Resonances of the Laplace operator on homogeneous vector bundles on symmetric spaces of real rank one

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Résonances du Laplacien sur les fibrés vectoriels homogènes sur des espaces symétriques de rang réel un

THÈSE

présentée publiquement le 29 juin 2021

Pour l'obtention du

Doctorat de l'Université de Lorraine

(Mention Mathématiques)

par

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Remerciements :

Bien sûr, toute thèse de doctorat commence par le remerciement de son (ses) directeur(s) de thèse. Mais ici et même avec le recul, je ne les échangerai contre aucun(s) autre(s). Angela et Tomasz on été toujours présents pour mes questions mathématiques (et rapidement). Ils ont su très brillament, et tout au long de ma thèse, me donner un peu de l'importante étendue de leurs connaissances, sans jamais me faire sentir inférieur. Aussi humainement, ils ont été là à tout moment pour m'aider à gérer ma vie extra-universitaire. Les doctorants et docteurs savent que ce n'est pas toujours facile, et parfois les avis de personnes, qui sont déjà passées par là, sont vraiment primordiaux pour prendre des décisions.

J'exprime toute ma gratitude à Brigit Speh et Werner Müller d'avoir accepté de rapporter ma thèse. Je me doute combien cette tâche doit coûter du temps, précieux pour tout enseignant chercheur. Je remercie aussi Roberto Camporesi, Jan Frahm, Roberto Miatello, Anke Pohl et Matteï Zworski d'avoir accepté de faire parti de mon jury de thèse.

Plus particulièrement, je remercie Salah qui a accepté de présider le jury et qui s'est soucié, en tant qu'ancien enseignant et membre de mon comité de suivi de thèse, de mon devenir tout au long de ma thèse. Merci à tous mes professeurs qui m'ont fait découvrir et aimer les mathématiques durant mon cursus à l'université de Lorraine, en particulier à Benjamin, Camille, Chakib, Dong, Jean, Jean-Louis, Julien, Patrick, Philippe, Saïd, Tillman et Victor, qui a aussi accepté d'être membre de mon comité de suivi de thèse. Merci aux secrétaires du département, de l'IECL, de L'UFR MIM et de l'école doctorale IAEM sans qui rien n'est jamais possible : Christelle, Claude, Sabrina, Sedat. Merci à Fabrice dont l'influence dépasse souvent le cadre de son travail.

Merci à mes collègues doctorants, avec qui on partage le meilleur et le pire. Merci à Allan fidèle frère académique, Gabriel excellent collègue depuis le master, Benjamin pour sa joie de vivre permanente, Spyros avec qui le partage fût toujours intéressant et Fabien pour m'avoir supporté dans notre bureau et ailleurs.

C'est toujours quand une page se tourne que l'on rend compte de la chance que l'on a eu de l'écrire. Donc merci à la région Lorraine, de m'avoir soutenu financièrement pour pouvoir effectuer cette thèse. Merci à la commission Franco-américaine Fulbright de m'avoir permis de faire un séjour de 4 mois (et même 8 mois prévus au départ...) à l'université d'Oklahoma. Merci à l'école doctorale IAEM de Lorraine d'avoir pris en charge mon voyage en Chine, pour une école d'été et une série de conférences qui resteront de très belles expériences. Sans ces bourses, rien n'est possible.

Merci à mes amis et à toute ma famille, de m'avoir supporté, soutenu, sermoné,

repris, aidé (et bien d'autres choses) : Merci à Alain, Christophe, Francois et Mohamed pour nos parties d'échecs pleines de rembondissements, à Quentin et Alex pour ces superbes années d'études passées ensembles, à Fabienne, Christophe et Manon pour votre bonne humeur permanente, à Rémy et Mathieu pour vos soirées qui vident la tête, à Clément et Louis pour ces parties de squash acharnées et pour le reste. Merci à mes parents pour leurs épaules et à mes grands parents pour leurs partages d'expériences.

Merci Lisa pour tout le reste.

Merci à tous pour m'avoir accordé de votre temps, quelqu'en soit la quantité.

"En mathématiques, nous sommes d'avantage des serviteurs que des maîtres."

"In mathematics, we are servants, much more than masters."

Charles Hermite (1822 - 1901)

Et plus on en sait, plus on a ce sentiment...

And the more we know, the more we have this feeling...

Contents

List of Figures

Abstract

We study the resonances of the Laplacian acting on the compactly supported sections of a homogeneous vector bundle over a Riemannian symmetric space of the non-compact type. The symmetric space is assumed to have rank-one but the irreducible representation τ of the maximal compact K defining the vector bundle is arbitrary. We determine the resonances. Under the additional assumption that τ occurs in the spherical principal series, we determine the resonance representations. They are all irreducible. We find their Langlands parameters, their wave front sets and determine which of them are unitarizable.

Résumé

On étudie les résonances de l'opérateur de Laplace agissant sur les sections d'un fibré vectoriel homogène sur un espace symétrique Riemannien de type non-compact. On suppose que l'espace symétrique est de rang un, mais la représentation irréductible τ du compact maximal K, qui définit le fibré vectoriel, est quelconque. On détermine alors les résonances. Si on suppose de plus que τ apparaît dans les représentations de la série principale sphérique, on détermine les représentations issues des résonances. Elles sont toutes irréductibles. On trouve leurs paramètres de Langlands, leurs fronts d'onde et lesquelles sont unitarisables.

Introduction

The introduction in english is given just after this one.

Dans cette introduction, nous esquissons d'abord une (très) brève histoire de l'analyse harmonique, et signalons quelques jalons de l'étude de la transformée de Fourier sur les groupes de Lie. Nous donnons ensuite quelques résultats de l'analyse harmonique sur un fibré vectoriel homogène au-dessus d'un espace symétrique riemannien de type non compact. Ensuite, nous introduisons la notion de résonances du Laplacien. Après une brève présentation du contexte général motivant leur définition et leur étude, nous passons au cas du Laplacien agissant sur les sections lisses à support compact d'un fibré vectoriel homogène sur un espace symétrique riemannien X de type non-compact. Notre étude est limitée à X de rang un, mais le fibré vectoriel homogène sur X est arbitraire. La dernière section rassemble les principaux résultats de cette thèse.

1 Tour d'horizon de l'analyse harmonique

En musique, les harmoniques sont les composants du son d'un instrument. Ils sont différents pour chaque instrument et sont précisément ce qui crée le *timbre*. Cela nous permet de distinguer différents instruments. Les sons sont des superpositions d'harmoniques, avec des fréquences multiples d'une fréquence fondamentale. Le processus de décomposition d'un son en ses harmoniques est appelé analyse harmonique.

C'est de là que vient le nom “analyse harmonique” en mathématiques. Le son est remplacé par une fonction que nous essayons de décomposer en composantes simples. Les outils les plus connus sont les extensions en série de Fourier et la transformée de Fourier. Si f est une fonction à valeurs complexes suffisamment régulière définie sur $\mathbb{R}/\mathbb{Z}$, alors

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n(f) e^{inx} \ . \tag{1}$$

Ici $c_n(f)$ est le coefficient de Fourier de f défini par

$$c_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} \, dt \ . \tag{2}$$

La formule (1) donne la décomposition en harmoniques $x \mapsto c_n(f)e^{inx}$.

De manière similaire, si F est une fonction "suffisamment régulière" sur $\mathbb{R}$, alors

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{F}(\xi) e^{i\xi x} \, d\xi \ , \tag{3}$$

où $\hat{F}$ est la transformée de Fourier de F

$$\hat{F}(\xi) = \int_{-\infty}^{+\infty} F(x) \ e^{-i\xi x} \, dx \ . \tag{4}$$

Ici $x \mapsto \hat{F}(\xi)e^{i\xi x}$ sont les "harmoniques". On arrive à "reconstruire" les fonctions f et F à partir de leurs harmoniques grâce aux formules d'inversion (1) et (3).

Le théorème d'inversion (3) est valable si toutes les intégrales convergent, c'est-à-dire pour $F \in L^1(\mathbb{R})$ et $\hat{F} \in L^1(\mathbb{R})$. De plus, le théorème de Plancherel (1910) stipule que la transformée de Fourier, définie sur $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, s'étend uniquement à un automorphisme de l'espace de Hilbert $L^2(\mathbb{R})$.

Les séries de Fourier ont été introduites par Joseph Fourier (1768 - 1830) pour résoudre une équation aux dérivées partielles, connue sous le nom de problème de la "corde vibrante"

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}$$

En physique, cette équation décrit le mouvement d'une corde qui vibre librement. Plus tard, en utilisant la même méthode, il a résolu l'équation de la chaleur

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} \ . \tag{5}$$

Ces travaux peuvent être trouvés dans sa “Théorie analytique de la chaleur” [Fou09].

On sait de nos jours, que nous avons ici les *représentations unitaires* unidimensionnelles $(e_n, \mathbb{C})$ du groupe commutatif $\mathbb{R}/\mathbb{Z}$ sur l'espace de Hilbert $\mathbb{C}$, où $n \in \mathbb{Z}$,

$$\begin{aligned} \mathbb{R}/\mathbb{Z} \times \mathbb{C} &\to \mathbb{C} \\ (t, z) &\mapsto e_n(t) \cdot z := e^{int} \cdot z \end{aligned} \tag{6}$$

On peut voir le n-ième coefficient de Fourier de f comme un produit de convolution :

$$c_n(f) = \frac{1}{2\pi} f * e_n(0) := c_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \overline{e_n(t, \cdot)} \, dt \ .$$

Ici $\mathbb{C}$ est appelé l'espace de la représentation $(e_n, \mathbb{C})$.

Une representation (π, V) d'un groupe G sur un espace de Hilbert V est appelée unitaire, si

$$\pi : G \to U(V)$$

est un homomorphisme de groupe de G dans $U(V)$ des opérateurs unitaires sur V, tel que pour tout vecteur $v \in V$ la fonction $G \ni g \to \pi(g)v \in V$ est continue.

Deux représentations sont dites équivalentes, s'il existe un isomorphisme de leurs espaces de Hilbert sous-jacents qui entrelace les actions de groupe correspondantes (voir par exemple [Hal15, Definition 4.3]). Une représentation est dite irréductible si aucun sous-espace de Hilbert propre non trivial de V n'est invariant sous $\pi(G)$ (voir par exemple [Hal15, Definition 4.2] pour la définition exacte). On note $\hat{G}$ l'ensemble de toutes les représentations irréductibles unitaires, en identifiant les représentations isomorphes. La formule (6) montre que pour $G = \mathbb{R}/\mathbb{Z}$, $\hat{G}$ coïncide avec $\mathbb{Z}$. Les coefficients de Fourier de f peuvent alors être vus aussi comme une "projection" de f sur chaque espace des représentations (6). La formule d'inversion de Fourier (1) montre que l'on peut reconstruire f en faisant la somme des projections de f sur chaque représentation irréductible.

Le cas $G = \mathbb{R}$ est analogue. En effet,

$$\begin{aligned} \mathbb{R} \times \mathbb{C} &\to \mathbb{C} \\ (t, z) &\mapsto e_\xi(t) \cdot z := e^{i\xi t} \cdot z \end{aligned} \tag{7}$$

est l'analogue de (6) pour $\mathbb{R}$. La somme laisse place à la "somme continue" (l'intégrale) sur $\mathbb{R}$ et l'on voit que les deux formules (1) et (3) n'ont plus de grandes différences, sauf que le dual unitaire $\hat{\mathbb{R}}$ est isomorphe à $\mathbb{R}$ lui-même.

Notons que

$$\hat{F}(\xi) e^{i\xi x} = \int_{\mathbb{R}} F(t) e_\xi(x - t) \, dt =: F * e_\xi(x) \ . \tag{8}$$

Cela nous permet de réécrire la formule d'inversion comme suit :

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F * e_\xi(x) \, d\xi \ . \tag{9}$$

La généralisation de l'analyse de Fourier aux groupes commutatifs localement compacts G arriva au début du XX^e siècle avec les travaux de Schur, Weil, H. Cartan, Godement et Pontryagin. Dans ce cas, $\hat{G}$ est composé de l'ensemble des homomorphismes continus $G \to \mathbb{S}^1$. Ici $\mathbb{S}^1$ est le cercle unité dans le plan complexe et $\hat{G}$, munit de la multiplication des fonctions, est un groupe. On définit la transformée de Fourier par analogie pour tout $f \in L^1(G)$ par la fonction

$$\hat{f}(\pi) = \int_G \overline{\pi(g)} f(g) dg \ .$$

La transformée de Fourier peut aussi être définie sur $L^2(G)$. Il fut montré qu'il existe une mesure (de Haar) $d\nu$ sur $\hat{G}$, telle que la transformée de Fourier est un isomorphisme unitaire de $L^2(G)$ dans $L^2(\hat{G})$. On a alors la formule de Plancherel

$$\|f\|^2_{L^2(G)} = \|\hat{f}\|^2_{L^2(\hat{G})}$$

et, si $f \in L^1(G)$ et $\hat{f} \in L^1(\hat{G})$, le formule d'inversion

$$f(g) = \int_{\hat{G}} \pi(g) \hat{f}(\pi) \ d\nu(\pi) \ .$$

Cette mesure $d\nu$ sur le groupe $\hat{G}$ est appelée la mesure de Plancherel.

Pour les groupes G compacts, on doit attendre le fameux théorème de Peter-Weyl (1927) qui dit que pour toute fonction $f \in C(G)$ et $g \in G$, on a

$$f(g) = \sum_{\pi \in \hat{G}} d_\pi \operatorname{Tr}\left(\pi(g) \hat{f}(\pi)\right),$$

où $\hat{f}(\pi) = \int_G \pi(g)^* f(g) dg$, et l'entier d_π est la dimension de l'espace de la représentation π. La formule de Plancherel-Parseval

$$\|f\|^2_{L^2(G)} = \|\hat{f}\|^2_{L^2(\hat{G})}$$

suit.

L'analyse harmonique non commutative s'est développée rapidement dans la seconde moitié du XX^e siècle. La notion de transformée de Fourier a été formulée pour des groupes de groupes de type I. Cette classe comprend une très grande partie de groupes (contenant des groupes de Lie). Cependant, les résultats, principalement dus à Mautner et Segal, souffrent de leur généralité et du manque de description explicite de la mesure Plancherel sur l'objet dual $\hat{G}$ (voir [Fol16, Paragraph 7.5]). La partie principale du travail de Harish-Chandra a été de trouver cette mesure explicitement dans le cas du groupe de Lie $\mathrm{SL}(2, \mathbb{R})$ de matrices de déterminant 1 à coefficients réels (voir [HC52]) et plus tard pour un groupe semi-simple arbitraire. Ce fut une percée majeure dans l'analyse harmonique et les méthodes de son travail sont encore largement utilisées aujourd'hui.

Supposons à présent que G soit un groupe semi-simple réel. Harish-Chandra fit apparaître deux grandes familles de représentations dans la formule de la mesure de Plancherel. Elles sont aujourd'hui nommées les *représentations de la série principale*, $\hat{G}_{cont}$, et les *représentations de la série discrète*, $\hat{G}_{discr}$. Leurs ensembles sont notés respectivement $\hat{G}_{cont}$ et $\hat{G}_{discr}$. En ces termes, la formule abstraite de Plancherel pour $L^2(G)$ se présente comme suit:

$$L^2(G) = \int_{\hat{G}_{cont}} \mathscr{H}_\pi \otimes \mathscr{H}_\pi^* d\nu(\pi) \quad + \quad \sum_{\hat{G}_{discr}} \mathscr{H}_{\pi'} \otimes \mathscr{H}_{\pi'}^* \ . \tag{10}$$

On note K un sous-groupe compact maximal de G. L'espace homogène G/K est alors un espace Riemmanien symétrique de type non-compact. Il existe une décomposition du groupe G, appelée *la décomposition d'Iwasawa* (voir Chapitre I). Ainsi pour tout élément x dans G, il y a des éléments uniques $\mathbf{k}(x) \in K$, $\mathbf{a}(x) \in A$ et $\mathbf{n}(x) \in N$ tels que

$$x = \mathbf{k}(x)\mathbf{a}(x)\mathbf{n}(x) \ .$$

où $\mathbf{k}(x) \in K$, $\mathbf{a}(x) \in A$ and $\mathbf{n}(x) \in N$. Le groupe A étant commutatif, l'application "exp" définit une bijection avec son algèbre de Lie $\mathfrak{a}$. On peut ainsi noter $\mathbf{a}(x) = \exp(\mathbf{H}(x))$. On note $\mathfrak{a}^*$ l'ensemble des formes linéaires sur l'espace vectoriel $\mathfrak{a}$. Un des autres nombreux travaux primordiaux de Harish-Chandra est le calcul explicite de la mesure de Plancherel dans le cas des fonctions K-bi-invariantes, soit telles que $f(k_1 g k_2) = f(g)$ pour tout $g \in G$ et $k_1, k_2 \in K$. Dans ce cas, la mesure de Plancherel s'écrit à partir de la fonction c d'Harish-Chandra :

$$d\nu(\lambda) = c(\lambda)^{-2} d\lambda \ ,$$

qui a été calculée explicitement par Gindikin et Karpelevich dans [GK62].

Helgason [Hel00, Hel08] généralisa les mêmes formules de transformées de Fourier sur les espaces G/K. Ses formules de transformées de Fourier s'écrivent comme suit. Soit M le centralisateur de A dans K. Pour des mesures bien choisies et toute fonction f sur G/K suffisamment régulières, définissons la transformée de Fourier $\widetilde{f} : \mathfrak{a}^* \times K/M \to \mathbb{C}$ de f par

$$\widetilde{f}(\lambda, k) = \int_G e^{-(i\lambda+\rho)(\mathbf{H}(g^{-1}k))} f(g) \ dg \ ,$$

où $\rho \in \mathfrak{a}^*$ est une constante qui dépend seulement de G et la formule d'inversion suivante est vraie :

$$f(x) = \int_{\mathfrak{a}^*} \int_K \hat{f}(\lambda, k) e^{(i\lambda-\rho)(\mathbf{H}(x^{-1}k))} |c(\lambda)|^{-2} \ d\lambda \ dk \ .$$

Les analogies avec les formules précédentes sont nombreuses. En effet, si on pose la fonction sphérique

$$\phi_\lambda(x) = \int_K e^{(i\lambda-\rho)(H(xk))} dk \ ,$$

la formule d'inversion peut s'écrire comme dans les autres cas comme une convolution de f avec cette fonction sphérique,

$$f(x) = C_G \int_{\mathfrak{a}^*} \int_{kM} (f * \phi_\lambda)(x) |c(\lambda)|^{-2} \ d\lambda \ dk \ ,$$

avec C_G une constante bien choisie. C'est l'analogue de (9).

Cette thèse utilise une généralisation des formules de Fourier aux sections des fibrés vectoriels au dessus de l'espace homogène G/K. Ces généralisations ont été réalisées par Camporesi dans [Cam97a] et sont basées sur la formule de Plancherel d'Harish-Chandra sur G.

2 Analyse Harmonique sur les fibrés vectoriels homogènes

Un fibré vectoriel E sur une variété associe un espace vectoriel de dimension finie à chaque point de la variété. L'exemple le plus révélateur est celui du fibré tangent, qui associe à chaque point de la variété l'espace des vecteurs tangents à la variété en ce point. Par exemple, le fibré tangent de la sphère dans l'espace est la collection de tous les plans tangents à la sphère. Ces espaces vectoriels sont appelés les fibres du fibré vectoriel. Le lecteur est renvoyé à la section I.4 pour des définitions plus précises.

Un fibré vectoriel E sur G/K est dit *homogène* si G agit sur E (par convention par la gauche) et que l'action translate les fibres (pour tout $x \in G/K$ et $g \in G$, $gE_x = E_{gx}$), et que cette translation soit un isomorphisme linéaire. Tout fibré vectoriel homogène est determiné par une représentation (τ, V_τ) de K. On note un tel fibré E_τ. Si cette représentation n'est pas irréductible, soit $V_\tau = \bigoplus V_{\tau_i}$, le fibré peut lui aussi être décomposé selon les composantes irrédutibles

$$E_\tau = \bigoplus E_{\tau_i} \ .$$

On peut donc considérer que τ est irréductible (donc de dimension finie).

Une section de ce fibré est une appplication qui a un point x de la variété G/K associe une vecteur de la fibre qui lui est associée. Les sections de ce fibré peuvent être vues comme les fonctions sur G à valeurs dans V_τ, qui vérifient

$$f(xk) = \tau(k^{-1})f(x) \quad \text{pour tout } x \in G \text{ et } k \in K \, .$$

On note $C_c^\infty(G,\tau) \simeq (G \otimes V_\tau)^K$ l'espace de ces fonctions qui sont lisses et à support compact. Contrairement aux fonctions sur G/K à valeurs complexes étudiées par Helgason, ces fonctions sont à valeurs dans un espace vectoriel avec des propriétés spécifiques sur les orbites du sous-groupe compact K. Les exemples les plus courants en physique théorique sont les formes différentielles (voir [Ped94]), les spineurs (voir [CP01]) et plus généralement n'importe quel champ tensoriel sur G/K : ces objets apparaissent naturellement dans les modèles physiques et c'est bien entendu naturel de regarder le spectre du Laplacien opérant dessus.

On considère le décomposition de la représentation τ sur M comme suit

$$\tau = \bigoplus_{i \in I} \sigma_i \ , \tag{11}$$

où I est un ensemble fini d'indices et σ_i sont des représentations irreductibles de M. Selon Camporesi, on définit la transformée de Fourier pour $f \in C_c^\infty(G,\tau)$ par

$$\widetilde{f}(\lambda, k) = \int_G e^{-(i\lambda+\rho)(\mathbf{H}(x^{-1}k))} \tau(\mathbf{k}(x^{-1}k)^{-1}) f(x) \, dx$$

pour $\lambda \in \mathfrak{a}^*$, $k \in K$. La formule d'inversion, donnée dans [Cam97a], peut se détailler de la façon suivante. On pose $P = MAN$. Le sous-groupe P est dit *sous-groupe* *parabolique minimal* de G. Soit P' un sous-groupe parabolique cuspidal de G tel que $P \subset P'$ et $A' \subset A$. Soit $P' = M'A'N'$ la décomposition de Langlands de P'. On note $K' = M' \cap K$ la partie "compacte" de M' et $M' = K'A_1N_1$ la décomposition d'Iwasawa correspondante telle que $A = A'A_1$ et $N = N'N_1$. Pour toute série discrète de M', il existe une représentation irréductible $\widetilde{\sigma}'$ de M et $\mu_1 \in \mathfrak{a}_1^*$ telles que σ' soit infinitésimallement équivalente à $\mathrm{Ind}_{MA_1N_1}^{M'}(\widetilde{\sigma}' \otimes \mu_1 \otimes 1)$. Ainsi pour tout $\nu' \in \mathfrak{a}'^*$, la représentation $\mathrm{Ind}_{P'}^{G}(\sigma' \otimes \nu' \otimes 1)$ de la serie principale généralisée de P' peut être vue comme une sous-représentation de $\mathrm{Ind}_{P}^{G}(\widetilde{\sigma}' \otimes (\nu' + \mu_1) \otimes 1)$ et vérifie $\mathrm{Ind}_{P'}^{G}(\sigma' \otimes \nu' \otimes 1)|_K \supset \tau$ si $\tau|_M \supset \widetilde{\sigma}'$. On a alors, pour tout $f \in C_c^\infty(G, \tau)$,

$$f(x) = \frac{1}{d_\tau} \sum_{P'} C_{P'} \sum_{\sigma'} \int_{\mathfrak{a}'^*} \int_K e^{(i\nu' + \mu_1 - \rho)(\mathbf{H}(x^{-1}k))} \tau(\mathbf{k}(x^{-1}k)) P_{\sigma'} \widetilde{f}(\nu' - i\mu_1, k) p_{\sigma'}(\nu')\, d\nu'\, dk$$

où les sommes sont sur l'ensemble des sous-groupes paraboliques cuspidaux P' de G décrit précédemment et sur l'ensemble des représentations σ' telles que $\mathrm{Ind}_{P'}^{G}(\sigma' \otimes \nu' \otimes 1)|_K \supset \tau$. L'opérateur $P_{\sigma'}$ est la projection sur la composante isotypique de $\widetilde{\sigma}'$ dans V_τ. La fonction $p_{\sigma'}$ est appelée la densité de Plancherel.

Si on pose $\varphi_\tau^{\sigma',\nu'}$ la fonction sphérique définie par

$$\varphi_\tau^{\sigma',\nu'} = \int_K \tau(\mathbf{k}(xk)) P_{\sigma'} \tau(k^{-1}) e^{(i\nu' + \mu_1 - \rho)(\mathbf{H}(x^{-1}k))} dk\ , \tag{12}$$

on obtient la formule d'inversion de même forme que dans les cas précédents :

$$f(x) = \frac{1}{d_\tau} \sum_{P'} C_{P'} \sum_{\sigma'} \int_{\mathfrak{a}'^*} (\varphi_\tau^{\sigma',\nu'} * f)(x) p_{\sigma'}(\nu')\, d\nu'\ .$$

Quelques remarques :

- Si τ est la représentation triviale alors $\widetilde{f}$ correspond à la transformée de Helgason-Fourier sur la fonction K-invariante à droite f.
- On peut montrer que $\varphi_\tau^{\sigma',\nu'} * f$ est une fonction lisse à support compact, si f l'est (voir Lemme I.9.3 et son corollaire).
- La formule de p_σ n'est pas connue en général et c'est un problème pour la généralisation en rang supérieur. Dans cette thèse, on utilise [Mia79] qui la donne pour les groupes G tels que la dimension de $\mathfrak{a}$ est 1.

3 Résonances

Les résonances sont des objets spectraux attachés à des opérateurs différentiels agissant sur des domaines non compacts et apparaissent comme des pôles de l'extension méromorphe de la résolvante de ces opérateurs. Ils sont liés à des objets géométriques, dynamiques et analytiques intéressants. Les questions fondamentales concernent l'existence de l'extension méromorphe de la résolvante, les propriétés de distribution et de répartition des résonances, le rang et l'interprétation des opérateurs dits résidus associés aux résonances. En conséquence, ils sont étudiés de manière intensive, dans de nombreux contextes différents, en utilisant une variété de techniques et de points de vue différents. Les références standard pour l'introduction des résonances en mathématiques sont [Agm86, Agm75]. Un aperçu récent, contenant également une longue liste de références, est [Zwo17]. Leur étude a dérivé de l'investigation des opérateurs de Schrödinger sur les espaces euclidiens comme $\mathbb{R}^n$, à une étude du Laplacien sur des espaces courbes, comme des variétés hyperboliques ou asymptotiquement hyperboliques, des espaces symétriques ou localement symétriques.

3.1 Motivations en physique quantique

Comme décrit dans l'introduction de [DZ19], les résonances de diffusion ont une relation directe avec la mécanique quantique. Ils surviennent dans l'étude de la diffusion des ondes et des particules. Les physiciens ont beaucoup appris sur les forces et les interactions dans les atomes et les noyaux grâce à des expériences de diffusion. Les atomes sont bombardés de faisceaux de particules. Ensuite, ces particules sont détectées par des dispositifs qui donnent l'intensité en fonction de l'angle de diffusion ou de l'énergie.

Le Laplacien positif $-\Delta$ sur $L^2(\mathbb{R}^d)$ représente en mécanique quantique l'opérateur d'énergie cinétique d'une particule quantique libre. Les nombreuses interactions, par exemple qui se produisent dans une expérience de diffusion, sont représentées par une fonction à valeur réelle, appelée *potentiel* $V \in L_c^\infty(\mathbb{R}^d)$. Nous supposons sans perte de généralité que V a un support compact, car à l'infini les interactions avec le centre de diffusion sont nulles. L'opérateur d'énergie dans ce cas n'est rien d'autre qu'un opérateur de Schrödinger, également appelé *hamiltonien perturbé* (perturbé par V)

$$H_V = -\Delta + V \ . \tag{13}$$

D'après le théorème de Kato-Rellich, H_V est auto-adjoint. Son spectre est donc réel. Sa résolvante

$$R(\lambda^2) = (H_V - \lambda^2)^{-1}$$

est bien définie sur le demi-plan supérieur $\Im(\lambda) > 0$ comme un opérateur borné sur $L^2(\mathbb{R})$. Lorsqu'il est restreint à $C_c^\infty(\mathbb{R})$, $R(\lambda^2)$ s'étend à une fonction méromorphe de λ sur le plan complexe. Les singularités de la résolvante étendue sont appelées les résonances de cet opérateur de Schrödinger. En général, il est très difficile de les calculer. C'est possible dans des cas très particuliers, et pour un potentiel très particulier. Un cas bien connu est celui de l'opérateur de Laplace sur $\mathbb{R}^n$ (voir section II.1.1). Le but de cette thèse est de faire les mêmes calculs, en remplaçant $\mathbb{R}^n$ par un fibré vectoriel homogène (sur un espace symétrique de type non compact et de rang réel un).

Faisons un calcul plus concret lié à l'équation d'onde. En fait, la solution de cette équation a un taux d'oscillation et de décroissance donné par les résonances de diffusion. En dimension un, l'équation d'onde est donnée par

$$(\partial_t^2 + H_V)\ u(t,x) = (\partial_t^2 - \partial_x^2 + V(x))\ u(t,x) = 0 \ .$$

Supposons les conditions initiales $u(0,x) = u_0(x)$ et $\partial_t w(0,x) = w_1(x)$, où $w_j \in C_c^\infty(\mathbb{R})$. Alors pour tout $A > 0$, on a l'expansion de résonance (séries de Fourier dans le cas où $V = 0$)

$$w(t,x) = \sum_{\Im\lambda_j > -A} e^{-il\lambda x} f_{\lambda,l}(x) + E_A(t,x)\,,$$

où les λ_j varient sur l'ensemble des pôles de la résolvante étendue $R(\lambda)$, la somme est finie, $(H_V - \lambda^2)^{l+1} f_{\lambda,l} = 0$ et le terme d'erreur $E_A(t,x)$ est borné par e^{-At} dans un sens bien défini, voir [DZ19, Théorème 2.9].

Les pôles λ_j, avec $\Im(\lambda_j) < 0$ de la résolvante étendue sont appelés résonances de diffusion ou simplement résonances de l'opérateur H_V. Les simulations numériques des solutions montrent que ces fonctions ont des pics décroissant à l'infini. On remarque que la partie réelle des résonances est la vitesse des oscillations et la partie imaginaire est l'opposé de la vitesse de la décroissance. La théorie qui en résulte attire de nombreux chercheurs.

Bien sûr, il s'agit d'un bref aperçu de l'utilité des résonances dans un cadre physique. Pour des applications dans différentes situations, voir [DZ19, Chapitre 1].

3.2 Résonances de l'opérateur de Laplace sur les espaces symétriques riemanniens non compacts

Les espaces symétriques riemanniens de type non compact ont des structures géométriques importantes pour étudier les résonances du Laplacien. En plus d'être des objets intrinsèquement intéressants, ils jouent le rôle d'espaces modèles pour comprendre des phénomènes sur des géométries plus compliquées ou moins régulières. Dans un cadre typique, on travaille sur une variété riemannienne complète X de géométrie finie, pour laquelle le Laplacien positif Δ est un opérateur essentiellement auto-adjoint sur l'espace de Hilbert $L^2(X)$ des fonctions de carré intégrable sur X. On suppose que Δ a un spectre continu $[\rho_X, +\infty[$, avec $\rho_X \geq 0$. Le spectre de Δ peut avoir des parties discrètes, mais ces parties ne jouent aucun rôle significatif dans la suite; nous les négligeons donc. Aussi, pour simplifier les explications, on translate le Laplacien de sorte que son spectre commence à 0, on change les variables $z \mapsto z^2$ pour la résolvante de sorte que, comme ci-dessus, la résolvante soit analytique en dehors de l'axe réel. La résolvante $R(z) = (\Delta - \rho_X - z^2)^{-1}$ du Laplacien translaté $\Delta - \rho_X$ est alors une fonction holomorphe de z sur le demi-plan complexe supérieur (et sur le demi-plan complexe inférieur). Pour chacun de ces complexes z, $R(z)$ est un opérateur linéaire borné de $L^2(X)$ dans lui-même. En tant que tel, il ne peut pas être étendu sur l'axe réel. Limitons la résolvante au sous-espace dense $C_c^\infty(X)$ des fonctions lisses à support compact sur X. Sur cet espace, l'application $z \mapsto R(z)$ peut admettre une extension méromorphe à travers l'axe réel vers un domaine plus grand dans $\mathbb{C}$ (vers une surface de Riemann au dessus de $\mathbb{C}$). Les pôles de cet extension méromorphe, s'ils existent, sont les résonances, également appelées résonances quantiques ou pôles de diffusion (comme expliqué dans le paragraphe précédent), de Δ.

Introduisons quelques notations. Un espace symétrique riemannien de type non compact est un espace homogène de la forme $X = G/K$ où G est un groupe de Lie semi-simple réel non compact connexe de centre fini et K est un sous-groupe compact maximal de G. Les exemples de base sont les espaces hyperboliques réels n-dimensionnels H^n. Dans ce cas, le groupe de Lie G est le groupe de Lorentz généralisé $\mathrm{SO_e}(n, 1)$ et $K = \mathrm{SO}(n)$. Un espace symétrique riemannien de type non compact X a des sous-espaces plats maximaux, tous de même dimension, appelée rang (réel) de X. Par exemple, le rang de H^n est 1. Puisque X est un espace symétrique du groupe de Lie G, tous les opérateurs naturels agissant sur X, comme le Laplacien et sa résolvante, sont G-invariants. Ils peuvent donc être étudiés en utilisant la théorie des représentations de G. L'étude analytique de la résolvante du Laplacien agissant sur les fonctions sur H^n, en particulier son extension méromorphe à travers son spectre, est classique et bien comprise. Elle joue un rôle central dans l'étude de la résolvante sur des variétés riemanniennes complètes plus générales pour lesquelles les espaces hyperboliques sont des modèles. Toujours dans le cas des fonctions sur un espace symétrique riemannien plus général $X = G/K$, l'étude des résonances peut en principe se faire à l'aide d'une analyse harmonique adaptée, dite analyse d'Helgason-Fourier, qui fournit une diagonalisation du Laplacien et donc une formule explicite pour sa résolvante en tant qu'opérateur intégral singulier sur le spectre. Cette formule a permis à Hilgert et Pasquale [HP09] de déterminer et d'étudier les résonances pour un X arbitraire de rang un. Le cas général de rang supérieur est toujours ouvert. Le cas de rang un a été précédemment résolu par Miatello et Will [MW00] avec une méthode différente, dans le contexte des espaces de Damek-Ricci. D'autres travaux pertinents dans ce contexte sont [MV05] et [Str05]. Les réponses aux problèmes de base concernant l'existence et la localisation des résonances du Laplacien, ainsi que l'interprétation théorique de la représentation des opérateurs dits résiduels aux résonances, ne sont connus que pour (la plupart) des espaces symétriques riemanniens de rang 2. Ces résultats sont apparus dans des articles communs de Hilgert, Pasquale et Przebinda (voir [HPP17a, HPP17b, HPP16]).

Tous les articles mentionnés ci-dessus considèrent le Laplacien agissant sur des fonctions scalaires sur X. Une question plus générale est de considérer le Laplacien agissant sur des sections d'un fibré vectoriel homogène sur X. Un tel fibré est déterminé par une représentation de dimension finie τ de K. Notons ce fibré E_τ. Les sections de E_τ peuvent être vues comme des fonctions vectorielles sur G, à valeurs dans l'espace de la représentation τ, telles que

$$f(xk) = \tau(k^{-1})f(x) \quad \text{pour tout } x \in G \text{ et } k \in K\,.$$

L'espace de ces fonctions qui sont lisses et à support compact est noté $C_c^\infty(G, \tau)$. Cela signifie que nous remplaçons les fonctions à valeurs complexes sur X par des fonctions à valeurs vectorielles qui ont des propriétés de transformations spécifiques sur les orbites du sous-groupe compact K. Ceci est une généralisation de l'étude précédente, dans le sens où les sections du fibré vectoriel homogène E_{triv_K}, pour la représentation triviale de K, sont exactement les fonctions à valeurs complexes sur X.

Des exemples de sections de fibrés vectoriels homogènes sur X sont les formes différentielles, les champs vectoriels, et plus généralement les champs tensoriels sur X:

tous ces objets apparaissent naturellement dans les modèles physiques, et il est donc naturel de rechercher des résonances du Laplacien dans ces contextes.

La résolvante du Laplacien des formes sur un espace symétrique riemannien de rang un de type non-compact a été étudiée par plusieurs auteurs; voir [Cam97a, Cam05b, Cam05a, Cam97b, Ped94, CP04, BO00]. En particulier, [CP04] donne (pour les formes différentielles) la liste des résonances et la surface de Riemann sur laquelle la résolvante admet l'extension méromorphe. Néanmoins, à notre connaissance, il n'existe qu'un seul article étudiant les résonances et les opérateurs résiduels du Laplacien agissant sur les sections d'un fibré vectoriel homogène sur X, à savoir [Wil03], où X est l'espace hyperbolique complexe et les fibres sont de dimension 1.

4 Résultats de la thèse

Le but de cette thèse est d'étudier les résonances du Laplacien agissant sur les sections d'un fibré vectoriel homogène sur un espace symétrique riemannien de type non compact $X = G/K$. L'espace symétrique est supposé avoir un rang réel égal à un mais la représentation de K est arbitraire. Puisque toute représentation de dimension finie de K se décompose en irréductibles, nous restreignons notre attention aux représentations irréductibles. Comme dans le cas des fonctions, les problèmes que nous soulevons dans cette thèse, sont de déterminer l'existence et la localisation des résonances et d'étudier les opérateurs de résidus qui leur sont associés.

Cette thèse est organisée comme suit. Dans le chapitre I, nous introduisons les notations et rappelons quelques faits de base sur la structure des espaces symétriques riemanniens de type non compact et de rang réel un. Il y a quatre cas, répertoriés dans le tableau suivant:

G	K	$X = G/K$
$\mathrm{Spin}(n,1)$	$\mathrm{Spin}(n)$	espace hyperbolique réel
$\mathrm{SU}(n,1)$	$\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1))$	espace hyperbolique complexe
$\mathrm{Sp}(n,1)$	$\mathrm{Sp}(n)$	espace hyperbolique quaternionique
$\mathscr{F}_4$	$\mathrm{Spin}(9)$	espace hyperbolique octonionique

On pose $\mathfrak{a}$ comme un sous-espace plat maximal dans $\mathfrak{p} = T_{eK}(X)$ l'espace tangent de X au point de base eK, et M le centralisateur de $\mathfrak{a}$ dans K. Dans la section I.9, nous rappelons quelques faits sur la généralisation de la transformée d'Helgason-Fourier aux fibrés vectoriels homogènes. Ils sont principalement dus à Camporesi (voir [Cam97a]). En particulier, le théorème de Plancherel pour les sections L^2 des fibrés vectoriels homogènes y est donné. Désignons la décomposition de τ sur M comme suit :

$$\tau = \bigoplus_{\sigma \in \hat{M}(\tau)} d_\sigma \sigma \tag{14}$$

où $\hat{M}(\tau)$ est l'ensemble des représentations unitaires irréductibles de M qui apparaissent dans la restriction de τ à M, d_σ est le degré de σ. Nous avons besoin de certaines propriétés des fonctions sphériques généralisées $\varphi_\tau^{\sigma,\lambda}$ associées aux représentations irréductibles $\sigma \in \hat{M}(\tau)$, qui sont détaillées dans la section I.9. La formule explicite de la densité de Plancherel p_σ correspondant à ces $\sigma \in \hat{M}$ est donnée dans la proposition II.1.4 (voir aussi l'annexe A). Le corollaire I.9.1 prouve la convergence de l'opérateur intégral singulier fournissant une formule explicite pour la résolvante R du Laplacien en utilisant la formule d'inversion de la transformée de Helgason-Fourier à valeurs vectorielles. Dans la section II.1, nous calculons les résonances, ce qui est le premier objectif principal de cette thèse. La fonction holomorphe $z \mapsto R(z)$ est étendue méromorphiquement du demi-plan supérieur complexe à l'espace entier, en utilisant le théorème des résidus. La résolvante étendue est une fonction méromorphe avec des pôles simples sur l'axe imaginaire: ces pôles sont les résonances. Cela conduit à notre premier théorème :

Theorem 1

Soit G un groupe de Lie semi-simple non compact connexe de centre fini et de décomposition d'Iwasawa $G = KAN$, où K est un sous-groupe compact maximal fixe de G. Supposons que $\dim A = 1$. *Soit M le centralisateur de A en K. Soit $(\tau, \mathscr{H}_\tau)$ une représentation unitaire irréductible de K, et soit E_τ le fibré vectoriel homogène sur G associé à τ. Pour chaque $\sigma \in \hat{M}(\tau)$, soit $\mathbb{N}_\sigma$ l'ensemble de $k \in \mathbb{Z}$ tel que*

$$\lambda_k := -i(B_{\max} + k)$$

est un pôle de la densité de Plancherel (voir (II.15) *pour la formule) et $B_{\max} + k \geq 0$. Ici $B_{\max}$ est une constante non négative qui ne dépend que de G et σ. On se réfère à* (II.17), (II.18) *et* (II.20) *pour la définition précise.*

Dans ce cadre, la continuation méromorphe de la résolvante R de l'opérateur de Laplace agissant sur les sections lisses et à support compact de E_τ peut être écrit comme la somme

$$R = \sum_{\sigma \in \hat{M}(\tau)} d_\sigma R_\sigma \,, \tag{15}$$

où R_σ est donné pour tout $f \in C_c^\infty(G,\tau)$ et pour tout $N \in \mathbb{N}$ par la formule suivante:

$$\begin{aligned}\left(R_\sigma(\zeta_\sigma) f\right)(x) = & \frac{1}{|\alpha|} \int_{\mathbb{R}-i(N+1/4)} \frac{1}{\zeta_\sigma - \lambda|\alpha|} \Big(\varphi_\tau^{\sigma,\lambda\alpha} * f\Big)(x) \; \frac{p_\sigma(\lambda\alpha)}{\lambda} \, d\lambda \\ & + \frac{2i\pi}{|\alpha|} \sum_{\substack{k \in \mathbb{N}_\sigma \\ \lambda_k > -i(N+1/4)}} \frac{1}{\zeta_\sigma - \lambda_k|\alpha|} \Big(\varphi_\tau^{\sigma,\lambda_k\alpha} * f\Big)(x) \; \operatorname*{Res}_{\lambda=\lambda_k} \frac{p_\sigma(\lambda\alpha)}{\lambda}\end{aligned} \tag{16}$$

Dans (16), *α est la racine restreinte la plus longue, $\varphi_\tau^{\lambda,\sigma}$ est une fonction sphérique de type σ et*

$$\zeta_\sigma := \sqrt{-z - \langle \rho, \rho \rangle + \langle \mu_\sigma + \rho_M, \mu_\sigma + \rho_M \rangle} \tag{17}$$

avec $z \in \mathbb{C}$ tel que $\Im(\zeta_\sigma) > -(N + 1/4)$. Ici, $\sqrt{\cdot}$ désigne la branche à valeur unique de la fonction racine carrée déterminée sur $\mathbb{C} \setminus [0, +\infty[$ par la condition $\sqrt{-1} = -i$.

