Opetopes: Syntactic and Algebraic Aspects

Cédric Ho Thanh

Opetopes

SYNTACTIC AND ALGEBRAIC ASPECTS

A doctoral thesis written by

Cédric Ho Thanh

under the direction of

Prof. **Pierre-Louis Curien**, *Université de Paris*
and
Prof. **Samuel Mimram**, *École Polytechnique*

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Denis-Charles Cisinski, *Universität Regensburg*	*president*
Nicola Gambino, *University of Leeds*	*referee*
Marek Zawadowski, *Uniwersytet Warszawski*	*referee*
Eugenia Cheng, *School of the Art Institute of Chicago*	*member*
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François Métayer, *Université Paris Nanterre*	*member*
Pierre-Louis Curien, *Université de Paris*	*director*
Samuel Mimram, *École Polytechnique*	*director*

Abstract

Opetopes are shapes (akin to globules, cubes, simplices, dendrices, etc.) introduced by Baez and Dolan to describe laws and coherence cells in higher-dimensional categories. In a nutshell, they are trees of trees of trees of trees of... These shapes are attractive because of their simple nature and easy to find "in nature", but their highly inductive definition makes them difficult to manipulate efficiently.

This thesis develops the theory of opetopes along three main axes. First, we give it clean and robust foundations by carefully detailing the approach of Kock–Joyal–Batanin–Mascari, based on polynomial monads and trees. Starting with the identity functor on sets, and repeatedly applying the Baez–Dolan construction, we obtain a sequence of polynomial monads whose operations are trees over previous monads. This process generates opetopes and captures their recursive nature. We then introduce the important formalism of higher addresses, which allows to "walk" through opetopes and all their lower-dimensional faces simultaneously, in order to reach a given node or edge. This allows a more thorough investigation of the structure of the generating polynomial monads, and gives us important insights on the tree calculi they encode, in this case the natural operations on opetopes, such as grafting and substitution.

The second part of this thesis deals with computerized manipulation of opetopes. We introduce two syntactical approaches to opetopes and opetopic sets. In each, opetopes are represented as syntactical constructs whose well-formation conditions are enforced by corresponding sequent calculi. In the first one, called the named approach, we express the compositional nature of opetopes using a specifically crafted kind of term. In the second, called unnamed approach, we just give a syntactical account of the tree structure of opetopes using higher addresses. This latter approach is closer to the definition of the first part of this thesis, whereas the former is more human-friendly.

Lastly, in the third part of this thesis, we focus on the algebraic structures that can naturally be described by opetopes. So-called opetopic algebras include categories, planar operads, and Loday's combinads over planar trees. We start by extending the generating polynomial monads to categories of truncated opetopic sets, and let opetopic algebras be simply algebras over these extensions. Later, we introduce the category of opetopic shapes, and using the theory of parametric right adjoint monads of Weber, we show that opetopic algebras can also be understood as presheaves over opetopic shapes having the unique lifting property against a certain set of maps. We then turn our attention to the notion of weak opetopic algebras and their homotopy theory. Following the literature in the simplicial and dendroidal cases, we introduce three models: ∞-opetopic algebras, complete Segal spaces, and homotopy-coherent opetopic algebras. We show that classical results of Rezk, Joyal–Tierney, and Horel (for ∞-categories), and Cisinski–Moerdijk (for ∞-operads) can be reformulated and generalized in our setting. In particular, these models are equivalent.

Keywords. Opetope, opetopic set, polynomial monad, polynomial tree, Baez–Dolan construction, sequent calculus, opetopic category, opetopic algebra, operad, combinad.

Résumé

Les opétopes sont des formes (tout comme les globes, les cubes, les simplex, les dendrex, etc.) inventées par Baez et Dolan afin de pouvoir décrire les cellules de cohérence des catégories supérieures faibles. Informellement, ce sont des arbres d'arbres d'arbres d'arbres... Ces formes sont séduisantes car elles sont intrinsèquement simples et apparaissent fréquemment en pratique. Cependant, leur nature inductive les rend difficile à manipuler efficacement.

Cette thèse développe la théorie des opétopes selon trois axes. Premièrement, nous formulons une définition propre et robuste, en suivant minutieusement l'approche de Kock–Joyal–Batanin–Mascari, basée sur la théorie des monades et des arbres polynomiaux. En itérant la construction de Baez–Dolan sur le foncteur identité sur la catégorie des ensembles, nous obtenons une suite de monades polynomiales, et leurs opérations sont des arbres sur des monades précédentes. Ce processus génère les opétopes et cerne leur structure récursive. Ensuite, nous présentons la notion d'adresse supérieure, qui nous permettent de "naviguer" dans les opétopes et leurs faces afin d'atteindre un nœud ou une arrête donné. Ce formalisme permet une étude plus poussée de la structure des monades polynomiales et des opérations sur les arbres qu'elles encapsulent. Dans notre cas, il s'agit des opérations naturelles sur les opétopes, par exemple les greffes et les substitutions.

Ensuite, nous introduisons deux systèmes syntaxiques pour décrire les opétopes et les ensembles opétopiques, avec pour objectif leur implémentation informatique. Dans chacune de ces deux approches, les opétopes sont encodés par des expressions dont la validité est assurée par des calculs des séquents correspondants. Dans la première, appelée approche nommée, nous décrivons la structure compositionnelle des opétopes en utilisant un certain type de terme. La seconde, appelée approche anonyme, se concentre sur une représentation syntaxique simple des arbres sous jacents aux opétopes. Bien que plus proche de la définition polynomiale, sa syntaxe est moins facile à lire que celle de l'approche nommée.

Enfin, dans la dernière partie de cette thèse, nous étudions les structures algébriques que les opétopes décrivent. Ces structures, que nous appelons algèbres opétopiques, généralisent les catégories, les opérades planaires, et les combinades des arbres planaires de Loday. Nous commençons pas étendre les monades génératrices à des catégories d'ensemble opétopiques tronqués, de sorte à ce que les algèbres opétopiques ne soient simplement que des algèbres sur ces extensions. Nous introduisons la catégories des formes opétopiques, et en mettant à contribution la théorie des adjoints à droite paramétriques de Weber, nous montrons que les algèbres opétopiques peuvent se comprendre comme des préfaisceaux satisfaisant certaines conditions de relèvement unique. Nous nous intéressons ensuite à la notion l'algèbre faible. En se basant sur les théories existantes dans le cas simplicial et dendroïdal, nous en donnons trois interprétations: les ∞-algèbres opétopiques, les espaces de Segal complets, et les algèbres opétopiques à homotopies cohérentes près. Nous montrons que certains résultats classiques de Rezk, Joyal–Tierney, et Horel (pour les ∞-catégories), et de Cisinski–Moerdijk (pour les ∞-opérades) peuvent être reformulés et généralisés dans ce cadre. En particulier, ces trois modèles sont équivalents.

Mots-clefs. Opétope, ensemble opétopique, monade polynomiale, arbre polynomial, construction de Baez–Dolan, calcul de séquent, catégorie opétopique, algèbre opétopique, opérade, combinade.

Contents

Introduction

HIGHER structures appear increasingly in a variety of contexts, such as mathematical physics, algebraic topology, knot theory, and representation theory, with the aim of providing finer and finer invariants for the objects under study, like spaces or groups. They arise naturally when *weakening structures*, for example relaxing a binary operation that is associative (typically, a category) into one where associativity holds only *up to homotopy* (typically, a bicategory). One then needs to consider the homotopies between homotopies etc., and what *coherence laws* need to be enforced.

This thesis takes place in one of the formalisms that have been proposed to define such higher structures: the *opetopes* and *opetopic sets* of Baez and Dolan [BD98], further studied by Cheng [Che03a] [Che04b], Leinster [Lei04], Kock et. al. [KJBM10], and others. Opetopes are shapes (akin to cubes, globules, simplices, etc.) originally introduced to describe laws and coherence in higher category theory. Their name reflects the fact that they encode the possible shapes for higher-dimensional operations: they are *ope*ration poly*topes*. More concretely, while commutative diagrams (e.g. commutative squares) are a convenient representation of relations in 1-categories, and commutative squares with 2-cells for 2-categories, opetopes provide a formal account of *pasting diagrams* of cells in every dimension.

TOWARDS HIGHER STRUCTURES

Let us take a leisurely dive into a classical example of higher structure. Let X be a topological space. The problem at hand is to construct an algebraic structure that fully captures the homotopical data of X, i.e. its structure up to continuous deformation.

Let us start by defining its *fundamental groupoid* $\Pi_1 X$. As a first attempt, the objects are the points of X, and if $x, y \in X$, a 1-*cell* $u : x \longrightarrow y$ is a *path* from x to y, i.e. a continuous map $u : [0,1] \longrightarrow X$ such that $u(0) = x$ and $u(1) = y$. If we are given two paths $f : x \longrightarrow y$ and $g : y \longrightarrow z$, then we can *concatenate* (or *compose*) them to form a new path $g \circ_0 f : x \longrightarrow z$ by means of the formula

$$(g \underset{0}{\circ} f)(t) \coloneqq \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ g(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

We shall abbreviate $gf \coloneqq g \circ_0 f$. Graphically, this is represented by the diagram

$$\underset{0}{\cdot} \xrightarrow{\quad f \quad} \underset{\frac{1}{2}}{\cdot} \xrightarrow{\quad g \quad} \underset{1}{\cdot}$$

Unfortunately, concatenation is not associative, in that if we have yet another path $e : w \longrightarrow x$, then $g(fe) \neq (gf)e$, as showed by the following formulas

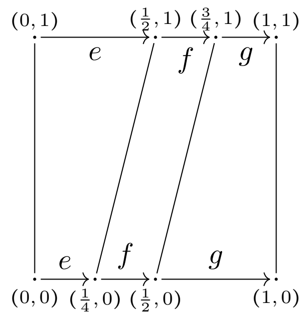
$$(g(fe))(t) = \begin{cases} e(4t) & \text{if } 0 \leq t \leq \frac{1}{4}, \\ f(4t-1) & \text{if } \frac{1}{4} \leq t \leq \frac{1}{2}, \\ g(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases} \qquad ((gf)e)(t) = \begin{cases} e(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ f(4t-2) & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4}, \\ g(4t-3) & \text{if } \frac{3}{4} \leq t \leq 1. \end{cases}$$

Graphically,

$$\underset{0}{\cdot} \xrightarrow{\quad e \quad} \underset{\frac{1}{4}}{\cdot} \xrightarrow{\quad f \quad} \underset{\frac{1}{2}}{\cdot} \xrightarrow{\qquad g \qquad} \underset{1}{\cdot} \qquad \neq \qquad \underset{0}{\cdot} \xrightarrow{\qquad e \qquad} \underset{\frac{1}{2}}{\cdot} \xrightarrow{\quad f \quad} \underset{\frac{3}{4}}{\cdot} \xrightarrow{\quad g \quad} \underset{1}{\cdot}$$

Although $g(fe)$ and $(gf)e$ have the same image in X, the "speed" at which they go through it is not the same. This can be mediated by the means of a homotopy $A_{e,f,g} : g(fe) \longrightarrow (gf)e$, called *coherence cell* (or simply *coherence*), which in essence "readjusts the speed":

$$A_{e,f,g}(t,u) = \begin{cases} e((4-2u)t) & \text{if } 0 \leq t \leq \frac{1+u}{4}, \\ f(4t-1-u) & \text{if } \frac{1+u}{4} \leq t \leq \frac{2+u}{4}, \\ g((2+2u)t-1-2u) & \text{if } \frac{2+u}{4} \leq t \leq 1. \end{cases}$$



In the right diagram, t and u are represented in the horizontal and vertical coordinates respectively. The vertical and slanted lines represents the points (t,u) for which $A_{e,f,g}(t,u) = e(0)$, $f(0)$, $g(0)$, or $g(1)$, respectively. As such, $A_{e,f,g}$ *witnesses* the weak associativity of the concatenation. Since we are studying associativity of concatenation, coherences are also called *associators*.

One would be tempted to "collapse" all these homotopies, i.e. consider homotopy classes of paths instead of individual paths. So we improve our attempt to define $\Pi_1 X$, and now, if $x, y \in X$, then

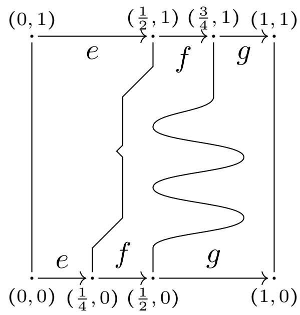
$$\Pi_1 X(x,y) := \{u : [0,1] \longrightarrow X \mid u(0) = x, u(1) = y\} /\simeq$$

where $\simeq$ is the homotopy relation. In this setting, the concatenation is strictly associative, as $A_{e,f,g}$ witnesses the fact that $g(fe)$ and $(gf)e$ are in the same homotopy class. One can also show that the constant path $1_x : [0,1] \longrightarrow X$ given by $1_x(t) = x$ for all $t \in [0,1]$ is a neutral element, and that given $u : [0,1] \longrightarrow X$, the path $\bar{u}$ such that $\bar{u}(t) = u(1-t)$ is a two sided inverse of u. Thus, $\Pi_1 X$ is a (1-)groupoid that encodes the 1-dimensional homotopical data of X. For example, if $x \in X$, then $\Pi_1(x,x)$ is the usual fundamental group $\pi_1(X,x)$ of X based at x. Formally, we have the following result:

Theorem 0.0.1 (Homotopy hypothesis for groupoids)**.** *1-groupoids model homotopy 1-types. More precisely, if* $\mathrm{ho}\,\mathcal{G}\mathrm{pd}_1$ *is the category of* 1*-groupoids localized at the equivalences of categories, and if* $\mathrm{ho}\,\mathcal{T}\mathrm{op}_1$ *is the category of topological spaces whose* n*-homotopy groups are trivial for all* $n \geq 2$*, localized at the weak homotopy equivalences, then* Π_1 *is an equivalence of categories* $\mathrm{ho}\,\mathcal{T}\mathrm{op}_1 \longrightarrow \mathrm{ho}\,\mathcal{G}\mathrm{pd}_1$.

This means that Π_1 completely and faithfully captures the 1-dimensional homotopical data of X. However, since we collapsed all higher homotopies, tremendous information is lost. For example, if $\mathbb{S}^m$ is the Euclidean m-sphere, then $\Pi_1\mathbb{S}^m$ is a contractible groupoid whenever $m \geq 2$, and thus Π_1 cannot distinguish higher-dimensional spheres.

In order to capture higher-dimensional information about the space X, let us instead construct its *fundamental* 2*-groupoid* Π_2X. As for Π_1X, its objects are the points of X, its morphisms are the paths, but this time, we do not collapse homotopies between paths, and instead consider them as 2-cells. Formally, as a first attempt, if u and v are two paths with the same endpoints, and if H is an endpoint-preserving homotopy from u to v, then we have a corresponding 2-cell $u \longrightarrow v$ in Π_2X. As previously stated, the composition of paths is not associative, but holds "up to homotopy"[1]: for e, f, and g as above, we have $g(fe) \simeq (gf)e$. There is a canonical witness of that homotopy, denoted by $A_{e,f,g}$, which we constructed above. Note that there is infinitely many such witnesses $g(fe) \longrightarrow (gf)e$, for example



but the construction of $A_{e,f,g}$ stands as the most natural, and more importantly, can be specified independently of e, f and g. Much like paths, homotopies can be concatenated, but yet again, this operation is not associative. It can be made so by collapsing all the homotopies above dimension 2, and the 2-cells of Π_2X should in fact be homotopy classes of 2-cells. This certainly seems like a reasonable definition, but in order to improve theorem 0.0.1, we need to define the kind of abstract structure Π_2X is. A natural attempt would be:

Definition 0.0.2 (Weak 2-groupoid, tentative)**.** A weak 2-groupoid[2] $\mathcal{G}$ is made of

Data.

(1) a set $\mathcal{G}_0$ of objects (or 0-cells);

(2) for $x, y \in \mathcal{G}_0$, a 1-groupoid $\mathcal{G}(x, y)$, whose objects are called 1-cells, and morphisms are called 2-cells; if f is an object in $\mathcal{G}(x, y)$, we write $f : x \longrightarrow y$; composition in $\mathcal{G}(x, y)$ is denoted by $\circ_1$, but omitted if the context allows;

Operations. for $x, y, z \in \mathcal{G}_0$,

(1) a functor $- \circ_0 - : \mathcal{G}(y, z) \times \mathcal{G}(x, y) \longrightarrow \mathcal{G}(x, z)$ such that for all 2-cells A, B, C, and D, we have $(A \circ_1 B) \circ_0 (C \circ_1 D) \cong (A \circ_0 C) \circ_1 (B \circ_0 D)$, provided that all composites are well-defined;

(2) a distinguished 1-cell $\mathrm{id}_x : x \longrightarrow x$, called the *identity* of x;

(3) a functor $(-)^{-1} : \mathcal{G}(x, y) \longrightarrow \mathcal{G}(y, x)$;

[1] In fact, unitality and invertibility only hold up to homotopy too, but we shall restrict our attention to associativity.

[2] The model we have chosen here is that of *bigroupoids* [HKK01], see also [Rob16].

Coherence cells. for $e : w \longrightarrow x$, $f : x \longrightarrow y$, and $g : y \longrightarrow z$,

(1) an isomorphism $A_{e,f,g} : g(fe) \longrightarrow (gf)e$ natural in all variables, called *associator*;
(2) isomorphisms $f \circ_0 \mathrm{id}_x \longrightarrow f$ and $\mathrm{id}_y \circ_0 f \longrightarrow f$, natural in all variables, called *unitors*;
(3) isomorphisms $f \circ_0 f^{-1} \longrightarrow \mathrm{id}_y$ and $f^{-1} \circ_0 f \longrightarrow \mathrm{id}_x$, natural in all variables, called *reversors.*

Unfortunately, this definition of 2-groupoid does not model homotopy 2-types, as there are crucial properties, called *coherence laws*, that all the $\Pi_2 X$ have, but which are not consequences of the axioms of definition 0.0.2. Consider a sequence of four concatenable paths:

$$\cdot \xrightarrow{d} \cdot \xrightarrow{e} \cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$$

Using the homotopies $A_{e,f,g}$ defined above, there are two natural ways to go from $g(f(ed))$ to $((gf)e)d$, written in left diagram, called *Mac Lane's pentagon*[3] [ML98, paragraph VII.1]:

$$\begin{tikzcd}
 & g(f(ed)) \arrow[dl, swap, "A_{d,e,f}"] \arrow[dr, "A_{d,e,gf}"] & \\
g((fe)d)) \arrow[d, swap, "A_{d,fe,g}"] & & (gf)(ed) \arrow[d, "A_{ed,f,g}"] \\
(g(fe))d \arrow[rr, swap, "A_{e,f,g}"] & & ((gf)e)d
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
 & g(f(ed)) \arrow[dl, swap, "A_{d,e,f}"] \arrow[dr, "A_{d,e,gf}"] \arrow[d, phantom, "P_{d,e,f,g}"] & \\
g((fe)d)) \arrow[d, swap, "A_{d,fe,g}"] & {} & (gf)(ed) \arrow[d, "A_{ed,f,g}"] \\
(g(fe))d \arrow[rr, swap, "A_{e,f,g}"] & & ((gf)e)d
\end{tikzcd}$$

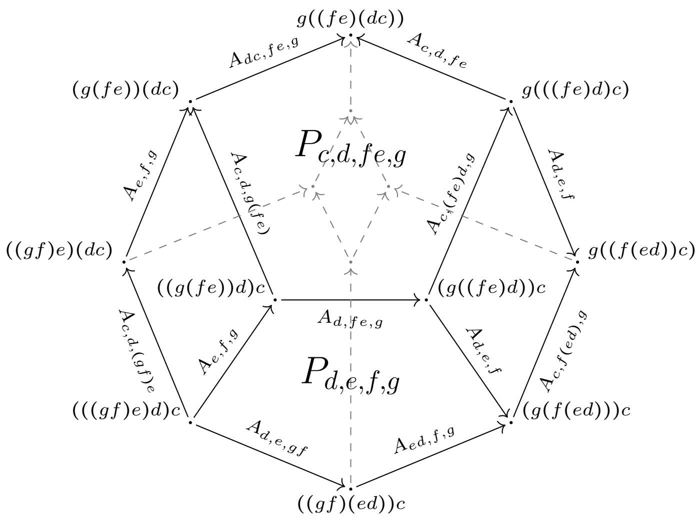
These two homotopies $g(f(ed)) \longrightarrow ((gf)e)d$ are not equal, but there is a canonical homotopy $P_{d,e,f,g} : A_{ed,f,g}A_{d,e,fg} \longrightarrow A_{e,f,g}A_{d,fe,g}A_{d,e,f}$ (the formula of which is pretty long), thus "filling" the pentagon as on the right above. Therefore, in our definition 0.0.2 of weak 2-groupoid, the associators need to be chosen so that Mac Lane's pentagon above *commutes.* Likewise, identities and inverses need their own coherence laws, which are fairly easy to find in the 2-dimensional case [HKK01, definitions 1.1 and 1.2]. It turns out that this is the only missing part of our definition, and the following result holds:

Theorem 0.0.3 (Homotopy hypothesis for 2-groupoids [CHR12, theorem 7.1])**.** *Weak 2-groupoids model homotopy 2-types.*

If one wants to consider 3-groupoids (called *Azumaya tricategory* in [GPS95, chapter 2]), then the $P_{d,e,f,g}$'s above need to become part of the structure, and adequate coherence

[3]Whiskering of homotopies have been omitted for simplicity.

conditions have to be found, such as the K_5 *associahedron*[4]:



As the value of n increases, the process goes on:

(1) in each dimension, we have coherence cells, i.e. canonical witnesses of the weak associativity of the concatenation operation (the A's, P's, etc.);
(2) using those coherences, we find that more complex "associativity problems" have natural but different solutions (akin to Mac Lane's pentagon and K_5);
(3) those different solutions can be mediated with even higher coherences.

The challenge is to find a general definition of n-groupoid and even ∞-groupoid, i.e. to find all these coherence cells and coherence conditions, or at least a tractable description thereof.

This approach stems from an intuitive idea but its complexity quickly becomes unmanageable. Other works use entirely different structures to model homotopy (n- or ∞-)types. For example, Kan complexes (which are simplicial sets that satisfy some lifting condition) are known to be models [GJ09, theorem 11.4]. The use of presheaves, using the underlying category as a description of "cell shapes", dates back to Grothendieck's *Pursuing Stacks* [Gro83] and the notion of *test category*. This program and the study of test categories has been largely pursued (!) since, see e.g. [Cis06] [Jar06] [Mal09] [CM11a] [Ara12] [ACM19].

More generally, one would like a definition of weak higher categories (i.e. where cells are not necessarily invertible) that encompasses the idea of higher groupoids. The approach we exposed, whereby a weak $(n+1)$-category is some sort of 1-category that is "weakly enriched" over weak n-categories, has been investigated and found applications for small values of n, see e.g. [B67] [Pow89] for $n = 2$, [GPS95] [Gur06] for $n = 3$, and [Tri06] for $n = 4$. Quasi-categories [BV73] [Joy08] [Lur09] are by far the most popular, and have been shown to be adequate models for $(\infty, 1)$-categories, i.e. categories whose cells of dimension ≥ 2 are weakly invertible. Reviews of the various approaches to higher categories can be found in [CL04] [CG07] [CG11].

[4] The back of the associahedron is represented by the dashed arrows, but not labeled for clarity. The pentagon faces are instances of the P coherence cells, and the diamond faces are just exchanges of independent instances of A (or the canonical witness thereof).

OPETOPES

We now present a different perspective on the coherence problems above, that will lead us to opetopes. First, let us represent a homotopy $H : u \longrightarrow v$ between two paths with the same endpoints as follows:

$$\begin{tikzcd} \cdot \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow H"] & \cdot \end{tikzcd}$$

In the same manner, if g and f are concatenable, and if gf and v have the same endpoints, then a homotopy $H : gf \longrightarrow v$ can be represented as

$$\begin{tikzcd} & \cdot \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow H"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "v"'] & {} & \cdot \end{tikzcd}$$

This diagram can be read as "H is a homotopy from a concatenation of g and f to v", although we do not choose an actual representative of the concatenation gf. Similarly, for suitable paths g, f, e, and v, one may consider a homotopy

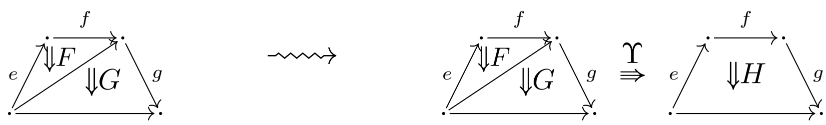
$$\begin{tikzcd} & \cdot \arrow[r, "f"] \arrow[dr, phantom, "\Downarrow H"] & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "e"] \arrow[rrr, "v"'] & & & \cdot \end{tikzcd}$$

going from a concatenation of g, f, and e, to v. But what *is* a good candidate for gfe? We shall say that v is *a* concatenation of g, f, and e, if the homotopy H above (or rather, if there exists a homotopy like H that) meets a certain universality criterion. Intuitively, for H to be a *concatenator*, i.e. a witness of the fact that v is a concatenation of g, f, and e, it should be weakly initial among all homotopies starting from g, f, and e. So if $G : gfe \longrightarrow v'$ is any homotopy, then there exists a $C : v \longrightarrow v'$ (itself satisfying some universality criteria) such that G is a concatenation of C and H:

$$G \simeq CH.$$

To summarize, in this approach, rather than trying to define concatenations directly (which as we saw gives rise to many candidates for the same problem, which all need to be mediated by higher coherences), we define them by universal property. This is what Hermida calls "coherence via universality" [Her01]. Note that in the equation above, we require that G is a concatenation of C and H, which according to the new definition of concatenation, means that there exists a witness $\Gamma : CH \longrightarrow G$ which is itself universal. The definitions of concatenation and universal cell are thus mutually coinductive.

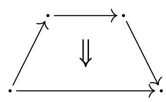
This process can be generalized to arbitrary dimensions. For example, a good notion of associator of e, f, and g can be retrieved as a concatenator Υ of the pasting diagram on the left:

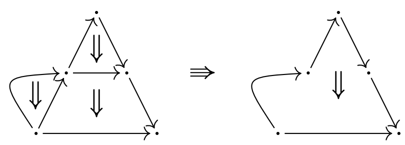


where F and G are both universal. With a more precise statement of this theory, one could prove that H is also universal, i.e. a concatenator of e, f, and g. So rather than having an associator $A : g(fe) \longrightarrow (gf)e$, we instead obtain a scheme to construct a universal cell $\Upsilon : g(fe) \longrightarrow gfe$, and similarly, $\Upsilon' : (gf)e \longrightarrow gfe$. Note that Υ and Υ' cannot be concatenated, and the Mac Lane pentagon (along with the associahedra) cannot appear as a result of competing solutions to universal problems. However, universal cells mediating the two do occur, and have to be studied. This method of producing coherence cells has two important properties:

(1) higher coherence cells are parametrized by pasting diagrams of lower coherence cells;
(2) all coherence cells are *many-to-one*, meaning that their codomain is always formed of a single lower cell.

Opetopes serve as combinatorial devices that describe the *shape* of those pasting diagrams of higher coherence cells. In other words, they are *generic* pasting diagrams. Here is an example of a 2-dimensional and 3-dimensional opetope, respectively:





In this thesis, we study the theory of opetopes: their subtle combinatorics, ways to describe and formalize them, the algebraic structures they encode, as well as related questions in homotopy theory.

PLAN

This work is divided in three parts. Each has its own introductory chapter, so we keep this overview short.

Part I: Opetopes. We begin in chapter 2 with some results regarding polynomial trees, polynomial monads, and the Baez–Dolan construction. Then, in chapter 3, we present the approach of Kock et. al. [KJBM10], which gives a very concise definition of opetopes, albeit a dreadfully inductive one. We also introduce the category $\mathbb{O}$ of opetopes, whose generating morphisms correspond to geometrical face embeddings. The formalism of higher addresses (section 3.3, which relies on notations and definitions introduced in chapter 2) allows us to transcribe this intuition in an elegant way. We make use of this in chapter 4, where we provide an alternative proof of the equivalence between opetopic sets and many-to-one polygraphs. Not only does it validate the geometrical intuition behind our definition of $\mathbb{O}$, but it is a first compelling application of this framework. The proof itself is not immediate, but significantly shorter than in the previous state of the art [HMP00] [HMZ02] [Che04b] [HMZ08].

Part II: Syntax. This thesis intends to promote opetopes as a convenient foundation for higher category theory. In particular, a representation which is adapted to computer manipulations and proofs is desirable. In the second part, we study syntactical descriptions of opetopes. The dichotomy between pasting diagrams and trees, discussed in chapter 3, gives rise to two very different approaches.

(1) In the *named approach* (chapters 6 and 7), we use the former point of view, and represent opetopes and cells in opetopic sets as well-typed terms (in some context). The well-typedness is enforced by the sequent calculus $\textsc{Opt}^!$ for opetopes, and $\textsc{OptSet}^!$ for opetopic sets.
(2) In the *unnamed approach* (chapters 8 and 9), we leverage the tree-like structure of opetopes, rather than considering them as pasting diagrams. Akin to the named approach, opetopes and cells of opetopic sets are represented by syntactical constructs we call *preopetopes*, that are constrained by the corresponding derivation systems: $\textsc{Opt}^?$ for opetopes, and $\textsc{OptSet}^?$ for opetopic sets. In this setting, we also describe those *universality conditions* mentioned earlier, by translating the rules of Baez–Dolan [BD98] and Finster [Fin16] in our syntax (see section 9.4).

Part III: Algebras. In the third part of this thesis, we demonstrate that the theory of opetopes is suitable for the study of higher structures. We start by introducing *opetopic algebras* in chapter 11, which are algebraic structures whose operations have *higher dimensional tree-like arities*, and whose underlying generators and relations are encoded by opetopic sets. This encoding is formalized by the means of a reflective adjunction

$$h : \mathcal{P}\mathrm{sh}(\mathbb{O}) \underset{\perp}{\rightleftarrows} \mathcal{A}\mathrm{lg} : M$$

between opetopic sets and the category $\mathcal{A}\mathrm{lg} = \mathcal{A}\mathrm{lg}_{k,n}$ of *k-colored n-dimensional opetopic algebras*. In particular, it exhibits $\mathcal{A}\mathrm{lg}$ as a localization of $\mathcal{P}\mathrm{sh}(\mathbb{O})$ (or equivalently, as a projective sketch over $\mathbb{O}$). This is the *nerve theorem for* $\mathbb{O}$ (see theorem 11.2.33). If $k = n = 1$ we recover the "free 1-category" adjunction $(-)^* : \mathcal{G}\mathrm{raph} \underset{\perp}{\rightleftarrows} \mathcal{C}\mathrm{at} : U$. However, when it comes to presheaf models of categories, simplicial sets are more adequate than graphs. Likewise, in the opetopic setting, we define the category $\mathbb{\Lambda} = \mathbb{\Lambda}_{k,n}$ of *opetopic shapes*, which turns out to be a more appropriate shape theory. The category $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ also enjoys a nerve theorem:

$$\tau : \mathcal{P}\mathrm{sh}(\mathbb{\Lambda}) \underset{\perp}{\rightleftarrows} \mathcal{A}\mathrm{lg} : N.$$

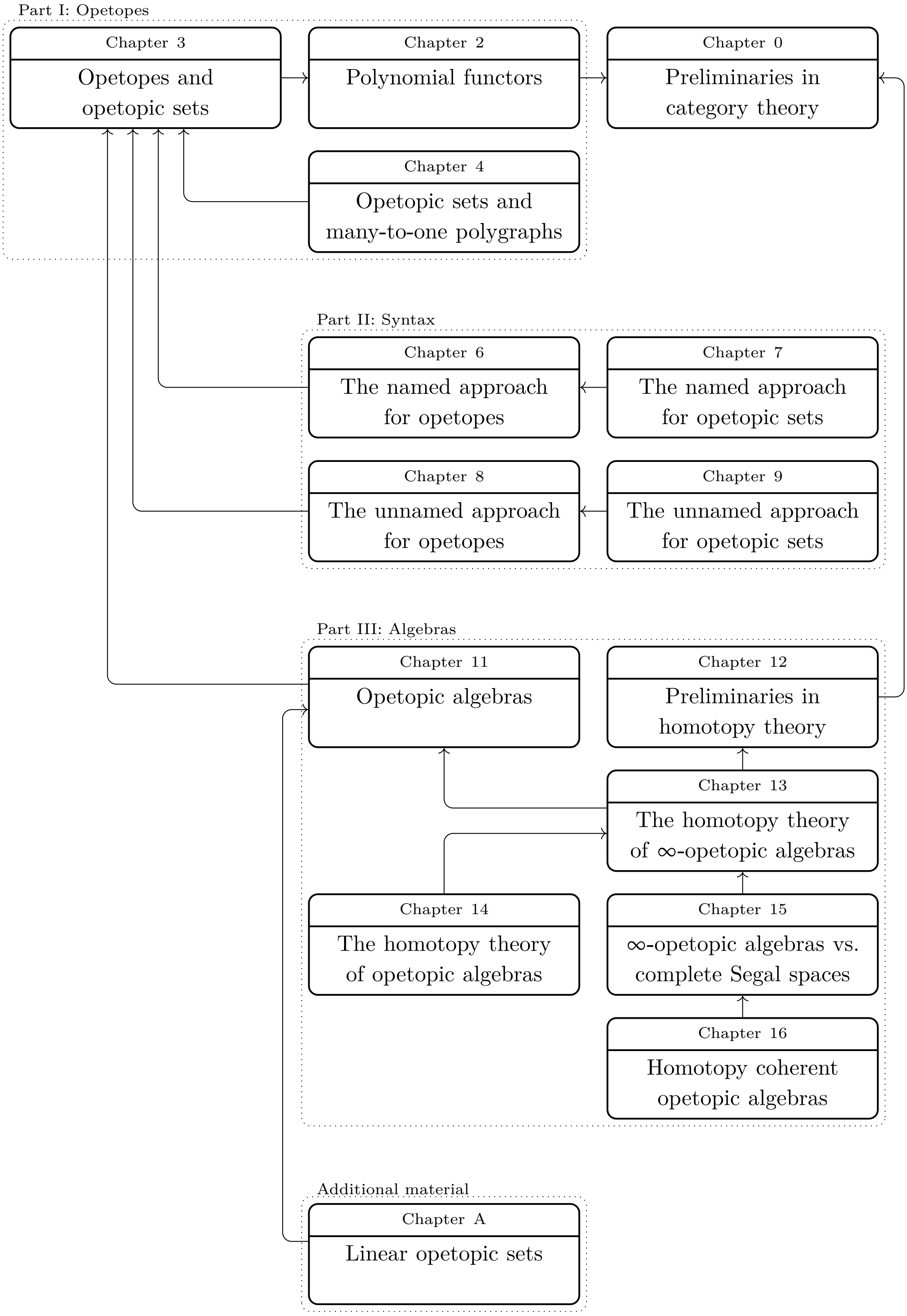
As promised, we recover simplicial sets in the case $k = n = 1$, i.e. $\mathbb{\Lambda}_{1,1} = \mathbb{\Delta}$.

We then turn our attention to weak opetopic algebra, where associativity and unitality only hold up to coherent homotopy (in this case, coherent higher cells). In chapter 13, we construct a model structure *à la Cisinski* [Cis06] on the category $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$, which subsumes Joyal's model structure for quasi-categories [Joy08] [Ber18] and Cisinski–Moerdijk model structure for ∞-operads [CM11b] in the planar case. We then show in chapter 15 that those coherent homotopies can be modeled by simplicial methods, i.e. that if $k = 1$, then there is a Quillen equivalence

$$\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})_\infty \underset{\perp}{\overset{\sim}{\rightleftarrows}} \mathcal{S}\mathrm{p}(\mathbb{\Lambda})_{\mathrm{Rezk}}$$

between the model structure of chapter 13 and the *Rezk structure* on the category $\mathcal{S}\mathrm{p}(\mathbb{\Lambda})$ of simplicial presheaves over $\mathbb{\Lambda}$. This generalizes the results of Joyal and Tierney [JT07] for quasi-categories, and of Cisinski and Moerdijk [CM13] for planar operads. Lastly, in chapter 16, we provide another model for ∞-algebras. Following the work of Horel [Hor15], it is based on the category $\mathbb{I}\mathcal{A}\mathrm{lg}$ of opetopic algebras internal to simplicial sets, instead of simplicial presheaves. We generalize the methods of this article in order to construct the *Horel structure* $\mathbb{I}\mathcal{A}\mathrm{lg}_{\mathrm{Horel}}$, that we then localize it to obtain the desired model $\mathbb{I}\mathcal{A}\mathrm{lg}_{\mathrm{Rezk}}$. It is related to $\mathcal{P}\mathrm{sh}(\mathbb{A})_\infty$ via a zig-zag of Quillen equivalences.

Figure 1: Chapter dependency graph



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I would not be where I am today without the love and unyielding support of my family. I am extremely grateful to my wonderful wife Amanda, my caring parents Christine and Tuyen, and my younger sister Delphine.

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Apparently I can write what I want here. I would like to thank some of the previous occupants of office 3033 for cleaning behind them like any reasonable adult would. I would like to thank my LaTeX compiler for its clear and easy to understand error messages. I extend my gratitude and admiration to Dr. Alexis Lemaire and his brilliant thesis [Lem10] for enlightening me about the real value of a PhD. I would like to thank Vinci for caring about my safety in the unlikely event of a fire.

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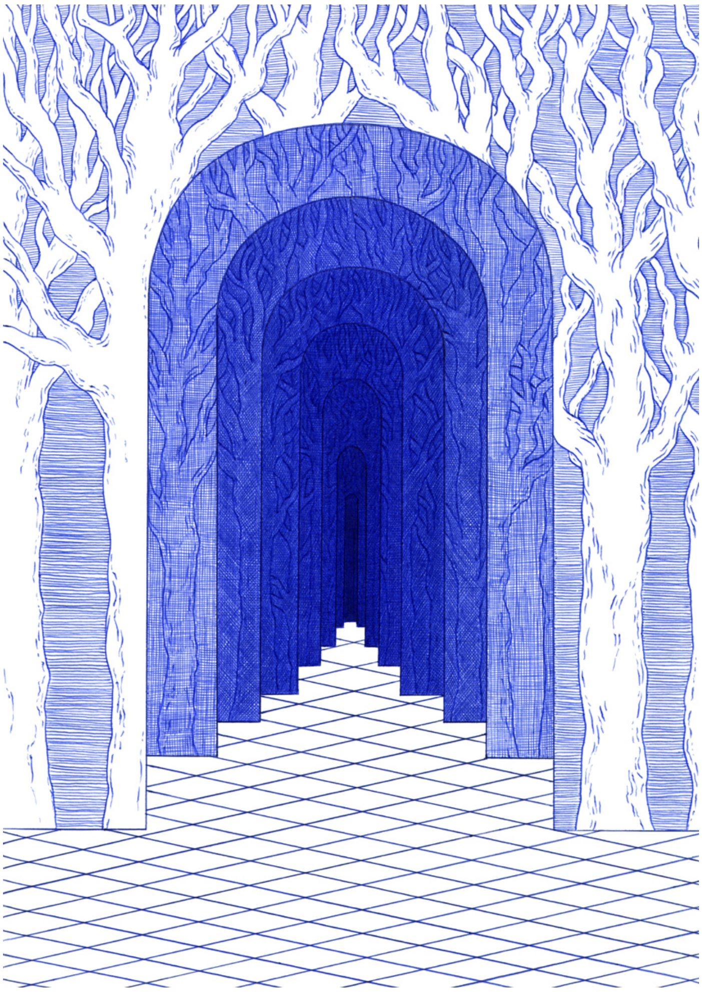


The Hall of Sleep by Kevin Lucbert

Chapter Zero

Preliminaries in category theory

IN this first chapter, we recall some notions and results in category theory, which shall be used implicitly most of the time. We assume that the reader is familiar with elementary category theory. Good references on the matter can be found in [ML98] [Awo10] [Bor94a] [Rie17].

0.1 GENERALITIES

Notation 0.1.1. We write $\mathcal{C}\mathrm{at}$ for the (large) category of small categories, i.e. that in which the class of objects and morphisms is a set, and $\mathcal{C}\mathrm{AT}$ for the large (in some universe) category of (smaller) categories. Let $\mathcal{C}$ and $\mathcal{D}$ be categories (of any size), and write $\mathcal{D}^{\mathcal{C}} := \mathcal{C}\mathrm{AT}(\mathcal{C}, \mathcal{D})$ for the category of functors from $\mathcal{C}$ to $\mathcal{D}$, and natural transformations.

Notation 0.1.2. For $n \in \mathbb{N}$, let $[n]$ be the linear order on $n+1$ elements, seen as a category:

$$[n] := \Big(0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n\Big).$$

In particular, if $\mathcal{C} \in \mathcal{C}\mathrm{AT}$, then $\mathcal{C}^{[1]}$ is the *arrow category* of $\mathcal{C}$, whose objects are the morphisms of $\mathcal{C}$, and morphisms are commutative squares.

Definition 0.1.3 (Comma category)**.** Consider two functors $F : \mathcal{A} \longrightarrow \mathcal{C}$ and $G : \mathcal{B} \longrightarrow \mathcal{C}$. The *comma category* F/G (also denoted by $(F \downarrow G)$ in the literature) is defined as the pullback

$$\begin{tikzcd}
F/G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{[1]} \arrow[d, "\partial"] \\
\mathcal{A} \times \mathcal{B} \arrow[r, "F\times G"] & \mathcal{C} \times \mathcal{C},
\end{tikzcd}$$

where ∂ maps a morphism $m : c \longrightarrow d$ to the tuple (c, d). If $\mathcal{A} = \mathcal{C}$ and F is the identity functor, then we write $F/G = \mathcal{C}/G$. If F is constant at an object $c \in \mathcal{C}$, then we write $F/G = c/G$. In particular, if $F = \mathrm{id}_{\mathcal{C}}$ and G is constant at c, then $F/G = \mathcal{C}/c$ is the usual slice category. Those notations transpose to the case where G is an identity or F is constant.

Definition 0.1.4 (3-for-2)**.** Let $\mathcal{C}$ be a category. We say that a class of morphisms $\mathsf{K} \subseteq \mathcal{C}^{[1]}$ has the *3-for-2 property* if for any pair (f, g) of composable morphisms, if two among f, g, and gf are in K, then so is the third.

Definition 0.1.5 (Cell complex)**.** Let K be a class of morphisms of $\mathcal{C}$. A *relative* K*-cell complex* is a transfinite composition of pushouts of morphisms of K. We write $\mathsf{Cell}_{\mathsf{K}}$ for the class of relative K-cell complexes. If $\mathcal{C}$ has an initial object $\varnothing$, then a K-*cell complex* is an object $X \in \mathcal{C}$ such that the initial map $\varnothing \longrightarrow X$ is a relative cell complex.

Definition 0.1.6 (Localization)**.** Let $\mathcal{C}$ be a category, and $\mathsf{K} \subseteq \mathcal{C}^{[1]}$ be a class of morphisms of $\mathcal{C}$. The *localization*, if it exists, is a category $\mathsf{K}^{-1}\mathcal{C}$ equipped with a functor $\gamma : \mathcal{C} \longrightarrow \mathsf{K}^{-1}\mathcal{C}$ mapping all morphisms of K to isomorphisms, and which is initial for this property, i.e. if $F : \mathcal{C} \longrightarrow \mathcal{D}$ also maps morphisms of K to isomorphisms, then there exists a unique $\bar{F}$ making the following triangle commute:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{D} \\ \mathsf{K}^{-1}\mathcal{C}. \arrow[ur, "\bar{F}"'] & \end{tikzcd}$$

0.2 LIFTING PROPERTIES

Definition 0.2.1 (Lifting property)**.** Let $\mathcal{C}$ be a category, and $l, r \in \mathcal{C}^{[1]}$. We say that l has the *left lifting property against* r (equivalently, r has the *right lifting property against* l), written $l \pitchfork r$, if for any solid commutative square as follows, there exists a (non necessarily unique) dashed arrow making the two triangles commute:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "l"'] & \cdot \arrow[d, "r"] \\ \cdot \arrow[r] \arrow[ur, dashed] & \cdot \end{tikzcd} \tag{0.2.2}$$

Notation 0.2.3. Let $c \in \mathcal{C}$ and $f \in \mathcal{C}^{[1]}$. If $\mathcal{C}$ has a terminal object 1, then we write $f \pitchfork c$ for $f \pitchfork (c \longrightarrow 1)$. Dually, if $\mathcal{C}$ has an initial object 0, we write $c \pitchfork f$ for $(0 \longrightarrow c) \pitchfork f$.

Let L and R be two classes of morphisms of $\mathcal{C}$. We write $\mathsf{L} \pitchfork \mathsf{R}$ if for all $l \in \mathsf{L}$ and $r \in \mathsf{R}$ we have $l \pitchfork r$. The class of all morphisms r (resp. l) such that $\mathsf{L} \pitchfork r$ (resp. $l \pitchfork \mathsf{R}$) is denoted $\mathsf{L}^{\pitchfork}$ (resp. ${}^{\pitchfork}\mathsf{R}$). We also write $\mathsf{L}^{\pitchfork}$ (resp. ${}^{\pitchfork}\mathsf{R}$) for the category spanned by objects $c \in \mathcal{C}$ such that $\mathsf{L} \pitchfork c$ (resp. $c \pitchfork \mathsf{R}$) and morphisms f such that $\mathsf{L} \pitchfork f$ (resp. $f \pitchfork \mathsf{R}$), and although this is ambiguous, the context shall always make it clear.

