we also refer to the recent work by Fornaro *et al.* [FMPS21] about full description of the spectrum of degenerate OU operators in L^p-spaces.
Independently from the constant preservation, we also emphasize that the L^p estimates in (1.15) for the perturbed operator seem, to the best of our knowledge, to be new and have some interest by their own.
Let us eventually mention that such stability results could turn out to be useful to investigate the well posedness of some related stochastic differential equations through the corresponding martingale problem.

We could actually derive more general estimates, possibly involving the degenerate directions as well, dependingly on the structure of A. Some results in that direction are gathered in Section 4. Anyhow, to illustrate our approach we now briefly present the various steps to derive (1.15).

1.1 Strategy of the proof for Estimates (1.15)

Fixed a classical bounded solution u to Cauchy Problem (1.10), let us introduce $v(t,z) := u(t, e^{-tA}z)$. This well-known transformation (cf. [DPL95]) precisely allows to get rid of the drift term in the PDE satisfied by v. Indeed, we have that $u(t,z) = v(t, e^{tA}z)$ and since u solves (1.10), it holds for any (t,z) in $(0,T)\times\mathbb{R}^N$, that:

$$\begin{aligned} f(t,z) &= \partial_t u(t,z) - L^{\rm ou}u(t,z) \\ &= v_t(t,e^{tA}z) + \langle Dv(t,e^{tA}z), Ae^{tA}z\rangle - \mathrm{Tr}\Big(e^{tA}Be^{tA^*}D^2v(t,e^{tA}z)\Big) \\ &\quad - \langle Dv(t,e^{tA}z), Ae^{tA}z\rangle \\ &= v_t(t,e^{tA}z) - \mathrm{Tr}\Big(e^{tA}Be^{tA^*}D^2v(t,e^{tA}z)\Big). \end{aligned} \tag{1.16}$$

Denoting $\tilde f(t,z) := f(t,e^{-tA}z)$, It now follows that v satisfies the PDE:

$$\begin{cases} \partial_t v(t,z) \;=\; \mathrm{Tr}\Big(e^{tA}Be^{tA^*}D^2v(t,z)\Big) + \tilde f(t,z) & \text{on } (0,T)\times\mathbb{R}^N; \\ v(0,z) = 0 & \text{on } \mathbb{R}^N. \end{cases} \tag{1.17}$$

In terms of the function v, the known estimates in (1.12) rewrites as:

$$\|B_I e^{tA^*}D^2v(t,e^{tA}\cdot)\,e^{tA}B_I\|_{L^p((0,T)\times\mathbb{R}^N)} \le C_p\|\tilde f(t,e^{tA}\cdot)\|_{L^p((0,T)\times\mathbb{R}^N)}, \tag{1.18}$$

where we used the notation $\|B_I e^{tA^*}D^2v(t,e^{tA}\cdot)\,e^{tA}B_I\|_{L^p((0,T)\times\mathbb{R}^N)}$ to stress the dependence on t instead of the more precise formulation

$$\|B_I e^{\cdot A^*}D^2v(\cdot,e^{\cdot A}\cdot)\,e^{\cdot A}B_I\|_{L^p((0,T)\times\mathbb{R}^N)}.$$

By changing variable in the integrals, Control (1.18) is equivalent to

$$\|B_I e^{tA^*}D^2v(t,\cdot)\,e^{tA}B_I\|_{L^p((0,T)\times\mathbb{R}^N,m)} \le C_p\|\tilde f\|_{L^p((0,T)\times\mathbb{R}^N,m)}, \tag{1.19}$$

where $L^p((0,T)\times\mathbb{R}^N,m)$ denotes the L^p norms w.r.t. the measure

$$m(dt,dx) \;:=\; \det(e^{-At})dtdx.$$

Considering now the following, more general equation

$$\begin{cases} \partial_t w(t,z) + \mathrm{Tr}\Big(e^{tA}Be^{tA^*}D^2w(t,z)\Big) + \mathrm{Tr}\Big(e^{tA}S(t)e^{tA^*}D^2w(t,z)\Big) \;=\; \tilde f(t,z); \\ w(0,z) = 0, \end{cases} \tag{1.20}$$

we can establish the well-posedness of the Cauchy problem (1.20), exploting, for instance, probabilistic arguments and the underlying Gaussian process.

Now, the crucial step consists in adapting some arguments from [KP17] based on the use of the Poisson process to derive that the same L^p-estimates in (1.19) still hold for w, independently from the non-negative definite, symmetric matrix $S(t)$. Precisely,

$$\|B_I e^{tA^*}D^2w(t,\cdot)\,e^{tA}B_I\|_{L^p((0,T)\times\mathbb{R}^N,m)} \le C_p\|\tilde f(t,\cdot)\|_{L^p((0,T)\times\mathbb{R}^N,m)}, \tag{1.21}$$

with the *same* constant C_p appearing in (1.19).
The last step then consists in coming back to the Ornstein-Uhlenbeck operators framework. Namely, we introduce $\tilde u(t,z) := w(t,e^{tA}z)$ which solves, by definition, the following equation:

$$\begin{cases} \partial_t \tilde u(t,z) + L^{\mathrm{ou},S}_t\tilde u(t,z) \;=\; f(t,z), & \text{on } (0,T)\times\mathbb{R}^N, \\ \tilde u(0,z) \;=\; 0, & \text{on } \mathbb{R}^N. \end{cases}$$

Hence, it holds that $\tilde{u} = u_S$. Noticing that

$$D^2 w(t, \cdot) \,=\, D^2[\tilde{u}(t, e^{-tA}\,\cdot)] \,=\, e^{-tA^*} D^2\tilde{u}(t, e^{-tA}\cdot)e^{-tA},$$

we thus get from (1.21) that the following estimates hold:

$$\|B_I D^2 \tilde{u}\, B_I\|_{L^p((0,T)\times\mathbb{R}^N)} \,\leq\, C_p \|f\|_{L^p((0,T)\times\mathbb{R}^N)}. \tag{1.22}$$

Through the previous steps we have then constructed a solution $\tilde{u}$ of Cauchy Problem (1.14) which indeed satisfies the estimates in (1.15) with the same C_p, associated with the *unperturbed* or *proxy* operator. The maximum principle will eventually provide uniqueness for the solution $\tilde{u}$.

Remark 1.3. i) We point out that we could also consider more general time-dependent Ornstein-Uhlenbeck operators like:

$$M\phi(z) \,=\, \mathrm{Tr}(B(t)D^2\phi(z)) + \langle Az, D\phi(z)\rangle.$$

Arguing as before starting from L^p-estimates (or Schauder estimates) for M we can derive the same L^p-estimates (or Schauder estimates) for a perturbation of M like (1.13).

ii) We could extend the L^p-estimates (or the Schauder estimates) related to L^{ou} to more general operators like

$$L_t^{\mathrm{ou},S}\phi(z) + \langle b(t), D\phi(z)\rangle$$

where $b : \mathbb{R}_+ \to \mathbb{R}^N$ is continuous. We can even add to L_t a possibly degenerate non-local perturbation (cf. Section 7 of [KP17]). The L^p-estimates (or Schauder estimates) are still preserved with the same constant. For the sake of simplicity in the sequel we will only consider $b(t) = 0$ and we will not deal with non-local perturbations of $L_t^{\mathrm{ou},S}$.

Organisation of the Paper. The article is organised as follows. At the end of the current section, we first give some useful notations. In Section 2 we will then focus on driftless second order Cauchy problems associated with a non-negative, possibly degenerate, diffusion matrix. We will also consider its relation with the Ornstein-Uhlenbeck dynamics. We will establish through the probabilistic perturbation approach of [KP17] that if some L^p-estimates hold for a particular diffusion matrix so does it, with the same associated constant, for a non-negative perturbation of the diffusion matrix (see Section 3). Finally by the argument of Section 1.1 we will obtain (1.22). Stability results in anisotropic Sobolev space and Schauder estimates are given in Section 4.

1.2 Definition of solution and useful notations

Let us consider the following Cauchy problem:

$$\begin{cases} \partial_t v(t,z) \,=\, \mathrm{tr}\,(Q(t)D^2 v(t,z)) + \langle b(t,z), Dv(t,z)\rangle + f(t,z), & \text{on } (0,T)\times\mathbb{R}^N; \\ v(0,z) \,=\, 0, & \text{on } \mathbb{R}^N; \end{cases} \tag{1.23}$$

where $Q\colon [0,T] \to \mathbb{R}^N \otimes \mathbb{R}^N$ is a *continuous, symmetric, non-negative definite* matrix and $b : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ is a *continuous* function such that $|b(t,z)| \leq K_T(1+|z|)$, $(t,z) \in [0,T] \times \mathbb{R}^N$, for some constant $K_T > 0$.