Les résonances de l'opérateur de Laplace agissant sur les sections de E_τ apparaissent dans des familles paramétrées par les éléments de $\hat{M}(\tau)$. Soit

$$S_\sigma = \Big\{(z,\zeta) \in \mathbb{C}^2 \mid \zeta^2 := -z - \langle \rho,\rho\rangle + \langle \mu_\sigma + \rho_M, \mu_\sigma + \rho_M\rangle)\Big\} \ .$$

Alors la resolvante R_σ s'étend méromorphiquement de $S_\sigma^+ = \{(z,\zeta) \in S \mid \Im(\zeta) > 0\}$ à S_σ. Les pôles (simples) de cette extension sont les couples

$$(z_{\sigma,k},\zeta_{\sigma,k}) = \Big((B_{\max}+k)^2|\alpha|^2 - \rho_\alpha^2|\alpha|^2 + \langle \mu_\sigma + \rho_M, \mu_\sigma + \rho_M\rangle \ , -i(B_{\max}+k)^2|\alpha|^2 \ \Big) \quad (18)$$

où μ_σ est le plus haut poids de la représentation σ, les nombres k sont dans $\mathbb{N}_\sigma$ et ρ_M est la demi-somme des racines pour M.

Nous renvoyons à la section I pour les définitions des différents objets apparaissant dans ce théorème et à la section II.1 pour sa démonstration.

Le deuxième problème que nous abordons dans cette thèse est l'interprétation théorique des représentations issues des résonances. Plus précisément, considérons la partie résiduelle de la suite méromorphe de la résolvante dans (16). Pour chaque pôle λ_k de la densité de Plancherel pour $\sigma \in \hat{M}(\tau)$, on peut introduire un opérateur, appelé opérateur résiduel en $\lambda_k\alpha$, défini comme suit:

$$\begin{array}{rccc} R_k^\sigma : & C_c^\infty(G,\tau) & \longrightarrow & C^\infty(G,\tau) \\ & f & \longmapsto & \varphi_\tau^{\sigma,\lambda_k\alpha} * f \end{array} \quad (19)$$

Puisque le produit de convolution est à gauche, il ne permute pas avec la translation à gauche de f (voir (I.22)). Comme G agit sur l'image de R_k^σ par translation à gauche, nous obtenons une représentation de G, appelée représentation résiduelle en $\lambda_k\alpha$.

Dans la section II.2 nous limitons notre attention aux représentations τ qui contiennent la représentation triviale de M. Dans ce cas, la structure de la série principale est bien connue (voir [HT93, JW77, Joh76]). La complexité du cas général (voir [Col85]) est particulièrement intéressante et pourrait conduire à des résultats beaucoup moins généralisables. Nous l'évitons donc dans cette thèse. On considère les résonances correspondant à $\sigma = \text{triv}$. Pour simplifier la notation, nous écrivons R_k au lieu de R_k^{triv}. Soit $\mathscr{E}_k$ la représentation résiduelle en $\lambda_k\alpha$. Nous montrons que les $\mathscr{E}_k$ sont irréductibles et équivalentes à un sous-quotient d'une représentation de la série principale sphérique de G. Nous déterminons lesquelles d'entre elles sont unitaires et lesquelles sont de dimension finie. De plus, nous identifions leurs paramètres de Langlands et calculons leurs fronts d'onde. Les paramètres de Langlands sont de la forme (MA,δ,ν) et désignent la représentation induite $\text{Ind}_{MAN}^G(\delta \otimes e^{i\nu} \otimes \text{triv})$. Un K-type minimal de la représentation induite de plus haut poids $\mu_{\min}$ identifie un sous-quotient irréductible unique de cette représentation induite (voir [Vog77]). L'ensemble du front d'onde d'une représentation a été introduit par Howe (voir [How81]). Lorsque G est semi-simple, c'est un ensemble fermé constitué d'orbites nilpotentes. Pour $\mathscr{E}_k$, il s'agit de la fermeture d'une seule orbite nilpotente. Dans le théorème suivant, α est la plus longue racine restreinte comme dans le théorème 1 et pour chaque racine restreinte β, l'espace correspondant dans $\mathfrak{g}$ est noté $\mathfrak{g}_\beta$. Le K-type minimal de $\mathscr{E}_k$ est donné dans la démonstration du théorème dans chaque cas: on les retrouve dans les tables II.1, II.2, II.3 et II.4 respectivement pour les espaces hyperboliques réel, complexe, quaternionique et octonionique.

Theorem 2
Supposons que la représentation τ contienne la représentation triviale de M. Les représentations de résidus $\mathscr{E}_k$ sont alors irréductibles.

1. *Si $G = \mathrm{Spin}(2n,1)$, alors τ a un plus haut poids de la forme $(N,0,\ldots,0)$, où N est un entier non négatif.*
   - *Si $N \geq k+1$, alors $\mathscr{E}_k$ a les paramètres de Langlands $\Big(MA, \mathscr{H}^{k+1}(\mathbb{R}^{2n-1}), \left(n-\frac{3}{2}\right)\alpha\Big)$ avec $(k+1,0,\ldots,0)$ comme plus haut poids du K-type minimal. Ici $\mathscr{H}^{k+1}(\mathbb{R}^{2n-1})$ sont les polynômes harmoniques de degré $k+1$ sur $\mathbb{R}^{2n-1}$. Cette représentation est unitaire. Son ensemble de front d'onde est l'orbite nilpotente générée par $\mathfrak{g}_\alpha$.*
   - *Si $N < k+1$, alors $\mathscr{E}_k$ a le paramètre de Langlands $\Big(MA, \mathrm{triv}, (\rho_\alpha + k)\Big)$ avec la représentation triviale comme K-type minimal. Il est de dimension finie. De plus, il n'est pas unitaire si $k \neq 0$.*

2. *Si $G = \mathrm{SU}(n,1)$, alors τ a un plus haut poids de la forme $(M_1,0,\ldots,0,-M_2,-L)$, où M_1 et M_2 sont des entiers positifs tels que $M_1 \geq M_2 \geq 0$, $L \in \mathbb{Z}$ et M_1+M_2+L est pair.*
   - *Si $M_1+M_2 \geq 2k+2$ et $|L| \leq -2k-2+M_1+M_2$, alors $\mathscr{E}_k$ est unitaire.*
     - *si $n > 2$, alors $\mathscr{E}_k$ a un K-type minimal de plus haut poids $\Big((k+1),0,\ldots,0,-(k+1),0\Big)$. Son paramètre de Langlands est $\Big(MA, \delta, \left(\frac{n}{2}-1\right)\alpha\Big)$ où le plus haut poids de δ est $\Big((k+1),0,\ldots,0,-(k+1),0\Big)$. Son front d'onde est l'orbite nilpotente générée par $\mathfrak{g}_{\alpha/2}$.*
     - *si $n = 2$, cette représentation est la série discrète avec le paramètre Blattner $((k+1),-(k+1),0,\ldots,0)$. Son front d'onde est l'orbite nilpotente générée par $\mathfrak{g}_{\alpha/2}$.*
   - *Si $L \geq |-2k-1+M_1+M_2|+1$, alors $\mathscr{E}_k$ est la représentation avec le paramètre de Langlands $\Big(MA, \delta, \left(\frac{k}{2}+\frac{n}{2}-\frac{1}{2}\right)\alpha\Big)$, où le plus haut poids de δ est $\Big(0,0,\ldots,0,-(k+1),(k+1)/2\Big)$. Cette représentation n'est pas unitaire. Son front d'onde est l'orbite nilpotente générée par l'élément n_2 de $\mathfrak{g}_\alpha$ (voir le lemme III.4.2 pour la définition).*
   - *Si $L \leq |-2k-1+M_1+M_2|-1$, alors $\mathscr{E}_k$ est la représentation avec le paramètre de Langlands $\Big(MA, \delta, \left(\frac{k}{2}+\frac{n}{2}-\frac{1}{2}\right)\alpha\Big)$, où le plus haut poids de δ est $\Big((k+1),0,\ldots,0,(k+1)/2\Big)$. Cette représentation n'est pas unitaire. Son front d'onde est l'orbite nilpotente générée par l'élément n_1 de $\mathfrak{g}_\alpha$ (voir le lemme III.4.2 pour la définition).*

- *Si* $M_1 + M_2 \in [0, 2k+2[$ *et* $L < |2k+2-M_1+M_2|$, *alors* $\mathscr{E}_k$ *est la représentation avec le paramètre de Langlands* $\Big(MA, \mathrm{triv}, (\rho_\alpha + k)\alpha\Big)$. *Il est de dimension finie et non unitaire (si* $k \neq 0$*).*

3. *Si* $G = \mathrm{Sp}(n,1)$, *alors* τ *a un plus haut poids de la forme* $(t_1, t_2, 0, \dots, 0, t_{n+1})$, *où* t_1, t_2 *et* t_{n+1} *sont des entiers positifs tels que* $t_1 \geq t_2$, $t_{n+1} \leq t_1 + t_2$ *et* $t_1 + t_2 + t_{n+1}$ *est pair.*

   - *Si* $t_{n+1} \leq t_1 + t_2 - 2k - 4$, *alors* $\mathscr{E}_k$ *est la représentation avec le paramètre de Langlands* $\Big(MA, \delta, \pm\left(n - \frac{3}{2}\right)\alpha\Big)$ *avec* τ *comme* K*-type minimal, où le plus haut poids de* δ *est* $(k+2, k+2, 0, \dots, 0)$. *Cette représentation n'est pas unitaire. Son front d'onde est l'orbite nilpotente générée par* $\mathfrak{g}_{\alpha/2}$.
   - *Si* $t_{n+1} \geq |t_1 + t_2 - 2k - 2|$ *la représentation résiduelle est unitaire. Son ensemble de front d'onde est l'orbite nilpotente générée par* $\mathfrak{g}_\alpha$.
     - *Si* $k \leq 2n - 4$, *alors* $\mathscr{E}_k$ *est la représentation avec le paramètre de Langlands* $\Big(MA, \delta, \pm\left(n - \frac{k}{2}\right)\alpha\Big)$ *avec le plus petit* K*-type* $(k+1, 0, \dots, 0, k+1)$, *où le plus haut poids de* δ *est* $(k+1, 0, \dots, 0, \frac{k+1}{2})$.
     - *Si* $k \geq 2n - 3$, *alors* $\mathscr{E}_k$ *est la représentation en série discrète avec le paramètre Blattner* $\mu_k = (k+1, 0, \dots, 0, k+1)$.
   - *Si* $t_1 + t_2 < 2k + 2 - t_1 - t_2$, *alors* $\mathscr{E}_k$ *est la représentation avec le paramètre de Langlands* $\Big(MA, \mathrm{triv}, (\rho_\alpha + k)\alpha\Big)$. *Il est de dimension finie et non unitaire (si* $k \neq 0$*).*

4. *Si* $G = F_4$, *alors* τ *a un plus haut poids de la forme* $(a/2, b/2, b/2, b/2)$, *où* a *et* b *sont entiers positifs tels que* $a \geq b$ *et* ab *sont pairs.*

   - *Si* $b \leq a - 2k - 8$, *alors* $\mathscr{E}_k$ *est la représentation avec le paramètre de Langlands* $\Big(MA, \delta, \frac{1}{2}(k+1)\alpha\Big)$, *où le plus élevé de* δ *est* $\frac{k+4}{4}(3,1,1,1)$. *Cette représentation n'est pas unitaire. Son front d'onde est l'orbite nilpotente générée par* $\mathfrak{g}_{\alpha/2}$.
   - *Si* $b > a - 2k - 8$ *et* $b \geq 2k + 2 - a$, *alors* $\mathscr{E}_k$ *est la représentation avec le paramètre de Langlands* $\Big(MA, \delta, \frac{1}{2}(k+10)\alpha\Big)$, *où le plus élevé de* δ *est* $\frac{k+1}{4}(3,1,1,1)$. *Cette représentation est unitaire. Son front d'onde est l'orbite nilpotente générée par* $\mathfrak{g}_\alpha$.
   - *Si* $b < 2k + 2$, *alors* $\mathscr{E}_k$ *est la représentation avec le paramètre de Langlands* $\Big(MA, \mathrm{triv}, (\rho_\alpha + k)\alpha\Big)$. *Il est de dimension finie et non unitaire (si* $k \neq 0$*).*

Une belle conséquence de ces résultats au cas par cas est la suivante.

Corollary 4.0.1
*Pour un $k \in \mathbb{N}$ fixe, il existe une correspondance biunivoque entre les sous-quotients irréductibles de $\mathscr{H}_{\lambda_k \alpha}$ et les orbites nilpotentes (réelles) de $\mathfrak{g}$ sous l'action adjointe. Cette correspondance envoie chaque sous-quotient dans l'orbite dont la fermeture est l'ensemble du front d'onde de ce sous-quotient. Comme nous l'avons montré, $\mathscr{E}_k$ est équivalent à l'un de ces sous-quotients. Son ensemble de front d'onde est alors la fermeture d'*une *orbite nilpotente dans* $\mathfrak{g}$.

Introduction (English version)

In this introduction, we first sketch a (very) brief history of harmonic analysis, and point out some milestones on the study of the Fourier transform on Lie groups. We then outline some results in the harmonic analysis on a homogeneous vector bundle over a Riemannian symmetric space of non-compact type. Next, we introduce the notion of resonances of the Laplacian. After a short presentation of the general context motivating their definition and study we pass to case of the Laplacian acting on the compactly supported smooth sections of a homogeneous vector bundle over a Riemannian symmetric space X of non-compact type. Our study is restricted to X of rank one , but the homogeneous vector bundle over X is arbitrary. The last section collects the main results of this thesis.

1 A brief overview of harmonic analysis

In music, harmonics are the components of the sound of an instrument. They are different for each instrument and are precisely what creates the *timbre*. This allows us to distinguish different instruments. Sounds are superpositions of harmonics, with frequencies that are multiples of a fundamental frequency. The process of breaking down a sound into its harmonics is referred to as harmonic analysis.

This is where the name “harmonic analysis” in mathematical originates. The sound is replaced by a function that we try to break down into simple components. The best known tools are expansions in Fourier series and Fourier transform. If f is a sufficiently regular complex-valued function defined on $\mathbb{R}/\mathbb{Z}$, then

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n(f) e^{inx} \ . \tag{1}$$

Here $c_n(f)$ is the Fourier coefficient of f, defined by

$$c_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} \, dt \ . \tag{2}$$

Formula (1) gives the decomposition into its harmonics $x \mapsto c_n(f)e^{inx}$.

Similarly, if F is a sufficiently regular, complex valued function on $\mathbb{R}$, then

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{F}(\xi) e^{i\xi x} \, d\xi \ , \tag{3}$$

where $\hat{F}$ is the Fourier transform of F

$$\hat{F}(\xi) = \int_{-\infty}^{+\infty} F(x) \ e^{-i\xi x} \, dx \ . \tag{4}$$

Here the harmonics are $x \mapsto \hat{F}(\xi)e^{i\xi x}$. We manage to “reconstruct” the functions f and F from their harmonics thanks to the inversion formulas (1) and (3). The inversion theorem (3) holds if all integrals converge, i.e. for $F \in L^1(\mathbb{R})$ and $\hat{F} \in L^1(\mathbb{R})$. Moreover, Plancherel’s theorem (1910) states that the Fourier transform, defined on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, uniquely extends to a Hilbert space isomorphism of $L^2(\mathbb{R})$ in itself.

The Fourier series were introduced by Joseph Fourier (1768 - 1830) to solve a partial differential equation, known as the “vibrating string” problem

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \, .$$

Physically, this equation describes the motion of a freely vibrating string. Later, using the same method, he solved the heat equation

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} \ . \tag{5}$$

These works can be found in his "Analytical Theory of Heat" [Fou09].

In contemporary terms, what we have here are the one dimensional *unitary representations* $(e_n, \mathbb{C})$ of the commutative group $\mathbb{R}/\mathbb{Z}$ on the Hilbert space $\mathbb{C}$, indexed by $n \in \mathbb{Z}$,

$$\begin{aligned} \mathbb{R}/\mathbb{Z} \times \mathbb{C} &\to \mathbb{C} \\ (t, z) &\mapsto e_n(t) \cdot z := e^{int} \cdot z \, . \end{aligned} \tag{6}$$

Here $\mathbb{C}$ is called *the space of the representation* $(e_n, \mathbb{C})$.

A representation (π, V) of a group G on a Hilbert space V is called *unitary*, if

$$\pi : G \to U(V)$$

is a group homomorphism from G into the group $U(V)$ of the unitary operators on V, such that for every vector $v \in V$ the function $G \ni g \to \pi(g)v \in V$ is continuous.

Two representations are said to be *equivalent*, if there exists an isomorphism of their underlying Hilbert spaces which intertwines the corresponding group actions (see for example [Hal15, Definition 4.3]). A representation is said to be *irreducible* if no non-trivial proper Hilbert subspace of V is invariant under $\pi(G)$ (see for example [Hal15, Definition 4.2] for the exact definition). We denote by $\hat{G}$ the set of all equivalence classes of the irreducible unitary representations of G. The formula (6) shows that for $G = \mathbb{R}/\mathbb{Z}$, $\hat{G}$ coincides with the integers. Fourier inversion (1) shows that we can reconstruct f by adding these projections of f on each irreducible representation.

The case $G = \mathbb{R}$ is analogous. Indeed,

$$\begin{aligned} \mathbb{R} \times \mathbb{C} &\to \mathbb{C} \\ (t, z) &\mapsto e_\xi(t) \cdot z := e^{i\xi t} \cdot z \end{aligned} \tag{7}$$

is the analogue of (6) for $\mathbb{R}$. The sum gives way to the "continuous sum" (the integral) over $\mathbb{R}$ and we see that the two formulas (1) and (3) have no longer significant differences, except that the unitary dual $\hat{\mathbb{R}}$ is identified with $\mathbb{R}$. Notice that

$$\hat{F}(\xi)e^{i\xi x} = \int_{\mathbb{R}} F(t) e_\xi(x - t) \, dt =: F * e_\xi(x) \, . \tag{8}$$

This allows us to rewrite the inversion formula as

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F * e_\xi(x) \, d\xi \, . \tag{9}$$

The generalization of Fourier analysis to a commutative locally compact group G arrived at the beginning of the XXth century with the works of Schur, Weil, H. Cartan, Godement and Pontryagin. In this case $\hat{G}$ consists of continuous homorphisms $G \to \mathbb{S}^1$. Here $\mathbb{S}^1$ is the unit circle in the complex plane and $\hat{G}$, with multiplication of functions, is a group. By analogy, one defines the Fourier transform of $f \in L^1(G)$ as the function of $\hat{G}$ given by

$$\hat{f}(\pi) = \int_G \overline{\pi(g)} f(g) dg \, .$$

The Fourier transform can also be defined on $L^2(G)$. It was shown that there is a Haar measure $d\nu$ on $\hat{G}$, such that the Fourier transform is a unitary isomorphism of $L^2(G)$ onto $L^2(\hat{G})$. We have the Plancherel formula

$$\|f\|^2_{L^2(G)} = \|\hat{f}\|^2_{L^2(\hat{G})}$$

and, if $f \in L^1(G)$ and $\hat{f} \in L^1(\hat{G})$, the inversion formula

$$f(g) = \int_{\hat{G}} \pi(g)\hat{f}(\pi)\, d\nu(\pi)\ .$$

One refers to this $d\nu$ as to the *the Plancherel measure.*

For compact G groups (not necessarily commutative) , we had to wait for the famous Peter-Weyl theorem (1927), which says that for any function $f \in C(G)$ and $g \in G$,

$$f(g) = \sum_{\pi \in \hat{G}} d_\pi \operatorname{Tr}\left(\pi(g)\hat{f}(\pi)\right),$$

where

$$\hat{f}(\pi) = \int_G \pi(g)^* f(g) dg\,,$$

and the integer d_π is the dimension of the space of the representation π. The Plancherel-Parseval formula

$$\|f\|^2_{L^2(G)} = \|\hat{f}\|^2_{L^2(\hat{G})}$$

follows. The non-commutative harmonic analysis developped rapidly in the second half of the twentieth century. The notion of the Fourier transform was formulated for groups of type I. This class includes a very large part of groups (containing Lie groups). However, the results, mainly due to Mautner and Segal, suffer from its generality and lack of explicit description of the Plancherel measure on the dual object $\hat{G}$ (see [Fol16, Paragraph 7.5]). The main part of Harish-Chandra's work was to find this measure explicitly in the case of the Lie group $\mathrm{SL}(2,\mathbb{R})$ of matrices of determinant 1 with real coefficients (see [HC52]) and later for an arbitrary real semisimple group. This was a major breakthrough in harmonic analysis and the methods of his work are still widely used today.

From now on we assume that G is a real semi-simple group with a finite center. Harish-Chandra revealed two large families of representations which appear in the Plancherel measure. They are called *representations of the principal series*, $\hat{G}_{cont}$, and the *representations of the discrete series*, $\hat{G}_{discr}$. In these terms the abstract Plancherel formula for $L^2(G)$ looks as follows,

$$L^2(G) = \int_{\hat{G}_{cont}} \mathscr{H}_\pi \otimes \mathscr{H}^*_\pi d\nu(\pi) \quad + \quad \sum_{\hat{G}_{discr}} \mathscr{H}_{\pi'} \otimes \mathscr{H}^*_{\pi'}\,. \tag{10}$$

Denote by K a maximal compact subgroup of G. The homogeneous space G/K is called a Riemmanian symmetric space of non-compact type. There exists a decomposition of the group G, $G = KAN$, called *the Iwasawa decomposition* (see Chapter

I). Thus for any element x in G, we have unique elements $\mathbf{k}(x) \in K$, $\mathbf{a}(x) \in A$ and $\mathbf{n}(x) \in N$ such that

$$x = \mathbf{k}(x)\mathbf{a}(x)\mathbf{n}(x) \ .$$

The map $\exp : \mathfrak{a} \to A$, from the Lie algebra to the group, is a bijection. Hence there is a unique $\mathbf{H}(x) \in \mathfrak{a}$ such that $\mathbf{a}(x) = \exp(\mathbf{H}(x))$. We denote by $\mathfrak{a}^*$ the space of linear forms on vector space $\mathfrak{a}$. One of Harish-Chandra's early works is the Plancherel theorem for the space of square integrable K-bi-invariant functions on G (i.e. $f(k_1 g k_2) = f(g)$ for all $g \in G$ and $k_1, k_2 \in K$). In this case, the Plancherel measure is written in term of Harish-Chandra's c function,

$$d\nu(\lambda) = |c(\lambda)|^{-2} d\lambda \ ,$$

and was explicitly computed by Gindikin and Karpelevich in [GK62].

Helgason [Hel00, Hel08] generalized the Plancherel theorem from the K-bi-invariant functions to functions defined on the symmetric space G/K. His Fourier transform formulas looks as follows. Let M be the centralizer of A in K. For suitably normalized measures and sufficiently regular functions f on G/K, define the Helgason-Fourier transform $\widetilde{f} : \mathfrak{a}^* \times K/M \to \mathbb{C}$ by

$$\widetilde{f}(\lambda, k) = \int_G e^{-(i\lambda+\rho)(\mathbf{H}(g^{-1}k))} f(g) \ dg \ ,$$

where $\rho \in \mathfrak{a}^*$ depends only on G, and the following inversion formula is true:

$$f(x) = \int_{\mathfrak{a}^*} \int_k \widetilde{f}(\lambda, k) e^{(i\lambda-\rho)(\mathbf{H}(x^{-1}k))} |c(\lambda)|^{-2} \ d\lambda \ d(k) \ .$$

There are similarities with the preceding Euclidean formulas. Indeed, if we define the spherical function

$$\phi_\lambda(x) = \int_K e^{(i\lambda-\rho)(H(xk))} dk \ ,$$

then the inversion formula can also be written in terms of a convolution of f with this spherical function,

$$f(x) = \int_{\mathfrak{a}^*} \int_K (f * \phi_\lambda)(x) |c(\lambda)|^{-2} \ d\lambda \ dk \ .$$

This is the analog of (9).

This thesis uses a generalization of Helgason's Fourier formulas to sections of vector bundles over the homogeneous space G/K. These generalizations are due to Camporesi (see [Cam97a]) and are based on Harish-Chandra's Plancherel formula on G.

2 Harmonic analysis on homogeneous vector bundles

A vector bundle E over a manifold associates a vector space with each point of the manifold. The most telling example is that of the tangent bundle, which associates with each point of the manifold the space of vectors tangent to the manifold at this point. For example, the tangent bundle of the sphere in $\mathbb{R}^3$ is the collection of all planes tangent to the sphere. These vector spaces are called the fibers of the vector bundle. The reader is referred to the I.4 for more precise definitions.

A vector bundle E over G/K is called *homogeneous* if G acts on E (by convention on the left) and the action translates the fibers (for all $x \in G/K$ and $g \in G$, $gE_x = E_{gx}$), and each translation is a linear isomorphism. Any homogeneous vector bundle is determined by a representation (τ, V_τ) of K. We denote a such bundle by E_τ. If $V_\tau = \bigoplus V_{\tau_i}$ is a sum of irreducible components, then the bundle can also be decomposed,

$$E_\tau = \bigoplus E_{\tau_i} \ .$$

We focus on the case when τ is irreducible (hence of finite dimension).

A section of the vector bundle is a function which to a point $x \in G/K$ associates a vector in the fiber over it. The sections can be seen as functions on G with values in V_τ satisfying the following equation,

$$f(xk) = \tau(k^{-1})f(x) \quad \text{ for all } x \in G \text{ and } k \in K \ .$$

We denote by $C_c^\infty(G,\tau) \simeq (C_c^\infty(G) \otimes V_\tau)^K$ the space of smooth and compactly supported functions as above. Compared to the complex valued functions on G/K studied by Helgason, these functions are valued in a vector space and have the above specified properties under the action of the compact subgroup K by right translations. The most common examples in theoretical physics are differential forms (see [Ped94]), spinors (see [CP01]) and more generally any tensor field on G/K: these objects appear in physical models and it is natural to study the spectrum of the Laplacian operating on them.

We consider the restriction of the representation τ to M,

$$\tau|_M = \bigoplus_{i \in I} \sigma_i \ , \tag{11}$$

where I is a finite index set and the σ_i are irreducible representations of M. Following Camporesi, we define the Fourier transform of $f \in C_c^\infty(G,\tau)$ by

$$\widetilde{f}(\lambda, k) = \int_G e^{-(i\lambda+\rho)(\mathbf{H}(x^{-1}k))} \tau(\mathbf{k}(x^{-1}k)^{-1}) f(x) \ dx \ ,$$

where $\lambda \in \mathfrak{a}^*$ and $k \in K$. The inversion formula, given in [Cam97a], can be explained as follows. We set $P = MAN$. The subgroup P is called a *minimal parabolic subgroup* of G. Let P' be a cuspidal parabolic subgroup of G such that $P \subset P'$ and $A' \subset A$. Let $P' = M'A'N'$ be the Langlands decomposition of P'. We denote by $K' = M' \cap K$ the "compact" part M' and $M' = K'A_1N_1$ the corresponding Iwasawa decomposition such that $A = A'A_1$ and $N = N'N_1$. For any discrete series of M', there is an irreducible representation $\widetilde{\sigma}'$ of M and $\mu_1 \in \mathfrak{a}_1^*$ such that σ' is infinitesimally equivalent to $\mathrm{Ind}_{MA_1N_1}^{M'}(\widetilde{\sigma}' \otimes \mu_1 \otimes 1)$. Thus for all $\nu' \in \mathfrak{a}'^*$, the representation $\mathrm{Ind}_{P'}^G(\sigma' \otimes \nu' \otimes 1)$ of the generalized principal series of P' can be seen as a subrepresentation of $\mathrm{Ind}_P^G(\widetilde{\sigma}' \otimes$

$\nu' + \mu_1 \otimes 1)$ and verify $\mathrm{Ind}_{P'}^{G}(\sigma' \otimes \nu' \otimes 1)|_K \supset \tau$ if $\tau|_M \supset \widetilde{\sigma}'$. So we have, for all $f \in C_c^\infty(G, \tau)$,

$$f(x) = \frac{1}{d_\tau} \sum_{P'} C_{P'} \sum_{\sigma'} \int_{\mathfrak{a}'^*} \int_K e^{(i\nu' + \mu_1 - \rho)(\mathbf{H}(x^{-1}k))} \tau(\mathbf{k}(x^{-1}k)) P_{\sigma'} \widetilde{f}(\nu' - i\mu_1, k) p_{\sigma'}(\nu') \, d\nu' \, dk$$

where the sums are over the set of cuspidal parabolic subgroups P' of G described previously and over the set of representations σ' such that $\mathrm{Ind}_{P'}^{G}(\sigma' \otimes \nu' \otimes 1)|_K \supset \tau$. The operator $P_{\sigma'}$ is the projection on the isotypic component of $\widetilde{\sigma}'$ in V_τ. The function $p_{\sigma'}$ is called the *Plancherel density.*

If we denote by $\varphi_\tau^{\sigma',\nu'}$ the corresponding spherical function

$$\varphi_\tau^{\sigma',\nu'} = \int_K \tau(\mathbf{k}(xk)) P_{\sigma'} \tau(k^{-1}) e^{(i\nu' + \mu_1 - \rho)(\mathbf{H}(x^{-1}k))} dk \ , \tag{12}$$

we obtain the same form for the inversion formula as in the previous cases:

$$f(x) = \frac{1}{d_\tau} \sum_{P'} C_{P'} \sum_{\sigma'} \int_{\mathfrak{a}'^*} (\varphi_\tau^{\sigma',\nu'} * f)(x) p_{\sigma'}(\nu') \, d\nu' \ .$$

Some remarks:

- If τ is the trivial representation then $\widetilde{f}$ agrees with the Helgason-Fourier transform of the right-K-invariant function f.
- If f is a smooth function with compact support then $\varphi_\tau^{\sigma',\nu'} * f$ is a Paley-Wiener function (see Lemma I.9.3 and its Corollary).
- The formula of p_σ is not known in general and this causes problems. In this thesis, we use [Mia79] which gives the formula for groups G of real rank one, that is the dimension of $\mathfrak{a}$ is 1.

3 Resonances

Resonances are spectral objects attached to differential operators acting on non-compact domains and appear as poles of the meromorphic continuation of the resolvent of these operators. They are linked to interesting geometric, dynamical, and analytic objects. The basic questions concern the existence of the meromorphic extension of the resolvent, the distribution and counting properties of the resonances, the rank and interpretation of the so-called residue operators associated with the resonances. As a consequence, they are intensively studied, in many different settings, using a variety of techniques and different viewpoints. Standard references for the introduction of resonances in mathematics are [Agm86, Agm75]. A recent overview, also containing an extensive list of references, is [Zwo17]. Their study evolved from an investigation of the Schrödinger operators on the Euclidean spaces like $\mathbb{R}^n$, to a study of the Laplacian on curved spaces, like hyperbolic or asymptotically hyperbolic manifolds, symmetric or locally symmetric spaces.

3.1 Motivations from quantum physics

As described in the introduction of [DZ19], scattering resonances have a direct relation to quantum mechanics. They arise in the study of scattering of waves and particles. Physicists learned a lot about the forces and interactions in atoms and nuclei from scattering experiments. Atoms are bombarded with beams of particles. Then these particles are detected by devices that give the intensity as a function of the scattering angle or energy.

The positive Laplacian $-\Delta$ on $L^2(\mathbb{R}^d)$ represents in quantum mechanics the kinetic energy operator of a free quantum particle. The many interactions, for instance which occurs in a scattering experiment, are represented by a real-valued function, called *potential* $V \in L_c^\infty(\mathbb{R}^d)$. We are assuming without loss of generality that V has compact support, because at infinity the interactions with the scattering center are zero. The energy operator in this case is nothing else but a Schrödinger operator, also called *perturbed Hamiltonian* (perturbed by V)

$$H_V = -\Delta + V \ . \tag{13}$$

By the Kato-Rellich Theorem, H_V is self-adjoint. Hence its spectrum is real. Its resolvent

$$R(\lambda^2) = (H_V - \lambda^2)^{-1}$$

is well defined on the upper half plane $\Im(\lambda) > 0$ as a bounded operator on $L^2(\mathbb{R})$. When restricted to $C_c^\infty(\mathbb{R})$, $R(\lambda^2)$ extends to a meromorphic function of λ on the complex plane. The singularities of the extended resolvent are called the resonances of this Schrödinger operator. In general, it's very difficult to compute them. It is possible in very special cases, and for very special potentials. A well known case correspond to the case of the Laplace operator on $\mathbb{R}^n$ (see section II.1.1). The goal of this thesis is to do the same computations, replacing $\mathbb{R}^n$ by a homogeneous vector bundle (on a symmetric space of non-compact type and real rank one).

Let us do a more concrete computation related to the wave equation. In fact, the solution of this equation has a rate of oscillation and decay given by the scattering resonances. In dimension one, the wave equation is given by

$$(\partial_t^2 + H_V)\ u(t,x) = (\partial_t^2 - \partial_x^2 + V(x))\ u(t,x) = 0 \ .$$

Assume the initial conditions $u(0,x) = u_0(x)$ and $\partial_t w(0,x) = w_1(x)$, where $w_j \in C_c^\infty(\mathbb{R})$. Then for any $A > 0$, we have the resonance expansion (Fourier series in the case where $V = 0$)

$$w(t,x) = \sum_{\Im\lambda_j > -A} e^{-il\lambda x} f_{\lambda,l}(x) + E_A(t,x) \,,$$

where the λ_j vary over the set of the poles of the extended resolvent $R(\lambda)$, the sum is finite, $(H_V - \lambda^2)^{l+1} f_{\lambda,l} = 0$ and the error term $E_A(t,x)$ is bounded by e^{-At} in a well defined sense, see [DZ19, Theorem 2.9].

The poles λ_j with $\Im(\lambda_j) < 0$ of the extended resolvent are called scattering resonances or just resonances of the operator H_V. Numerical simulations of the solutions show these functions have peaks decreasing at infinity. One remarks that the real part of the resonances is the rate of oscillations and the imaginary part is the opposite of the rate of the decay. The resulting rapidly growing theory attracts many researchers.

Of course, this is a brief overlook of the utility of the resonances in one physical setting. For applications to different situations we refer to [DZ19, Chapter 1].

3.2 Resonances of the Laplace operator on non-compact Riemannian symmetric spaces

Riemannian symmetric spaces of the non-compact type are important geometrical settings to study resonances of the Laplacian. Besides being intrinsically interesting objects, they play the role of model spaces to understand phenomena on more complicated or less regular geometries. In a typical setting, one works on a complete Riemannian manifold X with a finite geometry, for which the positive Laplacian Δ is an essentially self-adjoint operator on the Hilbert space $L^2(X)$ of square integrable functions on X. We suppose that Δ has a continuous spectrum $[\rho_X, +\infty[$, with $\rho_X \geq 0$. The spectrum of Δ might have some discrete parts, but these parts do not play any significant role, so we neglect them. Also, for simplicity, we assume to have shifted the Laplacian so that its spectrum has bottom at 0, and have changed variables $z \mapsto z^2$ for the resolvent so that, as above, the resolvent is analytic away from the real axis. The resolvent $R(z) = (\Delta - \rho_X - z^2)^{-1}$ of the shifted Laplacian $\Delta - \rho_X$ is then a holomorphic function of z on the upper (and on the lower) complex half plane. For each such z, $R(z)$ is a bounded linear operator from $L^2(X)$ to itself. As such, it cannot be extended across the real axis. However, let us restrict the resolvent to the dense subspace $C_c^\infty(X)$ of compactly supported smooth functions on X. Then the map $z \mapsto R(z)$ might admit a meromorphic extension across the real axis to a larger domain in $\mathbb{C}$ or to a cover of such a domain. The poles, if they exist, are the resonances, also called quantum resonances or scattering poles, of Δ.

Let us introduce some notations. A Riemannian symmetric space of the non-compact type is a homogeneous space of the form $X = G/K$ where G is a connected non-compact real semisimple Lie group with finite center and K is a maximal compact subgroup of G. The basic examples are the n-dimensional real hyperbolic spaces H^n. In this case, the Lie group G is the Lorentz group $\mathrm{SO}_{\mathrm{e}}(n, 1)$ and $K = \mathrm{SO}(n)$. A Riemannian symmetric space of the non-compact type X has maximal flat subspaces, all of the same dimension, called the (real) rank of X. For instance, the rank of H^n is 1. Since X is a symmetric space of the Lie group G, all natural operators acting on X, like the Laplacian and its resolvent, are G-invariant. They can therefore be studied using the representation theory of G. The analytic study of the resolvent of the Laplacian acting on functions on H^n, in particular its meromorphic continuation across its spectrum, is classical and well-understood. It plays a central role when studying the resolvent on more general complete Riemannian manifolds for which the hyperbolic spaces are models. Still in the case of functions on a more general Riemannian symmetric space $X = G/K$, the study of resonances can in principle be done using an adapted harmonic analysis, the so called Helgason-Fourier analysis, which provides a diagonalization of the Laplacian and hence an explicit formula for its resolvent as a singular integral operator over the spectrum. This formula allowed Hilgert and Pasquale [HP09] to determine and study the resonances for an arbitrary X of rank one. The rank-one case was previously solved by Miatello and Will [MW00] with different methods, in the context of Damek-Ricci spaces. The general higher-rank case is still open. Relevant works in this context are [MV05] and [Str05]. Complete answers to the basic problems concerning the existence and location of the resonances of the Laplacian, as well as the representation-theoretic interpretation of the so-called residue operators at the resonances, are available only for (most of the) Riemannian symmetric spaces of rank 2. These results appeared in joint articles by Hilgert, Pasquale and Przebinda (see [HPP17a, HPP17b, HPP16]).

All the articles mentioned above consider the Laplacian acting on scalar functions on X. A more general question is to consider the Laplacian acting on sections of a homogeneous vector bundle on X. Such a bundle is determined by a finite-dimensional representation τ of K. Let us denote this bundle by E_τ. The sections of E_τ can be seen as vector-valued functions on G, with values in the space of the representation τ, such that

$$f(xk) = \tau(k^{-1})f(x) \quad \text{for all } x \in G \text{ and } k \in K\,.$$

The space of such functions which are smooth and compactly supported is denoted by $C_c^\infty(G,\tau)$. This means that we are replacing complex-valued functions on X with vector-valued functions which have specific transformation properties on the orbits of the compact subgroup K. This is a generalisation of the previous study, in the sense that the section of the homogeneous vector bundle E_{triv_K}, for the trvial representation of K, are exactly the complex valued functions on X.

Examples of sections of homogeneous vector bundles on X are the differential forms, the vector fields, and more generally the tensor fields on X: all these objects naturally arise in physical models, and it is therefore natural to look for resonances of the Laplacian in these settings.

The resolvent of the Laplacian of forms on a rank-one Riemannian symmetric space of the non-compact type has been studied by several authors; see [Cam97a, Cam05b, Cam05a, Cam97b, Ped94, CP04, BO00]. In particular, [CP04] gives (for the differential forms on rank 1) the list of resonances and the Riemann surface on which the resolvent admits meromorphic extension. Nevertheless, to our knowledge, there is only one article studying the resonances and the residue operators of the Laplacian acting on the sections of a homogeneous vector bundle over X, namely [Wil03], where X is a complex hyperbolic space and the fibers have dimension one.

4 Results of the thesis

The goal of this thesis is to study the resonances of the Laplacian acting on the sections of a homogeneous vector bundle over a Riemannian symmetric space of the non-compact type $X = G/K$. The symmetric space is assumed to have rank one but the representation of K is arbitrary. Since every finite-dimensional representation of K decomposes into irreducibles, we restrict our attention to irreducible representations. As in the case of functions, the basic problems are to determine the existence, the localisation of the resonances and to study the residue operators associated with them.