Definition 0.2.4 (Saturated class)**.** We say that the class of morphisms K is *saturated* if $\mathsf{K} = {}^{\pitchfork}(\mathsf{K}^{\pitchfork})$. The *saturation* of K is the smallest saturated class containing K, i.e. ${}^{\pitchfork}(\mathsf{K}^{\pitchfork})$.

Lemma 0.2.5. *If* $\mathsf{K} \subseteq \mathcal{C}^{[1]}$ *is a class of morphisms, then* $({}^{\pitchfork}(\mathsf{K}^{\pitchfork}))^{\pitchfork} = \mathsf{K}^{\pitchfork}$ *and* ${}^{\pitchfork}(({}^{\pitchfork}\mathsf{K})^{\pitchfork}) = {}^{\pitchfork}\mathsf{K}$.

Definition 0.2.6 (Orthogonality)**.** We say that l is *left orthogonal to* r (equivalently, r is *right orthogonal* to l), written $l \perp r$, if for any solid commutative square as in equation (0.2.2), there exists a *unique* dashed arrow making the two triangles commute. The relation $\perp$ is also known as the *unique lifting property*. The notations for $\pitchfork$ presented above, e.g. $\mathsf{L}^{\pitchfork}$, still make sense when $\pitchfork$ is replaced by $\perp$.

Lemma 0.2.7. *If* $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ *is an adjunction,* $f \in \mathcal{C}^{[1]}$*, and* $g \in \mathcal{D}^{[1]}$*, then* $Lf \pitchfork g$ *if and only if* $f \pitchfork Rg$*. Likewise,* $Lf \perp g$ *if and only if* $f \perp Rg$*.*

Definition 0.2.8 (Local isomorphism)**.** Let K be a class of morphisms of $\mathcal{C}$. A morphism $f \in \mathcal{C}^{[1]}$ is a K-*local isomorphism* if for all $c \in \mathcal{C}$ such that $\mathsf{K} \perp c$, we have $f \perp c$. Note that the class of K-local isomorphism is *not* ${}^{\perp}(\mathsf{K}^{\perp})$.

0.3 PRESHEAVES

Notation 0.3.1. Let $\mathcal{C}$ be a small category, and let $\mathcal{P}\mathrm{sh}(\mathcal{C}) := \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$ be the category of ($\mathcal{S}$et-valued) presheaves over $\mathcal{C}$. The *Yoneda embedding* will be denoted by $\mathsf{y}_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{C})$, or just y is the context is clear. Recall that, as the name suggests, it is an embedding, and that it exhibits $\mathcal{P}\mathrm{sh}(\mathcal{C})$ as the free cocompletion of $\mathcal{C}$. As such, if $c \in \mathcal{C}$, we sometimes write c instead of $\mathsf{y}c$.

Definition 0.3.2 (Cell counting function)**.** If $X \in \mathcal{P}\mathrm{sh}(\mathcal{C})$ is a presheaf over a small category $\mathcal{C}$, let $\#X$ be the cardinality of the sum $\sum_{c \in \mathcal{C}} X_c$. As a shorthand, we write $\#c := \#\mathsf{y}c$, for $c \in \mathcal{C}$.

Definition 0.3.3 (Shape)**.** Let $X \in \mathcal{P}\mathrm{sh}(\mathcal{C})$. An element $x \in X_c$ for some $c \in \mathcal{C}$ is called a *cell* of X of *shape* c, which we write $x^\natural = c$.

Definition 0.3.4 (Category of elements)**.** For $f : c \longrightarrow d$ a morphism in $\mathcal{C}$, we write $f : X_d \longrightarrow X_c$ instead of X_f or f^*. The *category of elements* $\mathcal{C}/X$ (also denoted by $\int_{\mathcal{C}} X$) of X has objects all the cells of X, and a morphism $f : x \longrightarrow y$ in $\mathcal{C}/X$ is a morphism $f : y^\natural \longrightarrow x^\natural$ in $\mathcal{C}$ such that $f(x) = y$. Note that for all $c \in \mathcal{C}$, the category of elements of $\mathsf{y}c$ is simply the slice $\mathcal{C}/c$.

Remark 0.3.5. In the literature, $\mathcal{C}/X$ is rather defined as the comma category $\mathsf{y}_{\mathcal{C}}/X$ (see definition 0.1.3), i.e. as the category of pairs (x, c) where $c \in \mathcal{C}$ and $x \in X_c$. Here, we consider that the shape c of x is an intrinsic data that can be retrieved using $(-)^\natural$.

Proposition 0.3.6 ([Lei04, proposition 1.1.7])**.** *There is an equivalence of categories*

$$\Psi : \mathcal{P}\mathrm{sh}(\mathcal{C})/X \xrightarrow{\simeq} \mathcal{P}\mathrm{sh}(\mathcal{C}/X)$$

where if $p : Y \longrightarrow X$ is a presheaf over X, and $x \in \mathcal{C}/X$, then $\Psi Y_x = p^{-1}(x)$.

Definition 0.3.7 (The category of simplices)**.** Let $\mathbb{\Delta}$, the *category of simplices*, be the full subcategory of $\mathcal{C}\mathrm{at}$ spanned by categories of the form $[n]$, where n ranges over $\mathbb{N}$. It admits a well-known presentation [Jar06, section 1.1] [Hov99, section 3.1], where the generators, respectively called *coface* and *codegeneracy*, are denoted by $d^i : [n-1] \longrightarrow [n]$ and $s^i : [n+1] \longrightarrow [n]$, where $0 \leq i \leq n$. Specifically, d^i is the unique increasing map that does not have $i \in [n+1]$ in its image, while s^i has it twice.

Definition 0.3.8 (Lifting problem)**.** Let $\mathcal{C}$ be a small category, $c \in \mathcal{C}$, $x : X \hookrightarrow c$ be a subpresheaf of the representable c, and $Y \in \mathcal{P}\mathrm{sh}(\mathcal{C})$. An *$x$-lifting problem* of Y is simply a morphism $f : X \longrightarrow Y$. It is *solved* is there exists a $\bar{f} : c \longrightarrow Y$ (called a *solution*) extending f:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x"'] & Y \\ c \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

It is *unsolved* if such an $\bar{f}$ does not exist.

Definition 0.3.9. Let $\mathcal{C}$ be a small category, and $\mathcal{D}$ be a category with coproducts. There is a functor $- \boxtimes - : \mathcal{P}\mathrm{sh}(\mathcal{C}) \times \mathcal{D} \longrightarrow \mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$, called the *box product induced by the coproducts of* $\mathcal{D}$, where for $X \in \mathcal{P}\mathrm{sh}(\mathcal{C})$, $c \in \mathcal{C}$, and $d \in \mathcal{D}$,

$$(X \boxtimes d)_c := \sum_{X_c} d.$$

Note that if $\mathcal{C} = [0]$ is the discrete category with one object, then this construction gives a functor $\mathcal{S}\mathrm{et} \times \mathcal{D} \longrightarrow \mathcal{D}$, where for $X \in \mathcal{S}\mathrm{et}$ and $d \in \mathcal{D}$, we have $X \boxtimes d = \sum_X d$.

Dually, if $\mathcal{D}$ has all products, then there is a natural exponentiation functor $\mathcal{P}\mathrm{sh}(\mathcal{C})^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$, where for $X : \mathcal{C} \longrightarrow \mathcal{S}\mathrm{et}$, $c \in \mathcal{C}$, and $d \in \mathcal{D}$,

$$(d^X)_c := \prod_{X_c} d.$$

It $\mathcal{C} = [0]$, then this construction gives rise to a functor $\mathcal{S}\mathrm{et}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathcal{D}$, where for $X \in \mathcal{S}\mathrm{et}$ and $d \in \mathcal{D}$ we have $d^X = \prod_X d$.

Example 0.3.10 (Simplicial set)**.** A presheaf $X \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ is called a *simplicial set*. We write X_n instead of $X_{[n]}$, and $\Delta[n]$ instead of $\mathsf{y}[n]$.

Let $\partial\Delta[n]$, the *boundary* of $[n]$, be the maximal subpresheaf of $\Delta[n]$ not containing $\mathrm{id}_{[n]} \in \Delta[n]_n$. It is the smallest subpresheaf of $\Delta[n]$ containing the cofaces

$$\left([n-1] \xrightarrow{d^i} [n]\right) \in \Delta[n]_{n-1}$$

for $0 \leq i \leq n$. We also say that it is *spanned* by the d^i's. Write $\mathsf{b}_n : \partial\Delta[n] \longrightarrow \Delta[n]$ for the natural *boundary inclusion*, and B for the set of all boundary inclusions.

For $0 \leq k \leq n$, the *k-horn* $\Lambda^k[n]$ is the maximal subpresheaf of $\partial\Delta[n]$ not containing d^k. Let $\mathsf{h}_n^k : \Lambda^k[n] \longrightarrow \Delta[n]$ be the natural *horn inclusion*, and denote by H the set of all horn inclusions. If $0 < k < n$, the horn is called *inner*, and let $\mathsf{H}_{\mathrm{inner}}$ be the set of inner horn inclusions.

Definition 0.3.11 (Anodyne extension)**.** Let $\mathsf{An} := {}^{\pitchfork}(\mathsf{H}^{\pitchfork})$ be the saturation of the set of horn inclusions. An element of An is called an *anodyne extension*. Similarly, let $\mathsf{An}_{\mathrm{inner}}$, the class of *inner anodyne extensions*, be the saturation of $\mathsf{H}_{\mathrm{inner}}$.

Definition 0.3.12. A simplicial set $X \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ is a *quasi-category* [BV73] (or *inner Kan complex*) if $\mathsf{H}_{\mathrm{inner}} \pitchfork X$, i.e. all inner horn lifting problems of X are solved.

0.4 KAN EXTENSIONS

Theorem 0.4.1. *(1) [ML98, proposition IX.5.1] If $\mathcal{D}$ is complete, then every functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{D}$ has an end. Dually, if $\mathcal{D}$ is cocomplete, then every such F has a coend.*

(2) [Lor19, theorem 1.4.1] Let $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ be two functors, where $\mathcal{D}$ has all limits. We have

$$\mathcal{D}^{\mathcal{C}}(F, G) \cong \int_{c \in \mathcal{C}} \mathcal{D}(Fc, Gc).$$

(3) [Lor19, proposition 2.2.1] (density formula) Let $\mathcal{C}$ *be a small category, and* $X \in \mathcal{P}\mathrm{sh}(\mathcal{C})$. *We have*

$$X \cong \int^{c \in \mathcal{C}} X_c \boxtimes \mathrm{y}c$$

where $- \boxtimes - : \mathcal{S}\mathrm{et} \times \mathcal{P}\mathrm{sh}(\mathcal{C}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{C})$ *is defined in definition 0.3.9.*

Notation 0.4.2. Let $\mathcal{C}$ be a small category, $X : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{S}\mathrm{et}$, and $Y : \mathcal{C} \longrightarrow \mathcal{S}\mathrm{et}$. The coend $\int^c Xc \times Yc$ admits the following simple description as a quotient in $\mathcal{S}\mathrm{et}$:

$$\int^{c \in \mathcal{C}} Xc \times Yc = \frac{\sum_{c \in \mathcal{C}} Xc \times Yc}{\sim}$$

where for $f : c \longrightarrow d$, $x \in Xd$, $y \in Yc$, we have an identification

$$(x, Yf(y)) \sim (Xf(x), y).$$

The class of a pair $(u, v) \in Xc \times Yc$ will be denoted by $u \otimes v$. Abusing notations a little bit, the equivalence relation $\sim$ above then translates to the very familiar identity $x \otimes f(y) = f(x) \otimes y$.

Definition 0.4.3 (Left Kan extensions [ML98, section X.5])**.** Consider a diagram of categories

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{D} \\ \mathcal{E} \end{tikzcd}$$

where $\mathcal{D}$ is cocomplete. The *(pointwise) left Kan extension* $\mathrm{Lan}_K F$ of F along K is the functor $\mathcal{E} \longrightarrow \mathcal{D}$ given by

$$\mathrm{Lan}_K F(e) := \operatorname*{colim}_{Ka \to e} Fa = \int^{a \in \mathcal{C}} \mathcal{E}(Ka, e) \boxtimes Fa, \tag{0.4.4}$$

where $\boxtimes$ is defined in definition 0.3.9. Using notation 0.4.2, $\mathrm{Lan}_K F(e)$ is the set of tensors $f \otimes x$, where $f : Ka \longrightarrow e$ and $x \in Fa$, subject to the identity

$$(g \cdot K\phi) \otimes y = g \otimes (K\phi)(y),$$

where $g : Kb \longrightarrow e$, $y \in Fa$, and $\phi : a \longrightarrow b$. The left Kan extension of F comes with a natural transformation $\alpha : F \longrightarrow (\mathrm{Lan}_K F) \cdot K$ which is initial among all natural transformations of the form $F \longrightarrow G \cdot K$, where $G : \mathcal{E} \longrightarrow \mathcal{D}$. Dually, if $\mathcal{D}$ is complete, the *(pointwise) right Kan extension* $\mathrm{Ran}_K F$ of F along K is the functor $\mathcal{E} \longrightarrow \mathcal{D}$ given by

$$\mathrm{Ran}_K F(e) := \lim_{e \to Ka} Fa = \int_{a \in \mathcal{C}} Fa^{\mathcal{E}(e, Ka)}, \tag{0.4.5}$$

where the exponentiation is defined in definition 0.3.9. The right Kan extension of F comes with a natural transformation $\beta : (\mathrm{Ran}_K F) \cdot K \longrightarrow F$ which is terminal among all natural transformations of the form $GK \longrightarrow F$, where $G : \mathcal{E} \longrightarrow \mathcal{D}$.

Proposition 0.4.6 ([ML98, corollary X.3])**.** *Consider a diagram of categories*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{D} \\ \mathcal{E} \end{tikzcd}$$

where K is fully faithful. If $\mathcal{D}$ is cocomplete, then the universal natural transformation $\alpha : F \longrightarrow (\operatorname{Lan}_K F) \cdot K$ is a natural isomorphism. Dually, if $\mathcal{D}$ is complete, then the universal natural transformation $\beta : (\operatorname{Ran}_K F) \cdot K \longrightarrow F$ is a natural isomorphism.

Remark 0.4.7. Throughout this thesis, we shall mainly consider left Kan extensions along the Yoneda embedding. Given

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\mathsf{y}_{\mathcal{C}}"'] & \mathcal{D} \\ \mathcal{P}\mathrm{sh}(\mathcal{C}) & \end{tikzcd}$$

and $X \in \mathcal{P}\mathrm{sh}(\mathcal{C})$, the coend of equation (0.4.4) becomes

$$\operatorname{Lan}_{\mathsf{y}_{\mathcal{C}}} F(X) = \int^{c \in \mathcal{C}} X_c \boxtimes Fc. \tag{0.4.8}$$

Since $\mathsf{y}_{\mathcal{C}}$ is dense, by proposition 0.4.6, there is an isomorphism $Fc \cong \operatorname{Lan}_{\mathsf{y}_{\mathcal{C}}} F(\mathsf{y}_{\mathcal{C}} c)$ natural in $c \in \mathcal{C}$.

Definition 0.4.9 (Nerve of a functor)**.** A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ induces a *nerve* $N_F : \mathcal{D} \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{C})$, mapping $d \in \mathcal{D}$ to the presheaf $\mathcal{D}(F-, d) \in \mathcal{P}\mathrm{sh}(\mathcal{C})$.

Definition 0.4.10 (Dense functor [ML98, section X.6])**.** A functor $G : \mathcal{A} \longrightarrow \mathcal{B}$ is *dense* if for all $b \in \mathcal{B}$ we have

$$b \cong \operatorname*{colim}_{Ga \to b} Ga.$$

Equivalently, G is dense if $\operatorname{Lan}_G G \cong \mathrm{id}_{\mathcal{B}}$.

Proposition 0.4.11. *(1) The nerve N_F is fully faithful if and only if F is dense.*
(2) If $\mathcal{D}$ is cocomplete, we have an adjunction $\operatorname{Lan}_{\mathsf{y}} F : \mathcal{P}\mathrm{sh}(\mathcal{C}) \rightleftarrows \mathcal{D} : N_F$.

Notation 0.4.12 (The classical setting)**.** Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor. Using Kan extensions, we obtain two adjunctions

$$F_! : \mathcal{P}\mathrm{sh}(\mathcal{A}) \rightleftarrows \mathcal{P}\mathrm{sh}(\mathcal{B}) : F^*, \qquad F^* : \mathcal{P}\mathrm{sh}(\mathcal{B}) \rightleftarrows \mathcal{P}\mathrm{sh}(\mathcal{A}) : F_*,$$

where $F_!$ (resp. F_*) is the left (resp. the right) Kan extension of $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{\mathsf{y}_{\mathcal{B}}} \mathcal{P}\mathrm{sh}(\mathcal{B})$ along the Yoneda embedding $\mathsf{y}_{\mathcal{A}}$, and $F^* = N_{\mathsf{y}_{\mathcal{B}} \cdot F}$ is the precomposition by F.

Unfolding definitions, for $X \in \mathcal{P}\mathrm{sh}(\mathcal{A})$ and $b \in \mathcal{B}$, we have

$$F_! X_b = \int^{a \in \mathcal{A}} X_a \times \mathcal{B}(b, Fa), \tag{0.4.13}$$

and for $Y \in \mathcal{P}\mathrm{sh}(\mathcal{B})$ and $a \in \mathcal{A}$, we have $F^* Y_a = Y_{Fa}$. Therefore,

$$F_! F^* Y_b = \int^{a \in \mathcal{A}} Y_{Fa} \times \mathcal{B}(b, Fa)$$

and the counit $\varepsilon_Y : F_! F^* Y \longrightarrow Y$ of the adjunction $F_! \dashv F^*$ simply maps a tensor $y \otimes f$ to $f(y)$, where $y \in Y_{Fa}$ and $f : b \longrightarrow Fa$.

Lemma 0.4.14 ([SGA72, exposé I, proposition 5.6])**.** *If any among F, $F_!$, and F_* is fully faithful, then so are the other two.*

0.5 LOCALLY PRESENTABLE CATEGORIES

DEFINITION

Definition 0.5.1 (Filtered category)**.** For κ a regular cardinal, a small category $\mathcal{J}$ is κ-*filtered* if every diagram of less than κ morphisms has a cocone. We say that $\mathcal{J}$ is filtered if it is $\aleph_0$-filtered. A κ-*filtered colimit* is a colimit whose domain category is a κ-filtered category. A functor is *finitary* if it preserves filtered colimits.

Definition 0.5.2 (Presentable object [AR94, definition 1.13])**.** Let κ be a regular cardinal. An object $c \in \mathcal{C}$ is κ-*presentable* if $\mathcal{C}(c, -)$ preserves κ-filtered colimits. Equivalently, if $F : \mathcal{J} \longrightarrow \mathcal{C}$ is a functor, where $\mathcal{J}$ is κ-filtered, then a morphism $c \longrightarrow \operatorname{colim} F$ factors essentially uniquely through a Fj, for some $j \in \mathcal{J}$. We say that c is *presentable* if it is κ-presentable, for some regular cardinal κ, and *finitely presentable* if it is $\aleph_0$-presentable.

Example 0.5.3. In $\mathcal{S}\mathrm{et}$, the finitely presentable sets are exactly the finite sets. Indeed, let X be a finite set, $F : \mathcal{J} \longrightarrow \mathcal{S}\mathrm{et}$ be a filtered diagram, and take a map $f : X \longrightarrow \operatorname{colim} F$. For each $x \in X$, there exists a $j_x \in \mathcal{J}$ such that $f(x) \in Fj_x$. The discrete subcategory of $\mathcal{J}$ spanned by the j_x's is finite (since X is finite), thus has a cocone, say leading to an object $j \in \mathcal{J}$. Then f factors through Fj. More generally, κ-presentable sets are those whose cardinality is less than κ.

Definition 0.5.4 (Locally presentable category [AR94, definition 1.17])**.** Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is *locally κ-presentable* if it has all colimits, and if there exists a set of κ-presentable objects that generate $\mathcal{C}$ under κ-filtered colimits. It is *locally presentable* if it is locally κ-presentable, for some regular cardinal κ, and *finitely presentable* if it is $\aleph_0$-presentable. In the latter case, we write $\mathcal{C}_{\mathrm{fin}}$ for the full subcategory spanned by finitely presentable objects.

Example 0.5.5.

1. In $\mathcal{S}\mathrm{et}$, a set X is the union of its finite subsets, i.e. the colimit of the canonical diagram $\mathcal{S}\mathrm{et}_{\mathrm{fin}}/X$. Since finite unions of finite sets are still finite, this diagram is filtered. Thus the category $\mathcal{S}\mathrm{et}$ is locally finitely presentable. The set of generating finitely presentable object can be taken to be any skeleton $\mathcal{S}\mathrm{et}_{\mathrm{fin}}$.
2. More generally, the category $\mathcal{P}\mathrm{sh}(\mathcal{C})$ of presheaves over a small category $\mathcal{C}$ is locally finitely presentable, and the finitely presentable objects are the finite colimits of representable presheaves. Note that a finitely presentable presheaf $X \in \mathcal{P}\mathrm{sh}(\mathcal{C})_{\mathrm{fin}}$ need not be finite in the sense that $\#X < \aleph_0$. For instance, no nonempty simplicial set has finitely many cells. However, if $\mathcal{C}$ is *locally finite* (i.e. all its slices $\mathcal{C}/c$ have finitely many morphisms), then $\mathcal{P}\mathrm{sh}(\mathcal{C})_{\mathrm{fin}}$ is exactly the category spanned by the presheaves with finitely many cells.

THE GABRIEL–ULMER DUALITY

Theorem 0.5.6 (Gabriel–Ulmer duality [GU71] [LP09, theorem 8])**.** *Let $\mathcal{L}\mathrm{EX}$ be the 2-category of finitely complete small categories and finitely continuous functors, and $\mathcal{L}\mathrm{FP}$ be the 2-category of locally finitely presentable categories and finitary right adjoints. Then the functor $\mathcal{L}\mathrm{EX}(-, \mathcal{S}\mathrm{et}) : \mathcal{L}\mathrm{EX}^{\mathrm{op}} \longrightarrow \mathcal{L}\mathrm{FP}$ is an equivalence of 2-categories. Consider the* *functor $T : \mathcal{L}\text{FP} \longrightarrow \mathcal{L}\text{EX}$, that maps a locally finitely presentable category $\mathcal{C}$ to a skeleton of $\mathcal{C}_{\text{fin}}^{\text{op}}$. Then T is an inverse to $\mathcal{L}\text{EX}(-, \mathcal{S}\text{et})$ up to natural equivalence.*

Corollary 0.5.7 ([LP09, theorem 10])**.** *If $\mathcal{C}$ a finitely locally presentable category, then $\mathcal{C} \simeq \mathcal{L}\text{EX}(\mathcal{C}_{\text{fin}}^{\text{op}}, \mathcal{S}\text{et})$.*

THE REPRESENTATION THEOREM

Definition 0.5.8 (Orthogonality class [AR94, definitions 1.32 and 1.35])**.** Let $\mathcal{C}$ be a category, and κ be a regular cardinal. A subcategory $\mathcal{D}$ is a *κ-orthogonality class* if there exists a class K of morphisms of $\mathcal{C}$ whose domains and codomains are κ-presentable, such that $\mathcal{D}$ is the full subcategory spanned by those objects $c \in \mathcal{C}$ satisfying $\mathsf{K} \perp c$. We say that $\mathcal{D}$ is a *small κ-orthogonality class* if K is a set; it is an *orthogonality class* if it is a λ-orthogonality class for some regular cardinal λ.

Example 0.5.9. Let $n \in \mathbb{N}$. The *spine* $S[n] \subseteq \Delta[n]$ is the colimit of the following diagram in $\mathcal{P}\text{sh}(\mathbb{\Delta})$:

$$\begin{tikzcd}
& \Delta[1] & & \Delta[1] & & \Delta[1] & \cdots & \Delta[1] & & \Delta[1] \\
\Delta[0] \arrow[ur, "d_1"] & & \Delta[0] \arrow[ul, "d_0"'] \arrow[ur, "d_1"] & & \Delta[0] \arrow[ul, "d_0"'] \arrow[ur, "d_1"] & & & & \Delta[0], \arrow[ul, "d_0"'] \arrow[ur, "d_1"] &
\end{tikzcd}$$

where there are n instances of $\Delta[1]$. In other words, it is a chain of n copies of $\Delta[1]$ glued end to end. Let $\mathsf{S} := \left\{ S[n] \hookrightarrow \Delta[n] \mid n \in \mathbb{N} \right\}$ be the set of *spine inclusions*. A simplicial set $X \in \mathcal{P}\text{sh}(\mathbb{\Delta})$ such that $\mathsf{S} \perp X$ is said to satisfy the *Segal condition*. It is well-known that nerves of categories are exactly those simplicial sets that satisfy the Segal condition [Seg68]. Therefore, $\mathcal{C}\text{at}$ is equivalent to the orthogonality class of $\mathcal{P}\text{sh}(\mathbb{\Delta})$ induced by S. Since domains and codomains of morphisms of S are finitely presentable, it is an $\aleph_0$-orthogonality class.

Proposition 0.5.10 ([GZ67, proposition 1.3])**.** *Let $F : \mathcal{C} \underset{\perp}{\rightleftarrows} \mathcal{D} : U$ be an adjunction. The following are equivalent:*

(1) the counit $\varepsilon : FU \longrightarrow \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism;
(2) the right adjoint U is fully faithful;
(3) for $\mathsf{K} := \left\{ f \in \mathcal{C}^{[1]} \mid Ff \text{ is an iso.} \right\}$ the class of maps that F maps to isomorphisms, the canonical factorization $\bar{F} : \mathsf{K}^{-1}\mathcal{C} \longrightarrow \mathcal{D}$ is an equivalence of categories.

Theorem 0.5.11 ([GU71, theorem 8.5])**.** *Let $\mathcal{C}$ be a small category, and $\mathcal{D}$ be a small κ-orthogonality class of $\mathcal{P}\text{sh}(\mathcal{C})$ induced by a set K of maps.*

(1) The inclusion $U : \mathcal{D} \longrightarrow \mathcal{P}\text{sh}(\mathcal{C})$ preserves κ-filtered colimits and has a left adjoint[1].
(2) The class of K-local isomorphisms is the smallest class of morphisms that contains K, satisfies 3-for-2, and is closed under colimits (in $\mathcal{P}\text{sh}(\mathcal{C})^{[1]}$). Further, if F is the left adjoint of U, then a morphism $f \in \mathcal{P}\text{sh}(\mathcal{C})^{[1]}$ is a K-local isomorphism if and only if Ff is an isomorphism.

[1]In [AR94, section 1.37], the left adjoint is called the *orthogonal-reflection construction*. Much in the spirit of Quillen's small object argument [Hov99, theorem 2.1.14], it takes a presheaf $X \in \mathcal{P}\text{sh}(\mathcal{A})$ and iteratively adds and collapses cells, forming a sequence $X = X^{(0)} \longrightarrow X^{(1)} \longrightarrow \cdots \longrightarrow X^{(\alpha)} \longrightarrow \cdots$ that converges in κ steps to a presheaf orthogonal to K.

Corollary 0.5.12. *With $\mathcal{C}$, $\mathcal{D}$, and K as in theorem 0.5.11, and writing $F : \mathcal{P}\mathrm{sh}(\mathcal{C}) \longrightarrow \mathcal{D}$ for the left adjoint of U, the canonical factorization $\bar{F} : \mathsf{K}^{-1}\mathcal{P}\mathrm{sh}(\mathcal{C}) \longrightarrow \mathcal{D}$ is an equivalence of categories.*

Proof. Follows from proposition 0.5.10 and theorem 0.5.11. □

Definition 0.5.13 (Projective sketch [AR94, paragraph 1.49])**.** A *projective sketch* (also called a *limit sketch*) is the datum of a category $\mathcal{S}$, a class of *distinguished* diagrams $D_i : \mathcal{J}_i \longrightarrow \mathcal{S}$, and to each such diagram, a choice of a cone $\gamma_i : c_i \longrightarrow D_i$ over it (where $c_i \in \mathcal{S}$ is considered as a constand functor $\mathcal{J}_i \longrightarrow \mathcal{S}$). A projective sketch is *small* if there is only a *set* of distinguished diagrams.

Let $\mathcal{E}$ be a category with all limits. A *model* of $\mathcal{S}$ in $\mathcal{E}$ is a functor $M : \mathcal{S} \longrightarrow \mathcal{E}$ that maps each cone γ_i to a limit cone of MD_i. A morphism of models is simply a natural transformation. If the category $\mathcal{E}$ is omitted, it is assumed to be $\mathcal{S}\mathrm{et}$.

Example 0.5.14. Lawvere theories [Law04] [HP07, definition 2.2] are projective sketches, where the distinguished diagrams are finite and discrete, and where the cones are product cones (although not all such projective sketches are Lawvere theories).

Theorem 0.5.15 (Representation theorem [AR94, theorem 1.46, corollary 1.52])**.** *Let $\mathcal{C}$ be a category and κ be a regular cardinal. The following are equivalent:*

(1) $\mathcal{C}$ is locally κ-presentable;

(2) $\mathcal{C}$ is equivalent to a κ-orthogonality class[2] *in $\mathcal{P}\mathrm{sh}(\mathcal{A})$ for some small category $\mathcal{A}$;*

(3) $\mathcal{C}$ is equivalent to an accessibly embedded full reflective subcategory of $\mathcal{P}\mathrm{sh}(\mathcal{A})$ for some small category $\mathcal{A}$, i.e. $\mathcal{C}$ is a full subcategory of $\mathcal{P}\mathrm{sh}(\mathcal{A})$ closed under κ-filtered colimits, the embedding $\mathcal{C} \hookrightarrow \mathcal{P}\mathrm{sh}(\mathcal{A})$ preserves κ-filtered colimits, and has a left adjoint;

(4) $\mathcal{C}$ is equivalent to a category of models over a small projective sketch whose distinguished diagrams have less than κ morphisms.

Proof (sketch). (1) $\Longrightarrow$ (3) Let $\mathcal{C}_\kappa$ be the full subcategory spanned by a chosen set of κ-presentable objects generating $\mathcal{C}$ under κ-filtered colimits. For $i : \mathcal{C}_\kappa \hookrightarrow \mathcal{C}$ the inclusion, the adjunction $\mathrm{Lan}_{\mathsf{y}}\, i : \mathcal{P}\mathrm{sh}(\mathcal{C}_\kappa) \rightleftarrows \mathcal{C} : N_i$ satisfies the required properties.

(3) $\Longrightarrow$ (1) Up to equivalence, we have a reflective adjunction $F : \mathcal{P}\mathrm{sh}(\mathcal{A}) \rightleftarrows \mathcal{C} : U$ where $\mathcal{A}$ is a small category, and one can check that the set $\{Fa \mid a \in \mathcal{A}\}$ generates $\mathcal{C}$ by filtered colimits.

(2) $\Longrightarrow$ (3) This is theorem 0.5.11.

(3) $\Longrightarrow$ (4) Let $i : \mathcal{C} \hookrightarrow \mathcal{P}\mathrm{sh}(\mathcal{A})$ be the embedding, L be its left adjoint. Let $\mathsf{K} \subseteq \mathcal{P}\mathrm{sh}(\mathcal{A})^{[1]}$ be the set of all morphisms $f : X \longrightarrow a$ such that $a \in \mathcal{A}$ and Lf is an isomorphism. For such an f, let D_f be the forgetful functor $\mathcal{A}/X \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{A})$, and γ_f be the obvious cocone from D_f to a. Then $\mathcal{A}^{\mathrm{op}}$ endowed with all the cones γ_f, where f ranges over K, forms a small projective sketch, and the associated category of models is equivalent to $\mathcal{C}$.

[2] In a locally presentable category, a κ-orthogonality class is necessarily a small orthogonality class. Conversely, a small orthogonality class is a κ-orthogonality class for some regular cardinal κ.

(4) $\implies$ (2) Assume that $\mathcal{C}$ is the category of models of the small projective sketch $\mathcal{S}$, with distinguished diagrams $D_i : \mathcal{J}_i \longrightarrow \mathcal{S}$ and cones $\gamma_i : d_i \longrightarrow D_i$. In $\mathcal{P}\mathrm{sh}(\mathcal{S}^{\mathrm{op}})$, consider the canonical morphisms

$$\bar{\gamma}_i : \mathsf{y}d_i \longrightarrow \lim\left(\mathcal{J}_i \xrightarrow{D_i} \mathcal{S} \xrightarrow{\mathsf{y}} \mathcal{P}\mathrm{sh}(\mathcal{S})\right)$$

and let Γ be the set of all the $\bar{\gamma}_i$'s. Then domains and codomains of maps in Γ are κ-small, and $\mathcal{C}$ is equivalent to the orthogonality class induced by Γ [3]. $\square$

Example 0.5.16. Many classical algebraic structures, such as monoids, groups, abelian groups, rings, modules over a fixed ring, etc. are models of Lawvere theories, and in particular, models of projective sketches. Thus, those categories are all locally finitely presentable.

[3]Intuitively, we identify models of $\mathcal{S}$ (those functors mapping the γ_i's to limit cones) to presheaves over $\mathcal{S}^{\mathrm{op}}$ that "see" the canonical morphisms $d_i \longrightarrow \lim D_i$ as isomorphisms.

Part I

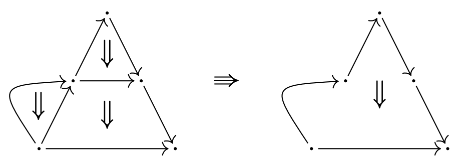
Opetopes

Chapter One

Introduction

This first part is dedicated to laying the foundations of opetope theory. Over the recent years, they have been the subject of many efforts towards providing a good definition that would allow exploring their combinatorics [Che03a] [HMP00] [Lei04]. In this thesis, we follow the approach of Kock–Joyal–Batanin–Mascari [KJBM10]. It is based on *polynomial functors* and *polynomial monads*, which we present in chapter 2. Polynomial functors are categorifications of classical polynomial functions, and have found important applications in type theory (where they are also called *containers* or *container functors*) [AAG05] [AGS12] [Koc12] and more recently in operad theory [Web14] [GHK17]. Here, they are relevant as polynomial functors that are also monads (a.k.a. polynomial monads) encapsulate the idea of "tree calculus". As we will see, opetopes are essentially trees, which makes this formalism especially adequate throughout this thesis.

In chapter 3, we present the definition of opetopes of [KJBM10]. Informally, an opetope is a higher dimensional geometrical shape that looks like this



A crucial feature is that all the cells (in any dimension) are *many-to-one*, i.e. have many inputs but a single output. In fact, the *source* of an opetope (on the left of the triple arrow above) is a well-formed pasting scheme of lower dimensional opetopes, which already hints that the definition is very inductive. Let us survey low dimensional cases.

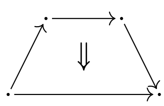
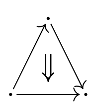
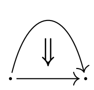
Dimension 0. By convention, there is a unique 0-dimensional opetope, called the *point*, denoted by ♦, and graphically represented by

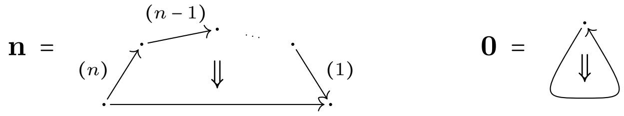
$$\cdot$$

Dimension 1. Still by convention, there is a unique 1-dimensional opetope, called the arrow, denoted by ▪, which can be represented as follows:

$$\cdot \longrightarrow \cdot$$

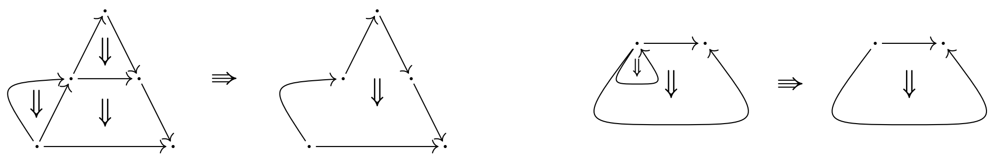
Dimension 2. Now, the induction starts. A 2-opetope is essentially a well-formed pasting diagram (or rather, a *filler* thereof) of 1-dimensional opetopes, i.e. a gluing of several instances of the arrow, glued end-to-end along the point. Examples include the following:

3 =  **2** =  **1** = 

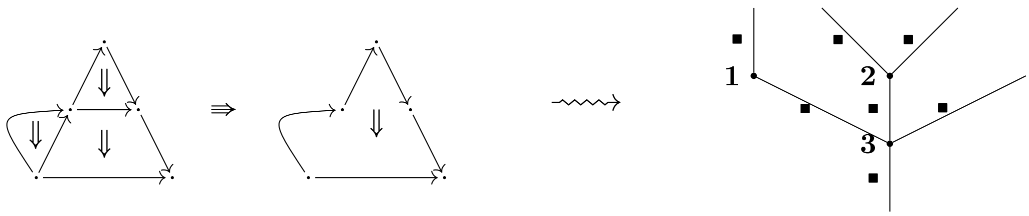


In the last one, the pasting diagram of arrows contains 0 arrows, which is perfectly valid.

Dimension 3. Likewise, a 3-opetope is completely determined by a pasting diagram of 2-opetopes:



Inspecting the 3-dimensional case more closely reveals that 3-opetopes are essentially trees whose nodes are decorated by 2-opetopes, and edges by 1-opetopes:

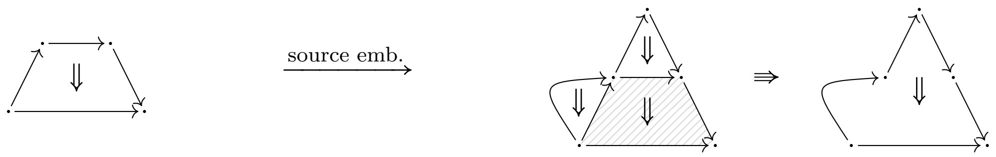


The tree on the right is essentially the Poincaré dual of the pasting diagram on the left of the triple arrow:

(1) the 2-cells become nodes, and the number of input faces of these 2-cells match the number of input edges of the corresponding nodes;
(2) the edges are decorated by the corresponding 1-dimensional cells of the pasting diagram; in this case, they are all arrows.

This is where the theory of polynomial functors and trees becomes relevant. By defining opetopes as trees, the theory presented in chapter 2 allows us to formally generate and conveniently manipulate them.

As the graphical representation suggests, opetopes carry a very geometrical notion of "face embedding". For example, the 2-opetope on the left (denoted by **3**) naturally embeds as a face of the 3-opetope on the right:



With the tree representation above, it just means that **3** decorates a node of the opetope on the right. Moreover, the face on the right of the triple arrow, which we call the *target*, is a geometrical feature as well, and has a corresponding embedding of a 2-opetope. For example:

Opetopes and these formal embeddings fit naturally in a category $\mathbb{O}$, and the language of polynomial functors gives us tools to write down the necessary relations. In this chapter, we also investigate presheaves over $\mathbb{O}$, called *opetopic sets*, and state preliminary results about their structure. The material of chapters 2 and 3 is present in various amount of details throughout the author's previous works [Ho 18b, CHM19b, CHM19a, HL19, HL20a, HL20b].

Lastly, in chapter 4, we "validate" the geometrical intuition behind the definition of the category $\mathbb{O}$ by relating them to *many-to-one polygraphs.* A polygraph (also called *computad*) is a strict ∞-category that is freely generated in every dimension; it is many-to-one if the target of a generating cell is also a generating cell. It is known that the category of all polygraphs is not a presheaf category [CJ04] [MZ08] [Che13], but that the subcategory $\mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ of many-to-one polygraphs is [Hen19]. Here, we construct an adjoint equivalence

$$|-| : \mathcal{P}\mathrm{sh}(\mathbb{O}) \underset{\perp}{\overset{\sim}{\rightleftarrows}} \mathcal{P}\mathrm{ol}^{\mathrm{mto}} : N,$$

providing an explicit description of the underlying shape theory of many-to-one polygraphs. Although this adjunction comes from the left Kan extension of a functor $\mathbb{O} \longrightarrow \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$, it is more enlightening to describe what the right adjoint N does. We call it the *opetopic nerve*, and informally, it strips a polygraph $\mathcal{P} \in \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ from its structure of ω-category, only retaining data about the adjacency relations among generators. For example, if the source of a generator $x \in \mathcal{P}_n$ is

$$\mathsf{s}x = b \underset{n-1}{\circ} a,$$

where $a, b \in \mathcal{P}_{n-1}$, then $N\mathcal{P}$ encodes the fact that a and b occur in the source of x, and that a is "below" b. Since the generators of $\mathcal{P}$ are many-to-one, those compositions schemes are in fact composition *trees*, and thus opetopic sets are a natural structure to store this data. The equivalence between opetopic sets and many-to-one polygraphs was already known from [HMP00] [HMZ02] [Che04b] [HMZ08], however the proof there is indirect and spans over multiple articles. Our clean formalism allows us to proceed directly. The recent work of Henry [Hen19] showed that the category of many-to-one polygraphs (among many others) is a presheaf category, but left the equivalence between "opetopic plexes" (serving as shapes for many-to-one polygraphs) and opetopes open. We establish this in our present work. The material of this chapter has been significantly reworked from its first appearance in [Ho 18b].

Chapter Two

Polynomial functors

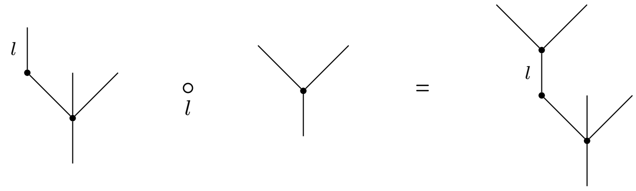
This chapter exposes elements of the theory of polynomial functors, trees, and monads. In a nutshell, a polynomial functor P has a set of "operations" B, and each $b \in B$ has a set of "inputs" $E(b)$ and one "output". Further, the inputs and the output of b are typed in some input set I and output set J. This data can concisely be summarized in a diagram in $\mathcal{S}\mathrm{et}$:

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$$

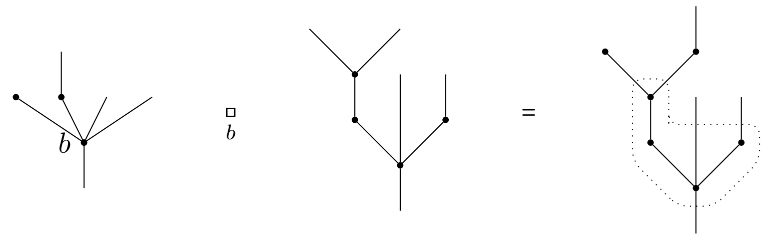
where $E = \sum_{b \in B} E(b)$, where s maps an input of some operation to its type, where p maps an input to the corresponding operation (i.e. an element of $E(b)$ to b), and where t specifies the output type.

The theory of polynomial functors provides a very convenient formalism to talk about trees and "decorated trees". Instead of considering a tree as a set of nodes and vertices satisfying the classical connectivity and acyclicity condition, a *polynomial tree* T is a finite polynomial endofunctor (i.e. $I = J$), where its types are understood as edges, and operations as nodes. The maps s and t then describe the adjacencies between nodes and edges, and the graph-theoretical tree conditions are implemented by requiring that all the operations of T fit together in a unique way: if $i \in I$ is an edge that is neither a leaf nor the root, then there exists a unique node $a \in B$ such that $t(a) = i$, and there exists a unique $b \in B$ and $e \in E(b)$ such that $s(e) = i$. Graphically, this means that there is exactly one node (here a) that has this edge as output edge, and exactly one node (here b) that has it as input edge.

There are two elementary operations that can be performed on trees: *grafting* and *substitution*. If T and U are two polynomial trees with the same set of types I, l is a leaf of T (i.e. an edge that is not the output of any node), such that the type of l and the type of the root edge of U are the same, then we may *graft* U onto T to form a bigger tree $T \circ_l U$, which is just T where U has been glued onto leaf l. For example, if T and U are as on the left and middle, respectively, and assuming that the type of leaf l and root edge l' match, then the grafting $T \circ_l U$ of T and U is depicted on the right:

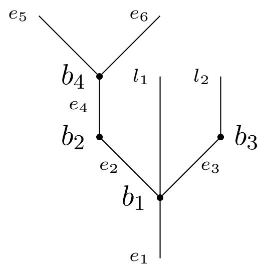


Substitution is more tricky. Given two trees T and U again, a *node* b of T, and a bijection between the input edges of b and the leaves of U (which can be understood as "rewiring instructions"), we may replace b by the whole tree U inside T, to form the substitution $T \circ_b U$. For example if T and U are as on the left and middle, and with the obvious rewiring, the substitution $T \circ_b U$ is depicted on the right:



Grafting and substitution are examples of operations with "tree shaped arities". In binary form, such an operation, say $\wedge$, takes two arguments x and y, and a *parameter* e which is an "input" of x, subject to well-formedness conditions. In the case of grafting (resp. substitution), x and y are trees, and the inputs of x are its leaves (resp. its nodes). Given all those parameters, we can evaluate the expression $x \wedge_e y$. Now, consider a *tree of arguments*, where arguments are represented as nodes, and inputs of arguments as input edges:

$T \quad =$



In the case of grafting and substitution, each b_i would itself be a tree. If $\wedge$ satisfies adequate associativity and unitality conditions, we may consider the evaluation

$$T^{\wedge} := (b_1 \underset{e_2}{\wedge} (b_2 \underset{e_4}{\wedge} b_4)) \underset{e_3}{\wedge} b_3$$

described by this tree. So in essence, $\wedge$ can take trees of arguments as parameters. This idea of operation with tree shaped arity is encapsulated in the notion of *polynomial monad*, which shall also be surveyed in this chapter.