The function f belongs to $B_b\Big(0,T;C_0^\infty(\mathbb{R}^N)\Big)$, the space of all Borel bounded functions $\phi\colon [0,T] \times \mathbb{R}^N \to \mathbb{R}$ such that $\phi(t,\cdot)$ is smooth and compactly supported for any t in $[0,T]$, for any k in $\mathbb{N}$ the $C^k(\mathbb{R}^N)$-norms of $\phi(t,\cdot)$ are bounded in time and the supports of the functions $\phi(t,\cdot)$ are contained in the same ball. Moreover, we require that, for any $z \in \mathbb{R}^N$, the mapping:

$$t \mapsto \phi(t,z) \tag{1.24}$$

is a *piece-wise continuous* function on $[0,T]$, i.e. it is continuous except for a finite number of points.

Remark 1.4. Note that to perform the technique used in [KP17] and based on the Poisson process we need to consider equations like (1.23) with a source f which is possibly discontinuous in time (cf. the proof in Section 2 of [KP17] and Section 3.2 below).

We interpret Cauchy Problem (1.23) in an *integral* form:

$$v(t,z) = \int_0^t \Big(f(s,z) + \mathrm{Tr}(Q(s)D^2v(s,z)) + \langle b(s,z), Dv(s,z)\rangle\Big)ds. \tag{1.25}$$

In particular, we say that a continuous and bounded function $v : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a solution to Equation (1.23) if $v(t,\cdot)$ belongs to $C^2(\mathbb{R}^N)$, for any $t \in [0,T]$, and (1.25) holds as well, for any (t,z) in $[0,T] \times \mathbb{R}^N$.
We finally note that, for any $z \in \mathbb{R}^N$, the function $t \mapsto v(t,z)$ is a C^1-piece-wise function on $[0,T]$.
By Theorem 4.1 in [KP10] we deduce in a quite standard way that if a solution v exists, then it is unique and the following maximum principle holds:

$$\sup_{(t,z)\in[0,T]\times\mathbb{R}^N} |v(t,z)| \leq T \sup_{(t,z)\in[0,T]\times\mathbb{R}^N} |f(t,z)|. \tag{1.26}$$

About the proof of (1.26), we only make some remarks. By considering v and $-v$, we see that it is enough to prove that $v(t,z) \leq T\|f\|_\infty$, for all $(t,z) \in [0,T] \times \mathbb{R}^N$. Moreover, setting $\tilde{v} = v - t\|f\|_\infty$, we note that $\tilde{v}$ verifies (1.25) with f replaced by $f - \|f\|_\infty \leq 0$. Finally, by considering the equation verified by $e^{-t}\tilde{v}$, we can apply Theorem 4.1 in [KP10] to obtain the result.

2 Estimates for driftess second order operators and related perturbation

Throughout this section, we consider the following Cauchy problem:

$$\begin{cases} \partial_t v(t,z) \, = \, \mathrm{Tr}\left(Q(t)D^2v(t,z)\right) + f(t,z), & \text{on } (0,T) \times \mathbb{R}^N; \\ v(0,z) \, = \, 0, & \text{on } \mathbb{R}^N, \end{cases} \tag{2.27}$$

which can be seen as a special case of (1.23) when $b = 0$. Moreover, we assume that Q is not identically zero.

2.1 Well-posedness

Proposition 2.1 (Well-posedness in integral form for the driftless Cauchy problem)**.** *Let f be in $B_b\Big(0,T;C_0^\infty(\mathbb{R}^N)\Big)$. Then, there exists a unique solution v to Cauchy problem* (2.27) *in an integral sense, i.e. it solves for $(t,z)\in[0,T]\times\mathbb{R}^N$:*

$$v(t,z)=\int_0^t\Big(f(s,z)+\mathrm{Tr}(Q(s)D^2v(s,z)\Big)ds. \tag{2.28}$$

We will denote in short $v=PDE(Q,f)$.

Proof. By the maximum principle (cf. Equation (1.26)) uniqueness holds for Cauchy Problem (2.27). We can then focus on proving the existence of a solution.
Let us introduce now

$$v(t,z)\;:=\;\int_0^t\mathbb{E}[f(s,z+I_{s,t})]\,ds$$

with the following notation: $I_{s,u}:=\sqrt{2}\int_s^u Q(r)^{1/2}dW_r$, where W is an N-dimensional Brownian motion on some probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq0},\mathbb{P})$ and $Q(r)^{1/2}$ stands for a square root of $Q(r)$, i.e. $Q(r)=Q(r)^{1/2}(Q(r)^{1/2})^*$.
Applying the Itô formula in space to $f(s,z+I_{s,u})_{u\in[s,t]}$, we get that

$$\mathbb{E}f(s,z+I_{s,t})\;=\;f(s,z)+\mathbb{E}\left[\int_s^t\mathrm{Tr}(Q(u)D^2f(s,z+I_{s,u}))\,du\right].$$

Hence,

$$v(t,z)\;=\;\int_0^t\left(f(s,z)+\mathbb{E}\left[\int_s^t\mathrm{Tr}(Q(u)D^2f(s,z+I_{s,u}))\,du\right]\right)ds,$$

from which it readily follows that

$$\begin{aligned}\partial_t v(t,z)\;&=\;f(t,z)+\int_0^t\mathbb{E}\Big[\mathrm{Tr}(Q(t)D^2f(s,z+I_{s,t}))\Big]\,ds\\&=\;f(t,z)+\mathrm{Tr}\Big(Q(t)D^2\int_0^t\mathbb{E}\left[f(s,z+I_{s,t})\right]\,ds\Big)\\&=\;f(t,z)+\mathrm{Tr}\Big(Q(t)D^2v(t,z)\Big),\end{aligned}$$

for almost every $t\in[0,T]$ and any $z\in\mathbb{R}^N$. □

2.2 Relation to the Ornstein-Uhlenbeck dynamics

If now in particular, $Q(t)$ has the particular form $Q(t)=e^{tA}Be^{tA^*}$ (cf. Equation (1.17)), we introduce

$$u(t,z):=v(t,e^{tA}z),$$

where v is the solution to (2.28) (see Proposition 2.1). Since we can differentiate with respect to t the function $u(\cdot,z)$ for a.e. $t\in[0,T]$, we can perform computations similar to (1.16) and get that $u(t,z)$ solves in integral form:

$$\begin{cases}\partial_t u(t,z)\;=\;L^{\mathrm{ou}}\,u(t,z)+\bar{f}(t,z), & \text{on } (0,T)\times\mathbb{R}^N;\\ u(0,z)\;=\;0, & \text{on } \mathbb{R}^N;\end{cases} \tag{2.29}$$

with L^{ou} as in (1.7), $\bar{f}(t,z) = f(t, e^{tA}z)$. Precisely, for all $(t,z) \in [0,T] \times \mathbb{R}^N$,

$$u(t,z) = \int_0^t \Big(\bar{f}(s,z) + L^{\mathrm{ou}} v(s,z)\Big) ds. \tag{2.30}$$

Hence, u is a solution to (2.29).
Let us also point out that the well-posedness of (2.29) could also have been obtained directly from Gaussian type calculations, similar to those in the proof of Proposition 2.1, introducing $u^{\mathrm{ou}}(t,z) := \int_0^t \mathbb{E}[\bar{f}(s, e^{(t-s)A}z + I^{\mathrm{ou}}_{s,t})]ds$ where $I^{\mathrm{ou}}_{s,u} := \sqrt{2}\int_s^u e^{(u-v)A} B dW_v$.