This thesis is organized as follows. In section I, we introduce the notation and recall some basic facts about the structure of Riemannian symmetric spaces of the non-compact type and real rank one. There are four cases, listed in the following table:

G	K	$X = G/K$
$\mathrm{Spin}(n,1)$	$\mathrm{Spin}(n)$	real hyperbolic space
$\mathrm{SU}(n,1)$	$\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1))$	complex hyperbolic space
$\mathrm{Sp}(n,1)$	$\mathrm{Sp}(n)$	quaternionic hyperbolic space
$\mathscr{F}_4$	$\mathrm{Spin}(9)$	octonion hyperbolic space

We set $\mathfrak{a}$ to be a maximal flat subspace in $\mathfrak{p} = T_{eK}(X)$ the tangent space of X at the base point eK, and M the centralizer of $\mathfrak{a}$ in K. In section I.9, we recall some facts on the generalisation of the Helgason-Fourier transform to homogeneous vector bundles. They are principally due to Camporesi [Cam97a]. In particular, the Plancherel Theorem for L^2-sections of the homogeneous vector bundles is given there. Denote the decomposition of τ over M as follows:

$$\tau = \bigoplus_{\sigma\in\hat{M}(\tau)} d_\sigma \sigma \tag{14}$$

where $\hat{M}(\tau)$ is the set of irreducible unitary representations of M which occurs in the restriction of τ to M, d_σ is the degree of σ. We need some properties of the generalised spherical functions $\varphi_\tau^{\sigma,\lambda}$ associated with the irreducible representations $\sigma \in \hat{M}(\tau)$, which is detailed in section I.9. The explicit formula for the Plancherel density p_σ corresponding to these $\sigma \in \hat{M}$ is given in Proposition II.1.4 (see also Appendix A). Corollary I.9.1 proves the convergence of the singular integral operator providing an explicit formula for the resolvent R of the Laplacian using the inversion formula of vector-valued Helgason-Fourier transform. In section II.1, we compute the resonances, which is the first main goal of this thesis. The holomorphic function $z \mapsto R(z)$ is meromorphically extended from the complex upper half-plane to the whole space, using the residue theorem. The extended resolvent is a meromorphic function with simple poles on the imaginary axis: these poles are the resonances. This leads to our first theorem:

Theorem 1

Let G be a connected non-compact semisimple Lie group with finite center and with

Iwasawa decomposition $G = KAN$, where K be a fixed maximal compact subgroup of G. Suppose $\dim A = 1$. Let M denote the centralizer of A in K. Let (τ, V_τ) be an irreducible unitary representation of K, and let E_τ be the homogeneous vector bundle over G associated with τ. For each $\sigma \in \hat{M}(\tau)$, let $\mathbb{N}_\sigma$ be the set of $k \in \mathbb{Z}$ such that

$$\lambda_k := -i(B_{\max} + k)$$

is a pole of the Plancherel density (see (II.15) for the formula) and $B_{\max} + k \geq 0$. Here $B_{\max}$ is a nonnegative constant depending only on G and σ. We refer to (II.17), (II.18) and (II.20) for the precise definition.

In this setting, the meromorphic continuation of the resolvent R of the Laplace operator acting on the smooth compactly supported sections of E_τ can be written as the sum

$$R = \sum_{\sigma \in \hat{M}(\tau)} d_\sigma R_\sigma \,, \tag{15}$$

where R_σ is given for all $f \in C_c^\infty(G, \tau)$ and for all $N \in \mathbb{N}$ by the following formula:

$$\begin{aligned} \left(R_\sigma(\zeta_\sigma) f\right)(x) = \frac{1}{|\alpha|} \int_{\mathbb{R}-i(N+1/4)} \frac{1}{\zeta_\sigma - \lambda|\alpha|} \Big(\varphi_\tau^{\sigma,\lambda\alpha} * f\Big)(x) \; \frac{p_\sigma(\lambda\alpha)}{\lambda} \; d\lambda \\ + \quad \frac{2i\pi}{|\alpha|} \sum_{\substack{k \in \mathbb{N}_\sigma \\ \lambda_k > -i(N+1/4)}} \frac{1}{\zeta_\sigma - \lambda_k|\alpha|} \Big(\varphi_\tau^{\sigma,\lambda_k\alpha} * f\Big)(x) \; \operatorname*{Res}_{\lambda=\lambda_k} \frac{p_\sigma(\lambda\alpha)}{\lambda} \end{aligned} \tag{16}$$

In (16), α is the longest restricted root, $\varphi_\tau^{\lambda,\sigma}$ is a spherical function of type σ and

$$\zeta_\sigma := \sqrt{-z - \langle\rho, \rho\rangle + \langle\mu_\sigma + \rho_M, \mu_\sigma + \rho_M\rangle)} \tag{17}$$

with $z \in \mathbb{C}$ such that $\Im(\zeta_\sigma) > -(N + 1/4)$. Here $\sqrt{\cdot}$ denotes the single-valued branch of the square root function determined on $\mathbb{C} \setminus [0, +\infty[$ by the condition $\sqrt{-1} = -i$.

The resonances of the Laplace operator acting on the sections of E_τ appear in families parametrized by the elements of $\hat{M}(\tau)$. Let

$$S_\sigma = \Big\{(z, \zeta) \in \mathbb{C}^2 \mid \zeta^2 := -z - \langle\rho, \rho\rangle + \langle\mu_\sigma + \rho_M, \mu_\sigma + \rho_M\rangle)\Big\} \;.$$

Then the resolvent R_σ extends meromorphically from $S_\sigma^+ = \{(z, \zeta) \in S \mid \Im(\zeta) > 0\}$ to S_σ. The (simple) poles of this extension are the pairs

$$(z_{\sigma,k}, \zeta_{\sigma,k}) = \Big((B_{\max}+k)^2|\alpha|^2 - \rho_\alpha^2|\alpha|^2 + \langle\mu_\sigma + \rho_M, \mu_\sigma + \rho_M\rangle \;, -i(B_{\max}+k)^2|\alpha|^2\Big) \tag{18}$$

where μ_σ is the highest weight of the representation σ, the numbers k are in $\mathbb{N}_\sigma$ and ρ_M is half sum of roots for M.

We refer to section I for the definitions of the various objects appearing in this theorem and to section II.1 for its proof.

The second problem we address in this article is the representation theoretic interpretation of the resonances. More precisely, consider the residual part of the meromorphic continuation of the resolvent in (16). For each pole λ_k of the Plancherel density for

$\sigma \in \hat{M}(\tau)$, one can introduce an operator, called the residue operator at $\lambda_k\alpha$, defined as follows:

$$\begin{array}{rrcl} R_k^\sigma: & C_c^\infty(G,\tau) & \longrightarrow & C^\infty(G,\tau) \\ & f & \longmapsto & \varphi_\tau^{\sigma,\lambda_k\alpha} * f \end{array} \tag{19}$$

Since the convolution product is on the left, it seems that it does not commute with the left translation of f. But it does, as we shall see in (I.22). As G acts on the image of R_k^σ by the left translations, we get a representation of G, called the residue representation at $\lambda_k\alpha$.

In section II.2 we restrict our attention to the representations τ which contains the trivial representation of M. In this case the structure of the principal series is well known [HT93, JW77, Joh76]. The complexity of the general case (see [Col85]) is formidable and might lead to much less pleasing results, thus we avoid it. We consider the family of resonances corresponding to $\sigma = \mathrm{triv}$. To simplify the notation, we write R_k instead of R_k^{triv}. Let $\mathscr{E}_k$ be the residue representation at $\lambda_k\alpha$. We show that the $\mathscr{E}_k$'s are irreducible and equivalent to a subquotient of a spherical principal series representation of G. We determine which of them are unitary and which are finite-dimensional. Also, we identify their Langlands parameters and compute their wave front sets. The Langlands parameters are of the form (MA, δ, ν) and denotes the induced representation $\mathrm{Ind}_{MAN}^G(\delta \otimes e^\nu \otimes \mathrm{triv})$. A lowest K-type of the induced representation with highest weight $\mu_{\min}$ identifies a unique irreducible subquotient of that induced representation (See [Vog77]). The wave front set of a representation has been introduced by Howe (see [How81]). When G is semisimple, it is a closed set consisting of nilpotents orbits. For $\mathscr{E}_k$ it turns out to be the closure of a single nilpotent orbit. In the following theorem, α is the longest restricted root as in Theorem 1 and for each restricted root β, the corresponding root space in $\mathfrak{g}$ is denoted by $\mathfrak{g}_\beta$. The minimal K-type of $\mathscr{E}_k$ is given in the proof of the theorem in each case: they can be found in tables II.1, II.2, II.3 and II.4 respectively for the real, complex, quaternionic and octonionic hyperbolic spaces.

Theorem 2

Suppose that the representation τ contains the trivial representation of M. The residue representations $\mathscr{E}_k$ are then irreducible.

1. *If $G = \mathrm{Spin}(2n,1)$, then τ has highest weight of the form $(N, 0, \ldots, 0)$, where N is a nonnegative integer.*
   - *If $N \geq k+1$, then $\mathscr{E}_k$ has Langlands parameters $\Big(MA, \mathscr{H}^{k+1}(\mathbb{R}^{2n-1}), \big(n - \frac{3}{2}\big)\alpha\Big)$ with $(k+1, 0, \ldots, 0)$ as a lowest K-type's highest weight. Here $\mathscr{H}^{k+1}(\mathbb{R}^{2n-1})$ are harmonics of degree $k+1$ on $\mathbb{R}^{2n-1}$. This representation is unitary. Its wave front set is the nilpotent orbit generated by $\mathfrak{g}_\alpha$.*
   - *If $N < k+1$, then $\mathscr{E}_k$ has Langlands parameter $\Big(MA, \mathrm{triv}, (\rho_\alpha + k)\alpha\Big)$ with the trivial representation as a lowest K-type. It is finite-dimensional. Also, it is non-unitary if $k \neq 0$.*

2. *If $G = \mathrm{SU}(n,1)$, then τ has highest weight of the form $(M_1, 0, \ldots, 0, -M_2, -L)$, where M_1 and M_2 are positive integers such that $M_1 \geq M_2 \geq 0$, $L \in \mathbb{Z}$ and $M_1 + M_2 + L$ is even.*

   - *If $M_1 + M_2 \geq 2k + 2$ and $|L| \leq -2k - 2 + M_1 + M_2$, then $\mathscr{E}_k$ is unitary.*
     - *if $n > 2$, then $\mathscr{E}_k$ has minimal K-type of highest weight $\left((k+1), 0, \ldots, 0, -(k+1), 0\right)$. Its Langlands parameter is $\left(MA, \delta, \left(\frac{n}{2} - 1\right)\alpha\right)$ where the highest weight of δ is $\left((k+1), 0, \ldots, 0, -(k+1), 0\right)$. Its wave front set is the nilpotent orbit generated by $\mathfrak{g}_{\alpha/2}$.*
     - *if $n = 2$, this representation is the discrete series with Blattner parameter $((k+1), -(k+1), 0, \ldots, 0)$. Its wave front set is the nilpotent orbit generated by $\mathfrak{g}_{\alpha/2}$.*
   - *If $L \geq |-2k-1+M_1+M_2|+1$, then $\mathscr{E}_k$ is the representation with Langlands parameter $\left(MA, \delta, \left(\frac{k}{2} + \frac{n}{2} - \frac{1}{2}\right)\alpha\right)$, where the highest weight of δ is $\left(0, 0, \ldots, 0, -(k+1), (k+1)/2\right)$. This representation is non-unitary. Its wave front set is the nilpotent orbit generated by the element n_2 of $\mathfrak{g}_\alpha$ (see Lemma III.4.2 for the definition).*
   - *If $L \leq |-2k-1+M_1+M_2|-1$, then $\mathscr{E}_k$ is the representation with Langlands parameter $\left(MA, \delta, \left(\frac{k}{2} + \frac{n}{2} - \frac{1}{2}\right)\alpha\right)$, where the highest weight of δ is $\left((k+1), 0, \ldots, 0, (k+1)/2\right)$. This representation is non-unitary. Its wave front set is the nilpotent orbit generated by the element n_1 of $\mathfrak{g}_\alpha$ (see Lemma III.4.2 for the definition).*
   - *If $M_1 + M_2 \in [0, 2k+2[$ and $L < |2k + 2 - M_1 + M_2|$, then $\mathscr{E}_k$ is the representation with Langlands parameter $\left(MA, \mathrm{triv}, (\rho_\alpha + k)\alpha\right)$. It is finite-dimensional and non unitary (if $k \neq 0$).*

3. *If $G = \mathrm{Sp}(n,1)$, then τ has a highest weight of the form $(t_1, t_2, 0, \ldots, 0, t_{n+1})$, where t_1, t_2 and t_{n+1} are positive integers such that $t_1 \geq t_2$, $t_{n+1} \leq t_1 + t_2$ and $t_1 + t_2 + t_{n+1}$ is even.*

   - *If $t_{n+1} \leq t_1 + t_2 - 2k - 4$, then $\mathscr{E}_k$ is the representation with Langlands parameter $\left(MA, \delta, \pm\left(n - \frac{3}{2}\right)\alpha\right)$ with τ as a lowest K-type, where the highest weight of δ is $(k+2, k+2, 0, \ldots, 0)$. This representation is non-unitary. Its wave front set is the nilpotent orbit generated by $\mathfrak{g}_{\alpha/2}$.*
   - *If $t_{n+1} \geq |t_1 + t_2 - 2k - 2|$ the residue representation is unitary. Its wave front set is the nilpotent orbit generated by $\mathfrak{g}_\alpha$.*

  - *If $k \leq 2n-4$, then $\mathscr{E}_k$ is the representation with Langlands parameter $\left(MA, \delta, \pm\left(n-\frac{k}{2}\right)\alpha\right)$ with lowest K-type $(k+1, 0, \ldots, 0, k+1)$, where the highest weight of δ is $(k+1, 0, \ldots, 0, \frac{k+1}{2})$.*
  - *If $k \geq 2n-3$, then $\mathscr{E}_k$ is the discrete series representation with Blattner parameter $\mu_k = (k+1, 0, \ldots, 0, k+1)$.*

- *If $t_1 + t_2 < 2k + 2 - t_1 - t_2$, then $\mathscr{E}_k$ is the representation with Langlands parameter $\left(MA, \mathrm{triv}, (\rho_\alpha + k)\alpha\right)$. It is finite-dimensional and non unitary (if $k \neq 0$).*

4. *If $G = F_4$, then τ has a highest weight of the form $(a/2, b/2, b/2, b/2)$, where a and b are positive integers such that $a \geq b$ and $a - b$ is even.*

   - *If $b \leq a - 2k - 8$, then $\mathscr{E}_k$ is the representation with Langlands parameter $\left(MA, \delta, \frac{1}{2}(k+1)\alpha\right)$, where the highest of δ is $\frac{k+4}{4}(3, 1, 1, 1)$. This representation is non-unitary. Its wave front set is the nilpotent orbit generated by $\mathfrak{g}_{\alpha/2}$.*
   - *If $b > a - 2k - 8$ and $b \geq 2k + 2 - a$, then $\mathscr{E}_k$ is the representation with Langlands parameter $\left(MA, \delta, \frac{1}{2}(k+10)\alpha\right)$, where the highest of δ is $\frac{k+1}{4}(3, 1, 1, 1)$. This representation is unitary. Its wave front set is the nilpotent orbit generated by $\mathfrak{g}_\alpha$.*
   - *If $b < 2k + 2$, then $\mathscr{E}_k$ is the representation with Langlands parameter $\left(MA, \mathrm{triv}, (\rho_\alpha + k)\alpha\right)$. It is finite-dimensional and non unitary (if $k \neq 0$).*

A nice consequence of our case-by-case results is the following.

Corollary 4.0.1
For a fixed $k \in \mathbb{N}$, there is one-to-one correspondence between the irreducible subquotients of $\mathscr{H}_{\lambda_k\alpha}$ and the (real) nilpotent orbits of $\mathfrak{g}$ under the adjoint action. This correspondence maps each subquotient into the orbit whose closure is the wave front set of that subquotient. As we showed, $\mathscr{E}_k$ is equivalent to one of these subquotients. Its wave front set is then the closure of one *nilpotent orbit in $\mathfrak{g}$.*

I

Preliminaries

We shall use the standard notations $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{C}^{\times}$ for the nonnegative integers, the integers, the real numbers, the complex numbers and the nonzero complex numbers. For a complex number $z \in \mathbb{C}$, we denote by $\Re(z)$ and $\Im(z)$ its real and imaginary parts. The positive constants in the Haar measures do not matter in our computations and equalities. Integrals have to be considered up to positive multiples.
We will denote by uppercase letters G, K, A, M, N Lie groups and by gothic letters their respecting Lie algebras $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{m}, \mathfrak{n}$.

In this section we fix some notations and we recall some basic facts.

I.1 Context

Let G be a connected non-compact real semisimple Lie group with finite center. We recall that a Lie group is called *semisimple* if it is a direct product of simple groups (Its Lie algebra is nonabelian and contains no nonzero proper ideals). Let $B(\cdot,\cdot)$ be the Killing form on the Lie algebra $\mathfrak{g}$ of G, that is a bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined for all $X, Y \in \mathfrak{g}$ by

$$B(X,Y) = \mathrm{Tr}\left(\mathsf{ad}(X) \circ \mathsf{ad}(Y)\right) .$$

Here ad is the adjoint action and can be defined through the Lie bracket

$$\mathsf{ad}(X)(Y) = [X,Y] ,$$

for all $X, Y \in \mathfrak{g}$. We recall that for a semisimple Lie algebra, the Killing form is non-degenerate. An involution on $\mathfrak{g}$ is a Lie algebra automorphism whose square is equal to the identity. Such an involution θ is called a *Cartan involution* on if $B_\theta(X,Y) := -B(X,\theta Y)$ is a positive definite bilinear form. We denote by $\mathfrak{k}$ the set of fixed points of θ and by $\mathfrak{p}$ the eigenspace of θ for the eigenvalue -1. In other words:

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta X = X\} \quad \text{and} \quad \mathfrak{p} = \{X \in \mathfrak{g} \mid \theta X = -X\} .$$

Then $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$. The corresponding connected Lie subgroup of G is maximal compact. We denote it by K. The Cartan decomposition of the Lie algebra $\mathfrak{g}$ is given by:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} .$$

We recall that here $\mathfrak{p}$ is not a Lie algebra, but we have the following inclusions:

$$[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p} , \quad [\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k} , \quad [\mathfrak{p},\mathfrak{p}] \in \mathfrak{k} .$$

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $A = \exp \mathfrak{a}$ its associated subgroup of G. The exponential map $\exp : \mathfrak{g} \to G$ restricts to a diffeomorphism between $\mathfrak{a}$ and A. The inverse map is the logarithm denoted "log". The group A can also be seen as the maximal torus in $\exp \mathfrak{p}$. The Cartan decomposition thus implies $G = KAK$, because $\mathfrak{p} = \bigcup_{k \in K} \mathsf{Ad}(k)\mathfrak{a}$.

I.2 Roots and restricted roots systems

Let $\mathfrak{a}^*$ be the vector space of linear forms on $\mathfrak{a}$ and $\mathfrak{a}^*_{\mathbb{C}}$ its complexification. The set Σ of restricted roots of the pair $(\mathfrak{g},\mathfrak{a})$ consists of all linear forms $\alpha \in \mathfrak{a}^*$ for which the vector space

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid [H,X] = \alpha(H)X \text{ , for every } H \in \mathfrak{a}\}$$

contains nonzero elements. The dimension of $\mathfrak{g}_\alpha$ is called the multiplicity of the root α and is denoted by m_α. In other words,

$$\Sigma = \{\alpha \in \mathfrak{a}^* \mid \mathfrak{g}_\alpha \neq 0\} \tag{I.1}$$

If $\alpha \in \Sigma$ then $-\alpha \in \Sigma$. We can choose an order: Let Σ_+ be a fixed set of positive restricted roots and let $\rho := \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ be the half sum of the positive roots counted with their multiplicities. Set $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma_+} \mathfrak{g}_\alpha$ and N the connected Lie subgroup of G having $\mathfrak{n}$ for Lie algebra. The Iwasawa decomposition of the Lie algebra can be then written $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$.
According to the Iwasawa decomposition at the group level $G = KAN$, every element x in G can be uniquely written as

$$x = \mathbf{k}(x) e^{\mathbf{H}(x)} \mathbf{n}(x) \tag{I.2}$$

where $\mathbf{k}(x) \in K$, $\mathbf{H}(x) \in \mathfrak{a}$ and $\mathbf{n}(x) = \mathbf{n} \in N$. In the following, we set

$$a^\lambda := \exp(\lambda(\log a)) \ \text{ for } a \in A \text{ and } \lambda \in \mathfrak{a}^*_\mathbb{C} \ . \tag{I.3}$$

Let M be the centralizer of $\mathfrak{a}$ in K, $\mathfrak{m}$ its Lie algebra. We recall that a *Cartan subalgebra* $\mathfrak{h}$ of a complex Lie algebra is a nilpotent algebra for which if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$, then $Y \in \mathfrak{h}$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{m}$. Then the Lie algebra $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. We denote by $\mathfrak{h}_\mathbb{C}$ its complexification. The set Π of roots of the pair $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ consists of all linear forms $\varepsilon \in \mathfrak{h}^*_\mathbb{C}$ for which the vector space

$$\mathfrak{g}_{\mathbb{C},\varepsilon} := \{X \in \mathfrak{g}_\mathbb{C} \mid [H, X] = \varepsilon(H) X \ , \text{for every } H \in \mathfrak{h}_\mathbb{C}\}$$

contains nonzero elements. These vector spaces are one-dimensional.
We choose a set Π_+ of positive roots in Π which is compatible with Σ_+, i.e. such that a root $\varepsilon \in \Pi$ is positive when $\varepsilon|_\mathfrak{a} \in \Sigma_+$. Let also $\Pi_\mathfrak{k}$ ($\Pi_{\mathfrak{k}\ +}$) be the set of (positive) roots of the pair $(\mathfrak{k}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$.
We have the decomposition

$$\mathfrak{g}_\mathbb{C} = \mathfrak{h}_\mathbb{C} \bigoplus_{\varepsilon \in \Pi} \mathfrak{g}_{\mathbb{C},\varepsilon} \ .$$

I.3 The rank-one case

In this thesis, we are restricting ourself to real rank-one groups G. In other words, we suppose that $\mathfrak{a}$ is one-dimensional and then isomorphic to $\mathbb{R}$.

Rank-one symmetric spaces of the non-compact type are classified into three infinite families – namely, the real, complex and quaternionic hyperbolic spaces – and one exceptional example, the octonionic hyperbolic plane.

Since G is of real rank one, the set Σ is either equal to $\{\pm\alpha\}$ or $\{\pm\alpha, \pm\alpha/2\}$. Among the groups listed in the table in the introduction, only $G = \mathrm{Spin}(n, 1)$ has restricted root system $\{\pm\alpha\}$. As a system of positive roots Σ_+ we choose $\{\alpha\}$ and $\{\alpha, \alpha/2\}$. Then $\rho = \frac{1}{2}\Big(m_\alpha + \frac{m_{\alpha/2}}{2}\Big)\alpha$, where we set $m_{\alpha/2} = 0$, if $\Sigma = \{\pm\alpha\}$.

The Killing form B is positive definite on $\mathfrak{p}$, so $\langle X, Y\rangle := B(X, Y)$ defines an Euclidean structure on $\mathfrak{p}$ and on $\mathfrak{a} \subset \mathfrak{p}$. For all $\lambda \in \mathfrak{a}^*$, let H_λ denote the unique element in $\mathfrak{a}$ such that $\langle H_\lambda, H\rangle = \lambda(H)$ for all $H \in \mathfrak{a}$. We extend the inner product to $\mathfrak{a}^*$ by setting $\langle \lambda, \mu\rangle := \langle H_\lambda, H_\mu\rangle$ for all $\lambda,\ \mu \in \mathfrak{a}^*$. Further, we denote the $\mathbb{C}$-bilinear extension of $\langle\cdot,\cdot\rangle$ on $\mathfrak{a}$ to $\mathfrak{a}^*_{\mathbb{C}}$ by the same symbol. We identify $\mathfrak{a}^*_{\mathbb{C}}$ to $\mathbb{C}$ by means of the isomorphism:

$$\begin{array}{ccc} \mathfrak{a}^*_{\mathbb{C}} & \longrightarrow & \mathbb{C} \\ \lambda & \longmapsto & \lambda_\alpha := \frac{\langle \lambda,\alpha\rangle}{\langle \alpha,\alpha\rangle} \end{array} \tag{I.4}$$

which identifies ρ with $\rho_\alpha := \frac{1}{2}\left(m_\alpha + \frac{m_{\alpha/2}}{2}\right)$.

We recall that the Lie algebra $\mathfrak{n} = \bigoplus\limits_{\alpha\in\Sigma} \mathfrak{g}_\alpha$ is nilpotent. More explicitly the property

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$$

for $\alpha, \beta \in \Sigma$, allow us to see that $\mathfrak{g}_{\alpha/2}$ is nilpotent of degree 2 when it exists and $\mathfrak{g}_\alpha$ is nilpotent of degree 1. It's important to consider the possible nilpotent orbits under $\mathsf{Ad}(G)$ in $\mathfrak{g}$ which are needed in section III.

I.4 Homogeneous vector bundles

We recall that a *vector bundle* over a topological space X is pair (E, π) such that

- E is a topological space such that $\pi : E \to X$ is continuous.
- For all $x \in X$, $E_x := \pi^{-1}(x)$ is a finite dimensional vector space. Its topology is that induced from E.
- For all $x \in X$, there exist a neighbourhood U of x in X, a topological vector space E_0 over $\mathbb{R}$ and $\alpha : \pi^{-1}(U) \to U \times V$, a homeomorphism such that:
  - $p_1 \circ \alpha = \pi$ where $p_1 : U \times E_0 \to U$,
  - for all $x \in U$, the map $\alpha|_{E_x} : E_x \to \{x\}\times E_0$ is an isomorphism of topological vector spaces.

The *rank* of the vector bundle is the dimension of the vector spaces E_x.

A *section* of E is a smooth map from X to E such that the image of x is in the fiber E_x.

Definition I.4.1
We say that E is a homogeneous vector bundle over G/K, if G acts on E by an smooth action such that:

1. $g \cdot E_x = E_{gx}$, for all $x \in G/K$ and $g \in G$, where gx is the action of G on G/K,
2. the map $g\cdot : E_x \to E_{gx}$, $(x, v) \mapsto g \cdot (x, v)$ is linear for all $g \in G$ and $x \in G/K$

Every homogeneous vector bundle can be constructed as follows. We fix a finite-dimensional unitary representation (τ, V_τ) of K. The group K acts on $G \times V_\tau$ on the right, such that for all $k \in K$, $g \in G$ and $v \in V_\tau$,

$$(g, v) \cdot k = (g, v)k := (gk, \tau(k^{-1})v)$$

This action of K is a homeomorphism on $G \times V_\tau$. We can then quotient the topological product space $G \times V_\tau$ by this action of K.
Let $E_\tau = G \times_\tau V_\tau := (G \times V_\tau)/H$ and $p : E_\tau \to G/H$ such that $p((g, v)H) = gH$. This map is well defined and surjective. Moreover, if we provide E_τ and G/K with their respective quotient topology, p is continue.
In these conditions, E_τ can be equiped with a structure of homogeneous vector bundle. In fact, one can prove that every homogeneous vector bundle on $X = G/K$ is isomorphic to E_τ for a well choosen $\tau \in \hat{K}$.

Lemma I.4.1

Let E a homogeneous vector bundle on G/K. Let $V_\tau := E_0 = E_{eK}$. Since G (and so K) acts on the fibers of E, we obtain a represention $\tau : K \to GL(E_{eK})$ which is continuous. Let $V_\tau := E_0 = E_{eK}$. Thus E is isomorphic to $G \times_\tau V_\tau = E_\tau$.

For the proofs and more properties on homogeneous vector bundles, we refer the reader to [Wal73, §5.2 p. 114].

So from now on, we fix a finite-dimensional unitary representation (τ, V_τ) of K. Let $E_\tau := X \times_\tau V_\tau$ denote the homogeneous vector bundle over X. We write $\Gamma^\infty(E_\tau)$ for the space of all smooth sections of E_τ. As proved in [Wal73, §5.4 p. 119], there is an isomorphism of vector spaces between $\Gamma^\infty(E_\tau)$ and

$$C^\infty(G, \tau) := \{f : G \overset{\text{smooth}}{\longrightarrow} V_\tau \mid f(xk) = \tau(k^{-1})f(x) \ \text{ for all } x \in G \text{ and } k \in K\}$$

Set

$$C^\infty(G, K, \tau, \tau) := \{F : G \overset{\text{smooth}}{\longrightarrow} \operatorname{End}(V_\tau) \mid F(k_1 x k_2) = \tau(k_2^{-1})F(x)\tau(k_1^{-1})\}$$

where $x \in G$ and $k_1, k_2 \in K$. The elements $F \in C^\infty(G, K, \tau, \tau)$ are sometimes called the radial systems of sections of E_τ. The link with the sections comes by the fact that for every $v \in V_\tau$, the function $F(\cdot)v$ is a smooth section of E_τ.

Remark

The three following algebras are $*$-isomorphic (as C^*-algebras, see [vD69]) :

- The algebra $C^\infty(G, K, \tau, \tau)$. The involution is given by $f^*(x) = \big(f(x^{-1})\big)^*$ where the star in the right-hand side is the Hilbert space adjoint.
- The algebra

  $$C^\infty_\chi(G) := \{f : G \overset{\text{smooth}}{\longrightarrow} \mathbb{C} \mid \chi * f * \chi = f \text{ and } f(k^{-1}xk) = f(x)\, \forall x \in G \text{ and } k \in K\}\,,$$

  where for all $k \in K$, $\chi(k) := d_\tau \operatorname{Tr} \tau(k^{-1})$. The involution is given by $f^*(x) = \overline{f(x^{-1})}$.

- For a fixed $i \in [\![1, d]\!]$. The algebra

$$C^\infty_{e_i}(G) := \{f : G \to \mathbb{C} \text{ smooth} \mid e_i * f * e_i = f\}$$

where for all $k \in K$, $e_i(k) := d_\tau \tau_{ii}(k^{-1})$. The involution is given by $f^*(x) = \overline{f(x^{-1})}$.

The "last" algebra is particularly interesting, because for each i the algebras are $*$-isomorphic. This is quite surprising at a first look. One could choose any vector in $v \in V_\tau$ and have $C^\infty_v(G) \simeq C^\infty(G, K, \tau, \tau)$.

Remark (When τ is not irreducible)
If τ is not irreducible then $E_\tau = \bigoplus_i E_{\tau_i}$, where τ_i are the irreducible components of τ. Studying the sections of E_τ amounts to studying the sections of each bundle E_{τ_i}. We can therefore suppose without loss of generality that τ is irreducible.
We notice that if τ is the trivial representation "triv" of K on the one-dimensional vector space $\mathbb{C}$. Then the sections of E_{triv} are the functions on G/K. They can be seen as right-K-invariant functions on G. Moreover, in this case, the radial systems of sections agree with the K-bi-invariant functions on G. We will refer to this situation as the scalar case.

I.5 Principal series representations

Let $\hat{M}$ be the set of all equivalence classes of irreducible unitary representations of M. For $(\sigma, \mathscr{H}_\sigma) \in \hat{M}$ and $\lambda \in \mathfrak{a}^*_\mathbb{C}$, we denote by

$$\tau_\lambda = \pi^\sigma_\lambda = \operatorname{Ind}^G_{MAN}(\sigma \otimes e^{i\lambda} \otimes \text{triv})$$

the induced representation from MAN to G by the representation $\sigma \otimes e^{i\lambda} \otimes \text{triv}$. We will use the same notation for its derived representation of $\mathfrak{g}$ too. The representation space $\mathscr{H}^\sigma_\lambda$ of π^σ_λ is the Hilbert space completion of

$$\{f : G \to \mathscr{H}_\sigma \mid f(xman) = a^{-i\lambda-\rho}\sigma(m^{-1})f(x) \text{ for all } x \in G,\ m \in M,\ a \in A \text{ and } n \in N\} \tag{I.5}$$

with respect of the L^2-inner product

$$\langle f, g\rangle_\sigma = \int_K \langle f(k), g(k)\rangle_{\mathscr{H}_\sigma}\, dk\,,$$

where $\langle \cdot, \cdot \rangle_{\mathscr{H}_\sigma}$ is an inner product on $\mathscr{H}_\sigma$ making σ unitary. The action of π^σ_λ on $\mathscr{H}^\sigma_\lambda$ is given by

$$\pi^\sigma_\lambda(g)f(x) := f(g^{-1}x)$$

for all $g, x \in G$ and $f \in \mathscr{H}^\sigma_\lambda$. The set $\{\pi^\sigma_\lambda \mid \lambda \in \mathfrak{a}^*_\mathbb{C}, \sigma \in \hat{M}\}$ is called the *minimal principal series* of G. If $\sigma = \text{triv}_M$ then it is called the *spherical principal series*.

The compact picture of the principal series representations is obtained by restriction of the elements of $\mathscr{H}_\lambda^\sigma$ to K. Its representation space, which we denote by $\mathscr{H}^\sigma$, is the Hilbert completion of:

$$\{f : K \to \mathscr{H}_\sigma \mid f(km) = \sigma(m^{-1})f(k) \ \text{ for all } k \in K,\ m \in M\}$$

with respect to L^2 inner product. It is independent of λ. The action is given by:

$$\pi_\lambda^\sigma(g)f(k) := e^{-(i\lambda+\rho)\mathbf{H}(g^{-1}k)} f(\mathbf{k}(g^{-1}k))$$

for all $g \in G$, $k \in K$ and $f \in \mathscr{H}^\sigma$. The representation π_λ^σ is unitary for $\lambda \in \mathfrak{a}^*$.

In the above definitions, we are following the notation of [Hel08, chapter VI, §3] and [Cam97a]. In the literature (for instance in [Vog81]), one often finds $(\pi_{i\lambda}^\sigma, \mathscr{H}_{i\lambda}^\sigma)$, instead of our $(\pi_\lambda^\sigma, \mathscr{H}_\lambda^\sigma)$ for the representation $\mathrm{Ind}_{MAN}^G(\sigma \otimes e^{i\lambda} \otimes \mathrm{triv})$.

In the following, when working with principal series, we actually work with their Harish-Chandra modules (see the next paragraph I.6 for the definition). The restriction of π_λ^σ to K is the representation $\mathrm{Ind}_M^K \sigma$ of K induced from σ. In particular, because of Frobenius reciprocity theorem, for any $\tau \in \hat{K}$:

$$m(\tau, \pi_\lambda^\sigma|_K) = m(\sigma, \tau|_M)\,.$$

Here the symbol $m(\beta, \alpha)$ denotes the multiplicity of the irreducible representation β in the representation α. We say that τ is a K-type of π_λ^σ if it occurs in $\pi_\lambda^\sigma|_K$. We say that τ is a minimal K-type of an admissible representation π of G if and only if its highest weight μ minimizes the Vogan norm

$$\|\mu\|_V = \langle \mu + 2\rho_K, \mu + 2\rho_K\rangle$$

in the set of K-types of π. Here $2\rho_K$ is the sum of positive roots of the pair $(\mathfrak{k}_\mathbb{C}, \mathfrak{h}_\mathbb{C}|_{\mathfrak{k}_\mathbb{C}})$. [Vog77, Theorem 1] ensures that each minimal K-type $\tau_{\min}$ has multiplicity one in π_λ^σ. Therefore, if we fix this $\tau_{\min}$ there exists a unique irreducible subquotient $J(\sigma, \lambda, \mu)$ of π_λ^σ containing $\tau_{\min}$.

These representations are so important because they appear in the continuous part of the Plancherel formula. Let G acts on $L^2(G)$ by left translation:

$$g \cdot f(x) = f(g^{-1}x)\ ,\ \text{for all } g, x \in G \text{ and } f \in L^2(G)\ .$$

Harish-Chandra proved the decomposition of $L^2(G)$ for any reductive group. In the rank one case, this can be written:

$$\begin{aligned} L^2(G) &= \int_{\hat{G}} \mathscr{H}_\pi \otimes \mathscr{H}_\pi^* d\nu(\pi) \\ &= \sum_{\sigma \in \hat{M}} \int_{i\mathfrak{a}^*} \mathscr{H}_\lambda^\sigma \otimes \mathscr{H}_\lambda^{\sigma *} d\nu(\lambda) \quad \oplus \quad \bigoplus_\omega d_\omega \mathscr{H}_\omega \otimes \mathscr{H}_\omega^*\ , \end{aligned} \tag{I.6}$$

where $d\nu$ is the Plancherel measure and the last direct sum is over the discrete series ω of G. For recall π is a discrete series of G if there exists a nonzero matrix coefficient of π which belongs to $L^2(G)$. The *matrix coefficients* of a representation π are the functions $\langle \pi(\cdot)u, v\rangle$ from G to $\mathbb{C}$, where u, v are in the Hermitian space V_π of the representation π and $\langle\cdot,\cdot\rangle$ is a Hermitian product on V_π. This is equivalent to ask that all matrix coefficients of π belong to $L^2(G)$. If (π, V_π) is a discrete series then there exist $d_\pi \in \mathbb{R}_+$, called the *formal degree of* π, such that

$$\int_G \langle \pi(g)u, v\rangle \overline{\langle \pi(g)u', v'\rangle}\, dg = d_\pi^{-1} \langle u, v\rangle \overline{\langle u', v'\rangle}\ , \text{ with } \ u, u', v, v' \in V_\pi\ . \tag{I.7}$$

In the rank one case, there is always discrete series, unless if $\mathfrak{g} = \mathfrak{so}(2n+1, 1)$. This is also the only case, when there are no resonances of the Laplace operator.
The abstract Plancherel formula (I.6) for $L^2(G, \tau)$ can be then written as follows (see [BOS94, page 4]):

$$L^2(G,\tau) = \sum_{\sigma\in\hat{M}} \int_{i\mathfrak{a}^*} \mathscr{H}_\lambda^\sigma \otimes (\mathscr{H}_\lambda^{\sigma *} \otimes V_\tau)^K d\nu(\sigma,\lambda) \quad \oplus \quad \bigoplus_\omega d_\omega \mathscr{H}_\omega \otimes (\mathscr{H}_\omega^* \otimes V_\tau)^K\ , \tag{I.8}$$

where ^{K} denotes the K-invariant vectors. In other words, $(\mathscr{H}^* \otimes V_\tau)^K = \mathrm{Hom}_K(\mathscr{H}, V_\tau)$, that is to say the space of intertwining operators between $\mathscr{H}|_K$ and V_τ.

I.6 Langlands parameters of a representation

The irreducible admissible representations of the group G can be classified by their Langlands parameter. Let us recall some facts and definitions.

A representation V of G is said *admissible*, if its restriction to K

$$V|_K = \bigoplus_{\gamma\in\hat{K}} V(\gamma)$$

is such that $\dim V(\gamma)$ is finite. Every unitary and/or irreducible representation is admissible.
We say that an admissible representation (π, V) is tempered if all K-finite matrix coefficients

$$\langle \pi(\cdot)u, v\rangle\ , \quad \text{with } u, v \in V$$

are in $L^p(G)$ for any $p > 2$. A G-module V is a $(\mathfrak{g}, K)$*-module* if:

- For all $v \in V$, $k \in K$ and $X \in \mathfrak{g}$, $k \cdot (X \cdot v) = (\mathsf{Ad}(k)X) \cdot k \cdot v$,
- For all $v \in V$, $K \cdot v$ spans a finite dimensional subspace of V on which K acts continuously,
- For any $v \in V$, $Y \in \mathfrak{k}$, $\left(\frac{d}{dt}\exp(tY)\cdot v\right)\Big|_{t=0} = Y \cdot v$.

We say that a $(\mathfrak{g}, K)$-module V is a *Harish-Chandra module* if it is admissible and finite dimensional as a $U(\mathfrak{g})$-module.

Theorem I.6.1 (Casselman subrepresentation theorem)
Every Harish-Chandra module can be embedded in a principal series representation.

However it's not clear how it is embedded.

The group MAN is called a *minimal parabolic subgroup* of G. The others minimal parabolic subgroups are the conjugates of MAN. A group P is called *parabolic* if it is closed and if it contains a minimal parabolic subgroup. We denote by $\mathfrak{s}$ its Lie algebra. There exists a canonical decomposition of $\mathfrak{s}$, called *Langlands decomposition*

$$\mathfrak{s} = \mathfrak{m}_P \oplus \mathfrak{a}_P \oplus \mathfrak{n}_P \ ,$$

such that:

- $\mathfrak{m}_P$, $\mathfrak{a}_P$ and $\mathfrak{n}_P$ three orthogonal Lie subalgebras of $\mathfrak{g}$ with respect to the inner product $\Re\Big(B(\cdot, \theta\cdot)\Big)$,
- $\mathfrak{m}_P \oplus \mathfrak{a}_P = \mathfrak{s} \cup \theta(\mathfrak{s})$,
- $\mathfrak{a} = \mathfrak{p} \cap Z_{\mathfrak{m}_P \oplus \mathfrak{a}_P}$

If we define $M_{0,P}$, A_P and N_P as the connected Lie subgroups with Lie algebras $\mathfrak{m}_P$, $\mathfrak{a}_P$ and $\mathfrak{n}_P$ respectively, and $M_P = Z_K(\mathfrak{a})M_{0,P}$. Then

$$P = M_P A_P N_P$$

is the Langlands decomposition of the group P.

Theorem I.6.2 ([Kna01, Theorem 7.24'])
Let $P = M_P A_P N_P$ be a parabolic subgroup of G. Let σ be an irreducible tempered representation of M and let $\nu \in \mathfrak{a}^*$ such that $\Re(\nu)$ is in the open positive Weyl chamber. Then $\mathrm{Ind}_{MAN}^{G}(\sigma \otimes e^{\nu} \otimes \mathrm{triv})$, as a Harish-Chandra module, has unique irreducible quotient $J(P, \sigma, \nu)$.

$J(P, \sigma, \nu)$ is called the *Langlands quotient* of the representation of the principal series.