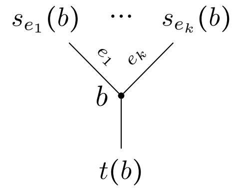
2.1 POLYNOMIAL FUNCTORS

Definition 2.1.1 (Polynomial functor [GK13, paragraph 1.4]). A *polynomial functor* P is a diagram in $\mathcal{S}et$ of the form

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J. \tag{2.1.2}$$

We say that P is a *polynomial endofunctor* if $I = J$. In this case, we also say that P is a *polynomial endofunctor over* I. We say that P is *finitary* if the fibres of $p : E \longrightarrow B$ are all finite sets. We will always assume polynomial functors and endofunctors to be finitary.

We use the following terminology for a polynomial functor P as in diagram diagram (2.1.2), which is motivated by the intuition that a polynomial functor encodes a multi-sorted signature of function symbols. The elements of B are called the *nodes* or *operations* of P, and for every node b, the elements of the fibre $E(b) := p^{-1}(b)$ are called the *input* of b. The elements of I are called the *input colors* or *input sorts* of P, and the elements of J are *output colors* or *output sort*. For every input e of a node b, we denote its color by $s_e(b) := s(e)$.



Definition 2.1.3 (Morphism of polynomial functors)**.** A morphism f from a polynomial functor P over I (on the first row) to a polynomial functor P' over I' (on the second row) is a commutative diagram of the form

$$\begin{tikzcd} I \arrow[d, "f_0"'] & E \arrow[l, "s"'] \arrow[r, "p"] \arrow[d, "f_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "t"] \arrow[d, "f_1"'] & I \arrow[d, "f_0"'] \\ I' & E' \arrow[l, "s'"'] \arrow[r, "p'"] & B' \arrow[r, "t'"] & I' \end{tikzcd}$$

where the middle square is cartesian[1]. If P and P' are both polynomial functors over I, then a morphism from P to P' *over I* is a commutative diagram as above, but where f_0 is required to be the identity [Koc11, paragraph 0.1.3] [KJBM10, section 2.5]. Let $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$ denote the category of polynomial functors and morphisms of polynomial functors, and $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$ the category of polynomial functors over I and morphisms of polynomial functors over I.

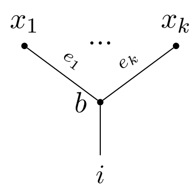
Remark 2.1.4 (Polynomial functors really are functors)*.* If P is as in diagram (2.1.2), then it induces a functor

$$\mathcal{S}\mathrm{et}/I \xrightarrow{s^*} \mathcal{S}\mathrm{et}/E \xrightarrow{p_*} \mathcal{S}\mathrm{et}/B \xrightarrow{t_!} \mathcal{S}\mathrm{et}/J.$$

Explicitly, for $(X_i \mid i \in I) \in \mathcal{S}\mathrm{et}/I$, $P(X)$ is given by the "polynomial"

$$PX = \left(\sum_{b \in B(j)} \prod_{e \in E(b)} X_{s(e)} \mid j \in I \right), \tag{2.1.5}$$

where $B(i) := t^{-1}(i)$ and $E(b) := p^{-1}(b)$. Visually, elements of PX_i are nodes $b \in B$ such that $t(b) = i$, and whose inputs are decorated by elements of $(X_i \mid i \in I)$ in a manner compatible with their colors:



with $x_j \in X_{s_{e_j}(b)}$ for $1 \leq j \leq k$. Moreover, the endofunctor $P : \mathcal{S}\mathrm{et}/I \longrightarrow \mathcal{S}\mathrm{et}/I$ preserves connected limits since s^* and p_* preserve all limits (as right adjoints), and $t_!$ preserves and reflects connected limits.

[1]This condition states that an operation $b \in B$ and its image $f_1(b)$ have the same number of inputs, i.e. that f_2 restricts and corestricts as a bijection $E(b) \longrightarrow E'(f_1(b))$.

Remark 2.1.6. Note that if P is finitary polynomial functor (in the sense of definition 2.1.1) as in diagram (2.1.2), then the products of equation (2.1.5) are finite, and thus $P : \mathcal{S}\mathrm{et}/I \longrightarrow \mathcal{S}\mathrm{et}/J$ preserves filtered colimits, i.e. is finitary in the sense of definition 0.5.1.

Example 2.1.7. (1) [GK13, example 1.6 (i)] The identity functor $\mathrm{id} : \mathcal{S}\mathrm{et}/I \longrightarrow \mathcal{S}\mathrm{et}/I$ is polynomial, and given by

$$I \xleftarrow{\mathrm{id}} I \xrightarrow{\mathrm{id}} I \xrightarrow{\mathrm{id}} I.$$

(2) [GK13, example 1.9] The free monoid monad M on $\mathcal{S}\mathrm{et} = \mathcal{S}\mathrm{et}/1$ maps a set X to $MX = \sum_{i \in \mathbb{N}} X^i$, and using equation (2.1.5), it is easy to see that M can be written down as the following polynomial functor:

$$1 \longleftarrow \mathbb{N}_< \xrightarrow{p} \mathbb{N} \longrightarrow 1,$$

where $\mathbb{N}_< := \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid a < b\}$, and $p(a, b) := b$.

(3) [GK13, section 1.11] If P and P' are as in

$$I \xleftarrow{s} E \longrightarrow B \xrightarrow{t} J, \qquad J \xleftarrow{u} F \longrightarrow C \xrightarrow{v} K,$$

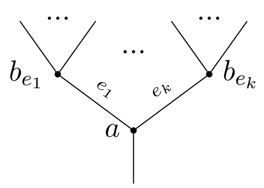
then the composite functor $P'P : \mathcal{S}\mathrm{et}/I \longrightarrow \mathcal{S}\mathrm{et}/K$ is also polynomial. Its underlying diagram is given by

$$I \xleftarrow{x} G \longrightarrow D \xrightarrow{y} K,$$

where

$$D := \{(a, (b_e \mid e \in F(a))) \mid a \in C, b_e \in B, t(b_e) = u_e(a)\},$$
$$G(a, (b_e \mid e)) := \sum_{e \in F(a)} E(b_e),$$

where y maps $(a, (b_e \mid e)) \in D$ to $v(a) \in K$, and if $f \in E(b_e)$, then x_f maps $(a, (b_e \mid e))$ to $s_f(b_e)$. Intuitively, D is just the set of "trees with two levels"



The set of inputs of such a tree is the set of all the inputs of all of the b_{e_i}'s.

Remark 2.1.8. The construction of remark 2.1.4 defines a fully faithful functor

$$\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I) \longrightarrow \mathcal{C}\mathrm{art}(\mathcal{S}\mathrm{et}/I),$$

the latter being the category of endofunctors of $\mathcal{S}\mathrm{et}/I$ and cartesian natural transformations[2]. In fact, the image of this full embedding consists precisely in those endofunctors that preserve connected limits [GK13, section 1.18]. The composition of endofunctors gives $\mathcal{C}\mathrm{art}(\mathcal{S}\mathrm{et}/I)$ the structure of a monoidal category, and $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$ is stable under this monoidal product [GK13, proposition 1.12]. The identity functor is also in $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$ (see example 2.1.7 (1)), thus $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$ is a monoidal subcategory of $\mathcal{C}\mathrm{art}(\mathcal{S}\mathrm{et}/I)$.

[2] We recall that a natural transformation is *cartesian* if all its naturality squares are cartesian.

In this section, we refine the notion of polynomial endofunctor of definition 2.1.1 to introduce polynomial trees. Briefly, a polynomial tree is a polynomial functors whose operations have very specific adjacencies (by the means of input and output types) which essentially implements the familiar connectivity and acyclicity conditions from graph theory, see e.g. [Die17, section 1.5].

Definition 2.2.1 (Polynomial tree [Koc11, section 1.0.3])**.** A polynomial functor T given by

$$T_0 \xleftarrow{s} T_2 \xrightarrow{p} T_1 \xrightarrow{t} T_0$$

is a *polynomial tree* (or just *tree*) if

(1) the sets T_0, T_1 and T_2 are finite (in particular, each node has finitely many inputs); by convention we assume $T_0 \neq \varnothing$;
(2) the map t is injective;
(3) the map s is injective, and the complement of its image $T_0 - \operatorname{im} s$ has a single element, called the *root*;
(4) write $T_0 \cong T_2 + \{r\}$, with r the root, and define the *walk-to-root* function σ by $\sigma(r) := r$, and otherwise $\sigma(e) := tp(e)$; then we ask that for all $x \in T_0$, there exists $k \in \mathbb{N}$ such that $\sigma^k(x) = r$.

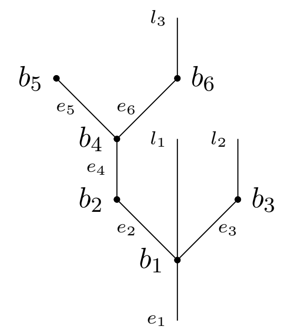
We call the colors of a tree its *edges* and the inputs of a node the *input edges* of that node.

Let $\mathcal{T}\mathrm{ree}$ be the full subcategory of $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$ whose objects are trees. Note that it is the category of *symmetric* or *non-planar* trees (the automorphism group of a tree is in general non-trivial) and that its morphisms correspond to inclusions of non-planar subtrees. An *elementary tree* is a tree with at most one node, and we write $\mathcal{T}\mathrm{ree}_{\mathrm{elem}}$ for the full subcategory of $\mathcal{T}\mathrm{ree}$ spanned by elementary trees.

Remark 2.2.2. Let us unfold definition 2.2.1 a little bit. Condition (1) asserts that a tree is made of finitely many nodes and edges, and that nodes have finitely many inputs (in fact, this follows from finiteness of T_0 and condition (3)). Condition (2) states that different nodes have different output edges. Condition (3) states that an edge is an input edge of exactly one node, except for a unique edge, called the root, which is not an input. In condition (4), we define the walk-to-root function σ as follows:

(1) for r the root, let $\sigma(r) := r$;
(2) if an edge e is not the root, then it is the input edge of a unique node, say b, and let $\sigma(e)$ be the output edge of b. Informally, $\sigma(e)$ is edge directly "below" e, and the sequence $e, \sigma(e), \sigma^2(e), \ldots$ is a sequence of consecutively adjacent edges. Condition (4) states that this sequence eventually reaches the root edge, and stabilizes there.

Example 2.2.3. Consider the following graphical tree:



This corresponds to the polynomial tree T

$$T_0 \xleftarrow{s} T_2 \xrightarrow{p} T_1 \xrightarrow{t} T_0$$

with

(1) $T_0 \coloneqq \{e_1, e_2, e_3, e_4, e_5, e_6, l_1, l_2, l_3\}$,
(2) $T_1 \coloneqq \{b_1, b_2, b_3, b_4, b_5, b_6\}$,
(3) $T_2(b_1) \coloneqq \{e_2, e_3, l_1\}$, $T_2(b_2) \coloneqq \{e_4\}$, $T_2(b_3) \coloneqq \{l_2\}$, $T_2(b_4) \coloneqq \{e_5, e_6\}$, $T_2(b_5) \coloneqq \varnothing$, $T_2(b_6) \coloneqq \{l_3\}$,
(4) $t(b_1) \coloneqq e_1$, $t(b_2) \coloneqq e_2$, $t(b_3) \coloneqq e_3$, $t(b_4) \coloneqq e_4$, $t(b_5) \coloneqq e_5$, $t(b_6) \coloneqq e_6$,

and thus, where the walk-to-root function is given by $\sigma(e_2) = e_1$, $\sigma(e_3) = e_1$, $\sigma(e_4) = e_2$, $\sigma(e_5) = e_4$, $\sigma(l_1) = e_1$, $\sigma(l_2) = e_3$, $\sigma(l_3) = e_6$. Note that s is injective, that that the unique element not in its image is e_1, the root edge.

Definition 2.2.4 (Category of elements)**.** For $P \in \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$, its *category of elements*[3] $\mathrm{elt}\, P$ is the slice category $\mathcal{T}\mathrm{ree}_{\mathrm{elem}}/P$. It describes the adjacencies between the colors and operations of P. Explicitly, for P as in

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I, \tag{2.2.5}$$

the set of objects of $\mathrm{elt}\, P$ is $I + B$, and for each $b \in B$, there is a morphism $\mathsf{t} : t(b) \longrightarrow b$, and a morphism $\mathsf{s}_e : s_e(b) \longrightarrow b$ for each $e \in E(b)$. Note that there is no non-trivial composition of arrows in $\mathrm{elt}\, P$.

The terminology is motivated by the following result.

Proposition 2.2.6 ([Koc11, proposition 2.1.3])**.** *Similar to proposition 0.3.6, there is an equivalence of categories* $\mathcal{P}\mathrm{sh}(\mathrm{elt}\, P) \simeq \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}/P$.

Proof. For $X \in \mathcal{P}\mathrm{sh}(\mathrm{elt}\, P)$, construct the following polynomial functor over P:

$$\begin{tikzcd} \sum_{i \in I} X_i \arrow[d] & E_X \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \sum_{b \in B} X_b \arrow[r] \arrow[d] & \sum_{i \in I} X_i \arrow[d] \\ I & E \arrow[l] \arrow[r] & B \arrow[r] & I, \end{tikzcd}$$

[3]Not to be confused with the category of elements of a presheaf, see definition 0.3.4.

where $E_X \longrightarrow \sum_{i \in I} X_i$ is given by the maps $\mathsf{s}_e : X_b \longrightarrow X_{\mathsf{s}_e b}$, for $b \in B$ and $e \in E(b)$. In the other direction, let $f : P' \longrightarrow P$ be a morphism of polynomial endofunctors, and write it down as

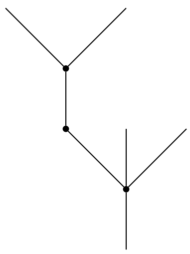
$$\begin{tikzcd} I' \arrow[d, "f_0"'] & E' \arrow[l, "s'"'] \arrow[r, "p'"] \arrow[d, "f_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[r, "t'"] \arrow[d, "f_1"'] & I' \arrow[d, "f_0"'] \\ I & E \arrow[l, "s"'] \arrow[r, "p"] & B \arrow[r, "t"] & I. \end{tikzcd}$$

We define a corresponding presheaf $Y : (\mathrm{elt}\, P)^{\mathrm{op}} \longrightarrow \mathcal{S}\mathrm{et}$ as follows. On objects, if $i \in I$, let $Y_i := f_0^{-1}(i)$, and if $b \in B$, let $Y_b := f_1^{-1}(b)$. On morphisms, let $x \in Y_b = f_1^{-1}(b)$, and simply define $\mathsf{t}\, x := t'(x) \in Y_{t(b)}$. For $e \in E(b)$, since the middle square is cartesian, there exists a unique $e' \in E'(x)$ such that $f_2(e') = e$, and let $\mathsf{s}_e\, x := s'(e') \in Y_{s(e)}$.

Finally, the two constructions are easily seen to define mutually inverse equivalences of categories. $\square$

Definition 2.2.7 (P-tree)**.** For $P \in \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$, the category $\mathrm{tr}\, P$ of *P-trees* is the slice $\mathcal{T}\mathrm{ree}/P$. If $f : P \longrightarrow Q$ is a morphism of polynomial functors, then it induces a natural functor $f_* : \mathrm{tr}\, P \longrightarrow \mathrm{tr}\, Q$ by postcomposition.

Remark 2.2.8. A fundamental difference between $\mathcal{T}\mathrm{ree}$ and $\mathrm{tr}\, P$ is that the latter is always *rigid* i.e. it has no non-trivial automorphisms [Koc11, proposition 1.2.3]. In particular, this implies that $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$ does not have a terminal object. For example, the automorphism group of the following tree is $\mathfrak{S}_2 \times \mathfrak{S}_2$:

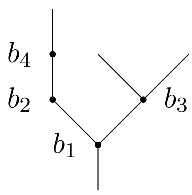


Notation 2.2.9. A P-tree $T \in \mathrm{tr}\, P$ is a morphism from a polynomial tree, which we shall denote by $\langle T \rangle$, to P, as in $T : \langle T \rangle \longrightarrow P$. We point out that $\langle T \rangle_1$ is the set of nodes of the P-tree T, while $T_1 : \langle T \rangle_1 \longrightarrow P_1$ provides a *decoration* of the nodes of $\langle T \rangle$ by operations of P, and likewise for edges.

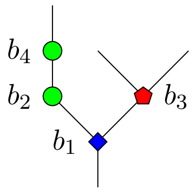
Example 2.2.10. Consider the following polynomial endofunctor P

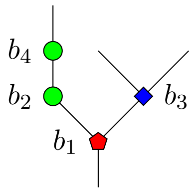
$$\{*\} \longleftarrow E \xrightarrow{p} \{●, ◆, ⬟, ✹\} \longrightarrow \{*\}$$

where $\#E(●) = 1$ (i.e. ● has one input), $\#E(◆) = 2$, $\#E(⬟) = 2$, and $\#E(✹) = 3$. Then the tree T represented by



can be made into a P-tree by the means of the morphism $T \longrightarrow P$ mapping b_1, b_2, b_3 and b_4 to ●, ●, ◆, and ⬟ respectively, which is represented on the left.



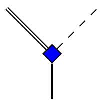


An alternative decoration, and so a different P-tree, is given on the right. Here, decoration of edges is trivial, as P_0 is a singleton. Note that the decoration of b_2 and b_4 must be ●, since it is the unique operation of P with 1 input, while no node of T can be decorated by ✹, since it has 3 inputs. For the same reason, if a tree U has a node with 0 or more than 3 inputs, then it cannot be the underlying tree of a P-tree.

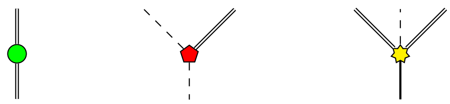
Consider now a more complicated polynomial endofunctor Q given by

$$\{\|, \text{¦}, |\} \xleftarrow{s} E' \xrightarrow{p} \{\text{●}, \text{◆}, \text{⬟}, \text{✹}\} \xrightarrow{t} \{\|, \text{¦}, |\}$$

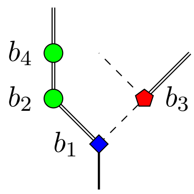
where $t(\text{◆}) = |$, $E'(\text{◆}) = \{i_1, i_2\}$, $s_{i_1}(\text{◆}) = \|$, and $s_{i_2}(\text{◆}) = \text{¦}$, which can be represented as



Similarly, the other operations of Q are represented as



Then the polynomial tree above can be made into a Q-tree as follows:



This is just the operations ●, ◆, ⬟, ✹ of Q, seen as corollas, assembled into a well-typed tree. Note that this time, decorating b_1 by ⬟ is not possible, as b_3 would have to be decorated by a binary operation outputting $\|$, which Q does not have.

Definition 2.2.11 (Address)**.** Let $T \in \mathfrak{T}\mathrm{ree}$ be a polynomial tree and σ be its walk-to-root function (definition 2.2.1). We define the *address* function & on edges inductively as follows:

(1) if r is the root edge, let $\&r := []$,

(2) if $i \in T_0 - \{r\}$ and if $\&\sigma(i) = [x]$, define $\&i := [xe]$, where $e \in T_2$ is the unique element such that $s(e) = i$.

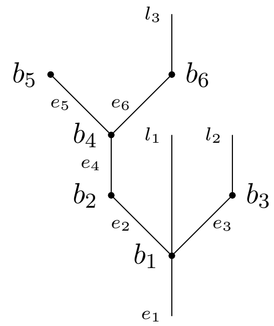
Thus an address is a sequence of elements of T_2, enclosed by brackets. Informally, $\&e$ gives "walking instructions" to go from the root edge to e (see example 2.2.12). The address of a node $b \in T_1$ is simply $\&b := \&t(b)$. Note that this function is injective since t is. Let

$T^\bullet$ be the set of *node addresses* of T, and let $T^{|}$ be the set of addresses of leaf edges, i.e. those edges not in the image of t.

Assume now that $U : \langle U \rangle \longrightarrow P$ is a P-tree. If $b \in \langle U \rangle_1$ has address $\& b = [p]$, write $\mathsf{s}_{[p]} U := U_1(b) \in B$ for the decoration of the node at address $[p]$. For convenience, we let $U^\bullet := \langle U \rangle^\bullet$, and $U^{|} := \langle U \rangle^{|}$.

The formalism of addresses is a useful bookkeeping syntax for the operations of grafting and substitution on trees. The syntax of addresses will extend to the category of opetopes and will allow us to give a precise description of the composition of morphisms in the category of opetopes (see definition 3.4.2) as well as certain constructions on opetopic sets.

Example 2.2.12. Consider the tree of example 2.2.3:



The addresses of the nodes are given by: $\& b_1 = []$ since b_1 is the root node, $\& b_2 = [e_2]$ since to get to b_2 from the root node b_1, one only needs to walk along the edge e_2, $\& b_3 = [e_3]$, $\& b_4 = [e_2 e_4]$ since to get to b_4 from the root node b_1, one needs to walk along the edge e_2, then e_4, $\& b_5 = [e_2 e_4 e_5]$, and $\& b_6 = [e_2 e_4 e_6]$. Therefore,

$$\begin{aligned} T^\bullet &= \{[], [e_2], [e_3], [e_2 e_4], [e_2 e_4 e_5], [e_2 e_4 e_6]\} , \\ T^{|} &= \{[l_1], [e_3 l_2], [e_2 e_4 e_6 l_3]\} . \end{aligned}$$

Definition 2.2.13 (Prefix)**.** Let T be a tree and $[p], [q] \in T^\bullet$. We say that $[p]$ is a *prefix* of $[q]$, denoted by $[p] \sqsubseteq [q]$, if the sequence p is a prefix of q. We call $\sqsubseteq$ be the *prefix order*. This definition transposes to $T^{|}$ *mutadis mutandis.*

Notation 2.2.14. We denote by $\operatorname{tr}^{|} P$ the set of P-trees with a marked leaf, i.e. endowed with the address of one of its leaves. Similarly, we denote by $\operatorname{tr}^\bullet P$ the set of P-trees with a marked node.

Definition 2.2.15 (Elementary P-trees)**.** Let P be a polynomial endofunctor as in equation equation (2.2.5). For $i \in I$, define $\mathsf{I}_i \in \operatorname{tr} P$ as having underlying tree

$$\{i\} \longleftarrow \varnothing \longrightarrow \varnothing \longrightarrow \{i\}, \tag{2.2.16}$$

along with the obvious morphism to P, that which maps i to $i \in I$. This corresponds to a tree with no nodes and a unique edge, decorated by i. Given an operation $b \in B$, define $\mathsf{Y}_b \in \operatorname{tr} P$, the *corolla* at b, as having underlying tree

$$s(E(b)) + \{*\} \xleftarrow{\ s\ } E(b) \longrightarrow \{b\} \longrightarrow s(E(b)) + \{*\}, \tag{2.2.17}$$

where the right map sends b to $*$, and where the morphism $\mathsf{Y}_b \longrightarrow P$ is the identity on $s(E(b)) \subseteq I$, maps $*$ to $t(b) \in I$, is the identity on $E(b) \subseteq E$, and maps b to $b \in B$. This corresponds to a P-tree with a unique node, decorated by b. Observe that for $T \in \operatorname{tr} P$, giving a morphism $\mathsf{I}_i \longrightarrow T$ is equivalent to specifying the address $[p]$ of an edge of T decorated by i. Likewise, morphisms of the form $\mathsf{Y}_b \longrightarrow T$ are in bijection with addresses of nodes of T decorated by b.

Remark 2.2.18. Let P be a polynomial endofunctor as in equation (2.2.5).

(1) Let $i \in I$ be a color of P. Since I_i does not have any node, the set $\mathsf{I}_i^\bullet$ of node addresses is empty. On the other hand, the set of its leaf addresses is $\mathsf{I}_i^{|} = \{[]\}$, since the unique leaf is the root edge.

(2) Let $b \in B$ be an operation of P. Then $\mathsf{Y}_b^\bullet = \{[]\}$ since the only node is the one above the root edge. For leaves, we have $\mathsf{Y}_b^{|} = \{[e] \mid e \in E(b)\}$.

Definition 2.2.19 (Grafting)**.** For $S, T \in \operatorname{tr} P$, $[l] \in S^{|}$ such that the leaf of S at $[l]$ and the root edge of T are decorated by the same $i \in I$, define the *grafting* $S \circ_{[l]} T$ of S and T on $[l]$ by the following pushout (in $\operatorname{tr} P$):

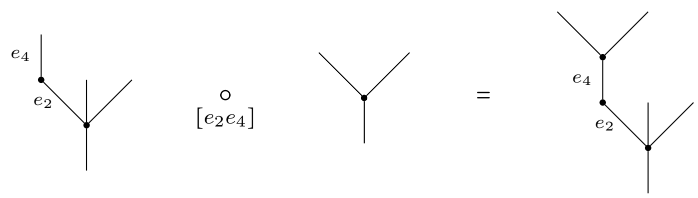
$$
\begin{tikzcd}
\mathsf{I}_i \arrow[r, "{[]}"] \arrow[d, "{[l]}"'] & T \arrow[d] \\
S \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & S \circ_{[l]} T.
\end{tikzcd}
\tag{2.2.20}
$$

In particular,

$$
\begin{aligned}
(S \underset{[l]}{\circ} T)^\bullet &\cong S^\bullet + \{[lp] \mid [p] \in T^\bullet\}, \\
(S \underset{[l]}{\circ} T)^{|} &\cong S^{|} - \{[l]\} + \{[lp] \mid [p] \in T^{|}\}.
\end{aligned}
$$

In particular $\mathsf{I}_i \circ_{[]} T \cong T$ and $S \circ_{[l]} \mathsf{I}_i \cong S$. We assume, by convention, that the grafting operator $\circ$ associates to the right.

Example 2.2.21. Consider S and T two P-trees (where the decorations are omitted from the picture) as on the left and the middle, respectively.



Assuming that the decorations of e_4 and the root edge of T match, we can define the grafting $S \circ_{[e_2e_4]} T$, which is given on the right.

Proposition 2.2.22 ([Koc11, proposition 1.1.21])**.** *Every P-tree is either of the form I_i, for some $i \in I$, or obtained by iterated graftings of corollas (i.e. P-trees of the form Y_b for $b \in B$).*

Proof. This can easily be proved by induction on the number of nodes. $\square$

Notation 2.2.23 (Total grafting). Let $T, U_1, \ldots, U_k \in \operatorname{tr} P$, write $T^{|} = \{[l_1], \ldots, [l_k]\}$, and assume the grafting $T \circ_{[l_i]} U_i$ is defined for all i. Then the *total grafting* will be denoted concisely by

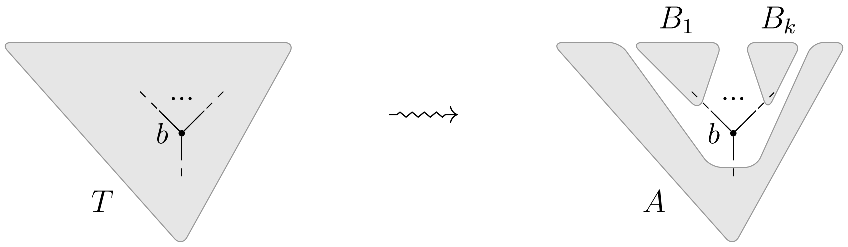
$$T \bigcirc_{[l_i]} U_i = (\cdots(T \underset{[l_1]}{\circ} U_1) \underset{[l_2]}{\circ} U_2 \cdots) \underset{[l_k]}{\circ} U_k. \tag{2.2.24}$$

It is easy to see that the result does not depend on the order in which the graftings are performed.

Definition 2.2.25 (Substitution)**.** Let T be a P-tree, $[p] \in T^{\bullet}$, and $b = \mathsf{s}_{[p]} T$. Then T can be decomposed so as to isolate the node of T at address $[p]$:

$$T = A \underset{[p]}{\circ} \mathsf{Y}_b \bigcirc_{[e_i]} B_i, \tag{2.2.26}$$

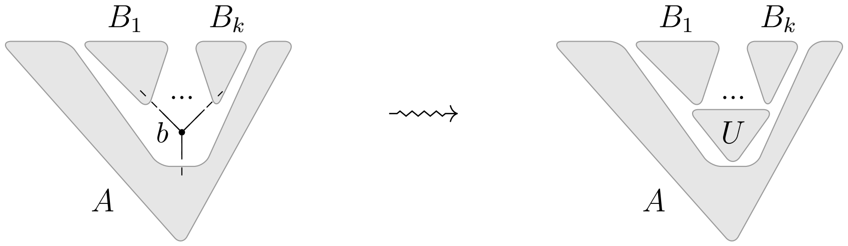
where $E(b) = \{e_1, \ldots, e_k\}$, and $A, B_1, \ldots, B_k \in \operatorname{tr} P$. Graphically:



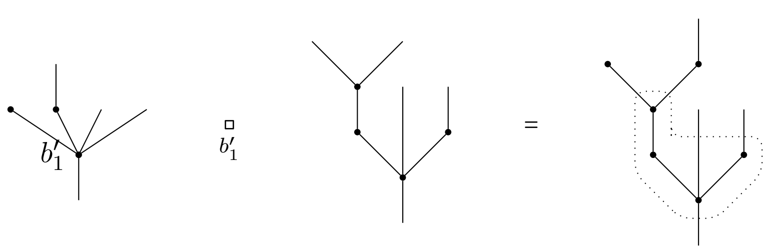
For U a P-tree with a bijection $\wp : U^{|} \longrightarrow E(b)$ over I, we define the *substitution* $T \square_{[p]} U$ (leaving $\wp$ implicit) as

$$T \underset{[p]}{\square} U := A \underset{[p]}{\circ} U \bigcirc_{\wp^{-1} e_i} B_i. \tag{2.2.27}$$

In other words, the node at address $[p]$ in T has been replaced by U, and the map $\wp$ provided "rewiring instructions" to connect the leaves of U to the rest of T:



Example 2.2.28. Consider T and U two P-trees (where the decorations are omitted from the picture) as on the left and middle, respectively.



Assume that $T_0(e'_1) = U_0(e_1)$, i.e. that the decoration of edges e'_1 and e_1 match, and similarly, that $T_0(e'_5) = U_0(e_5)$, $T_0(e'_6) = U_0(e_6)$, $T_0(l'_1) = U_0(l_1)$, $T_0(l'_2) = U_0(l_2)$. Then the obvious bijection $\wp : U^{|} \longrightarrow T_2(b'_1)$ that maps e_5 to e'_5 etc. allow us to compute the substitution $T \square_{[]} U$, which is given on the right.

2.3 POLYNOMIAL MONADS

We now discuss polynomial monads, which are polynomial functors with a structure of cartesian monad. Recall from example 2.1.7 that if $P \in \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$, then PP is a polynomial endofunctor. Thus, if P is a monoid object, then it induces a monad over $\mathcal{S}\mathrm{et}/I$. In this section, we show how one may reason on polynomial monads in terms of devices that "compose operations" in an adequate way. This idea is formalized in theorems 2.3.5 and 2.3.10, which presents polynomial monads as "compositors of tree-shaped arity", and refined in theorem 2.3.6, which discusses the biased vs. unbiased dichotomy.

Definition 2.3.1 (Polynomial monad, classical definition)**.** A *polynomial monad over I* is a monoid in $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$. Let $\mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}(I)$ be the category of polynomial monads over I and morphisms of polynomial functors over I that are also monad morphisms.

Remark 2.3.2. A polynomial monad over I is necessarily a *cartesian monad* on $\mathcal{S}\mathrm{et}/I$, i.e. its underlying endofunctor preserves pullbacks and its unit and multiplication are cartesian natural transformations.

Definition 2.3.3 ($(-)^\star$ construction)**.** Given a polynomial endofunctor P as in equation (2.2.5), we define a new polynomial endofunctor $P^\star$ as

$$I \xleftarrow{s} \operatorname{tr}^{\shortmid} P \xrightarrow{p} \operatorname{tr} P \xrightarrow{t} I \tag{2.3.4}$$

where s maps a P-tree with a marked leaf to the decoration of that leaf, p forgets the marking, and t maps a P-tree to the decoration of its root. Remark that for $T \in \operatorname{tr} P$ we have $p^{-1}(T) \cong T^{\shortmid}$. Clearly, there is an inclusion $P \longrightarrow P^\star$, mapping $b \in B$ to $\mathsf{Y}_b \in \operatorname{tr} P$, and $e \in E(b)$ to $[e] \in \mathsf{Y}_b^{\shortmid}$ (see remark 2.2.18).

Theorem 2.3.5 ([Koc11, proposition 1.2.8])**.** *The polynomial functor $P^\star$ has a canonical structure of polynomial monad. Further, the functor $(-)^\star$ is left adjoint to the forgetful functor $\mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}(I) \longrightarrow \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$. In other words, $P^\star$ is the free polynomial monad over P.*

Proof (sketch).

1. By definition, operations of $P^\star$ are P-trees, thus operations of $P^\star P^\star$ are trees of P-trees, of uniform height 2, i.e. of the form

   $$\mathsf{Y}_T \underset{[[l_i]]}{\bigcirc} \mathsf{Y}_{S_i}$$

   where $[l_i]$ ranges over $T^{\shortmid}$. There is a natural law $\mu : P^\star P^\star \longrightarrow P^\star$ such that

   $$\mu_1 \left(\mathsf{Y}_T \underset{[[l_i]]}{\bigcirc} \mathsf{Y}_{S_i} \right) = T \underset{[l_i]}{\bigcirc} S_i .$$

   It implements the fact that a tree of P-tree (where the input edges of Y_T are in bijective correspondence with $T^{\shortmid}$) can simply be seen as a P-tree.
2. For the adjunction, let $f : P \longrightarrow M$ be a morphism in $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$, where P and M are given by

   $$I \xleftarrow{s} E \longrightarrow B \xrightarrow{t} I, \qquad I \xleftarrow{u} F \longrightarrow C \xrightarrow{v} I,$$

and where M is a polynomial monad with laws $\eta : \mathrm{id} \longrightarrow M$ and $\mu : MM \longrightarrow M$. The morphism f maps operations of P to operations of M, so we can extend it by mapping trees of operations of P, i.e. P-trees, to M-trees, and then reduce those M-trees using the monad laws of M. Formally, we define $\bar{f} : P^\star \longrightarrow M$ by induction (see proposition 2.2.22) as follows. If $i \in I$, then $\bar{f}_1(\mathsf{I}_i) := \eta_1(i)$. If $b \in B$, then $\bar{f}_1(\mathsf{Y}_b) = f_1(b)$. Consider now a P-tree with at least two nodes

$$T = \mathsf{Y}_b \bigcirc_{\substack{[e] \\ e \in X}} U_e,$$

where $b \in B$, $\varnothing \neq X \subseteq E(b)$, and U_e is a P-tree with at least one node. First, we "complete" T by considering

$$T' := \mathsf{Y}_b \bigcirc_{\substack{[e] \\ e \in E(b)}} U_e,$$

where if $e \in E(b) - X$, we let $U_e := \mathsf{Y}_{\eta_1(s(e))}$. By induction, $\bar{f}_1$ is defined on the U_e's, and the tree

$$T'' := \mathsf{Y}_{f_1(b)} \bigcirc_{\substack{[f_2(e)] \\ e \in E(b)}} \bar{f}_1(U_e)$$

is an M-tree of uniform height 2. Let then $\bar{f}_1(T) := \mu_1(T'')$. This gives a morphism $\bar{f} : P^\star \longrightarrow M$ extending f. It is easy to check that it is a monad morphism, and unique for this property. □

We abuse notation and let $(-)^\star$ be the "free polynomial monad" monad on $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$.

Theorem 2.3.6 ($(-)^\star$-algebras via partial laws)**.** *Let P be a polynomial endofunctor, say*

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I.$$

A $(-)^\star$-algebra structure on P is equivalent to the following data:

- **(Unit)** *a map $\eta : I \longrightarrow B$;*
- **(Partial multiplication)** *a map $\wedge : E \times_I B \longrightarrow B$, where for $(e, b) \in E \times_I B$ and $a := p(e)$ we write $a \wedge_e b$ instead of $\wedge(e, b)$, and say that $a \wedge_e b$ is an* admissible expression *(or just* admissible*);*
- **(Partial readdressing)** *for $a \wedge_e b$ an admissible expression, a bijective map $\rho_{a \wedge_e b} : E(a) + E(b) - \{e\} \longrightarrow E(a \wedge_e b)$ over I;*

such that the following conditions are satisfied:

- **(Trivial)** *for $i \in I$, we have $t(\eta(i)) = i$, and $E(\eta(i))$ is a singleton whose unique element e is such that $s(e) = i$;*
- **(Left unit)** *for $i \in I$, $b \in B$, and e the unique element of $E(\eta(i))$, we have $\eta(i) \wedge_e b = b$, and $\rho_{\eta(i) \wedge_e b} : E(b) \longrightarrow E(b)$ is the identity;*
- **(Right unit)** *for $i \in I$, $b \in B$, and $e \in E(b)$, we have $b \wedge_e \eta(i) = b$, and $\rho_{b \wedge_e \eta(i)}$ is given by*

$$\begin{aligned} E(b) + E(\eta(i)) - \{e\} &\longrightarrow E(b) \\ x \in E(b) - \{e\} &\longmapsto x \\ y \in E(\eta(i)) &\longmapsto e; \end{aligned}$$

(Disjoint multiplication) *for $a \wedge_e b$ and $a \wedge_f c$ admissible, where $e \neq f$, we have*

$$(a \underset{e}{\wedge} b) \underset{f'}{\wedge} c = (a \underset{f}{\wedge} c) \underset{e'}{\wedge} b,$$

where $f' := \rho_{a \wedge_e b}(f)$ and $e' := \rho_{a \wedge_f c}(e)$, and the following coherence diagram commutes:

$$\begin{tikzcd} E(a) + E(b) + E(c) - \{e, f\} \arrow[r, "\rho_{a \wedge_e b}"] \arrow[d, "\rho_{a \wedge_f c}"'] & E(a \wedge_e b) + E(c) - \{f\} \arrow[d, "\rho_{(a \wedge_f c) \wedge_{e'} b}"] \\ E(a \wedge_f c) + E(b) - \{e\} \arrow[r, "\rho_{(a \wedge_e b) \wedge_{f'} c}"] & E((a \wedge_e b) \wedge_{f'} c); \end{tikzcd} \tag{2.3.7}$$

(Nested multiplication) *for $a \wedge_e b$ and $b \wedge_f c$ admissible, we have*

$$(a \underset{e}{\wedge} b) \underset{f'}{\wedge} c = a \underset{e}{\wedge} (b \underset{f}{\wedge} c),$$

where $f' := \rho_{a \wedge_e b}(f)$, and the following coherence diagram commutes:

$$\begin{tikzcd} E(a) + E(b) + E(c) - \{e, f\} \arrow[r, "\rho_{a \wedge_e b}"] \arrow[d, "\rho_{b \wedge_f c}"'] & E(a \wedge_e b) + E(c) - \{f\} \arrow[d, "\rho_{a \wedge_e (b \wedge_f c)}"] \\ E(a \wedge_f c) + E(b) - \{e\} \arrow[r, "\rho_{(a \wedge_e b) \wedge_{f'} c}"] & E((a \wedge_e b) \wedge_{f'} c). \end{tikzcd} \tag{2.3.8}$$

Moreover with the data above, the components of the structure map $P^\star \longrightarrow P$ as in

$$\begin{tikzcd} I \arrow[d, equal] & \operatorname{tr}^{\shortmid} P \arrow[l] \arrow[r] \arrow[d, "\wp"] \arrow[dr, phantom, "\lrcorner" very near start] & \operatorname{tr} P \arrow[r] \arrow[d, "\mathsf{t}"] & I \arrow[d, equal] \\ I & E \arrow[l] \arrow[r] & B \arrow[r] & I \end{tikzcd}$$

are inductively given by:

(1) *for $i \in I$, $\mathsf{t}\,\mathsf{I}_i = \eta(i)$; for $b \in B$, $\mathsf{t}\,\mathsf{Y}_b = b$; for $T \in \operatorname{tr} P$, $[l] \in T^{\shortmid}$, and $b \in B$ such that the grafting $T \circ_{[l]} \mathsf{Y}_b$ is defined:*

$$\mathsf{t}(T \underset{[l]}{\circ} \mathsf{Y}_b) = (\mathsf{t}\,T) \underset{\wp_T[l]}{\wedge} b, \tag{2.3.9}$$

where $\wp_T$ is defined next;

(2) *for $i \in I$,*

$$\begin{aligned} \wp_{\mathsf{I}_i} : \mathsf{I}_i^{\shortmid} = \{[]\} &\longrightarrow E(\eta(i)) \\ [] &\longmapsto e, \end{aligned}$$

where e is the unique element of $E(\eta(i))$ (see **(Trivial)***); for $b \in B$,*

$$\begin{aligned} \wp_{\mathsf{Y}_b} : \mathsf{Y}_b^{\shortmid} &\longrightarrow E(b) \\ [e] &\longmapsto e; \end{aligned}$$

for $T \in \operatorname{tr} P$, $[l] \in T^{\shortmid}$, $b \in B$ such that the grafting $T \circ_{[l]} \mathsf{Y}_b$ is defined, and letting $c := (\mathsf{t}\,T) \wedge_{\wp_T[l]} b$, the readdressing $\wp_{T \circ_{[l]} \mathsf{Y}_b}$ is given by

$$\begin{aligned} (T \underset{[l]}{\circ} \mathsf{Y}_b)^{\shortmid} \cong T^{\shortmid} + \mathsf{Y}_b^{\shortmid} - \{[l]\} &\longrightarrow E((\mathsf{t}\,T) \underset{\wp_T[l]}{\wedge} b) \\ [p] \in T^{\shortmid} - \{[l]\} &\longmapsto \rho_c(\wp_T[l]) \\ [e] \in \mathsf{Y}_b^{\shortmid} &\longmapsto \rho_c(e). \end{aligned}$$

Proof. (1) Assume that P is a $(-)^\star$-algebra, and write $m : P^\star \longrightarrow P$ for its structure map. Let the function η of **(Unit)** map $i \in I$ to $m_1(\mathsf{I}_i) \in B$. For $(e, b) \in E \times_I B$ and $a := p(e)$, let $T := \mathsf{Y}_a \circ_{[e]} \mathsf{Y}_b$, and $a \wedge_e b := m_1(T)$. Note that

$$T^{|} = \{[f] \mid f \in E(a), f \neq e\} \cup \{[ef] \mid f \in E(b)\}.$$

From there, the partial readdressing is simply given by

$$\begin{aligned} \rho_{a \wedge_e b} : E(a) + E(b) - \{e\} &\longrightarrow E(a \underset{e}{\wedge} b) \\ f \in E(a) - \{e\} &\longmapsto \wp_T[f] \\ f \in E(b) &\longmapsto \wp_T[ef]. \end{aligned}$$

Conditions **(Trivial)**, to **(Nested multiplication)** hold since P is a $(-)^\star$-algebra.

(2) Assume that P is a polynomial functor endowed with a unit, a partial multiplication, and a partial readdressing map as in the statement of the theorem. Then it is straightforward to check that the construction of $m : P^\star \longrightarrow P$ at the end of the statement gives a $(-)^\star$-algebra structure to P. □

Theorem 2.3.10. *The forgetful functor* $\mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}(I) \longrightarrow \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$ *is monadic.*

Proof. We show that every polynomial monad is canonically a $(-)^\star$-algebra. Let M be a polynomial monad whose underlying polynomial endofunctor is

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I,$$

and $\eta : \mathrm{id} \longrightarrow M$ and $\mu : MM \longrightarrow M$ be its monad laws. We define a structure map $(-)^\star$-algebra by the means of partial laws (theorem 2.3.6) as follows.

(1) The unit map is simply the map $\eta_1 : I \longrightarrow B$ induced by η on operations.