2.3 About the Lp-estimates in (1.11)

The aim of this section is to fully justify the estimates in (1.11). This is a consequence of the previous probabilistic representation and of Theorem 3 in [BCLP10]. For u solving (1.10), it holds that for all $(t,z) \in [0,T] \times \mathbb{R}^N$,

$$u(t,z) \;=\; \int_0^t \mathbb{E}\left[f(s, e^{A(t-s)}z + I^{\mathrm{ou}}_{s,t})\right] ds \;=\; \int_0^t \int_{\mathbb{R}^N} f(s,z') p^{\mathrm{ou}}(t-s,z,z')\, dz' ds, \tag{2.31}$$

where for $v > 0$, $p^{\mathrm{ou}}(v,z,\cdot)$ stands for the density at time v of the stochastic process

$$X^{\mathrm{ou}}_u \;:=\; e^{Au}z + \sqrt{2}\int_0^u e^{A(u-w)} B dW_w \;=\; z + \int_0^u A X^{\mathrm{ou}}_w dw + BW_u,\ u \geq 0.$$

We recall from [LP94] that assumption **[K]** is equivalent to the fact that there exist $n \in \mathbb{N}$ and positive integers $\{d_i \colon i \in 1, \cdots, n\}$ such that $d = d_1$, $\sum_{i=1}^n d_i = N$ and for all $i \in \{2, \cdots, n\}$ the matrixes

$$\mathscr{A}^i \;:=\; (A_{j,\ell})_{(j,\ell) \in \{\sum_{m=1}^{i-1} d_m + 1, \cdots, \sum_{m=1}^{i} d_i\} \times \{\sum_{m=1}^{i-2} d_m + 1, \cdots, \sum_{m=1}^{i-1} d_m\}},$$

with the natural notation $\sum_{m=1}^0 = 0$, have rank d_i. The matrix A writes:

$$A \;=\; \begin{pmatrix} * & * & \dots & \dots & * \\ \mathscr{A}^2 & * & \ddots & \ddots & \vdots \\ 0_{d_3,d} & \mathscr{A}^3 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ 0_{d_n,d} & \dots & 0_{d_n,d_{n-2}} & \mathscr{A}^n & * \end{pmatrix}. \tag{2.32}$$

Following the proof of Lemma 5.5 in [DM10], where the case $d_i = d$ for any i in $[\![1,n]\!]$ is addressed, it can be derived that there exists $C \geq 1$ s.t. for all $(v,z,z') \in (0,T] \times (\mathbb{R}^N)^2$,

$$|D_x^2 p^{\mathrm{ou}}(v,z,z')| \;\leq\; \frac{C}{v^{\sum_{i=1}^n d_i(i-\frac{1}{2})+1}} \exp\left(-C^{-1} v |\overline{\mathbb{M}}_v^{-1}(e^{Av}z - z')|^2\right), \tag{2.33}$$

where

$$\overline{\mathbb{M}}_v \;:=\; \mathrm{diag}(vI_{d\times d}, v^2 I_{d_2\times d_2}, \dots, v^n I_{d_n \times d_n}), \quad v \geq 0,$$

reflects the various scales of the system. For a given function $f \in B_b\left(0,T; C_0^\infty(\mathbb{R}^N)\right)$, it is then clear from (2.31) and (2.33) that for all $(t,z) \in (0,T] \times \mathbb{R}^N$:

$$D_x^2 u(t,z) \;=\; \mathrm{p.v.} \int_0^t \int_{\mathbb{R}^N} f(s,z') D_x^2 p^{\mathrm{ou}}(t-s,z,z') dz' ds. \tag{2.34}$$

It indeed suffices to observe that:

$$\begin{aligned}
&\Big|\mathrm{p.v.}\int_0^t\int_{\mathbb{R}^N} f(s,z')D_x^2 p^{\mathrm{ou}}(t-s,z,z')\,dz'ds\Big|\\
&\quad=\Big|\mathrm{p.v.}\int_0^t\int_{\mathbb{R}^N}[f(s,z')-f(s,e^{A(t-s)}x)]D_x^2 p^{\mathrm{ou}}(t-s,z,z')\,dz'ds\Big|\\
&\quad\underset{(2.33)}{\le}\sup_{s\in[0,T]}\|Df(s,\cdot)\|_\infty\\
&\qquad\times\int_0^t\int_{\mathbb{R}^N}\frac{C}{(t-s)^{\sum_{i=1}^n d_i(i-\frac12)+\frac12}}\exp\Big(-C^{-1}(t-s)|\mathbb{T}_{t-s}^{-1}(e^{A(t-s)}z-z')|^2\Big)\,dz'ds\\
&\quad\le C\sup_{s\in[0,T]}\|Df(s,\cdot)\|_\infty T^{\frac12}.
\end{aligned}$$

The estimates in (1.11) now follows from the proof of Theorem 3 in [BCLP10], starting from (2.34) instead of (16) therein. The strategy is clear. It is necessary to introduce a cut-off function which separates the points (s,z') which do not induce any singularity in (2.34) for the derivatives of the density, namely such that $t-s\ge c_0$ or $|e^{A(t-s)}z-z'|\ge c_0$, for some fixed constant $c_0>0$, from those who are close to the singularity. For the non-singular part of the integral the expected L^p-control readily follows from (2.33) and the Young inequality (see also Proposition 5 in [BCLP10]), whereas the derivation of the bound for the singular part requires some involved harmonic analysis, see Section 4 on the same reference. We can also refer to Theorem 11 and its proof in [Pri15b] for similar issues linked with the corresponding L^p-estimates for degenerate Ornstein-Uhlenbeck operators in an elliptic setting.

2.4 The main result for Equation (2.27)

Let us fix p in $(1,+\infty)$ and assume that there exist $R(t)\in\mathbb{R}^N\otimes\mathbb{R}^N$ depending continuously on $t\ge 0$ and a constant $C_p>0$, such that for any f in $B_b\Big(0,T;C_0^\infty(\mathbb{R}^N)\Big)$, the unique solution $v=PDE(Q,f)$ to equation (2.27) satisfies

$$\|R(t)^*D^2v\,R(t)\,\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})}\le C_p\|f\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})},\tag{2.35}$$

for some absolutely continuous measure $\mathfrak{m}$ w.r.t. the Lebesgue measure on $[0,T]\times\mathbb{R}^N$ such that $\mathfrak{m}(dt,dx)=g(t)dtdx$ for some borel bounded function g (note that in (1.19) we have $R(t)=e^{tA}B^{1/2}$, $\mathfrak{m}(dt,dx)=g(t)dtdx=\det(e^{-At})dtdx$).
We would like to exhibit that a control like (2.35) also holds for the solution w to the following Cauchy Problem:

$$\begin{cases}\partial_t w(t,z)\;=\;\mathrm{tr}\,(Q(t)D^2w(t,z))+\mathrm{tr}\,(Q'(t)D^2w(t,z))+f(t,z), & \text{on } (0,T)\times\mathbb{R}^N;\\ w(0,z)\;=\;0, & \text{on } \mathbb{R}^N,\end{cases}\tag{2.36}$$

Namely we have to prove the following result.

Theorem 2.2. *Let us consider equations* (2.27) *and* (2.36) *where $Q(t)$, $Q'(t)$ are two continuous in time, non-negative definite matrices in $\mathbb{R}^N\otimes\mathbb{R}^N$ and $f\in B_b\Big(0,T;C_0^\infty(\mathbb{R}^N)\Big)$. Assume that estimate* (2.35) *holds as explained above.*

Then the solution w *to* (2.36) *verifies*

$$\|R(t)^* D^2 w\, R(t)\,\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})} \leq C_p \|f\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})}, \tag{2.37}$$

$p \in (1, \infty)$ *with the same constant* C_p *as in* (2.35).

From Theorem 2.2 using the argument of Section 1.1 we can easily derive Theorem 1.1.

3 Perturbation argument for proving Theorem 2.2

We aim here at applying the probabilistic perturbative approach considered in [KP17]. The key idea in that work was, for a well-posed PDE which enjoys some quantitative given estimates, to introduce a *small* random perturbation in the source f through a suitable Poisson type process and to investigate the properties of the associated PDE involving an unknown function v. After considering a small random perturbation of v, we arrive at the useful integral formula (3.45). Taking the expectation, the contributions associated with the jumps yield, for an appropriate intensity of the underlying Poisson process, a finite difference operator. For the PDE satisfied by the expectation, involving the finite difference operator, *the initial estimates are preserved.* Repeating the previous argument we can obtain a PDE involving the composition of two finite difference operators.
Compactness arguments then allow to derive that, the initial estimates still hold at the limit with the composition of two finite difference operators replaced by the corresponding differential operator of order two. Iterating this procedure we can obtain the result.

Below, we start recalling basic properties of Poisson type processes and corresponding stochastic integrals, which are needed for our approach.

3.1 Poisson stochastic integrals

We briefly recall here the very definition of the stochastic integral driven by a Poisson process. We start reminding the construction of such processes.
Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to be fixed from this point further, we start considering a sequence of independent real-valued random variables $\{\tau_m\}_{m\in\mathbb{N}}$ on Ω whose distribution is exponential of parameter $\lambda > 0$:

$$\mathbb{P}(\tau_m > r) \;=\; e^{-r\lambda}, \quad r \geq 0.$$

We can then define the partial sums sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ as follows:

$$\sigma_0 \;=\; 0; \quad \sigma_m = \sum_{i=1}^{m} \tau_i, \quad n = 1, 2, \ldots$$

For any fixed $t \geq 0$, π_t now denotes the number of consecutive sums of τ_i which lie on $[0, t]$, i.e.

$$\pi_t = \sum_{n=0}^{\infty} \mathbb{1}_{\sigma_m \leq t}, \tag{3.38}$$

where $\mathbb{1}_{\sigma_m \le t}$ represents the indicator function of the event $\{\sigma_m \le t\}$. The process $\{\pi_t\}_{t\ge 0}$ we have just constructed is usually known in the literature as a *Poisson process* with intensity λ (see, for instance, [Pro05]).