We are now able to cite the Theorem of Langlands parameters :

Theorem I.6.3 ([Kna01, Theorem 8.54])
Fix a minimal parabolic subgroup MAN of G. There is a one to one correspondence between the equivalence classes of irreducible admissible representations π of G and the triples $(P, [\gamma], \nu)$ such that:

- *$P = M_P A_P N_P$ is a parabolic subgroup containing MAN*
- *γ is an irreducible tempered representation of M and $[\gamma]$ is its equivalence class*

- *ν is a member of $\mathfrak{a}_{\mathbb{C}}^*$.*

The correspondence is that $(P, [\gamma], \nu)$ corresponds to the class of $J(P, \gamma, \nu)$. The data $(P, [\gamma], \nu)$ are called the Langlands parameters *of the admissible representation π and π is equivalent as a Harish-Chandra module to $J(P, \gamma, \nu)$.*

Another invariant of the representation π is its minimal K-type $\tau_{\min}$.

Suppose now that G is of real rank-one. Then up to conjugation there are just two parabolic subgroups containing the minimal parabolic subgroup MAN: G and MAN. Then for any irreducible admissible representation π of G, π belongs to the discrete series (when the parabolic subgroup is G) or there exists a pair (δ, ν) as follows: Let $\tau_{\min}$ be a minimal K-type in π. Then there exists δ such that $\tau_{\min}$ is also the minimal K-type in the induced representation of δ to K. The parameter ν is the element of $\mathfrak{a}_{\mathbb{C}}$ such that π is infinitesimally equivalent to $J(MAN, \delta, \nu)$ in the principal series $\mathrm{Ind}_{MAN}^{G}(\delta \otimes e^{\nu} \otimes \mathrm{triv})$. This is the unique subquotient containing $\tau_{\min}$ as lowest K-type. For more explanations the reader can follow [Vog81, Sections 4.1, 4.2, 6.5, 6.6].

I.7 Homogeneous differential operators

We recall the definition of differential operator on vector bundles.

Definition I.7.1 (Differential operator)
Let M a manifold and E, F vector bundles over M of rank k and l respectively. we say that D is a *differential operator of order m from E to F*, if $D : \Gamma^\infty E \to \Gamma^\infty F$ is a linear map such that for all $x \in M$ there exists a neighbourhood U of x in M with local coordinates $x_1, \ldots, x_n$ and local trivialisations, $\psi : E|_U \to U \times \mathbb{K}^k$ and $\phi : F|_U \to U \times \mathbb{K}^l$ of E and F respectively such that if $f \in \Gamma^\infty E$, $y \in U$, if $(\psi \circ f)(y) = (y, S(y))$ and if $(\phi \circ Df)(y) = (y, T(y))$ then :

$$T(y) = \sum_{|I| \leq m} A_I(y) \frac{\partial^{|I|}}{\partial x^I} S(y)$$

where A_I is a map C^∞ from U to $\mathrm{Hom}(\mathbb{K}^k, \mathbb{K}^l)$.

A homogeneous differential operator D on E_τ is a linear differential operator from $\Gamma^\infty(E_\tau)$ to itself which is invariant under the G-action by left translations, that is

$$L(g)D = DL(g) \quad \text{for all } g \in G\,. \tag{I.9}$$

The set of homogeneous differential operators on E_τ is an algebra with respect to composition. We denote it by $\mathbb{D}(E_\tau)$. It acts on $C^\infty(G, \tau)$ because of the isomorphism with the space smooth sections $\Gamma^\infty(E_\tau)$. Unlike in the scalar case, i.e. when τ is the trivial representation, this algebra need not be commutative. Conditions equivalent to the commutativity of $\mathbb{D}(E_\tau)$ are stated in [Cam97b, Proposition 2.2] and [RS18, Proposition 3.1]. In the rank one case, this algebra is always commutative when G is $\mathrm{Spin}(n,1)$ or $\mathrm{SU}(n,1)$. See for instance [Cam97b, Theorem 2.3]. The structure of $\mathbb{D}(E_\tau)$ can be found in [Olb94, Section 2.2]. When τ is the trivial representation, in the so-called "scalar" case, the algebra is always commutative. This can be found in Helgason (see Theorem 4.9 in [Hel00])

Let $U(\mathfrak{g}_\mathbb{C})$ be the universal enveloping algebra of the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$. Each element of $U(\mathfrak{g}_\mathbb{C})$ induces a left-invariant differential operator on G by:

$$\Big(X_1\cdots X_k\cdot f\Big)(g) := \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\cdots\frac{\partial}{\partial t_k} f(g\exp t_1X_1\exp t_2X_2\cdots\exp t_kX_k)\Big|_{t_1=\ldots=t_k=0} \tag{I.10}$$

for all $X = X_1\cdots X_n \in U(\mathfrak{g}_\mathbb{C})$, $f\in C^\infty(G)$ and $g\in G$.
Let $U(\mathfrak{g}_\mathbb{C})^K$ denote the subalgebra of the elements in $U(\mathfrak{g}_\mathbb{C})$ which are invariant under the adjoint action Ad of K. The elements of $U(\mathfrak{g}_\mathbb{C})^K$ act on $C^\infty(G,\tau)$ as homogeneous differential operators. Since K is compact, Theorem 1.3 in [Min92] ensures that all elements of $\mathbb{D}(E_\tau)$ can be written as an element of $U(\mathfrak{g}_\mathbb{C})^K$. But there is no isomorphism in general. More explicitly, there exists a map from $U(\mathfrak{g}_\mathbb{C})^K$ to $\mathbb{D}(E_\tau)$ such that its kernel is $U(\mathfrak{g}_\mathbb{C})\mathcal{I}$, where $\mathcal{I}$ is the kernel of τ in $U(\mathfrak{h}_\mathbb{C})$.
We can extend this action to the set of radial system of section $C^\infty(G,K,\tau,\tau)$ by setting:

$$\Big(D\cdot\phi\Big)v := D\cdot(\phi\cdot v)$$

for all $D\in U(\mathfrak{g}_\mathbb{C})$, $\phi\in C^\infty(G,K,\tau,\tau)$ and $v\in V_\tau$.

I.8 The Laplace operator

Let $\{X_1,\ldots,X_{\dim\mathfrak{g}}\}$ be any basis of $\mathfrak{g}$. We denote by g^{ij} the ij-th coefficient of the inverse of the matrix $\Big(B(X_i,X_j)\Big)_{1\le i,j\le\dim\mathfrak{g}}$, where B is the Killing form. The Casimir operator is defined by

$$\Omega := \sum_{1\le i,j\le\dim\mathfrak{g}} g^{ij}X_jX_i\ .$$

If $\Big(X_k\Big)_{k=1,\ldots,\dim\mathfrak{k}}$ and $\Big(X_k\Big)_{k=\dim\mathfrak{k}+1,\ldots,\dim\mathfrak{g}}$ are, respectively, orthonormal bases of $\mathfrak{k}$ and $\mathfrak{p}$ with respect to B_θ, then:

$$\Omega = -\sum_{i=1}^{\dim\mathfrak{k}} X_i^2 + \sum_{i=\dim\mathfrak{k}+1}^{\dim\mathfrak{g}} X_i^2\ .$$

In fact, Ω is in the center of $U(\mathfrak{g}_\mathbb{C})$. The invariant differential operator corresponding $-\Omega$ is the positive Laplacian Δ.

We can extend any representation of $\mathfrak{g}$ to $\mathfrak{g}_\mathbb{C}$ by linearity and to a representation of the associative algebra $U(\mathfrak{g}_\mathbb{C})$. These representations will always be denoted by the same symbol. Since Ω is in the center of $U(\mathfrak{g}_\mathbb{C})$, the linear operator $\pi_\lambda^\sigma(\Omega)$ is an interwining operator of the representation π_λ^σ for all $\lambda\in\mathfrak{a}_\mathbb{C}^*$ and $\sigma\in\hat{M}$. Lemma 4.1.8

in [Vog81] ensures that $\pi_\lambda^\sigma(\Omega)$ acts by a scalar. To compute this scalar, one can use [Kna01, Proposition 8.22 and Lemma 12.28], and get that:

$$\pi_\lambda^\sigma(\Omega) = -\langle\lambda,\lambda\rangle - \langle\rho,\rho\rangle + \langle\mu_\sigma+\rho_{\mathfrak{m}},\mu_\sigma+\rho_{\mathfrak{m}}\rangle \,\mathrm{Id}\ . \tag{I.11}$$

Here μ_σ is the highest weight of σ and $\rho_{\mathfrak{m}}$ is half sum of the positive roots $\varepsilon \in \Pi^+$ such that $\varepsilon|_{\mathfrak{a}} = 0$.

I.9 The vector-valued Helgason-Fourier transform and spherical functions of type τ

In this section we review some basic facts on Camporesi's extension of the Helgason-Fourier transform to homogeneous vector bundles. We refer the reader to [Cam97a] for more information.

We keep the notations of the introduction. In particular, since we suppose that G is of real rank one, in the Plancherel formula only (minimal) principal series representations, and if $G \neq \mathrm{Spin}(2n+1,1)$, discrete series representations occur.

Let (τ, V_τ) be an irreducible unitary representation of K. Let

$$\hat{M}(\tau) = \{\sigma \in \hat{M} \mid m(\sigma, \tau|_M) \geq 1\}$$

denote the set of unitary irreducible representations of M which occur in the restriction of τ to M. We will denote by d_γ the dimension of a representation γ. For $\sigma \in \hat{M}(\tau)$, let P_σ be the projection of V_τ onto the subspace of vectors of V_τ which transform under M according to σ. Explicitly,

$$P_\sigma = d_\sigma \int_M \tau(m^{-1})\chi_\sigma(m)\, dm\,, \tag{I.12}$$

where χ_σ denotes the character of σ.
We denote by $p_\sigma(\lambda)$ the Plancherel density associated to the principal series representation π_λ^σ. Recall the Iwasawa decomposition (I.2) of $x \in G$. According to [Cam97a, Theorem 1.1], the vector-valued Helgason-Fourier transform of $f \in C_c^\infty(G,\tau)$ is the function from $\mathfrak{a}_{\mathbb{C}}^* \times K$ to V_τ defined by

$$\widetilde{f}(\lambda, k) = \int_G F^{i\bar{\lambda}-\rho}(x^{-1}k)^* f(x)\ dx\ . \tag{I.13}$$

Here, for $\mu \in \mathfrak{a}_{\mathbb{C}}$ and $x \in G$,

$$F^\mu(x) = e^{\mu(H(x))}\tau(\mathbf{k}(x)) \tag{I.14}$$

and * denotes the Hilbert space adjoint.

In the rank-one case, the inversion formula is

$$f(x) = \frac{1}{d_\tau} \sum_{\sigma \in \hat{M}(\tau)} \int_{\mathfrak{a}^*} \int_K F^{i\lambda-\rho}(x^{-1}k)\ P_\sigma\ \widetilde{f}(\lambda, k)\ p_\sigma(\lambda)\ \ d\lambda\ dk \\ + \sum_{\gamma \in D_G} C_\gamma \int_K F^{-i\mu-\rho}(x^{-1}k)\ P_{\gamma'}\ \widetilde{f}(i\mu, k)\ dk \quad \text{(I.15)}$$

Here $\mu \in \mathfrak{a}^*$ and $\gamma' \in \hat{M}$ are chosen so that γ is infinitesimally equivalent to a subrepresentation of $\mathrm{Ind}_{MAN}^G(\gamma' \otimes e^\mu \otimes \mathrm{triv})$. Moreover, C_γ is a suitable constant depending on γ. The set D_G is the set of discrete series of G. It consists of all irreducible unitary representations of G whose matrix coefficients are in $L^2(G)$.
The second sum term is called the discrete part of the Plancherel formula. In the following, we will disregard this term. In fact, only the first term, i.e. the continuous part of the Plancherel formula, can contribute to the resonances, by means of the poles of the Plancherel density.

As a consequence not involving discrete series, Parseval's formula for the continuous spectrum reads as follows: for $f, h \in C_c^\infty(G, \tau)$

$$\langle f, h\rangle_c = \sum_{\sigma \in \hat{M}(\tau)} \frac{1}{d_\sigma} \int_{\mathfrak{a}^*} \int_K \langle P_\sigma \widetilde{f}(\lambda, k), P_\sigma \widetilde{h}(\lambda, k)\rangle\ p_\sigma(\lambda)\ d\lambda\ dk\ , \quad \text{(I.16)}$$

where the index c underlines that we are only considering the contribution from the continuous spectrum. See [Cam97a, p. 286]. On the right-end side of (I.16), $\langle \cdot, \cdot \rangle$ denotes the inner product on V_τ making τ unitary. The corresponding norm will be denoted by $\|u\| = \sqrt{\langle u, u\rangle}$.

We will also need a vector-valued analogue of Harish-Chandra's spherical functions on a non-compact reductive Lie group. These vector-valued functions were introduced by Godement [God52] and Harish-Chandra [HC84]. They depend on the fixed representation τ of K and on a representation of the principal series indexed by $\sigma \in \hat{M}(\tau)$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

Keep the above notation for the principal series. Let P_τ denote the projection of $\mathscr{H}_\lambda^\sigma$ onto its subspace of vectors which transform under K according to τ, that is,

$$P_\tau = d_\tau \int_K \pi_\lambda^\sigma(k) \chi_\tau(k^{-1})\, dk\,. \quad \text{(I.17)}$$

Definition I.9.1
The spherical function $\varphi_\tau^{\sigma,\lambda}$ is defined as the $\mathrm{End}(V_\tau)$-valued function on G given by

$$\varphi_\tau^{\sigma,\lambda}(x) := \varphi_\tau^{\pi_\lambda^\sigma}(x) := d_\tau \int_K \tau(k) \psi_\tau^{\sigma,\lambda}(xk^{-1})\, dk\,, \quad \text{(I.18)}$$

where

$$\psi_\tau^{\sigma,\lambda}(x) = \mathrm{Tr}\left(P_\tau \pi_\lambda^\sigma(x) P_\tau\right). \quad \text{(I.19)}$$

Let $\mathrm{Hom}_K(\mathscr{H}^\sigma_\lambda, V_\tau)$ be the space of K-intertwining operators between $\pi^\sigma_\lambda|_K$ and τ. We equip this space with the scalar product $\langle P, Q\rangle = \frac{1}{d_\tau}\,\mathrm{Tr}(PQ^*)$, where $*$ denotes the adjoint. We fix an orthonormal basis $\{P_\xi\}_{\xi=1,\dots,m(\sigma,\tau|_M)}$ of this space. Then

$$\varphi^{\sigma,\lambda}_\tau(g) = \sum_{\xi=1}^{m(\sigma,\tau|_M)} P_\xi\ \pi^\sigma_\lambda(g)\ P^*_\xi\ . \tag{I.20}$$

See pp. 268–269 and 273 in [Cam97a].

Lemma I.9.1
The spherical functions $\varphi^{\sigma,\lambda}_\tau$ are even functions of $\lambda \in \mathfrak{a}^*$ for all σ.

Proof. Due to Lemma 3.1 in [Cam97b], the spherical functions $\varphi^{\sigma,\lambda}_\tau$ are in one-to-one correspondence with their traces. Now $\mathrm{Tr}(\varphi^{\sigma,\lambda}_\tau) = m(\sigma,\tau|_M)\ \chi_\tau * \Theta^\sigma_\lambda$, where χ_τ and Θ^σ_λ are the respective characters of τ and π^σ_λ. Lemma 4, page 162, in [HC76] gives us that $\Theta^\sigma_\lambda = \Theta^{-\sigma}_{-\lambda}$ because -1 is in the Weyl group. As -1 acts trivially on M (so on σ), the lemma follows.

■

The spherical function $\varphi^{\sigma,\lambda}_\tau$ can be described as an Eisenstein integral (see [Cam97a, Lemma 3.2]),

$$\varphi^{\sigma,\lambda}_\tau(x) = \frac{d_\tau}{d_\sigma}\int_K \tau(\mathbf{k}(xk))\ P_\sigma\ \tau(k^{-1})\ e^{(i\lambda-\rho)(\mathbf{H}(xk))}\ dk\ . \tag{I.21}$$

Notice that $\varphi^{\sigma,\lambda}_\tau$ satisfies $\varphi^{\sigma,\lambda}_\tau(k_1xk_2) = \tau(k_1)\varphi^{\sigma,\lambda}_\tau(x)\tau(k_2)$ for every $x \in G$ and $k_1, k_2 \in K$. The convolution with a function $f \in C^\infty_c(G,\tau)$ is defined by:

$$(\varphi^{\sigma,\lambda}_\tau * f)(x) := \frac{d_\tau}{d_\sigma}\int_G \varphi^{\sigma,\lambda}_\tau(x^{-1}g)f(g)\ dg\ . \tag{I.22}$$

According to [Cam97a, Proposition 3.3], it can be expressed in terms of the vector-valued Helgason-Fourier transform of f:

$$(\varphi^{\sigma,\lambda}_\tau * f)(x) = \frac{d_\tau}{d_\sigma}\int_K F^{i\lambda-\rho}(x^{-1}k)\ P_\sigma\ \widetilde{f}(\lambda,k)\ dk\ . \tag{I.23}$$

Lemma I.9.2
The spherical functions $\varphi^{\sigma,\lambda}_\tau$ are joint eigenfunctions of the homogeneous differential operators on E_τ. Moreover, for all $z \in Z(\mathfrak{g}_\mathbb{C})$, the center of $U(\mathfrak{g}_\mathbb{C})$, we have

$$z \cdot \varphi^{\sigma,\lambda}_\tau = \gamma(z)(i\lambda - \mu_\sigma - \rho_\mathfrak{m})\ \varphi^{\sigma,\lambda}_\tau\ . \tag{I.24}$$

Here γ is the Harish-Chandra homomorphism described in [Kna01, Chapter VIII, paragraph 5] and μ_σ is the highest weight of σ.

Proof. In fact, the function Ψ_λ defined in [Yan98] by

$$\Psi_\lambda(nak) := \tau(k^{-1})a^{\lambda+\rho}, \text{ with } k \in K, \ a \in A, \ n \in N,$$

is nothing but the function $x \mapsto F^{-\lambda-\rho}(x^{-1})$ defined in (I.14). Hence [Yan98, Proposition 1.3, Corollary 1.4 and Theorem 1.6] allows us to prove (I.24).

■

Remark

For the Casimir operator, the eigenvalue is given by (I.11) :

$$\gamma(\Omega)(i\lambda - \mu_\sigma - \rho_\mathfrak{m}) = -\langle \lambda, \lambda \rangle - \langle \rho, \rho \rangle + \langle \mu_\sigma + \rho_\mathfrak{m}, \mu_\sigma + \rho_\mathfrak{m} \rangle \tag{I.25}$$

To compute the resonances of the Laplacian, we need to describe the vector-valued Helgason-Fourier transform of functions on $C_c^\infty(G, \tau)$.

By the Cartan decomposition, we can uniquely write an element $x \in G$ as $x = k \exp X$, with $k \in K$ and $X \in \mathfrak{p}$. For $X \in \mathfrak{p}$, we write $|X| = B(X, X)^{1/2}$. Define the open ball centered at 0 and of radius $R > 0$ in $\mathfrak{a} \subset \mathfrak{p}$ by

$$B_\mathfrak{a}^R = \{X \in \mathfrak{a} \,\big|\, |X| < R\} \ .$$

Moreover, we denote the geodesic distance between $o = eK$ in xK by $d(o, xK)$. Let

$$B_R = \{x \in G \,\big|\, d(o, xK) \leq R\} \ .$$

We fix an orthonormal basis $\{e_1, \dots, e_{d_\tau}\}$ of V_τ, then we define

$$||\widetilde{f}(\lambda, k)||^2 := \sum_{i=1}^{d_\tau} \langle \widetilde{f}(\lambda, k), e_i \rangle^2 \ .$$

The direct implication of the Paley-Wiener theorem for $C_c^\infty(G, \tau)$ is given by the following Lemma.

Lemma I.9.3

Let f be in $C_c^\infty(G, \tau)$ and $R > 0$. If $\operatorname{supp} f \subset B_R$, then $\widetilde{f}(\lambda, k)$ is an entire function of $\lambda \in \mathfrak{a}_\mathbb{C}^*$ for all $N \in \mathbb{N}$:

$$\sup_{\lambda \in \mathfrak{a}_\mathbb{C}^*, k \in K} e^{-R\,|\Im(\lambda)|}(1 + |\lambda|)^N ||\widetilde{f}(\lambda, k)|| < \infty \tag{$*$}$$

Proof. Using the integration formula with respect to the Iwasawa decomposition $G = ANK$ (for example in [Hel00, Ch.I §5 Corollary 5.3]), one can prove that for all $f \in C_c^\infty(G, \tau)$

$$\widetilde{f}(\lambda, k) = \mathscr{F}_\mathfrak{a}\left(L_{k^{-1}} \hat{f}\right) \tag{I.26}$$

Here $\lambda \in \mathfrak{a}^*$ and $k \in K$,

$$\hat{f}(g) := e^{\rho(H(g))} \int_N f(gn) dn$$

is called the Radon transform of f and

$$\mathscr{F}_{\mathfrak{a}}(\phi)(\lambda) = \int_{\mathfrak{a}} \phi(X) e^{i\lambda(X)}\, dX \tag{I.27}$$

the Fourier transform on $\mathfrak{a}$ for $\phi \in C_c^\infty(\mathfrak{a})$ and $\lambda \in \mathfrak{a}^*$. Recall that the Euclidean Paley-Wiener theorem ensures that $\mathscr{F}_{\mathfrak{a}}(\phi)$ is an entire function of the exponential type and rapidly decreasing, i.e. if $\operatorname{supp}\phi \subset B_R^{\mathfrak{a}}$ then

$$\forall N \in \mathbb{N},\ \exists C_N \geq 0,\ \text{so that } |\mathscr{F}_{\mathfrak{a}}(\phi)(\lambda)| \leq C_N(1+|\lambda|)^{-N} e^{R\ |\Im(\lambda)|}$$

Define $\hat{f}_i(\cdot) := \langle \hat{f}(\cdot), e_i\rangle$. Then $\hat{f}_i$ is a smooth function.

The idea of the proof is as follows:

$$\operatorname{supp} f \subset B_{\mathbb{R}} \overset{\text{(2)}}{\Longrightarrow} \operatorname{supp} L_{k^{-1}}\hat{f}_i \subset B_{\mathbb{R}}^{\mathfrak{a}} \overset{\text{(1)}}{\Longleftrightarrow} (*)$$

(1) Since $\sup\limits_{i=1,\ldots,d_\tau} |\langle\,\cdot\,, e_i\rangle|$ is a norm on V_τ and since all norms on V_τ are equivalent, $(*)$ is equivalent to

$$\sup_{\lambda\in\mathfrak{a}_{\mathbb{C}}^*,\ k\in K,\ i=1,\ldots,d_\tau} e^{-r|\Im\lambda|}(1+|\lambda|)^N |\langle \widetilde{f}(\lambda,k), e_i\rangle| < \infty\,.$$

Moreover, by (I.26), this is also equivalent to

$$\sup_{\lambda\in\mathfrak{a}_{\mathbb{C}}^*,\ k\in K,\ i=1,\ldots,d_\tau} e^{-r|\Im\lambda|}(1+|\lambda|)^N \left|\mathscr{F}_A\Big(L_{k^{-1}}\hat{f}_i\Big)(\lambda)\right| < \infty\,.$$

In turn, by the Paley-Wiener theorem for the Fourier transform on $\mathfrak{a}$, this is equivalent to $\operatorname{supp} L_{k^{-1}}\hat{f}_i \subset B_R^{\mathfrak{a}}$ as a function on $\mathfrak{a}$ for every i and for every $k \in K$.

(2) Suppose $\operatorname{supp} f \subset B_R$ and let $X \notin B_R^{\mathfrak{a}}$. For all $k \in K$ and $n \in N$, due to [Hel00, Chapter IV, (13)]:

$$d(o, ke^X nK) > \left|\mathbf{H}(ke^X n)\right| = |X| \geq R$$

So $ke^X n \notin B_R \supset \operatorname{supp} f$ and then $\hat{f}_i(ke^X) = 0$ for every i which implies that $X \notin \operatorname{supp} L_{k^{-1}}\hat{f}_i$.

■

From Lemma I.9.3 using (I.23) and the fact that K is compact, we obtain the following corollary.

Corollary I.9.1
For every function $f \in C_c(G,\tau)$, the convolution product $\varphi_\tau^{\sigma,\lambda} * f$ is an even entire function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with the property that there exists a constant $r > 0$ such that for all $N \in \mathbb{N}$ the following inequality holds:

$$\sup_{\lambda\in\mathfrak{a}_{\mathbb{C}}^*} e^{-r|\Im\lambda|}(1+|\lambda|)^N ||\varphi_\tau^{\sigma,\lambda} * f|| < \infty\ . \tag{I.28}$$

Remark

Campoli ([Cam80]) proves the Paley-Wiener theorem for the case of functions $f \in C_c^\infty(G)$ in (real) rank one which are invariant by convolution on K with τ's character $\chi_\tau := d_\tau \operatorname{Tr} \tau$, i.e. $f = \overline{\chi_\tau} *_K f *_K \chi_\tau$. We denote this algebra by $C_{c,\chi_\tau}^\infty(G)$.

There is a well known-isomorphism between $C_{c,\chi_\tau}^\infty(G)$ and the space $C_c^\infty(G,K,\tau,\tau)$ of radial systems of sections. It's given by $\Xi : C_{c,\chi}^0(G) \to C_c(G,K,\tau,\tau)$ defined for all $f \in C_{c,\chi}^0(G)$ by

$$\Xi(f)(x) := \int_K f(k^{-1}x)\tau(k^{-1})dk \ .$$

Its inverse is given for all $\Xi(f) \in C_c(G,K,\tau,\tau)$ by

$$\Xi(f) \mapsto d_\tau \operatorname{Tr} \Xi(f)$$

This yields a Paley-Wiener theorem for $C_c^\infty(G,K,\tau,\tau)$. As pointed out in the previous section, there is a link between $C_c(G,K,\tau,\tau)$ and $C_c(G,\tau)$, but no isomorphism.

A general Paley-Wiener theorem was proved by Arthur in a general setting, without restricting to the rank one case ([Art83, Th.3.3]). It concerns functions $f : G \to V_\tau$ such that $f(k_1xk_2) = \tau(k_1)f(x)\tau(k_2)$ where τ is a two side representation of K.

Remark

One can prove the Paley-Wiener theorem for the compactly supported sections of the homogeneous vector bundle E_τ following the proof of [BOS94, Lemma 2.3], modified so that it applies to the Helgason-Fourier transform of Camporesi. The theorem is cited as follows.

Let f be in $C_c^\infty(G,\tau)$ and $r > 0$. Consider the following conditions:

(1) $\operatorname{supp} f \subset B_r$

(2) $\operatorname{supp} \hat{f} \subset B_r^{\mathfrak{a}}$

(3) $\displaystyle\sup_{\lambda\in\mathfrak{a}_{\mathbb{C}}^*, k\in K} e^{-r|\Im(\lambda)|}(1+|\lambda|)^N ||\widetilde{f}(\lambda,k)|| < \infty$, for all $N \in \mathbb{N}$

Then (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3). If we suppose that G has one conjugacy class of Cartan subgroups then (3) $\Rightarrow$ (1).

For the computations in this thesis we only need one direction of the Paley-Wiener type Theorem, i.e. Lemma I.9.3. The theorem above is built from a theorem of Delorme [Del82] which supposes that G has one conjugacy class of Cartan subgroups. In the rank one case, this theorem holds only for $G = \operatorname{Spin}(2n+1,1)$ (when there are no discrete series). Recall that in that case, there are no resonances because the Plancherel measure has no poles.

II

Computation of the resonances and the residue representations

In this chapter, we prove Theorem 1, which lists the resonances of the Laplace operator on homogeneous vector bundles on rank-one Riemannian symmetric spaces of non-compact type. As preliminary examples, we present the cases of the Laplace operator on $\mathbb{R}^n$ and on the homogeneous line bundles over $H^2(\mathbb{R}) \simeq {\mathrm{SL}(2,\mathbb{R})}\big/{\mathrm{SO}(2)}$. These cases are known, but are complementary because they are not included in our general rank-one case treated in section II.1.3.

Each resonance produces a representation which we describe below. Let R be the resolvent of the Laplace operator. For each resonance z_k, G acts by left translation on the space

$$\{R(z_k)f \mid f \in C_c^\infty(G,\tau)\} \ .$$

This is a representation attached to z_k, which we call in this thesis *residue representation at z_k*. We show that all these representations are irreducible, find their Langlands parameters, and determine which of them are unitarizable.

II.1 Computation of the resonances

II.1.1 Example of $\mathbb{R}^n$

Let $X = \mathbb{R}^n$, with $n \geq 2$. The Laplace operator

$$\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

is self-adjoint on $L^2(\mathbb{R}^n)$ and positive. Its spectrum is $[0, +\infty[$. Then the resolvent of Δ, $R(z) = (\Delta - z)^{-1}$ is a bounded linear operator which depends holomorphicaly on $z \in \mathbb{C} \setminus [0, +\infty[$. The restriction of $R(z)$ to a dense subspace of $L^2(\mathbb{R}^n)$, for example the space of compactly supported smooth functions $C_c^\infty(\mathbb{R}^n)$, has a meromorphic extension accross $[0, +\infty[$:

$$z \mapsto R(z) \in \mathrm{Hom}(C_c^\infty(\mathbb{R}^n), C_c^\infty(\mathbb{R}^n)') ,$$

where $C_c^\infty(\mathbb{R}^n)'$ is the dual of $C_c^\infty(\mathbb{R}^n)$. This meromorphic extension is defined on a Riemannian surface over $\mathbb{C}$ which depends only on the parity of n. The poles of this extension (if they exist) are the resonances of the Laplace operator.

Since

$$\Delta e^{i\lambda \cdot x} = |\lambda|^2 e^{i\lambda \cdot x} , \tag{II.1}$$

the Plancherel Theorem on $\mathbb{R}^n$ implies that, for every $z \in \mathbb{C} \setminus [0, +\infty[$, $f \in C_c^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$(\Delta - z)^{-1} f(x) = (2\pi)^n \int_{\mathbb{R}^n} \hat{f}(\lambda)(|\lambda|^2 - z)^{-1} e^{i\lambda \cdot x} \, d\lambda . \tag{II.2}$$

Hence,

$$\begin{aligned}\Big(R(z) f\Big)(x) &= (2\pi)^n \int_{\mathbb{R}^n} \hat{f}(\lambda)(|\lambda|^2 - z)^{-1} e^{i\lambda x} \, d\lambda \\ &= (2\pi)^n \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y) \; e^{i\lambda(x-y)} \; (|\lambda|^2 - z)^{-1} \, dy \right) \, d\lambda\end{aligned}$$

If $\underline{n = 1}$ (which corresponds to the "rank-one case"), we change the variable $\zeta = \sqrt{z}$, choosing $\sqrt{-1} = i$. Hence $\Im(\zeta) > 0$. Then

$$\Big(R(\zeta^2) f\Big)(x) = 2\pi \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) \; e^{i\lambda(x-y)} \; (\lambda^2 - \zeta^2)^{-1} \, dy \, d\lambda .$$

Recall the convolution $f * e_\lambda(x) = \int_{\mathbb{R}} f(y) \; e^{i\lambda(x-y)} \, dy$. This is the Fourier transform of a smooth compactly supported function. It extends to a Paley-Wiener function on the complex plane. The function

$$\frac{f * e_\lambda(x)}{(\lambda^2 - \zeta^2)}$$

has two singularities: at $\lambda = \zeta$ and $\lambda = -\zeta$. For any $N > \Im(\zeta)$ only one of them, $\lambda = -\zeta$, is between the two horizontal lines, $\mathbb{R}$ and $\mathbb{R} - iN$. Therefore the residue theorem implies that for any $N > \Im(\zeta)$,

$$\int_{\mathbb{R}} \frac{f * e_\lambda(x)}{(\lambda^2 - \zeta^2)} \, d\lambda = \int_{\mathbb{R} - iN} \frac{f * e_\lambda(x)}{(\lambda + \zeta)(\lambda - \zeta)} \, d\lambda - 2\pi i \operatorname*{Res}_{\lambda = -\zeta} \frac{f * e_\lambda(x)}{(\lambda + \zeta)(\lambda - \zeta)} .$$

The integral on the right is equal to

$$\int_{\mathbb{R}-iN} \frac{f * e_\lambda(x)}{2\lambda(\lambda+\zeta)}\, d\lambda + \int_{\mathbb{R}-iN} \frac{f * e_\lambda(x)}{2\lambda(\lambda-\zeta)}\, d\lambda\,.$$

Furthermore,

$$\begin{aligned}\int_{\mathbb{R}-iN} &\frac{f * e_\lambda(x)}{2\lambda(\lambda+\zeta)}\, d\lambda \\ &= \int_{\mathbb{R}+iN} \frac{f * e_\lambda(x)}{2\lambda(\lambda+\zeta)}\, d\lambda + 2\pi i \operatorname*{Res}_{\lambda=0} \frac{f * e_\lambda(x)}{2\lambda(\lambda+\zeta)} + 2\pi i \operatorname*{Res}_{\lambda=-\zeta} \frac{f * e_\lambda(x)}{2\lambda(\lambda+\zeta)}\,.\end{aligned}$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}} \frac{f * e_\lambda(x)}{(\lambda^2-\zeta^2)}\, d\lambda = \int_{\mathbb{R}+iN} \frac{f * e_\lambda(x)}{2\lambda(\lambda+\zeta)}\, d\lambda + \int_{\mathbb{R}-iN} \frac{f * e_\lambda(x)}{2\lambda(\lambda-\zeta)}\, d\lambda + \pi i \frac{f * e_0(x)}{\zeta} \\ - \pi i \frac{f * e_{-\zeta}(x)}{\zeta} + \pi i \frac{f * e_{-\zeta}(x)}{\zeta}\,,\end{aligned}$$

where

$$f * e_0(x) = \int_{\mathbb{R}} f(y)\, dy\,.$$

Both integrals on the right-hand side extend to a holomorphic function of ζ for $\Im(\zeta) > -N$, then the following meromorphic continuation of the resolvent of the Laplace operator is

$$\frac{1}{2\pi}\Big(R(\zeta^2)f\Big)(x) = \pi i \frac{f * e_0(x)}{\zeta} + \int_{\mathbb{R}+iN} \frac{f * e_\lambda(x)}{2\lambda(\lambda+\zeta)}\, d\lambda + \int_{\mathbb{R}-iN} \frac{f * e_\lambda(x)}{2\lambda(\lambda-\zeta)}\, d\lambda$$

So there is just one resonance at $\zeta = 0$, which is a simple pole.

For $n > 1$, we work in spherical coordinates on λ-coordinate in $\mathbb{R}^n$,

$$\Big(R(z)f\Big)(x) = (2\pi)^n \int_0^{+\infty} (r^2 - z)^{-1} \int_{\mathbb{S}^{n-1}} \left(\int_{\mathbb{R}^n} f(y) e^{irw(x-y)} dy\right)\, dw\ r^{n-1} dr\,.$$

For a fixed $x \in \mathbb{R}^n$, let

$$F(r) = \int_{\mathbb{S}^{n-1}} \underbrace{\left(\int_{\mathbb{R}^n} f(y) e^{-irwy} dy\right)}_{\substack{=\hat{f}(rw) \text{ is rapidly decreasing in } r \\ \text{by the Paley-Wiener theorem}}} e^{irwx}\, dw\ r^{n-2}\,.$$

Then $F(r)$ stays rapidly decreasing in r when integrating in $w \in \mathbb{S}^{n-1}$.

- Suppose $n > 2$ is odd. Then F is odd. We set $\zeta = \sqrt{z}$, where the holomorphic branch of the square root function is such that $\sqrt{-1} = i$. Then $\Im(\zeta) > 0$ and

$$
\begin{aligned}
(2\pi)^{-n}\Big(R(\zeta^2)f\Big)(x) &= \int_0^{+\infty} \left(\frac{1}{r^2-\zeta^2}\right)\ F(r)\ r\ dr \\
&= \frac{1}{2}\int_0^{+\infty} \left(\frac{1}{r-\zeta} + \frac{1}{r+\zeta}\right) F(r)\ dr \\
&= \frac{1}{2}\int_0^{+\infty} \frac{F(r)}{r-\zeta}\ dr\ +\ \frac{1}{2}\int_0^{+\infty} \frac{F(r)}{r+\zeta}\ dr \\
&= \frac{1}{2}\int_0^{+\infty} \frac{F(r)}{r-\zeta}\ dr\ +\ \frac{1}{2}\int_{-\infty}^{0} \frac{F(r)}{r-\zeta}\ dr \\
&\uparrow \text{ because } F \text{ is odd. We cannot do this step when } n \text{ is even} \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{F(r)}{r-\zeta}\ dr\ .
\end{aligned}
$$

By Morera's theorem, the above formula extends the function $\zeta \mapsto R(\zeta^2)f(x)$ from the upper half plane to a holomorphic function on $\mathbb{C} \setminus \mathbb{R}$. We want to extend it beyond the upper half plane. We shift the contour of integration in the direction of the negative imaginary axis. By the residue theorem, for $R > 0$ and $N > 0$:

$$
\int_{-R}^{R} \frac{F(r)}{r-\zeta}\ dr\ =\ \int_{-R}^{-R-iN} \frac{F(r)}{r-\zeta}\ dr\ +\ \int_{-R-iN}^{R-iN} \frac{F(r)}{r-\zeta}\ dr\ +\ \int_{R-iN}^{R} \frac{F(r)}{r-\zeta}\ dr\ .
$$

As F is rapidly decreasing, the first and the third integrals on the right-hand side tend to 0 when R goes to infinity. As N is arbitrary, one gets a continuation of $\zeta \mapsto R(\zeta^2)f(x)$ to $\mathbb{C}$ without singularities. There are no resonances. The original resolvent $z \mapsto R(z)$ has an entire extension to a Riemannian surface which is the branched cover of $\mathbb{C}$ associated to the square root.

- Suppose n even. We set $r = e^t$ and $\zeta = \log(z)$, where $\log(z)$ is such that its imaginary part is in $]0, \pi[$. We have then

$$
\Big(R(e^\zeta)f\Big)(x) = (2\pi)^n \int_0^{+\infty} \frac{F(e^t)\ e^{2t}}{e^{2t}-e^{2\zeta}} dt\ .
$$

The same method as before allows us to extend $\zeta \mapsto R(e^\zeta)f(x)$ as an entire function on $\mathbb{C}$. There are also no resonances. The original resolvent $z \mapsto R(z)$ has an entire extension to a Riemannian surface which is the branched cover of $\mathbb{C}$ associated to the logarithm.

Proposition II.1.1

There are no resonances of the Laplace operator

$$
\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}
$$

acting on smooth compactly support functions on $\mathbb{R}^n$, when $n > 1$. For $n = 1$, there is one resonance at 0.

II.1.2 Example of $G = \mathrm{SL}(2, \mathbb{R})$

In this part we compute the resonances for $G = \mathrm{SL}(2, \mathbb{R})$, which will not be considered later. The harmonic analysis is well-known - thanks to the work of Harish-Chandra. We recall that $\mathrm{SL}(2, \mathbb{R})$ is isomorphic to $\mathrm{SU}(1, 1)$ and locally isomorphic to $\mathrm{SO}_{\mathrm{e}}(2, 1)$. We use a direct method based on the original Harish-Chandra - Plancherel formula for $\mathrm{SL}(2, \mathbb{R})$. Will already studied the resonances in this case for $\mathrm{SU}(1, 1)$ in [Wil03, section 5] via a different method.

Fix the maximal compact $K = \mathrm{SO}(2)$ consisting of elements

$$k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The irreducible unitary representations of K are the characters $\chi_n : K \mapsto \mathbb{C}$ defined by

$$\chi_n(k(\theta)) = e^{in\theta}$$

with $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$. Let A and N be defined as follows

$$A = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \,\middle|\, t > 0 \right\}, \quad N = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \,\middle|\, y \in \mathbb{R} \right\}.$$

with Lie algebras

$$\mathfrak{a} = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\}, \quad \mathfrak{n} = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \,\middle|\, y \in \mathbb{R} \right\}.$$

The system of restricted roots is $\{\pm\alpha\}$ where $\alpha \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} = 2x$ and the half sum of positive roots is $\rho = -\frac{1}{2}\alpha$. We have chosen $\{-\alpha\}$ as positive system for the computations. The centralizer of A in K is $M = \{\pm I_2\}$. We denote by $(\pi^{\pm}_{i\lambda}, \mathscr{H}^{\pm}_{i\lambda})$ the principal series representation $\mathrm{Ind}^G_{MAN}(\delta^{\pm} \otimes e^{i\lambda\alpha} \otimes \mathrm{triv}_N)$, where $\lambda \in \mathbb{C}$ and $\hat{M} = \{\delta^+, \delta^-\}$, where δ^+ is the trivial representation and δ^- is the non-trivial representation of M. The space $H^{\delta^{\pm}}_{i\lambda}$ is the Hilbert space completion of the space of smooth functions $f : G \to \mathbb{C}$ such that

$$f(manx) = a^{-(1+i\lambda)\rho}\delta^{\pm}(m)f(x) = t^{1+i\lambda}\delta^{\pm}(m)f(x),$$

where $a = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in A$, $n \in N$, $x \in G$.

For $\lambda \in \mathbb{C}$, we denote by $(\pi_{i\lambda}, \mathscr{H}_{i\lambda})$ the representation

$$\mathscr{H}_{i\lambda} = \mathscr{H}^+_{i\lambda} \oplus \mathscr{H}^-_{i\lambda}.$$

Proposition II.1.2 **([Lan85], VI. §6, Theorem 8, pages 119-124)**
If $i\lambda$ is not an integer, then

$$\mathscr{H}^{+}_{i\lambda}|_K = \bigoplus_{n\in 2\mathbb{Z}} \operatorname{Ind}_K^G(\chi_n) \qquad \text{and} \qquad \mathscr{H}^{-}_{i\lambda}|_K = \bigoplus_{n\in 2\mathbb{Z}+1} \operatorname{Ind}_K^G(\chi_n)$$

are irreducible submodules of $\mathscr{H}_{i\lambda}$ and $\mathscr{H}_{i\lambda}$ is the direct sum of them.