(2) Recall that operations of MM are M-trees of uniform height 2 (see example 2.1.7). Let $(e, b) \in E \times_I B$, write $a := p(e)$, and let

$$a \underset{e}{\wedge} b := \mu_1 \left(T_{a \wedge_e b}\right), \qquad T_{a \wedge_e b} := \left(\mathsf{Y}_a \underset{[e]}{\circ} \mathsf{Y}_b\right) \underset{\substack{[f] \\ f \in E(b) - \{e\}}}{\bigcirc} \mathsf{Y}_{\eta_1(s_f(a))}$$

In other words, $T_{a \wedge_e b}$ is the tree of uniform height 2, where Y_b has been grafted onto Y_a at the leaf corresponding to e, and "completed with unit corollas" (much like in the proof of theorem 2.3.5) provided by $\eta_1 : I \longrightarrow B$. Note that for $i \in I$, the set $E(\eta_1(i))$ is a singleton, thus

$$\begin{aligned} T^{|}_{a \wedge_e b} &\cong \mathsf{Y}^{|}_b + \sum_{f \in E(b) - \{e\}} \mathsf{Y}^{|}_{\eta_1(s_f(a))} \\ &\cong E(b) + \sum_{f \in E(b) - \{e\}} E(s_f(a)) \\ &\cong E(a) + E(b) - \{e\}. \end{aligned}$$

(3) For $a \wedge_e b$ an admissible expression as above, the partial readdressing map is given by

$$E(a) + E(b) - \{e\} \xrightarrow{\cong} T^{|}_{a \wedge_e b} \xrightarrow{\mu_2} E(a \underset{e}{\wedge} b).$$

From fact that M is a polynomial monad, it is straightforward to check that the conditions of theorem 2.3.6 hold. □

Definition 2.3.11 (Target and readdressing map)**.** Let M be a polynomial monad as in

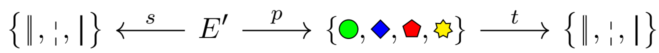
$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I.$$

By theorem 2.3.10, M is a $(-)^\star$-algebra, and we will write its structure map $M^\star \longrightarrow M$ as

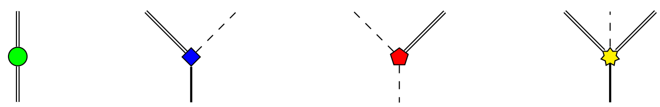
$$\begin{tikzcd} I \arrow[d, equal] & \operatorname{tr}^{|} M \arrow[l] \arrow[r] \arrow[d, "\wp"] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{tr} M \arrow[r] \arrow[d, "\mathsf{t}"] & I \arrow[d, equal] \\ I & E \arrow[l] \arrow[r] & B \arrow[r] & I. \end{tikzcd} \tag{2.3.12}$$

For $T \in \operatorname{tr} M$, we call $\wp_T : T^{|} \xrightarrow{\cong} E(\mathsf{t}T)$ the *readdressing* function of T, and $\mathsf{t}T \in B$ is called the *target*[4] of T. If we think of an element $b \in B$ as the corolla Y_b, then the target map t "contracts" (see remark 2.3.13) a tree to a corolla, and since the middle square is a pullback, the number of leaf is preserved. The map $\wp_T$ establishes a coherent correspondence between the set $T^{|}$ of leaf addresses of a tree T and the elements of $E(\mathsf{t}T)$.

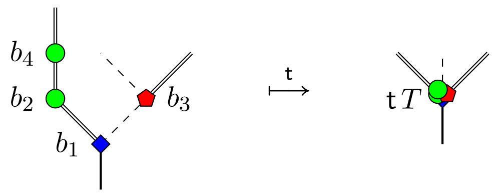
Remark 2.3.13. Let M be a polynomial monad as in



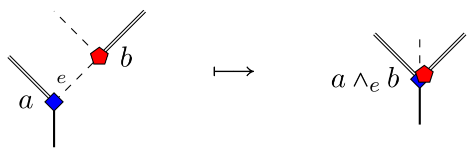
where its operations are represented as



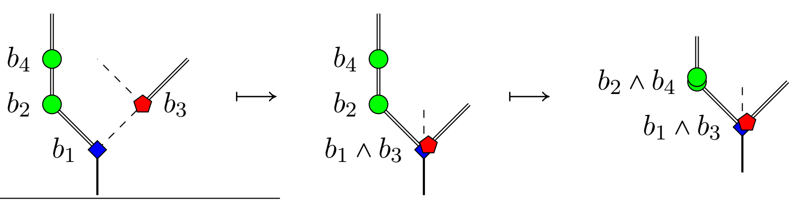
The target map of definition 2.3.11 associates to every M-tree T its target $\mathsf{t}T$, which we think of as the *contraction* of all inner edges of T:



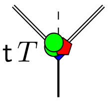
In theorem 2.3.6, we focus on contractions of trees with two nodes, which are called *admissible expressions*:



What this result asserts is that inner edge contractions of T can be performed one by one, and in any order:



[4] This terminology will become meaningful when we deal with opetopes in chapter 3.

$\longmapsto$ $\mathsf{t}T$ 

Remark that in both cases, the leaves of the trees are in bijective correspondence with the inputs of the targets, and that correspondence preserves the types.

Proposition 2.3.14 (Contraction associativity formula). *Let M be a polynomial monad as in equation* (2.3.12)*, let t and $\wp$ be as in definition 2.3.11, and write $u : \mathrm{id} \longrightarrow M$ for its unit law.*

(1) If $i \in I$, then $\mathsf{t}\,\mathsf{I}_i = u(i)$, and $\wp_{\mathsf{I}_i}$ maps the only leaf address $[]$ of the tree I_i to the only element of $E(u(i))$.

(2) If $b \in B$, then $\mathsf{t}\,\mathsf{Y}_b = b$. Recall from remark 2.2.18 that the set of leaf address of Y_b is simply $\{[e] \mid e \in E(b)\}$, and $\wp_{\mathsf{Y}_b}$ maps $[e]$ to e.

(3) If we have a grafting $T \circ_{[l]} U$ if M-trees, then

$$\mathsf{t}(T \underset{[l]}{\circ} U) \;=\; \mathsf{t}(\mathsf{Y}_{\mathsf{t}T} \underset{[\wp_T[l]]}{\circ} \mathsf{Y}_{\mathsf{t}U}). \tag{2.3.15}$$

Further, for $[r] \in U^{\mathsf{l}}$, we have a leaf $[l] \cdot [r] = [lr] \in \left(T \circ_{[l]} U\right)^{\mathsf{l}}$, and writing $V := \mathsf{Y}_{\mathsf{t}T} \circ_{[\wp_T[l]]} \mathsf{Y}_{\mathsf{t}U}$, we have

$$\wp_{T \circ_{[l]} U}[lr] \;=\; \wp_V \left(\wp_T[l] \cdot \wp_U[r]\right). \tag{2.3.16}$$

Proof. Since M is an algebra over the monad $(-)^\star$, the following two diagrams commute:

$$\begin{tikzcd} M \arrow[r, "\eta"] \arrow[dr, equal] & M^\star \arrow[d, "m"] \\ & M, \end{tikzcd} \qquad \begin{tikzcd} M^{\star\star} \arrow[r, "m^\star"] \arrow[d, "\mu"'] & M^\star \arrow[d, "m"] \\ M^\star \arrow[r, "m"] & M, \end{tikzcd}$$

where $m : M^\star \longrightarrow M$ is the structure map of M. □

Remark 2.3.17. Let M be a polynomial monad with partial multiplication $\wedge : E \times_I B \longrightarrow B$. Combining equations (2.3.9) and (2.3.15), we have that for a well-defined grafting of M-tree $T \circ_{[l]} U$,

$$\mathsf{t}(T \underset{[l]}{\circ} U) \;=\; (\mathsf{t}T) \underset{\wp_T[l]}{\wedge} (\mathsf{t}U).$$

More generally,

$$\mathsf{t}\left(T \bigcirc_{[l_i]} U_i\right) \;=\; \mathsf{t}((T \underset{[l_1]}{\circ} U_1) \underset{[l_2]}{\circ} U_2 \cdots) \;=\; ((\mathsf{t}T) \underset{[p_1]}{\wedge} (\mathsf{t}U_1)) \underset{[p_2]}{\wedge} (\mathsf{t}U_2) \cdots$$

but computing each $[p_i]$ can be a daunting task. Just with 2 graftings, the result becomes

$$\mathsf{t}\left(T \bigcirc_{[l_i]} U_i\right) \;=\; \begin{cases} ((\mathsf{t}T) \underset{\wp_T[l_1]}{\wedge} (\mathsf{t}U_1)) \underset{\wp_T[l_1] \cdot \wp_{U_1}^{-1}[p] \cdot [q]}{\wedge} (\mathsf{t}U_2) \\ \qquad \text{if } \wp_T[l_1] \sqsubseteq \wp_T[l_2], \text{ say } \wp_T[l_2] = \wp_T[l_1] \cdot [[p]q] \\ ((\mathsf{t}T) \underset{\wp_T[l_1]}{\wedge} (\mathsf{t}U_1)) \underset{\wp_T[l_2]}{\wedge} (\mathsf{t}U_2) \\ \qquad \text{if } \wp_T[l_1] \not\sqsubseteq \wp_T[l_2]. \end{cases}$$

With definition 2.3.18, we will simply write

$$\mathsf{t}\left(T \bigcirc_{[l_i]} U_i \right) = (\mathsf{t}\, T) \bigwedge_{\wp_T[l_i]} (\mathsf{t}\, U_i).$$

Definition 2.3.18. Let M be a polynomial monad as in

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I,$$

with partial multiplication $\wedge : E \times_I B \longrightarrow B$. Let $b \in B$, and for each $e \in E(b)$, let $c_e \in B$ be such that $t(c_e) = s_e(b)$. Define

$$b \bigwedge_{e \in E(b)} c_e := \mathsf{t}\left(\mathsf{Y}_b \bigcirc_{[e]} \mathsf{Y}_{c_e} \right)$$

2.4 THE BAEZ–DOLAN CONSTRUCTION

As we saw in section 2.3, for $P \in \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$, the polynomial monad $P^\star$ is the algebraic structure describing graftings of P-trees (definition 2.3.3). In this section, we present a polynomial monad that describes *substitutions* of P-trees.

Definition 2.4.1 (Baez–Dolan $(-)^+$ construction)**.** Let M be a polynomial monad as in equation (2.2.5), and define its *Baez–Dolan construction* M^+ to be the polynomial functor

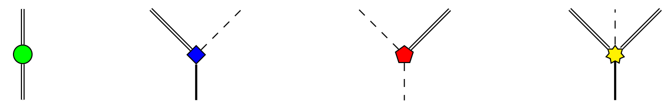
$$B \xleftarrow{\mathsf{s}} \operatorname{tr}^\bullet M \xrightarrow{p} \operatorname{tr} M \xrightarrow{\mathsf{t}} B \tag{2.4.2}$$

where s maps an M-tree with a marked node to the decoration of that node, p forgets the marking, and t is the target map of definition 2.3.11. If $T \in \operatorname{tr} M$, remark that $p^{-1}T = T^\bullet$ is the set of node addresses of T.

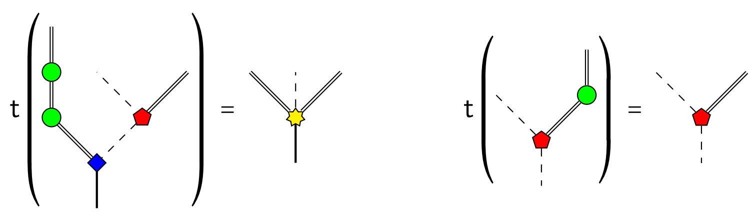
Example 2.4.3. Recall that a P-tree is simply a polynomial tree whose nodes are *decorated* by operations of P, and edges by colors of P (see example 2.2.10 for examples). In the case $P = M^+$, those operations are now M-trees, so a M^+-tree is a *tree of M-trees*. For example, let M be

$$\{\Vert, ¦, |\} \xleftarrow{s} E \xrightarrow{p} \{●, ◆, ⬟, ✸\} \xrightarrow{t} \{\Vert, ¦, |\}$$

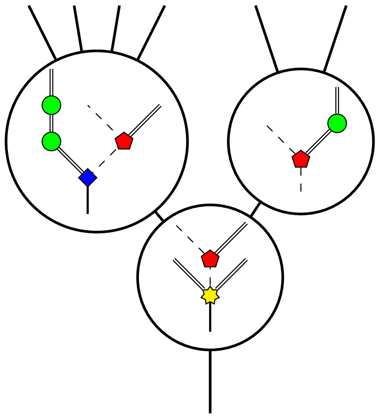
where, in the same fashion as example 2.2.10, its operations are represented as



Assume further that M has a structure of polynomial monad where



Then the following is a valid M^+-tree:



where the inputs of the root node correspond to the ✪ and ⬟ nodes of the decorating tree, respectively.

Remark 2.4.4 (Nested addresses). Let M be a polynomial monad, and $T \in \operatorname{tr} M^+$. Then the nodes of T are decorated in M-trees, and its edges by operations of M. Assume that $U \in \operatorname{tr} M$ decorates some node of T, say $U = \mathsf{s}_{[p]} T$ for some node address $[p] \in T^\bullet$.

(1) The input edges of that node are in bijection with $U^\bullet$. In particular, the address of those input edges are of the form $[p[q]]$, where $[q]$ ranges over $U^\bullet$. This really motivates enclosing addresses in brackets.

(2) On the other hand, the output edge of that node is decorated by $\mathsf{t} U$ (where t is defined in definition 2.3.11).

Notation 2.4.5. Let M be a polynomial monad, and $T \in \operatorname{tr} M^+$. For $[a]$ the addresses of an edge of T, let $\mathsf{e}_{[a]} T$ be the color of M^+ (i.e. operation of M) decorating that edge. Explicitly, if $[a] = []$, then $\mathsf{e}_{[]} T := \mathsf{t}\, \mathsf{s}_{[]} T$. Otherwise, $[a] = [p[q]]$ for some $[p] \in T^\bullet$ (the node below the edge) and $[q] \in (\mathsf{s}_{[p]} T)^\bullet$, and let $\mathsf{e}_{[p[q]]} T := \mathsf{s}_{[q]}\, \mathsf{s}_{[p]} T$.

Theorem 2.4.6 ([KJBM10, section 3.2])**.** *The polynomial functor M^+ has a canonical structure of a polynomial monad. Using the definition by partial laws (theorem 2.3.6):*

(Unit) *the unit $B \longrightarrow \operatorname{tr} M$ maps b to Y_b;*

(Partial multiplication) *the partial multiplication $\wedge : \operatorname{tr}^\bullet M \times_B \operatorname{tr} M \longrightarrow \operatorname{tr} M$ is given by substitution of trees (see definition 2.2.25), i.e. for $U \wedge_{[p]} T$ an admissible expression,*

$$U \underset{[p]}{\wedge} T := U \underset{[p]}{\square} T;$$

(Partial readdressing) *for $U \square_{[p]} T$ admissible, define $\rho_{U \square_{[p]} T}$ by (see remark 2.4.10 for a graphical explanation)*

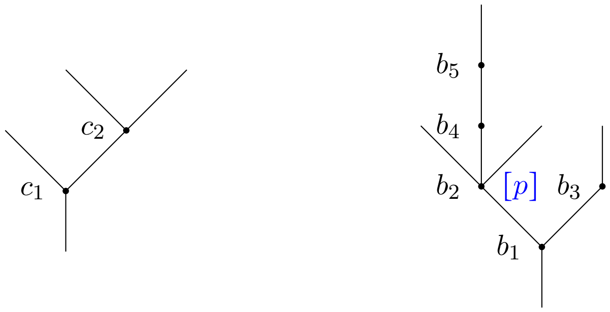
$$U^\bullet + T^\bullet - \{[p]\} \longrightarrow (U \underset{[p]}{\square} T)^\bullet$$

$$[q] \in T^\bullet \longmapsto [pq] \tag{2.4.7}$$

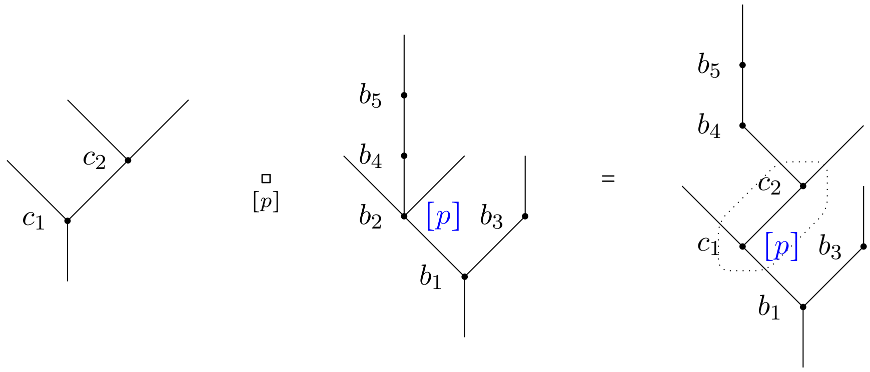
$$[p[e]p'] \in U^\bullet \longmapsto [p] \cdot \wp_T^{-1}(e) \cdot [p'] \tag{2.4.8}$$

$$[q] \in U^\bullet \longmapsto [q] \qquad \textit{if } [p] \not\sqsubseteq [q]. \tag{2.4.9}$$

Remark 2.4.10. Let T and U be M^+-trees as below (we omit the decorations for simplicity), and $[p] \in U^\bullet$ the address of a node of U, say b_2, written in blue on the right:

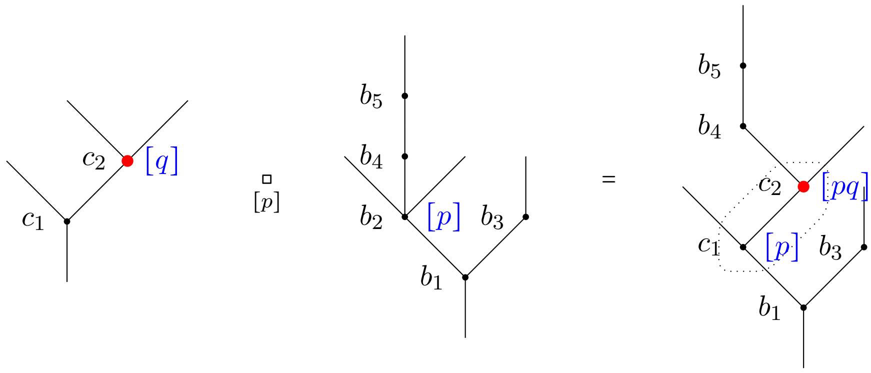


Assuming $\mathsf{t}\, T = \mathsf{s}_{[p]}\, U$, the expression $U \,\square_{[p]}\, T$ is admissible, and its evaluation gives



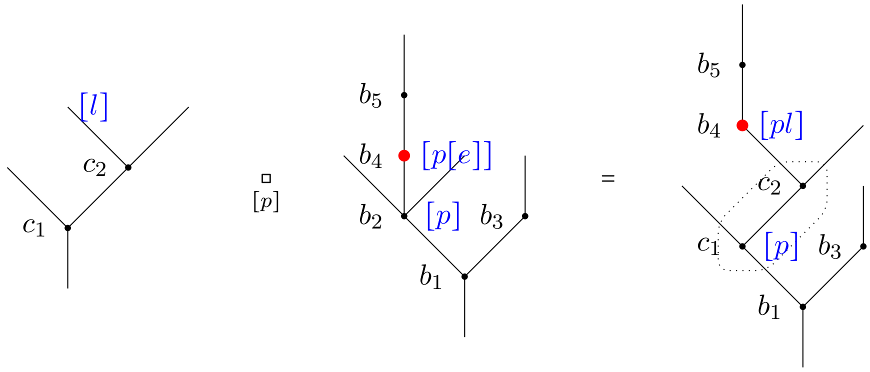
The map $\rho_{U \square_{[p]} T}$ establishes a bijection between $U^\bullet + T^\bullet - \{[p]\}$ and the set of node addresses of $U \,\square_{[p]}\, T$. Its definition is based on three cases.

Equation (2.4.7). If $[q] \in T^\bullet$, then the address of the corresponding node in $U \,\square_{[p]}\, T$ is simply $[pq]$. This reflects that the tree T has been inserted in U at address $[p]$.

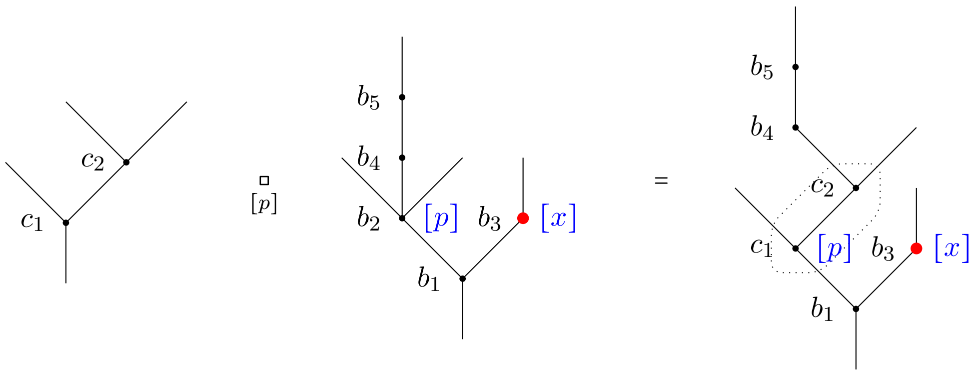


Equation (2.4.8). If a node d of U is located "above" the node that will be replaced by T, i.e. if $[p] \subsetneq \& d$, then $\& d$ necessarily decomposes as $\& d = [p[e]p']$, where $[e] \in (\mathsf{s}_{[p]}\, U)^\bullet$. In the example below, $d = b_4$, so that $[p'] = []$. On the other hand, there is a bijection $\wp_T$ between the leaves of T and the input edges of b_2. Assuming

$\wp_T^{-1}[e] = [l]$, the new address of b_4 in $U \square_{[p]} T$ is $[p] \cdot \wp_T^{-1}(e) \cdot [p'] = [plp']$.



Equation (2.4.9). The last case concerns nodes of U that are not located above b_2, and states that their address does not change.



Proof theorem 2.4.6, **(Trivial)**. For $b \in B$ we have $\mathsf{t}\,\mathsf{Y}_b = b$. On the other hand, $\mathsf{Y}_b^\bullet = \{[]\}$ and $\mathsf{s}_{[]}\,\mathsf{Y}_b = b$. □

Proof of **(Left unit)**. Let $b \in B$ and $T \in \operatorname{tr} M$ be such that $\mathsf{Y}_b \square_{[]} T$ is admissible. Then, by definition, $\mathsf{Y}_b \square_{[]} T = T$. Then, $\mathsf{Y}_b^\bullet + T^\bullet - \{[]\} = T^\bullet$, and $\rho_{\mathsf{Y}_b \square_{[]} T}$ maps $[q] \in T^\bullet$ to $[q]$, so it is indeed the identity. □

Proof of **(Right unit)**. Let $b \in B$, $T \in \operatorname{tr} M$, and $[p] \in T^\bullet$ be such that $T \square_{[p]} \mathsf{Y}_b$ is admissible. By definition, $T \square_{[p]} \mathsf{Y}_b = T$. Then, $\rho_{T \square_{[p]} \mathsf{Y}_b}$ is given by

$$\begin{aligned}
T^\bullet + \{[]\} - \{[p]\} &\longrightarrow T^\bullet \\
[p[e]p'] \in T^\bullet - \{[p]\} &\longmapsto [p] \cdot \rho_{\mathsf{Y}_b}^{-1} e \cdot [p'] \;=\; [pep'] \\
[p'] \in T^\bullet - \{[p]\} \text{ not as above} &\longmapsto [p'] \\
[] \in \mathsf{Y}_b^\bullet &\longmapsto [p]
\end{aligned}$$

as indeed ρ_{Y_b} maps $[e] \in (\mathsf{Y}_b)^{|}$ to $e \in E(b)$. □

Proof of **(Disjoint multiplication)**. Let $A \square_{[e]} B$ and $A \square_{[f]} C$ be two admissible expressions, where $[e] \neq [f]$. Without loss of generality, we distinguish two cases: one where $[e] \sqsubseteq [f]$, and one where $[e]$ and $[f]$ are $\sqsubseteq$-incomparable.

(1) Assume $[e] \sqsubseteq [f]$, so that $[f] = [eqr]$ for some e and r, and decompose A as

$$A \;=\; X \underset{[e]}{\circ} \mathsf{Y}_{\mathsf{t}B} \underset{[[q]]}{\circ} Y \underset{[r]}{\circ} \mathsf{Y}_{\mathsf{t}C} \underset{[[v_i]]}{\bigcirc} Z_i,$$

where $q \in E(\mathsf{t}B)$ and $\{v_i\}_i \subseteq E(\mathsf{t}C)$. Then

$$A \underset{[e]}{\square} B = X \underset{[e]}{\circ} B \underset{\rho_B^{-1}[q]}{\circ} Y \underset{[r]}{\circ} \mathsf{Y}_{\mathsf{t}C} \underset{[[v_i]]}{\bigcirc} Z_i,$$

and $\rho_{A \square_{[e]} B}[f] = \rho_{A \square_{[e]} B}[e[q]r] = [e] \cdot \rho_B^{-1}[q] \cdot [r]$. Thus,

$$(A \underset{[e]}{\square} B) \underset{[e]\cdot\rho_B^{-1}[q]\cdot[r]}{\square} C = X \underset{[e]}{\circ} B \underset{\rho_B^{-1}[q]}{\circ} Y \underset{[r]}{\circ} C \underset{\rho_C^{-1}[v_i]}{\bigcirc} Z_i,$$

and the reindexing $\rho_{(A \square_{[e]} B) \square_{\rho_{A \square_{[e]} B}[f]} C}\, \rho_{A \square_{[e]} B}$ is given by

$$\begin{array}{llllll}
[p] \in B^\bullet & \longmapsto & [ep] & & \longmapsto & [ep] \\
 & & [p] \in C^\bullet & & \longmapsto & [e] \cdot \rho_B^{-1}[q] \cdot [rp] \\
[eqs] \not\sqsubseteq [f] & \longmapsto & [e] \cdot \rho_B^{-1}[q] \cdot [s] & & \longmapsto & [e] \cdot \rho_B^{-1}[q] \cdot [s] \\
[fv_is] & \longmapsto & [e] \cdot \rho_B^{-1}[q] \cdot [r] \cdot [v_is] & & \longmapsto & [e] \cdot \rho_B^{-1}[q] \cdot [r] \cdot \rho_C^{-1}[v_i] \cdot [s] \\
[p] \in A^\bullet \ \text{ n.a.a.} & \longmapsto & [p] & & \longmapsto & [p]
\end{array}$$

(n.a.a. is an acronym for "not as above"). On the other hand, we have

$$A \underset{[f]}{\square} C = X \underset{[e]}{\circ} \mathsf{Y}_{\mathsf{t}B} \underset{[q]}{\circ} Y \underset{[r]}{\circ} C \underset{\rho_C^{-1}[v_i]}{\bigcirc} Z_i.$$

The reindexing gives $\rho_{A \square_{[f]} C}[e] = [e]$, and

$$\begin{aligned}
(A \underset{[f]}{\square} C) \underset{[e]}{\square} B &= X \underset{[e]}{\circ} B \underset{\rho_B^{-1}[q]}{\circ} Y \underset{[r]}{\circ} C \underset{\rho_C^{-1}[v_i]}{\bigcirc} Z_i \\
&= (A \underset{[e]}{\square} B) \underset{[e]\cdot\rho_B^{-1}[q]\cdot[r]}{\square} C.
\end{aligned}$$

The reindexing $\rho_{(A \square_{[f]} C) \square_{[e]} B}\, \rho_{A \square_{[f]} C}$ is given by

$$\begin{array}{lllll}
 & & [p] \in B^\bullet & \longmapsto & [ep] \\
[p] \in C^\bullet & \longmapsto & [fp] & \longmapsto & [e] \cdot \rho_B^{-1}[q] \cdot [rp] \\
[eqs] \not\sqsubseteq [f] & \longmapsto & [eqs] & \longmapsto & [e] \cdot \rho_B^{-1}[q] \cdot [s] \\
[fv_is] & \longmapsto & [f] \cdot \rho_C^{-1}[v_i] \cdot [s] & \longmapsto & [e] \cdot \rho_B^{-1}[q] \cdot [r] \cdot \rho_C^{-1}[v_i] \cdot [s] \\
[p] \in A^\bullet \ \text{ n.a.a.} & \longmapsto & [p] & \longmapsto & [p]
\end{array}$$

We see that the square equation (2.3.7) commutes in the case $[e] \sqsubseteq [f]$.

(2) Assume $[e]$ and $[f]$ are $\sqsubseteq$-incomparable, and write A as

$$A = (X \underset{[e]}{\circ} \mathsf{Y}_{\mathsf{t}B} \underset{[v_i]}{\bigcirc} Y_i) \underset{[f]}{\circ} \mathsf{Y}_{\mathsf{t}C} \underset{[w_j]}{\bigcirc} Z_j.$$

Then

$$A \underset{[e]}{\square} B = (X \underset{[e]}{\circ} B \underset{\rho_B^{-1}[v_i]}{\bigcirc} Y_i) \underset{[f]}{\circ} \mathsf{Y}_{\mathsf{t}C} \underset{[w_j]}{\bigcirc} Z_j,$$

the reindexing gives $\rho_{A \square_{[e]} B}[f] = [f]$,

$$(A \underset{[e]}{\square} B) \underset{[f]}{\square} C = (X \underset{[e]}{\circ} B \underset{\rho_B^{-1}[v_i]}{\bigcirc} Y_i) \underset{[f]}{\circ} C \underset{\rho_C^{-1}[w_j]}{\bigcirc} Z_j,$$

and the complete reindexing $\rho_{(A \square_{[e]} B) \square_{[f]} C}\, \rho_{A \square_{[e]} B}$ is given by

$$\begin{array}{lllll}
[p] \in B^\bullet & \longmapsto & [ep] & \longmapsto & [ep] \\
 & & [p] \in C^\bullet & \longmapsto & [fp] \\
[ev_is] & \longmapsto & [e] \cdot \rho_B^{-1}[v_i] \cdot [s] & \longmapsto & [e] \cdot \rho_B^{-1}[v_i] \cdot [s] \\
[fw_js] & \longmapsto & [fw_js] & \longmapsto & [f] \cdot \rho_C^{-1}[w_j] \cdot [s] \\
[p] \in A^\bullet \ \text{ n.a.a.} & \longmapsto & [p] & \longmapsto & [p]
\end{array}$$

On the other hand,

$$A \underset{[f]}{\square} C = (X \underset{[e]}{\circ} \mathsf{Y}_{\mathsf{t}B} \bigcirc_{[v_i]} Y_i) \underset{[f]}{\circ} C \bigcirc_{\rho_C^{-1}[w_j]} Z_j,$$

we have $\rho_{A \square_{[f]} C}[e] = [e]$,

$$\begin{aligned}(A \underset{[f]}{\square} C) \underset{[e]}{\square} B &= (X \underset{[e]}{\circ} B \bigcirc_{\rho_B^{-1}[v_i]} Y_i) \underset{[f]}{\circ} C \bigcirc_{\rho_C^{-1}[w_j]} Z_j \\ &= (A \underset{[e]}{\square} B) \underset{[f]}{\square} C,\end{aligned}$$

and further

$$\begin{array}{llllll} & & [p] \in B^\bullet & \longmapsto & [ep] \\ [p] \in C^\bullet & \longmapsto & [fp] & \longmapsto & [fp] \\ [ev_i s] & \longmapsto & [ev_i s] & \longmapsto & [e] \cdot \rho_B^{-1}[v_i] \cdot [s] \\ [fw_j s] & \longmapsto & [f] \cdot \rho_C^{-1}[w_j] \cdot [s] & \longmapsto & [f] \cdot \rho_C^{-1}[w_j] \cdot [s] \\ [p] \in A^\bullet \ \text{ n.a.a.} & \longmapsto & [p] & \longmapsto & [p] \end{array}$$

so that the square equation (2.3.7) commutes in the case where $[e]$ and $[f]$ are $\sqsubseteq$-incomparable too. Finally, the monad structure of M^+ satisfies condition **(Disjoint multiplication)** of theorem 2.3.6. $\square$

Proof of **(Nested multiplication)**. Let $A, B, C \in \operatorname{tr} M$, $[e] \in A^\bullet$, $[f] \in B^\bullet$, such that $AA \square_{[e]} B$ and $B \square_{[f]} C$ are admissible. Write A and B as:

$$A = (X \underset{[e]}{\circ} \mathsf{Y}_{\mathsf{t}B} \bigcirc_{[v_i]} Y_i), \qquad B = Z \underset{[f]}{\circ} \mathsf{Y}_{\mathsf{t}C} \bigcirc_{[w_j]} T_j.$$

Then,

$$A \underset{[e]}{\square} B = X \underset{[e]}{\circ} B \bigcirc_{\rho_B^{-1}[v_i]} Y_i,$$

we have $\rho_{A \square_{[e]} B}[f] = [ef]$, and

$$(A \underset{[e]}{\square} B) \underset{[ef]}{\square} C = X \underset{[e]}{\circ} (Z \underset{[f]}{\circ} \mathsf{Y}_{\mathsf{t}C} \bigcirc_{[w_j]} T_j) \bigcirc_{\alpha(\rho_B^{-1}[v_i])} Y_i,$$

where

$$\alpha(\rho_B^{-1}[v_i]) = \begin{cases} [f] \cdot \rho_C^{-1}[w_j] \cdot [r] & \text{if } \rho_B^{-1}[v_i] \text{ of the form } [fw_j r], \\ \rho_B^{-1}[v_i] & \text{otherwise.} \end{cases}$$

Remark that $\alpha(\rho_B^{-1}[v_i]) = \rho_{B \square_{[f]} C}^{-1} v_i$. The reindexing $\rho_{(A \square_{[e]} B) \square_{[ef]} C}\, \rho_{A \square_{[e]} B}$ is given by:

$$\begin{array}{lllll} & & [p] \in C^\bullet & \longmapsto & [efp] \\ [fw_j r] \in B^\bullet & \longmapsto & [efw_j r] & \longmapsto & [ef] \cdot \rho_C^{-1}[w_j] \cdot [r] \\ [p] \in B^\bullet,\ [f] \not\sqsubseteq [p] & \longmapsto & [ep] & \longmapsto & [ep] \\ [ev_i r] \in A^\bullet & \longmapsto & [e] \cdot \rho_B^{-1} v_i \cdot [r] & \longmapsto & [e] \cdot \rho_{B \square_{[f]} C}^{-1} v_i \cdot [r] \end{array}$$

On the other hand, we have

$$B \underset{[f]}{\square} C = Z \underset{[f]}{\circ} C \bigcirc_{\rho_C^{-1}[w_j]} T_j,$$

$$
\begin{aligned}
A \underset{[e]}{\square} (B \underset{[f]}{\square} C) &= X \underset{[e]}{\circ} (Z \underset{[f]}{\circ} \mathsf{Y}_{\mathsf{t}C} \underset{[w_j]}{\bigcirc} T_j) \underset{\rho^{-1}_{B \square_{[f]} C} v_i}{\bigcirc} Y_i \\
&= (A \underset{[e]}{\square} B) \underset{[ef]}{\square} C
\end{aligned}
$$

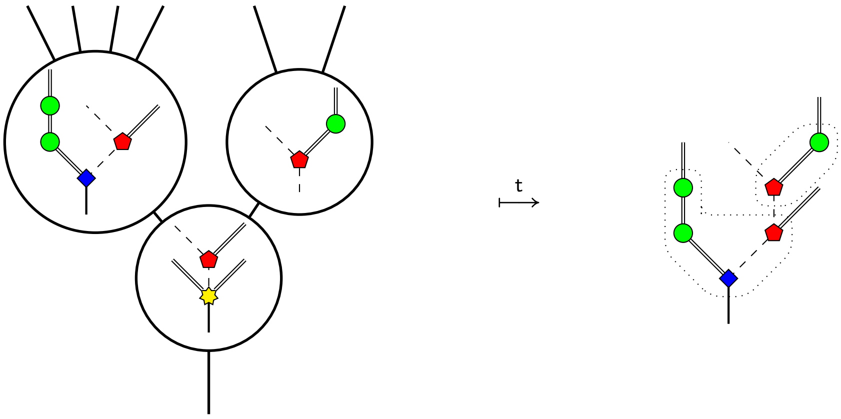
and the reindexing is given by

$$
\begin{array}{llllll}
[p] \in C^\bullet & \longmapsto & [fp] & \longmapsto & [efp] \\
[fw_j r] \in B^\bullet & \longmapsto & [f] \cdot \rho_C^{-1}[w_j] \cdot [r] & \longmapsto & [ef] \cdot \rho_C^{-1}[w_j] \cdot [r] \\
[p] \in B^\bullet,\ [f] \subsetneq [p] & \longmapsto & [p] & \longmapsto & [ep] \\
 & & [ev_i r] \in A^\bullet & \longmapsto & [e] \cdot \rho^{-1}_{B \square_{[f]} C}[v_i] \cdot [r]
\end{array}
$$

We thus see that the square equation (2.3.8) commutes, and that the monad structure of M^+ satisfies condition **(Nested multiplication)**. $\square$

This completes the proof of theorem 2.4.6, endowing M^+ with a canonical monad structure, whose partial law is given by substitution of M-trees.

Example 2.4.11. Consider the M^+-tree T of example example 2.4.3 on the left below, where the inputs of the root node correspond to the ✩ and ⬟ nodes of the decorating tree, respectively.



According to the monad law on M^+ of theorem 2.4.6, the target $\mathsf{t}T \in \operatorname{tr} M$ on the right is the substitution of the lower decorating M-tree by the two upper decorating M-trees, following the scheme of the outer thick tree. Since M is itself a polynomial monad, we may compute the iterated target $\mathsf{tt}T$, which would be an operation of M.

Definition 2.4.12 (Simultaneous substitution)**.** We specify definition 2.3.18 in the case of M^+. Let $T \in \operatorname{tr} M$, and for each $[p] \in T^\bullet$, let $U_{[p]} \in \operatorname{tr} M$ be such that $\mathsf{t}U_{[p]} = \mathsf{s}_{[p]} T$. Define the *simultaneous substitution*

$$
T \underset{[p]}{\square} U_{[p]} \coloneqq \mathsf{t}\left(\mathsf{Y}_T \underset{[[p]]}{\bigcirc} \mathsf{Y}_{U_{[p]}} \right).
$$

Intuitively, it consists of simultaneously substituting the $U_{[p]}$'s in T, and writing it down this way spares the tedious case analysis of remark 2.3.17.

The following fact is at the heart of the Baez–Dolan construction. Indeed, it is even the original *definition* of the construction, see [BD98, definition 15].

Proposition 2.4.13. *For $M \in \mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}(I)$ a polynomial monad, there is an equivalence of categories $\mathcal{A}\mathrm{lg}(M^+) \simeq \mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}(I)/M$.*

Proof. We construct mutually weakly inverse equivalences between the two categories. First, write M as

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I.$$

Let $x : X \longrightarrow B$ be a set over B, and $m : M^+X \longrightarrow X$ be an M^+-algebra strucure on X. Let E_X be the pullback on the left, and define $\Phi X \in \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)/M$ as the morphisms of polynomial functors on the right:

$$\begin{tikzcd} E_X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "x"] \\ E \arrow[r, "p"] & B, \end{tikzcd} \qquad \begin{tikzcd} I \arrow[d, equal] & E_X \arrow[l, "s\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[r, "tx"] \arrow[d, "x"'] & I \arrow[d, equal] \\ I & E \arrow[l, "s"] \arrow[r, "p"] & B \arrow[r, "t"] & I. \end{tikzcd}$$

There is an evident bijection $\mathrm{tr}\,\Phi X \cong M^+X$ in $\mathcal{S}\mathrm{et}/I$, and the structure map m extends by pullback along $E_X \longrightarrow X$ to a map $(\Phi X)^\star \longrightarrow \Phi X$ in $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$. It is easy to verify that this determines a $(-)^\star$-algebra structure on ΦX, and that the map $\Phi X \longrightarrow M$ in $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$ is a morphism of $(-)^\star$-algebras. Conversely, given an $N \in \mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}(I)/M$ whose underlying polynomial functor is

$$I \longleftarrow E' \longrightarrow B' \longrightarrow I,$$

then the bijection $\mathrm{tr}\,N \cong M^+B'$ in $\mathcal{S}\mathrm{et}/I$ and the $(-)^\star$-algebra map $N^\star \longrightarrow N$ provide a map $M^+B' \xrightarrow{\Psi N} B'$ in $\mathcal{S}\mathrm{et}/I$. It is easy to verify that ΨN is the structure map of a M^+-algebra and that the constructions Φ and Ψ are functorial and mutually inverse. $\square$

Chapter Three

Opetopes and opetopic sets

This chapter deals with the main notion of this thesis: opetopes. Opetopes were first introduced by Baez and Dolan in order to formulate a definition of weak ω-categories [BD98]. Their name reflects the fact that they encode the possible shapes for higher-dimensional operations: they are *ope*ration poly*topes*. Informally, an opetope is either a point, or a tree of opetopes of lower dimension, i.e. a tree of trees of trees of...

Over the recent years, they have been the subject of many efforts to provide a good definition that would allow exploring their combinatorics. The original definition of [BD98] relies on an operadic version of the $(-)^+$ construction of section 2.4. A notion of opetopic set is also introduced, but not as presheaves over a category of opetopes. This gap is filled in [Che03a], where Cheng reformulates the definition of Baez and Dolan in terms of symmetric multicategories and slicing (an analogue to the $(-)^+$ construction, see proposition 2.4.13), and using this more explicit formalism, introduces a category of opetopes. Non symmetrical approaches have also been explored. In [Lei04, chapter 7], Leinster proposes a definition based on the T-operads of Burroni [Bur71]. Here, the $(-)^+$ construction corresponds to the "free T-operad" monad. Cheng proved that this definition is equivalent to the symmetric one [Che04a, corollary 2.6].

In this thesis, we shall rely on the approach of Kock et. al. [KJBM10]. There, the authors tackle the subtle tree-like structure of opetopes directly, using the formalism of polynomial functors and trees, presented in chapter 2. Note that this definition is equivalent to all the above ones [KJBM10, theorem 3.16].

It is worthwile to mention that the formalism of [KJBM10] has been used for yet another approach to opetopes by Steiner [Ste12], where trees are replaced by *opetopic chain complexes*, a combinatorial device encoding the internal structure of opetopes.

3.1 DEFINITION

In this section, we formulate the definition of opetopes using the formalism of polynomial functors, monads and trees, surveyed in chapter 2.

Definition 3.1.1 (The $\mathfrak{Z}^n$ monad)**.** Let $\mathfrak{Z}^0$ be the identity polynomial monad on $\mathcal{S}\mathrm{et}$, which has one color and one operation with one input (see example 2.1.7). We write it as

$$\{\blacklozenge\} \longleftarrow \{*\} \longrightarrow \{\blacksquare\} \longrightarrow \{\blacklozenge\}.$$

For $n \geq 1$, let $\mathfrak{Z}^n := (\mathfrak{Z}^{n-1})^+$ (see definition 2.4.1), and write it as $\mathfrak{Z}^n$ as

$$\mathbb{O}_n \xleftarrow{\mathsf{s}} E_{n+1} \xrightarrow{p} \mathbb{O}_{n+1} \xrightarrow{\mathsf{t}} \mathbb{O}_n, \tag{3.1.2}$$

i.e. for all $n \in \mathbb{N}$, $\mathbb{O}_n$ is the set of colors of $\mathfrak{Z}^n$ (or equivalently, the set of operations of $\mathfrak{Z}^{n-1}$ if $n \geq 1$).

Definition 3.1.3 (Opetope)**.** An *n-dimensional opetope* (or *n-opetope* for short) ω is simply an element of $\mathbb{O}_n$, and we write $\dim \omega = n$. If $n \geq 2$, then by definitions 2.4.1 and 3.1.1, n-opetopes are exactly the $\mathfrak{Z}^{n-2}$-trees. In this case, an opetope $\omega \in \mathbb{O}_n$ is called *degenerate* if its underlying tree has no nodes (and thus consists of a unique edge), so that $\omega = \mathsf{I}_\phi$ for some $\phi \in \mathbb{O}_{n-2}$. We say that ω it is an *endotope* it its underlying tree has exactly one node, i.e. $\omega = \mathsf{Y}_\psi$ for some $\psi \in \mathbb{O}_{n-1}$.

Following equation (2.3.12), for $n \geq 2$ and $\omega \in \mathbb{O}_n$, the structure of polynomial monad $(\mathfrak{Z}^{n-2})^\star \longrightarrow \mathfrak{Z}^{n-2}$ gives a bijection $\wp_\omega : \omega^| \longrightarrow (\mathsf{t}\omega)^\bullet$ between the leaves of ω and the nodes of $\mathsf{t}\omega$, preserving the decoration by $(n-2)$-opetopes.

Example 3.1.4.
1. The unique 0-opetope is denoted by $\blacklozenge$ and called the *point*.
2. The unique 1-opetope is denoted by $\blacksquare$ and called the *arrow*.
3. If $n \geq 2$, then $\omega \in \mathbb{O}_n$ is a $\mathfrak{Z}^{n-2}$-tree, i.e. a tree whose nodes are labeled in $(n-1)$-opetopes, and edges are labeled in $(n-2)$-opetopes. In particular, 2-opetopes are $\mathfrak{Z}^0$-trees, i.e. linear trees, and thus in bijection with $\mathbb{N}$. We will refer to them as *opetopic integers*, and write $\mathbf{n}$ for the 2-opetope having exactly n nodes.
4. A 3-opetope is a $\mathfrak{Z}^1$-tree, i.e. a planar tree.
5. A 4-opetope is a $\mathfrak{Z}^2$-tree. Unfolding definitions, if $\omega : \langle\omega\rangle \longrightarrow \mathfrak{Z}^2$, then nodes of ω are decorated by elements of $\mathbb{O}_3$, i.e. planar trees. Further, if $x \in \langle\omega\rangle_1$ is a node of ω, then ω_2 exhibits a bijection between the input edges of x and the nodes of $\omega_1(x) \in \mathbb{O}_3$.

Proposition 3.1.5. *For small values of n, the category $\mathcal{A}\mathrm{lg}_{0,n}$ of $\mathfrak{Z}^n$-algebras (the first index shall become relevant in chapter 11) is given by the following table*

n	0	1	2	3
$\mathcal{A}\mathrm{lg}_{0,n}$	$\mathcal{S}\mathrm{et}$	$\mathcal{M}\mathrm{on}$	$\mathcal{O}\mathrm{p}$	$\mathcal{C}\mathrm{omb}_{\mathbb{PT}}$

where $\mathcal{M}\mathrm{on}$ is the category of monoids, $\mathcal{O}\mathrm{p}$ of non colored planar operads, and $\mathcal{C}\mathrm{omb}_{\mathbb{PT}}$ of combinads over the combinatorial pattern of planar trees [Lod12].