Now, let $c : [0,T] \to \mathbb{R}^N$ be a continuous function. We can define the Poisson stochastic integral as

$$b_t := \int_0^t c(s) d\pi_s = \sum_{\sigma_k \le t,\ k\ge 1} c(\sigma_k) = \sum_{0<s\le t} c(s)(\pi_s - \pi_{s-}) \tag{3.39}$$

$b_0 = 0$ (as usual $\pi_{s-}(\omega)$ denotes the left limit at s, for any ω, $\mathbb{P}$-a.s.). We now recall the following formula for the expectation of the stochastic integral:

$$\mathbb{E}[\int_0^t c(s) d\pi_s] = \lambda \int_0^t c(s) ds. \tag{3.40}$$

(cf. Lemma 2.1 in [KP17] for a direct proof; see also Theorem 16 in [Pro05] and Theorem 5.3 in [KP17] for a more general formula involving stochastic integrals with predictable processes against the Poisson process). We also recall the following more general result.

Lemma 3.1. *Let $\{\pi_t\}_{t\ge 0}$ be a Poisson Process of intensity λ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let us consider a stochastic process $(\xi_t)_{t\in[0,T]}$ with values in $\mathbb{R}$ which has cadlag paths ($\mathbb{P}$-a.s.) and is $\mathcal{F}_t$-adapted where $\mathcal{F}_t$ is the augmented σ-algebra generated by the random variables π_s, $0 \le s \le t$. Suppose that $\sup_{\omega\in\Omega,\ s\in[0,T]} |\xi_s(\omega)| < \infty$. Then,*

$$\mathbb{E} \int_0^t \xi_{s-}\, d\pi_s = \lambda \int_0^t \mathbb{E}\xi_s ds. \tag{3.41}$$

3.2 Proof of Theorem 2.2

According to the notations appeared in Proposition 2.1, let $v = PDE(Q, f)$ and $w = PDE(Q + Q', f)$ be the unique solutions of equations (2.27) and (2.36), respectively. The proof of Theorem 2.2 will be obtained adapting the method developed in [KP17] (see in particular Section 3, therein). Let e_1 be the first unit vector in $\mathbb{R}^N$. We define

$$X_t = \int_0^t \sqrt{Q'(r)}\, e_1 d\pi_r$$

where $\sqrt{Q'(t)}$ is the unique $N \times N$ symmetric, non-negative definite square root of $Q'(t)$ and $\{\pi_t\}_{t\ge 0}$ is a Poisson Process of intensity λ (cf. Equation (3.39)). The parameter λ will be chosen appropriately later on.
Recall that the solution v to (2.27) is given by

$$v(t, z) = \int_0^t ds \int_{\mathbb{R}^N} [f(s, z + z')\mu_{s,t}(dz'), \tag{3.42}$$

where $\mu_{s,t}$ is the Gaussian law of the stochastic integral $I_{s,t} := \sqrt{2} \int_s^t Q(v)^{1/2} dW_v$ (see the proof of Proposition 2.1).

Let us fix $\epsilon > 0$. We notice that the shifted source $f_\epsilon(t, z) := f(t, z - \epsilon X_t)$ (which also depends on ω; we have omitted to write such dependence on ω) is again in

$B_b\Big(0,T;C_0^\infty(\mathbb{R}^N)\Big)$. This is the reason why we considered such a function space for the source. It precisely allows to take into account the time discontinuities coming from the jumps of the Poisson process.
For any fixed ω in Ω, Proposition 2.1 readily gives that there exists a unique solution $v_\epsilon = \mathrm{PDE}(Q, f(t, z-\epsilon X_t))$, depending also on ϵ and ω as parameters, such that

$$\sup_{(t,z)\in[0,T]\times\mathbb{R}^N} |v_\epsilon(t,z)| \;\le\; T \sup_{(t,z)\in[0,T]\times\mathbb{R}^N} |f(t,z)|. \tag{3.43}$$

Moreover, thanks to the invariance for translations of the L^p-norms, it follows from (2.35) that

$$\|R(t)^* D^2 v_\epsilon R(t)\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})} \;\le\; C_p\|f_\epsilon\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})} \;\le\; C_p\|f\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})}. \tag{3.44}$$

By Equation (3.42), we know that v_ϵ is given by

$$v_\epsilon(t,z) = \int_0^t \int_{\mathbb{R}^N} [f(s, z-\epsilon X_s + z')\,\mu_{s,t}(dz')ds.$$

For each $z\in\mathbb{R}^N$, the stochastic process $(v_\epsilon(t,z))_{t\in[0,T]}$ has continuous paths ($\mathbb{P}$-a.s.) and is $\mathcal{F}_t$-adapted where $\mathcal{F}_t$ is the completed σ-algebra generated by the random variables π_s, $0\le s\le t$.
For fixed $z\in\mathbb{R}^N$, let us introduce the process $(v_\epsilon(t,z+\epsilon X_t))_{t\in[0,T]}$ which is given by

$$v_\epsilon(t,z+\epsilon X_t) = \int_0^t \int_{\mathbb{R}^N} [f(s, z+\epsilon X_t-\epsilon X_s + z')\,\mu_{s,t}(dz')ds.$$

It is not difficut to check that it is is $\mathcal{F}_t$-adapted and it has bounded and càdlàg paths. Applying (2.28) on each interval $[\sigma_m, \sigma_{m+1}\wedge t), m\in\{0,\cdots,\pi_t]$ on which X_s is constant, one then derives that:

$$v_\epsilon(t,z+\epsilon X_t) \;=\; \int_0^t \Big(\mathrm{tr}(Q(s)D_z^2 v_\epsilon(s,z+\epsilon X_s)) + f(s,z)\Big)\,ds + \int_0^t g_\epsilon(s,z)\,d\pi_s, \tag{3.45}$$

where $g_\epsilon(s,z) = v_\epsilon(s, z+\epsilon\sqrt{Q'(s)}\,e_1+\epsilon X_{s-}) - v_\epsilon(s,z+\epsilon X_{s-})$ is precisely the contribution associated with the jump times. It is clear that $g_\epsilon(s,z)\neq 0$ if and only if π_s has a jump at time s. We then have by Lemma 3.1:

$$\mathbb{E}\int_0^t g_\epsilon(s,z)\,d\pi_s \;=\; \lambda\int_0^t \Big(\bar{v}_\epsilon(s,z+\epsilon\sqrt{Q'(s)}\,e_1) - v_\epsilon(s,z)\Big)\,ds, \tag{3.46}$$

where $\bar{v}_\epsilon(s,z) = \mathbb{E}[v_\epsilon(s,z+\epsilon X_s)]$. Let us denote

$$l(t) := \sqrt{Q'(t)}\,e_1.$$

Taking the expectation on both sides of equation (3.45), we find out that $\bar{v}_\epsilon$ is an integral solution of the following PDE:

$$\partial_t\bar{v}_\epsilon(t,z) \;=\; \mathrm{tr}(Q(t)D_z^2\bar{v}_\epsilon(t,z)) + \lambda\,(\bar{v}_\epsilon(t,z+\epsilon l(t)) - \bar{v}_\epsilon(t,z)) + f(t,z), \tag{3.47}$$

with zero initial condition. Remark that uniqueness of bounded continuous solutions to (3.47) follows by the maximum principle, arguing as in the proof of Lemma 2.2 in

[KP17] (first one considers the case $\lambda T \le 1/4$ and then one iterates the procedure by steps of size $1/(4\lambda)$). Moreover, by (3.44) we obtain (using also the Jensen inequality and the Fubini theorem) that

$$\begin{aligned}
\|R(t)^* D^2 \bar{v}_\epsilon \, R(t)\|^p_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})} &= \int_{(0,T)\times\mathbb{R}^N} |R(t)^* D^2 \bar{v}_\epsilon(t,z) R(t)|^p \mathfrak{m}(dt,dz) \\
&= \int_0^T \int_{\mathbb{R}^N} |\mathbb{E}[R(t)^* D^2 v_\epsilon(t, z+\epsilon X_t) R(t)]\,|^p dz g(t) dt \\
&\le \int_0^T \int_{\mathbb{R}^N} \mathbb{E}[|R(t)^* D^2 v_\epsilon(t, z+\epsilon X_t) R(t)\,|^p] dz g(t) dt \\
&= \mathbb{E} \int_0^T \int_{\mathbb{R}^N} |R(t)^* D^2 v_\epsilon(t, z+\epsilon X_t) R(t)\,|^p dz g(t) dt \\
&= \mathbb{E} \int_0^T \int_{\mathbb{R}^N} |R(t)^* D^2 v_\epsilon(t, \bar{z}) R(t)\,|^p d\bar{z} g(t) dt \\
&\le C_p^p \|f\|^p_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})},
\end{aligned}$$

using (3.44) for the last inequality (L^p-estimate for the PDE with random source). Choosing $\lambda = \epsilon^{-2}$ we have from (3.47) that

$$\partial_t \bar{v}_\epsilon(t,z) \;=\; \mathrm{tr}(Q(t) D_z^2 \bar{v}_\epsilon(t,z)) + \; \epsilon^{-2} \left(\bar{v}(t, z+\epsilon l(t)) - \bar{v}_\epsilon(t,z)\right) + f(t,z), \tag{3.48}$$

with zero initial condition and moreover

$$\|R(t)^* D^2 \bar{v}_\epsilon R(t)\|^p_{L^p((0,T)\times\mathbb{R}^N)} \;\le\; C_p^p \|f\|^p_{L^p((0,T)\times\mathbb{R}^N)}. \tag{3.49}$$