If $s=0$, then $\mathscr{H}_0$ is the direct sum of three irreducible submodules:

$$\mathscr{H}_0 = \bigoplus_{n\in 2\mathbb{Z}} \operatorname{Ind}_K^G(\chi_n) \ \oplus \bigoplus_{\substack{n\in 2\mathbb{Z}+1\\ n\geq 1}} \operatorname{Ind}_K^G(\chi_n) \ \oplus \bigoplus_{\substack{n\in 2\mathbb{Z}+1\\ n\leq -1}} \operatorname{Ind}_K^G(\chi_n)\,.$$

If $m\geq 2$ is an integer and $i\lambda = m-1$, then $\mathscr{H}_{m-1}$ contains three irreducible submodules

$$X^m = \bigoplus_{\substack{m\leq n\\ n-m\in 2\mathbb{Z}}} \operatorname{Ind}_K^G(\chi_n)\,, \qquad X^{-m} = \bigoplus_{\substack{-m\geq n\\ n-m\in 2\mathbb{Z}}} \operatorname{Ind}_K^G(\chi_n)\,, \qquad \bigoplus_{n-m\in 2\mathbb{Z}+1} \operatorname{Ind}_K^G(\chi_n)\,.$$

The quotient module ($\mathscr{H}_{m-1}$ divided by the three submodules) is irreducible, finite dimensional of dimension $m-1$. The $(\mathfrak{g},K)$-modules $\mathscr{H}_{m-1}$ and $\mathscr{H}_{-m+1}$ are dual to each other.

If $m\geq 2$ is an integer and $i\lambda = -m+1$, then $\mathscr{H}_{m-1}$ contains the finite dimensional submodule

$$\mathbb{C}e_{-m+2} \oplus \mathbb{C}e_{-m+4} \oplus \ldots \oplus \mathbb{C}e_{m-2}\,.$$

where e_n is the function on G defined by

$$e_n\left(\begin{pmatrix} u & 0\\ 0 & u\end{pmatrix}\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\right) = y^{1+i\lambda}e^{in\theta}\,. \tag{II.3}$$

Here u, y and x are real such that the indicated product is in G. Thus we have the highest weight modules, lowest weight modules, finite dimensional modules and modules with unbounded $\mathbb{K}$-types on both side.

Here is the complete list of the irreducible unitarizable $(\mathfrak{g},K)$-modules

1. *Lowest weight module X^m with lowest weight $m\geq 1$ and the highest weight module X^m with highest weight $m\leq -1$*
2. *Principal series $\mathscr{H}^{+}_{i\lambda}$ and $\mathscr{H}^{-}_{i\lambda}$, $\lambda\in\mathbb{R}\setminus\{0\}$;*
3. *Principal series $\mathscr{H}^{+}_{0}$;*
4. *Complementary series $\mathscr{H}^{+}_{s}$, $0<s<1$;*
5. *Trivial representation.*

In this thesis, we identify representations if they are infinitesimally equivalent. So we work with $(\mathfrak{g},K)$-modules, when convenient.

The Plancherel theorem of Harish-Chandra for $G=\mathrm{SL}(2,\mathbb{R})$ may be stated as follows:

Theorem II.1.1 (Inversion formula for SL$(2,\mathbb{R})$)
For any $f \in C_c^\infty(G)$,

$$2\pi f(g) = \sum_{n=1}^{\infty} n \operatorname{Tr}\left(X^{(n+1)} + X^{(-n-1)}\right) * f(g) + \frac{1}{2}\int_0^\infty \left(\operatorname{Tr}(\pi_{i\lambda}^+) * f\right)(g) \quad \lambda \tanh\left(\frac{\pi\lambda}{2}\right) d\lambda + \frac{1}{2}\int_0^\infty \left(\operatorname{Tr}(\pi_{i\lambda}^-) * f\right)(g) \quad \lambda \coth\left(\frac{\pi\lambda}{2}\right) d\lambda \ , \quad \text{(II.4)}$$

where

$$\left(\operatorname{Tr}(\pi) * f\right)(g) = \int_G \operatorname{Tr}\left(\pi(x)\right) f(g^{-1}x) \ dx$$

The first term on the right-hand side of (II.4) is the discrete part of the Plancherel formula. We note that this inversion formula is of the same form as (I.15) if we rewrite it in terms of the Fourier transform on G, i.e.

$$\left(\operatorname{Tr}(\pi_{i\lambda}^+) * f\right)(g) = \operatorname{Tr}\left(\pi_{i\lambda}^+(g) \ \int_G \pi_{i\lambda}^+(x)\right) f(x) \ dx\Big) = \operatorname{Tr}\left(\pi_{i\lambda}^+(g) \ \widetilde{f}(\lambda)\right) \ .$$

A direct computation (see [Lan85, page 195]) shows that in the universal enveloping algebra $U(\mathfrak{g}_c)$

$$\Omega = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^2 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2$$

and

$$\pi_{i\lambda}(\Omega) = -(\lambda^2 + 1) \operatorname{Id} \ .$$

Let $n \in \mathbb{Z}$ and $\chi_n \in \hat{K}$ be fixed. Consider the space $C_c^\infty(G, \chi_n)$ of functions f on G such that

$$f(gk) = \chi_n(k^{-1}) f(g) \ , \quad \text{(II.5)}$$

for all $k \in K$ and $g \in G$. We denote by $\phi_{i\lambda,\pm}^{n,n}$ the *spherical functions* defined by

$$\phi_{i\lambda,\pm}^{n,n}(x) = \langle \pi_{i\lambda}^{\pm}(x) e_n, e_n \rangle \ , \quad \text{(II.6)}$$

where $x \in G$ and e_n is as in (II.3). Then the Plancherel formula for SL$(2,\mathbb{R})$ can be rewritten for $f \in C_c^\infty(G, \tau)$ as follows

$$2\pi f(g) = \sum_{l=1}^{\infty} l \left(\phi_l^{n,n} * f\right)(g) + \frac{1}{2}\int_0^\infty \left(\phi_{i\lambda,\pm}^{n,n} * f\right)(g) \quad \lambda \tanh\left(\frac{\pi\lambda}{2}\right) d\lambda + \frac{1}{2}\int_0^\infty \left(\phi_{i\lambda,\pm}^{n,n} * f\right)(g) \quad \lambda \coth\left(\frac{\pi\lambda}{2}\right) d\lambda \ . \quad \text{(II.7)}$$

This comes from the identification

$$f(x) = \int_K \chi_n(k) f(xk)\, dk \qquad \text{for } x \in G\,,$$

and the proof is similar to the following proof of (II.10).

As the discrete part of the formula does not give resonances of the Laplace operator, we will omit it in the computations from now.

We consider $-\Omega$ acting by the left on $C_c^\infty(G,\chi_n)$. This action gives a homogeneous differential operator, which is the Laplace operator Δ. This corresponds to the construction of the positive Laplacian Δ in this thesis in section I.8. One has, for $f \in C_c^\infty(G,\chi_n)$,

$$\Delta(\phi^{n,n}_{i\lambda,\pm} * f) = (\lambda^2+1)\left(\phi^{n,n}_{i\lambda,\pm} * f\right)\ . \tag{II.8}$$

The resolvent R of Δ is defined for $z \in \mathbb{C}$ by

$$R(z) = (\Delta - z)^{-1}$$

We have then, up to a constant, for $f \in C_c^\infty(G,\chi_n)$ and for all $z \in \mathbb{C} \setminus [0,+\infty[$ and $g \in G$

$$\begin{aligned}
(R(z)f)(g) &= \int_0^\infty \left((\Delta - z)^{-1}\phi^{n,n}_{i\lambda,+} * f\right)(g)\ \ \lambda \tanh\left(\frac{\pi\lambda}{2}\right) d\lambda \\
&\quad + \int_0^\infty \left((\Delta - z)^{-1}\phi^{n,n}_{i\lambda,-} * f\right)(g)\ \ \lambda \coth\left(\frac{\pi\lambda}{2}\right) d\lambda \\
&= \int_0^\infty (\lambda^2+1-z)^{-1}\ \left(\phi^{n,n}_{i\lambda,+} * f\right)(g)\ \ \lambda \tanh\left(\frac{\pi\lambda}{2}\right) d\lambda \\
&\quad + \int_0^\infty (\lambda^2+1-z)^{-1}\left(\phi^{n,n}_{i\lambda,-} * f\right)(g)\ \ \lambda \coth\left(\frac{\pi\lambda}{2}\right) d\lambda
\end{aligned}$$

We change the variable $\zeta = \sqrt{z-1}$, where the holomorphic branch of the square root function is such that $\sqrt{-1} = i$. Then $\Im(\zeta) > 0$ and

$$\begin{aligned}
\left(R(\zeta^2+1)f\right)(g) &= \int_0^\infty (\lambda^2-\zeta^2)^{-1}\ \left(\phi^{n,n}_{i\lambda,+} * f\right)(g)\ \ \lambda \tanh\left(\frac{\pi\lambda}{2}\right) d\lambda \\
&\quad + \int_0^\infty (\lambda^2-\zeta^2)^{-1}\left(\phi^{n,n}_{i\lambda,-} * f\right)(g)\ \ \lambda \coth\left(\frac{\pi\lambda}{2}\right) d\lambda\ .
\end{aligned}$$

Moreover, we have

$$2\lambda(\lambda^2-\zeta^2)^{-1} = \frac{1}{\lambda-\zeta} + \frac{1}{\lambda+\zeta}\ .$$

Hence, up to a constant,

$$\begin{aligned}
\left(R(\zeta^2+1)f\right)(g) &= \int_0^\infty \left(\frac{1}{\lambda-\zeta} + \frac{1}{\lambda+\zeta}\right)\ \left(\phi^{n,n}_{i\lambda,+} * f\right)(g)\ \ \tanh\left(\frac{\pi\lambda}{2}\right) d\lambda \\
&\quad + \int_0^\infty \left(\frac{1}{\lambda-\zeta} + \frac{1}{\lambda+\zeta}\right)\left(\phi^{n,n}_{i\lambda,-} * f\right)(g)\ \ \coth\left(\frac{\pi\lambda}{2}\right) d\lambda\ .
\end{aligned}$$

As for $\mathbb{R}^n$:

$$\begin{aligned}
&\int_0^\infty \left(\frac{1}{\lambda-\zeta}+\frac{1}{\lambda+\zeta}\right) \left(\phi^{n,n}_{i\lambda,+} * f\right)(g) \quad \tanh\left(\frac{\pi\lambda}{2}\right) \, d\lambda \\
&= \int_0^\infty \frac{1}{\lambda-\zeta} \left(\phi^{n,n}_{i\lambda,+} * f\right)(g) \quad \tanh\left(\frac{\pi\lambda}{2}\right) \, d\lambda \quad + \quad \int_0^\infty \frac{1}{\lambda+\zeta} \left(\phi^{n,n}_{i\lambda,+} * f\right)(g) \quad \tanh\left(\frac{\pi\lambda}{2}\right) \, d\lambda \\
&= \int_0^\infty \frac{1}{\lambda-\zeta} \left(\phi^{n,n}_{i\lambda,+} * f\right)(g) \quad \tanh\left(\frac{\pi\lambda}{2}\right) \, d\lambda \quad + \quad \int_{-\infty}^0 \frac{1}{\lambda-\zeta} \left(\phi^{n,n}_{i\lambda,+} * f\right)(g) \quad \tanh\left(\frac{\pi\lambda}{2}\right) \, d\lambda \\
&= \int_{\mathbb{R}} \frac{1}{\lambda-\zeta} \left(\phi^{n,n}_{i\lambda,+} * f\right)(g) \quad \tanh\left(\frac{\pi\lambda}{2}\right) \, d\lambda \ ,
\end{aligned}$$

Up to a constant,

$$\begin{aligned}
\left(R(\zeta^2+1)f\right)(g) = \int_{\mathbb{R}} \frac{1}{\lambda-\zeta} & \left(\phi^{n,n}_{i\lambda,+} * f\right)(g) \quad \tanh\left(\frac{\pi\lambda}{2}\right) \, d\lambda \\
& + \int_{\mathbb{R}} \frac{1}{\lambda-\zeta} \left(\phi^{n,n}_{i\lambda,-} * f\right)(g) \quad \coth\left(\frac{\pi\lambda}{2}\right) \, d\lambda \ . \qquad \text{(II.9)}
\end{aligned}$$

By Morera's theorem, the function $\zeta \mapsto R(\zeta^2+1)f(g)$ is holomorphic in $\zeta \in \mathbb{C} \setminus \mathbb{R}$. We want to extend it meromorphically beyond the upper half-plane. For this, we shift the contour of integration in the direction of the negative imaginary axis and apply the residue theorem. The singularities of the meromorphic continuation depend on n being even or odd. For n even, we have:

$$\begin{aligned}
\left(\phi^{n,n}_{i\lambda,-} * f\right)(g) &= \int_G \operatorname{Tr}\left(\pi^-_{i\lambda}(x)\right) f(g^{-1}x) \, dx \\
&= \int_G \int_K \operatorname{Tr}\left(\pi^-_{i\lambda}(x)\right) f(g^{-1}xkk^{-1}) \, dk \, dx \\
&= \int_G \int_K \operatorname{Tr}\left(\pi^-_{i\lambda}(xk^{-1})\right) f(g^{-1}x)\chi_n(k) \, dk \, dx \\
&= \int_G \operatorname{Tr}\left(\pi^-_{i\lambda}(x) \int_K \pi^-_{i\lambda}(k^{-1})\chi_n(k) \, dk \right) f(g^{-1}x) \, dx
\end{aligned}$$

But $\int_K \pi^-_{i\lambda}(k^{-1})\chi_n(k) \, dk = 0$ because

$$\pi^-_{i\lambda}|_K = \bigoplus_{j \in 2\mathbb{Z}+1} \chi_j \ .$$

In the same way, if n is odd,

$$\left(\phi^{n,n}_{i\lambda,+} * f\right)(g) = 0 \ . \qquad \text{(II.10)}$$

Hence only one integral in (II.4) is non-zero for $f \in C^\infty_c(G, \chi_n)$, the one which corresponds to the principal series which contains the K-type χ_n.

Suppose n is even: The formula (II.9) becomes

$$\left(R(\zeta^2+1)f\right)(g) = \int_{\mathbb{R}} \frac{1}{\lambda-\zeta} \ \left(\phi^{n,n}_{i\lambda,+} * f\right)(g) \ \ \tanh\left(\frac{\pi\lambda}{2}\right) \, d\lambda \tag{II.11}$$

For every $n \in \mathbb{N}$, the complex number $-in$ is a pole of $\tanh\left(\frac{\pi\lambda}{2}\right)$. Then by the residue theorem, for every $N \in \mathbb{N}+1/2$ and $r \in \mathbb{R}$:

$$\begin{aligned}
&\int_{-r}^{r} \frac{1}{\lambda-\zeta} \ \left(\phi^{n,n}_{i\lambda,+} * f\right)(g) \ \ \tanh\left(\frac{\pi\lambda}{2}\right) d\lambda + \int_{r}^{r-iN} \frac{1}{\lambda-\zeta} \ \left(\phi^{n,n}_{i\lambda,+} * f\right)(g) \ \ \tanh\left(\frac{\pi\lambda}{2}\right) d\lambda \\
&+ \int_{r-iN}^{-r-iN} \frac{1}{\lambda-\zeta} \ \left(\phi^{n,n}_{i\lambda,+} * f\right)(g) \ \ \tanh\left(\frac{\pi\lambda}{2}\right) d\lambda + \int_{-r-iN}^{-r} \frac{1}{\lambda-\zeta} \ \left(\phi^{n,n}_{i\lambda,+} * f\right)(g) \ \ \tanh\left(\frac{\pi\lambda}{2}\right) d\lambda \\
&= \sum_{\substack{n\in\mathbb{N} \\ n<N}} \frac{1}{-in-\zeta} \ \left(\mathrm{Tr}(\pi^{+}_{-n}) * f\right)(g) \ \ \operatorname*{Res}_{\lambda=-in} \tanh\left(\frac{\pi\lambda}{2}\right) .
\end{aligned}$$

So we have the following result for $r \to \infty$.

Proposition II.1.3

When n is even :

$$\begin{aligned}
\left(R(\zeta^2+1)f\right)(g) = \int_{\mathbb{R}-iN} \frac{1}{\lambda-\zeta} \ \left(\phi^{n,n}_{i\lambda,+} * f\right)(g) \ \ \tanh\left(\frac{\pi\lambda}{2}\right) d\lambda \\
+ \sum_{\substack{l\in\mathbb{Z}^*_+ \\ l<N}} \frac{1}{-il-\zeta} \ \left(\phi^{+}_{-l} * f\right)(g) \ \ \operatorname*{Res}_{\lambda=-il} \tanh\left(\frac{\pi\lambda}{2}\right) .
\end{aligned}$$

When n is odd :

$$\begin{aligned}
\left(R(\zeta^2+1)f\right)(g) = \int_{\mathbb{R}-iN} \frac{1}{\lambda-\zeta} \ \left(\phi^{n,n}_{i\lambda,-} * f\right)(g) \ \ \coth\left(\frac{\pi\lambda}{2}\right) d\lambda \\
+ \sum_{\substack{l\in\mathbb{Z}^*_+ \\ l<N}} \frac{1}{-il-\zeta} \ \left(\phi^{-}_{-l} * f\right)(g) \ \ \operatorname*{Res}_{\lambda=-il} \coth\left(\frac{\pi\lambda}{2}\right) .
\end{aligned}$$

The above formulas give a meromorphic continuation of the resolvent of the positive Laplace operator on the half-plane above $\mathbb{R}-iN$ with simple poles at $\zeta = -il$ where

- *$l \in \mathbb{Z}^*_+$, $l < N$ and l odd for n even*
- *$l \in \mathbb{Z}^*_+$, $l < N$ and l even for n odd*

We make $N \to +\infty$. Let $M = \{(z,\zeta) \in \mathbb{C}^2 \mid z = \zeta^2+1\}$. Then the resolvent extends meromorphically from $M^+ = \{(z,\zeta) \in M \mid \Im(\zeta) > 0\}$ to M and with simple poles at $(-l+1, -il)$, $l \in \mathbb{Z}^*_+$. These poles are the resonances of the Laplace operator.

II.1.3 Rank-one case (excluding $\mathrm{SL}(2,\mathbb{R})$)

In this section, we prove Theorem 1. We recall that the resonances of the positive Laplace operator Δ are defined as the poles of the meromorphic continuation of its resolvent $(\Delta - z)^{-1}$ considered as an operator defined on $C_c^\infty(G,\tau)$. We know thanks to Lemma I.9.2, that the spherical functions $\varphi_\tau^{\sigma,\lambda}$ are eigenfunctions of Δ with eigenvalues $M(\sigma,\lambda) := \langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle - \langle\mu_\sigma+\rho_{\mathfrak{m}},\mu_\sigma+\rho_{\mathfrak{m}}\rangle$. The Plancherel theorem (I.8) (or more explicitly in (I.15)) gives us a decomposition of $L^2(G,\tau)$ into a continuous and, possibly, a discrete part:

$$L^2(G,\tau) = L^2_{\mathrm{cont}}(G,\tau) \oplus L^2_{\mathrm{dicr}}(G,\tau)$$

By a slight abuse of notation, we identify $R(z)$ with its restriction to the set $L^2_{\mathrm{cont}}(G,\tau) \cap C_c^\infty(G,\tau)$. The reader will notice that the discrete part we are omitting brings no additional resonances.

Lemma II.1.1
Let $z \in \mathbb{C} \setminus [\langle\rho,\rho\rangle, +\infty[$. The function $R(z)$ can be written

$$R(z) = \frac{1}{d_\tau} \sum_{\sigma\in\hat{M}(\tau)} R_\sigma(\zeta_\sigma) \tag{II.12}$$

for all $f \in C_c^\infty(G,\tau)$ where

$$R_\sigma(\zeta_\sigma)f(x) = \frac{1}{|\alpha|}\int_{\mathbb{R}} \frac{1}{\lambda|\alpha| - \zeta_\sigma}\Big(\varphi_\tau^{\sigma,\lambda\alpha} * f\Big)(x)\ \frac{p_\sigma(\lambda\alpha)}{\lambda}\ d\lambda \tag{II.13}$$

and ζ_σ is defined by

$$\zeta_\sigma := \sqrt{z - \langle\rho,\rho\rangle + \langle\mu_\sigma+\rho_M, \mu_\sigma+\rho_M\rangle}\ . \tag{II.14}$$

Proof. Let $f \in C_c^\infty(G,\tau)$. Due to the inversion of the vector-valued Helgason-Fourier transform (I.15) and the formula (I.23) for the convolution product between $\varphi_\tau^{\sigma,\lambda}$ and $f \in L^2_{\mathrm{cont}}(G,\tau) \cap C_c^\infty(G,\tau)$, we have

$$f(x) = \frac{1}{d_\tau}\sum_{\sigma\in\hat{M}(\tau)} \int_{\mathfrak{a}^*} \Big(\varphi_\tau^{\sigma,\lambda} * f\Big)(x)\ p_\sigma(\lambda)\ \ d\lambda$$

Because of Lemma I.9.3, we know that $\varphi_\tau^{\sigma,\lambda} * f$ is a rapidly decreasing smooth function. So by Lemma 2.2,

$$\begin{aligned} R(z)f &= \frac{1}{d_\tau}\sum_{\sigma\in\hat{M}(\tau)} \int_{\mathfrak{a}^*} \Big((\Delta - z)^{-1}\varphi_\tau^{\sigma,\lambda} * f\Big)(x)\ p_\sigma(\lambda)\ \ d\lambda \\ &= \frac{1}{d_\tau}\sum_{\sigma\in\hat{M}(\tau)} \int_{\mathfrak{a}^*} (M(\sigma,\lambda) - z)^{-1}\Big(\varphi_\tau^{\sigma,\lambda} * f\Big)(x)\ p_\sigma(\lambda)\ d\lambda\ . \end{aligned}$$

Computing $R(z)$ is then equivalent to compute for each $\sigma \in \hat{M}(\tau)$

$$R_\sigma(z) := \int_{\mathfrak{a}^*} (M(\sigma,\lambda) - z)^{-1} \left(\varphi_\tau^{\sigma,\lambda} * f\right)(x)\ p_\sigma(\lambda)\ d\lambda$$

As the values of μ_σ, ρ and ρ_M are constant we can introduce the variable ζ_σ defined in (II.22). Finally, changing the variable with the isomorphism (I.4) between $\mathfrak{a}^*_\mathbb{C}$ and $\mathbb{C}$:

$$R_\sigma(\zeta_\sigma)f(x) := \int_\mathbb{R} (\lambda^2|\alpha|^2 - \zeta_\sigma^2)^{-1} \left(\varphi_\tau^{\sigma,\lambda\alpha} * f\right)(x)\ p_\sigma(\lambda\alpha)\ d\lambda$$

Notice that we are using the same symbol λ for the variable in $\mathfrak{a}^*$ and the corresponding value $\lambda_\alpha \in \mathbb{R}$. Since the function $\varphi_\tau^{\sigma,\lambda\alpha}$ is even in λ the same computation as in [HP09, Lemma 2.4] yields the formula (II.13):

$$\begin{aligned}
R_\sigma(\zeta_\sigma)f(x) &= \int_\mathbb{R} \frac{1}{2\lambda|\alpha|} \left(\frac{1}{\lambda|\alpha| - \zeta_\sigma} + \frac{1}{\lambda|\alpha| + \zeta_\sigma}\right) \left(\varphi_\tau^{\sigma,\lambda\alpha} * f\right)(x)\ p_\sigma(\lambda\alpha)\ d\lambda \\
&= \frac{1}{2|\alpha|} \int_\mathbb{R} \left(\frac{1}{\lambda|\alpha| - \zeta_\sigma} + \frac{1}{\lambda|\alpha| + \zeta_\sigma}\right) \left(\varphi_\tau^{\sigma,\lambda\alpha} * f\right)(x)\ \frac{p_\sigma(\lambda\alpha)}{\lambda}\ d\lambda \\
&= \frac{1}{|\alpha|} \int_0^{+\infty} \left(\frac{1}{\lambda|\alpha| - \zeta_\sigma} + \frac{1}{\lambda|\alpha| + \zeta_\sigma}\right) \left(\varphi_\tau^{\sigma,\lambda\alpha} * f\right)(x)\ \frac{p_\sigma(\lambda\alpha)}{\lambda}\ d\lambda \\
&= \frac{1}{|\alpha|} \int_\mathbb{R} \frac{1}{\lambda|\alpha| - \zeta_\sigma} \left(\varphi_\tau^{\sigma,\lambda\alpha} * f\right)(x)\ \frac{p_\sigma(\lambda\alpha)}{\lambda}\ d\lambda
\end{aligned}$$

■

Morera's theorem ensures that the function $R_\sigma(\,\cdot\,)f(x)$ is holomorphic in $\zeta_\sigma \in \mathbb{C}\backslash\mathbb{R}$. We want to extend meromorphically $R_\sigma(\,\cdot\,)f(x)$ from one half-plane to the other. To fix notation, we will consider $R_\sigma(\,\cdot\,)f(x)$ as a function defined on the upper half plane $\Im(\zeta) > 0$. We will find the meromorphic continuation of this function to $\mathbb{C}$ by shifting the contour of integration in the direction of the negative imaginary axis and by applying the residue theorem. The poles of the Plancherel density give then poles of the meromorphic continuation. That is why, if the Plancherel density has no poles, so does the meromorphic continuation, and the Laplacian has no resonances (as for $G = \mathrm{Spin}(2n+1,1)$).

The formula of Plancherel density is given in Appendix A (equations (A.1), (A.2), (A.4) and (A.5)) for each of the rank-one groups G. The following proposition unifies the case-by-case formulas found in the literature. The proof is a direct computation remplacing the values in each case thanks to the table in the following Remark.

Proposition II.1.4
Let G be of real rank-one. Set

$$m^\alpha := \frac{1}{2}(m_{\alpha/2} + m_\alpha - 1)\ .$$

Then the Plancherel density is given by the following formula:

$$p_\sigma(\lambda\alpha) = (-1)^s\ \lambda\ \tanh\left(\pi\lambda + \frac{3\pi s i}{2}\right) \prod_{j=1}^{m^\alpha} \left(\lambda^2 + (B_j + \rho - j)^2\right) \tag{II.15}$$

where

$$s = \begin{cases} 2b_1 & \textit{if } G = \mathrm{Spin}(2n,1) \\ 2b_0 + n - 1 & \textit{if } G = \mathrm{SU}(n,1) \\ 2b_0 & \textit{if } G = \mathrm{Sp}(n,1) \\ 2b_1 & \textit{if } G = \mathscr{F}_4 \end{cases} \tag{II.16}$$

and

$$B_j = \begin{cases} b_j & \textit{if } G = \mathrm{Spin}(2n,1) \\ b_{j+1} - b_0 & \textit{if } G = \mathrm{SU}(n,1) \\ \begin{pmatrix} b_{1+j} - b_0 - 1 & \textit{for } j = 1, \dots, n \\ b_0 - b_{2n+1-j} - 1 & \textit{for } j = n+1, \dots, 2n-1 \end{pmatrix} & \textit{if } G = \mathrm{Sp}(n,1) \end{cases} \tag{II.17}$$

If $G = \mathscr{F}_4$ the exceptional case, then

$$(B_1, \dots, B_7) = (b_1+b_2+b_3,\ b_1+b_2-b_3,\ b_1/2,\ b_2/2,\ b_3/2,\ -b_1+b_2+b_3,\ -b_1+b_2-b_3) \tag{II.18}$$

Here the b_j's are the coefficients of the highest weight of σ. We refer the reader to Appendix A to see the computations of this formula.

If one wants to use this formula, one has to find which representations $\sigma \in \hat{M}$ appear in τ, identifying their highest weight thanks to Appendix A, and then compute the Plancherel measure for each of these σ.

Remark

To make the computations in each of the four cases, one needs the following table:

G	K	Σ^+	$m_{\alpha/2}$	m_α	ρ_α	m^α
$\mathrm{Spin}(2n,1)$	$\mathrm{Spin}(2n)$	$\{\alpha\}$	0	$2n-1$	$n-\frac{1}{2}$	$n-1$
$\mathrm{SU}(n,1)$	$\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1))$	$\{\alpha/2,\alpha\}$	$2n-2$	1	$\frac{n}{2}$	$n-1$
$\mathrm{Sp}(n,1)$	$\mathrm{Sp}(n)$	$\{\alpha/2,\alpha\}$	$4n-4$	3	$n+\frac{1}{2}$	$2n-1$
$\mathscr{F}_4$	$\mathrm{Spin}(9)$	$\{\alpha/2,\alpha\}$	8	7	$\frac{11}{2}$	7

Suppose s even: The formula above goes in $\tanh(\pi\lambda)$, which has first order poles at $\lambda \in i\left(\mathbb{Z}+\frac{1}{2}\right)$ however the singular points of this function are the imaginary numbers of the form $i\left(\mathbb{Z}+\frac{1}{2}\right)$. Since s is even, the $B_j+\rho-j$ are in $\mathbb{Z}+\frac{1}{2}$. Hence the zeros of the polynomial part of $\frac{p_\sigma(\lambda\alpha)}{\lambda}$ are the element

$$\{\pm i(B_j+\rho-j) \mid j=1,\dots,m^\alpha\} \ . \tag{II.19}$$

Thus the Plancherel density has simples poles in the complement of this set in $i(\mathbb{Z}+\frac{1}{2})$.

Suppose s odd: The formula above goes in $\coth(\pi\lambda)$, with simple poles at $\lambda \in i\mathbb{Z}$. Since s is odd, the $B_j+\rho-j$ are in $\mathbb{Z}$. Thus the poles of the Plancherel density are simple and located in the complement of the set (II.19) in $i\mathbb{Z}$.

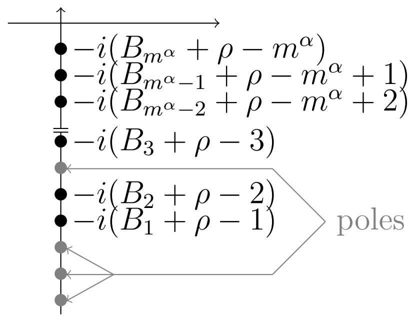


Figure II.1: Example of poles with $B_2 = B_3+1$ for $\mathrm{SO}(2n,1)$

Let

$$B_{\max} = \max(|B_1+\rho-1|, |B_{m^\alpha}+\rho-m^\alpha|) \ . \tag{II.20}$$

All the values of the form $-i(B_{\max}+k)$, with $k \in \mathbb{N}$, are poles of the Plancherel density. Let us view R_σ as the $D'(X)$-valued holomorphic function on $\Im(\zeta_\sigma) > 0$ defined in (II.13). We want to determine the meromorphic continuation of this function through the real axis. For this we shift the contour of integration in the direction of the negative imaginary axis as in Figure II.2 below. Due to Corollary I.9.1, the convolution product is rapidly decreasing in λ. Then we just need to upper bound the expression $\left|(\zeta_\sigma-\lambda|\alpha|)^{-1}\ \frac{p_\sigma(\lambda\alpha)}{\lambda}\right|$ by a polynomial in $|\lambda|$, to make the two integrals along the vertical segments between $-R$ and $-R-i(\rho_1+N+1/2)$ and between R and $R-i(\rho_1+N+1/2)$ tend to 0 when R goes near infinity. For $|\Re(\lambda)|$ near to infinity and $\Im(\zeta_\sigma)>0$, we have $|(\zeta_\sigma-\lambda|\alpha|)^{-1}| < \Im(\zeta_\sigma)^{-1}$ and $\left|\frac{p_\sigma(\lambda\alpha)}{\lambda}\right| < (1+|\lambda|)^{\deg(p_\alpha)+1}$, where p_α is the polynomial part of p_σ. So, the shift is allowed for all $N \in \mathbb{N}$.

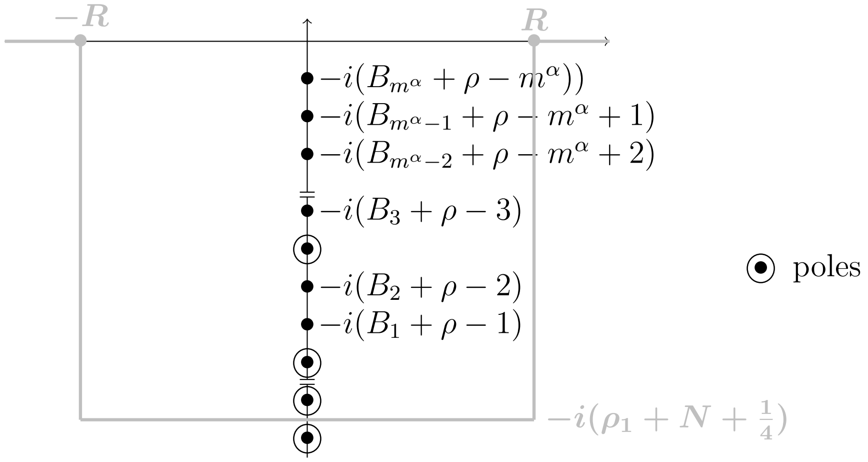


Figure II.2: Shift of contour and residue theorem for SO$(2n,1)$

Let $\mathbb{N}_\sigma$ be the set of $k \in \mathbb{Z}$ such that

$$\lambda_k := -i(B_{\max}+k)$$

is a pole of the Plancherel density (II.15) and $B_{\max}+k \geq 0$. The residue theorem ensures us that for all $N \in \mathbb{N}$ and ζ_σ with $\Im(\zeta) > 0$:

$$\begin{aligned}(R_\sigma(\zeta_\sigma)f)(x) &= \frac{1}{|\alpha|}\int_{\mathbb{R}-i(N+1/4)} \frac{1}{\lambda|\alpha|-\zeta_\sigma}\left(\varphi_\tau^{\sigma,\lambda\alpha} * f\right)(x)\,\frac{p_\sigma(\lambda\alpha)}{\lambda}\,d\lambda \\ &+ \quad \frac{2i\pi}{|\alpha|} \sum_{\substack{k\in\mathbb{N}_\sigma \\ \lambda_k > -i(N+1/4)}} \frac{1}{\lambda_k|\alpha|-\zeta_\sigma}\left(\varphi_\tau^{\sigma,\lambda_k\alpha} * f\right)(x)\ \operatorname*{Res}_{\lambda=\lambda_k}\frac{p_\sigma(\lambda\alpha)}{\lambda} \end{aligned} \tag{II.21}$$

The right-hand side of this formula yields a meromorphic continuation of the resolvent of the Laplace operator on $\Im(\zeta) > -(N+1/4)$. The singular values of the Plancherel density induce then poles of the meromorphic continuation of the resolvent of the Laplace operator. If we resume the notation in the expression (II.22): $z = -\zeta_\sigma^2 - \langle\rho,\rho\rangle + \langle\mu_\sigma+\rho_M, \mu_\sigma+\rho_M\rangle$. This proves the theorem.

Theorem II.1.2. *Let G be a connected non-compact semisimple Lie group with finite center and with Iwasawa decomposition $G = KAN$, where K is a fixed maximal compact subgroup of G. Suppose* $\dim A = 1$. *Let M denote the centralizer of A in K. Let (τ, V_τ) be an irreducible unitary representation of K, and let E_τ be the homogeneous vector bundle over G associated with τ.*

$$\zeta_\sigma := \sqrt{-z - \langle\rho, \rho\rangle + \langle\mu_\sigma + \rho_M, \mu_\sigma + \rho_M\rangle)} \tag{II.22}$$

with $z \in \mathbb{C}$ such that $\Im(\zeta_\sigma) > -(N + 1/4)$. Here $\sqrt{\cdot}$ denotes the single-valued branch of the square root function determined on $\mathbb{C} \setminus [0, +\infty[$ by the condition $\sqrt{-1} = -i$.

The resonances of the Laplace operator acting on the sections of E_τ appear in families parametrized by the elements of $\hat{M}(\tau)$. Let

$$S_\sigma = \left\{(z, \zeta) \in \mathbb{C}^2 \mid \zeta^2 := -z - \langle\rho, \rho\rangle + \langle\mu_\sigma + \rho_M, \mu_\sigma + \rho_M\rangle)\right\} .$$

Then the resolvent extends meromorphically from $S_\sigma^+ = \{(z, \zeta) \in S \mid \Im(\zeta) > 0\}$ to S_σ. The (simple) poles of this extension are the pairs

$$(z_{\sigma,k}, \zeta_{\sigma,k}) = \left((B_{\max}+k)^2|\alpha|^2 - \rho_\alpha^2|\alpha|^2 + \langle\mu_\sigma+\rho_M, \mu_\sigma+\rho_M\rangle , -i(B_{\max}+k)^2|\alpha|^2\right) \tag{II.23}$$

where $k \in \mathbb{N}_\sigma$ (defined below the figure II.2), μ_σ is the highest weight of the representation σ, $B_{\max}$ is defined in (II.20) *and ρ_M is half sum of roots of M.*

II.2 Residue representations

We consider $C^\infty(G, \tau)$ as a G-module by left-translations. One can see that for each $\sigma \in \hat{M}(\tau)$ and for each $k \in N_\sigma$, the residues at λ_k in the meromorphic continuation (II.21) span a G-invariant subspace of $C^\infty(G, \tau)$ if $f \in C_c^\infty(G, \tau)$. This is exactly the image of the G-intertwining map :

$$\begin{array}{rccc} R_k^\sigma : & C_c^\infty(G, \tau) & \longrightarrow & C^\infty(G, \tau) \\ & f & \longmapsto & \varphi_\tau^{\sigma,\lambda_k} * f \end{array} . \tag{II.24}$$

We denote this space by

$$\mathscr{E}_k^\sigma := \{\varphi_\tau^{\sigma,\lambda_k} * f \mid f \in C_c^\infty(G, \tau)\} . \tag{II.25}$$

We want to identify these representations in terms of their Langlands parameters, decide which of them are unitarizable and compute their wave front sets. The idea is to decompose R_k^σ as follows:
Recall the notation $(\pi_{\lambda_k}^\sigma, \mathscr{H}_{\lambda_k\alpha}^\sigma)$ for the principal series representation corresponding to σ and $i\lambda_k\alpha$. For all $l = 1, \ldots, m(\sigma, \tau|_M)$, we denote by P_l the projection of $\mathscr{H}_{\lambda_k\alpha}^\sigma$ on its l-th K-isotypic component, which we identify with (τ, V_τ). We get the following

decomposition of the residue operator R_k^σ as a composition of two G-intertwining maps:

$$\begin{array}{llll} R_k^\sigma : C_c^\infty(G,\tau) & \to \mathscr{H}^\sigma_{\lambda_k\alpha} & \to C^\infty(G,\tau) \\ f & \mapsto T_l(f) & \mapsto \sum_l P_l \, \pi^\sigma_{\lambda_k\alpha}(\,\cdot\,^{-1})\big(T_l(f)\big) \end{array} \tag{II.26}$$

where T_l is the map from $C_c^\infty(G,\tau)$ to $\mathscr{H}^\sigma_{\lambda_k\alpha}$ defined by

$$T_l(f) = \int_G \pi^\sigma_{\lambda_k}(g)\Big(P_l^* f(g)\Big)\, dg\ . \tag{II.27}$$

Here $*$ denotes the Hermitian adjoint. Hence, P_l^* maps V_τ into the principal series and is K-equivariant.

Lemma II.2.1

T_l is an intertwining operator between the left regular representation of G on $C_c^\infty(G,\tau)$ and the principal series representation $(\pi^\sigma_{\lambda_k}, \mathscr{H}^\sigma_{\lambda_k\alpha})$. Moreover, for each l the range of the map T_l is the closed subspace of $\mathscr{H}^\sigma_{\lambda_k\alpha}$ spanned by the left translates of $P_l^*V_\tau$. We will denote this space by $\langle \pi^\sigma_{\lambda_k}(G)P_l^*V_\tau\rangle$.

Proof. First of all, by definition, $T_l\Big(C_c^\infty(G,\tau)\Big)$ is contained in $\langle \pi^\sigma_{\lambda_k}(G)P_l^*V_\tau\rangle$. Fix an element g_0 of G and a vector v_0 in V_τ. Let $\delta_{g_0} \in C_c(G)^*$ be the (scalar) Dirac delta at g_0 and set $\delta_{g_0,v_0} = \delta_{g_0}v_0$. Consider the operator on $C_c^\infty(G,\tau)$ defined by as the distribution $f \mapsto \int_G\int_K \tau(k)\delta_{g_0,v_0}(gk)dk\ f(g)dg$. Since P_l^* is a K-intertwining operator between τ and π_{λ_k}, calculations show that

$$T_l\left(\int_K \tau(k)\delta_{g_0,v_0}(\,\cdot\, k)\ dk\right) = \pi^\sigma_{\lambda_k}(g_0)P_l^*v_0\ .$$

Since the Dirac delta can be approximated by smooth compactly supported functions, each element of $\pi^\sigma_{\lambda_k}(G)\Big(P_l^*V_\tau\Big)$ can be written as limit of elements of $T_l\Big(C_c^\infty(G,\tau)\Big)$. This proves the lemma.

■

Remark

The map from $\mathscr{H}^\sigma_{\lambda_k\alpha}$ to $C_c^\infty(G,\tau)$ defined by $\phi \mapsto P_l\pi^\sigma_{\lambda_k}(\,\cdot\,^{-1})\phi$ is a G-intertwining operator between $(\pi^\sigma_{\lambda_k}, \mathscr{H}^\sigma_{\lambda_k\alpha})$ and the left regular representation on $C_c^\infty(G,\tau)$. It is known as the Poisson transform (see [Olb94, Yan98]).