Proof (sketch).
1. If $n = 0$, then $\mathfrak{Z}^0$ is by definition the identity functor on $\mathcal{S}\mathrm{et}/\mathbb{O}_0 = \mathcal{S}\mathrm{et}$, thus $\mathfrak{Z}^0$-algebras bear no structure, and are simply sets.
2. The polynomial monad $\mathfrak{Z}^1 = (\mathfrak{Z}^0)^+$ is isomorphic to

$$\{\blacksquare\} \xleftarrow{\mathsf{s}} \mathbb{N}_< \longrightarrow \mathbb{N} \xrightarrow{\mathsf{t}} \{\blacksquare\}$$

where for $m \in \mathbb{N}$, $\mathbb{N}_<(m) := \{0, 1, \ldots, m-1\}$, and this case is already treated in example 2.1.7.
3. The functor $\mathfrak{Z}^2 : \mathcal{S}\mathrm{et}/\mathbb{N} \longrightarrow \mathcal{S}\mathrm{et}/\mathbb{N}$ maps a signature $X = (X_m \mid m \in \mathbb{N}) \in \mathcal{S}\mathrm{et}/\mathbb{N}$ to the set of trees whose nodes are adequately decorated by elements of X, i.e. it is the free planar operad monad.
4. A $\mathfrak{Z}^4$-algebra is a set of "planar trees" (i.e. an element of $\mathcal{S}\mathrm{et}/\mathbb{O}_3$) with an suitable notion of substitution, which is structure encapsulated in the notion of $\mathbb{PT}$-combinad. $\square$

Proposition 3.1.6. *Let $\omega \in \mathbb{O}_n$ with $n \geq 2$. We have the following.*

(1) *If ω is degenerate, say $\omega = \mathsf{I}_\phi$ for some $\phi \in \mathbb{O}_{n-2}$, then $\mathsf{t}\omega = \mathsf{Y}_\phi$, and $\wp_\omega : \omega^{|} = \{[]\} \longrightarrow \mathsf{Y}_\phi^\bullet = \{[]\}$ obviously maps $[]$ to $[]$.*

(2) *If ω is an endotope, say $\omega = \mathsf{Y}_\psi$ for some $\psi \in \mathbb{O}_{n-1}$, then $\mathsf{t}\omega = \psi$. Further, $\omega^{|} = \{[[q]] \mid [q] \in \psi^\bullet\}$, and $\wp_\omega$ maps $[[q]]$ to $[q]$.*

(3) *Otherwise, ω decomposes as $\omega = \nu \circ_{[l]} \mathsf{Y}_\psi$, for some $\nu \in \mathbb{O}_n$, $\psi \in \mathbb{O}_{n-1}$, and $[l] \in \nu^{|}$, and*

$$\mathsf{t}\omega = (\mathsf{t}\nu) \underset{\wp_\nu [l]}{\square} \psi.$$

The readdressing function $\wp_\omega : \omega^{|} \longrightarrow (\mathsf{t}\omega)^\bullet$ is given as follows. Let $[j] \in \omega^{|}$.

  a) If $[l] \sqsubseteq [j]$, then $[j] = [l[q]]$ for some $[q] \in \psi^\bullet$, and $\wp_\omega[l[q]] = (\wp_\nu[l]) \cdot [q]$.

  b) If $[l] \not\sqsubseteq [j]$, then $[j] \in \nu^{|}$. Assume $\wp_\nu[l] \sqsubseteq \wp_\nu[j]$. Then $\wp_\nu[j] = (\wp_\nu[l]) \cdot [[q]] \cdot [a]$, for some $[q] \in (\mathsf{s}_{\wp_\nu[l]} \mathsf{t}\nu)^\bullet = (\mathsf{t}\psi)^\bullet$, and let $\wp_\omega[j] = (\wp_\nu[l]) \cdot (\wp_\psi^{-1}[q]) \cdot [a]$.

  c) If $\wp_\nu[l] \not\sqsubseteq \wp_\nu[j]$, then $\wp_\omega[j] = \wp_\nu[j]$.

Proof. Direct consequence of proposition 2.3.14 and theorem 2.4.6. □

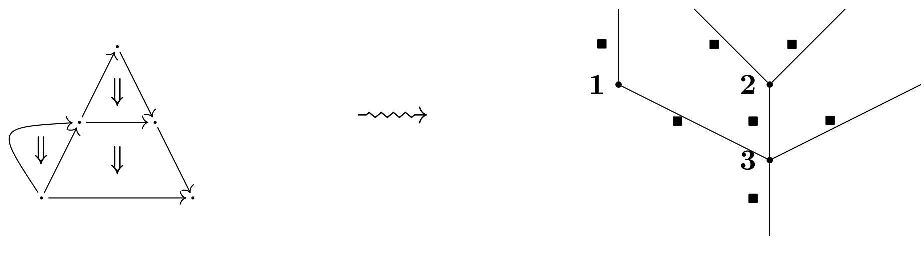
Definition 3.1.7 (Partition of opetopes)**.** Let $\mathbb{O}_{n+2}^{(2)}$ be the set of $(n+2)$-opetopes *of uniform height* 2, i.e. of the form

$$\mathsf{Y}_\alpha \bigcirc_{[[p]]} \mathsf{Y}_{\beta_{[p]}},$$

with $\alpha, \beta_{[p]} \in \mathbb{O}_{n+1}$ and $[p]$ ranging over $\alpha^\bullet$. If $\xi \in \mathbb{O}_{n+2}^{(2)}$ is as above, and $\nu = \mathsf{t}\xi = \alpha \square_{[p]} \beta_{[p]}$, then the $\beta_{[p]}$'s are disjoint subtrees of ν, which jointly cover ν. Thus, ξ exhibits a partition of ν into subtrees.

3.2 OPETOPES VS. PASTING DIAGRAMS

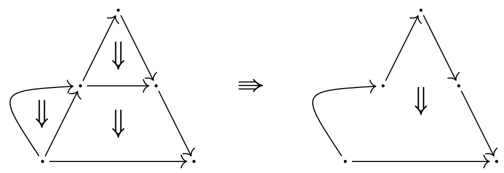
Opetopes are closely related to the notion of *pasting diagram* commonly used in higher category theory to describe composition of higher cells. Informally, a pasting diagram[1] of dimension n is a tree whose nodes are decorated with n-*cells*, edges with $(n-1)$-cells, and where the output edge of a node corresponds to its target, and the input edges to the cells in its source. For instance, the figure on the left below is a graphical representation of a 2-pasting diagram, and the corresponding decorated tree is drawn on the right:



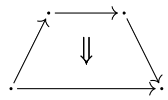
If we consider the k-cells as k-opetopes for all k, then an n-pasting diagram P with $n \geq 1$ induces a $\mathfrak{Z}^{n-1}$-tree, i.e. a $(n+1)$-opetope, say ω. We say that P is the *source pasting diagram* of ω. The opetope ω also has a target $\mathsf{t}\omega \in \mathbb{O}_{n-1}$, and in the sequel, graphical

[1] In this thesis, we only consider *many-to-one* pasting diagrams, i.e. those whose output consist of a single cell.

representations of opetopes include both the source pasting diagram and the target. For instance, if P is the pasting diagram above, then ω is represented by



The dichotomy between pasting diagrams and opetopes can lead to ambiguities. For example,



represents a 2-opetope (the opetopic integer **3**), but also a 2-pasting diagram having a unique 2-cell. In this work, such ambiguities shall be resolved by the surrounding context.

3.3 HIGHER ADDRESSES

By definition, an opetope ω of dimension $n \geq 2$ is a $\mathfrak{Z}^{n-2}$-tree, and thus the formalism of tree addresses (definition 2.2.11 and remark 2.4.4) can be applied to reference nodes of ω, also called its *source faces* or simply *sources*. In this section, we iterate this formalism into the concept of *higher dimensional address*.

Definition 3.3.1 (Higher address)**.** The set $\mathbb{A}_n$ of *n-addresses* is defined as follows. The unique 0-address is $*$ (also written $[]$ by convention), while an $(n+1)$-address is a sequence of n-addresses, enclosed by brackets. Note that the address $[]$, associated to the empty sequence, is in $\mathbb{A}_n$ for all $n \geq 0$. However, the surrounding context will almost always make the notation unambiguous.

Example 3.3.2. Since the unique element of $\mathbb{A}_0$ is $*$, the set of 1-addresses is

$$\mathbb{A}_1 = \{[\underbrace{* * \cdots *}_{k}] \mid k \in \mathbb{N}\}.$$

A 1-address $[* * \cdots *]$ where $*$ occurs k times will be more concisely written $[*^k]$. The following are higher addresses[2]:

$$[[][*][]] \in \mathbb{A}_2, \qquad [[[][*][**]][[* * *]]] \in \mathbb{A}_3, \qquad [[[[*]]]] \in \mathbb{A}_4.$$

The expression $[*[*]]$ is not a valid higher address, as $*$ and $[*]$ do not have the same dimension.

For $\omega \in \mathbb{O}$ an opetope, nodes of ω can be specified uniquely using higher addresses, as we now show. Recall that E_{n-1} is the set of inputs of $\mathfrak{Z}^{n-2}$ (equation (3.1.2)). In $\mathfrak{Z}^0$, set $E_1(\blacksquare) = \{*\}$, so that the unique "node address"[3] of $\blacksquare$ is $* \in \mathbb{A}_0$.

[2] Ambiguity with the dimension of addresses could be lifted altogether by indicating the dimension as e.g. $[]^1 \in \mathbb{A}_1$, $[* * * *]^1 \in \mathbb{A}_1$, $[[]^1[*]^1[]^1]^2 \in \mathbb{A}_2$, $[[[[*]^1]^2]^3]^4 \in \mathbb{A}_4$. However, this makes notations significantly heavier, so we avoid using this convention.

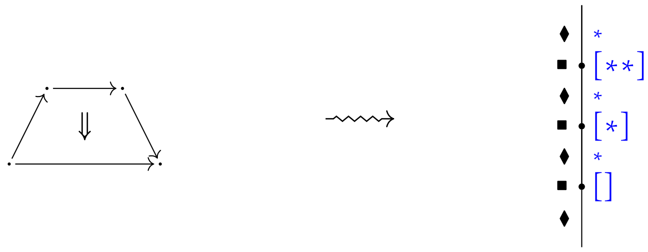
[3] Of course, $\blacksquare$ is not a tree, but this abuse of terminology allows to talk about higher addresses and opetopes in a uniform manner.

For $n \geq 2$, recall that an opetope $\omega \in \mathbb{O}_n$ is a $\mathfrak{Z}^{n-2}$-tree $\omega : \langle \omega \rangle \longrightarrow \mathfrak{Z}^{n-2}$ (definition 2.2.7), and write $\langle \omega \rangle$ as

$$I_\omega \longleftarrow E_\omega \longrightarrow B_\omega \longrightarrow I_\omega.$$

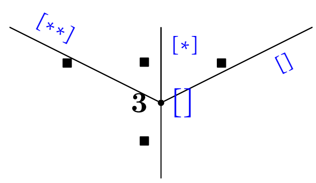
A node $b \in B_\omega$ has an address $\& b = [e_1 \cdots e_k]$ which is a list of elements of E_ω. By $\omega_2 : E_\omega \longrightarrow E_{n-1}$, it corresponds to a list $[p] := [\omega_2(e_1) \cdots \omega_2(e_k)]$ of elements of E_{n-1}, and since by induction elements of E_{n-1} are $(n-2)$-addresses, $[p] \in \mathbb{A}_{n-1}$. We now identify the nodes of ω to their addresses, which completes the induction process. In particular, for $[p] \in \omega^\bullet$ a node address of ω, we make use of the notation $\mathsf{s}_{[p]}\,\omega$ of section 2.2 to refer to the decoration of the node at address $[p]$, which is an $(n-1)$-opetope.

Example 3.3.3. Consider the 2-opetope on the left, called **3**:

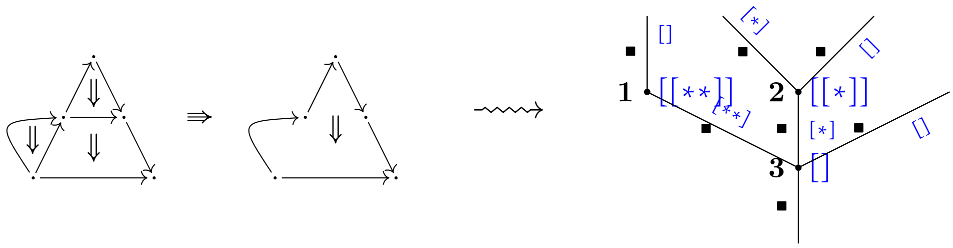


Its underlying pasting diagram consists of 3 arrows $\blacksquare$ grafted linearly. Since the only node address of $\blacksquare$ is $* \in \mathbb{A}_0$, the underlying tree of **3** can be depicted as on the right. On the left of that tree are the decorations: nodes are decorated with $\blacksquare \in \mathbb{O}_1$, while edges are decorated with $\blacklozenge \in \mathbb{O}_0$. For each node in the tree, the set of input edges of that node is in bijective correspondence with the node addresses of the decorating opetope, written on the right of each edge. In this low dimensional example, those addresses can only be $*$. Finally, on the right of each node of the tree is its 1-address, which is just a sequence of 0-addresses giving "walking instructions" to get from the root to that node.

The 2-opetope **3** can then be seen as a corolla in some 3-opetope as follows:



As previously mentioned, the set of input edges is in bijective correspondence with the set of node addresses of **3**. Here is now an example of a 3-opetope (already studied in section 3.2), with its annotated underlying tree on the right (the 2-opetopes **1** and **2** are analogous to **3**):



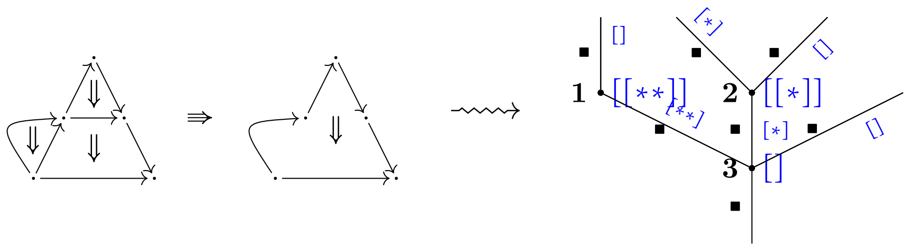
Further, the leaf addresses of this opetope are $[[]]$, $[[*][]]$, $[[*][*]]$, and $[[**][]]$.

Definition 3.3.4. (1) If $[p_1], [p_2] \in \mathbb{A}_n$, then their *concatenation* is $[p_1] \cdot [p_2] := [p_1 p_2]$. In particular, $[p_1] \cdot [] = [] \cdot [p_1] = [p_1]$.

(2) Let $\sqsubseteq$ be the *prefix order* on $\mathbb{A}_n$, i.e. $[p_1] \sqsubseteq [p_2]$ if and only if there exists $[p_3] \in \mathbb{A}_n$ such that $[p_1] \cdot [p_3] = [p_2]$.

(3) Let $\leq$ be the *lexicographical order* on $\mathbb{A}_n$. It is trivial on $\mathbb{A}_0$, given by the prefix order on $\mathbb{A}_1$, and on $\mathbb{A}_n$, it is induced by lexicographical order on $\mathbb{A}_{n-1}$.

Example 3.3.5. Consider the 3-opetope of example 3.3.3:



On nodes, we have the ordering $[] < [[*]] < [[**]]$, and on edges, $[] < [[]] < [[*]] < [[*][]] < [[*][*]] < [[**]] < [[**][]]$. In particular, leaves are ordered as $[[]] < [[*][]] < [[*][*]] < [[**][]]$.

3.4 THE CATEGORY OF OPETOPES

In this section we define the category $\mathbb{O}$ of opetopes by generators and relations, akin to the works of Hermida–Makkai–Power [HMP02, section 2] and Cheng [Che03a].

Lemma 3.4.1 (Opetopic identities)**.** *Let $\omega \in \mathbb{O}_n$ with $n \geq 2$.*

Inner edge. For an inner edge $[p[q]] \in \omega^\bullet$ (the fact that ω has an inner edge implies that it is non degenerate), we have $\mathsf{t}\,\mathsf{s}_{[p[q]]}\,\omega = \mathsf{s}_{[q]}\,\mathsf{s}_{[p]}\,\omega$.

Globularity 1. If ω is non degenerate, we have $\mathsf{t}\,\mathsf{s}_{[]}\,\omega = \mathsf{t}\,\mathsf{t}\,\omega$.

Globularity 2. If ω is non degenerate, and $[p[q]] \in \omega^{|}$, we have $\mathsf{s}_{[q]}\,\mathsf{s}_{[p]}\,\omega = \mathsf{s}_{\wp_\omega [p[q]]}\,\mathsf{t}\,\omega$.

Degeneracy. If ω is degenerate, we have $\mathsf{s}_{[]}\,\mathsf{t}\,\omega = \mathsf{t}\,\mathsf{t}\,\omega$.

Proof. *Inner edge.* By definition of a $\mathfrak{Z}^{n-2}$-tree.

Globularity 1 and 2. By theorem 2.3.10, the monad structure on $\mathfrak{Z}^{n-2}$ amounts to a structure map $(\mathfrak{Z}^{n-2})^\star \longrightarrow \mathfrak{Z}^{n-2}$, which, taking the notations of definition 2.3.11, is written as

$$\begin{tikzcd}
\mathbb{O}_{n-2} \arrow[d, equal] & \operatorname{tr}^{|} \mathfrak{Z}^{n-2} \arrow[l, "\mathsf{e}"'] \arrow[r, "p"] \arrow[d, "\wp"'] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{tr} \mathfrak{Z}^{n-2} \arrow[r, "\mathsf{e}_{[]}"] \arrow[d, "\mathsf{t}"'] & \mathbb{O}_{n-2} \arrow[d, equal] \\
\mathbb{O}_{n-2} & \mathbb{O}^\bullet_{n-1} \arrow[l, "\mathsf{s}"'] \arrow[r, "p"] & \mathbb{O}_{n-1} \arrow[r, "\mathsf{t}"] & \mathbb{O}_{n-2}.
\end{tikzcd}$$

The claims follow from the commutativity of the right and left square respectively.

Degeneracy. Let $\omega = \mathsf{I}_\phi$, for $\phi \in \mathbb{O}_{n-2}$. By proposition 3.1.6, $\mathsf{t}\,\mathsf{t}\,\omega = \mathsf{t}\,\mathsf{Y}_\phi = \phi$, and clearly, $\phi = \mathsf{s}_{[]}\,\mathsf{Y}_\phi = \mathsf{s}_{[]}\,\mathsf{t}\,\omega$. $\square$

Definition 3.4.2 (The category $\mathbb{O}$ of opetopes)**.** With the identities of lemma 3.4.1, we define the category $\mathbb{O}$ of opetopes by generators and relations as follows.

Objects. We set $\operatorname{ob}\mathbb{O} = \sum_{n\in\mathbb{N}} \mathbb{O}_n$.

Generating morphisms. Let $\omega \in \mathbb{O}_n$ with $n \geq 1$. We introduce a generator $\mathsf{t} : \mathsf{t}\,\omega \longrightarrow \omega$, called the *target embedding*. If $[p] \in \omega^\bullet$, then we introduce a generator $\mathsf{s}_{[p]} : \mathsf{s}_{[p]}\,\omega \longrightarrow \omega$, called a *source embedding*. An *elementary face embedding* is either a source or the target embedding.

Relations. We impose 4 relations described by the following commutative squares, which just enforce the identities of lemma 3.4.1. Let $\omega \in \mathbb{O}_n$ with $n \geq 2$.

(Inner) For $[p[q]] \in \omega^\bullet$ (forcing ω to be non degenerate), the following square must commute:

$$\begin{tikzcd}
\mathsf{s}_{[q]}\,\mathsf{s}_{[p]}\,\omega \arrow[r, "\mathsf{s}_{[q]}"] \arrow[d, "\mathsf{t}"'] & \mathsf{s}_{[p]}\,\omega \arrow[d, "\mathsf{s}_{[p]}"] \\
\mathsf{s}_{[p[q]]}\,\omega \arrow[r, "\mathsf{s}_{[p[q]]}"] & \omega
\end{tikzcd}$$

(Glob1) If ω is non degenerate, the following square must commute:

$$\begin{tikzcd}
\mathsf{t}\,\mathsf{t}\,\omega \arrow[r, "\mathsf{t}"] \arrow[d, "\mathsf{t}"'] & \mathsf{t}\,\omega \arrow[d, "\mathsf{t}"] \\
\mathsf{s}_{[]}\,\omega \arrow[r, "\mathsf{s}_{[]}"] & \omega .
\end{tikzcd}$$

(Glob2) If ω is non degenerate, and for $[p[q]] \in \omega^|$, the following square must commute:

$$\begin{tikzcd}
\mathsf{s}_{\wp_\omega[p[q]]}\,\mathsf{t}\,\omega \arrow[r, "\mathsf{s}_{\wp_\omega[p[q]]}"] \arrow[d, "\mathsf{s}_{[q]}"'] & \mathsf{t}\,\omega \arrow[d, "\mathsf{t}"] \\
\mathsf{s}_{[p]}\,\omega \arrow[r, "\mathsf{s}_{[p]}"] & \omega .
\end{tikzcd}$$

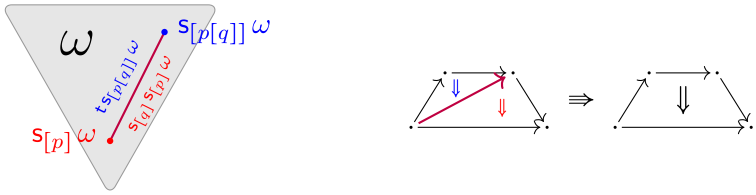
(Degen) If ω is degenerate, the following square must commute:

$$\begin{tikzcd}
\mathsf{t}\,\mathsf{t}\,\omega \arrow[r, "\mathsf{t}"] \arrow[d, "\mathsf{s}_{[]}"'] & \mathsf{t}\,\omega \arrow[d, "\mathsf{t}"] \\
\mathsf{t}\,\omega \arrow[r, "\mathsf{t}"] & \omega .
\end{tikzcd}$$

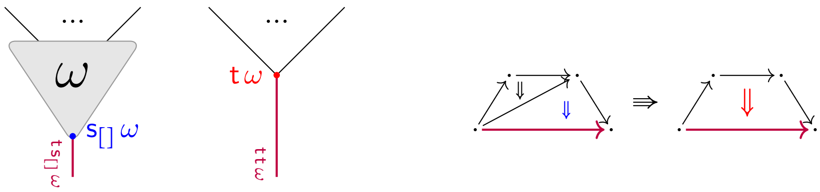
Remark 3.4.3. It is immediate from the definition that $\mathbb{O}$ is a direct category (i.e. a Reedy category where the only decreasing morphisms are identities, see definition 12.1.21) and rigid.

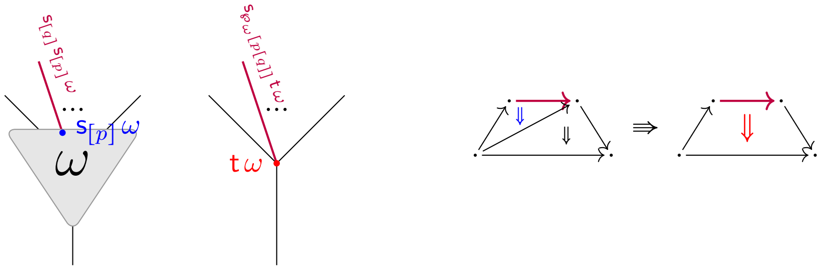
Remark 3.4.4. Let us explain this definition a little more. Opetopes are trees whose nodes (and edges) are decorated by opetopes. The decoration is now interpreted as a geometrical feature, namely as an embedding of a lower dimensional opetope. Further, the target of an opetope, while not an intrinsic data, is also represented as an embedding. The relations can be understood as follows.

(Inner) The inner edge at $[p[q]] \in \omega^\bullet$ is decorated by the target of the decoration of the node "above" it (here $\mathsf{s}_{[p[q]]}\,\omega$), and in the $[q]$-source of the node "below" it (here $\mathsf{s}_{[p]}\,\omega$). By construction, those two decorations match, and this relation makes the two corresponding embeddings $\mathsf{s}_{[q]}\,\mathsf{s}_{[p]}\,\omega \longrightarrow \omega$ match as well. On the left is an informal diagram about ω as a tree (reversed gray triangle), and on the right is an example of pasting diagram represented by an opetope, with the relevant features of the **(Inner)** relation colored or thickened.

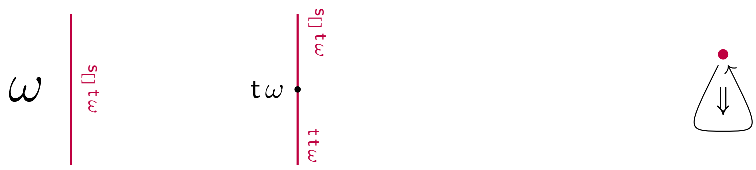


(Glob1-2) If we consider the underlying tree of ω as its “geometrical source”, and the corolla $\mathsf{Y}_{\mathsf{t}\omega}$ as its “geometrical target”, then they should be parallel. The relation **(Glob1)** expresses this idea by “gluing” the root edges of ω and $\mathsf{Y}_{\mathsf{t}\omega}$ together, while **(Glob2)** glues the leaves according to $\wp_\omega$.





(Degen) If ω is a degenerate opetope, depicted as on the right, then its target should be a “loop”, i.e. its only source and its target should be glued together.



Notation 3.4.5. For $n \in \mathbb{N}$, let $\mathbb{O}_{\leq n}$ be the full subcategory of $\mathbb{O}$ spanned by opetopes of dimension at most n. We define $\mathbb{O}_{<n}$, $\mathbb{O}_{\geq n}$, and $\mathbb{O}_{>n}$ similarly. For $m \leq n$, let $\mathbb{O}_{m,n} \coloneqq \mathbb{O}_{\geq m} \cap \mathbb{O}_{\leq n}$. We will tacitly consider the set $\mathbb{O}_n$ as a discrete category.

Remark 3.4.6 (Symmetric opetopes). In [BD98] [Che04b] [Che03a], opetopes and the category of opetopes are defined following a different approach. There, polynomial monads are replaced by *symmetric multicategories* (simply called *C-operads* in [BD98]), and the $(-)^+$ construction of definition 2.4.1 is replaced by the *slice construction*. The idea is the same: if $\mathcal{M}$ is a symmetric multicategory, then the objects of its slice $\mathcal{M}^+$ are the multimorphisms of $\mathcal{M}$. The opetopes defined in this way admit a natural action of the symmetric group on their source faces, inherited from the symmetric multicategories at play. Thus, the resulting category $\mathbb{O}_{\mathrm{sym}}$ of *symmetric opetopes* has non-trivial isomorphism classes, unlike $\mathbb{O}$. However, like $\mathbb{O}$, the category $\mathbb{O}_{\mathrm{sym}}$ is rigid, which hints an equivalence between the symmetric approach and that of [KJBM10] (used in this thesis, and equivalent to Leinster’s opetopes [Lei04]). This is formally proved in [Che04a].

Recall that $\mathcal{P}\mathrm{sh}(\mathbb{O})$ is the category of presheaves over $\mathbb{O}$, which we call *opetopic sets*. In this section, we expose some definitions and technical results about them.

Definition 3.5.1 (Remarkable opetopic sets)**.** (1) The representable presheaf at $\omega \in \mathbb{O}_n$ is denoted $O[\omega]$, or if it is not ambiguous, just ω. Its cells are morphisms of $\mathbb{O}$ of the form $\mathsf{f} : \psi \longrightarrow \omega$, for f a sequence of elementary face embeddings, which we write $\mathsf{f}\omega \in O[\omega]_\psi$ for short. For instance, the cell of maximal dimension is simply ω (as the corresponding sequence of face embeddings is empty), its $(n-1)$-cells are $\left\{\mathsf{s}_{[p]}\,\omega \mid [p] \in \omega^\bullet\right\} \cup \{\mathsf{t}\,\omega\}$.

(2) The *boundary* $\partial O[\omega]$ of ω is the maximal subpresheaf of $O[\omega]$ not containing the maximal cell ω. We write $\mathsf{b}_\omega : \partial O[\omega] \hookrightarrow O[\omega]$ for the *boundary inclusion* of ω. The set of boundary inclusions is denoted by B.

(3) The *spine* $S[\omega]$ is the maximal subpresheaf of $\partial O[\omega]$ not containing the cell $\mathsf{t}\omega$. We write $\mathsf{s}_\omega : S[\omega] \hookrightarrow O[\omega]$ for the *spine inclusion*. The set of spine inclusions is denoted by S.

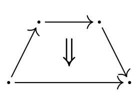
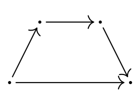
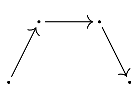
Remark 3.5.2. Recall from definition 3.4.2 that $\mathbb{O}$ is a direct category (i.e. does not have "degeneracy maps"). Therefore, the representable opetopic set $O[\omega]$ of an opetope $\omega \in \mathbb{O}_n$ is empty in dimension $> n$. Similarly, the boundary $\partial O[\omega]$ of ω is equal to $O[\omega]$ in dimension $< n$, and empty otherwise:

$$\partial O[\omega]_\psi = \begin{cases} \varnothing & \text{if } \dim \psi \geq n, \\ O[\omega]_\psi & \text{if } \dim \psi < n. \end{cases}$$

The spine $S[\omega]$ can be described similarly:

$$S[\omega]_\psi = \begin{cases} \varnothing & \text{if } \dim \psi \geq n, \\ O[\omega]_\psi - \{\mathsf{t}\,\omega\} & \text{if } \dim \psi = n-1, \\ O[\omega]_\psi & \text{if } \dim \psi < n-1. \end{cases}$$

Example 3.5.3. Consider the opetopic integer $\mathbf{3}$ of example 3.1.4. Graphically, we have

$O[\mathbf{3}]$ =  $\partial O[\mathbf{3}]$ =  $S[\mathbf{3}]$ = 

Notation 3.5.4. Let $F : \mathbb{O} \longrightarrow \mathcal{C}^{[1]}$ be a function that maps opetopes to morphisms in some category $\mathcal{C}$, and M the set $\{F(\omega) \mid \omega \in \mathbb{O}\}$. Then for $n \in \mathbb{N}$, let $\mathsf{M}_{\geq n} := \{F(\omega) \mid \omega \in \mathbb{O}_{\geq n}\}$, and similarly for $\mathsf{M}_{>n}$, $\mathsf{M}_{\leq n}$, $\mathsf{M}_{<n}$, and $\mathsf{M}_{=n}$. For convenience, the latter is abbreviated M_n. If $m \leq n$, we also let $\mathsf{M}_{m,n} = \mathsf{M}_{\geq m} \cap \mathsf{M}_{\leq n}$. By convention, $\mathsf{M}_{\leq n} = \varnothing$ if $n < 0$. For example, $\mathsf{S}_{\geq 2} = \{\mathsf{s}_\omega \mid \omega \in \mathbb{O}_{\geq 2}\}$, and $\mathsf{S}_{n,n+1} = \mathsf{S}_n \cup \mathsf{S}_{n+1}$.

Lemma 3.5.5. *Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$ be a finite opetopic set. Then it is the quotient of a sum of finitely many opetopic sets, i.e. there is an epimorphism*

$$\sum_{i \in I} O[\omega_i] \longrightarrow X$$

where I is a finite set.

Proof. Since $\mathbb{O}$ is directed and well-founded, all its slices are finite. Thus, X has finitely many cells, say $\{x_1, \ldots, x_k\}$. Simply consider the following quotient (recall the $(-)^\natural$ notation from definition 0.3.3)

$$\sum_i O[x_i^\natural] \xrightarrow{x_i^\natural \mapsto x_i} X.$$

□

Lemma 3.5.6. *For $\omega \in \mathbb{O}$, with $\dim \omega \geq 1$ the following square is a pushout and a pullback, where all arrows are canonical inclusions:*

$$\begin{tikzcd} \partial O[\mathsf{t}\,\omega] \arrow[r, "i"] \arrow[d, "\mathsf{b}_{\mathsf{t}\,\omega}"'] & S[\omega] \arrow[d] \\ O[\mathsf{t}\,\omega] \arrow[r] & \partial O[\omega]. \end{tikzcd}$$

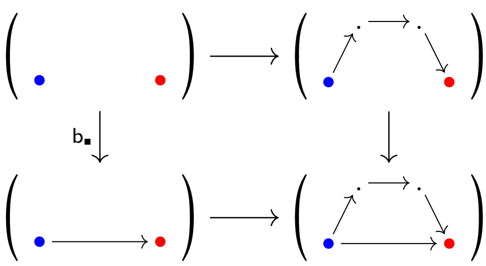
Proof. Let $n := \dim \omega$, and consider the pushout

$$\begin{tikzcd} \partial O[\mathsf{t}\,\omega] \arrow[r, "i"] \arrow[d, "\mathsf{b}_{\mathsf{t}\,\omega}"'] & S[\omega] \arrow[d] \\ O[\mathsf{t}\,\omega] \arrow[r] & P. \end{tikzcd}$$

In dimension $< n-1$ the boundary inclusion $\mathsf{b}_{\mathsf{t}\,\omega}$ is an identity, hence for all $\phi \in \mathbb{O}_{<n-1}$, $P_\phi = S[\omega]_\phi = \partial O[\omega]_\phi$. In dimension $(n-1)$, the boundary $\partial O[\mathsf{t}\,\omega]$ is empty, hence for all $\psi \in \mathbb{O}_{n-1}$, $P_\psi = O[\mathsf{t}\,\omega]_\psi + S[\omega]_\psi = \partial O[\omega]_\psi$. Finally, in dimension $\geq n$, all involved opetopic sets are empty. In conclusion, $P = \partial O[\omega]$.

The category $\mathcal{P}\mathrm{sh}(\mathbb{O})$ is a topos, and by [Lac11, theorem 3.1], it is an *adhesive category*. So by [Lac11, proposition 2.1], pushout squares of monomorphisms (in this case $\mathsf{b}_{\mathsf{t}\,\omega}$ along i) are also pullback squares. □

Example 3.5.7. With $\omega = \mathbf{3}$, the square of lemma 3.5.6 becomes

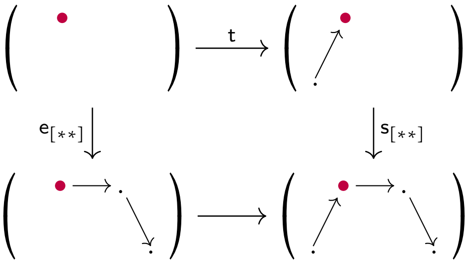


Lemma 3.5.8. *Let $n \geq 1$, $\nu \in \mathbb{O}_n$, $[l] \in \nu^{\shortmid}$, and $\psi \in \mathbb{O}_{n-1}$ be such that $\mathsf{e}_{[l]}\,\nu = \mathsf{t}\,\psi$, so that the grafting $\nu \circ_{[l]} \mathsf{Y}_\psi$ is well-defined. Then the following square is a pushout and a pullback:*

$$\begin{tikzcd} O[\mathsf{e}_{[l]}\,\nu] \arrow[r, "\mathsf{t}"] \arrow[d, "\mathsf{e}_{[l]}"'] & O[\psi] \arrow[d, "\mathsf{s}_{[l]}"] \\ S[\nu] \arrow[r] & S[\nu \circ_{[l]} \mathsf{Y}_\psi]. \end{tikzcd}$$

Proof. Similar to the proof of lemma 3.5.6. □

Example 3.5.9. With $\nu = \mathbf{2}$, $\psi = \blacksquare$, and $[l] = [**]$, we have $\nu \circ_{[**]} \mathsf{Y}_{\blacksquare} = \mathbf{3}$, and the square of lemma 3.5.8 becomes



Lemma 3.5.10. *Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$ be an opetopic set.*

(1) If $\mathsf{S}_{n,n+1} \perp X$, then $\mathsf{B}_{n+1} \perp X$. In particular, every morphism in $\mathsf{B}_{\geq n+1}$ is an $\mathsf{S}_{\geq n}$-local isomorphism.

(2) If $\mathsf{S}_{n,n+1} \perp X$ and $\mathsf{B}_{n+2} \perp X$, then $\mathsf{S}_{n+2} \perp X$. In particular, if $\mathsf{S}_{n,n+1} \perp X$ and $\mathsf{B}_{\geq n+2} \perp X$, then $\mathsf{S}_{\geq n} \perp X$.

Proof. (1) Let $\omega \in \mathbb{O}_{n+1}$. Note that the following triangle commutes

$$\begin{tikzcd} S[\omega] \arrow[r, "i"] \arrow[dr, "\mathsf{s}_\omega"'] & \partial O[\omega] \arrow[d, "\mathsf{b}_\omega"] \\ & O[\omega]. \end{tikzcd}$$

Since the class of maps ${}^{\perp}X$ has the 3-for-2 property, in order to show that $\mathsf{b}_\omega \perp X$, it is enough to show that $i \perp X$. Take a morphism $f : S[\omega] \longrightarrow X$. The existence of a lift $\partial O[\omega] \longrightarrow X$ follows from the existence of a lift $O[\omega] \longrightarrow X$, since $\mathsf{s}_\omega \perp X$. For unicity, consider two lifts $g, h : \partial O[\omega] \longrightarrow X$ of f. By lemma 3.5.6, in order to show that they are equal, it suffices to show that they coincide on $O[\mathsf{t}\,\omega]$. But since they coincide on $S[\omega]$ (as they extend f), they must coincide on the subpresheaf $S[\mathsf{t}\,\omega] \subseteq S[\omega]$. Since $\mathsf{S}_n \perp X$, g and h coincide on $O[\mathsf{t}\,\omega]$, and are thus equal.

(2) Let $\omega \in \mathbb{O}_{n+2}$ and $f : S[\omega] \longrightarrow X$. By assumption, the restriction $f|_{S[\mathsf{t}\,\omega]}$ of f to $S[\mathsf{t}\,\omega]$ extends to a unique $g : O[\mathsf{t}\,\omega] \longrightarrow X$. We now show that the following square commutes:

$$\begin{tikzcd} \partial O[\mathsf{t}\,\omega] \arrow[r] \arrow[d] & S[\omega] \arrow[d, "f"] \\ O[\mathsf{t}\,\omega] \arrow[r, "g"] & X. \end{tikzcd}$$

By lemma 3.5.6, it suffices to show that f and g coincide on $S[\mathsf{t}\,\omega]$ and on $O[\mathsf{t}\,\mathsf{t}\,\omega]$. The former is tautological, and the latter follows from the hypothesis that $\mathsf{s}_{\mathsf{t}\mathsf{t}\omega} \perp X$ and that f and g coincide on $S[\mathsf{t}\,\mathsf{t}\,\omega] \subseteq S[\mathsf{t}\,\omega]$. Therefore, the square above commutes, and by lemma 3.5.6 again, f and g extend to a morphism $h : \partial O[\omega] \longrightarrow X$, which in turn extends to a morphism $i : O[\omega] \longrightarrow X$, since by assumption $\mathsf{B}_{n+2} \perp X$.

For unicity, consider two lifts $i, i' : O[\omega] \longrightarrow X$ of f. By lemma 3.5.6, they are equal if and only if their restriction $g, g' : O[\mathsf{t}\,\omega] \longrightarrow X$ are equal. Since $g|_{S[\mathsf{t}\,\omega]} = f|_{S[\mathsf{t}\,\omega]} = g'|_{S[\mathsf{t}\,\omega]}$, and since by assumption $\mathsf{S}_{n+1} \perp X$, we have $g = g'$, and thus $i = i'$. $\square$

Corollary 3.5.11. *Let X be an opetopic set such that $\mathsf{S}_{n,n+1} \perp X$. Then $\mathsf{S}_{\geq n} \perp X$ if and only if $\mathsf{B}_{\geq n+2} \perp X$.*

Proof. Direct consequence of lemma 3.5.10. □

Lemma 3.5.12. *Let $n \in \mathbb{N}$, and $\omega \in \mathbb{O}_{n+2}$. Then the inclusion $S[\mathsf{t}\,\omega] \hookrightarrow S[\omega]$ is a relative S_{n+1}-cell complex.*

Proof. We show that the morphism $S[\mathsf{t}\,\omega] \hookrightarrow S[\omega]$ is a composite of pushouts of elements of S_{n+1}. If ω is degenerate, say $\omega = \mathsf{I}_\phi$ for some $\phi \in \mathbb{O}_n$, then $S[\mathsf{t}\,\omega] = S[\mathsf{Y}_\phi] = O[\phi] = S[\omega]$, so the result trivially holds.

Assume that ω is not degenerate, let $X^{(0)} := S[\mathsf{t}\,\omega]$, and $[p_1] \succ \cdots \succ [p_k]$ be the node addresses of ω, sorted in reverse lexicographical order. By induction, assume that $X^{(i-1)}$ is a subpresheaf of $S[\omega]$ containing the $(n+1)$-cells $\mathsf{s}_{[p_1]}\,\omega, \ldots, \mathsf{s}_{[p_{i-1}]}\,\omega \in S[\omega]$. Clearly, this holds when $i = 1$, as $S[\mathsf{t}\,\omega]$ does not contain any $(n+1)$-cell.

Take $[q] \in (\mathsf{s}_{[p_i]}\,\omega)^\bullet$. By induction, and since $[p_i[q]] \succ [p_i]$, the $(n+1)$-cell $\mathsf{s}_{[p_i[q]]}\,\omega$ is in $X^{(i-1)}$. Further, the n-cell $\mathsf{s}_{[q]}\,\mathsf{s}_{[p_i]}\,\omega$ is present in $X^{(i-1)}$, since by **(Inner)**, $\mathsf{s}_{[q]}\,\mathsf{s}_{[p_i]}\,\omega = \mathsf{t}\,\mathsf{s}_{[p_i[q]]}\,\omega$. Therefore, we have an inclusion $u_i : S[\mathsf{s}_{[p_i]}\,\omega] \longrightarrow X^{(i-1)}$ mapping $\mathsf{s}_{[q]}\,\mathsf{s}_{[p_i]}\,\omega$ to $\mathsf{s}_{[q]}\,\mathsf{s}_{[p_i]}\,\omega$, and let $X^{(i)}$ be the pushout

$$\begin{tikzcd} S[\mathsf{s}_{[p_i]}\,\omega] \arrow[r, "u_i"] \arrow[d, "\mathsf{s}_{\mathsf{s}_{[p_i]}\,\omega}"'] & X^{(i-1)} \arrow[d] \\ O[\mathsf{s}_{[p_i]}\,\omega] \arrow[r] & X^{(i)} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Clearly, $X^{(i)}$ is a subpresheaf of $S[\omega]$ containing the $(n+1)$-cell $\mathsf{s}_{[p_j]}\,\omega$ for $1 \leq j \leq i$, and the induction hypothesis is satisfied.

Finally, $X^{(k)} \subseteq S[\omega]$ contains all the $(n+1)$-cells of $S[\omega]$, whence $X^{(k)} = S[\omega]$. By construction, the chain of inclusions $S[\mathsf{t}\,\omega] = X^{(0)} \hookrightarrow X^{(1)} \hookrightarrow \cdots \hookrightarrow X^{(k)} = S[\omega]$ is a relative S_{n+1}-cell complex. □

Corollary 3.5.13. *Let $n \in \mathbb{N}$, and $\omega \in \mathbb{O}_{n+2}$. Then the target embedding $\mathsf{t}\,\omega \longrightarrow \omega$ of ω is an $\mathsf{S}_{n+1,n+2}$-local isomorphism.*

Proof. In the square below

$$\begin{tikzcd} S[\mathsf{t}\,\omega] \arrow[r, "\mathsf{s}_{\mathsf{t}\,\omega}"] \arrow[d, "r"'] & O[\mathsf{t}\,\omega] \arrow[d, "\mathsf{t}"] \\ S[\omega] \arrow[r, "\mathsf{s}_\omega"] & O[\omega] \end{tikzcd}$$

the map r is an S_{n+1}-local isomorphism by lemma 3.5.12, and the horizontal maps are in $\mathsf{S}_{n+1,n+2}$. The result follows by 3-for-2. □

Corollary 3.5.14. *Let $\psi \in \mathbb{O}_n$.*

(1) The morphism $\mathsf{t}\,\mathsf{t} = \mathsf{s}_{[]}\,\mathsf{t} : O[\psi] \longrightarrow O[\mathsf{I}_\psi]$ is in S_{n+2}. [4]

(2) The morphisms $\mathsf{s}_{[]}, \mathsf{t} : O[\psi] \longrightarrow O[\mathsf{Y}_\psi]$ are $\mathsf{S}_{n+1,n+2}$-local isomorphisms.

Proof. (1) This is clear, since the only $(n+1)$-cell of $O[\mathsf{I}_\psi]$ is $\mathsf{t}\,\mathsf{I}_\psi$.

[4] This improves corollary 3.5.13, where it was shown only to be a $\mathsf{S}_{n,n+2}$-local isomorphism.