Now, the idea is to apply again the same reasoning above to Equation (3.48) with respect to $\bar{v}_\epsilon$ and $f(t, z+\epsilon X_t)$ again with $\lambda = \epsilon^{-2}$. We obtain first a solution p_ϵ to (3.48) corresponding to $f(t, z+\epsilon X_t)$ and we then derive that

$$w_\epsilon(t,z) = \mathbb{E}[p_\epsilon(t, z-\epsilon X_t)]$$

is the unique bounded continuous (integral) solution w_ϵ of the following problem:

$$\begin{aligned}
\partial_t w_\epsilon(t,z) \;=\; & \mathrm{tr}(Q(t) D^2 w_\epsilon(t,z)) \\
& + \epsilon^{-2} \left[w_\epsilon(t, z+\epsilon l(t)) - 2 w_\epsilon(t,z) + w_\epsilon(t, z-\epsilon l(t))\right] + f(t,z),
\end{aligned} \tag{3.50}$$

with initial condition $w_\epsilon(0,z) = 0$. Moreover, the previous estimates still hold with w_ϵ instead of v_ϵ, i.e.,

$$\sup_{(t,z)\in[0,T]\times\mathbb{R}^N} |w_\epsilon(t,z)| \;\le\; T \sup_{(t,z)\in[0,T]\times\mathbb{R}^N} |f(t,z)|; \tag{3.51}$$

$$\|R(t)^* D^2 w_\epsilon \, R(t)\,\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})} \;\le\; C_p \|f\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})}. \tag{3.52}$$

We would like now to let ϵ goes to zero, possibly passing to a subsequence $\epsilon_n \to 0$, and prove that the associated limit w solves

$$\begin{cases} \partial_t w(t,z) \;=\; \mathrm{tr}(Q(t) D^2 w(t,z)) + \langle D^2 w(t,z) \sqrt{Q'(t)} e_1, \sqrt{Q'(t)} e_1 \rangle + f(t,z), \\ w(0,z) \;=\; 0 \end{cases} \tag{3.53}$$

and estimates (3.51) and (3.52) hold with w_ϵ replaced by w.
To do so we will proceed by compactness. Namely, we are going to prove that the family of solutions w_ϵ solving (3.50), indexed by the parameter ϵ, is equi-Lipschitz on any compact subset of $[0,T]\times\mathbb{R}^N$ and the same holds for any derivative in space of w_ϵ.
Indeed, one can apply the finite difference operators with respect to z at any order in (3.50). We recall that for a *smooth* function $\phi\colon\mathbb{R}^N\to\mathbb{R}$, the first finite difference $\delta_{h,i}\phi$, $i\in\{1,\cdots,N\}$ of step h in the direction e_i (i^{th} basis vector) is given by

$$\delta_{h,i}\phi(z) \;=\; \frac{\phi(z+he_i)-\phi(z)}{h}, \quad z\in\mathbb{R}^N.$$

For a given multi-index $\gamma\in\mathbb{N}^N$, the γ-th order finite difference operator $\delta_{h,\gamma}$, is then defined, for any $h>0$, through composition. Namely,

$$\delta_{h,\gamma}\phi(z) \;=\; \delta^{\gamma_1}_{h,1}\delta^{\gamma_2}_{h,2}\dots\delta^{\gamma_N}_{h,N}\phi(z),$$

where $\delta^{\gamma_i}_{h,i}$ denotes the γ_i-th times composition of $\delta_{h,i}$ with itself.
Since any spatial derivative of f belongs to $B_b\left(0,T;C^\infty_0(\mathbb{R}^N)\right)$, using (3.51) we deduce first that any finite difference of any order of w_ϵ is bounded. Consequently, w_ϵ is infinitely differentiable in space with bounded derivatives on $[0,T]\times\mathbb{R}^N$. Equation (3.50), to be understood in its integral form similarly to (2.28), then gives that those derivatives are themselves Lipschitz continuous in time. This precisely gives the equi-Lipschitz on any compact subset of $[0,T]\times\mathbb{R}^N$ of the family w_ϵ and any spatial derivative.
We can now apply the Arzelà-Ascoli theorem to w_ϵ showing the existence of a subsequence $\{w_{\epsilon_n}\}_{n\in\mathbb{N}}$ which converges uniformly on any compact set to a function $w\colon[0,T]\times\mathbb{R}^N\to\mathbb{R}$. Similarly, any derivative in space of w_{ϵ_n} tend to the respective derivatives of w, uniformly on the compact sets.
Passing to the limit as $n\to\infty$ along the sequence $(\epsilon_n)_n$ in Equation (3.50) (written in the integral form), we can then conclude that w solves (3.53).
Moreover, estimates (3.51) and (3.52) holds with w_ϵ replaced by w. Iterating the previous argument in N steps we finally prove that the unique solution w to

$$\begin{cases}\partial_t w(t,z) \;=\; \mathrm{tr}(Q(t)D^2w(t,z))+\sum_{k=1}^N\langle D^2w(t,z)\sqrt{Q'(t)}e_k,\sqrt{Q'(t)}e_k\rangle+f(t,z),\\ w(0,z)\;=\;0\end{cases} \tag{3.54}$$

verifies estimates (3.51) and (3.52) with w_ϵ replaced by w. The proof is complete. □

4 Additional stability results

In this section we extend the previous approach to derive the stability with respect to a second order perturbation of the Ornstein-Uhlenbeck operator in (1.7) under the Kalman condition [**K**]. Here, we consider also L^p-estimates involving the degenerate components of the operator and some associated Schauder estimates.

4.1 Anisotropic Sobolev spaces and maximal Lp-regularity.

With the notations of Section 2.3 we write $z\in\mathbb{R}^N$ as $z=(x,y_2,\cdots,y_n)$ with $x\in\mathbb{R}^d$, $y_i\in\mathbb{R}^{d_i}$, $i\in\{2,\cdots,n\}$, recalling also that $\sum_{i=2}^n d_i=d'$.

Given β in $(0,1)$ and i in $[\![2,n]\!]$, we want to introduce the β-fractional Laplacian $\Delta^{\beta}_{y_i}$ along the *component* y_i. To do so, we follow [HMP19] by considering the orthogonal projection $p_i\colon \mathbb{R}^N \to \mathbb{R}^{d_i}$ such that $p_i(z) = p_i\Big((x,y_2,\dots,y_n)\Big) = y_i$ and denoting its adjoint by $E_i\colon \mathbb{R}^{d_i} \to \mathbb{R}^N$. We can now define the β-fractional Laplacian $\Delta^{\beta}_{y_i}$ as:

$$\Delta^{\beta}_{y_i}\phi(z) \;:=\; \text{p.v.}\int_{\mathbb{R}^{d_i}} [\phi(z+E_iw)-\phi(z)]\,\frac{dw}{|w|^{d_i+2\beta}}, \quad z\in\mathbb{R}^N,$$

for any sufficiently regular function $\phi\colon \mathbb{R}^N\to\mathbb{R}$.
Fixed p in $(1,+\infty)$, we recall that we have denoted by $L^p((0,T)\times\mathbb{R}^N)$ the standard L^p-space with respect to the Lebesgue measure.
We can now define the appropriate anisotropic Sobolev space to state our results. For notational simplicity, let us denote

$$\alpha_i \;:=\; \frac{1}{2i-1}. \tag{4.55}$$

Set now $\alpha := (\alpha_1,\cdots,\alpha_k)\in\mathbb{R}^k$. The *homogeneous* space $\dot{W}^{2,p}_{\alpha}([0,T]\times\mathbb{R}^N)$ is composed by all the functions $\varphi\colon [0,T]\times\mathbb{R}^N\to\mathbb{R}$ in $L^p([0,T]\times\mathbb{R}^N)$ such that $(t,z)\in[0,T]\times\mathbb{R}^N\mapsto \Delta_x\varphi(t,z)\in L^p([0,T]\times\mathbb{R}^N)$, where $\Delta_x\varphi$ is intended in distributional sense, and for any i in $[\![2,n]\!]$, $\Delta^{\alpha_i}_{y_i}\varphi(t,z)$ is well defined for almost every (t,z) and

$$\Delta^{\alpha_i}_{y_i}\varphi(t,z) \;:=\; \Delta^{\alpha_i}_{y_i}\varphi(t,\cdot)(z) \text{ belongs to } L^p([0,T]\times\mathbb{R}^N).$$