The idea on how to identify $\mathscr{E}_k^\sigma$ is to compute the range of the map T_l, for each l, using Lemma II.2.1. Knowing the composition series of $\mathscr{H}^\sigma_{\lambda_k\alpha}$, we can identify this range in this principal series and then project back with the Poisson transform, the second part of the map in (II.26). The main issue is that, even in rank-one, the structure of the principal series representations is very complicated in general. See [Col85]. In this thesis we will consider only the case when σ is the trivial representation of M. In this case the composition series is more transparent and the result have a pleasant uniform

form. For example, τ occurs in $\pi^{\sigma}_{\lambda_k}$ with multiplicity one. So, from now on, we fix an irreducible representation $\tau \in \hat{K}$ which contains the trivial representation of M. We study only the residue representations which arise from $\sigma = \mathrm{triv}_M$ and we will denote them $\mathscr{E}_k := \mathscr{E}_k^{\mathrm{triv}_M}$. Similarly, we set $R_k := R_k^{\mathrm{triv}}$ and write T and P instead of T_l and P_l respectively. The map in (II.26) becomes:

$$\begin{array}{rclcl} R_k : C_c^{\infty}(G,\tau) & \to & \mathscr{H}^{\mathrm{triv}}_{\lambda_k\alpha} & \to & C^{\infty}(G,\tau) \\ f & \mapsto & T(f) & \mapsto & P\ \pi^{\mathrm{triv}}_{\lambda_k\alpha}(\,\cdot\,^{-1})\big(T(f)\big) \end{array} \tag{II.28}$$

where P is the projection onto the τ-isotypic component. The structure of the spherical principal series representations $(\pi_\lambda, \mathscr{H}_\lambda) := (\pi^{\mathrm{triv}}_\lambda, \mathscr{H}^{\mathrm{triv}}_\lambda)$ of our groups G has been studied by different authors (see e.g. [HT93, JW77, Joh76, Mol99]) for every G we are studying. Our main reference will be the paper of Howe and Tan [HT93] which provides an explicit description of the subquotients of this principal series representation. We will treat the residue representations case by case. In each case, we compute the Langlands parameters of $\mathscr{E}_k$, $k \in \mathbb{N}_\sigma$.

Method to compute explicitly the Langlands parameters: First of all we have to compute the minimal K-type of $\mathscr{E}_k$ in the sense of Vogan [Vog81, 5.4.18.]. It can be done by minimizing Vogan's norm on the set of K-types in $\mathscr{E}_k$. Using the branching rules in [BS79] we find the representation δ. In fact, from the highest weight of the minimal K-type $\tau_{\min}$ we know what are the representations in its restriction to M. Then we induce back from these representations and we check if $\tau_{\min}$ is the minimal K-type in these induced representations. For $G = \mathrm{Sp}(n,1)$ this work is already done in Proposition 3.1 in [BS81]. Sometimes such δ does not exist. This implies that π is a discrete series representation. [Vog81, Theorem 6.6.15] ensures us that the two representations $\mathscr{H}^{\mathrm{triv}}_{\lambda_k}$ and $\mathscr{H}^{\delta}_{\nu}$ are conjugate under K. This means that the infinitesimal characters of $\pi^{\mathrm{triv}}_{\lambda_k\alpha}$ and π^{δ}_{ν} are the same up to one element of the Weyl group of the pair $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. They can be computed depending on ν. Recall that π^{σ}_{λ} has infinitesimal character $\mu_\sigma + \rho_\mathfrak{m} + \lambda$ where μ_σ is the highest weight of σ and $\rho_\mathfrak{m}$ is half sum of positive roots orthogonal to the restricted root α. Up to a sign change, there is a unique possible value of ν.
The residue representation $\mathscr{E}_k$ can then be identified with the unique Langlands subquotient containing the same minimal K-type $\tau_{\min}$. To summarize:

1. Find a minimal K-type $\tau_{\min}$ of π.

2. Find δ such that $\tau_{\min}$ is minimal in $\mathrm{Ind}_M^K(\delta)$. If this doesn't work then π is a discrete series representation.

3. Find ν such that the infinitesimal character of $\mathscr{H}^{\mathrm{triv}}_{\lambda_k}$ is the same as the infinitesimal character of $\mathscr{H}^{\delta}_{\nu}$. This means that $\mu_{\mathrm{triv}} + \rho_m + \lambda_k$ is conjugate to $\mu_\sigma + \rho_m + \nu$ by an element of the complex Weyl group.

II.2.1 Example of $\mathbb{R}$

Recall that in the case of the $\mathbb{R}$ the Laplace operator (or the double derivative in this case) has only one resonance at $\lambda = 0$ (see section II.1.1). In this case, the residue representation takes place is the space

$$\left\{x \mapsto \int_{\mathbb{R}} f(y)\, dy \ \mid f \in C_c^\infty(\mathbb{R})\right\} \ ,$$

which is the set of constant functions. $\mathbb{R}$ acts trivially on this set. So the residue representation here is the trivial representation of $\mathbb{R}$.

II.2.2 Example of $\mathrm{SL}(2, \mathbb{R})$

Recall (see section II.1.2) that here

$$\phi_l^{n,n}(x) = \langle \pi_l(x) e_n, e_n\rangle \ , \tag{II.29}$$

where $x \in G$, e_n is as in (II.3) and $(\pi_l, \mathscr{H}_l)$ is defined in Proposition II.1.2. Then we obtain the following residue representations

When n is even :

$$\{\phi_{-l} * f \mid l \text{ is odd}, \ f \in C_c^\infty(G, \chi_n)\} \ .$$

When n is odd :

$$\{\phi_{-l} * f \mid l \text{ is even}, \ f \in C_c^\infty(G, \chi_n)\} \ .$$

So $l-1$ and n have the same parity. Then $\langle \pi_{-l}(G) P^* V_{\chi_n}\rangle$ is one of the following irreducible components of π_{-l} (see also Proposition II.1.2)

$$X^{-l-1} = \bigoplus_{\substack{p \leq -l-1 \\ p+l-1 \in 2\mathbb{Z}}} \mathrm{Ind}_K^G(\chi_p)\,, \qquad X^{l+1} = \bigoplus_{\substack{p \geq l+1 \\ p+l-1 \in 2\mathbb{Z}}} \mathrm{Ind}_K^G(\chi_p)\,, \qquad \bigoplus_{\substack{-l-1<p<l+1 \\ p+l-1 \in 2\mathbb{Z}}} \mathrm{Ind}_K^G(\chi_p)\,. \tag{II.30}$$

In particular, $\langle \pi_{-l}(G) P^* V_{\chi_n}\rangle$ is the component where χ_n belongs:

1. $\langle \pi_{-l}(G) P^* V_{\chi_n}\rangle = X^{-l-1}$ if $n \leq -l-1$,
2. $\langle \pi_{-l}(G) P^* V_{\chi_n}\rangle = X^{l+1}$ if $n \geq l+1$
3. $\langle \pi_{-l}(G) P^* V_{\chi_n}\rangle$ is the final dimensional constituent above if $-l-1 < n < l+1$

For a fixed number n, we have $\left\lfloor \dfrac{|n|}{2} \right\rfloor - 1$ infinite dimensional representations which are isomorphic to the holomorphic discrete series X^{-l-1} or X^{l+1} (as in [Wil03, Theorem 5.1]). The other representations are finite dimensional.

II.2.3 Case of $\mathrm{SO}(2n,1)$

Let $G = \mathrm{SO}(2n,1)$. The K-types of the spherical principal series representations are parametrized by $m \in \mathbb{N}$ and known as the space of spherical harmonics on $\mathbb{R}^{2n}$ of homogeneous degree m, denoted by $\mathscr{H}^m(\mathbb{R}^{2n})$. Their highest weight of the form $m\epsilon_1$ with respect to the fundamental weights described in section A.1. A representation τ containing the trivial representations of M is of this form. From now on, let τ act on the harmonic polynomials on $\mathbb{R}^{2n}$ of fixed homogeneous degree N:

$$V_\tau \simeq \mathscr{H}^N(\mathbb{R}^{2n}) \ .$$

As σ is trivial the poles of the Plancherel density p_{triv_M} are the

$$\lambda_k = -i(\rho_\alpha + k) = -i(n + k - 1/2)$$

with $k \in \mathbb{N}$. The composition series of $\mathscr{H}_{\lambda_k\alpha}$ described in [HT93] is the following:

$$\mathscr{H}_{\lambda_k\alpha} \simeq \overbrace{\sum_{m=0}^{k} P_m^*\Big(\mathscr{H}^m(\mathbb{R}^{2n})\Big)}^{1} \oplus \overbrace{\sum_{m>k} P_m^*\Big(\mathscr{H}^m(\mathbb{R}^{2n})\Big)}^{2} \tag{II.31}$$

where P_m is the projection of $\mathscr{H}_{\lambda_k\alpha}$ onto the K-isotypic component isomorphic to $\mathscr{H}^m(\mathbb{R}^{2n})$. The action of G cannot send a K-type from the second summand to the first one.

Langlands parameters of $\mathscr{E}_k$. In Figure II.3 below, each bullet corresponds to one K-type of the representation $\mathscr{H}_{\lambda_k\alpha}$, the abscissa of the bullet being the coefficient appearing in the highest weight of the K-type. The figure describes the two cases $N \geq k$ and $N < k$. The barrier at $k+1$ and the arrows mean that the action of $\mathfrak{g}$ cannot send a K-type which is on the right of $k+1$ to a K-type which on the left of $k+1$.

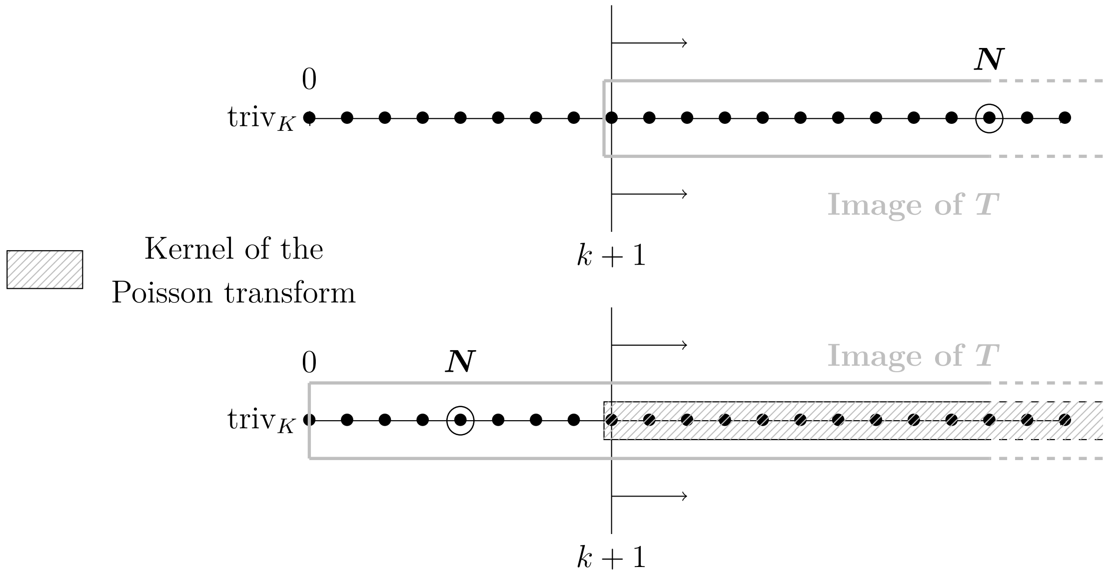


Figure II.3: K-types in $\mathscr{H}_{\lambda_k\alpha}$ and composition series for $k=7$

Suppose that $N \geq k+1$: The image of T is the space spanned by the action of $\mathfrak{g}$ on $P_m^*\Big(\mathscr{H}^N(\mathbb{R}^{2n})\Big)$. Since that action cannot cross the barrier at $k+1$, the image of T is the infinite-dimensional irreducible representation

$$\sum_{m>k} j_{-2n-k+1}\Big(\mathscr{H}^m(\mathbb{R}^{2n})\Big)$$

The second map in (II.26) (the Poisson transform) is an intertwining operator. So its kernel is either 0 or the entire representation. Choosing a nonzero $h \in \mathscr{H}^N(\mathbb{R}^{2n})$ nonzero, the function $P_{\tau_N}\ \pi_{\lambda_k\alpha}(\ \cdot\ ^{-1})j_{-2n-k+1}\Big(h\Big)$ has value h at e_G. So the kernel of the Poisson transform is not the entire space, thus it is 0. Consequently $\mathscr{E}_k$ is 2 in (II.31).

Suppose that $N \leq k$: Here the image of T is the entire representation $\mathscr{H}_{\lambda_k\alpha}$ because the action of $\mathfrak{g}$ can cross the barrier at $k+1$. The kernel of the second intertwining map in (II.28) is 2 in (II.31) by the same argument as before. It follows that $\mathscr{E}_k$ is the subquotient isomorphic to 1 in (II.31).

The unitarity is given in [HT93, Diagram 3.15]. To identify the representations we have determined, we compute their Langlands parameters. They are collected in Table II.1.

Case	Minimal K-type	δ	values of ν
$N > k+1$	$\mathscr{H}^{k+1}(\mathbb{R}^{2n})$	$\mathscr{H}^{k+1}(\mathbb{R}^{2n-1})$	$\pm\left(n-\frac{3}{2}\right)\alpha$
$N \leq k$	triv_K	triv_M	$(\rho_\alpha+k)\alpha$

Table II.1: Langlands parameters of $\mathscr{E}_k$ when $G=\mathrm{SO}(2n,1)$

The entries of Table II.1 are computed as follows. Let μ_l be the highest weight of $\mathscr{H}^l(\mathbb{R}^{2n})$. A minimal K-type minimizes the Vogan norm of the highest weight in $\mathscr{E}_k$:

$$\|\mu_l\|_V = \langle \mu_l + 2\rho_K, \mu_l + 2\rho_K\rangle$$

where ρ_K is half sum of positive roots in $S_+^{\mathfrak{k}_\mathbb{C}}$ (see Appendix A.1 for the definition). One can check that $\|\mu_l\|_V$ is minimal when l is minimal. This yields the first column of the table.

Finally, to find δ, one can use [BS79, Theorem 3.4]. The M-types in $\mathscr{H}^l(\mathbb{R}^{2n})$ have highest weight

$$\mu_\delta(a) = a\epsilon_1$$

where $a \in [\![0, k+1]\!]$. The only one which has the same minimal K-type is $\begin{cases} \mu_\delta(0) & \text{, for } 1 \\ \mu_\delta(k+1) & \text{, for } 2 \end{cases}$.

To find ν one has to compare the infinitesimal character of

$$\mathrm{Ind}_{MAN}^G(\mathrm{triv}\otimes e^{(\rho_\alpha+k)\alpha}\otimes\mathrm{triv}) \quad \text{and} \quad \mathrm{Ind}_{MAN}^G(\delta\otimes e^{\nu}\otimes\mathrm{triv})\ .$$

They have to agree up to the action of the Weyl group of $(\mathfrak{g}_\mathbb{C}, \mathfrak{k}_\mathbb{C})$. Theorem 2, 1., follows from these computations.

■

Theorem II.2.1. *Suppose that the representation τ contains the trivial representation of M. The residue representations $\mathscr{E}_k$ are then irreducible.*

- *If $N \geq k+1$, then $\mathscr{E}_k$ has Langlands parameters $\left(MA, \mathscr{H}^{k+1}(\mathbb{R}^{2n-1}), \left(n-\frac{3}{2}\right)\alpha\right)$ with $(k+1, 0, \ldots, 0)$ as a lowest K-type's highest weight. Here $\mathscr{H}^{k+1}(\mathbb{R}^{2n-1})$ are harmonics of degree $k+1$ on $\mathbb{R}^{2n-1}$. This representation is unitary.*
- *If $N < k+1$, then $\mathscr{E}_k$ has Langlands parameter $\left(MA, \mathrm{triv}, (\rho_\alpha + k)\right)$ with the trivial representation as a lowest K-type. It is finite-dimensional. Also, it is non-unitary if $k \neq 0$.*

II.2.4 Case of $\mathrm{SU}(n,1)$

Let now $G = \mathrm{SU}(n,1)$. The structure of the spherical principal series is described for $\mathrm{U}(n,1)$ in [HT93] but Molchanov find the same result for $\mathrm{SU}(n,1)$ in [Mol99]. Choose a basis $\{z_1, \ldots, z_n, z_{n+1}\}$ of $\mathbb{C}^{n+1}$ as a vector space and $\{z_1, \ldots, z_n, z_{n+1}, \overline{z_1}, \ldots, \overline{z_n}, \overline{z_{n+1}}\}$ the corresponding basis of $\mathbb{R}^{2n+2}$. The K-types of the spherical principal series representations are the spaces

$$\mathscr{H}^{m_1,m_2}(\mathbb{C}^n) \otimes \mathscr{H}^l(\mathbb{C})$$

where $\mathscr{H}^{m_1,m_2}(\mathbb{C}^n)$ are the spherical harmonics on $\mathbb{C}^{2n}$ of homogeneous degree m_1 and m_2 in the variables $z_1, \ldots, z_{n+1}$ and $\overline{z_1}, \ldots, \overline{z_n}, \overline{z_{n+1}}$ respectively. Moreover, the space $\mathscr{H}^l(\mathbb{C})$ is defined for all integer l by:

$$\mathscr{H}^l(\mathbb{C}) = \begin{cases} \mathscr{H}^{l,0}(\mathbb{C}) & \text{if } l \geq 0 \\ \mathscr{H}^{0,-l}(\mathbb{C}) & \text{if } l \leq 0 \end{cases}$$

Their highest weights are of the form $m_1\epsilon_1 - m_2\epsilon_n - l\epsilon_{n+1}$ with respect to the fundamental weights described in section A.2. From now on, let

$$V_\tau \simeq \mathscr{H}^{M_1,M_2}(\mathbb{C}^n) \otimes \mathscr{H}^L(\mathbb{C}) \ . \tag{II.32}$$

so the highest weight of τ is

$$\mu_\tau = M_1\epsilon_1 - M_2\epsilon_n - L\epsilon_{n+1} \tag{II.33}$$

for fixed nonnegative integers M_1, M_2, a fixed integer L. From now on we call this representation $\tau_{M_1,M_2,L}$. As σ is trivial, the poles of the Plancherel density are the

$$\lambda_k = \pm i(\frac{n}{2} + k)$$

where k is a non-negative integer. Hence

$$\mathscr{H}_{\lambda_k\alpha} \simeq \sum_{\substack{l \in \mathbb{Z} \\ m_1, m_2 \geq 0 \\ m_1 - m_2 + l = 0}} P^*\left(\mathscr{H}^{m_1,m_2}(\mathbb{C}^n) \otimes \mathscr{H}^l(\mathbb{C})\right) \tag{II.34}$$

where $P := P_{m_1,m_2,l}$ is the projection of $\mathscr{H}_{\lambda_k\alpha}$ onto the K-isotypic component isomorphic to $\mathscr{H}^{m_1,m_2}(\mathbb{C}^n) \otimes \mathscr{H}^l(\mathbb{C})$. One can see that the conditions in the sum in (II.34) imply that the pair (m_1+m_2, l) determines the triple (m_1, m_2, l) and since m_1 and m_2 are nonnegative, $-l \le m_1+m_2 \le l$ for all $l \in \mathbb{Z}$.

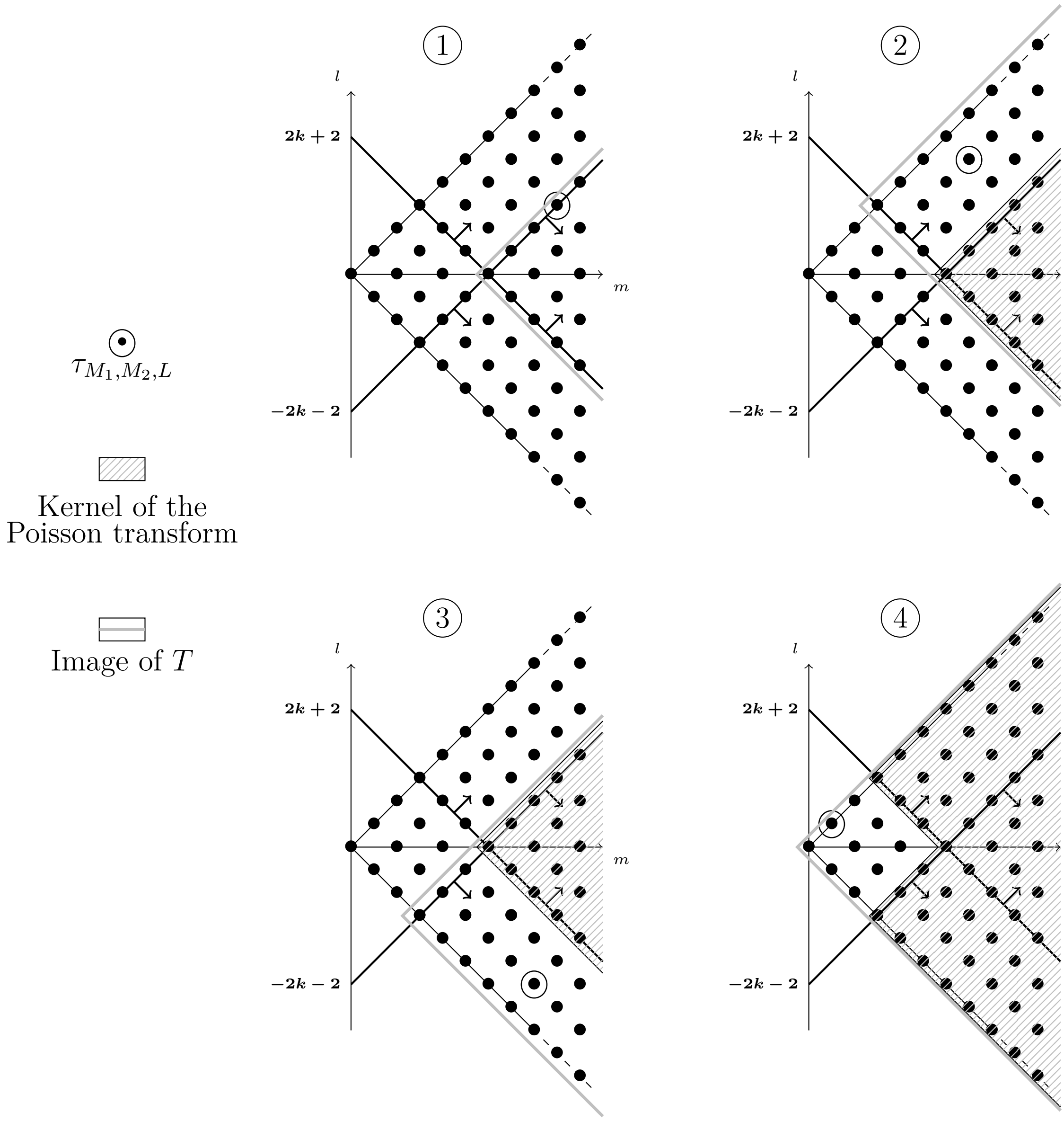


Figure II.4: K-types in $\mathscr{H}_{\lambda_k\alpha}$ and composition series for $k=1$

Langlands parameters of $\mathscr{E}_k$. In Figure II.4, each bullet corresponds to one K-type of $\mathscr{H}_{\lambda_k\alpha}$, the coordinates of the point being the pair (m_1+m_2, l) for the K-type $\mathscr{H}^{m_1,m_2}(\mathbb{C}^n) \otimes \mathscr{H}^l(\mathbb{C})$. [HT93, Lemma 4.4] ensures us that, the action of G cannot send a K-type to another constituent if it doesn't follow the sense of the arrows. Thus the space is separated in four constituents. We denote these constituents North - East - South - West depending on their position. The figure describe the four cases when the K-type $\tau_{M_1,M_2,L}$ is in each constituent and is represented by the bullet (M_1+M_2, L).

The reader should recall Lemma II.2.1 and (II.28).

Case 1 : $M_1 + M_2 \geq 2k+2$ and $|L| \leq -2k-2+M_1+M_2$
Since the action of G on the K-types cannot cross the barriers from the side out, the image of T is the entire East-component

$$\sum_{\substack{m_1,m_2\geq 0,\ m_1+m_2\geq 2k+2 \\ l\in\mathbb{Z},\ |l|\leq -2k-2+m_1+m_2 \\ m_1-m_2+l=0}} P^*\Big(\mathscr{H}^{m_1,m_2}(\mathbb{C}^n)\otimes\mathscr{H}^l(\mathbb{C})\Big)\ .$$

The Poisson transform is an intertwining operator. Hence its kernel, contained in an irreducible representation, is either 0 or the entire representation. Choosing a nonzero $h \in \mathscr{H}^N(\mathbb{R}^{2n})$, the function $P_{\tau_{M_1,M_2,L}}\ \pi_{\lambda_k\alpha}(\ \cdot\ ^{-1})P^*(h)$ has value h at e_G. Thus $\mathscr{E}_k$ is equivalent as a representation to the image T.

Case 2 : $L \geq |-2k-1+M_1+M_2|+1$
From the North-constituent, the action of G can cross the barrier $l=-2k-2+m$ but not the barrier $l=2k+2-m$. Thus the image of T is composed of two North and East-components:

$$\sum_{\substack{m_1,m_2\geq 0 \\ l\in\mathbb{Z},\ l>2k+2-m_1-m_2 \\ m_1-m_2+l=0}} P^*\Big(\mathscr{H}^{m_1,m_2}(\mathbb{C}^n)\otimes\mathscr{H}^l(\mathbb{C})\Big)\ .$$

Following the proof in the previous case, one can conclude that the image of the Poisson transform is nonzero and that the North-constituent, where $\tau_{M_1,M_2,L}$ is, is not in the kernel of this map. Because of the barrier $l=-2k-2+m$ the action of G cannot bring a K-type from the East-constituent into the North-constituent. Thus the East-constituent is the kernel of the Poisson transform and $\mathscr{E}_k$ is equivalent to the subquotient of the North-East constituents modulo the East-constituent.

Case 3 : $L \leq |-2k-1+M_1+M_2|-1$
This case is completely symmetric to the previous one. The figure explains the result.

Case 4 : $M_1+M_2 \in [0, 2k+2[$ and $L < |2k+2-M_1+M_2|$
$\tau_{M_1,M_2,L}$ is in the finite-dimensional West-constituent. From there, the action of G can send a K-type onto any other K-type in $\mathscr{H}_{\lambda_k\alpha}$. Thus the image of T is the entire spherical principal series representation. As in the previous cases, one can prove that the image of the Poisson transform is nonzero and that the West-constituent, where $\tau_{M_1,M_2,L}$ is, is not in the kernel of this map. Because of the two barriers, the action of G cannot bring a K-type from the other constituents into the West-constituent. This means that the kernel of the Poisson transform

is the quotient

$$\mathscr{H}_{\lambda_k\alpha} \Big/ \sum_{\substack{m_1,m_2\geq 0 \\ l\in\mathbb{Z},\ |l|\geq +2k+2-m_1-m_2 \\ m_1-m_2+l=0}} j_{-k-n}\Big(\mathscr{H}^{m_1,m_2}(\mathbb{C}^n)\otimes\mathscr{H}^l(\mathbb{C})\Big)$$

The four representations $\mathscr{E}_k$ we have found above are irreducible subquotients of $\mathscr{H}_{\lambda_k\alpha}$. Their unitarity is given in [HT93, Diagram 3.15]. We list their Langlands parameter in Table II.2.

Case	Minimal K-type	δ	values of ν
1	$(k+1,0,\dots,0,-(k+1),0)$	$\big((k+1),0,\dots,0,-(k+1),0\big)$ if $n>2$ $\emptyset$ if $n=2$	$\pm\left(\frac{n}{2}-1\right)\alpha$ if $n>2$
2	$(0,\dots,0,-(k+1),k+1)$ if $k+1\leq n-1$ $(\left\lfloor\frac{k+2-n}{2}\right\rfloor,0,\dots,0,-(k+1),\left\lceil\frac{k+n}{2}\right\rceil)$ if $k+1>n-1$	$\big(0,0,\dots,0,-(k+1),(k+1)/2\big)$	$\pm\left(\frac{k}{2}+\frac{n}{2}-\frac{1}{2}\right)\alpha$
3	$((k+1),0,\dots,0,k+1)$ if $k+1\leq n-1$ $((k+1),0,\dots,0,-\left\lfloor\frac{k+2-n}{2}\right\rfloor,\left\lceil\frac{k+n}{2}\right\rceil)$ if $k+1>n-1$	$\big((k+1),0,\dots,0,(k+1)/2\big)$	$\pm\left(\frac{k}{2}+\frac{n}{2}-\frac{1}{2}\right)\alpha$
4	triv_K	triv_M	$(\rho_\alpha+k)\alpha$

Table II.2: Langlands parameters of $\mathscr{E}_k$ when $G=\mathrm{SU}(n,1)$

To compute the entries of Table II.2, let $\mu_{m_1,m_2,l}$ be the highest weight of $\mathscr{H}^{m_1,m_2}(\mathbb{C}^n)\otimes\mathscr{H}^l(\mathbb{C})$. The minimal K-type is obtained after minimising the Vogan norm of their highest weight in $\mathscr{E}_k$:

$$\|\mu_{m_1,m_2,l}\|_V = \langle\mu_{m_1,m_2,l}+2\rho_K,\mu_{m_1,m_2,l}+2\rho_K\rangle$$

where $\rho_{\mathfrak{k}}$ is half sum of positive roots in $S^+_{\mathfrak{k}_\mathbb{C}}$ (see Appendix A.2 for the definition). Computations show that $\|\mu_{m_1,m_2,l}\|_V$ is minimal when $(m_1+n-1)^2+(m_2+n-1)^2+l^2$ is minimal.

To find δ, one can use [BS79, Theorem 4.4]. We show the reasoning in case 1. The minimal K-type $\tau_{\min}$ has highest weight

$$(k+1)\epsilon_1-(k+1)\epsilon_n$$

Suppose $n>2$: The branching rules imply that

$$\delta\in\hat{M}(\tau_{\min})\Leftrightarrow\mu_\delta=a\epsilon_2+b\epsilon_n-\frac{a+b}{2}(\epsilon_{n+1}+\epsilon_1)\ ,$$

where $0\leq a\leq k+1$ and $0\leq -b\leq k+1$. But the minimal K-type of $\mathrm{Ind}_M^K(\delta)$ is $\tau_{\min}$ only when $a=k+1$ and $b=-k-1$. So

$$\mu_\delta=(k+1)\epsilon_2+(k+1)\epsilon_n$$

Suppose $n=2$: The difference with the case $n>2$ is that there is no integer between 1 and n. Here

$$\delta\in\hat{M}(\tau_{\min})\Leftrightarrow\mu_\delta=a\epsilon_2+\frac{-a}{2}(\epsilon_1+\epsilon_3)\ ,$$

where $-(k+1) \leq a \leq k+1$. One can prove that $\mu' = a\epsilon_1 - a\epsilon_3$ is always the highest weight of a K-type in $\mathrm{Ind}_M^K(\delta)$ with a smaller Vogan norm than $\tau_{\min}$. So we get then a discrete series representation. The Blattner parameter of the discrete series is the highest weight $\mu_{\tau_{\min}}$ of its minimal K-type $\tau_{\min}$. Its Harish-Chandra parameter is

$$\Lambda_k = \mu_{\tau_{\min}} + 2\rho_{\mathfrak{k}} - \rho_{\mathfrak{g}} = (k+1)\epsilon_1 - k\epsilon_2 - \epsilon_3 \ .$$

To find ν, one has to compare the infinitesimal characters of

$$\mathrm{Ind}_{MAN}^G(\mathrm{triv} \otimes e^{(\rho_\alpha + k)\alpha} \otimes \mathrm{triv}) \quad \text{and} \quad \mathrm{Ind}_{MAN}^G(\delta \otimes e^{\nu} \otimes \mathrm{triv}) \ .$$

They have to coincide up to the action of the Weyl group of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$. For Case 1, one gets respectively the infinitesimal characters

$$\left(\frac{n}{2} + k\right)(\epsilon_1 - \epsilon_{n+1}) + \frac{1}{2}\sum_{j=2}^{n}(n - 2i + 2)\epsilon_i$$

and

$$\nu_\alpha(\epsilon_1 - \epsilon_{n+1}) + \left(\frac{n}{2} + k\right)\epsilon_2 + \frac{1}{2}\sum_{j=3}^{n-1}(n - 2i + 2)\epsilon_i + \left(-\frac{n}{2} - k\right)\epsilon_n$$

The (complex) Weyl group permutes the $\pm\epsilon_i$'s, so $\nu = \pm\left(\frac{n}{2} - 1\right)\alpha$. The theorem follows from similar computations, in other cases.

■

Theorem II.2.2. *We suppose that τ is a K-type containing the trivial representation of M. τ has a highest weight of the form $(M_1, 0, \dots, 0, -M_2, -L)$, where M_1 and M_2 are positive integers such that $M_1 \geq M_2 \geq 0$, $L \in \mathbb{Z}$ and $M_1 + M_2 + L$ is even. The residue representations $\mathscr{E}_k$, defined in* (II.25)*, are irreducible.*

1. *If $M_1 + M_2 \geq 2k + 2$ and $|L| \leq -2k - 2 + M_1 + M_2$, the residue representation is unitary.*
   - *if $n > 2$, this representation has Langlands parameter $\left(MA, \delta, \pm\left(\frac{n}{2} - 1\right)\alpha\right)$ where the highest weight of δ is $\big((k+1), 0, \dots, 0, -(k+1), 0\big)$.*
   - *if $n = 2$, this representation is the discrete series with Blattner parameter $((k+1), (k+1), 0)$.*
2. *If $L \geq |-2k - 1 + M_1 + M_2| + 1$, the residue R_k is the representation with Langlands parameter $\left(MA, \delta, \pm\left(\frac{k}{2} + \frac{n}{2} - \frac{1}{2}\right)\alpha\right)$, where the highest weight of δ is $\big(0, 0, \dots, 0, -(k+1), (k+1)/2\big)$. This representation is non-unitary.*
3. *If $L \leq |-2k - 1 + M_1 + M_2| - 1$, the residue R_k is the representation with Langlands parameter $\left(MA, \delta, \pm\left(\frac{k}{2} + \frac{n}{2} - \frac{1}{2}\right)\alpha\right)$, where the highest weight of δ is $\big((k+1), 0, \dots, 0, (k+1)/2\big)$. This representation is non-unitary.*
4. *If $M_1 + M_2 \in [0, 2k+2[$ and $L < |2k + 2 - M_1 + M_2|$, the residue R_k is the representation with Langlands parameter $\big(MA, \mathrm{triv}, (\rho_\alpha + k)\alpha\big)$. It is finite-dimensional and non unitary (if $k \neq 0$).*

Remark

Since the multiplicity of τ in $\mathscr{H}_\nu^\delta$ is 1 (see for instance [Koo82]) for these first two cases, the residue representation can be identified as the unique irreducible subquotient of $\mathscr{H}_\nu^\delta$ containing τ.

II.2.5 Case of $\mathrm{Sp}(n,1)$

Let $G = \mathrm{Sp}(n,1)$. We recall that $K = \mathrm{Sp}(1) \times \mathrm{Sp}(n)$. Up to equivalence there a unique representation of $\mathrm{Sp}(1) = \mathrm{SU}(2)$ of dimension $j+1$ for all nonnegative integers j. Denote this representation by θ_j, acting on the space V_1^j. Denote by $V_p^{m_1,m_2}$ the irreducible representation of $\mathrm{Sp}(p)$ with highest weight $(m_1, m_2, 0, \dots, 0)$ where $m_1 \geq m_2 \geq 0$ (see [Ž73, Theorem 6 page 327]).

As σ is trivial, the poles are the points

$$\lambda_k = \pm i(n + \frac{1}{2} + k)$$

with $k \in \mathbb{N}$. The spherical principal series representation decomposes over K as follows:

$$\mathscr{H} \simeq \sum P^*\left(V_p^{m_1,m_2} \otimes V_1^l\right) \tag{II.35}$$

where $P := P_{m_1,m_2,l}$ is the projector in $\mathscr{H}_{\lambda_k\alpha}$ on the K-isotopic component isomorphic to $V_p^{m_1,m_2} \otimes V_1^l$ and the sum is over the following set :

$$\begin{array}{cc} m_1 \geq m_2 \geq 0, & l \geq 0, \\ m_1 + m_2 \geq l, & m_1 + m_2 + l \in 2\mathbb{Z}\ . \end{array} \tag{II.36}$$

These conditions imply that a fixed pair $(m, l) = (m_1 + m_2, l)$ represents a fiber of K-types $V_p^{m_1,m_2} \otimes V_1^l$, one for each value of $m_1 - m_2$. The highest weight of $V_p^{m_1,m_2} \otimes V_1^l$ can be computed as $m_1\epsilon_1 + m_2\epsilon_2 + |l|\epsilon_{n+1}$ with respect to the sets of roots in (A.3). Let τ be $V_p^{a_1,a_2} \otimes V_1^{a_n+1}$. From now on we call this representation $\tau_{a_1,a_2,a_{n+1}}$.

Langlands parameters of $\mathscr{E}_k$. To understand the composition series of the representation $\mathscr{E}_k$, we have now to know how $\mathfrak{p}$ acts on its K-types. This action is given in [HT93, Lemma 5.3]. The diagram 5.18 in that paper gives us the different cases.
One can prove that the action of $\mathfrak{g}$ can bring one K-type in each fiber to every other in the same fiber. Then we can follow the method used for $G = \mathrm{SO}(2n, 1)$ or $\mathrm{SU}(n, 1)$ putting the K-types of $\mathscr{H}_{\lambda_k\alpha}$ in the same two-dimensional space corresponding to the points with coordinates $(m_1 + m_2, l)$. Notice that here each point in Figure II.5 corresponds then to a fiber of K-types. We refer to [HT93, Lemma 5.4] for more details.
Figure II.5 illustrates the computation of the residue representation $\mathscr{E}_k$. Two barriers cross the set of K-types. These separate the space $\mathscr{H}_{\lambda_k\alpha}$ in three constituents. The figure shows the three cases where the K-type $\tau_{a_1,a_2,a_{n+1}}$ is in each constituent. The arguments are the same as in the complex case. Naming the constituents North, West ant East according to their position, one gets the following equivalences:

$$\mathscr{E}_k \simeq \begin{cases} \text{East-constituent} & \text{if } a_{n+1} \leq -2k - 4 + a_1 + a_2 \\ \text{North-East-constituents} \Big/ \text{East-constituent} & \text{if } a_{n+1} \geq |-2k - 2 + a_1 + a_2| \\ \mathscr{H}_{\lambda_k\alpha} \Big/ \text{North-East-constituents} & \text{if } a_{n+1} \geq 2k + 2 - a_1 - a_2 \end{cases} \tag{II.37}$$

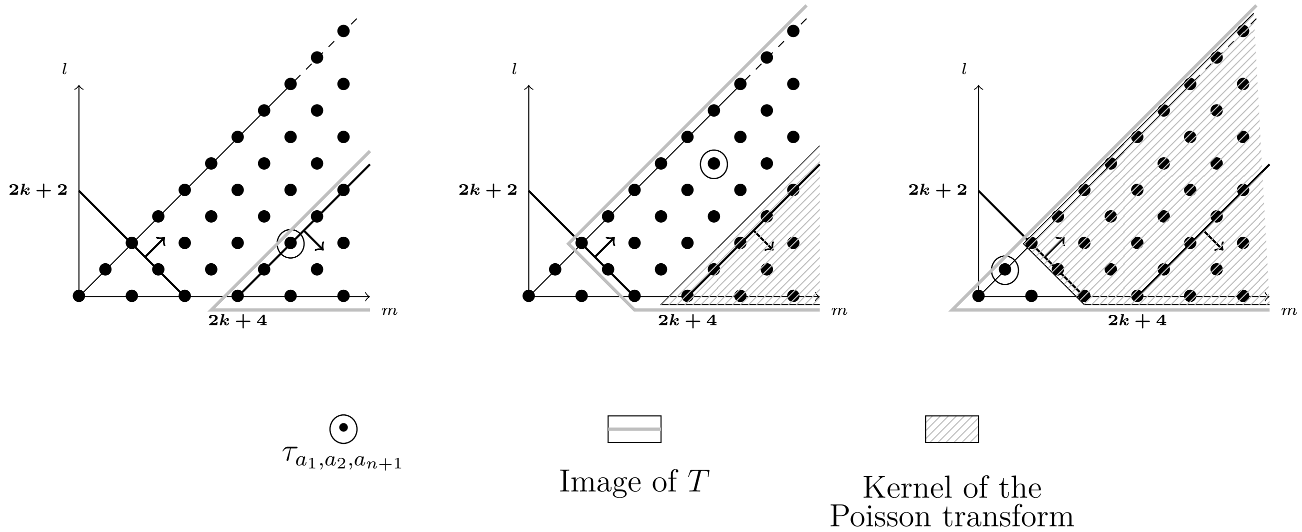


Figure II.5: K-types in $\mathscr{H}_{\lambda_k \alpha}$ and decomposition series for $k=1$

The three representations $\mathscr{E}_k$ we have found above are irreducible subquotients of $\mathscr{H}_{\lambda_k \alpha}$. Their unitarity is given in [HT93, Diagram 3.15]. We find the following results using Proposition 3.1 in [BS81]:

Case	Minimal K-type	δ	values of ν
1	$(k+2)\epsilon_1 + (k+2)\epsilon_2$	$(k+2)e_3 + (k+2)e_4$	$\pm\left(n - \frac{3}{2}\right)\alpha$
2	$(k+1)\epsilon_1 + (k+1)\epsilon_{n+1}$	$(k+1)e_3 + \frac{k+1}{2}(e_1 - e_2)$ if $k \leq 2n-4$ $\emptyset$ if $k > 2n-4$	$\pm\left(\frac{k}{2} + n\right)\alpha$ if $k \leq 2n-4$
3	triv_K	triv_M	$(\rho_\alpha + k)\alpha$

Table II.3: Langlands parameters of $\mathscr{E}_k$ when $G = \mathrm{Sp}(n,1)$

We indicate how to compute the entries of this table. In case 2, for $k > 2n-4$, one cannot find a representation δ following the conditions of the Langlands parameters. We conclude that $\mathscr{E}_k$ is in the discrete series in these cases. Their Blattner parameter is the highest weight of the minimal K-type, which is $(k+1)\epsilon_1 + (k+1)\epsilon_{n+1}$. The Harish-Chandra parameter of the discrete series is

$$(k+n)\epsilon_1 + (k+3-n)\epsilon_{n+1} + \sum_{j=2}^{n} 2(n-i+1)\epsilon_i \ .$$

■

Theorem II.2.3. *We suppose that τ is a K-type containing the trivial representation of M. τ has a highest weight of the form $(t_1, t_2, 0, \ldots, 0, t_{n+1})$, where t_1, t_2 and t_{n+1} are non-negative integers such that $t_1 \geq t_2$, $t_{n+1} = t_1 - t_2$ and $t_1 + t_2 + t_{n+1}$ is even. The residue representations $\mathscr{E}_k$, defined in* (II.25)*, are irreducible.*

1. *If $t_{n+1} \leq t_1 + t_2 - 2k - 4$, the residue $\mathscr{E}_k$ is the representation with Langlands parameter $\left(MA, \delta, \pm\left(n - \frac{3}{2}\right)\alpha\right)$ with lowest K-type's highest weight $(k+2, k+2, 0, \ldots, 0)$ and where the highest weight of δ is $(k+2, k+2, 0, \ldots, 0)$. This representation is non-unitary.*

2. *If $t_{n+1} \geq |t_1 + t_2 - 2k - 2|$ the residue representation is unitary.*
   - *If $k \leq 2n-4$, the residue $\mathscr{E}_k$ is the representation with Langlands parameter $\left(MA, \delta, \pm\left(n - \frac{k}{2}\right)\alpha\right)$ with lowest K-type $(k+1, 0, \ldots, 0, k+1)$, where the highest weight of δ is $(k+1, 0, \ldots, 0, \frac{k+1}{2})$.*
   - *If $k \geq 2n-3$, the residue $\mathscr{E}_k$ is the discrete series representation with Blattner parameter $\mu_k = (k+1, 0, \ldots, 0, k+1)$.*

3. *If $t_1 + t_2 < 2k + 2 - t_1 - t_2$, the residue $\mathscr{E}_k$ is the representation with Langlands parameter $\left(MA, \mathrm{triv}, (\rho_\alpha + k)\alpha\right)$. It is finite-dimensional and non unitary (if $k \neq 0$).*

Remark

One can verify that Theorem 1 in [Par15] gives the same conclusion as ours: there are discrete series representations in case 2 for $k > 2n - 4$. In fact, to verify the hypotheses of the theorem, we take the set of non-compact positive roots being

$$\Delta_n^+ = \{\epsilon_1 \pm \epsilon_{n+1},\ \epsilon_{n+1} \pm \epsilon_i \text{ for } i > 1\}\ .$$

We get $2\rho_n = 2\epsilon_1 + (2n-2)\epsilon_{n+1}$, such that,

$$\lambda = \mu_k - 2\rho_n = (k-1)\epsilon_1 + (k+3-2n)\epsilon_{n+1}\ ,$$

which is dominant if and only if $(k+3-2n) \geq 0$. This gives the same condition over k.