(2) The source embedding $\mathsf{s}_{[]} : O[\psi] \longrightarrow O[\mathsf{Y}_\psi]$ is precisely the spine inclusion $\mathsf{s}_{\mathsf{Y}_\psi}$ of the $(n+1)$-opetope Y_ψ. The target embedding $\mathsf{t} : O[\psi] \longrightarrow O[\mathsf{Y}_\psi]$ is the morphism $\mathsf{t} : O[\mathsf{t}\,\mathsf{t}\,\mathsf{I}_\psi] \longrightarrow O[\mathsf{t}\,\mathsf{I}_\psi]$ and is the vertical arrow in the diagram below.

$$\begin{tikzcd} \psi = S[\mathsf{I}_\psi] \arrow[d, "\mathsf{t}"'] \arrow[dr, "\mathsf{s}_{\mathsf{I}_\psi}"] & \\ \mathsf{Y}_\psi = \mathsf{t}\,\mathsf{I}_\psi \arrow[r, "\mathsf{t}"'] & \mathsf{I}_\psi. \end{tikzcd}$$

The bottom arrow is an $\mathsf{S}_{n+1,n+2}$-local isomorphism by corollary 3.5.13 and the diagonal arrow is in S_{n+2} by point (1). The result follows by 3-for-2. $\square$

Definition 3.5.15. Let $\mathsf{O} := \left\{ \varnothing \hookrightarrow O[\omega] \mid \omega \in \mathbb{O} \right\}$.

Definition 3.5.16 (Truncation and extension)**.** (1) Consider the inclusion $\iota^{\geq m} : \mathbb{O}_{m,n} \longrightarrow \mathbb{O}_{\geq m}$. It induces a *truncation* functor $(-)_{m,n} : \mathcal{P}\mathrm{sh}(\mathbb{O}_{\geq m}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{m,n})$ that has both a left adjoint $\iota_!^{\geq m}$ and a right adjoint $\iota_*^{\geq m}$. Explicitly, for $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{m,n})$, the presheaf $\iota_!^{\geq m} X$ is the "extension by 0", i.e. $(\iota_!^{\geq m} X)_{m,n} = X$, and $(\iota_!^{\geq m} X)_\psi = \varnothing$ for all $\psi \in \mathbb{O}_{>n}$. On the other hand, $\iota_*^{\geq m} X$ is the "canonical extension" of X into a presheaf over $\mathbb{O}_{\geq m}$: we have $(\iota_*^{\geq m} X)_{m,n} = X$, and $\mathsf{B}_{>n} \perp \iota_*^{\geq m} X$, which uniquely determines $\iota_*^{\geq m} X$.

(2) Likewise, precomposing by $\iota^{\leq n}$ gives a functor $\mathcal{P}\mathrm{sh}(\mathbb{O}_{\leq n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{m,n})$, also denoted by $(-)_{m,n}$ and again called *truncation*, that has both a left adjoint $\iota_!^{\leq n}$ and a right adjoint $\iota_*^{\leq n}$. Explicitly, for $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{m,n})$, the presheaf $\iota_!^{\leq n} X$ is

$$\iota_!^{\leq n} X = \operatorname*{colim}_{O[\psi]_{m,n} \to X} O[\psi].$$

On the other hand, $\iota_*^{\leq n} X$ is the "terminal extension" of X in that $(\iota_*^{\leq n} X)_{m,n} = X$, and $(\iota_*^{\leq n} X)_\psi$ is a singleton, for all $\psi \in \mathbb{O}_{<m}$. Note that $\mathsf{O}_{<m} \perp \iota_*^{\leq n} X$, and that it is uniquely determined by this property.

Notation 3.5.17. For $n < \infty$, we write $(-)_{\leq n} : \mathcal{P}\mathrm{sh}(\mathbb{O}_{\geq 0}) = \mathcal{P}\mathrm{sh}(\mathbb{O}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{\leq n})$ instead of $(-)_{0,n}$, and let $(-)_{<n} := (-)_{\leq n-1}$ if $n \geq 0$. Similarly, we write $(-)_{\geq m}$ instead of $(-)_{m,\infty}$, and let $(-)_{>m} := (-)_{\geq m+1}$.

Proposition 3.5.18. *The functors $\iota_!^{\geq m}$, $\iota_*^{\geq m}$, $\iota_!^{\leq n}$, and $\iota_*^{\leq n}$ are fully faithful.*

Proof. Note that $\iota^{\geq m}$ and $\iota^{\leq n}$ are fully faithful, and apply lemma 0.4.14. $\square$

Proposition 3.5.19. *Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$ be an opetopic set.*

(1) The presheaf X is in the essential image of $\iota_!^{\geq 0} : \mathcal{P}\mathrm{sh}(\mathbb{O}_{\leq n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O})$ if and only if $X_{>n} = \varnothing$.

(2) The presheaf X is in the essential image of $\iota_*^{\geq 0} : \mathcal{P}\mathrm{sh}(\mathbb{O}_{\leq n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O})$ if and only if $\mathsf{B}_{>n} \perp X$.

(3) The presheaf X is in the essential image of $\iota_*^{\leq n} : \mathcal{P}\mathrm{sh}(\mathbb{O}_{\geq m}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O})$ if and only if $\mathsf{O}_{<m} \perp X$.

(4) Consider $\iota_*^{\leq n} \iota_*^{\geq m}$, the right adjoint to the truncation $(-)_{m,n} : \mathcal{P}\mathrm{sh}(\mathbb{O}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{m,n})$ Then X is in the essential image of $\iota_*^{\leq n} \iota_*^{\geq m}$ if and only of $(\mathsf{O}_{<m} \cup \mathsf{B}_{>n}) \perp X$. In particular, $(-)_{m,n}$ is the localization with at $\mathsf{O}_{<m} \cup \mathsf{B}_{>n}$.

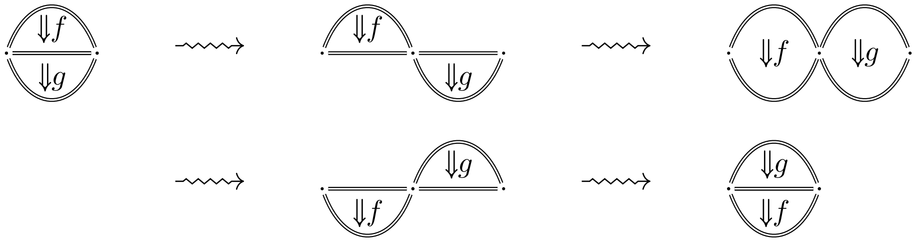
Proof. Points (1) to (3) are straightforward verifications. The first claim of point (4) follows from (2) and (3), and the second claim is an application of corollary 0.5.12. $\square$

Convention 3.5.20. To ease notations, we often leave truncations implicit, e.g. point (3) of last proposition can be reworded as: a presheaf $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{\geq m})$ is in the essential image of $\iota_*^{\geq m}$ if and only if $\mathsf{B}_{>m} \perp X$.

Chapter Four

Opetopic sets and many-to-one polygraphs

POLYGRAPHS where originally introduced by Street [Str76, section 2] under the name of *computad*. They are to (strict) ω-categories what graphs are to 1-categories: a combinatorial device that freely generates them. However, unlike graphs, the category $\mathcal{P}\mathrm{ol}$ of polygraphs fails to be a presheaf category [CJ04] [MZ08] [Che13]. The obstruction for it to be the case is an unpleasant corollary of the *exchange law*: if f and g are endomorphisms of an identity cell, then $fg = gf$.



Nonetheless, the recent work of Henry [Hen19] characterized many subcategories of $\mathcal{P}\mathrm{ol}$ to be presheaf categories. Among them, the category $\mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ of *many-to-one* polygraphs, in which the target (or codomain) of generating cells are themselves generating cells. In this chapter, we relate opetopes and many-to-one polygraphs in a formal way. Namely, we construct an equivalence of categories $|-| : \mathcal{P}\mathrm{sh}(\mathbb{O}) \longrightarrow \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$, called *polygraphic realization*.

Recall from [BD86, theorem 1] that if $\mathcal{A}$ and $\mathcal{B}$ are two Cauchy-complete categories such that $\mathcal{P}\mathrm{sh}(\mathcal{A}) \simeq \mathcal{P}\mathrm{sh}(\mathcal{B})$, then $\mathcal{A} \simeq \mathcal{B}$ (see theorem 4.3.27 for more details). In particular, any Cauchy-complete category $\mathcal{A}$ that acts as a "shape theory for many-to-one polygraphs", i.e. such that $\mathcal{P}\mathrm{sh}(\mathcal{A}) \simeq \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$, is equivalent to $\mathbb{O}$ (since it doesn't have any non-identity endomorphism, it is Cauchy-complete). This shows that the geometrical intuition behind the definition of $\mathbb{O}$ (remark 3.4.4) is essentially the unique way to faithfully implement the combinatorics of pasting diagrams.

The fact that many-to-one polygraphs are equivalent to opetopic sets was already known from [HMP00] [HMZ02] [Che04b] [HMZ08], however the proof there is indirect and spanned over multiple articles. The formalism we developed so far allows us to establish this result directly.

Definition 4.1.1 (ω-category)**.** An *ω-category* $\mathcal{C}$ (also called *strict ∞-category*) is the datum of a diagram of sets

$$\begin{tikzcd} \mathcal{C}_0 & \mathcal{C}_1 \arrow[l, "\mathsf{s},\mathsf{t}"'] \arrow[l, bend left, leftarrow, "\mathrm{id}"] & \cdots \arrow[l, "\mathsf{s},\mathsf{t}"'] \arrow[l, bend left, leftarrow, "\mathrm{id}"] & \mathcal{C}_n \arrow[l, "\mathsf{s},\mathsf{t}"'] \arrow[l, bend left, leftarrow, "\mathrm{id}"] & \cdots \arrow[l, "\mathsf{s},\mathsf{t}"'] \arrow[l, bend left, leftarrow, "\mathrm{id}"] \end{tikzcd}$$

with *composition maps* $\circ_k : \mathcal{C}_{n,k} \longrightarrow \mathcal{C}_n$, where $k < n$ and $\mathcal{C}_{n,k}$ is the pullback

$$\begin{tikzcd} \mathcal{C}_{n,k} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}_n \arrow[d, "\mathsf{t}^{n-k}"] \\ \mathcal{C}_n \arrow[r, "\mathsf{s}^{n-k}"] & \mathcal{C}_k, \end{tikzcd}$$

such that the following conditions hold:

(1) for all $k < n$, the diagram

$$\begin{tikzcd} \mathcal{C}_k & \mathcal{C}_n \arrow[l, "\mathsf{s}^{n-k},\mathsf{t}^{n-k}"'] \arrow[l, bend left, leftarrow, "\mathrm{id}^{n-k}"] \end{tikzcd}$$

with the composition map $\circ_k : \mathcal{C}_{n,k} \longrightarrow \mathcal{C}_n$ is a 1-category;

(2) for all $l < k < n$, the diagram

$$\begin{tikzcd} \mathcal{C}_l & \mathcal{C}_k \arrow[l, "\mathsf{s}^{k-l},\mathsf{t}^{k-l}"'] \arrow[l, bend left, leftarrow, "\mathrm{id}^{k-l}"] & \mathcal{C}_n \arrow[l, "\mathsf{s}^{n-k},\mathsf{t}^{n-k}"'] \arrow[l, bend left, leftarrow, "\mathrm{id}^{n-k}"] \end{tikzcd}$$

with the composition maps $\circ_l : \mathcal{C}_{k,l} \longrightarrow \mathcal{C}_k$, $\circ_l : \mathcal{C}_{n,l} \longrightarrow \mathcal{C}_n$ and $\circ_k : \mathcal{C}_{n,k} \longrightarrow \mathcal{C}_n$ is a strict 2-category.

The maps s and t are called *source* and *target* maps, respectively, and if $x \in \mathcal{C}_k$, then $\mathrm{id}_x := \mathrm{id}(x)$ is the *identity cell of x*. Note that by definition, the following equalities hold:

$$\mathsf{ss}x = \mathsf{st}x, \qquad \mathsf{ts}x = \mathsf{tt}x, \qquad \mathsf{s}\,\mathrm{id}_x = x = \mathsf{t}\,\mathrm{id}_x\,.$$

The first two are called the *globular identities.* Still be definition, for $0 \le l < k < n$ and $w, x, y, z \in \mathcal{C}_n$ the following *exchange law* holds:

$$(w \underset{k}{\circ} x) \underset{l}{\circ} (y \underset{k}{\circ} z) = (w \underset{l}{\circ} y) \underset{k}{\circ} (x \underset{l}{\circ} z),$$

assuming both sides are well-defined. Note that a strict n-category $\mathcal{C}$ is simply an ω-category where $\mathcal{C}_m$ only has identities for all $m > n$. Given an ω-category $\mathcal{C}$, we write $\mathcal{C}_{\le n}$ for the underlying strict n-category

$$\begin{tikzcd} \mathcal{C}_0 & \mathcal{C}_1 \arrow[l, "\mathsf{s},\mathsf{t}"'] \arrow[l, bend left, leftarrow, "\mathrm{id}"] & \cdots \arrow[l, "\mathsf{s},\mathsf{t}"'] \arrow[l, bend left, leftarrow, "\mathrm{id}"] & \mathcal{C}_n. \arrow[l, "\mathsf{s},\mathsf{t}"'] \arrow[l, bend left, leftarrow, "\mathrm{id}"] \end{tikzcd}$$

An *ω-functor* $f : \mathcal{B} \longrightarrow \mathcal{C}$ between two ω-categories is just a sequence of maps $f_n : \mathcal{B}_n \longrightarrow \mathcal{C}_n$ that induces a n-functor $f_{\le n} : \mathcal{B}_{\le n} \longrightarrow \mathcal{C}_{\le n}$ for all $n \in \mathbb{N}$. If the context is clear, we simply write f for $f_n : \mathcal{B}_n \longrightarrow \mathcal{C}_n$.

Definition 4.1.2 (Parallel cells [M03])**.** Let $\mathcal{D}$ be a strict ω-category, $n \in \mathbb{N}$. Two n-cells $x, y \in \mathcal{D}_n$ are *parallel*, denoted by $x \parallel y$, if $\mathsf{s}x = \mathsf{s}y$ and $\mathsf{t}x = \mathsf{t}y$. By convention, 0-cells are pairwise parallel.

Definition 4.1.3 (Cellular extension)**.** Let $\mathcal{D}$ be a strict $(n-1)$-category. A *cellular extension* of $\mathcal{D}$ consists in a set X and two maps $\mathsf{s}, \mathsf{t} : X \longrightarrow \mathcal{D}_{n-1}$ such that the *globular identities* hold, i.e. for all $x \in X$, we have $\mathsf{s}x \parallel \mathsf{t}x$. We also denote such a cellular extension by

$$\mathcal{D} \xleftarrow{\mathsf{s},\mathsf{t}} X.$$

Definition 4.1.4 (Free n-category)**.** Let $\mathcal{D}$ be a strict $(n-1)$-category and $\mathcal{D} \xleftarrow{\mathsf{s},\mathsf{t}} X$ be a cellular extension of $\mathcal{D}$. The *free strict n-category* generated by this cellular extension is the strict n-category $\mathcal{D}[X]$ such that

(1) as strict $(n-1)$-categories, $\mathcal{D} = \mathcal{D}[X]_{\le n-1}$;
(2) there is an inclusion $X \hookrightarrow \mathcal{D}[X]_n$, and the following diagrams commute:

$$\begin{tikzcd} & X \arrow[dl, "\mathsf{s}"'] \arrow[d, hook] \\ \mathcal{D}_{n-1} & \mathcal{D}[X]_n, \arrow[l, "\mathsf{s}"] \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, "\mathsf{t}"'] \arrow[d, hook] \\ \mathcal{D}_{n-1} & \mathcal{D}[X]_n; \arrow[l, "\mathsf{t}"] \end{tikzcd}$$

(3) if $\mathcal{E}$ is a strict n-category, $f : \mathcal{D} \longrightarrow \mathcal{E}_{\le n-1}$ is an $(n-1)$-functor, and $f_n : X \longrightarrow \mathcal{E}_n$ is a map such that for all $x \in X$, $f(\mathsf{s}x) = \mathsf{s}f_n(x)$ and $f(\mathsf{t}x) = \mathsf{t}f_n(x)$, then f and f_n extend uniquely to an n-functor $\mathcal{D}[X] \longrightarrow \mathcal{E}$.

The free extension $\mathcal{D}[X]$ always exists and is unique up to isomorphism, see [HMZ08, section 1].

For the rest of this section, let $\mathcal{D}$ be a strict $(n-1)$-category, $\mathcal{D} \xleftarrow{\mathsf{s},\mathsf{t}} X$ be a cellular extension of $\mathcal{D}$, and $\mathcal{E} := \mathcal{D}[X]$ be the n-category freely generated by the cellular extension.

Definition 4.1.5 (Counting function)**.** Define an n-category $\mathbb{N}^{(n)}$ by

$$\{0\} \xleftarrow{\mathsf{s},\mathsf{t}} \{0\} \xleftarrow{\mathsf{s},\mathsf{t}} \cdots \xleftarrow{\mathsf{s},\mathsf{t}} \{0\} \xleftarrow{\mathsf{s},\mathsf{t}} \mathbb{N},$$

where all compositions correspond to the addition of integers. For $x \in X$, define a *counting function* $\#_x : X \longrightarrow \mathbb{N}$ that maps x to 1, and all other elements of X to 0. This extends to a n-functor $\mathcal{E} \longrightarrow \mathbb{N}^{(n)}$. Similarly, let $\# : X \longrightarrow \mathbb{N}$ be the map sending all elements to 1, and extend it as $\# : \mathcal{E} \longrightarrow \mathbb{N}^{(n)}$.

Definition 4.1.6 (Context [GM09, definition 2.1.1])**.** Consider another cellular extension

$$\mathcal{D} \xleftarrow{\mathsf{s},\mathsf{t}} (X + \{\square\})$$

of $\mathcal{D}$, where $\mathsf{s}\square$ and $\mathsf{t}\square$ are chosen arbitrarily. A *n-context* of $\mathcal{E}$ is a cell $C \in \mathcal{D}[X + \{\square\}]_n$ such that $\#_\square C = 1$. One may think of C as a cell of $\mathcal{E}_n$ with a "hole", and we sometime write $C = C[\square]$. If $u \in \mathcal{E}_n$ is parallel to $\square$ in $\mathcal{D}[X + \{\square\}]$, let $C[u]$, be $C[\square]$ where $\square$ has been replaced by u.

Definition 4.1.7 (Category of contexts [GM09, definition 2.1.2])**.** The category $\mathcal{C}\mathrm{tx}_n\mathcal{E}$ of *n-contexts* of $\mathcal{E}$ has objects the n-cells of $\mathcal{E}$, and a morphism $C : x \longrightarrow y$ is an n-context $C = C[\square]$ such that $C[x] = y$. If $D : y \longrightarrow z$ is another context, then the composite of C and D is $DC := D[C[\square]] : x \longrightarrow z$, as indeed, $D[C[x]] = D[y] = z$.

Definition 4.1.8 (Primitive context)**.** A context is *primitive* over a cell $y \in \mathcal{E}_n$ if it is of the form $C : x \longrightarrow y$ with $x \in X$.

4.2 MANY-TO-ONE POLYGRAPHS

POLYGRAPHS

Definition 4.2.1 (Polygraph [HMZ08, definition 7.1])**.** A *polygraph* (also called a *computad*) $\mathcal{P}$ consists of a small ω-category $\mathcal{C}$ and sets $\mathcal{P}_n \subseteq \mathcal{C}_n$ for all $n \in \mathbb{N}$, such that $\mathcal{P}_0 = \mathcal{C}_0$, and such that $\mathcal{C}_{\leq n+1} = \mathcal{C}_{\leq n}[\mathcal{P}_{n+1}]$, i.e. the underlying $(n+1)$-category of $\mathcal{C}$ is freely generated by $\mathcal{P}_{n+1}$ over its underlying n-category. We usually write $\mathcal{P}^*$ instead of $\mathcal{C}$. A polygraph $\mathcal{P}$ is an *n-polygraph* if $\mathcal{P}_k = \varnothing$ whenever $k > n$. A *morphism of polygraphs* is an ω-functor mapping generators to generators. Let $\mathcal{P}\mathrm{ol}$ be the category of polygraphs and morphisms between them.

Example 4.2.2. A 1-polygraph $\mathcal{P}$ is simply a free 1-category generated by the graph $\mathcal{P}_0 \xleftarrow{\mathsf{s,t}} \mathcal{P}_1$.

Proposition 4.2.3. *The category* $\mathcal{P}\mathrm{ol}$ *is cocomplete. If* $F : \mathcal{J} \longrightarrow \mathcal{P}\mathrm{ol}$ *is diagram, and* $n \in \mathbb{N}$*, then* $(\mathrm{colim}_{i\in\mathcal{J}} Fi)_n \cong \mathrm{colim}_{i\in\mathcal{J}}(Fi)_n$.

Notation 4.2.4. If $\mathcal{P} \in \mathcal{P}\mathrm{ol}$, we write $\mathcal{C}\mathrm{tx}_n\mathcal{P}$ instead of $\mathcal{C}\mathrm{tx}_n\mathcal{P}_{\leq n-1}[\mathcal{P}_n]$ (see definition 4.1.7).

Proposition 4.2.5 ([GM09, proposition 2.1.3])**.** *Let* $\mathcal{P}$ *be a polygraph, and* $C \in \mathcal{C}\mathrm{tx}_n\mathcal{P}$. *Then* C *decomposes as*

$$C = d_n \underset{n-1}{\circ} (d_{n-1} \underset{n-2}{\circ} \cdots (d_1 \underset{0}{\circ} \square \underset{0}{\circ} e_1) \cdots \underset{n-2}{\circ} e_{n-1}) \underset{n-1}{\circ} e_n,$$

where $d_n, e_n \in \mathcal{P}_n^*$*, and for* $1 \leq i < n$*,* d_i *and* e_i *are identities of* i*-cells.*

Definition 4.2.6 (Whisker [GM09, paragraph 2.1.4])**.** Let $\mathcal{P}$ be a polygraph. an *n-whisker* of $\mathcal{P}$ is an n-context of the form $d_{n-1} \circ_{n-2} \cdots (d_1 \circ_0 \square \circ_0 e_1) \cdots \circ_{n-2} e_{n-1}$, where for $1 \leq i \leq n-1$, d_i and e_i are identities of i-cells.

Remark 4.2.7. If C is an $(n-1)$-context, then by proposition 4.2.5, it decomposes as $C[\square] = d_{n-1} \circ_{n-2} \cdots (d_1 \circ_0 \square \circ_0 e_1) \cdots \circ_{n-2} e_{n-1}$, and it induces an n-whisker

$$\mathrm{id}_{d_{n-1}} \underset{n-2}{\circ} \cdots (\mathrm{id}_{d_1} \underset{0}{\circ} \square \underset{0}{\circ} \mathrm{id}_{e_1}) \cdots \underset{n-2}{\circ} \mathrm{id}_{e_{n-1}}$$

which we shall also denote by C.

Proposition 4.2.8 ([GM09, proposition 2.1.5])**.** *Let* $\mathcal{P}$ *be a polygraph,* $u \in \mathcal{P}_n^*$*,* $k := \#u$*, and assume* $k \geq 1$*. Then* u *decomposes as*

$$u = C_1[x_1] \underset{n-1}{\circ} C_2[x_2] \underset{n-1}{\circ} \cdots \underset{n-1}{\circ} C_k[x_k],$$

where all of the C_i*'s are* n*-whiskers, and* $x_1, \ldots, x_k \in \mathcal{P}_n$.

Definition 4.2.9 (Partial composition [HMZ08, definition 3.8]). Let $\mathcal{P}$ be a polygraph, $x, y \in \mathcal{P}_n^*$ be n-cells, and $C : \mathsf{t}\, y \longrightarrow \mathsf{s}\, x$ be a context. The *partial composition* (called *placed composition* in [HMZ08, definition 3.8]) $x \circ_C y$ is defined as

$$x \underset{C}{\circ} y := x \underset{n-1}{\circ} C[y],$$

where the notation $C[y]$ follows remark 4.2.7. Note that this decomposition is in general not unique.

Lemma 4.2.10. *With x, y, and C as in definition 4.2.9, we have $\mathsf{s}(x \circ_C y) = C[\mathsf{s}\, y]$ and $\mathsf{t}(x \circ_C y) = \mathsf{t}\, x$.*

Proof. We have $\mathsf{t}(x \circ_C y) = \mathsf{t}(x \circ_{n-1} C[y]) = \mathsf{t}\, x$. On the other hand, $\mathsf{s}(x \circ_C y) = \mathsf{s}\, C[y]$. By proposition 4.2.5 and remark 4.2.7, $C[y]$ decomposes as

$$C[y] \;=\; \mathrm{id}_{d_{n-1}} \underset{n-2}{\circ} \cdots (\mathrm{id}_{d_1} \underset{0}{\circ} y \underset{0}{\circ} \mathrm{id}_{e_1}) \cdots \underset{n-2}{\circ} \mathrm{id}_{e_{n-1}},$$

where for $1 \leq k \leq n$, d_k and e_k are identities of k-cells. Thus,

$$\mathsf{s}\, C[y] \;=\; d_{n-1} \underset{n-2}{\circ} \cdots (d_1 \underset{0}{\circ} (\mathsf{s}\, y) \underset{0}{\circ} e_1) \cdots \underset{n-2}{\circ} e_{n-1} \;=\; C[\mathsf{s}\, y].$$

□

MANY-TO-ONE POLYGRAPHS

Definition 4.2.11 (Many-to-one polygraph [HMZ08, definition 7.4]). Let $\mathcal{P} \in \mathcal{P}\mathrm{ol}$ be a polygraph. For $n \geq 1$, an n-cell $x \in \mathcal{P}_n^*$ is said *many-to-one* if $\mathsf{t}\, x \in \mathcal{P}_{n-1}$, and we write $\mathcal{P}_n^{\mathrm{mto}}$ for the set of many-to-one n-cells of $\mathcal{P}$. By convention, all 0-cells are many-to-one. In turn, the polygraph $\mathcal{P}$ is called *many-to-one* (or *opetopic*) if all its generators are many-to-one. Let $\mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ be the full subcategory of $\mathcal{P}\mathrm{ol}$ spanned by many-to-one polygraphs.

Lemma 4.2.12. *Let $u \in \mathcal{P}_n^*$ be such that $\# u \geq 1$. By proposition 4.2.8, it decomposes as*

$$u \;=\; C_1[x_1] \underset{n-1}{\circ} C_2[x_2] \underset{n-1}{\circ} \cdots \underset{n-1}{\circ} C_k[x_k],$$

where $k := \# u$, where all of the C_i's are n-whiskers, and $x_1, \ldots, x_k \in \mathcal{P}_n$. Then u is a many-to-one cell if and only if $C_1 = \square$, i.e. if C_1 is the trivial context.

Proof. First, note that $\mathsf{t}\, u = \mathsf{t}\, C_1[x_1]$. Write C_1 as

$$C_1[\square] \;=\; d_{n-1} \underset{n-2}{\circ} \cdots (d_1 \underset{0}{\circ} \square \underset{0}{\circ} e_1) \cdots \underset{n-2}{\circ} e_{n-1},$$

where for $1 \leq i \leq n-1$, d_i and e_i are identities of i-cells (see definition 4.2.6). We have

$$\mathsf{t}\, u \;=\; \mathsf{t}\, C_1[x_1] \;=\; (\mathsf{t}\, d_{n-1}) \underset{n-2}{\circ} \cdots ((\mathsf{t}\, d_1) \underset{0}{\circ} (\mathsf{t}\, x_1) \underset{0}{\circ} (\mathsf{t}\, e_1)) \cdots \underset{n-2}{\circ} (\mathsf{t}\, e_{n-1}).$$

Thus, $\mathsf{t}\, u$ is a generator if and only if $\mathsf{t}\, d_i$ and $\mathsf{t}\, e_i$ are identities, for all $0 \leq i \leq n-1$. In this case, d_i and e_i are identity cells of $(i-1)$-cells, thus $C_1 = \square$. Conversely, if $C_1 = \square$, then $\mathsf{t}\, u = \mathsf{t}\, x_1$ is a generator since $\mathcal{P}$ is a many-to-one polygraph, thus u is a many-to-one cell.

□

The following result comes as a polygraphic analogue to proposition 2.2.22.

Proposition 4.2.13. *A many-to-one n-cell of $\mathcal{P}$ is of either of the following forms:*

(1) id_a *for* $a \in \mathcal{P}_{n-1}$*,*

(2) $x \in \mathcal{P}_n$*,*

(3) $v \circ_C x = v \circ_{n-1} C[x]$*, for some* $v \in \mathcal{P}_n^{\mathrm{mto}}$ *with* $\#v \geq 1$*,* $x \in \mathcal{P}_n$*, and* $C : \mathsf{t}\, x \longrightarrow \mathsf{s}\, v$*.*

Proof. Let $u \in \mathcal{P}_n^{\mathrm{mto}}$. If $\#u = 0$, then $u = \mathrm{id}_a$ for some $a \in P_{n-1}^*$. Further, $a = \mathsf{t}\, u \in \mathcal{P}_n$. If $\#u = 1$, then u is necessarily a generator. If $\#u = k \geq 2$, then by proposition 4.2.8, u decomposes as

$$u = C_1[x_1] \underset{n-1}{\circ} C_2[x_2] \underset{n-1}{\circ} \cdots \underset{n-1}{\circ} C_k[x_k],$$

where all of the C_i's are n-whiskers, and $x_1, \ldots, x_k \in \mathcal{P}_n$. Let

$$v \coloneqq C_1[x_1] \underset{n-1}{\circ} C_2[x_2] \underset{n-1}{\circ} \cdots \underset{n-1}{\circ} C_{k-1}[x_{k-1}].$$

Then C_k is a context $\mathsf{t}\, x_k \longrightarrow \mathsf{s}\, v$, and $u = v \circ_{C_k} x_k$. By lemma 4.2.12, and since u is many-to-one, $C_1 = \square$. By lemma 4.2.12 again, v is many-to-one, and $\#v = k - 1 \geq 1$, finishing the proof. □

Notation 4.2.14. For $\mathcal{P} \in \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$, let $\mathcal{C}\mathrm{tx}_n^{\mathrm{mto}}\mathcal{P}$ be the full subcategory of $\mathcal{C}\mathrm{tx}_n\mathcal{P}$ generated by many-to-one n-cells. In other words, an n-context $C : u \longrightarrow v$ is in $\mathcal{C}\mathrm{tx}_n^{\mathrm{mto}}\mathcal{P}$ if $u, v \in \mathcal{P}_n^{\mathrm{mto}}$. Necessarily, such a context is itself a many-to-one cell, as $\mathsf{t}\, C[\square] = \mathsf{t}\, C[u] = \mathsf{t}\, v$ is a generator.

Definition 4.2.15. We now define a polygraph $\mathcal{T} \in \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$. First, set $\mathcal{T}_0 \coloneqq \{\blacklozenge\}$, and inductively, let $\mathcal{T}_{n+1} \coloneqq \left\{ (u, v) \in \mathcal{T}_n^{\mathrm{mto}} \times \mathcal{T}_n \mid u \parallel v \right\}$ (see definitions 4.1.2 and 4.2.11) with $\mathsf{s}(u, v) \coloneqq u$ and $\mathsf{t}(u, v) \coloneqq v$.

Proposition 4.2.16. *The polygraph* $\mathcal{T}$ *is terminal in* $\mathcal{P}\mathrm{ol}^{\mathrm{mto}}$*.*

Proof. For $\mathcal{P} \in \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$, we show that there exists a unique morphism $f : \mathcal{P} \longrightarrow \mathcal{T}$.

Existence. If $x \in \mathcal{P}_0$, let $f(x) \coloneqq \blacklozenge$, and if $x \in \mathcal{P}_n$ with $n \geq 1$, let $f(x) = (f(\mathsf{s}\, x), f(\mathsf{t}\, x))$. The source and target compatibility is trivial.

Uniqueness. Let $g : \mathcal{P} \longrightarrow \mathcal{T}$ be a morphism different from f defined above. Necessarily $g_0 = f_0$ as $\mathcal{T}_0$ is a singleton. Let $x \in \mathcal{P}_n$ be such that $g(x) \neq f(x)$, with n minimal. Necessarily, $n \geq 1$, and we have

$$\begin{aligned} f(x) &= (f(\mathsf{s}\, x), f(\mathsf{t}\, x)) && \text{by definition of } f \\ &= (g(\mathsf{s}\, x), g(\mathsf{t}\, x)) && \text{by minimality of } n \\ &= (\mathsf{s}\, g(x), \mathsf{t}\, g(x)) && \\ &= g(x) && \text{see definition 4.2.15,} \end{aligned}$$

a contradiction. □

Notation 4.2.17. If $\mathcal{P}$ is a many-to-one polygraph, we write $! : \mathcal{P} \longrightarrow \mathcal{T}$ for the terminal map.

Definition 4.2.18. (1) An *effective category* is a category $\mathcal{C}$ equipped with a functor $F : \mathcal{C} \longrightarrow \mathcal{S}\mathrm{et}$. For example:

a) if $\mathcal{A}$ is a small category, then $\mathcal{P}\mathrm{sh}(\mathcal{A})$ (or any subcategory thereof) is naturally an effective category with the functor $\mathcal{P}\mathrm{sh}(\mathcal{A}) \longrightarrow \mathcal{S}\mathrm{et}$ mapping a presheaf X to $\sum_{a \in \mathcal{A}} X_a$;
b) $\mathcal{P}\mathrm{ol}$ (or any subcategory thereof) is an effective category, where the functor $\mathcal{P}\mathrm{ol} \longrightarrow \mathcal{S}\mathrm{et}$ maps a polygraph $\mathcal{P}$ to $\sum_{n \in \mathbb{N}} \mathcal{P}_n$.

(2) A category $\mathcal{C}$ is an *effective presheaf category* if it is effective, and equivalent, as an effective category, to a presheaf category.

(3) A functor $F : \mathcal{C} \longrightarrow \mathcal{S}\mathrm{et}$ is *familially representable* if $F \cong \sum_{i \in I} \mathcal{C}(c_i, -)$ for some family $\{c_i \mid i \in I\}$ of objects of $\mathcal{C}$.

Theorem 4.2.19. *(1) [Hen19, corollary 2.4.9] The category* $\mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ *is a* good class of polygraphs *[Hen19, definition 2.2.2], and in particular*

a) it is an effective presheaf category;

b) for all $n \in \mathbb{N}$, the functor $(-)^*_n : \mathcal{P}\mathrm{ol}^{\mathrm{mto}} \longrightarrow \mathcal{S}\mathrm{et}$ that maps a polygraph $\mathcal{P} \in \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ to its set $\mathcal{P}^*_n$ of n-cells is familially representable, and the representing objects are called the opetopic n-polyplexes *(or just* n-polyplexes*):*

$$\mathcal{P}^*_n \cong \sum_{\substack{\omega \text{ is an opetopic} \\ n\text{-polyplex}}} \mathcal{P}\mathrm{ol}^{\mathrm{mto}}(\omega, \mathcal{P}).$$

(2) [Hen19, proposition 2.2.6] The opetopic n-polyplexes are in bijective correspondence with $\mathcal{T}^*_n$. Isomorphic polyplexes are equal. If $u \in \mathcal{T}^*_n$, let $\underline{u}$ be the associated polyplex (refer to [Hen19, section 2.3] for the precise construction). The isomorphism above can be reformulated as

$$\mathcal{P}^*_n \cong \sum_{u \in \mathcal{T}^*_n} \mathcal{P}\mathrm{ol}^{\mathrm{mto}}(\underline{u}, \mathcal{P}).$$

More precisely, a cell $v \in \mathcal{P}^*_n$ corresponds to a unique morphism of the form $\underline{u} \longrightarrow \mathcal{P}$, and $u = {!}\,v$ (see notation 4.2.17).

(3) [Hen19, lemma 2.4.4, corollary 2.3.13] Let $0 \leq k < n$, $a \in \mathcal{T}^*_k$, and $u, v \in \mathcal{T}^*_n$ be such that $\mathsf{t}^{n-k} v = a = \mathsf{s}^{n-k} u$. Then we have natural maps $\mathsf{s}^{n-k} : \underline{a} \longrightarrow \underline{u}$ and $\mathsf{t}^{n-k} : \underline{a} \longrightarrow \underline{v}$, and $\underline{u \circ_k v}$ is obtained as the pushout

$$\begin{tikzcd} \underline{a} \arrow[r, "\mathsf{t}^{n-k}"] \arrow[d, "\mathsf{s}^{n-k}"'] & \underline{v} \arrow[d, "\iota_v"] \\ \underline{u} \arrow[r, "\iota_u"] & \underline{u \circ_k v}. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Furthermore, the maps ι_u and ι_v are injective on $(n-1)$- and n-cells.

Example 4.2.20. By definition 4.2.15, $\mathcal{T}$ has a unique 0-cell $\blacklozenge$, and the corresponding polyplex $\underline{\blacklozenge}$ is simply the polygraph with a single 0-generator. Indeed, $\mathcal{P}\mathrm{ol}^{\mathrm{mto}}(\underline{\blacklozenge}, -)$ maps a polygraph $\mathcal{P} \in \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ to its set of 0-cells $\mathcal{P}_0$.

If we write $\blacksquare := (\blacklozenge, \blacklozenge)$ for the unique 1-generator of $\mathcal{T}$, then

$$\mathcal{T}^*_1 = \left\{ \mathrm{id}_{\blacklozenge},\ \blacksquare,\ \blacksquare \underset{0}{\circ} \blacksquare,\ \blacksquare \underset{0}{\circ} \blacksquare \underset{0}{\circ} \blacksquare,\ \blacksquare \underset{0}{\circ} \blacksquare \underset{0}{\circ} \blacksquare \underset{0}{\circ} \blacksquare,\ \ldots \right\}.$$

Write $l_0 := \mathrm{id}_{\blacklozenge}$, and l_k for the composite $\blacksquare \circ_0 \cdots \circ_0 \blacksquare$ of k instances of $\blacksquare$. Then the polyplex $\underline{l_0}$ is simply $\underline{\blacklozenge}$, and $\underline{l_k}$ the free category on the linear graph with k vertices:

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet$$

Indeed, let $\mathcal{P} \in \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$. Then a 1-cell u of $\mathcal{P}$ is either

(1) an identity of a 0-cell;

(2) a sequence of k composable 1-generators of $\mathcal{P}$.

If $u = \mathrm{id}_a$ for some $a \in \mathcal{P}_0$, then it is uniquely identified (as a 1-cell) by a morphism $\underline{l_0} \longrightarrow \mathcal{P}$ mapping the unique 0-cell of $\underline{l_0}$ to a. If u is a composite of k generators, then uniquely identified (as a 1-cell) by a morphism $\underline{l_k} \longrightarrow \mathcal{P}$. In conclusion,

$$\mathcal{P}_1^* \cong \sum_{k \in \mathbb{N}} \mathcal{P}\mathrm{ol}^{\mathrm{mto}}(\underline{l_k}, \mathcal{P}) = \sum_{v \in \mathcal{T}_1^*} \mathcal{P}\mathrm{ol}^{\mathrm{mto}}(\underline{v}, \mathcal{P}).$$

Remark 4.2.21. If $\underline{u}$ is an n-polyplex, then under the isomorphism of theorem 4.2.19 (2), the identity morphism $\underline{u} \longrightarrow \underline{u}$ corresponds to an n-cell, which we call the *fundamental cell* of $\underline{u}$, and following [Hen19], denote by u. If $\mathcal{P}$ is a many-to-one polygraph, $v \in \mathcal{P}^*$, and $u = !\, v$, then the map $\underline{u} \longrightarrow \mathcal{P}$ corresponding to v maps the fundamental cell u to v.

Lemma 4.2.22 ([Hen19, lemma 2.4.5]). *Let $n \geq 1$, $\underline{u}$ be an n-polyplex, and u be its fundamental cell. For $a \in \underline{u}_{n-1}$, exactly one of the following two possibilities happens:*

(1) a occurs in $\mathsf{s}\, u$;

(2) a is the target of a n-generator of $\underline{u}$.

Furthermore, in the second case, the n-generator in question is unique.

Lemma 4.2.23 ([Hen19, remark 2.2.9, corollary 2.2.13]). *Let $f : \mathcal{P} \longrightarrow \mathcal{Q}$ a morphism of many-to-one polygraphs, and recall from definition 4.1.1 that $\mathcal{P}_{n,k}^* = \mathcal{P}_n^* \times_{\mathcal{P}_k^*} \mathcal{P}_n^*$. The following square is cartesian*

$$\begin{tikzcd} \mathcal{P}_{n,k}^* \arrow[r, "\circ_k"] \arrow[d, "f"'] & \mathcal{P}_n^* \arrow[d, "f"] \\ \mathcal{Q}_{n,k}^* \arrow[r, "\circ_k"] & \mathcal{Q}_n^*. \end{tikzcd}$$

In other words, for $u_1, u_2, v_1, v_2 \in \mathcal{P}_n^*$, if $u_1 \circ_k v_1 = u_2 \circ_k v_2$, $f(u_1) = f(u_2)$, and $f(v_1) = f(v_2)$, then $u_1 = u_2$ and $v_1 = v_2$.

Proof (sketch). Let us first consider the case $\mathcal{Q} = \mathcal{T}$, and let $u, v \in \mathcal{P}_n^*$ be such that $\mathsf{t}^{n-k}\, v = \mathsf{s}^{n-k}\, u = a$. In particular, pair (u, v) is in $\mathcal{P}_{n,k}^*$. We have a series of correspondences

$$\cfrac{\cfrac{\text{A tuple } (u,v) \in \mathcal{P}_{n,k}^*}{\text{A map } \underline{!\,u} \coprod_{\underline{!\,a}} \underline{!\,v} \longrightarrow \mathcal{P}} \text{ theorem 4.2.19 (2)}}{\cfrac{\text{A map } \underline{!(u \circ_k v)} \longrightarrow \mathcal{P} \text{ with a decomposition of } !(u \circ_k v) \text{ as } x \circ_k y}{\text{An element } u \circ_k v \in \mathcal{P}_n^* \text{ with a decomposition of } !(u \circ_k v) \text{ as } x \circ_k y.} \text{ theorem 4.2.19 (2)}} \text{ theorem 4.2.19 (3)}$$

In other words, the following square is a pullback

$$\begin{tikzcd} \mathcal{P}_{n,k}^* \arrow[r, "\circ_k"] \arrow[d, "!"'] & \mathcal{P}_n^* \arrow[d, "!"] \\ \mathcal{T}_{n,k}^* \arrow[r, "\circ_k"] & \mathcal{T}_n^*. \end{tikzcd}$$

For the general case, note that in the following diagram, the lower and outer squares are cartesian, and by the pasting lemma, so is the upper one:

$$\begin{tikzcd}
\mathcal{P}^*_{n,k} \arrow[r, "\circ_k"] \arrow[d, "f"] \arrow[dd, bend right, "!"'] & \mathcal{P}^*_n \arrow[d, "f"'] \arrow[dd, bend left, "!"] \\
\mathcal{Q}^*_{n,k} \arrow[r, "\circ_k"] \arrow[d, "!"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Q}^*_n \arrow[d, "!"'] \\
\mathcal{T}^*_{n,k} \arrow[r, "\circ_k"] & \mathcal{T}^*_n .
\end{tikzcd}$$

□

COMPOSITION TREES

Let $\mathcal{P} \in \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ and $v \in \mathcal{P}^{\mathrm{mto}}_n$. Then v is a composition of n-generators of $\mathcal{P}$, which are many-to-one, so intuitively, v is a "tree of n-generators". In this section, we make this idea formal. We first define a polynomial functor $\nabla_n \mathcal{P}$ whose operations are the n-generators of $\mathcal{P}$ (definition 4.2.24), and then construct the *composition* $T^\circ \in \mathcal{P}^{\mathrm{mto}}_n$ of a $\nabla_n\mathcal{P}$-tree T. In proposition 4.2.31, we show that this construction is bijective.

Definition 4.2.24 (The ∇ construction)**.** For $\mathcal{P} \in \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ and $n \geq 1$, let $\nabla_n\mathcal{P}$ be the following polynomial endofunctor:

$$\mathcal{P}_{n-1} \xleftarrow{\mathsf{s}} \mathcal{P}^\bullet_n \xrightarrow{p} \mathcal{P}_n \xrightarrow{\mathsf{t}} \mathcal{P}_{n-1},$$

where for $x \in \mathcal{P}_n$, the fiber $\mathcal{P}^\bullet_n(x)$ is the set of primitive contexts over $\mathsf{s}\,x$, and for $C : a \longrightarrow \mathsf{s}\,x$ in $\mathcal{P}^\bullet_n(x)$, $\mathsf{s}(C) := a$, $p(C) := x$, and t is the target map of $\mathcal{P}$.

Lemma 4.2.25. *Let $\mathcal{P}, \mathcal{Q} \in \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$, and $f : \mathcal{P} \longrightarrow \mathcal{Q}$. For $v \in \mathcal{P}^{\mathrm{mto}}_n$, write $E(v)$ for the set of primitive contexts over v, and likewise for many-to-one cells of $\mathcal{Q}$. Then f induces a bijection $E(v) \longrightarrow E(f(v))$.*

Proof. We proceed by induction on v (see proposition 4.2.13).

(1) If v is an identity (resp. a generator), then so is $f(v)$, thus $E(v)$ and $E(f(v))$ are both empty (resp. singletons). Trivially, $f : E(v) \longrightarrow E(f(v))$ is a bijection.