It is endowed with the natural *semi*-norm $\|\varphi\|_{\dot{W}^{2,p}_{\alpha}}$ where

$$\|\varphi\|^p_{\dot{W}^{2,p}_{\alpha}} \;=\; \|\Delta_x\varphi\|^p_{L^p} + \sum_{i=2}^{n}\|\Delta^{\alpha_i}_{y_i}\varphi\|^p_{L^p}. \tag{4.56}$$

The thresholds in (4.55) might seem awkward at first sight. They actually correspond to the indexes needed to get stability of the harmonic functions associated with the principal part of (1.7), that is considering A_0 consisting in the sub-diagonal part of A only (i.e. considering (2.32) when the diagonal elements and the strictly upper diagonal elements are equal to zero), along an associated dilation operator. Namely, setting

$$L^{\text{ou}}_0 f(z) = \text{Tr}(BD^2f(z)) + \langle A_0z, Df(z)\rangle, \quad z\in\mathbb{R}^N, \tag{4.57}$$

so that A_0, B satisfy **[K]**, if it holds that

$$(\partial_t - L^{\text{ou}}_0)u(t,z) = 0$$

then for all $\lambda>0$, we have that

$$(\partial_t - L^{\text{ou}}_0)u\Big(\delta_\lambda(t,z)\Big) \;=\; 0$$

where the dilation operator

$$\delta_\lambda(t,z) \;=\; (\lambda^{1/2}t, \lambda x, \lambda^{1/3}y_1, \cdots, y_k^{1/(2n-1)})$$

precisely exhibits the exponents in (4.55) for the degenerate components.
In [HMP19], see also [CZ19] and [Men18] where time inhomogeneous coefficients are considered as well, it has been proven that if A, B satisfy **[K]** and the diagonal and the strictly upper diagonal elements of A in (1.7) are equal to zero (i.e. when $A = A_0$), then the following Sobolev estimates hold:

$$\|u\|_{\dot{W}^{2,p}_{\alpha}} \le C_p \|f\|_{L^p}, \tag{4.58}$$

with $C_p := C_p(\nu, A, d, d')$, where again u is the unique bounded solution to the corresponding Cauchy problem (1.10). In particular we get also the maximal smoothing effects with respect to the degenerate directions. Note that the solution u to (1.1) verifies (4.58). The specific structure assumed on A is actually due to the fact that for such matrices there is an underlying homogeneous space structure which makes easier to establish maximal regularity estimates (see e.g. [CW71] in this general setting).

If A, B satisfy the Kalman condition **[K]** with a general A as in (2.32), having non zero strictly upper diagonal entries (non zero entries in the diagonal should not create difficulties), we believe that the approach in [BCLP10] could extend to show that Estimates (4.58) still hold in this general setting. However, such estimates have not been, up to our best knowledge, proven yet.

L^p-estimates for the degenerate directions of Ornstein-Uhlenbeck operators. Setting, as in Section 1.1, $u(t,z) = v(t, e^{tA}z)$ and since u solves (1.10) we have that v in turn solves (1.17). From the previous computations and setting

$$B_I := \begin{pmatrix} I_{d,d} & 0_{d,d'} \\ 0_{d',d} & 0_{d',d'} \end{pmatrix},$$

and considering A as in [HMP19], with the diagonal and strictly upper diagonal elements of A equal to zero in (2.32), we derive that

$$\begin{aligned}\|D_x^2 u\|_{L^p((0,T)\times\mathbb{R}^N)} &= \|B_I e^{tA^*} D^2 v(t, e^{tA}\cdot) e^{tA} B_I\|_{L^p((0,T)\times\mathbb{R}^N)} \\ &\le C_p \|\tilde{f}(t, e^{tA}\cdot)\|_{L^p((0,T)\times\mathbb{R}^N)}.\end{aligned}$$

On the other hand, for all $i \in \{2, \cdots, n\}$ and with α_i as in (4.55),

$$\begin{aligned}\|\Delta_{y_i}^{\alpha_i} u\|^p_{L^p((0,T)\times\mathbb{R}^N)} &= \int_0^T \int_{\mathbb{R}^N} \left| \text{p.v.} \int_{\mathbb{R}^{d_i}} [u(t, z + E_i w) - u(t,z)] \frac{dw}{|w|^{d_i + 2\alpha_i}} \right|^p dz dt \\ &= \int_0^T \int_{\mathbb{R}^N} \left| \text{p.v.} \int_{\mathbb{R}^{d_i}} [v(t, e^{tA}(z + E_i w)) - v(t, e^{tA}z)] \frac{dw}{|w|^{d_i + 2\alpha_i}} \right|^p dz dt \\ &=: \|\Delta^{\alpha_i, i, A} v\|^p_{L^p((0,T)\times\mathbb{R}^N)},\end{aligned}$$

using that $\text{Tr}(A) = 0$. Hence, setting

$$\begin{aligned}\|\Delta^{\alpha_0,0,A} v\|^p_{L^p((0,T)\times\mathbb{R}^N)} &:= \|\text{Tr}\Big(B_I e^{tA^*} D^2 v(t, e^{tA}\cdot) e^{tA} B_I^{1/2}\Big)\|^p_{L^p((0,T)\times\mathbb{R}^N)} \\ &= \|\text{Tr}\Big(B_I e^{tA^*} D^2 v e^{tA} B_I^{1/2}\Big)\|^p_{L^p((0,T)\times\mathbb{R}^N)},\end{aligned}$$

we get from Definition (4.56) that the Estimates (4.58) rewrite in term of v as:

$$\|v\|^p_{\dot W^{2,p,A}_\alpha} := \sum_{i=1}^n \|\Delta^{\alpha_i,i,A} v\|^p_{L^p((0,T)\times\mathbb{R}^N)} \le \tilde C_p \|f\|^p_{L^p((0,T)\times\mathbb{R}^N)}, \tag{4.59}$$

with $\tilde C_p = C_p^p$. We now want to prove that for w solving (1.20), namely

$$\begin{cases}\partial_t w(t,z) + \mathrm{Tr}\Big(e^{tA}Be^{tA^*}D^2 w(t,z)\Big) + \mathrm{Tr}\Big(e^{tA}S(t)e^{tA^*}D^2 w(t,z)\Big) \;=\; \tilde f(t,z),\\ w(0,z) \;=\; 0,\end{cases}$$

it also holds that

$$\|w\|^p_{\dot W^{2,p,A}_\alpha} := \sum_{i=1}^n \|\Delta^{\alpha_i,i,A} w\|^p_{L^p((0,T)\times\mathbb{R}^N)} \;\le\; \tilde C_p \|f\|^p_{L^p((0,T)\times\mathbb{R}^N)}, \tag{4.60}$$

with the same constants $\tilde C_p$ as in (4.59). This can be done through the previous perturbative approach of Section 3.2 employed to prove Theorem 2.2, which actually gives the expected control for the the second order derivatives contribution of the semi-norm $\|\cdot\|_{\dot W^{2,p,A}_\alpha}$.
For the other contributions and with the notations of Section 3.2, with $Q'(s) = e^{sA}S(s)e^{sA^*}$ and with m which is the Lebesgue measure on $[0,T]\times\mathbb{R}^N$ (indeed in the present case $g(t) = \det(e^{-At}) = 1$, for all t), we would get that