II.2.6 Case of F_4

The paper of Johnson [Joh76] gives the results we need in this case. Let G be the exceptional Lie group F_4. We recall that here $K = \mathrm{Spin}(9)$ and $M = \mathrm{Spin}(7)$. As σ is trivial, the poles of the meromorphically extended resolvent are the points

$$\lambda_k = \pm i\left(\frac{11}{2} + k\right)$$

with $k \in \mathbb{N}$.

The K-types of $\mathscr{H}_{\lambda_k\alpha}$ are the $V^{p,q}$ with $p \geq q \geq 0$ and $p+q \in 2\mathbb{Z}$ (see [Joh76, Theorem 3.1]), with highest weight

$$\mu_{p,q} = \frac{p}{2}\epsilon_1 + \frac{q}{2}\epsilon_2 + \frac{q}{2}\epsilon_3 + \frac{q}{2}\epsilon_4$$

with respect to the sets of roots in Appendix (A.3). Let τ be $V^{a,b}$, for $a \geq b \geq 0$ and $a+b \in 2\mathbb{Z}$. In the following, we call this representation $\tau_{a,b}$.
We can follow the method of the cases $G = \mathrm{SO}(2n,1)$ or $\mathrm{SU}(n,1)$, putting the K-types of $\mathscr{H}_{\lambda_k\alpha}$ in the same two-dimensional space corresponding to the points with coordinates (p,q). Figure II.6 illustrates the computations of $\mathscr{E}_k$. There are again two barriers. They separate the space $\mathscr{H}_{\lambda_k\alpha}$ in three constituents. The figure describes the three cases where the K-type $\tau_{p,q}$ is in different constituents. The arguments are the same as in the complex case.
Naming the constituent North, West ant East according to their position, one gets the following equivalences:

$$\mathscr{E}_k \simeq \begin{cases} \text{East-constituent} & \text{if } a_{n+1} \leq -2k-4+a_1+a_2 \\ \text{North-East-constituents} \Big/ \text{East-constituent} & \text{if } a_{n+1} \geq |-2k-2+a_1+a_2| \\ \mathscr{H}_{\lambda_k\alpha} \Big/ \text{North-East-constituents} & \text{if } a_{n+1} \geq 2k+2-a_1-a_2 \end{cases} \tag{II.38}$$

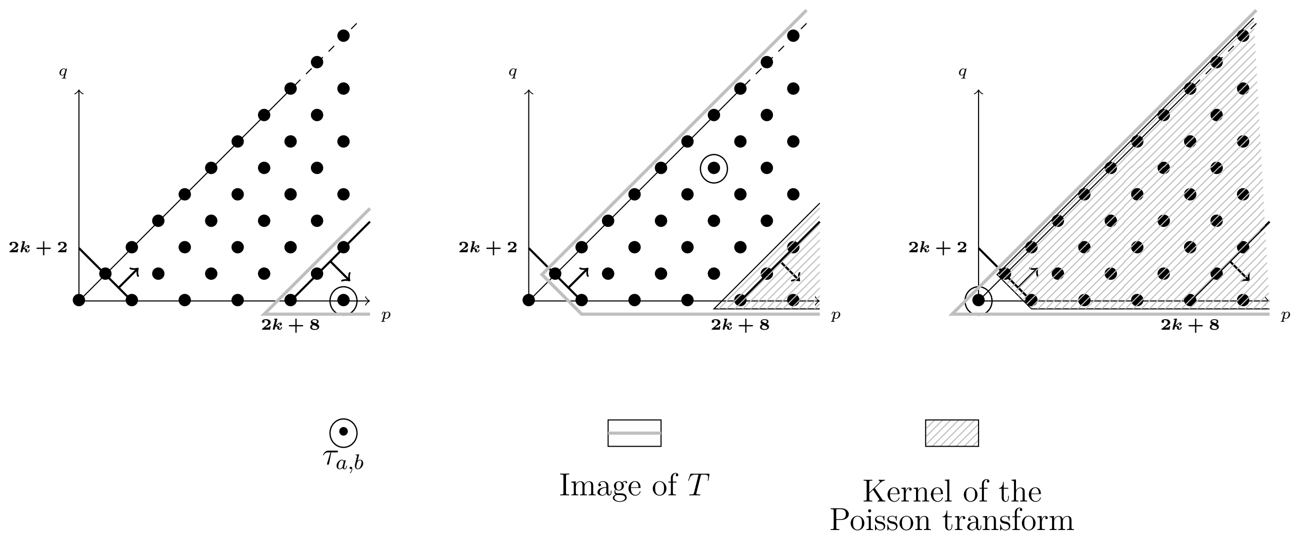


Figure II.6: K-types in $\mathscr{H}_{\lambda_k\alpha}$ and decomposition series for $k = 0$

The three representations $\mathscr{E}_k$ we have found above are irreducible subquotients of $\mathscr{H}_{\lambda_k\alpha}$. We find the following results using [BS79, Theorem 3.4]:

Case	Minimal K-type	δ	values of ν
1	$(k+4)\epsilon_1$	$\frac{(k+4)}{4}(3\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)$	$\pm\frac{1}{2}\left(k+1\right)\alpha$
2	$\frac{\lfloor\frac{3k+5}{2}\rfloor}{2}\epsilon_1+\frac{\lceil\frac{k-1}{2}\rceil}{2}(\epsilon_2+\epsilon_3+\epsilon_4)$	$\frac{(k+4)}{4}(3\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)$	$\frac{1}{2}\left(k+10\right)\alpha$
3	triv_K	triv_M	$(\rho_\alpha+k)\alpha$

Table II.4: Langlands parameters of $\mathscr{E}_k$ when $G=\mathscr{F}_4$

In the table $\lfloor\cdot\rfloor$ denotes the integer part and $\lceil\cdot\rceil$ is the upper integer part. Here the method is exactly the same as when $G=\mathrm{Spin}(2n,1)$. One has just to take care of the embedding of M in K which is not standard (see [BS79, section 6]). We used [CHH+12] to know how the (complex) Weyl group acts on the infinitesimal characters of the principal series.

Theorem II.2.4. *Let τ contain the trivial representation of M, it has a highest weight of the form $(a/2,b/2,b/2,b/2)$, where a and b are positive integers such that $a\geq b$ and $a-b$ is even.*

- *If $b\leq a-2k-8$, then $\mathscr{E}_k$ is the representation with Langlands parameter $\left(MA,\delta,\frac{1}{2}\left(k+1\right)\alpha\right)$, where the highest of δ is $\frac{k+4}{4}(3,1,1,1)$. This representation is non-unitary.*
- *If $b>a-2k-8$ and $b\geq 2k+2-a$, then $\mathscr{E}_k$ is the representation with Langlands parameter $\left(MA,\delta,\frac{1}{2}\left(k+10\right)\alpha\right)$, where the highest of δ is $\frac{k+1}{4}(3,1,1,1)$. This representation is unitary.*
- *If $b<2k+2$, then $\mathscr{E}_k$ is the representation with Langlands parameter $\left(MA,\mathrm{triv},(\rho_\alpha+k)\alpha\right)$. It is finite-dimensional and non unitary (if $k\neq 0$).*

III

Wave front set of the residue representations

In this section, we continue to assume that (τ, V_τ) contains the trivial representation of M. The residue representations $\mathscr{E}_k$ are those listed in Theorem 2 and defined in section II.2. The purpose of this part is to compute the wave front set $\mathrm{WF}(\mathscr{E}_k)$ of these representations. For this, we need some results on the Gelfand-Kirillov dimension [Vog78] and on the nilpotent orbits in semisimple Lie algebras [CM93].

III.1 The wave front set of a representation

First of all, wave front sets were introduced in P.D.E.'s. They allow to quantify the smoothness of a distribution. They were adapted to representation theory by Roger Howe in [How81]. Proposition 2.4 in [How81] tells us that the wave front set of $\mathscr{E}_k$ is equal to a closed union of nilpotent orbits of $\mathfrak{g}$. To know which occur, we use the Gelfand-Kirillov dimension of $\mathscr{E}_k$, computed by [Vog78, Theorem 1.2] in terms of K-types. This dimension is half of the dimension of the wave front set seen as a union of nilpotent orbits (see [Vog78, Ros95, BV80]). Combining this information with the list of nilpotent orbits in semisimple Lie algebras and their dimensions in [CM93], we identify the wave front set in Theorem 2.

In general, a compactly supported distribution f is smooth if and only if its Fourier transform $\hat{f}$ is rapidly decreasing and of exponential type. This is the *Paley-Wiener theorem*, well known for functions on open subsets of $\mathbb{R}^n$.

Lemma III.1.1 (Paley-Wiener Theorem for $\mathbb{R}^n$)
Let f be compactly supported distribution on $\mathbb{R}^n$. The following assertions are equivalent:

1. $f \in C_c^\infty(\mathbb{R}^n)$,
2. *for all $N \in \mathbb{N}^*$ and $\xi \in \mathbb{R}^n$, we have*

$$|\hat{f}(\xi)| \leq C_N(1+|\xi|)^{-N} \tag{III.1}$$

where C_N is a constant depending on N.

See [HÖ3][8.1 p.852].
This means that if f is not smooth, there exist some directions in $\mathbb{R}^n$, where $\hat{f}$ is not rapidly decreasing. Let us then introduce the set $\Sigma(f)$ of all $\eta \in \mathbb{R}^n \setminus \{0\}$ having no conic neighborhood V on which (III.1) is true. $\Sigma(f)$ is then a closed cone in $\mathbb{R}^n \setminus \{0\}$ and as soon as f is not smooth, this cone is not empty. It describes only the directions of high frequencies causing singularities of f. For each x in an open subset U of $\mathbb{R}^n$, we can define

$$\Sigma_x(f) = \bigcap_{\psi} \Sigma(\psi f) \ ,$$

where the intersection is over all $\psi \in C_c^\infty(U)$ such that $\psi(x) \neq 0$. The *wave front set* of a function f on an open subset U of $\mathbb{R}^n$ can then be defined as follows

$$\mathrm{WF}(f) := \{(x,\eta) \in U \times (\mathbb{R}^n \setminus \{0\}) \mid \eta \in \Sigma_x(f)\} \ .$$

This is a closed cone of $U \times (\mathbb{R}^n \setminus \{0\})$.

For functions on a Lie group, the wave front set is a closed cone of the cotangent bundle $T^*(G) = G \times \mathfrak{g}^*$. Howe introduced this notion to representation theory [How81, page 118], as follows.
Let (π, V_π) be a representation of G. Let $\mathcal{T}$ the space of all trace class operators on the Hilbert space V_π. For $T \in \mathcal{T}$, we define the continuous function

$$\mathrm{Tr}_\pi(T)(g) = \mathrm{Tr}\left(\pi(g)T\right) \qquad \text{for all } g \in G. \tag{III.2}$$

Then Tr_π is a bounded norm-decreasing function. Its image is usually called the *space of (continuous) matrix coefficients of* π. We also define the distribution

$$\mathrm{Tr}_\pi(T)(f) = \int_G f(g)\mathrm{Tr}_\pi(T)(g)\, dg \qquad \text{for all } f \in C_c^\infty(G) \tag{III.3}$$

Definition III.1.1 (Wave front set of a representation)
We define the wave front set $\mathrm{WF}(\pi)$ of a representation (π, V_π) of G as the closure in $T^*(G)$ of the union of $\mathrm{WF}\left(\mathrm{Tr}_\pi(T)\right)$ as T varies over $\mathcal{T}$.

$\mathrm{WF}(\pi)$ is a closed conical set of $T^*(G) = G \times \mathfrak{g}^*$. As it is invariant under left and right translation of G, we identify $\mathrm{WF}(\pi)$ with its projection on $\mathfrak{g}^*$. It turns out to be a closed conical Ad^*G-invariant subset.
The wave front set of a representation is an interesting object, in the sense that this is not always $\mathfrak{g}^*$ (it could be). Among others, it gives information about the differential and the "size" of the representation.
We recall some important results of [How81], we are using in this thesis:

Proposition III.1.1 (1.5 in [How81])
Let H be a closed subgroup of G with Lie algebra $\mathfrak{h}$. Let us denote by q the restriction map from $\mathfrak{g}^*$ to $\mathfrak{h}^*$. We have then the inclusion:

$$q(\mathrm{WF}(\pi)) \subset \mathrm{WF}(\pi|_H)\ . \tag{III.4}$$

Using this proposition for $H = K$, the maximal compact of G, and for an irreducible representation π of G, this proposition becomes more precise:

Proposition III.1.2 (2.5(a) in [How81])
Let us denote by q the restriction map from $\mathfrak{g}^*$ to $\mathfrak{k}^*$. Then

$$q(\mathrm{WF}(\pi)) = \mathrm{WF}(\pi|_K)\ . \tag{III.5}$$

Let V be a vector space. We recall that the *asymptotic cone* **AC** of a nonempty set $A \subset V$ is defined by the two equivalent conditions:

1. $\mathbf{AC}(A) = \{y \in X : \forall x \in A, \forall \lambda \geq 0 : x + \lambda y \in A\}$,
2. For all $u \in U$, if any cone containing a neighbourhood of u intersects A in an unbounded set, then $u \in \mathbf{AC}(A)$.

If $\sigma \in \hat{K}$, we denote the set of highest weights of σ by $\operatorname{supp}\sigma$.

Proposition III.1.3 (2.3 in [How81])
For a unitary representation σ of K, we have

$$\mathrm{WF}(\sigma) = \mathsf{Ad}^*(K)\Big(- \mathbf{AC}(\operatorname{supp}\sigma)\Big) \ . \tag{III.6}$$

Using these propositions, we are going to decide what is the wave front set between two candidates with the same dimension (knowing what is the dimension of the wave front set: see section III.4).
Let N be the set of all nilpotent elements of $\mathfrak{g}$. The following result gives us candidates for the wave front set.

Proposition III.1.4 (2.4 in [How81])
If G is a semisimple group and π an irreducible representation of G, then

$$\mathrm{WF}(\pi) \subset N \tag{III.7}$$

It is known ([CM93] for example) that there are only finitely many conjugacy classes of nilpotent elements in $\mathfrak{g}$. So the same is true for the wave front sets.

One step is missing: how to compute the dimension of the wave front set ?

Theorem III.1.1 ([Vog78, Ros95, BV80])
The dimension of the wave front set of a representation is twice its Gelfand-Kirillov dimension.

We will compute the Gelfand-Kirillov dimension using the following theorem.

Theorem III.1.2 (Theorem 1.2 [Vog78])
Suppose $\pi \in \hat{G}$. Let X be the space of K-finite vectors in π, and let d be the Gelfand-Kirillov dimension of X. Let Ω be the Casimir operator of K. We write $N_X(t)$ for the sum of the dimensions of the eigenspaces of Ω with eigenvalue less than or equal to t^2. Then there is a constant $A > 0$ depending on X, such that for $t > 1$,

$$A^{-1}t^d \leq N_X(t) \leq At^d$$

III.2 Nilpotent elements of $\mathfrak{g}$

The nilpotent set $\mathcal{N}$ of $\mathfrak{g}$ is the set of all nilpotent elements.
Here $\mathfrak{g}$ is of real rank one. So

$$\mathcal{N} = \mathsf{Ad}(G)\Big(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}\Big)$$

We know where we can find the generators of the nilpotent orbits. But how many orbit are there and what are their dimension?

(1) The question of the dimension is answered by [CM93, Corollary 6.1.4.]. It gives the dimensions for the complex nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$. The dimensions of the real nilpotent orbits are the same (but there is not the same number of orbits).

(2) Theorems 9.3.3, 9.3.4 and 9.3.5 in [CM93] allow us to decide how many orbits there are in each case. The exceptional case is treated in section 9.6.

III.3 Case of SO$(2n, 1)$

In this section, we prove the results about wave front sets stated in Theorem 2 for $G = \mathrm{SO}(2n, 1)$. First we compute the Gelfand-Kirillov dimension of the Harish-Chandra module of $\mathscr{E}_k$ defined in section II.2. This case contains the case of $\mathrm{SL}(2, \mathbb{R})$.

Lemma III.3.1

The Gelfand-Kirillov dimension of the residue representation $\mathscr{E}_k$ *is* $\begin{cases} 0 & \text{if } N \leq k \\ 2n-1 & \text{if } N > k \end{cases}$

Proof. First of all, if $N \leq k$, the residue representation is finite-dimensional, so the Gelfand-Kirillov dimension is 0. It follows from definition of the Gelfand-Kirillov dimension. For the case $N > k$, the residue representation is infinite-dimensional. One can compute the eigenvalue of each K-type τ_m in $\mathscr{H}_{\lambda_k \alpha}$ for the Casimir operator $\Omega_{\mathfrak{k}}$ of $\mathfrak{k}$ (using for example [Hal15, Proposition 10.6]):

$$\begin{aligned}\tau_m(\Omega_{\mathfrak{k}}) &= \Big(\langle \mu_{\tau_m} + \rho_{\mathfrak{k}}, \mu_{\tau_m} + \rho_{\mathfrak{k}} \rangle - \langle \rho_{\mathfrak{k}}, \rho_{\mathfrak{k}} \rangle\Big)\,\mathrm{Id} \\ &= \Big((m+n-1)^2 - (n-1)^2\Big)\,\mathrm{Id}\ ,\end{aligned}$$

where $\rho_{\mathfrak{k}} = \sum_{j=1}^{n} (n-j)\epsilon_j$ is half sum of positive roots in $(\mathfrak{k}, \mathfrak{h}|_{\mathfrak{k}})$. We recall that here τ_m is the space of harmonic polynomial of homogeneous degree m. One can compute its dimension (see [Ž73] for example)

$$d_m = \binom{m+2n-1}{2n-1} - \binom{m+2n-3}{2n-1}\ .$$

According to Vogan's Theorem (Proposition III.1.2), we compute the sum $N_{\mathscr{E}_k}(t)$ of the dimensions d_m of τ_m until its eigenvalue exceeds a fixed real number t^2. We have

$$(m+n-1)^2 - (n-1)^2 \leq t^2 \quad \underset{t\to\infty}{\Longleftrightarrow} \quad m+n-1 \underset{\infty}{\leq} t$$

So we will sum the dimensions up to $N_t = \lfloor t \rfloor - n + 1$. This sum of dimensions (depending on t) is then written as follows:

$$N_{\mathscr{E}_k}(t) = \sum_{m=k+1}^{N_t} d_m = \binom{N_t+2n-1}{2n-1} + \binom{N_t+2n-2}{2n-1} - \binom{k+2n-2}{2n-1} - \binom{k+2n-1}{2n-1}\ , \tag{III.8}$$

because the sum is telescopic. If t goes to infinity, we have :

$$N_{\mathscr{E}_k}(t) \overset{\infty}{\approx} \binom{N_t + 2n - 1}{2n - 1} \overset{\infty}{\approx} \frac{(N_t + 2n - 1)!}{N_t!} \overset{\infty}{\approx} t^{2n-1} \tag{III.9}$$

We denote above by $\overset{\infty}{\approx}$ the fact that the higher degree in t is the same on the right and the left-hand side.
Hence the Gelfand-Kirillov dimension is equal to $2n-1$.

■

Lemma III.3.2
Let

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \tag{III.10}$$

be the restricted root space decomposition of $\mathfrak{g}$. Then there are 2 nilpotent orbits in $\mathfrak{g}$, namely the zero orbit and the orbit generated by any non-zero element of $\mathfrak{g}_\alpha$.

Proof. The non-zero elements of $\mathfrak{g}_\alpha$ are conjugate by the elements of MA. Moreover, $\mathfrak{g}_{-\alpha} = \theta(\mathfrak{g}_\alpha) = \mathsf{Ad}(k_\theta)(\mathfrak{g}_\alpha)$ for a suitable element $k_\theta \in K$. Then all non zero nilpotent elements in the restricted root spaces above are conjugate.
This can also be found using Theorem 9.3.4 in [CM93]. For each real nilpotent orbit in $\mathfrak{so}(2n,1)$, it is written that there is one signed (alternating '+' and '-' on each row) Young diagram with $2n$ '+' and one '-' such that the rows of even length occurs with even multiplicity. Here even length is not possible because there is just one '-'. There are only two possible Young diagrams for $\mathfrak{so}(2n,1)$:

$$\begin{array}{|c|}\hline + \\ \hline + \\ \hline + \\ \hline \end{array} \qquad \qquad \begin{array}{|c|c|c|}\hline + & - & + \\ \hline + \\ \cline{1-1} + \\ \cline{1-1} + \\ \cline{1-1} \end{array}$$

$$\begin{array}{c} \vdots \end{array} \qquad \qquad \qquad \begin{array}{c} \vdots \end{array}$$

$$\begin{array}{|c|}\hline + \\ \hline - \\ \hline \end{array} \qquad \qquad \qquad \begin{array}{|c|}\hline + \\ \hline \end{array}$$

with $2n$ '+', which respectively correspond respectively to the 0 orbit and the orbit generated by $\mathfrak{g}_\alpha$.

■

We are now able to prove the results about the wave front set of $\mathscr{E}_k$ in Theorem 2 for $G = \mathrm{SO}(2n,1)$.

Theorem III.3.1. *The wave front set of the residue representation $\mathscr{E}_k$ is either the zero orbit or the nilpotent orbit generated by $\mathfrak{g}_\alpha$, respectively when $k \geq N$ or $k < N$. We recall that N is the nonnegative integer which defines the highest weight $(N, 0, \ldots, 0)$ of τ.*

Wave front set of $\mathscr{E}_k$. We have just two cases. When $N \leq k$, $\mathscr{E}_k$ is finite-dimensional and the Gelfand-Kirillov dimension is 0. The wave front set is then the zero orbit. If $N > k$, as $\mathscr{E}_k$ is infinite-dimensional, only the nilpotent orbit generated by $\mathfrak{g}_\alpha$ can correspond.
This can be checked using the formula for the dimension of nilpotent orbits in $\mathfrak{g}_\mathbb{C}$, the complexification of $\mathfrak{g}$ ([CM93, Corollary 6.1.4]). In fact if $N > k$, the Gelfand-Kirillov dimension is $2n-1$ because of Lemma III.3.1. The dimension of the wave front set is then $4n-2$. Because of the corollary cited above, we have:

$$4n-2 = \dim \mathrm{WF}(\mathscr{E}_k) = (2n+1)^2 - \frac{1}{2}\sum s_i^2 - \frac{1}{2}\sum_{\text{odd}} r_i \tag{III.11}$$

where $s_i = |\{j \mid d_j \geq i\}|$ and $r_i = |\{j \mid d_j = i\}|$ in a partition $[d_1, \ldots, d_k]$ of $2n+1$.
Thus, only one complex nilpotent orbit has this dimension: the one matching to the partition $[3, 1, 1, \ldots, 1]$ of $2n+1$. This corresponds indeed to the Young diagram of the (real) nilpotent orbit generated by $\mathfrak{g}_\alpha$.

■

III.4 Case of SU$(n, 1)$

In this section, we prove the results about wave front sets in Theorem 2 for $G = \mathrm{SU}(n, 1)$. First of all, we compute the Gelfand-Kirillov dimension of the Harish-Chandra module of $\mathscr{E}_k$ defined in Section II.2. Recall that there are 4 possibilities for the residue representation (see II.2.4). We label each case as in Figure III.1.

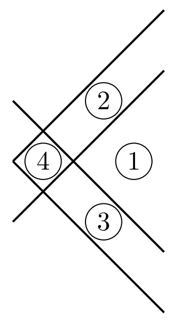


Figure III.1: Cases for $\mathscr{E}_k$ - SU$(n, 1)$

Lemma III.4.1
The Gelfand-Kirillov dimension of the residue representation $\mathscr{E}_k$ is $\begin{cases} 2n-1 & \textit{in case } 1 \\ n & \textit{in case } 2 \\ n & \textit{in case } 3 \\ 0 & \textit{in case } 4 \end{cases}$.

Proof. In case 4, the representation $\mathscr{E}_k$ is finite-dimensional. The Gelfand-Kirillov dimension is then 0. For the three other cases, we need some computations.
First of all, one can compute the eigenvalue of each K-type $\tau_{m,l}$ in $\mathscr{H}_{\lambda_k \alpha}$ for the Casimir operator $\Omega_\mathfrak{k}$ of $\mathfrak{k}$ (using for example [Hal15, Proposition 10.6]). Recall from II.2.4 that

$$\tau_{m,l} \simeq \mathscr{H}^{\frac{m-l}{2}, \frac{m+l}{2}}(\mathbb{C}^n) \times \mathscr{H}^l(\mathbb{C}) \tag{III.12}$$

with highest weight $\mu_{m,l} = \frac{m-l}{2}\epsilon_1 - \frac{m+l}{2}\epsilon_n + l\epsilon_{n+1}$. Then one finds:

$$\begin{aligned} \tau_{m,l}(\Omega_{\mathfrak{k}}) &= \Big(\langle \mu_{\tau_{m,l}} + \rho_{\mathfrak{k}}, \mu_{\tau_{m,l}} + \rho_{\mathfrak{k}} \rangle - \langle \rho_{\mathfrak{k}}, \rho_{\mathfrak{k}} \rangle \Big) \,\mathrm{Id} \\ &= \tfrac{1}{2}\Big((m+n-1)^2 - (n-1)^2 + 3l^2 \Big) \,\mathrm{Id} \ , \end{aligned} \tag{III.13}$$

where $\rho_{\mathfrak{k}} = \sum_{j=1}^{n} \frac{(n-2i+1}{2}\epsilon_i$ is half sum of positive roots in $(\mathfrak{k}, \mathfrak{h}|_{\mathfrak{k}})$. The dimension of the space of $\tau_{m,l}$ follows for example from the Weyl dimension formula [Kna02, Theorem 5.84]:

$$\begin{aligned} d_{m,l} &= \frac{\prod\limits_{\alpha\in\Sigma^+} \langle \mu_{m,l} + \rho_{\mathfrak{k}}, \alpha \rangle}{\prod\limits_{\alpha\in\Sigma^+} \langle \rho_{\mathfrak{k}}, \alpha \rangle} \\ &= \binom{\frac{m-l}{2} + n - 2}{n-1} \binom{\frac{m+l}{2} + n - 2}{n-1} \frac{(n-1)(m+n-1)}{\frac{m-l}{2}\frac{m+l}{2}} \end{aligned}$$

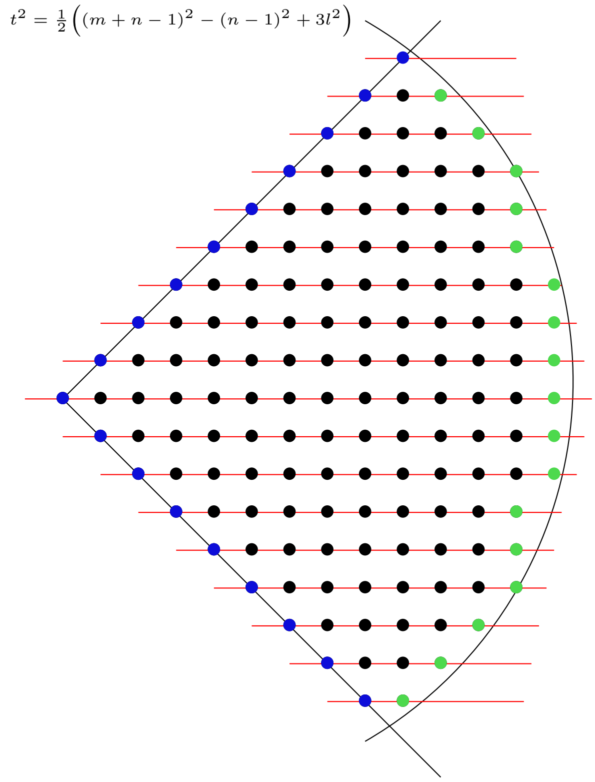
As we want to sum the dimensions, we will make appear a telescopic sum over m, as in the previous case. One can verify that the dimension can be rewrite as

$$d_{m,l} = \binom{\frac{m-l}{2} + n - 1}{n-1} \binom{\frac{m+l}{2} + n - 1}{n-1} - \binom{\frac{m-l}{2} + n - 2}{n-1} \binom{\frac{m+l}{2} + n - 2}{n-1}$$

The sum over m is now telescopic each 2 integers. Let $\delta(m,l,N) := \binom{\frac{m-l}{2}+N-1}{n-1}\binom{\frac{m+l}{2}+N-1}{n-1}$. So

$$d_{m,l} = \delta(m,l,n) - \delta(m,l,n-1) = \delta(m,l,n) - \delta(m-2,l,n) \ . \tag{III.14}$$

We recall that for a fixed l the values of m change by 2 integers because $l+m$ has to be even. The inequality given by the eigenvalue of the Casimir operator (III.13) limits the area of points of K-types by an ellipsis. The goal is to know the behaviour in t of the sum over the K-types inside the ellipsis when this t-ellipsis is going to infinity.



First one can see, that for a fixed l, the sum between two values $m_{\min}$ and $m_{\max}$ of m is telescopic:

$$\sum_{m=m_{\min}}^{m_{\max}} d_{m,l} = \sum_{m=m_{\min}}^{m_{\max}} \delta(m,l,n) - \delta(m-2,l,n) = \delta(m_{\max},l,n) - \delta(m_{\min},l,n-1) \ . \tag{III.15}$$

So on each (red) l-line, the sum of the dimensions of each representation (point) is just the difference between values depending on the first (blue) and the last (green) representation. these colored points represent respectively the representations $\tau_{m_{\min},l}$ and $\tau_{m_{\max},l}$. We compute the sum $N_{\mathscr{E}_k}(t)$ of the dimensions d_m of τ_m until the eigenvalue (III.13) exceeds a fixed real number t^2. We have for the eigenvalues

$$(m+n-1)^2 - (n-1)^2 + 3l^2 \leq t^2$$

So the values of l and m are bounded by a certain $l_{\max}$ and $m_{\max}$ respectively, depending on t. In term of power of t, the sum can be then written :

$$N_{\mathscr{E}_k}(t) \overset{\infty}{\approx} \sum_{l=-l_{\max}(t)}^{l_{\max}(t)} \delta(m_{\max}(l,t),l,n)$$

In fact, we can forget the term $\delta(m_{\min},l,n-1)$ because it is a constant in t so it will bring strictly lower powers of t than $\delta(m_{\max},l,n)$ in the sum. We recall that we denote above by $\overset{\infty}{\approx}$ the fact that the higher degree in t is the same on the right and the left-hand side. We will now compute case by case.

Case 1: The sum $N_{\mathscr{E}_k}(t)$ is then over the ellipsis (figure above), whose 'radius' is going to infinity in t^2. The number $l_{\max}(t)$ is roughly equal to the ordinate of the intersection between the ellipsis and the line "limiting" the representation (the 45°-line). Computing the intersection, we have that $l_{\max}(t)$ is asymptotic at infinity to $\frac{t}{2}$. As $m_{\max}(l,t)^2$ is roughly on the ellipsis, we also have that $m_{\max}(l,t)$ is asymptotic at infinity to $t^2 - 3l^2$ on the line with ordinate l. We get then

$$N_{\mathscr{E}_k}(t) \approx \sum_{l=-l_{\max}(t)}^{l_{\max}(t)} \delta(m_{\max}(l,t), l, n) \approx \sum_{|l|\leq t/2} \frac{\prod_{j=1}^{n-1}\left(\frac{(m_{\max}(l,t)+n-1-j)^2}{4} - \frac{l^2}{4}\right)}{(n-1)!}$$

And replacing $m_{\max}$:

$$N_{\mathscr{E}_k}(t) \approx \sum_{0\leq l\leq t/2} (t^2 - 4l^2)^{n-1} \approx t^{2n-1}$$

The Gelfand-Kirillov dimension follows from Proposition III.1.2.
The cases 2 and 3 are symmetric.
Case 2: The sum $N_{\mathscr{E}_k}(t)$ is then over the ellipsis , with 'radius' going to infinity in t^2, intersected with the area between 2 parallel lines "limiting" the representation (the 45°-lines). The number $l_{\max}(t)$ is roughly equal to the ordinate of the intersection between the ellipsis and the line "limiting" the representation (the 45°-line). Computing the intersection, we have that $l_{\max}(t)$ is asymptotic at infinity to $\frac{t}{2}$. As $m_{\max}(l,t)$ is roughly on the ellipsis or on the second "limiting" line, we also have that $m_{\max}(l,t)^2$ is asymptotic at infinity to $t^2 - 3l^2$ or to l on the line with ordinate l. So $m_{\max}(l,t)$ is of the same power of t at infinity. The difference with the preceding case is that the binomial coefficient in $m-l$ in $\delta(m_{\max}(l,t), l, n)$ is always a constant on the second "limiting" line. We get then

$$N_{\mathscr{E}_k}(t) \approx \sum_{l=-l_{\max}(t)}^{l_{\max}(t)} \delta(m_{\max}(l,t), l, n) \approx \sum_{|l|\leq t/2} t^{n-1} \approx t^n \ .$$

The Gelfand-Kirillov dimension follows from Proposition III.1.2.

■

The following lemma gives us the different nilpotent orbits in the nilradical of $\mathfrak{g}$.

Lemma III.4.2
There are four nilpotent orbits in SU$(n,1)$*:*

1. *the trivial orbit,*
2. *the one generated by $\mathfrak{g}_{\alpha/2}$, of dimension $4n-2$,*
3. *the one generated by $n_1 = i\begin{pmatrix}1 & 0 & -1\\ 0 & \mathbf{0} & 0\\ 1 & 0 & -1\end{pmatrix}$, of dimension $2n$,*

④ *the one generated by* $n_2 = i \begin{pmatrix} 1 & 0 & -1 \\ 0 & \mathbf{0} & 0 \\ 1 & 0 & -1 \end{pmatrix}$*, of dimension* $2n$.

Here the $\mathbf{0}$ *in the center of the matrix is the* $(n-1) \times (n-1)$ *zero matrix.*

Proof. Theorem 9.3.3 in [CM93] ensures us that (real) nilpotent orbits in $\mathfrak{su}(n, 1)$ are parametrized by signed (alternating '+' and '-' on each row) Young diagram with $2n$ '+' and one '-'. One finds that there are four possible Young diagrams corresponding to the nilpotent orbits:

$$
\begin{array}{|c|}\hline + \\ \hline + \\ \hline + \\ \hline \multicolumn{1}{c}{\vdots} \\ \hline + \\ \hline - \\ \hline \end{array}
\qquad\qquad
\begin{array}{|c|c|c|}\hline + & - & + \\ \hline + \\ \cline{1-1} + \\ \cline{1-1} + \\ \cline{1-1} \multicolumn{1}{c}{\vdots} \\ \cline{1-1} + \\ \cline{1-1} \end{array}
\qquad\qquad
\begin{array}{|c|c|}\hline + & - \\ \hline + \\ \cline{1-1} + \\ \cline{1-1} + \\ \cline{1-1} \multicolumn{1}{c}{\vdots} \\ \cline{1-1} + \\ \cline{1-1} \end{array}
\qquad\qquad
\begin{array}{|c|c|}\hline - & + \\ \hline + \\ \cline{1-1} + \\ \cline{1-1} + \\ \cline{1-1} \multicolumn{1}{c}{\vdots} \\ \cline{1-1} + \\ \cline{1-1} \end{array}
$$

The first one corresponds to the zero orbit. The second one is the only one which is nilpotent of degree 2. It corresponds to the orbit generated by any element of $\mathfrak{g}_{\alpha/2}$. In fact, the proof of the fact that $\mathfrak{g}_{\alpha/2}$ and $\mathfrak{g}_{-\alpha/2}$ are in the same orbit is exactly the same as the one in Lemma III.3.2.
The two last ones are nilpotent of degree 1. Computations show that n_1 and n_2 (defined in Lemma III.3.2) cannot be conjugated by an element of K. In particular, K cannot change the last number on the last line and the last column. In fact, let $k = \begin{pmatrix} B & 0 \\ 0 & e^{i\theta} \end{pmatrix}$, where $B \in SU(2n)$, $\theta \in \mathbb{R}$ and $\det(B)e^{i\theta} = 1$. We have:

$$Ad(k)n_1 = \begin{pmatrix} i|b_{11}|^2 & * & -ie^{-i\theta} \\ * & * & * \\ ie^{i\theta}\overline{b_{11}} & * & -i \end{pmatrix} ,$$

where b_{11} is the first coefficient in B. The conjugation by A is a multiplication by a positive factor and the conjugation by N doesn't affect n_1 or n_2. So n_1 and n_2 are each representative of one nilpotent orbit of degree 1.
The dimensions are given by Corollary 6.1.4 in [CM93]. Recall that the partition of the complex nilpotent orbit coming from a real nilpotent orbit is given by the boxes of the corresponding Young diagram. One can compute, for the first one $[3, 1, 1, \dots, 1]$. The numbers $s_i = |\{j \mid d_j \geq i\}|$ are $(n-1)$, 1, 1, 0, …, 0. So the dimension of the orbit is

$$(n+1)^2 - \sum_{i=1}^{n} s_i^2 = 4n - 2 .$$

For the partition $[2, 1, 1, \dots, 1]$, the dimension is

$$(n+1)^2 - \sum_{i=1}^{n} s_i^2 = (n+1)^2 - n^2 - 1 = 2n .$$

This completes the dimensions of the nilpotent orbits.

■

We are now able to prove the results about the wave front set of $\mathscr{E}_k$ in Theorem 2 for $\mathrm{SU}(n,1)$.