(2) Assume that v decomposes as $v = w \circ_C x$ with $\#w \geq 1$ and $x \in \mathcal{P}_n$, and $C : \mathsf{t}\,x \longrightarrow \mathsf{s}\,w$. Then a primitive context over v is either $w \circ_C \square$ or of the form $D[\square] \circ_C x$ for $D \in E(w)$, and $E(v) \cong 1 + E(w)$. Likewise, $E(f(v)) \cong 1 + E(f(w))$, and it is straightforward to check that $f : E(v) \longrightarrow E(f(v))$ is indeed a bijection. □

Proposition 4.2.26. *Let $f : \mathcal{P} \longrightarrow \mathcal{Q}$ be a morphism of many-to-one polygraphs. For all $n \geq 1$, it induces a morphism of polynomial functors $\nabla_n f : \nabla_n\mathcal{P} \longrightarrow \nabla_n\mathcal{Q}$, where $(\nabla_n f)_1 = f_n : \mathcal{P}_n \longrightarrow \mathcal{Q}_n$.*

Proof. Consider

$$\begin{tikzcd}
\mathcal{P}_{n-1} \arrow[d, "f"'] & \mathcal{P}^\bullet_n \arrow[l, "\mathsf{s}"'] \arrow[r, "p"] \arrow[d, "f^\bullet"'] & \mathcal{P}_n \arrow[r, "\mathsf{t}"] \arrow[d, "f"'] & \mathcal{P}_{n-1} \arrow[d, "f"'] \\
\mathcal{Q}_{n-1} & \mathcal{Q}^\bullet_n \arrow[l, "\mathsf{s}"'] \arrow[r, "p"] & \mathcal{Q}_n \arrow[r, "\mathsf{t}"] & \mathcal{Q}_{n-1}
\end{tikzcd}$$

where $f_n^\bullet$ maps a context $C : a \longrightarrow \mathsf{s}x$ to $f(C) : f(a) \longrightarrow f(\mathsf{s}x)$. Clearly, all squares commute, and by lemma 4.2.25, the middle one is cartesian. □

Definition 4.2.27 (Composition)**.** We define the *composition* operation $(-)^\circ : \operatorname{tr} \nabla_n \mathcal{P} \longrightarrow \mathcal{P}_n^{\mathrm{mto}}$. At the same time, we establish a bijection between $T^{|}$ and the primitive contexts over $\mathsf{s}T^\circ$, where $T \in \operatorname{tr} \nabla_n \mathcal{P}$.

(1) If $a \in \mathcal{P}_{n-1}$, then $(\mathsf{I}_a)^\circ := \mathrm{id}_a$. Note that the only primitive context over $\mathsf{s}\,\mathrm{id}_a$ is $\square : a \longrightarrow a$, and let $C_{[]} := \square$.

(2) If $x \in \mathcal{P}_n$, then $(\mathsf{Y}_x)^\circ := x$. Note that by definition of $\nabla_n \mathcal{P}$ (definition 4.2.24) we have $\mathsf{Y}_x^{|} = \{[D] \mid D \in x^\bullet\}$ (see remark 2.2.18), and let $C_{[D]} := D$.

(3) Consider a tree of the form $S = T \circ_{[l]} \mathsf{Y}_x$, with $T \in \operatorname{tr} \nabla_n \mathcal{P}$ having at least one node, $[l] \in T^{|}$ and $x \in \mathcal{P}_n$. By induction, the leaf $[l]$ corresponds to a primitive context $C_{[l]} : a \longrightarrow \mathsf{s}T^\circ$, and moreover, $a = \mathsf{t}x$. Let $S^\circ := T^\circ \circ_{C_{[l]}} x$. Let $[l'] \in S^{|}$. If $[l']$ is of the form $[lD]$, for some $[D] \in \mathsf{Y}_x^{|}$, let $C_{[l']} := C_{[l]}[D]$. Otherwise, $[l']$ is a leaf of T, and so $C_{[l']}$ is already defined.

Definition 4.2.28 (Composition tree)**.** A *composition tree* is simply a $\nabla_n \mathcal{P}$-tree. If $v \in \mathcal{P}_n^{\mathrm{mto}}$ and T is a composition tree such that $T^\circ = v$, then we say that T is a composition tree of v.

The following result generalizes lemma 4.2.23.

Proposition 4.2.29. *Let $f : \mathcal{P} \longrightarrow \mathcal{Q}$ a morphism of many-to-one polygraphs. The following square is cartesian*

$$\begin{tikzcd}
\operatorname{tr} \nabla_n \mathcal{P} \arrow[r, "(-)^\circ"] \arrow[d, "f"'] & \mathcal{P}_n^{\mathrm{mto}} \arrow[d, "f"] \\
\operatorname{tr} \nabla_n \mathcal{Q} \arrow[r, "(-)^\circ"] & \mathcal{Q}_n^{\mathrm{mto}}.
\end{tikzcd}$$

Proof. This amounts to showing that if $v \in \mathcal{P}_n^{\mathrm{mto}}$, then f establishes a bijective correspondence between the composition trees of v and $f(v)$. This is clear if $\# v \leq 1$, so let us assume $\# v \geq 2$, and let T be a composition tree of v. Then T decomposes as

$$T = \mathsf{Y}_a \bigcirc_{[C]} T_C,$$

where $a \in \mathcal{Q}_n$, C ranges over the primitive contexts over $\mathsf{s}a$, and $T_C \in \operatorname{tr} \nabla_n \mathcal{Q}$. Then, by lemma 4.2.23, there exists a unique $b \in \mathcal{P}_n$ such that $f(b) = a$, and unique v_C's such that $f(v_C) = T_C^\circ$. Further, f exhibits a bijection between the primitive contexts over a and over b. Note that $\# v_C < \# v$, so by induction, there exists a unique $S_C \in \operatorname{tr} \nabla_n \mathcal{P}$ such that $f(S_C) = T_C$. Finally,

$$S := \mathsf{Y}_b \bigcirc_{[D]} S_{f(D)}$$

is the unique tree such that $f(S) = T$. □

Lemma 4.2.30. *Let $\underline{u}$ be a polyplex. The fundamental cell $u \in \underline{u}_n$ has at most one composition tree.*

Proof. Assume that S and T are two composition trees of u. Necessarily, $\mathsf{e}_{[]}\, S = \mathsf{t}\, u = \mathsf{e}_{[]}\, T$, i.e. the root edge of S and T are both decorated by the target of u. By lemma 4.2.22, exactly one of the following two possibilities happens.

(1) If $\mathsf{t}\, u$ occurs in $\mathsf{s}\, u$, then u is an identity cell, and $S = \mathsf{I}_{\mathsf{t}\, u} = T$. We are done.

(2) Otherwise $\mathsf{t}\, u$ is the target of a unique n-generator of $\underline{u}$, say x. Since $\mathsf{t}\,\mathsf{s}_{[]}\, S = \mathsf{e}_{[]}\, S = x$, we have $\mathsf{s}_{[]}\, S = x$, in other words, the root node of S is decorated by x. Likewise, $\mathsf{s}_{[]}\, T = x$.
Let a be an $(n-1)$-generator occurring in $\mathsf{s}\, x$. It decorates an input edge of the root node of S and of T. If a occurs in $\mathsf{s}\, u$, then by lemma 4.2.22 again, there is no n-generator whose target is a. So in S and T, there cannot be a node above the corresponding edges, i.e. these edges are leaves. Otherwise, a is the target of a unique n-generator y, and necessarily, $\mathsf{s}_{[a]}\, S = y = \mathsf{s}_{[a]}\, T$. Applying lemma 4.2.22 repeatedly, we show that $S = T$. □

Proposition 4.2.31. *The composition map* $(-)^\circ : \operatorname{tr} \nabla_n \mathcal{P} \longrightarrow \mathcal{P}_n^{\mathrm{mto}}$ *is a bijection.*

Proof. *Surjectivity.* This clearly holds if $n \leq 1$. Assume $n \geq 2$ and let $v \in \mathcal{P}_n^{\mathrm{mto}}$. If $v = \mathrm{id}_a$ for some $a \in \mathcal{P}_{n-1}$, then $v = \mathsf{I}_a^\circ$. If $v \in \mathcal{P}_n$, then $v = \mathsf{Y}_v^\circ$. Otherwise, by proposition 4.2.13, v decomposes as $w \circ_C x$, where $w \in \mathcal{P}_n^{\mathrm{mto}}$, $x \in \mathcal{P}_n$, and $C : \mathsf{t}\, x \longrightarrow \mathsf{s}\, w$ is a context. By induction, there exist $T \in \operatorname{tr} \nabla_n \mathcal{P}$ such that $T^\circ = w$. By construction, C corresponds to a unique leaf address $[l] \in T^{\mathsf{l}}$. Finally, $v = \left(T \circ_{[l]} \mathsf{Y}_x\right)^\circ$.

Injectivity. Let $v \in \mathcal{P}_n^{\mathrm{mto}}$, $u := {!}\, v$, and $f : \underline{y} \longrightarrow \mathcal{P}$ map the fundamental cell u to v. By proposition 4.2.29, the following square is cartesian

$$\begin{tikzcd} \operatorname{tr} \nabla_n \underline{u} \arrow[r, "(-)^\circ"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{u}_n^{\mathrm{mto}} \arrow[d, "f"] \\ \operatorname{tr} \nabla_n \mathcal{P} \arrow[r, "(-)^\circ"] & \mathcal{P}_n^{\mathrm{mto}}, \end{tikzcd}$$

and in particular, f induces a bijection between the composition trees of $u \in \underline{u}_n^{\mathrm{mto}}$ and $v \in \mathcal{P}_n^{\mathrm{mto}}$. By lemma 4.2.30 and the previous point, u has exactly one composition tree, and thus, so does v. □

Notation 4.2.32. Let the *composition tree* operation $\mathrm{ct} : \mathcal{P}_n^{\mathrm{mto}} \longrightarrow \operatorname{tr} \nabla_n \mathcal{P}$ be the inverse of the composition operation $(-)^\circ$ of definition 4.2.27.

Corollary 4.2.33. *Let* $\mathcal{P} \in \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ *and* $v \in \mathcal{P}_n^{\mathrm{mto}}$ *with* $\# v \geq 1$. *Then* v *uniquely decomposes as*

$$v = x \bigcirc_{C} v_C$$

where $x \in \mathcal{P}_n$, C *ranges over the primitive contexts over* $\mathsf{s}\, x$, *and* $v_C \in \mathcal{P}_n^{\mathrm{mto}}$.

Proof. The composition tree of v decomposes uniquely as

$$\mathrm{ct}\, v = \mathsf{Y}_x \bigcirc_{[C]} T_C,$$

and applying back $(-)^\circ$ gives the desired decomposition of v. □

Corollary 4.2.34. *Let $\mathcal{P} \in \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$. A many-to-one context $C \in \mathcal{C}\mathrm{tx}_n^{\mathrm{mto}}\mathcal{P}$ decomposes uniquely as*

$$C = C[\square] = x \underset{D}{\circ} \square \underset{E}{\bigcirc} v_E$$

where $x, v_E \in \mathcal{P}_n^{\mathrm{mto}}$.

Proof. We proceed by induction on $\#C$. Since $\#_\square C = 1$ (i.e. C has only one occurrence of $\square$, see definition 4.1.5), we have $\#C \geq 1$. By corollary 4.2.33, C uniquely decomposes as

$$C = y \underset{F}{\bigcirc} u_F.$$

Either one of two cases occur.

(1) If $y = \square$, then consider

$$C = \mathrm{id}_{\mathtt{t}\square} \underset{\boxdot}{\circ} \square \underset{F}{\bigcirc} u_F,$$

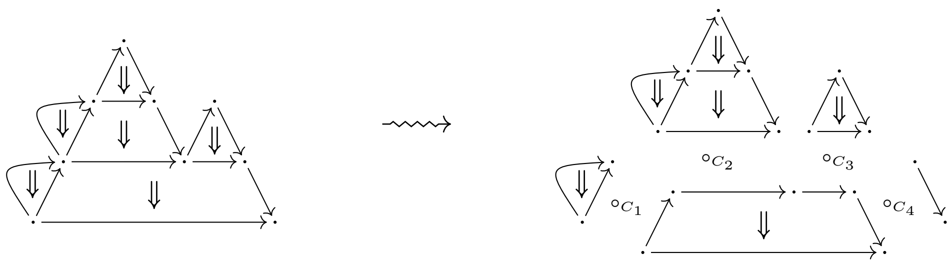
where $\boxdot$ is the trivial context $\mathtt{t}\square \longrightarrow \mathtt{t}\square$.

(2) Otherwise, there exists a unique $F \in y^\bullet$ such that $\#_\square u_F = 1$. By induction, u_F decomposes as on the left, and consider the decomposition of C as on the right:

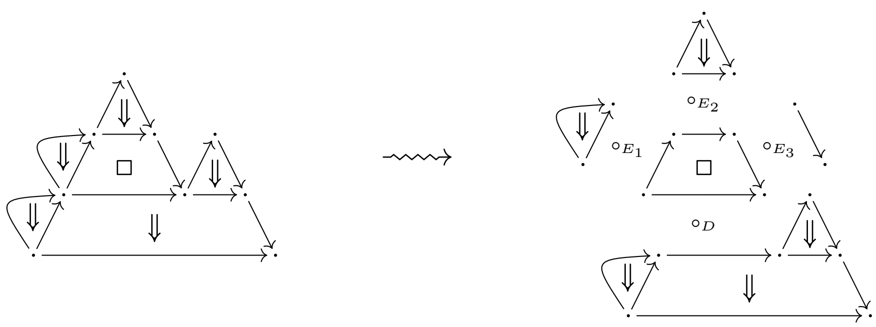
$$u_F = z \underset{G}{\circ} \square \underset{H}{\bigcirc} w_H, \qquad C = \left(y \underset{F}{\circ} z \underset{G \neq F}{\bigcirc} u_G \right) \underset{F}{\circ} \square \underset{H}{\bigcirc} w_H.$$

□

Remark 4.2.35. Given a many-to-one cell as on the left below, corollary 4.2.33 decomposes it as on the right.



Given a context as on the left below, corollary 4.2.34 detaches $\square$ from the cell "below" and the cells "above" it:



Remark 4.2.36. Let $C : x \longrightarrow y$ be an n-context of $\mathcal{P} \in \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ between many-to-one cells. In particular, C is a many-to-one cell in the extended category $\mathcal{Q}$ of definition 4.1.6, so by virtue of corollary 4.2.34, it uniquely decomposes as

$$C = z \underset{D}{\circ} \square \bigcirc_{E} v_E.$$

where $z, v_E \in \mathcal{P}_n^*$. Since $C[x] = y$, we have

$$y = z \underset{D}{\circ} x \bigcirc_{E} v_E.$$

Conversely, any decomposition of y of the form above induces a context $C : x \longrightarrow y$. In particular, the number of primitive contexts over y is $\# y$, and $\nabla_n \mathcal{P}$ is finitary.

Definition 4.2.37. We now extend $(-)^\circ$ and ct to functors between $\operatorname{tr} \nabla_n \mathcal{P}$ and $\mathcal{C}\mathrm{tx}_n^{\mathrm{mto}} \mathcal{P}$. On object, they are respectively defined in definitions 4.2.27 and 4.2.28.

(1) Let $f : T \longrightarrow S$ be a morphism in $\operatorname{tr} \nabla_n \mathcal{P}$. It corresponds to a decomposition of S as on the left, and let f° be the context $T^\circ \longrightarrow S^\circ$ on the right:

$$S = U \underset{[p]}{\circ} T \bigcirc_{[l]} V_{[l]}, \qquad f^\circ := U^\circ \underset{C_{[p]}}{\circ} \square \bigcirc_{C_{[l]}} V_{[l]}^\circ,$$

where $[l]$ ranges over $T^{|}$.

(2) Let $C : x \longrightarrow y$ be an n-context. By corollary 4.2.34, it decomposes uniquely as on the left, and let $\operatorname{ct} C$ correspond to the decomposition of $\operatorname{ct} y$ on the right:

$$C[\square] = z \underset{D}{\circ} \square \bigcirc_{E} t_E, \qquad \operatorname{ct} y = (\operatorname{ct} z) \underset{[l_D]}{\circ} (\operatorname{ct} x) \bigcirc_{[l_E]} (\operatorname{ct} t_E),$$

where E ranges over all primitive contexts over $\mathsf{s}\, x$.

One readily checks the following:

Proposition 4.2.38 (Composition tree duality)**.** *The functors* $(-)^\circ$ *and* ct *are mutually inverse isomorphisms of categories.*

Corollary 4.2.39. *For* $n \geq 2$ *and* $x \in \mathcal{P}_n$*, the functor* ct *induces a natural bijection*

$$\mathcal{P}_n^\bullet(x) \cong \sum_{a \in \mathcal{P}_{n-1}} (\operatorname{tr} \nabla_{n-1} \mathcal{P})(\mathsf{Y}_a, \operatorname{ct} \mathsf{s}\, x).$$

Proof. Direct consequence of proposition 4.2.38. $\square$

Notation 4.2.40. If $x \in \mathcal{P}_n$ and $[p] \in (\operatorname{ct} \mathsf{s}\, x)^\bullet$, then we write $\mathsf{s}_{[p]}\, x$ instead of $\mathsf{s}_{[p]} \operatorname{ct} \mathsf{s}\, x \in \mathcal{P}_{n-1}$.

4.3 THE EQUIVALENCE

We now aim at proving that the category of opetopic sets, i.e. $\mathcal{S}\mathrm{et}$-presheaves over the category $\mathbb{O}$ defined previously, is equivalent to the category of many-to-one polygraphs $\mathcal{P}\mathrm{ol}^{\mathrm{mto}}$. We achieve this by first constructing the *polygraphic realization* functor $|-| : \mathbb{O} \longrightarrow \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$. This functor "realizes" an opetope as a polygraph that freely implements all its tree structure by the means of adequately chosen generators in each dimension. Secondly, we consider the left Kan extension $|-| : \mathcal{P}\mathrm{sh}(\mathbb{O}) \longrightarrow \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ along the Yoneda embedding. This functor has a right adjoint, the "opetopic nerve" $N : \mathcal{P}\mathrm{ol}^{\mathrm{mto}} \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O})$, and we prove this adjunction to be an adjoint equivalence. This is done using the *shape function*, defined in section 4.3, which to any generator x of a many-to-one polygraph $\mathcal{P}$ associates an opetope $x^\natural$ along with a canonical morphism $\tilde{x} : |x^\natural| \longrightarrow \mathcal{P}$.

POLYGRAPHIC REALIZATION

An opetope $\omega \in \mathbb{O}_n$, with $n \geq 1$, has one target $\mathsf{t}\,\omega$, and sources $\mathsf{s}_{[p]}\,\omega$ laid out in a tree. If the sources $\mathsf{s}_{[p]}\,\omega$ happened to be generators in some polygraph, then that tree would describe a way to compose them. With this in mind, we define a many-to-one n-polygraph $|\omega|$, whose generators are essentially iterated faces (i.e. sources or targets) of ω (hypothesis **(PR1)** below). Moreover, $|\omega|$ will be "maximally unfolded" (or "free"), in that two (iterated) faces that are the same opetope, but located at different addresses, will correspond to distinct generators.

The rest of this section is devoted to inductively define the realization functor $|-| : \mathbb{O} \longrightarrow \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ together with its *boundary* $\partial|-|$. We bootstrap the process with definition 4.3.1 and state our induction hypotheses in 4.3.3.

Definition 4.3.1 (Low dimensional cases)**.** For $\blacklozenge$ the unique 0-opetope, let $\partial|\blacklozenge|$ be the empty polygraph, and $|\blacklozenge|$ be the polygraph with a unique generator in dimension 0, which we denote by $\blacklozenge$. For $\blacksquare$ the unique 1-opetope, let $\partial|\blacksquare| := |\blacklozenge| + |\blacklozenge|$, and let $|\blacksquare|$ be induced by the cellular extension

$$\partial|\blacksquare| \xleftarrow{\mathsf{s},\mathsf{t}} \{\blacksquare\},$$

where s and t map $\blacksquare$ to distinct 0-generators. There are obvious functors $|\mathsf{s}_{[]}|, |\mathsf{t}| : |\blacklozenge| \longrightarrow |\blacksquare|$, mapping $\blacklozenge$ to $\mathsf{s}\,\blacksquare$ and $\mathsf{t}\,\blacksquare$, respectively.

Definition 4.3.2 (Dimension 2)**.** For the reader's convenience, we construct $\partial|\mathbf{k}|$ and $|\mathbf{k}|$ for every opetopic integer $\mathbf{k}$, although this case already falls under the inductive definition (see definitions 4.3.4 and 4.3.11).

(1) Let $\partial|\mathbf{0}|$ be the 1-polygraph given by the following coequalizer:

$$|\blacklozenge| \overset{\mathsf{s}_{[]}}{\underset{\mathsf{t}}{\rightrightarrows}} |\blacksquare| \longrightarrow \partial|\mathbf{0}|.$$

In other words, $\partial|\mathbf{0}|$ has one object x and one generating endomorphism $f : x \longrightarrow x$. The 2-polygraph $|\mathbf{0}|$ is obtained by adjoining a generating 2-cell $\alpha : \mathrm{id}_x \longrightarrow f$ to $\partial|\mathbf{0}|$.

(2) Let $k \geq 1$, and consider the 1-polygraph

$$\mathcal{P} := \left(|\blacksquare| \coprod_{|\blacklozenge|} |\blacksquare| \coprod_{|\blacklozenge|} |\blacksquare| \coprod_{|\blacklozenge|} \cdots \coprod_{|\blacklozenge|} |\blacksquare| \right),$$

where there are k instances of $|\blacksquare|$. In other words, $\mathcal{P}$ is generated by a chain of k composable 1-cells, which we denote by

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_k} x_k.$$

Alternatively, $\mathcal{P}$ is the polyplex $\underline{l_k}$ of example 4.2.20. Let $\partial|\mathbf{k}|$ be the obvious pushout

$$\begin{tikzcd} \partial|\blacksquare| \arrow[r, "(x_0,x_k)"] \arrow[d] & \mathcal{P} \arrow[d] \\ |\blacksquare| \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \partial|\mathbf{k}|, \end{tikzcd}$$

i.e., $\partial|\mathbf{k}|$ is $\mathcal{P}$ with an additional generating 1-cell $g : x_0 \longrightarrow x_k$. Finally, $|\mathbf{k}|$ is obtained from $\partial|\mathbf{k}|$ by adjoining a generating 2-cell $\alpha : f_k \cdots f_2 f_1 \longrightarrow g$.

In both cases, there is a bijective correspondence between the generating cells of $|\mathbf{k}|$ and the objects of $\mathbb{O}/\mathbf{k}$, and the obvious inclusions induce functors

$$\partial|-|, |-| : \mathbb{O}_{\leq 2} \longrightarrow \mathcal{P}\mathrm{ol}^{\mathrm{mto}}.$$

Let $n \geq 2$ and assume by induction that $\partial|-|$ and $|-|$ are defined on $\mathbb{O}_{<n}$. Assume further that the following induction hypotheses hold (they are easily verified for $n = 2$).

Assumptions 4.3.3. For all $\psi \in \mathbb{O}_k$ with $k < n$, the following hold:

(PR1) for all $j \in \mathbb{N}$, the set $|\psi|_j$ of j-generators of $|\psi|$ is in bijection with the set of objects of the slice $\mathbb{O}_j/\psi$, i.e. of the form $\left(\phi \xrightarrow{\mathsf{a}} \psi\right)$ for $\phi \in \mathbb{O}_j$ and $\mathsf{a} : \phi \longrightarrow \psi$ a morphism in $\mathbb{O}$;

(PR2) for $\left(\phi \xrightarrow{\mathsf{a}} \psi\right)$ a generator of $|\psi|$, its target is $\left(\mathsf{t}\,\phi \xrightarrow{\mathsf{t}} \phi \xrightarrow{\mathsf{a}} \psi\right)$;

(PR3) for $l \leq k$, and for $\left(\phi \xrightarrow{\mathsf{a}} \psi\right)$ a l-generator of $|\psi|$, the composition tree of its source $\mathrm{ct}\,\mathsf{s}\left(\phi \xrightarrow{\mathsf{a}} \psi\right) \in \mathrm{tr}\,\nabla_{l-1}|\psi|$ is

$$\begin{aligned} \langle \phi \rangle &\longrightarrow \nabla_{l-1}|\psi| \\ [p] &\longmapsto \left(\mathsf{s}_{[p]}\,\phi \xrightarrow{\mathsf{s}_{[p]}} \phi \xrightarrow{\mathsf{a}} \psi\right). \end{aligned}$$

Recall that by proposition 4.2.38, this completely determines $\mathsf{s}\left(\phi \xrightarrow{\mathsf{a}} \psi\right) \in |\psi|^*_{l-1}$.

We now define $\partial|\omega|$ and $|\omega|$ when $\omega \in \mathbb{O}_N$. Defining the former is easy, and done in definition 4.3.4. The latter is defined in definition 4.3.11 as generated by a cellular extension

$$\partial|\omega| \xleftarrow{\mathsf{s},\mathsf{t}} \{\omega\}$$

of $\partial|\omega|$, where the target and source of the new generator are given by **(PR2)** and **(PR3)**. Lastly, we check the inductive hypotheses in proposition 4.3.12.

Definition 4.3.4 (Inductive step for $\partial|-|$)**.** For $\omega \in \mathbb{O}_N$, let $\partial|\omega|$ be the following many-to-one $(n-1)$-polygraph:

$$\partial|\omega| := \operatorname*{colim}_{\substack{\mathsf{a}:\psi\to\omega \\ \dim\psi<n}} |\psi|.$$

For $\mathsf{a} : \psi \longrightarrow \omega$ in $\mathbb{O}_{<n}/\omega$, this colimit comes with a corresponding coprojection $|\mathsf{a}| : |\psi| \hookrightarrow \partial|\omega|$.

Remark 4.3.5. Let $0 \le k < n$. By **(PR1)**, the set of k-generators of $\partial|\omega|$ is $\mathbb{O}_k/\omega$.

Lemma 4.3.6. *For $\omega \in \mathbb{O}_N$, and $j < n$, the set $\partial|\omega|_j$ of j-generators of $\partial|\omega|$ is the slice $\mathbb{O}_j/\omega$.*

Proof. Follows from the induction hypothesis **(PR1)** and proposition 4.2.3. □

Corollary 4.3.7. *For $\omega \in \mathbb{O}_N$ and $1 \le k < n$, the polynomial functor $\nabla_k \partial|\omega|$ is described as follows:*

$$\mathbb{O}_{k-1}/\omega \xleftarrow{\mathsf{s}} E \xrightarrow{p} \mathbb{O}_k/\omega \xrightarrow{\mathsf{t}} \mathbb{O}_{k-1}/\omega$$

where for $\left(\psi \xrightarrow{\mathsf{a}} \omega\right) \in \mathbb{O}_k/\omega$,

(1) the fiber $E\left(\psi \xrightarrow{\mathsf{a}} \omega\right)$ is simply $\psi^\bullet$;

(2) for $[p] \in E\left(\psi \xrightarrow{\mathsf{a}} \omega\right) \cong \psi^\bullet$, we have $\mathsf{s}[p] = \left(\mathsf{s}_{[p]}\psi \xrightarrow{\mathsf{s}_{[p]}} \psi \xrightarrow{\mathsf{a}} \omega\right)$;

(3) $\mathsf{t}\left(\psi \xrightarrow{\mathsf{a}} \omega\right) = \left(\mathsf{t}\psi \xrightarrow{\mathsf{t}} \psi \xrightarrow{\mathsf{a}} \omega\right)$.

Proof. Direct consequence of lemma 4.3.6 and **(PR1)**, **(PR2)**, and **(PR3)**. □

Definition 4.3.8. For $\omega \in \mathbb{O}_N$ and $1 \le k < n$, we have a morphism $u : \nabla_k \partial|\omega| \longrightarrow \mathfrak{Z}^{k-1}$

$$\begin{tikzcd}
\mathbb{O}_{k-1}/\omega \arrow[d, "u_0"'] & E \arrow[l, "\mathsf{s}"'] \arrow[r, "p"] \arrow[d, "u_2"'] & \mathbb{O}_k/\omega \arrow[r, "\mathsf{t}"] \arrow[d, "u_1"'] & \mathbb{O}_{k-1}/\omega \arrow[d, "u_0"'] \\
\mathbb{O}_{k-1} & \mathbb{O}_k^\bullet \arrow[l, "\mathsf{s}"'] \arrow[r, "p"] & \mathbb{O}_k \arrow[r, "\mathsf{t}"] & \mathbb{O}_{k-1}
\end{tikzcd}$$

induced by the forgetful maps $\mathbb{O}_{k-1}/\omega \longrightarrow \mathbb{O}_{k-1}$ and $\mathbb{O}_k/\omega \longrightarrow \mathbb{O}_k$.

Lemma 4.3.9. *Let $\omega \in \mathbb{O}_N = \operatorname{tr}\mathfrak{Z}^{n-2}$. The map $\omega : \langle\omega\rangle \longrightarrow \mathfrak{Z}^{n-2}$ factors through $u : \nabla_{n-1}\partial|\omega| \longrightarrow \mathfrak{Z}^{n-2}$ (definition 4.3.8):*

$$\begin{tikzcd}
 & \nabla_{n-1}\partial|\omega| \arrow[d, "u"] \\
\langle\omega\rangle \arrow[ur, "\bar{\omega}"] \arrow[r, "\omega"'] & \mathfrak{Z}^{n-2}.
\end{tikzcd}$$

Proof. Let $\bar{\omega}$ map a node $[p] \in \omega^\bullet$ to the cell $\left(\mathsf{s}_{[p]}\omega \xrightarrow{\mathsf{s}_{[p]}} \omega\right) \in \mathbb{O}_{n-1}/\omega$, and map an edge $[l]$ to the cell $\left(\mathsf{e}_{[l]}\omega \xrightarrow{\mathsf{e}_{[l]}} \omega\right) \in \mathbb{O}_{n-2}/\omega$ (see notation 2.4.5). □

Proposition 4.3.10. *On the one hand, consider the tree $\nabla_{n-1}\partial|\omega|$-tree $\bar{\omega}$ of lemma 4.3.9, and on the other hand, recall from remark 4.3.5 that there is a $(n-1)$-generator $\left(\mathsf{t}\omega \xrightarrow{\mathsf{t}} \omega\right)$ of $\partial|\omega|$ corresponding to the target embedding of ω. Then, in $\partial|\omega|$, the composite $\bar{\omega}^\circ$ (definition 4.2.27) and the generator $\left(\mathsf{t}\omega \xrightarrow{\mathsf{t}} \omega\right)$ are parallel.*

Proof. If ω is degenerate, say $\omega = \mathsf{I}_\phi$ for some $\phi \in \mathbb{O}_{n-2}$, then $\bar{\omega}^\circ = \mathrm{id}_{\left(\phi \xrightarrow{\mathsf{tt}} \omega\right)}$, while $\left(\mathsf{t}\omega \xrightarrow{\mathsf{t}} \omega\right) = \left(\mathsf{Y}_\phi \xrightarrow{\mathsf{t}} \omega\right)$. By **(Degen)**, those two cells are parallel.

For the rest of the proof, we assume that ω is not degenerate. First, we have

$$\mathsf{t}\bar{\omega}^\circ = \mathsf{t}\mathsf{s}_{[]}\bar{\omega} = \mathsf{t}\left(\mathsf{t}\mathsf{s}_{[]}\omega \xrightarrow{\mathsf{t}\mathsf{s}_{[]}} \omega\right) = \left(\mathsf{t}\mathsf{t}\omega \xrightarrow{\mathsf{t}\mathsf{t}} \omega\right) = \mathsf{t}\left(\mathsf{t}\omega \xrightarrow{\mathsf{t}} \omega\right).$$

Then, in order to show that $\mathsf{s}\bar{\omega}^\circ = \mathsf{s}(\mathsf{t}\omega \xrightarrow{\mathsf{t}} \omega)$, we show that the $(n-2)$-generators occurring on both sides are the same, and that the way to compose them is unique.

(1) Generators in $\mathsf{s}\bar{\omega}^\circ$ are of the form $\left(\phi \xrightarrow{[q]} \psi \xrightarrow{[p]} \omega\right)$, for $[p[q]] \in \omega^{\shortmid}$. By **(Glob2)**, those are equal to $\left(\phi \xrightarrow{\wp_\omega[p[q]]} \mathsf{t}\omega \xrightarrow{\mathsf{t}} \omega\right)$, which are exactly the generators in the cell $\mathsf{s}\left(\mathsf{t}\omega \xrightarrow{\mathsf{t}} \omega\right)$.

(2) To show that there is a unique way to compose all the $(n-2)$-generators of the form $(\phi \xrightarrow{\mathsf{s}_{[q]}} \psi \xrightarrow{\mathsf{s}_{[p]}} \omega)$, where $[p[q]]$ ranges over $\omega^{\shortmid}$, it is enough to show that no two have the same target. Assume $(\phi_i \xrightarrow{\mathsf{s}_{[q_i]}} \psi_i \xrightarrow{\mathsf{s}_{[p_i]}} \omega)$, with $i = 1, 2$, are $(n-2)$-generators occuring in $\mathsf{s}\bar{\omega}^\circ$ with the same target. Consider the following diagram:

$$\begin{tikzcd}
 & \phi_1 \arrow[rr, "\mathsf{s}_{[q_1]}"] \arrow[drr, "\mathsf{s}_{[r_1]}"'] & & \psi_1 \arrow[dr, "\mathsf{s}_{[p_1]}"] & \\
\rho \arrow[ur, "\mathsf{t}"] \arrow[dr, "\mathsf{t}"'] & & & \mathsf{t}\omega \arrow[r, "\mathsf{t}"] & \omega \\
 & \phi_2 \arrow[rr, "\mathsf{s}_{[q_2]}"] \arrow[urr, "\mathsf{s}_{[r_2]}"] & & \psi_2 \arrow[ur, "\mathsf{s}_{[p_2]}"'] &
\end{tikzcd}$$

where $[r_i] := \wp_\omega[p_i[q_i]] \in \mathsf{t}\omega^\bullet$. The outer hexagon commutes by assumption, the two squares on the right are instances of **(Glob2)**, and the left square commutes as $\mathsf{t} : \mathsf{t}\omega \longrightarrow \omega$ is a monomorphism, since ω is non degenerate. By inspection of the opetopic identities (see definition 3.4.2), the only way for the left square to commute is the trivial way, i.e. $[r_1] = [r_2]$. Since $\wp_\omega$ is a bijection, we have $[p_1[q_1]] = [p_2[q_2]]$, thus $[p_1] = [p_2]$ and $[q_1] = [q_2]$. □

Definition 4.3.11 (Inductive step for $|-|$)**.** For $\omega \in \mathbb{O}_N$, let $|\omega|$ be the cellular extension

$$\partial|\omega| \xleftarrow{\mathsf{s},\mathsf{t}} \{\omega\},$$

where t maps ω to the $(n-1)$-generator $\left(\mathsf{t}\omega \xrightarrow{\mathsf{t}} \omega\right)$, and where the composition tree of $\mathsf{s}\omega$ is $\bar{\omega}$ (lemma 4.3.9). For consistency, we also write $\left(\omega \xrightarrow{\mathrm{id}} \omega\right)$ for the unique n-generator of $|\omega|$. This is well-defined by proposition 4.3.10, and gives a functor $|-| : \mathbb{O}_{\leq n} \longrightarrow \mathcal{P}\mathrm{ol}$.

Proposition 4.3.12. *For $\omega \in \mathbb{O}_N$, the polygraphs $\partial|\omega|$ (definition 4.3.4) and $|\omega|$ (definition 4.3.11) satisfy the assumptions 4.3.3.*

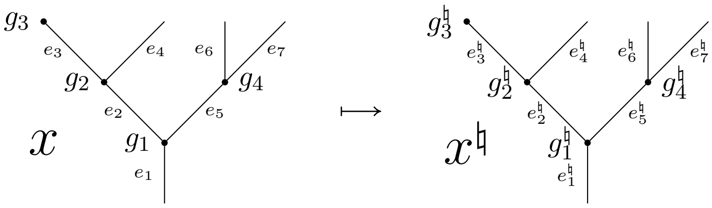
Proof. **(PR1)** For $j < n$, by lemma 4.3.6, we already have $|\omega|_j = \partial|\omega|_j = \mathbb{O}_j/\omega$. In dimension n, the only element of $\mathbb{O}_N/\omega$ is $\mathrm{id} : \omega \longrightarrow \omega$, which corresponds to the unique n-generator of $|\omega|$. If $j > n$, then both $\mathbb{O}_j/\omega$ and $|\omega|_j$ are empty.

(PR2) *and* **(PR3)** By definition, those hypotheses hold for the unique n-generator $\left(\omega \xrightarrow{\mathrm{id}} \omega\right)$ of $|\omega|$. By induction, they also hold on the other generators. $\square$

To conclude, we have defined a functor $|-| : \mathbb{O} \longrightarrow \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ which satisfies the assumptions 4.3.3 for all $n \in \mathbb{N}$.

THE SHAPE FUNCTION

This subsection is devoted to the definition of the *shape function* $(-)^\natural$. We first sketch the idea. Take $\mathcal{P} \in \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ and define $(-)^\natural : \mathcal{P}_n \longrightarrow \mathbb{O}_n$ by induction. The cases $n = 0, 1$ are trivial, since there is a unique 0-opetope and a unique 1-opetope. Assume $n \geq 2$, and take $x \in \mathcal{P}_n$. Then the composition tree of $\mathsf{s}\,x$ is a coherent tree whose nodes are $(n-1)$-generators, and edges are $(n-2)$-generators. Replacing those $(n-1)$ and $(n-2)$-generators by their respective shape, we obtain a coherent tree whose nodes are $(n-1)$-opetopes, and edges are $(n-2)$-opetopes, in other words, we obtain an n-opetope, which we shall denote by $x^\natural$.



The fact that $x^\natural$ corresponds to the intuitive notion of "shape" of x is justified by theorem 4.3.16. If P is the set of all the generators of $\mathcal{P}$, then the shape function $(-)^\natural : P \longrightarrow \mathbb{O}$ makes it into a set over $\mathbb{O}$. We shall see that P can be further promoted to an opetopic set, and that it completely determines $\mathcal{P}$. Furthermore, the notation $(-)^\natural$ will then coincide with definition 0.3.3.

Lemma 4.3.13. *If $x, y \in \mathcal{T}_n$ are two parallel generators, then they are equal.*

Proof. We have $x = (\mathsf{s}\,x, \mathsf{t}\,x) = (\mathsf{s}\,y, \mathsf{t}\,y) = y$. $\square$

Proposition 4.3.14. *For $x \in \mathcal{T}_n$ there exists a unique $x^\natural \in \mathbb{O}_n$ such that the terminal morphism $! : |x^\natural| \longrightarrow \mathcal{T}$ maps $x^\natural$ (the unique n-generator of $|x^\natural|$) to x. In particular, the map $(-)^\natural : \mathcal{T}_n \longrightarrow \mathbb{O}_n$ is a bijection.*

Proof. *Uniqueness.* Assume that there exists two distinct opetopes $\phi, \phi' \in \mathbb{O}_k$ such that $!\,\phi = !\,\phi'$, with k minimal for this property. Then necessarily, $k \geq 2$. On the one hand, we have $\langle \phi \rangle = \langle \mathrm{ct}\,\mathsf{s}\,\phi \rangle = \langle \mathrm{ct}\,\mathsf{s}\,\phi' \rangle = \langle \phi' \rangle$. On the other hand, for $[p] \in \phi^\bullet = (\phi')^\bullet$, we have

$$
\begin{aligned}
!\,\mathsf{s}_{[p]}\,\phi &= !\,\mathsf{s}_{[p]}\,\phi && \text{since } ! \text{ is also } |\mathsf{s}_{[p]}\,\phi| \hookrightarrow |\phi| \to \mathcal{T} \\
&= \mathsf{s}_{[p]}\,!\,\phi && \text{since } ! \text{ is a morphism of polygraphs} \\
&= \mathsf{s}_{[p]}\,!\,\phi' && \text{by assumption}
\end{aligned}
$$

$$
\begin{aligned}
&= \,!\,\mathsf{s}_{[p]}\,\phi' && \text{since ! is a morphism of polygraphs} \\
&= \,!\,\mathsf{s}_{[p]}\,\phi' && \text{since ! is also } |\mathsf{s}_{[p]}\,\phi'| \hookrightarrow |\phi'| \to \mathfrak{T},
\end{aligned}
$$

and by minimality of k, we have $\mathsf{s}_{[p]}\,\phi = \mathsf{s}_{[p]}\,\phi'$, for any address $[p]$. Consequently, $\phi = \phi'$, a contradiction.

Existence. The cases $n = 0, 1$ are trivial, so assume $n \geq 2$, and that by induction, the result holds for all $k < n$, i.e. that for $g \in \mathcal{P}_k$, there is a unique opetope $g^\natural \in \mathbb{O}_k$ such that $!(g^\natural) = g$. In particular the following two triangles commute:

$$
\begin{tikzcd}
|\mathsf{s}_{[p]}\,g^\natural| \arrow[r, "|\mathsf{s}_{[p]}|"] \arrow[dr, "!"'] & |g^\natural| \arrow[d, "!"] \\
& \mathcal{P},
\end{tikzcd}
\qquad
\begin{tikzcd}
|\mathsf{t}\,g^\natural| \arrow[r, "|\mathsf{t}|"] \arrow[dr, "!"'] & |g^\natural| \arrow[d, "!"] \\
& \mathcal{P},
\end{tikzcd}
$$

where $[p] \in (g^\natural)^\bullet$. Consequently, $(\mathsf{s}_{[p]}\,g)^\natural = \mathsf{s}_{[p]}(g^\natural)$ and $(\mathsf{t}\,g)^\natural = \mathsf{t}(g^\natural)$, and the following displays an isomorphism $\nabla_{n-1}\mathfrak{T} \longrightarrow \mathfrak{Z}^{n-2}$:

$$
\begin{tikzcd}
\mathfrak{T}_{n-2} \arrow[d, "(-)^\natural"'] & \mathfrak{T}^\bullet_{n-1} \arrow[l, "\mathsf{s}"'] \arrow[r, "p"] \arrow[d] & \mathfrak{T}_{n-1} \arrow[r, "\mathsf{t}"] \arrow[d, "(-)^\natural"'] & \mathfrak{T}_{n-2} \arrow[d, "(-)^\natural"'] \\
\mathbb{O}_{n-2} & \mathbb{O}^\bullet_{n-1} \arrow[l, "\mathsf{s}"'] \arrow[r, "p"] & \mathbb{O}_{n-1} \arrow[r, "\mathsf{t}"] & \mathbb{O}_{n-2}.
\end{tikzcd}
$$

Hence, the composite $\langle \mathrm{ct}\,\mathsf{s}\,x\rangle \xrightarrow{\mathrm{ct}\,\mathsf{s}\,x} \nabla_{n-1}\mathfrak{T} \xrightarrow{(-)^\natural} \mathfrak{Z}^{n-2}$ defines an n-opetope $x^\natural$ with $\langle x^\natural\rangle = \langle \mathrm{ct}\,\mathsf{s}\,x\rangle$. We claim that $!(x^\natural) = x$. We first show that $!\,\mathsf{s}(x^\natural) = \mathsf{s}\,x$. We have

$$
\begin{aligned}
\langle \mathrm{ct}\,\mathsf{s}\,x\rangle &= \langle x^\natural\rangle && \text{by definition} \\
&= \langle \mathrm{ct}\,\mathsf{s}(x^\natural)\rangle && \text{by } \textbf{(PR3)} \\
&= \langle \mathrm{ct}\,!\,\mathsf{s}(x^\natural)\rangle && \text{since ! is a morphism of polygraphs.}
\end{aligned}
$$

Then, for any address $[p]$ in $\langle \mathrm{ct}\,\mathsf{s}\,x\rangle$, we have

$$
\begin{aligned}
\mathsf{s}_{[p]}\,x &= !((\mathsf{s}_{[p]}\,x)^\natural) && \text{by induction} \\
&= !\,\mathsf{s}_{[p]}(x^\natural) && \text{by definition of } x^\natural \\
&= \mathsf{s}_{[p]}\,!(x^\natural) && \text{since ! is a morphism of polygraphs,}
\end{aligned}
$$

and therefore, by proposition 4.2.38, $\mathsf{s}\,x = \mathsf{s}\,!(x^\natural)$. Next,

$$
\begin{aligned}
\mathsf{t}\,!(x^\natural) &= \,!\,\mathsf{t}(x^\natural) && \text{by induction} \\
&\parallel \,!\,\mathsf{s}(x^\natural) && \text{since } \mathsf{s}(x^\natural) \parallel \mathsf{t}(x^\natural) \\
&= \mathsf{s}\,x && \text{showed above} \\
&\parallel \mathsf{t}\,x,
\end{aligned}
$$

and therefore, $\mathsf{t}\,!(x^\natural) = \mathsf{t}\,x$. Finally, $!(x^\natural) \parallel x$, and by lemma 4.3.13, $!(x^\natural) = x$. □

Notation 4.3.15. In the light of proposition 4.3.14, we identify $\mathfrak{T}_n$ with $\mathbb{O}_n$. This identification is compatible with faces, i.e. $\mathsf{s}_{[p]}$ and t. Then, $! : |\omega| \longrightarrow \mathfrak{T}$ maps a generator $(\phi \to \omega)$ to ϕ.

Theorem 4.3.16. *For $\mathcal{P} \in \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ and $x \in \mathcal{P}_n$, there exists a unique pair*[1]

$$\left(x^\natural, |x^\natural| \xrightarrow{\tilde{x}} \mathcal{P}\right) \in |-|/\mathcal{P}$$

(see definition 0.1.3) such that $\tilde{x}_n(x^\natural) = x$. Further, the shape function $(-)^\natural : \mathcal{P}_n \longrightarrow \mathbb{O}_n$ *maps an n-generator x to $x^\natural = {!}\,x$, where $!$ is the terminal morphism $\mathcal{P} \longrightarrow \mathcal{T}$, and the map*

$$\widetilde{(-)} : \mathcal{P}_n \longrightarrow \sum_{\omega \in \mathbb{O}_n} \mathcal{P}\mathrm{ol}^{\mathrm{mto}}(|\omega|, \mathcal{P}) \tag{4.3.17}$$

is a bijection[2].