$$\begin{aligned}
\|\bar v_\epsilon\|^p_{\dot W^{2,p,A}_\alpha} &= \sum_{i=1}^n \|\Delta^{\alpha_i,i,A}\bar v_\epsilon\|^p_{L^p((0,T)\times\mathbb{R}^N)}\\
&= \sum_{i=1}^n \int_0^T\int_{\mathbb{R}^N} |\Delta^{\alpha_i,i,A}\bar v_\epsilon(t,z)|^p\, dzdt\\
&= \sum_{i=1}^n \int_0^T\int_{\mathbb{R}^N} \Big|\mathbb{E}[\Delta^{\alpha_i,i,A} v_\epsilon(t,z+\epsilon X_t)]\Big|^p\, dzdt\\
&\le \sum_{i=1}^n \int_0^T\int_{\mathbb{R}^N} \mathbb{E}\Big[|\Delta^{\alpha_i,i,A} v_\epsilon(t,z+\epsilon X_t)\,|^p\Big]\, dzdt\\
&= \sum_{i=1}^n \mathbb{E}\left[\int_0^T\int_{\mathbb{R}^N} |\Delta^{\alpha_i,i,A} v_\epsilon(t,z+\epsilon X_t)\,|^p\, dzdt\right]\\
&= \sum_{i=1}^n \mathbb{E}\left[\int_0^T\int_{\mathbb{R}^N} |\Delta^{\alpha_i,i,A} v_\epsilon(t,\bar z)\,|^p\, d\bar zdt\right]\\
&\le \tilde C_p \|f\|^p_{L^p((0,T)\times\mathbb{R}^N)},
\end{aligned}$$

using for the last inequality that v_ϵ also satisfies (4.59) (similarly to what had been established in (3.44)). The same previous compactness argument then yields (4.60).
Setting eventually $\tilde u(t,z) := w(t,e^{tA}z)$, which is the unique integral solution (smooth in space) of

$$\begin{cases}\partial_t \tilde u(t,z) + L_t^{\mathrm{ou},S}\tilde u(t,z) = f(t,z), & \text{on } (0,T)\times\mathbb{R}^N,\\ \tilde u(0,z) = 0, & \text{on } \mathbb{R}^N,\end{cases}$$

where $L_t^{\mathrm{ou},S}$ introduced in (1.13) is the Ornstein-Uhlenbeck operator perturbed at second order, we derive that

$$\|\tilde{u}\|_{\dot{W}^{2,p}_\alpha} \leq C_p \|f\|_{L^p}, \tag{4.61}$$

with C_p as in (4.58). We have thus extended the results of Theorem 1.1 for the anisotropic Sobolev semi-norm in (4.56). The estimate (4.58) is stable for a continuous, non-negative definite, second order perturbation of the underlying degenerate Ornstein-Uhlenbeck operator.

4.2 Anisotropic Schauder estimates

Following Krylov [Kry96], for some fixed ℓ in $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and β in $(0,1]$, we introduce for a function $\phi\colon \mathbb{R}^N \to \mathbb{R}$ the Zygmund-Hölder semi-norm as

$$[\phi]_{C^{\ell+\beta}} := \begin{cases} \sup_{|\vartheta|=\ell} \sup_{x\neq y} \frac{|D^\vartheta \phi(x) - D^\vartheta\phi(y)|}{|x-y|^\beta}, & \text{if } \beta \neq 1; \\ \sup_{|\vartheta|=\ell} \sup_{x\neq y} \frac{\left|D^\vartheta \phi(x) + D^\vartheta\phi(y) - 2D^\vartheta\phi(\frac{x+y}{2})\right|}{|x-y|}, & \text{if } \beta = 1 \end{cases}$$

(we are using usual multi-indices ϑ for the partial derivatives). Consequently, the Zygmund-Hölder space $C^{\ell+\beta}_b(\mathbb{R}^N)$ is the family of bounded functions $\phi\colon \mathbb{R}^N \to \mathbb{R}$ such that ϕ and its derivatives up to order ℓ are continuous and the norm

$$\|\phi\|_{C^{\ell+\beta}_b} := \sum_{i=0}^{\ell} \sup_{|\vartheta|=i} \|D^\vartheta\phi\|_\infty + [\phi]_{C^{\ell+\beta}} \text{ is finite.}$$

We can now define the anisotropic Zygmund-Hölder spaces associated with the current setting and which again reflect the various scales already introduced in (4.55). Fixed $\gamma \in (0,3)$, the space $C^\gamma_{b,d}(\mathbb{R}^N)$ is the family of functions $\phi\colon \mathbb{R}^N \to \mathbb{R}$ such that for any i in $[\![1,n]\!]$ and any z_0 in $\mathbb{R}^N$, the function

$$w \in \mathbb{R}^{d_i} \to \phi(z_0 + E_i(w)) \in \mathbb{R} \text{ belongs to } C_b^{\gamma/(2i-1)}\left(\mathbb{R}^{d_i}\right),$$

with a norm bounded by a constant independent from z_0. In the above expression, we recall that the $\{E_i \colon i \in \{2,\cdots,n\}\}$ have been defined in the previous paragraph, $d_1 = d$ and E_1 is the embedding matrix from $\mathbb{R}^d$ into $\mathbb{R}^N$. It is endowed with the norm

$$\|\phi\|_{C^\gamma_{b,d}} := \sup_{z_0\in\mathbb{R}^N} \|\phi\Big(z_0 + E_0(\cdot)\Big)\|_{C^\gamma_b(\mathbb{R}^d)} + \sum_{i=1}^{k} \sup_{z_0\in\mathbb{R}^N} [\phi\Big(z_0+E_i(\cdot)\Big)]_{C^{\gamma/(1+2i)}(\mathbb{R}^{d_i})}. \tag{4.62}$$

We denote by $C^\gamma_{b,d}$ this function space because the regularity exponents reflect again the multi-scale features of the system. The norm could equivalently be defined through the corresponding spatial parabolic distance d defined as follows. For all $z = (x,y)$, $z' = (x',y')$ in $\mathbb{R}^N = \mathbb{R}^d \times \mathbb{R}^{d'}$:

$$d(z,z') := |x-x'| + \sum_{i=2}^{n} |y_i - y_i'|^{\frac{1}{2i-1}},$$

where the exponents are again those who appeared in (4.55).

Let as before f be in $B_b\left(0,T; C_0^\infty(\mathbb{R}^N)\right)$. Under **[K]**, by the results of Lunardi [Lun97], it follows that the unique bounded solution of the Cauchy Problem (1.10) (written in integral form) verifies the following anisotropic Schauder estimates:

$$\|u\|_{L^\infty((0,T),C^{2+\beta}_{b,d})} \le C_\beta \|f\|_{L^\infty((0,T),C^{\beta}_{b,d})}, \tag{4.63}$$

for some constant C_β independent from f, i.e.,

$$\sup_{0\le t\le T} \|u(t,\cdot)\|_{C^{2+\beta}_{b,d}} \le C_\beta \sup_{0\le t\le T} \|f(t,\cdot)\|_{C^{\beta}_{b,d}},$$

We again set as in the previous paragraph $u(t,z) = v(t, e^{tA}z)$ and since u solves (1.10) we have that v in turn solves (1.17). Write:

$$\begin{aligned} \|u\|_{L^\infty((0,T),C^{2+\beta}_{b,d})} &= \|v(t,e^{tA}\cdot)\|_{L^\infty((0,T),C^{2+\beta}_{b,d})} =: \|v\|_{L^\infty((0,T),C^{2+\beta}_{b,d,A})} \\ &\le C_\beta \|f\|_{L^\infty((0,T),C^{\beta}_{b,d})} \le C_\beta \|\tilde f(t,e^{tA}\cdot)\|_{L^\infty((0,T),C^{\beta}_{b,d})} \\ &= C_\beta \|\tilde f\|_{L^\infty((0,T),C^{\beta}_{b,d,A})}, \end{aligned} \tag{4.64}$$

denoting $\tilde f(t,z) := f(t, e^{-tA}z)$. We again want to prove, as in Section 1.1 , that for w solving (1.20),

$$\|w\|_{L^\infty((0,T),C^{2+\beta}_{b,d,A})} \le C_\beta \|\tilde f\|_{L^\infty((0,T),C^{\beta}_{b,d,A})} \tag{4.65}$$

with the same constant C_β as in (4.64). We proceed one more time through the previous perturbative approach of Section 3.2. With the notations employed therein, we deduce that there exists a unique solution $v_\epsilon = \mathrm{PDE}(Q, \tilde f(t, z-\epsilon X_t))$, depending also on ϵ and ω as parameters such that

$$\sup_{(t,z)\in[0,T]\times\mathbb{R}^N} |v_\epsilon(t,z)| \le T \sup_{(t,z)\in[0,T]\times\mathbb{R}^N} |\tilde f(t,z)|. \tag{4.66}$$

By the translation invariance of the Hölder-norms, using also that $X_t = e^{tA}e^{-tA}X_t$, it is not difficult to prove that, for any ω, $\mathbb{P}$-a.s.,

$$\|\tilde f\|_{L^\infty((0,T),C^{\beta}_{b,d,A})} = \|\tilde f(\cdot, \cdot - \epsilon X_\cdot)\|_{L^\infty((0,T),C^{\beta}_{b,d,A})}. \tag{4.67}$$