Theorem III.4.1. *The representation τ has highest weight of the form $(M_1, 0, \ldots, 0, -M_2, -L)$, where M_1 and M_2 are positive integers such that $M_1 \geq M_2 \geq 0$, $L \in \mathbb{Z}$ and $M_1 + M_2 + L$ is even.*

1. *If $M_1 + M_2 \geq 2k+2$ and $|L| \leq -2k-2+M_1+M_2$, then the wave front set of $\mathscr{E}_k$ is the nilpotent orbit generated by $\mathfrak{g}_{\alpha/2}$.*
2. *If $L \geq |-2k-1+M_1+M_2|+1$, then the wave front set of $\mathscr{E}_k$ is the nilpotent orbit generated by the element n_2 of $\mathfrak{g}_\alpha$ (see Lemma III.4.2 for the definition).*
3. *If $L \leq |-2k-1+M_1+M_2|-1$, then the wave front set of $\mathscr{E}_k$ is the nilpotent orbit generated n_1 of $\mathfrak{g}_\alpha$ (see Lemma III.4.2 for the definition).*
4. *If $M_1 + M_2 \in [0, 2k+2[$ and $L < |2k+2-M_1+M_2|$, then the wave front set of $\mathscr{E}_k$ is the zero orbit.*

Wave front set of $\mathscr{E}_k$. The two lemmas above conclude the cases 1 and 4. In fact, there is one possibility of dimension of nilpotent orbit corresponding to each Gelfand-Kirillov dimension.
Now we have to know which of the cases 2 and 3 correspond to the nilpotent orbit generated by n_1 or to the one generated by n_2. They have the same dimension, so we need to have more information on wave front sets. We will use the projection over K to figure it out. In fact, Proposition III.1.3 ensures us that

$$\mathrm{WF}(\mathscr{E}_k|_K) = \mathsf{Ad}^*(K)\Big(-AC(\operatorname{supp}\mathscr{E}_k)\Big) \tag{III.16}$$

where we recall that AC the asymptotic cone and $\operatorname{supp}\rho$ is the set of highest weights of $\mathscr{E}_k$. Proposition 2.5 gives us:

$$\mathrm{WF}(\mathscr{E}_k|_K) = q(\mathrm{WF}\,\mathscr{E}_k) \tag{III.17}$$

where q is the projection from $\mathfrak{g}^*$ onto $\mathfrak{k}^*$. We write $\mathscr{E}_k^2$ or $\mathscr{E}_k^3$ if $\mathscr{E}_k$ is respectively in case 2 or in case 3 (see figure III.1). We have:

$$-AC(\operatorname{supp}\mathscr{E}_k^2) \ni -\mu_{1,1} = \epsilon_n - \epsilon_{n+1} \text{ and } -AC(\operatorname{supp}\mathscr{E}_k^3) \ni \mu_{1,-1} = -\epsilon_1 + \epsilon_{n+1}$$

Moreover,

$$\begin{aligned} q\circ B:\ & n_1 \mapsto (2n+2)(-\epsilon_1+\epsilon_{n+1}) \\ & n_2 \mapsto (2n+2)(\epsilon_1-\epsilon_{n+1}) \end{aligned}$$

This gives the result, as K sends by the coadjoint action $\epsilon_n - \epsilon_{n+1}$ to $\epsilon_1 - \epsilon_{n+1}$.

■

III.5 Case of $\mathrm{Sp}(n,1)$

In this section, we prove the results on wave front sets, in Theorem 2 for $G = \mathrm{Sp}(n,1)$. First of all, we compute the Gelfand-Kirillov dimension of the Harish-Chandra module of $\mathscr{E}_k$ defined in section II.2. Recall that there are 3 possibilities of residue representation (see II.2.5). We label each case as in Figure III.2. The proofs are very similar to the complex case. So we will be less specific.

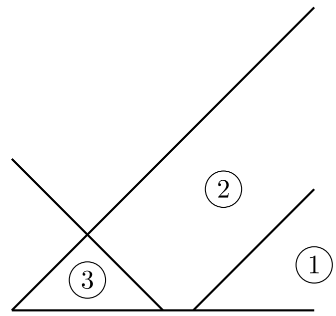


Figure III.2: Cases for $\mathscr{E}_k$ - $\mathrm{Sp}(n,1)$

Lemma III.5.1

The Gelfand-Kirillov dimension of the residue representation $\mathscr{E}_k$ *is*
$$\begin{cases} 4n-1 & \textit{in case 1} \\ 2n+1 & \textit{in case 2} \\ 0 & \textit{in case 3} \end{cases}$$

Proof. First of all, one can compute the eigenvalue on each K-type $\tau_{m,l}$ in $\mathscr{H}_{\lambda_k\alpha}$ for the Casimir operator $\Omega_{\mathfrak{k}}$ of $\mathfrak{k}$ (using for example [Hal15, Proposition 10.6]). Recall from II.2.5 that $\tau_{m,l}$ has highest weight $\mu_{m,l} = \frac{m+l}{2}\epsilon_1 - \frac{m-l}{2}\epsilon_n + l\epsilon_{n+1}$. Then one finds:

$$\tau_{m,l}(\Omega_{\mathfrak{k}}) = \frac{1}{2}\Big((m+2n-1)^2 - (2n-1)^2 + 3(l+1)^2 - 1\Big)\,\mathrm{Id} \tag{III.18}$$

The dimension of the space of $\tau_{m,l}$ follows for example from the Weyl dimension formula [Kna02, Theorem 5.84]:

$$d_{m,l} = \frac{(l+1)^2}{2n-1}\left(\binom{\frac{m-l}{2}+2n-2}{2n-2}\binom{\frac{m+l}{2}+2n-1}{2n-2} - \binom{\frac{m-l}{2}+2n-3}{2n-2}\binom{\frac{m+l}{2}+2n-2}{2n-2}\right)$$

Setting $\delta(m,l,N) := \binom{\frac{m-l}{2}+N}{2n-2}\binom{\frac{m+l}{2}+N+1}{2n-2}$ we have

$$d_{m,l} = \frac{(l+1)^2}{2n-1}\Big(\delta(m,l,2n-2) - \delta(m,l,2n-1)\Big) \tag{III.19}$$

As before, for a fixed l, the sum of $d_{m,l}$ between the values $m_{\min}$ and $m_{\max}$ of m is telescopic:

$$\sum_{m=m_{\min}}^{m_{\max}} d_{m,l} = \frac{(l+1)^2}{2n-1}\Big(\delta(m_{\max},l,n) - \delta(m_{\min},l,n-1)\Big) \tag{III.20}$$

The inequality given by the eigenvalue of the Casimir operator III.13 limits the area of points of K-types by an ellipsis. The goal is to know the behaviour in t of the sum over the K-types delimited by this ellipsis when this ellipsis is going to infinity. The method to compute the $N_{\mathscr{E}_k}(t)$ out of [Vog78, Theorem 1.2] is exactly the same as the $\mathrm{SU}(n,1)$ case and left to the reader.

■

The following lemma gives us the different nilpotent orbits in the nilradical of $\mathfrak{g}$.

Lemma III.5.2

There are three nilpotent orbits in $\mathfrak{sp}(n, 1)$*:*

1. *the trivial orbit,*
2. *the orbit generated by* $\mathfrak{g}_{\alpha/2}$*, of dimension* $8n - 2$*,*
3. *the orbit generated by* $\mathfrak{g}_{\alpha}$*, of dimension* $4n + 2$*.*

Proof. Using Theorem 9.3.5 in [CM93], one finds that there are three possible Young diagrams corresponding to the nilpotent orbits for $\mathfrak{sp}(n, 1)$:

$$\begin{array}{|c|} \hline + \\ \hline + \\ \hline + \\ \hline \vdots \\ \hline + \\ \hline - \\ \hline \end{array} \qquad \begin{array}{|c|c|c|} \hline + & - & + \\ \hline + \\ \cline{1-1} + \\ \cline{1-1} + \\ \cline{1-1} \vdots \\ \cline{1-1} + \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|} \hline + & - \\ \hline + \\ \cline{1-1} + \\ \cline{1-1} + \\ \cline{1-1} \vdots \\ \cline{1-1} + \\ \cline{1-1} \end{array}$$

The first one corresponds to the zero orbit. The second is the only one which is nilpotent of degree 2. It corresponds to the orbit generated by any element of $\mathfrak{g}_{\alpha/2}$. The last one is the only one nilpotent of degree 1. So it is generated by any element of $\mathfrak{g}_{\alpha}$.
The dimensions are given by Corollary 6.1.4 in [CM93]. Also here we have to recall that the partition of the complex nilpotent orbit coming from a real nilpotent orbit is given by the boxes of the corresponding Young diagram.

■

The combination of the two lemmas is sufficient to conclude the results about wave front set of $\mathscr{E}_k$ in Theorem 2 for Sp$(n, 1)$.

Theorem III.5.1. *The representation* τ *has a highest weight of the form* $(t_1, t_2, 0, \dots, 0, t_{n+1})$*, where* t_1*,* t_2 *and* t_{n+1} *are positive integers such that* $t_1 \geq t_2$*,* $t_{n+1} \leq t_1 + t_2$ *and* $t_1 + t_2 + t_{n+1}$ *is even.*

1. *If* $t_{n+1} \leq t_1 + t_2 - 2k - 4$ *, then the wave front set of* $\mathscr{E}_k$ *is the nilpotent orbit generated by* $\mathfrak{g}_{\alpha/2}$*.*
2. *If* $t_{n+1} \geq |t_1 + t_2 - 2k - 2|$ *the residue representation is unitary. Its wave front set is the nilpotent orbit generated by* $\mathfrak{g}_{\alpha}$*.*
3. *If* $t_1 + t_2 < 2k + 2 - t_1 - t_2$*, then the wave front set of* $\mathscr{E}_k$ *is the zero orbit.*

III.6 Case of $\mathscr{F}_4$

In this section, we prove the results about wave front set in Theorem 2 for $G = F_4$. We first compute the Gelfand-Kirillov dimension of the Harish-Chandra module of $\mathscr{E}_k$ defined in section II.2. Recall that there are three possible residue representations (see II.2.6) which we label as in Figure III.3. The proofs are very similar to the complex case.

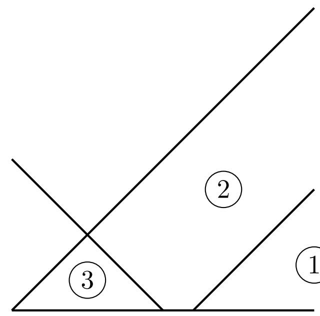


Figure III.3: Cases for $\mathscr{E}_k$ - $\mathscr{F}_4$

Lemma III.6.1

The Gelfand-Kirillov dimension of the residue representation $\mathscr{E}_k$ *is* $\begin{cases} 15 & \text{in case 1} \\ 11 & \text{in case 2} \\ 0 & \text{in case 3} \end{cases}$.

Proof. As in the complex case, one can compute the eigenvalue on each K-type $\tau_{p,q}$ in $\mathscr{H}_{\lambda_k\alpha}$ for the Casimir operator $\Omega_{\mathfrak{k}}$ of $\mathfrak{k}$ (using for example [Hal15, Proposition 10.6]). Recall from II.2.6 that $\tau_{p,q}$ has highest weight $\mu_{p,q} = \frac{p}{2}\epsilon_1 + \frac{q}{2}(\epsilon_2 + \epsilon_3 + \epsilon_4)$. Then one finds:

$$\tau_{p,q}(\Omega_{\mathfrak{k}}) = \left(\left(\frac{p+7}{2}\right)^2 + \frac{3(q+3)^2}{4} - 19\right) \mathrm{Id} \tag{III.21}$$

The dimension of the space of $\tau_{p,q}$ follows from the Weyl dimension formula [Kna02, Theorem 5.84]:

$$d_{p,q} = C(q)(\delta(p+2,q) - \delta(p,q))$$

where $\delta(p,q) := \frac{\left(\frac{p+q}{2}+6\right)!}{\left(\frac{p+q}{2}+2\right)!}\frac{\left(\frac{p-q}{2}+3\right)!}{\left(\frac{p-q}{2}-1\right)!}$ and $C(q) := \frac{(q+1)(q+3)(q+5)(2q+8)(2q+6)(2q+4)}{2^{12}\ 3^4\ 5^2\ 7}$. For a fixed q, the sum of $d_{p,q}$ between 2 values $p_{\min}$ and $p_{\max}$ of p is telescopic:

$$\sum_{p=p_{\min}}^{p_{\max}} d_{p,q} = C(q)\Big(\delta(p_{\max}+2,q) - \delta(p_{\min},q)\Big) . \tag{III.22}$$

The inequality given by the eigenvalue of the Casimir operator (III.21) bounds a region of K-types by an ellipsis. As in the previous case, we need to know the behaviour in t of the sum limited by this ellipsis when the ellipsis goes to infinity. The method to compute the $N_{\mathscr{E}_k}(t)$ of the [Vog78, Theorem 1.2] is exactly the same as the $\mathrm{SU}(n,1)$ case and is left to the reader.

■

The following lemma gives us the different nilpotent orbits in the nilradical of $\mathfrak{g}$.

Lemma III.6.2
There are three nilpotent orbits in $\mathfrak{g}$*:*

1. *the trivial orbit,*
2. *the orbit generated by* $\mathfrak{g}_{\alpha/2}$*, of dimension* 30*,*
3. *the orbit generated by* $\mathfrak{g}_{\alpha}$*, of dimension* 22*.*

Proof. The table on page 151 in [CM93] ensures us that there are three possible Dynkin diagrams corresponding to the nilpotent orbits. Relating these diagrams to the table on page 128 allows us to find the dimensions associated to these (real) nilpotent orbit. Now we want to relate the orbits generated respectively by $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{\alpha/2}$ and the two nilpotent orbits of $\mathfrak{g}$. First choose an element $X_{\alpha/2}$ in $\mathfrak{g}_{\alpha/2}$ such that $B\Big(X_{\alpha/2}, \theta(X_{\alpha/2})\Big) \neq 0$. Lemma 3.2 in [Hel01, Chapter IX] proves that for all $X_\alpha \in \mathfrak{g}_\alpha$,

$$\Big[X_{\alpha/2}, [\theta(X_{\alpha/2}), X_\alpha]\Big] = 2\langle \alpha, \alpha\rangle \; B\Big(X_{\alpha/2}, \theta(X_{\alpha/2})\Big) \; X_\alpha \; .$$

This proves that

$$\mathfrak{g}_\alpha = [\mathfrak{g}_{\alpha/2}, \mathfrak{g}_{\alpha/2}] \; .$$

The Jacobi identity allows us to conclude that $Z(\mathfrak{g}_{\alpha/2}) \subset Z(\mathfrak{g}_\alpha)$. This proves that the (nilpotent) orbit generated by $\mathfrak{g}_\alpha$ has lower dimension than that of $\mathfrak{g}_{\alpha/2}$.

■

The combination of the two lemmas above proves the results for F_4 about wave front set of $\mathscr{E}_k$ in Theorem 2.

Theorem III.6.1. *The representation* τ *has a highest weight of the form* $(a/2, b/2, b/2, b/2)$*, where* a *and* b *are positive integers such that* $a \geq b$ *and* $a - b$ *is even.*

1. *If* $b \leq a - 2k - 8$*, then the wave front set of* $\mathscr{E}_k$ *is the nilpotent orbit generated by* $\mathfrak{g}_{\alpha/2}$*.*
2. *If* $b > a - 2k - 8$ *and* $b \geq 2k + 2 - a$*, then the wave front set of* $\mathscr{E}_k$ *is the nilpotent orbit generated by* $\mathfrak{g}_\alpha$*.*
3. *If* $b < 2k + 2$*, then the wave front set of* $\mathscr{E}_k$ *is the zero orbit.*

A

The Plancherel densities

Recall that to determine the resolvent of the Laplacian, we use the inversion formula (I.15) for vector-valued Helgason-Fourier transform. The resonances arise from the singularities of the Plancherel measure. To find them, we need an explicit formula for the Plancherel density p_σ. Even if such an explicit formula is not known for an arbitrary group G, in the rank-one case, several authors have computed it ([Oka65], [Mia79]; see also [War72, Epilogue, pp. 414 ff.]). Our reference in the following is Miatello's article [Mia79]. The formula depends on the group G and on the highest weight of σ. According to this formula, p_σ is the product of two factors. The first one is a polynomial function in λ, denoted q_σ. The second factor is either a hyperbolic tangent or a hyperbolic cotangent (or 1 if $G = \mathrm{Spin}(2n+1,1)$). We denote it by ϕ_σ. The goal of this section is to explain how one can compute p_σ with $\sigma \in \hat{M}(\tau)$, where $\tau \in \hat{K}$ is arbitrarily fixed. For this, we are going to use the branching rules given by Baldoni Silva in [BS79]. Remark 1.3 in [Mia79] ensures us that the Plancherel formula of one of the four groups listed in the table in the introduction gives us the Plancherel formula for every group of real rank one. As we do not care about the constants in our computations in this thesis, p_σ is given up to a constant.

A.1 Case of Spin$(n, 1)$

Here G/K is the real hyperbolic space. Recall that K is Spin(n) and M is Spin$(n-1)$. In this case, the parity of n plays a role in the Plancherel measure. In fact, when n is odd, $\phi_\sigma(\lambda) = 1$. So it is non-singular and there are no resonances. In the following we therefore disregard the case of Spin$(2n+1, 1)$ and suppose that $G = \text{Spin}(2n, 1)$.
Let us recall some Lie algebraic structure. Our reference is [BS79, Section 3]. The Lie algebra of G is $\mathfrak{g} = \mathfrak{so}(2n, 1)$. Its complexification is $\mathfrak{g}_\mathbb{C} = \mathfrak{so}(2n+1, \mathbb{C})$. We have also $Lie(K)_\mathbb{C} = \mathfrak{k}_\mathbb{C} = \mathfrak{so}(2n, \mathbb{C})$ and $Lie(M)_\mathbb{C} = \mathfrak{m}_\mathbb{C} = \mathfrak{so}(2n-1, \mathbb{C})$.
We choose the following Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$ is:

$$\mathfrak{h}_\mathbb{C} = \left\{ H \in \mathfrak{so}(2n+1, \mathbb{C}) \mid H = \mathsf{diag}\left[\begin{pmatrix} 0 & ih_1 \\ -ih_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & ih_2 \\ -ih_2 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & ih_n \\ -ih_n & 0 \end{pmatrix}, 0 \right] \right\}$$

Let $\{\epsilon_j\}_{j=1,\dots,n}$ be the elementary weights defined by $\epsilon_j(H) = h_j$. We denote S, S^+ and S^0 respectively the set of roots, positive roots and simple roots of $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. We index the sets of roots by the Lie algebra, to which they correspond.

$$S = \left\{ \pm\epsilon_i \pm \epsilon_j \mid i \neq j \right\} \cup \left\{ \pm\epsilon_k \mid k \in \{1, \dots, n\} \right\} ; \; S^0 = \left\{ \alpha_i = \epsilon_i - \epsilon_{i+1} \mid i \in \{1, \dots, n-1\} \right\} \cup \left\{ \alpha_n = \epsilon_n \right\}$$

$$S_{\mathfrak{k}_\mathbb{C}} = \left\{ \pm \epsilon_i \pm \epsilon_j \mid i \neq j \right\} \; ; \; S_{\mathfrak{m}_\mathbb{C}} = \left\{ \pm \epsilon_i \pm \epsilon_j \mid i \neq j \right\} \cup \left\{ \pm \epsilon_k \mid k \in \{1, \dots, n-1\} \right\}$$
$$S^0_{\mathfrak{k}_\mathbb{C}} = \left\{ \epsilon_i - \epsilon_{i+1} \mid i \in \{1, \dots, n-1\} \right\} \; ; \; S^0_{\mathfrak{m}_\mathbb{C}} = \left\{ \epsilon_i - \epsilon_{i+1} \mid i \in \{1, \dots, n-2\} \right\} \cup \left\{ \epsilon_{n-1} \right\}$$

Notice that, unlike [BS79, Theorem 3.4], we keep the same notation for the ϵ_j's and their projections on $\mathfrak{m}_\mathbb{C}$. As in [BS79, Lemma 3.2], the fixed $(\tau, V_\tau) \in \hat{K}$ has highest weight of the form:

$$\mu_\tau = \sum_{j=1}^{n} a_j \epsilon_j$$

where $a_1 \geq \ldots \geq a_{n-1} \geq \mid a_n \mid \geq 0$, $a_i - a_j \in \mathbb{Z}$ and $2a_j \in \mathbb{Z}$ for all $i, j = 1, \dots, n$.
Let $\sigma \in \hat{M}(\tau)$. According to [BS79, Theorem 3.4], the highest weight μ_σ of a representation $\sigma \in \hat{M}(\tau)$ has the form:

$$\mu_\sigma = \sum_{i=1}^{n-1} b_j \epsilon_j$$

where for all $i, j = 1, \dots, n-1$ we have $a_j - b_i \in \mathbb{Z}$ and $a_1 \geq b_1 \geq a_2 \geq \ldots \geq a_{n-1} \geq b_{n-1} \geq \mid a_n \mid \geq 0$. Furthermore, $m(\sigma, \tau|_M) = 1$ for every $\sigma \in \hat{M}(\tau)$.
For $\sigma \in \hat{M}(\tau)$, the Plancherel density is given in [Mia79, pp. 256-257]:

$$p_\sigma(\lambda) = \left\{ \begin{array}{c} \tanh(\pi\lambda_\alpha) \text{ , if } b_j \in \mathbb{Z} \\ \coth(\pi\lambda_\alpha) \text{ , if } b_j \in \frac{1}{2} + \mathbb{Z} \end{array} \right\} \lambda_\alpha \prod_{j=1}^{n-1} \left(\lambda_\alpha^2 + (\rho + b_j - j)^2 \right) \tag{A.1}$$

We recall that λ_α is the complex number associated to $\lambda \in \mathfrak{a}^*_\mathbb{C}$ by (I.4).

Remark
The above formula agrees with the Plancherel density for the real hyperbolic space for $\tau = \mathrm{triv}_K$ (and the, $\sigma = \mathrm{triv}_M$). See e.g. [HP09]. It also gives the different irreducible continuous constituents of the Plancherel density for the p-forms on the real hyperbolic space, as computed in [Ped94].

A.2 Case of $\mathrm{SU}(n,1)$

Here G/K is the complex hyperbolic space. Recall that K is $\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1))$ and M is $\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n-1)\times\mathrm{U}(1))$. Our notations follows [BS79, Section 4]. The Lie algebra of G is $\mathfrak{g}=\mathfrak{su}(n,1)$. Its complexification is $\mathfrak{g}_{\mathbb{C}}=\mathfrak{sl}(n+1,\mathbb{C})$. We have also $Lie(K)_{\mathbb{C}}=\mathfrak{k}_{\mathbb{C}}=\mathfrak{sl}(n,\mathbb{C})$ and $Lie(M)_{\mathbb{C}}=\mathfrak{m}_{\mathbb{C}}=\mathfrak{sl}(n-1,\mathbb{C})$.
The elliptic Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$ consists of the diagonal matrices in $\mathfrak{sl}(n+1,\mathbb{C})$. Let $\{\epsilon_j\}_{j=1,\dots,n}$ be the elementary weights defined by $\epsilon_j(H)=h_j$ where $H=\mathsf{diag}(h_1,h_2,\dots,h_{n+1})$ described the elements of $\mathfrak{h}_{\mathbb{C}}$. We denote S, S^+ and S^0 respectively the set of roots, positive roots and simple roots of $(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})$. We index the sets of roots by the Lie algebra, to which they correspond.

$$S=\Big\{\pm(\epsilon_i-\epsilon_j)\mid 1\le i<j\le n+1\Big\}\ ;\ S^0=\Big\{\alpha_i=\epsilon_i-\epsilon_{i+1}\mid 1\le i\le n\Big\}$$

$$\begin{aligned} S_{\mathfrak{k}_{\mathbb{C}}}&=\Big\{\pm(\epsilon_i-\epsilon_j)\mid 1\le i<j\le n\Big\} & S_{\mathfrak{m}_{\mathbb{C}}}&=\Big\{\pm(\epsilon_i-\epsilon_j)\mid 2\le i<j\le n\Big\}\\ S^0_{\mathfrak{k}_{\mathbb{C}}}&=\Big\{\alpha_i=\epsilon_i-\epsilon_{i+1}\mid 1\le i\le n-1\Big\} & S^0_{\mathfrak{m}_{\mathbb{C}}}&=\Big\{\alpha_i=\epsilon_i-\epsilon_{i+1}\mid 2\le i\le n-1\Big\}\end{aligned}$$

As in [BS79, Lemma 4.2], the fixed $(\tau,V_\tau)\in\hat{K}$ has highest weight of the form

$$\mu_\tau=\sum_{j=1}^{n+1}a_j\epsilon_j$$

where $a_1\ge\ldots\ge a_{n-1}\ge a_n$ and $a_i\in\mathbb{Z}$ for all $i=1,\dots,n+1$.
Let $\sigma\in\hat{M}(\tau)$. By [BS79, Theorem 4.4]. The highest weight μ_σ of $\sigma\in\hat{M}(\tau)$ has the form

$$\mu_\sigma=b_0(\epsilon_1+\epsilon_{n+1})+\sum_{j=2}^{n}b_j\epsilon_j$$

where for all j, we have $b_j\in\mathbb{Z}$, $a_1\ge b_2\ge a_2\ge\ldots\ge a_{n-1}\ge b_n\ge a_n$ and $b_0=\frac{\sum_{j=1}^{n+1}a_j-\sum_{j=2}^{n}b_j}{2}$.
The fact that we are working with zero trace matrices implies that $\sum_{j=1}^{n+1}\epsilon_j=0$. Thus we can relate to Miatello's form for the weights. We get the same formula as in [Mia79, p. 258]:

$$\mu_\sigma=\sum_{j=2}^{n-1}(b_j-b_n)\epsilon_j+(b_n-b_0)\sum_{j=2}^{n}\epsilon_j$$

Miatello gives this following Plancherel density:

$$p_\sigma(\lambda) = \left\{ \begin{array}{l} \tanh(\pi\lambda_\alpha) \text{ , if } 2b_0+n \text{ is odd} \\ \coth(\pi\lambda_\alpha) \text{ , if } 2b_0+n \text{ is even} \end{array} \right\} \lambda_\alpha \prod_{j=1}^{n-1} \left(\lambda_\alpha^2 + (b_{j+1} - b_0 + \frac{n}{2} - j)^2\right) \quad \text{(A.2)}$$

We recall that λ_α is the complex number associated to $\lambda \in \mathfrak{a}_\mathbb{C}^*$ by (I.4).

A.3 Case of $\mathrm{Sp}(n,1)$

Here G/K is the quaternionic hyperbolic space. In this case, K is $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ and M is $\mathrm{Sp}(1) \times \mathrm{Sp}(n-1) \times \mathrm{Sp}(1)$.
Let us recall some Lie algebraic structure. The Lie algebra of G is $\mathfrak{g} = \mathfrak{sp}(n,1)$. Its complexification is $\mathfrak{g}_\mathbb{C} = \mathfrak{sp}(n+1,\mathbb{C})$. We have also $Lie(K)_\mathbb{C} = \mathfrak{k}_\mathbb{C} = \mathfrak{sp}(n,\mathbb{C})$ and $Lie(M)_\mathbb{C} = \mathfrak{m}_\mathbb{C} = \mathfrak{sp}(n-1,\mathbb{C})$.
Let $\mathfrak{t}$ denote the set of diagonal matrices in $\mathfrak{g}$. So $\mathfrak{t}_\mathbb{C}$ is the elliptic Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$. Let $\mathfrak{h}^-$ be a maximal abelian algebra of $\mathfrak{m}$. Then $\mathfrak{h} := \mathfrak{h}^- + \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{h}_\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$.
Let $\{\epsilon_j\}_{j=1,\dots,n}$ be the elementary weights defined by $\epsilon_j(H) = h_j$, where $H = \mathsf{diag}(h_1, h_2, \dots, h_{n+1}) \in \mathfrak{t}_\mathbb{C}$. We denote S, S^+ and S^0 respectively the set of roots, positive roots and simple roots of $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. We index the sets of roots by the Lie algebra, to which they correspond.

$$\begin{aligned} S &= \Big\{ \pm\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le n+1 \Big\} \cup \Big\{ \pm 2\epsilon_i \mid 1 \le i \le n+1 \Big\} \ ; \\ S^0 &= \Big\{ \epsilon_i - \epsilon_{i+1} \mid 1 \le i \le n \Big\} \cup \Big\{ 2\epsilon_{n+1} \Big\} \ ; \\ S_{\mathfrak{k}_\mathbb{C}} &= \Big\{ \pm\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le n \Big\} \cup \Big\{ \pm 2\epsilon_i \mid 1 \le i \le n \Big\} \ ; \\ S^0_{\mathfrak{k}_\mathbb{C}} &= \Big\{ \epsilon_i - \epsilon_{i+1} \mid 1 \le i \le n-1 \Big\} \cup \Big\{ 2\epsilon_n \Big\} \ ; \\ S_{\mathfrak{m}_\mathbb{C}} &= \Big\{ \pm(\epsilon_i - \epsilon_j) \mid 1 \le i < j \le n-1 \Big\} \cup \Big\{ \pm 2\epsilon_i \mid 1 \le i \le n-1 \Big\} \ ; \\ S^0_{\mathfrak{m}_\mathbb{C}} &= \Big\{ \epsilon_i - \epsilon_{i+1} \mid 1 \le i \le n-2 \Big\} \cup \Big\{ 2\epsilon_{n-1} \Big\} \end{aligned}$$

The fixed $(\tau, V_\tau) \in \hat{K}$ has highest weight of the form:

$$\mu_\tau = \sum_{j=1}^{n+1} a_j \epsilon_j$$

where $a_1 \ge \ldots \ge a_{n-1} \ge a_n \ge 0$, $a_{n+1} \ge 0$ and $a_i \in \mathbb{Z}$ for all $i = 1, \dots, n+1$.
See [BS79, Lemma 5.2]. Theorem 5.5 in [BS79] gives us the form of the highest weight μ_σ of $\sigma \in \hat{M}(\tau)$ as follows:

$$\mu_\sigma = b_0(\epsilon_1 + \epsilon_{n+1}) + \sum_{j=2}^{n} b_j \epsilon_j \quad \text{(A.3)}$$

where we have:

1. $a_j \ge b_{j+1}$, for all $j = 1, \dots, n-1$

② $b_j \geq a_{j+1}$, for all $j = 2, \ldots, n-1$

③ $b_0 = \frac{a_{n+1}+b_1-2j}{2}$, for some $j = 0, \ldots, \min(a_{n+1}, b_1)$

and b_1 satisfies $b_1 \in \mathbb{Z}_+$ and $\sum_{j=1}^{n}(a_j - b_j) \in 2\mathbb{Z}$.
We want to use the Plancherel formula given by Miatello [Mia79]. But the Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ used in this thesis is not the same as the Cartan algebra $\mathfrak{t}_{\mathbb{C}}$ used by Baldoni-Silva for the branching rules. So we have to use a Cayley transform to find the highest weight of σ relative to $\mathfrak{h}_{\mathbb{C}}$. This computation was already done in [BSK80]. Namely (see [Hel01, p155-156]) one knows that for each $\beta \in S$ we can select a root vector X_β such that $B(X_a, X_{-a}) = \frac{2}{\langle \beta,\beta \rangle}$ and $\theta\overline{X_\beta} = -X_\beta$. Choose $\beta = \epsilon_1 - \epsilon_{n+1}$. Then if we denote by $e_1, \ldots, e_{n+1}$ the fundamental weights of the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, we have :

$$e_1 = \epsilon_1 \circ \mathsf{Ad}u_\beta^{-1} \quad , \quad e_2 = -\epsilon_{n+1} \circ \mathsf{Ad}u_\beta^{-1} \quad , \quad e_i = -\epsilon_{i-1} \circ \mathsf{Ad}u_\beta^{-1} \text{ , for all } i = 3, \ldots n+1$$

where $u_\beta = \exp(\pi/4)(X_\beta - X_{-\beta})$.
So μ_σ is written in the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ as:

$$\mu_\sigma = b_0(e_1 - e_2) + \sum_{j=3}^{n+1} b_{j-1} e_j$$

and [Mia79] gives the following Plancherel density:

$$p_\sigma(\lambda) = \left\{ \begin{array}{c} \tanh(\pi\lambda_\alpha) \text{ , if } b_0 \in \mathbb{Z} \\ \coth(\pi\lambda_\alpha) \text{ , if } b_0 \in \mathbb{Z} + \frac{1}{2} \end{array} \right\} \lambda_\alpha \left(\lambda_\alpha^2 + (b_0 + \frac{1}{2})^2\right)$$
$$\prod_{j=3}^{n+1} \left(\lambda_\alpha^2 + (b_{j-1} - b_0 + n - j + \frac{3}{2})^2\right) \left(\lambda_\alpha^2 + (b_{j-1} - b_0 + n - j + \frac{5}{2})^2\right) \quad \text{(A.4)}$$

We recall that λ_α is the complex number associated to $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ by (I.4).

A.4 Case of F_4

Here, $G = F_4$. We recall $\mathfrak{g} = f_4^{-20}$, K is $Spin(9)$ and its Lie algebra is $\mathfrak{k} = \mathfrak{so}(9)$. Let $\mathfrak{t} \subset \mathfrak{k}$ be the compact Cartan subalgebra for both $\mathfrak{g}$ and $\mathfrak{k}$. The branching rules for the fixed τ are determined in [BS79, Paragraph 6]. For a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ its centralizer in $\mathfrak{k}$ is $\mathfrak{m} = \mathfrak{so}(7)$. The problem is that this Lie algebra $\mathfrak{m}$ is not contained in the standard way in $\mathfrak{k}$. So, we cannot use twice the branching rules for SO(n) directly. Let $K_1 = \text{Spin}(8)$ a subgroup of K contained in the standard way. We denote by $\mathfrak{k}_1$ its Lie algebra.
Let $\{\epsilon_j\}_{j=1,\ldots,n}$ be the elementary weights in $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. We denote S, S^+ and S^0 respectively the set of roots, positive roots and simple roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. We index the sets of roots by the Lie algebra, to which they correspond.

$$S = \Big\{ \pm \epsilon_i \pm \epsilon_j \mid 1 \le i < j \le 4 \Big\} \cup \Big\{ \pm \epsilon_i \mid 1 \le i \le 4 \Big\} \cup \Big\{ \tfrac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) \Big\}$$
$$S^0 = \Big\{ \alpha_1 = \epsilon_2 - \epsilon_3 \ , \ \alpha_2 = \epsilon_3 - \epsilon_4 \ , \ \alpha_3 = \epsilon_4 \ , \ \alpha_4 = \tfrac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4) \Big\}$$
$$S_{\mathfrak{k}_\mathbb{C}} = \Big\{ \pm \epsilon_i \pm \epsilon_j \mid 1 \le i < j \le 4 \Big\} \cup \Big\{ \pm \epsilon_i \mid 1 \le i \le 4 \Big\} \quad S_{(\mathfrak{k}_1)_\mathbb{C}} = \Big\{ \pm (\epsilon_i \pm \epsilon_j) \mid 1 \le i < j \le 4 \Big\}$$
$$S^0_{\mathfrak{k}_\mathbb{C}} = \Big\{ \alpha_1 \ , \ \alpha_2 \ , \ \alpha_3 \ , \ \alpha_2 + 2\alpha_3 + 2\alpha_4 = \epsilon_1 - \epsilon_2 \Big\} \quad S^0_{(\mathfrak{k}_1)_\mathbb{C}} = \Big\{ \epsilon_1 - \epsilon_2 \ , \ \epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \epsilon_3 + \epsilon_4 \Big\}$$

Let H_{α_4} be the unique element of $\mathfrak{a}$ such that $\alpha_4 = B(H_{\alpha_4}, \ \cdot \ )$. Choose the root vectors X_{α_4} and $X_{-\alpha_4}$ such that $[X_{\alpha_4}, X_{-\alpha_4}] = H_{\alpha_4}$ and $X_{\alpha_4} + X_{-\alpha_4} \in \mathfrak{p}$. Define $\mathfrak{a}$ to be the one-dimensional space spanned by $X_{\alpha_4} + X_{-\alpha_4}$. Let $\mathfrak{h} = \mathfrak{h}^- \oplus \mathfrak{a}$ where $\mathfrak{h}^-$ is a Cartan subalgebra of $\mathfrak{m}$. Then the Cayley transform $\mathsf{Ad}\left(\exp \frac{\pi}{4}(X_{\alpha_4} - X_{-\alpha_4}\right))$ maps $\mathfrak{t}_\mathbb{C}$ onto $\mathfrak{h}_\mathbb{C}$. The set of roots $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ is the following root system on $\mathfrak{m}$

$$\Phi_\mathfrak{m} = \Big\{ \pm (\epsilon_i - \epsilon_j) \mid 1 \le i < j \le n-1 \Big\} \cup \Big\{ \pm 2\epsilon_i \mid 1 \le i \le n-1 \Big\}$$
$$\Phi^0_\mathfrak{m} = \Big\{ \alpha_1 \ , \ \alpha_2 \ , \ \alpha_2 + 2\alpha_3 + \alpha_4 = \tfrac{1}{2}(\epsilon_1 - \epsilon_2 + \epsilon_3 + \epsilon_4) \Big\}$$

As said before, $\mathfrak{m}$ is not contained in the standard way in $\mathfrak{k}_1$. Let ϕ be the automorphism of $\mathfrak{k}_1$ which keeps the roots and such that $\phi(M)$ is contained in the standard way in $\phi(K_1)$. This automorphism is given by

$$\begin{aligned} \phi(\epsilon_1 - \epsilon_2) &= \epsilon_3 - \epsilon_4 \ , & \phi(\epsilon_3 - \epsilon_4) &= \epsilon_1 - \epsilon_2 \\ \phi(\epsilon_2 - \epsilon_3) &= \epsilon_2 - \epsilon_3 \ , & \phi(\epsilon_3 + \epsilon_4) &= \epsilon_3 + \epsilon_4 \end{aligned}$$

The fixed K-type τ has highest weight of the form:

$$\mu_\tau = a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3 + a_4\epsilon_4$$

where $a_1 \ge \ldots \ge a_4 \ge 0$, $2a_i \in \mathbb{Z}$ and $a_i - a_j \in \mathbb{Z}$ for all $i, j = 1, \ldots, 4$.
Because of the branching rules of $\mathrm{Spin}(n)$, the representations of K_1 which are contained in the restriction of τ to K_1 have the following highest weights:

$$c_1\epsilon_1 + c_2\epsilon_2 + c_3\epsilon_3 + c_4\epsilon_4$$

where $a_1 \ge c_1 \ldots \ge a_4 \ge \ \mid c_4 \mid$, $2c_i \in \mathbb{Z}$ and $a_i - c_j \in \mathbb{Z}$ for all $i, j = 1, \ldots, 4$.
We have now to apply ϕ to this highest weight to use the branching rules of $\mathfrak{so}(n)$. We get a highest weight of the form

$$\begin{aligned} d_1\epsilon_1 + d_2\epsilon_2 + d_3\epsilon_3 + d_4\epsilon_4 := \frac{1}{2}(c_1 + c_2 + c_3 - c4)\epsilon_1 + \frac{1}{2}(c_1 + c_2 - c_3 + c4)\epsilon_2 \\ + \frac{1}{2}(c_1 - c_2 + c_3 + c4)\epsilon_3 + \frac{1}{2}(-c_1 + c_2 + c_3 + c4)\epsilon_4 \end{aligned}$$

A representation of $\widehat{\phi(M)}$ is of the form $\sigma \circ \phi$ where $\sigma \in \hat{M}$. The $\phi(M)$-types which appear in the restriction of K_1-type (d_1, d_2, d_3, d_4) to $\phi(M)$ have highest weight of the form:

$$\mu_{\sigma\circ\phi} = b_1\epsilon_1 + b_2\epsilon_2 + b_3\epsilon_3$$

where $d_1 \geq b_1 \ldots \geq b_3 \geq |d_4|$, $2c_i \in \mathbb{Z}$ and $a_i - c_j \in \mathbb{Z}$ for all $i, j = 1, \ldots, 4$. Applying ϕ, one gets

$$\mu_\sigma = b_1\alpha_2 + (b_1 + b_2)\alpha_1 + \frac{1}{2}(b_1 + b_2 + b_3)(\alpha_2 + 2\alpha_3 + \alpha_4)$$

The Cartan subalgebra used in [Mia79] is neither $\mathfrak{t}$ nor $\mathfrak{h}$ because the maximal abelian subalgebra of $\mathfrak{p}$ is not the same $\mathfrak{a}$ here. Define the map

$$\begin{array}{rccc} \Psi: & \alpha_1 & \mapsto & \alpha_2 \\ & \alpha_2 & \mapsto & \alpha_1 \\ & \alpha_2 + 2\alpha_3 + \alpha_4 & \mapsto & \alpha_3 \\ & \alpha_3 & \mapsto & -(\alpha_2 + 2\alpha_3 + \alpha_4) \end{array}$$

This map Ψ sends α_4 on $-\epsilon_1$, so $\mathfrak{a}$ onto the maximal abelian subspace used in [Mia79]. The Cartan subalgebra $\Psi(\mathfrak{h})$ is that used in that paper. Applying Ψ we get

$$\mu_{\sigma\circ\Psi} = b_1\epsilon_1 + b_2\epsilon_2 + b_3\epsilon_3$$

and ϵ_1 becomes the real root, i.e. $\epsilon_1|_{\Psi(\mathfrak{a})} = -\alpha \circ \Psi$ where α is the longest restricted root. Now we can apply the Plancherel formula from [Mia79] and get

$$\begin{aligned} p_\sigma(\lambda) = &\left\{ \begin{array}{c} \tanh(\pi\lambda_\alpha) \text{ , if } b_i \in \mathbb{Z} \\ \coth(\pi\lambda_\alpha) \text{ , if } b_i \in \mathbb{Z} + \frac{1}{2} \end{array} \right\} \lambda_\alpha \left(\lambda_\alpha^2 + (\frac{b_3+1}{2})^2\right) \left(\lambda_\alpha^2 + (\frac{b_2+3}{2})^2\right) \\ &\left(\lambda_\alpha^2 + (\frac{b_1+5}{2})^2\right) \left(\lambda_\alpha^2 + (b_1 - b_2 - b_3 + \frac{1}{2})^2\right) \\ &\left(\lambda_\alpha^2 + (b_1 - b_2 + b_3 + \frac{3}{2})^2\right) \left(\lambda_\alpha^2 + (b_1 + b_2 - b_3 + \frac{7}{2})^2\right) \left(\lambda_\alpha^2 + (b_1 + b_2 + b_3 + \frac{9}{2})^2\right) \end{aligned} \tag{A.5}$$

We recall that λ_α is the complex number associated to $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ by (I.4).

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