Thus it also holds from (4.64)

$$\|v_\epsilon\|_{L^\infty((0,T),C^{2+\beta}_{b,d,A})} \le C_\beta \|\tilde f\|_{L^\infty((0,T),C^{\beta}_{b,d,A})}. \tag{4.68}$$

Recalling now that $\bar v_\epsilon(s,z) = \mathbb{E}[v_\epsilon(s, z+\epsilon X_s)]$ is an integral solution of

$$\partial_t \bar v_\epsilon(t,z) = \mathrm{tr}(Q(t) D_z^2 \bar v_\epsilon(t,z)) + \lambda\left(\bar v_\epsilon(t, z+\epsilon l(t)) - \bar v_\epsilon(t,z)\right) + \tilde f(t,z),$$

with zero initial condition, we write for any i in $\{2,\cdots,n\}$, w, w' in $\mathbb{R}^{d_i}$ e (t,z_0) in $[0,T]\times\mathbb{R}^N$, that

$$\begin{aligned} &|\bar v_\epsilon(t, e^{At}(z_0 + E_i(w)) - \bar v_\epsilon(t, e^{At}(z_0 + E_i(w'))| \\ &\quad\le \mathbb{E}\left[|v_\epsilon(t, e^{At}(z_0+E_i(w)) + \epsilon e^{At}e^{-At}X_t) - v_\epsilon(t, e^{At}(z_0+E_i(w')) + \epsilon e^{At}e^{-At}X_t)|\right] \\ &\quad\le \mathbb{E}\left[[v_\epsilon(t, e^{At}(z_0+E_i(\cdot)))]_{C^{\frac{2+\beta}{2i-1}}}\right] |w-w'|^{\frac{2+\beta}{2i-1}}. \end{aligned}$$

Hence,

$$\left[\bar{v}_\epsilon(t, e^{-At}(z_0 + E_i(\cdot)))\right]_{C^{\frac{2+\beta}{2i-1}}} \leq \mathbb{E}\left[[v_\epsilon(t, e^{-At}(z_0 + E_i(\cdot)))]_{C^{\frac{2+\beta}{2i-1}}}\right].$$

We would get similarly, that

$$\left[D_x^2 \bar{v}_\epsilon(t, e^{At}(z_0 + E_1(\cdot)))\right]_{C^\beta} \leq \mathbb{E}\left[[D_x^2 v_\epsilon(t, e^{At}(z_0 + E_1(\cdot)))]_{C^\beta}\right],$$

and for all $k \in \{0, 1, 2\}$,

$$\left|D_x^k \bar{v}_\epsilon(t, e^{At}(z_0 + E_1(\cdot)))\right|_\infty \leq \mathbb{E}\left[|D_x^k v_\epsilon(t, e^{At}(z_0 + E_1(\cdot)))|_\infty\right].$$

Summing all those contributions, we thus derive from (4.62), (4.64) that:

$$\|\bar{v}_\epsilon\|_{L^\infty((0,T),C^{2+\beta}_{b,d,A})} \leq \sup_{0\leq t\leq T} \mathbb{E}\left[\|v_\epsilon(t, \cdot)\|_{C^{2+\beta}_{b,d,A})}\right] \leq C_\beta \|\tilde{f}\|_{L^\infty((0,T),C^\beta_{b,d,A})}, \tag{4.69}$$

using (4.68) for the last inequality. Now, continuing as in Section 3.2, using also a compactness argument, one would derive that (4.65) indeed holds.

Going backwards, setting $\tilde{u}(t, z) := w(t, e^{tA}z)$, we find that $\tilde{u}$ is the unique (integral) solution u_S to (1.14): we finally derive that

$$\|u_S\|_{L^\infty((0,T),C^{2+\beta}_{b,d})} \leq C_\beta \|f\|_{L^\infty((0,T),C^\beta_{b,d})}, \tag{4.70}$$

where C_β is the same constant as in (4.63). Estimate (4.70) provides the extension of Theorem 1.1 for the anisotropic Schauder estimates.

Remark 4.1. Let us mention that for the perturbative method to work, roughly speaking, few properties were actually needed on the underlying norm. Namely, we used the translation invariance and some kind of commutation between the norm (or a function of the norm in the L^p-case) and expectation. Hence, this approach could possibly be applied to a much wider class of estimates in other function spaces (like e.g. Besov spaces). This will concern further research.

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Titre: Régularisation faible par un bruit de Lévy dégénéré et applications

Mots clés: EDOs mal posées, Solutions faibles via régularisations, Équations dégénérées de Kolmogorov, Estimées de Schauder pour opérateurs intégro-différentiels, Méthodes de la parametrix, Processus de Lévy

Résumé: Après une introduction générale sur le phénomène de régularisation par le bruit dans le cadre dégénéré, la première partie de cette thèse est consacrée à l'obtention d'estimées de Schauder, un outil analytique utile pour établir le caractère bien posé des équations différentielles stochastiques (EDS), pour deux différents classes d'équations de Kolmogorov sous une condition de type Hörmander faible, dont les coefficients appartiennent à des espaces de Holder anisotropes appropriés avec des multi-indices de régularité. La première classe considère un système non linéaire dirigé par un opérateur α-stable symétrique agissant uniquement sur certaines composantes. Notre méthode de preuve repose sur une approche perturbative basée sur des développements parametrix progressifs par des formules de type Duhamel. En raison des faibles propriétés de régularisation données par le cadre dégénéré, nous exploitons également certains contrôles sur les normes de Besov, afin de traiter la perturbation non-linéaire. Dans le prolongement de la première, nous présentons également les estimées de Schauder pour un opérateur dégénéré d'Ornstein-Uhlenbeck associé à une classe plus large d'opérateurs de type α-stable, comme l'opérateur stable relativiste ou de Lamperti. La preuve de ce résultat repose plutôt sur une analyse précise du comportement du semi-groupe de Markov correspondante entre les espaces de Hölder anisotropes et quelques techniques d'interpolation. En exploitant une approche parametrix rétrograde, la deuxième partie de cette thèse cherche à établir le caractère bien-posé au sens faible d'une chaîne dégénérée de EDS dirigées par la même classe de processus de type α-stable, sous des hypothèses de régularité de Hölder minimale sur les coefficients. Comme corollaire de notre méthode, nous présentons également des estimations de type Krylov d'intérêt indépendant pour le processus canonique sous-jacent. Enfin, nous soulignons à travers des contre-exemples appropriés qu'il existe bien un seuil (presque) optimal sur les exposants de régularité assurant le caractère faiblement bien posé pour l'EDS. En lien avec quelques applications mécaniques pour des dynamiques cinétique avec frottement, nous concluons en étudiant la stabilité des perturbations du second ordre pour des opérateurs de Kolmogorov dégénérés en normes Lp et Hölder.

Title: Weak regularization by degenerate Lévy noise and its applications

Keywords: Ill posed ODEs, Weak solutions through noise regularization, Kolmogorov degenerate equations, Schauder estimates for integro-differential operators, Parametrix Methods, Lévy processes

Abstract: After a general introduction about the regularization by noise phenomenon in the degenerate setting, the first part of this thesis focuses at establishing the Schauder estimates, a useful analytical tool to prove also the well-posedness of stochastic differential equations (SDEs), for two different classes of Kolmogorov equations under a weak Hörmander-like condition, whose coefficients lie in suitable anisotropic Hölder spaces with multi-indices of regularity. The first class considers a nonlinear system controlled by a symmetric α-stable operator acting only on some components. Our method of proof relies on a perturbative approach based on forward parametrix expansions through Duhamel-type formulas. Due to the low regularizing properties given by the degenerate setting, we also exploit some controls on Besov norms, in order to deal with the non-linear perturbation. As an extension of the first one, we also present Schauder estimates associated with a degenerate Ornstein-Uhlenbeck operator driven by a larger class of α-stable-like operators, like the relativistic or the Lamperti stable one. The proof of this result relies instead on a precise analysis of the behaviour of the associated Markov semigroup between anisotropic Hölder spaces and some interpolation techniques. Exploiting a backward parametrix approach, the second part of this thesis aims at establishing the well-posedness in a weak sense of a degenerate chain of SDEs driven by the same class of α-stable-like processes, under the assumptions of the minimal Hölder regularity on the coefficients. As a by-product of our method, we also present Krylov-type estimates of independent interest for the associated canonical process. Finally, we emphasize through suitable counter-examples that there exists indeed an (almost) sharp threshold on the regularity exponents ensuring the weak well-posedness for the SDE. In connection with some mechanical applications for kinetic dynamics with friction, we conclude by investigating the stability of second-order perturbations for degenerate Kolmogorov operators in Lp and Hölder norms.

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