Proof. *Uniqueness.* Assume $|\omega| \xrightarrow{f} \mathcal{P} \xleftarrow{f'} |\omega'|$ are different morphisms such that $f(\omega) = x = f'(\omega')$. Then ${!}\,\omega = {!}\,f(\omega) = {!}\,f'(\omega') = {!}\,\omega'$, and by proposition 4.3.14, $\omega = \omega'$. Let $\left(\phi \xrightarrow{\mathsf{a}} \omega\right) \in |\omega|_k$ be such that $f\left(\phi \xrightarrow{\mathsf{a}} \omega\right) \neq f'\left(\phi \xrightarrow{\mathsf{a}} \omega\right)$, with k minimal for this property. Then $k < n$ (since by assumption $f(\omega) = x = f'(\omega')$), and a factorizes as $\left(\phi \xrightarrow{\mathsf{j}} \psi \xrightarrow{\mathsf{b}} \omega\right)$, where j is a face embedding, i.e. either t or $\mathsf{s}_{[p]}$ for some $[p] \in \omega^\bullet$. Then by assumption,

$$\begin{aligned} f\left(\phi \xrightarrow{\mathsf{a}} \omega\right) &= \mathsf{j}f\left(\psi \xrightarrow{\mathsf{b}} \omega\right) \\ &= \mathsf{j}f'\left(\psi \xrightarrow{\mathsf{b}} \omega\right) && \text{by minimality of } k \\ &= f'\left(\phi \xrightarrow{\mathsf{a}} \omega\right), \end{aligned}$$

a contradiction.

Existence. The cases $n = 0, 1$ are trivial, so assume $n \geq 2$, and that by induction, the result holds for all $k < n$. Let $x^\natural := {!}\,x \in \mathbb{O}_n$. We wish to construct a morphism $O[x^\natural] \xrightarrow{\tilde{x}} \mathcal{P}$ having x in its image. For $\left(\psi \xrightarrow{\mathsf{j}} x^\natural\right)$ a face of $x^\natural$ (i.e. t or $\mathsf{s}_{[p]}$ for some $[p] \in (x^\natural)^\bullet$), we have $(\mathsf{j}\,x)^\natural = \psi$, so that by induction, there exists a morphism $|\psi| \xrightarrow{\widetilde{\mathsf{j}x}} \mathcal{P}$ having $\mathsf{j}\,x$ in its image, providing a commutative square

$$\begin{tikzcd} {|\psi|} \arrow[r, "\widetilde{\mathsf{j}x}"] \arrow[d, "|\mathsf{j}|"'] & \mathcal{P} \arrow[d, "!"] \\ {|x^\natural|} \arrow[r, "!"] & \mathcal{T}. \end{tikzcd}$$

To alleviate upcoming notations, write $\bar{\mathsf{j}} := \widetilde{\mathsf{j}x} : |\psi| \longrightarrow \mathcal{P}$. Let $\left(\phi \xrightarrow{\mathsf{a}} x^\natural\right) \in \mathbb{O}_{<n}/x^\natural$. If a is a face embedding, define $\bar{\mathsf{a}}$ as before. If not, then it factors through a face embedding as $\mathsf{a} = \left(\phi \xrightarrow{\mathsf{j}} \psi \xrightarrow{\mathsf{b}} \omega\right)$, and let $\bar{\mathsf{a}} := \bar{\mathsf{b}} \cdot |\mathsf{j}|$. Then the left square commutes,

[1] In [Hen19, proposition 2.2.3 (2)], $x^\natural$ is written $\underline{x}$ and called the *universal cell* (or *top cell*) of x.

[2] In other words, the functor $\mathcal{P}\mathrm{ol}^{\mathrm{mto}} \longrightarrow \mathcal{S}\mathrm{et}$ that maps a polygraph $\mathcal{P}$ to $\mathcal{P}_n$ is *familially representable* [CJ95, definition 2.4] with $\{|\omega| \mid \omega \in \mathbb{O}_n\}$ as representing family.

and passing to the colimit over $\mathbb{O}_{<n}/x^\natural$, we obtain the right square:

$$\begin{tikzcd} {|\phi|} \arrow[r, "\bar{\mathsf{a}}"] \arrow[d, "|a|"'] & \mathcal{P} \arrow[d, "!"] \\ {|x^\natural|} \arrow[r, "!"] & \mathcal{T}, \end{tikzcd} \qquad \begin{tikzcd} \partial|x^\natural| \arrow[r, "f"] \arrow[d] & \mathcal{P} \arrow[d, "!"] \\ {|x^\natural|} \arrow[r, "!"] & \mathcal{T}. \end{tikzcd}$$

We want a diagonal filler of the right square. Since $|x^\natural|$ is a one-generator cellular extension of $\partial|x^\natural|$ (definition 4.3.11), it is enough to check that $f\,\mathsf{s}\,x^\natural = \mathsf{s}\,x$, and $f\,\mathsf{t}\,x^\natural = \mathsf{t}\,x$. The latter is clear, as f extends $\bar{\mathsf{t}} : |\mathsf{t}\,x^\natural| \longrightarrow \mathcal{P}$, and $f\,\mathsf{t}\,x^\natural = \bar{\mathsf{t}}\,\mathsf{t}\,x^\natural = \mathsf{t}\,x$ by definition. We now proceed to prove the former. First, $\langle \mathrm{ct}\,\mathsf{s}\,x^\natural \rangle = \langle \mathrm{ct}\,\mathsf{s}\,x \rangle$ since both are mapped to the same element of $\mathcal{T}_n$. Then, for $[p]$ a node address of $\mathrm{ct}\,\mathsf{s}\,x^\natural$, we have $f\,\mathsf{s}_{[p]}\,x^\natural = \overline{\mathsf{s}_{[p]}}\,\mathsf{s}_{[p]}\,x^\natural = \mathsf{s}_{[p]}\,x$. Hence $f\,\mathsf{s}\,x^\natural = \mathsf{s}\,x$. □

Notation 4.3.18. For $\mathcal{P} \in \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ and $\omega \in \mathbb{O}_n$, let $\mathcal{P}_\omega := \left\{ x \in \mathcal{P}_n \mid x^\natural = \omega \right\}$. If $f : \mathcal{P} \longrightarrow \mathcal{Q}$ is a morphism of polygraphs, then it restricts and corestricts as a map $f : \mathcal{P}_\omega \longrightarrow \mathcal{Q}_\omega$.

THE ADJOINT EQUIVALENCE

Definition 4.3.19 (Polygraphic realization-nerve adjunction)**.** The polygraphic realization functor $|-| : \mathbb{O} \longrightarrow \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ extends to a left adjoint

$$|-| : \mathcal{P}\mathrm{sh}(\mathbb{O}) \underset{\perp}{\rightleftarrows} \mathcal{P}\mathrm{ol}^{\mathrm{mto}} : N,$$

by left Kan extension of $|-| : \mathbb{O} \longrightarrow \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ along the Yoneda embedding $\mathsf{y} : \mathbb{O} \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O})$. Explicitly, the polygraphic realization of an opetopic set $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$ can be computed with the coend on the left, while the *polygraphic nerve* $N\mathcal{P}$ of a polygraph $\mathcal{P} \in \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ is given on the right:

$$|X| = \int^{\omega \in \mathbb{O}} X_\omega \times |\omega|, \qquad N\mathcal{P} = \mathcal{P}\mathrm{ol}^{\mathrm{mto}}(|-|, \mathcal{P}) : \mathbb{O}^{\mathrm{op}} \longrightarrow \mathcal{S}\mathrm{et}.$$

We note $\eta : \mathrm{id}_{\mathcal{P}\mathrm{sh}(\mathbb{O})} \longrightarrow N|-|$ the unit of the adjunction, $\varepsilon : |N-| \longrightarrow \mathrm{id}_{\mathcal{P}\mathrm{ol}^{\mathrm{mto}}}$ the counit, and $\Phi : \mathcal{P}\mathrm{sh}(\mathbb{O})(-, N) \longrightarrow \mathcal{P}\mathrm{ol}^{\mathrm{mto}}(|-|, -)$ the natural hom-set isomorphism.

Notation 4.3.20. Using notation 0.4.2, the realization of an opetopic set $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$ is a polygraph whose generators are tensors of the form $x \otimes g$, for $x \in X_\omega$ and $g = (\psi \longrightarrow \omega)$ a generator of $|\omega|$. Further, the following equality holds:

$$y \otimes \left(\phi \xrightarrow{\mathsf{f}} \psi \xrightarrow{\mathsf{g}} \omega \right) = \mathsf{g}(y) \otimes \left(\phi \xrightarrow{\mathsf{f}} \psi \right),$$

where $y \in X_\omega$, and f and g are morphisms of $\mathbb{O}$. Note that all such tensors are equal to a unique one of the form $z \otimes \mathrm{id}$.

Remark 4.3.21. Note that with the nerve functor of definition 4.3.19, the bijection of equation (4.3.17) becomes

$$\widetilde{(-)} : \mathcal{P}_n \longrightarrow \sum_{\omega \in \mathbb{O}_n} N\mathcal{P}_\omega.$$

An n-generator $x \in \mathcal{P}_n$ then corresponds to a cell $\tilde{x} \in N\mathcal{P}_\omega$, where $\omega := x^\natural$, in other words, the shape function partitions the set of n-generators to form an opetopic set $N\mathcal{P}$. In the converse direction, for $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$,

$$\begin{aligned}
|X|_n &= \int^{\omega \in \mathbb{O}} X_\omega \times |\omega|_n && \text{see definition 4.3.19} \\
&\cong \int^{\omega \in \mathbb{O}} X_\omega \times \mathbb{O}_n/\omega && \text{see assumptions 4.3.3, } \textbf{(PR1)} \\
&\cong \int^{\omega \in \mathbb{O}} X_\omega \times \sum_{\psi \in \mathbb{O}_n} \mathbb{O}(\psi, \omega) && \text{by definition} \\
&\cong \sum_{\psi \in \mathbb{O}_n} \int^{\omega \in \mathbb{O}} X_\omega \times \mathbb{O}(\psi, \omega) \\
&\cong \sum_{\psi \in \mathbb{O}_n} X_\psi && \text{by theorem 0.4.1,}
\end{aligned}$$

so the set of n-generators of $|X|$ is the set of n-cells of X. In short, the polygraphic realization-nerve adjunction of definition 4.3.19 is “a duality between cells and generators”. Showing that it is an equivalence amounts to arguing that the cells of an opetopic set X can act as generators of a many-to-one polygraph in a unique way, and conversely, that a many-to-one polygraph is completely determined by its generators. Sadly this conceptual approach to theorem 4.3.23 turns out to be longer than the “brute-force” method we have adopted here.

Proposition 4.3.22. *Take $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$, $\mathcal{P} \in \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$, and $f : X \longrightarrow N\mathcal{P}$. The unit η at X, the transpose Φf of f, and the counit ε at $\mathcal{P}$ are respectively given by:*

$$\begin{aligned}
\eta : X_\omega &\longrightarrow N|X|_\omega & \Phi f : |X|_\omega &\longrightarrow \mathcal{P}_\omega & \varepsilon : |N\mathcal{P}|_\omega &\longrightarrow \mathcal{P}_\omega \\
x &\longmapsto \widetilde{x \otimes \mathrm{id}_\omega}, & x \otimes \mathrm{id}_\omega &\longmapsto f(x)(\omega), & \tilde{x} \otimes \mathrm{id}_\omega &\longmapsto x.
\end{aligned}$$

Proof. *Unit and transpose.* We have to check that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[d, "f"'] & N|X| \arrow[dl, "N\Phi f"] \\
N\mathcal{P},
\end{tikzcd}$$

and that f is unique for that property. For $x \in X_\omega$ we have

$$(N\Phi f)(\eta(x)) = (N\Phi f)(\widetilde{x \otimes \mathrm{id}_\omega}) = (\Phi f) \cdot (\widetilde{x \otimes \mathrm{id}_\omega}),$$

which maps ω to $f(x)(\omega)$. Since a map $|\omega| \longrightarrow \mathcal{P}$ is uniquely determined by the image of ω, we have $(N\Phi f)\eta = f$. Let $g : X \longrightarrow N\mathcal{P}$ be another morphism such that $(N\Phi g) \cdot \eta = f$. Then for $x \in X_\omega$ we have

$$f(x)(\omega) = ((N\Phi g) \cdot \eta)(x)(\omega) = (N\Phi g)((\widetilde{x \otimes \mathrm{id}_\omega})(\omega)) = g(x)(\omega),$$

whence $f = g$.

Counit. The counit is given by $\varepsilon = \Phi(\mathrm{id}_{N\mathcal{P}})$, so that

$$\varepsilon(\tilde{x} \otimes \mathrm{id}_\omega) = (\Phi(\mathrm{id}_{N\mathcal{P}}))(\tilde{x} \otimes \mathrm{id}_\omega) = \tilde{x}(\omega) = x.$$

□

Theorem 4.3.23. *The unit and counit are natural isomorphisms. Consequently, the polygraphic realization-nerve adjunction of definition 4.3.19 is an adjoint equivalence between* $\mathcal{P}\mathrm{sh}(\mathbb{O})$ *and* $\mathcal{P}\mathrm{ol}^{\mathrm{mto}}$.

Proof. *Unit.* Remark that for $x, y \in X_\omega$, if $x \otimes \mathrm{id}_\omega = y \otimes \mathrm{id}_\omega$, then $x = y$, which shows that η is injective. Take $f \in N|X|_\omega$. Then $f(\omega)$ is of the form $x \otimes \mathrm{id}_\omega$, hence $f = \eta(x)$, and η is surjective.

Counit. The following triangle identity shows that $N\varepsilon$ is a natural isomorphism:

$$\begin{tikzcd} N \arrow[rr, equal] \arrow[dr, "\eta N"'] & & N \\ & N|N-|. \arrow[ur, "N\varepsilon"'] & \end{tikzcd}$$

It is easy to check that the following square commutes, and since $\widetilde{(-)}$ is a bijection by theorem 4.3.16, ε is a natural isomorphism:

$$\begin{tikzcd} |N\mathcal{P}| \arrow[r, "\varepsilon"] \arrow[d, "\widetilde{(-)}"'] & \mathcal{P} \arrow[d, "\widetilde{(-)}"] \\ N|N\mathcal{P}| \arrow[r, "N\varepsilon"] & N\mathcal{P}. \end{tikzcd}$$

□

Many-to-one polygraphs have been the subject of other work [HMZ02] [HMZ08], and proved to be equivalent to the notion of *multitopic sets*. This, together with theorem 4.3.23, prove the following:

Corollary 4.3.24. *The category* $\mathcal{P}\mathrm{sh}(\mathbb{O})$ *of opetopic sets is equivalent to the category of multitopic sets.*

An *opetopic plex* is an opetopic polyplex of the form $\underline{u}$, where $u \in \mathfrak{T}_n$ (as opposed to $\mathfrak{T}_n^*$). In [Hen19, corollary 2.4.9 and remark 2.5.1], Henry shows that $\mathcal{P}\mathrm{ol}^{\mathrm{mto}}$ is a presheaf category over some category $\mathbb{O}\mathrm{plex}$ of opetopic plexes, and asks wether they are the same as opetopes. We now answer this question positively.

Definition 4.3.25 (Cauchy-complete category)**.** An idempotent morphism $e : a \longrightarrow a$ *splits* if it decomposes as $e = ir$ with $ri = \mathrm{id}_a$. A category is *Cauchy-complete* if all its idempotent morphisms split.

Example 4.3.26.
(1) Let $e : a \longrightarrow a$ be an idempotent map of sets, and $b := \mathrm{im}\, e \subseteq a$. Then the corestriction $\bar{e} : a \longrightarrow b$ of e is a retraction of the inclusion $i : b \hookrightarrow a$, and $e = i\bar{e}$. Therefore, the category $\mathcal{S}\mathrm{et}$ is Cauchy-complete.
(2) Let $\mathcal{F} \subseteq \mathcal{M}\mathrm{od}_{\mathbb{Z}/6\mathbb{Z}}$ be the full subcategory of free $\mathbb{Z}/6\mathbb{Z}$-modules. Then the map $- \times 3 : \mathbb{Z}/6\mathbb{Z} \longrightarrow \mathbb{Z}/6\mathbb{Z}$ is idempotent but not split, as its image $\mathbb{Z}/3\mathbb{Z}$ is not free. Therefore, $\mathcal{F}$ is not Cauchy-complete.

Theorem 4.3.27 ([BD86, theorem 1])**.** *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be Cauchy-complete categories. If we have an equivalence* $\mathcal{P}\mathrm{sh}(\mathcal{A}) \xrightarrow{\simeq} \mathcal{P}\mathrm{sh}(\mathcal{B})$*, then restricting to the representable presheaves gives an equivalence* $\mathcal{A} \xrightarrow{\simeq} \mathcal{B}$.

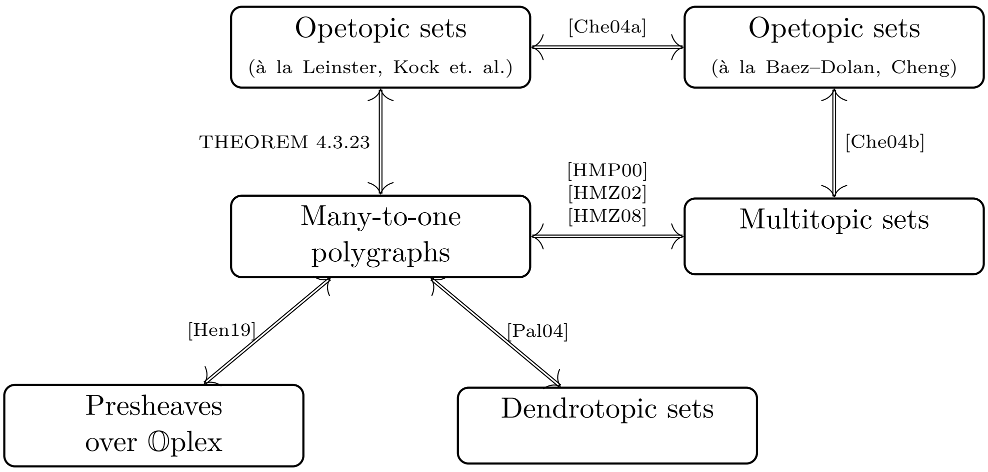
Corollary 4.3.28. *The category* $\mathbb{O}$plex *of opetopic plexes is equivalent to* $\mathbb{O}$.

Proof. By definition, $\mathbb{O}$ is a directed category, and by [Hen19, proposition 2.2.3 (4)], so is $\mathbb{O}$plex. In particular, they are both Cauchy-complete. On the other hand, $\mathcal{P}\mathrm{sh}(\mathbb{O}) \simeq \mathcal{P}\mathrm{ol}^{\mathrm{mto}} \simeq \mathcal{P}\mathrm{sh}(\mathbb{O}\mathrm{plex})$, and we conclude using theorem 4.3.27. □

In [Pal04], Palm studies another approach to weak higher-dimensional categories, based on *dendrotopic sets*. In particular, he show that dendrotopic sets are equivalent to many-to-one polygraphs. Therefore,

Corollary 4.3.29. *Opetopic sets are equivalent to dendrotopic sets.*

Figure 4.1: To summarize...



Part II

Syntax

Chapter Five

Introduction

We now explore two syntactical representations of opetopes and opetopic sets. Recall that opetopes are completely described by their source pasting diagram. In the *named approach* (chapters 6 and 7), those pasting diagrams are represented as terms describing the adjacencies of their cells. For example, the pasting diagram on the left shall be described by the term on the right:

$$\begin{tikzcd} & b \arrow[r, "g"] \arrow[d, phantom, "\Downarrow\alpha"] & c \arrow[r, "h"] \arrow[d, phantom, "\Downarrow\gamma"] & d \arrow[dr, "i"] \arrow[d, phantom, "\Downarrow\beta"] & \\ a \arrow[ur, "f"] \arrow[urr, "j"'] \arrow[rrrr, "l"'] & & & & e \arrow[from=ull, "k"'] \end{tikzcd} \qquad \rightsquigarrow \qquad \gamma(j \leftarrow \alpha, k \leftarrow \beta).$$

In the *unnamed approach* (chapters 8 and 9), those source pasting diagrams are treated as trees of opetopes. In general, a decorated tree $T \in \operatorname{tr} P$ for some polynomial endofunctor P, can be represented by a map from the set of node addresses of T to the set of operations of P (we omit the case where T is degenerate in this introduction). For example, if $T^\bullet = \{[x_1], \ldots, [x_k]\}$, then T can be described as

$$\begin{cases} [x_1] \leftarrow \text{decoration of the node at address } [x_1] \\ \vdots \\ [x_k] \leftarrow \text{decoration of the node at address } [x_k]. \end{cases}$$

Since an n-opetope is a tree over $\mathfrak{Z}^{n-2}$, the same idea applies. Further, the operations of $\mathfrak{Z}^{n-2}$ are $(n-1)$-opetopes, so this syntactical representation can be recursively applied

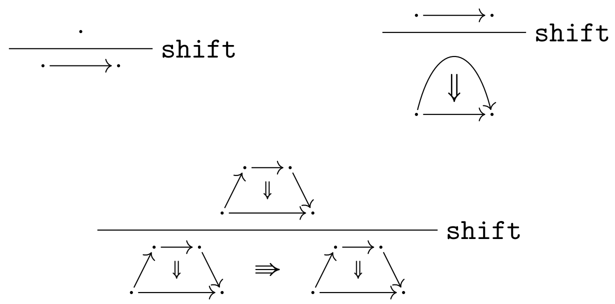
$$\begin{cases} [x_1] \leftarrow \begin{cases} [y_1] \leftarrow \cdots \\ \vdots \end{cases} \\ \vdots \\ [x_k] \leftarrow \{ \cdots \end{cases}$$

Our two derivation systems for opetopes, $\textsc{Opt}^!$ for the named approach and $\textsc{Opt}^?$ for the unnamed approach, leverage proposition 2.2.22, stating that if $n \geq 2$, then an n-opetope is either of the form I_ϕ for some $\phi \in \mathbb{O}_{n-2}$, Y_ψ for some $\psi \in \mathbb{O}_{n-1}$, or a grafting $\nu \circ_{[l]} \mathsf{Y}_\psi$ for some adequate $\nu \in \mathbb{O}_n$, $\psi \in \mathbb{O}_{n-1}$, and $[l] \in \nu^|$. Graphically, this means that opetopes are precisely all the shapes one can generate with the following operations:

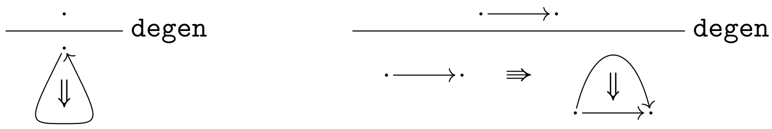
Introduction of a point. There is a unique 0-opetope (the *point*).

$$\frac{}{\cdot}\ \texttt{point}$$

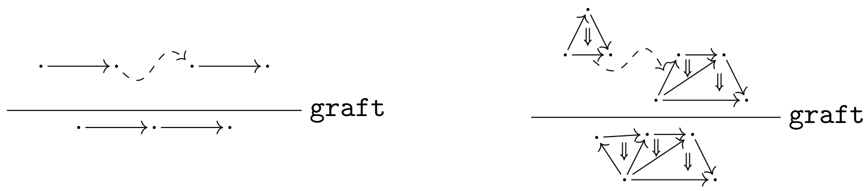
Shift to the next dimension. Given an n-opetope ω, we can form the $(n+1)$-dimensional endotope whose source and target are ω, as illustrated below. It can geometrically be thought of as the “extrusion” of ω.



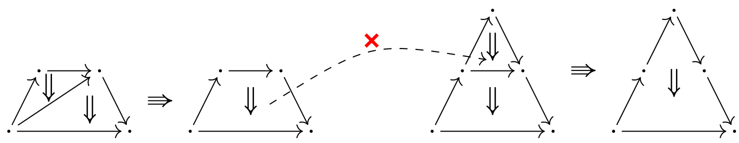
Introduction of degeneracies. Given an n-opetope ω, we can build a degenerate $(n+2)$-opetope whose target is the endotope at ω, as illustrated below for $n = 0$ and $n = 1$:



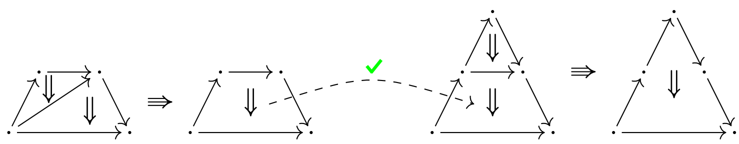
Grafting. Given an $(n+1)$-opetope ψ and an $(n+1)$-pasting diagram ω' such that the source of ω' contains an n-cell of the same shape as the target of ψ, we can obtain a new pasting diagram by *grafting* ψ onto ω':



Ill-formed graftings may occur with n-pasting diagrams, for $n \geq 3$, and a side condition is necessary to rule them out. Here is an example the `graft` rule will not allow: we deal with a 3-pasting diagram on the right of the dashed arrow, and the dashed arrow indicates that we attempt to graft the 3-opetope on the left (whose target shape is a trapezoid) onto the triangle shaped cell on the right

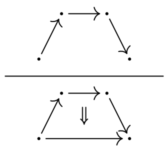


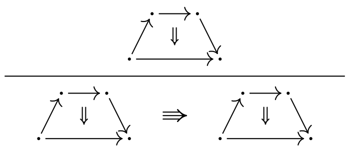
This is clearly not possible as the shapes on both ends of the dashed arrow do not match. However, grafting onto the lower trapezoid of the right opetope is acceptable:



As previously mentioned, an opetope is completely determined by its source pasting diagram, i.e. “arrangement” of source faces (the dichotomy between opetopes and pasting diagrams is more thoroughly discussed in section 3.2). We can observe the effect of rules `shift` and `graft` with this point of view to respectively obtain the following.

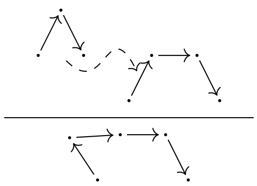
Filling of pasting diagrams. Given an n-pasting diagram, we may "fill" it by adding a target n-cell, and a top dimensional $(n+1)$-cell. We illustrate an instance of this rule on the left, and invite the reader to compare it with the instance of `shift` on the right:

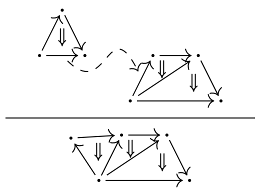

`fill`


`shift`

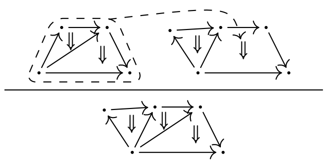
An endotope (as on the bottom right) is completely determined by its source (as any other opetope), but also completely determined by the source of its source. Therefore, the `fill` and `shift` rules are equivalent, in that the effect of one can be expressed in terms of the other.

Substitution. This consists in replacing a cell in a pasting diagram by another "parallel" pasting diagram. We illustrate an instance of this rule on the left below. If we consider the grafting of pasting diagrams on the right, then the substitution on the left operates on the *sources* of the pasting diagrams on the right, illustrating the motto "source of grafting is substitution of sources".


`subst`


`graft`

However, there is a loss of information, as filling the pasting diagram on the bottom left does not recover the pasting diagram on the bottom right. More generally, the source of a pasting diagram does not completely describe it. Therefore, the `subst` rule deals with strictly less data than `graft`. Here is another example of substitution


`subst`

Rules `point` and `degen` deal with opetopes whose source pasting diagram is empty, so observing the effect on these rules on sources is not very insightful.

So far we only discussed opetopes. As we will see, systems for describing finite opetopic sets can easily be derived from the respective systems for opetopes. In the named approach, systems $\textsc{OptSet}^!$ and $\textsc{OptSet}^!_{\mathrm{M}}$ (the latter being a more convenient variant) are obtained by adding rules to $\textsc{Opt}^!$, while in the unnamed approach, $\textsc{OptSet}^?$ is a sequent calculus completely *parametrized* by $\textsc{Opt}^?$. Finally, by the Gabriel–Ulmer duality corollary 0.5.7, all opetopic sets can be retrieved by passing to $\mathcal{S}et$-models, i.e. limit-preserving functors from the relevant category of contexts to $\mathcal{S}et$.

Syntactical methods in higher category theory have been explored in the literature. For example, a syntax for the closely related notion of multitope was proposed in [HMP02], but unfortunately, not all the desired computations (notably that of the *target* of an opetope) have been given algorithmic formulations there. The *Opetopic* proof assistant [Fin16] for weak higher categories relies on the notion of higher-dimensional tree. In that system, the notion of opetope is built-in, so that we have to trust the implementation. In contrast, the present approach allows us to reason about the construction of opetopes. We moreover believe that the ability to reason by induction on the proof trees, together with the very explicit nature of our syntaxes, will allow for optimizations in the automated manipulations of opetopes. Another proof assistant for weak higher categories, called *CaTT* [FM17], starts from the same idea of generating well-formed pasting schemes through inference rules. However, it is based on globular shapes instead of opetopic ones, making a comparison with the present work difficult. People have unsuccessfully tried to compare the resulting respective categorical formalisms; we hope that their formulation in a common logical language might be of help in this task. We should also mention here the *Globular* proof assistant [BKV16], also based on globular shapes, which is quite popular, notably thanks to its nice graphical interface.

The material of this part is taken from [CHM19b, CHM19a].

Chapter Six

The named approach for opetopes

In the *named approach*, we describe an opetope by naming its faces and describing their adjacencies using terms in a specifically crafted syntax. Not all such terms are geometrically meaningful however, and we introduce the $\mathrm{O{\small PT}}^!$ sequent calculus to select those that are.

6.1 THE OPT$^!$ SYSTEM

SYNTAX

In this section, we define the underlying syntax of $\mathrm{O{\small PT}}^!$, our named derivation system for opetopes. As explained in the introduction, a typical pasting diagram is pictured below:

$$\begin{tikzcd} & b \arrow[r, "g"] \arrow[d, phantom, "\Downarrow\alpha"] & c \arrow[r, "h"] \arrow[d, phantom, "\Downarrow\gamma"] \arrow[drr, "k"'] & d \arrow[d, phantom, "\Downarrow\beta"] \arrow[dr, "i"] & \\ a \arrow[ur, "f"] \arrow[urr, "j"'] \arrow[rrrr, "l"'] & & & & e \end{tikzcd}$$

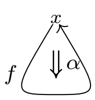
We shall use the names of the cells of this picture as variables, and encode the pasting diagram as the following expression:

$$\gamma(j \leftarrow \alpha, k \leftarrow \beta).$$

Here, j, k, α, β, and γ are now variables, equipped with a dimension (1 for j and k, and 2 for α , β, and γ), and the notation is meant to be read as "the variable γ in which α (resp. β) has been formally grafted on the input labeled by j (resp. k)". Such a term will be given a type:

$$i(t \leftarrow h(z \leftarrow g(y \leftarrow f))) \multimapdotbothA a \multimapdotbothA \varnothing,$$

which expresses the fact that the source is the "composite" $i \cdot h \cdot g \cdot f$, and that the source of the source is a. Since the pasting diagram is 2-dimensional, there is no further iterated source, and we conclude the sequence by a $\varnothing$ symbol. Similarly, the degenerate pasting diagram on the left below will be denoted by the typed term figured on the right:



$$\rightsquigarrow \qquad \alpha : \underline{x} \multimapdotbothA x \multimapdotbothA \varnothing$$

where the term $\underline{x}$ denotes a degenerate 1-dimensional pasting diagram with x as source.

Definition 6.1.1 (Term)**.** We assume that we have an $\mathbb{N}$-graded set $\mathbb{V}$ of variables. Elements of $\mathbb{V}_n$ represent n-dimensional cells. An *n-term* is constructed according to the following grammar[1]:

$$\begin{array}{rcllr}
\mathbb{T}_{-1} & ::= & \{\varnothing\} & & \text{by convention} \\
\mathbb{T}_0 & ::= & \mathbb{V}_0 & & \\
\mathbb{T}_{n+1} & ::= & \underline{\mathbb{V}_n} \quad | \quad \mathbb{T}'_{n+1} & & \\
\mathbb{T}'_{n+1} & ::= & \mathbb{V}_{n+1}(\mathbb{V}_n \leftarrow \mathbb{T}'_{n+1}, \ldots) & &
\end{array}$$

where the expression "$\mathbb{V}_n \leftarrow \mathbb{T}'_{n+1}, \ldots$" signifies that there is 0 or more instances of the "$\mathbb{V}_n \leftarrow \mathbb{T}'_{n+1}$" part between the parentheses. Terms of the form $\underline{u}$ are called *degenerate terms*, or *empty* syntactic pasting diagrams. Note that there are no degenerate 0-terms.

Example 6.1.2. If $f, g \in \mathbb{V}_1$, and $a \in \mathbb{V}_0$, then the following is an element of $\mathbb{T}_1$:

$$g(a \leftarrow f())$$

To make notations lighter, we omit empty parentheses "()", so the previous 1-term can be more concisely written as $g(a \leftarrow f)$. Since $f \in \mathbb{V}_1$, the expression $\underline{f}$ is a degenerate term in $\mathbb{T}_2$.

Notation 6.1.3. A term of the form $g(a_1 \leftarrow f_1, \ldots, a_k \leftarrow f_k)$ will oftentimes be abbreviated as $g(\overrightarrow{a_i \leftarrow f_i})$, leaving k implicit. By convention, the sequence $a_1 \leftarrow f_1, \ldots, a_k \leftarrow f_k$ above is always considered up to permutation, i.e. for σ a bijection of the set $\{1, \ldots, k\}$, the terms $g(a_1 \leftarrow f_1, \ldots, a_k \leftarrow f_k)$ and $g(a_{\sigma(1)} \leftarrow f_{\sigma(1)}, \ldots, a_{\sigma(k)} \leftarrow f_{\sigma(k)})$ are considered equal.

Notation 6.1.4. For $t \in \mathbb{T}_n$, write $t^\bullet$ for the set of n-variables occurring in t. In the previous example, $(g(a \leftarrow f))^\bullet = \{f, g\}$. Note that $x \in x^\bullet$ for all $x \in \mathbb{V}_n$.

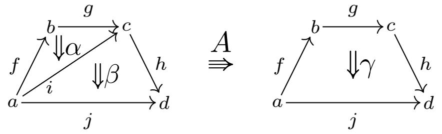
Definition 6.1.5 (Type)**.** An *n-type* T is a sequence of terms of the form

$$s_1 \multimap s_2 \multimap \cdots \multimap s_n \multimap \varnothing, \tag{6.1.6}$$

where $s_i \in \mathbb{T}_{n-i}$. As we will see (rule `shift` in definition 6.1.16, and theorem 6.1.26), this sequence of terms essentially describes a zoom complex in the sense of [KJBM10, section 1.6], which justifies the use of the $\multimap$ symbol.

Definition 6.1.7 (Typing)**.** A *typing* of a term $t \in \mathbb{T}_n$ is an expression of the form $t : T$, for T an n-type. If T is as in equation (6.1.6), then s_i is thought of as the i-th (iterated) source of t. We then write $\mathsf{s}\, t := s_1$, and more generally $\mathsf{s}^i\, t := s_i$. By convention, $\mathsf{s}^0\, t := t$.

Example 6.1.8. The pasting diagram on top will be described by the typing below:



[1] To be concise, we use a slightly unusual set-based notation. For example, an element of $\mathbb{T}'_{n+1}$ is of the form $v(a_1 \leftarrow t_1, a_2 \leftarrow t_2, \ldots)$, where $v \in \mathbb{V}_{n+1}$, $a_1, a_2, \ldots \in \mathbb{V}_n$, and $t_1, t_2, \ldots \in \mathbb{T}'_{n+1}$.

$$\rightsquigarrow \qquad A : \underbrace{\beta(i \leftarrow \alpha)}_{=\mathsf{s}\,A} \multimap \underbrace{h(c \leftarrow g(b \leftarrow f))}_{=\mathsf{s}^2\,A} \multimap \underbrace{a}_{=\mathsf{s}^3\,A} \multimap \underbrace{\varnothing}_{=\mathsf{s}^4\,A} .$$

Remark that the term $\mathsf{s}^2\,A = h(c \leftarrow g(b \leftarrow f))$ is the source pasting diagram of the *target* of A, as in $\mathsf{s}\,\mathsf{s}\,A = \mathsf{s}\,\mathsf{t}\,A$. Targets are not represented in this syntax, but already, we see that the information they carry is not lost. Refer to proposition 6.2.16 for a formal account of this observation.

Definition 6.1.9 (Context)**.** A *context* Γ is a set of typings, more commonly written as a list.

Notation 6.1.10. Write $\mathbb{V}_{\Gamma,k}$ for the set of k-variables typed in Γ, let $\mathbb{V}_\Gamma := \sum_{k\in\mathbb{N}} \mathbb{V}_{\Gamma,k}$, write $\mathbb{T}_{\Gamma,k}$ for the set of k-terms whose variables (in any dimension) are in $\mathbb{V}_\Gamma$, and $\mathbb{T}_\Gamma := \sum_{k\in\mathbb{N}} \mathbb{T}_{\Gamma,k}$.

Remark 6.1.11. As we will see (inference rules in definition 6.1.16), for a derivable context Γ, if x occurs in the typing of a variable of Γ, then $x \in \mathbb{V}_\Gamma$. Note that in any context Γ, if a variable $x \in \mathbb{V}_{\Gamma,k}$ occurs in the type of $y \in \mathbb{V}_{\Gamma,l}$, then $k < l$. In particular, there is no cyclic dependency among variables.

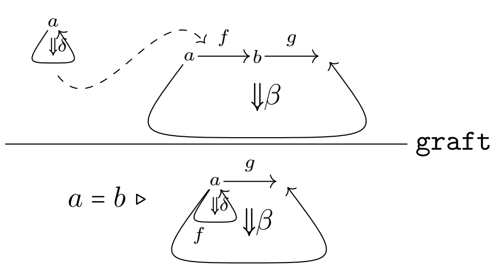
Definition 6.1.12 (Equational theory)**.** Let Γ be a context. An *equational theory* E on $\mathbb{V}_\Gamma$ is a set of formal equalities between variables of Γ. We write $=_E$ for the equivalence relation on $\mathbb{V}_\Gamma$ generated by E.

Definition 6.1.13 (Sequent)**.** A *sequent* is an expression of the form

$$E \triangleright \Gamma \vdash t : T$$

where Γ is a context, E is an equational theory on $\mathbb{V}_\Gamma$, and the right hand side is a typing. We may write $\vdash_n$ to signify that $t \in \mathbb{T}_n$. The equivalence relation $=_E$ on $\mathbb{V}_\Gamma$ extends to $\mathbb{T}_\Gamma$ in an obvious way. If $x =_E y$ and $y \in t^\bullet$, then by convention $x \in t^\bullet$, so that x and y really are interchangeable.

Remark 6.1.14. As illustrated below, grafting degenerate terms produces identifications of lower dimensional variables, which must be accounted for. This will be the role of the equational theories.



This example will be treated in details in example 6.3.5.

Definition 6.1.15 (Equivalence of sequents)**.** If $(E \triangleright \Gamma \vdash t : T)$ and $(F \triangleright \Upsilon \vdash u : U)$ are sequents, and if there exists a bijection $\sigma : \mathbb{V}_\Upsilon \longrightarrow \mathbb{V}_\Gamma$ such that

$$(E \triangleright \Gamma \vdash t : T) \;=\; (F^\sigma \triangleright \Upsilon^\sigma \vdash u^\sigma : U^\sigma) ,$$

where $(-)^\sigma$ is the substitution according to σ, then we say that both sequents are α-*equivalent* (or just *equivalent*), denoted by

$$(E \triangleright \Gamma \vdash t : T) \simeq (F \triangleright \Upsilon \vdash u : U)\,.$$

In the following, sequents are implicitly considered up to α-equivalence.

INFERENCE RULES

We present the inference rules of the $\textsc{Opt}^!$ system in definition 6.1.16. Rule `graft` requires the so-called graft notation and substitution operation, respectively introduced in definitions 6.1.17 and 6.1.19.

Definition 6.1.16 (The $\textsc{Opt}^!$ system)**.** *Introduction of points.* This rule introduces 0-cells, also called points. If $x \in \mathbb{V}_0$, then

$$\frac{}{\triangleright x : \varnothing \vdash_0 x : \varnothing}\ \texttt{point}$$

Introduction of degeneracies. This rule derives empty pasting diagrams. If $x \in \mathbb{V}_n$, then

$$\frac{E \triangleright \Gamma \vdash_n x : T}{E \triangleright \Gamma \vdash_{n+1} \underline{x} : x \multimap T}\ \texttt{degen}$$

Shift to the next dimension. This rule takes a term t and introduces a new cell x having t as source. If $x \in \mathbb{V}_{n+1} - \mathbb{V}_\Gamma$, then

$$\frac{E \triangleright \Gamma \vdash_n t : T}{E \triangleright \Gamma, x : t \multimap T \vdash_{n+1} x : t \multimap T}\ \texttt{shift}$$

Grafting. This rule glues an n-cell x onto an n-term t along a variable $a \in s_1^\bullet := (\mathsf{s}\,t)^\bullet$. We assume that Γ and Υ are compatible, in that for all $y \in \mathbb{V}$, if $y \in \mathbb{V}_\Gamma \cap \mathbb{V}_\Upsilon$, then the typing of y in both contexts match modulo the equational theory $E \cup F$. Further, the only variables typed in both Γ and Υ are a and the variables occurring in the sources of a (i.e. $\mathsf{s}^i\, a$, for $1 \leq i \leq n-1$).

If $x \in \mathbb{V}_n$, $t \in \mathbb{T}_n$ is not degenerate, $a \in (\mathsf{s}\,t)^\bullet$ (which ensures that a has not been used for grafting beforehand) is such that $\mathsf{s}\,a = \mathsf{s}\mathsf{s}\,x$ (recall that by example 6.1.8, we can understand this condition as $\mathsf{s}\,a = \mathsf{s}\mathsf{t}\,x$, so that x may indeed be glued onto a), then

$$\frac{E \triangleright \Gamma \vdash_n t : s_1 \multimap s_2 \multimap \cdots \qquad F \triangleright \Upsilon \vdash_n x : U}{G \triangleright \Gamma \cup \Upsilon \vdash_n t(a \leftarrow x) : s_1[\mathsf{s}\,x/a] \multimap s_2 \multimap \cdots}\ \texttt{graft}$$

where the notations $t(a \leftarrow x)$ and $s_1[\mathsf{s}\,x/a]$ are presented below, and where G is the union of E and F, and potentially a set of additional equalities incurred by the evaluation of $s_1[\mathsf{s}\,x/a]$. We also write `graft-`a to make explicit that we grafted onto a.

Definition 6.1.17 (Grafting of terms)**.** For a sequent $(E \triangleright \Gamma \vdash_n t : T)$, $a \in \mathbb{V}_{n-1}$, and $x \in \mathbb{V}_{\Upsilon,n}$, the graft notation $t(a \leftarrow x)$ of the `graft` rule can be simplified depending on the structure of t, according to the following rewriting rule. For $y \in \mathbb{V}_{\Gamma,n}$,

$$y(\overrightarrow{z_i \leftarrow v_i})(a \leftarrow x) \qquad \rightsquigarrow \qquad \begin{cases} y(\overrightarrow{z_i \leftarrow v_i(a \leftarrow x)}) & \text{if } a \notin (\mathsf{s}\,y)^\bullet, \\ y(\overrightarrow{z_i \leftarrow v_i}, a \leftarrow x) & \text{if } a \in (\mathsf{s}\,y)^\bullet. \end{cases} \tag{6.1.18}$$

In particular, note that if $a \notin (\mathsf{s}\,y)^\bullet$, then $y()(a \leftarrow x)$ rewrites to $y()$, and with the "empty parentheses convention", this gives $y(a \leftarrow x) \quad \rightsquigarrow \quad y$.

Definition 6.1.19 (Substitution in terms)**.** We now explain how to evaluate $u[w/a]$ for terms $u, v \in \mathbb{T}_n$, where u is of the form $u = y(\overrightarrow{z_i \leftarrow v_i})$.

(Subst1) If w is not degenerate, then

$$u[w/a] := \begin{cases} y(\overrightarrow{z_i \leftarrow v_i[w/a]}) & \text{if } a \neq_{E \cup F} y, \\ w(\overrightarrow{z_i \leftarrow v_i}) & \text{if } a =_{E \cup F} y. \end{cases} \tag{6.1.20}$$

(Subst2) If w is degenerate, say $w = \underline{b}$ for $b \in \mathbb{V}_{n-1}$, Then, by the hypothesis of the `graft` rule, we have $b =_E \mathsf{s}\,a$. Then, $u[\underline{b}/a]$ is defined by cases on the form of u:

- **(Subst2a)** if $u =_{E \cup F} a$, i.e. u is a variable, then $u[\underline{b}/a] := \underline{b}$;
- **(Subst2b)** if u is of the form $a(b \leftarrow r)$, then $u[\underline{b}/a] := r$;
- **(Subst2c)** if u is of the form $y(\ldots, z \leftarrow a, \ldots)$, then we remove the grafting $z \leftarrow a$, and we add the equality $b = z$ to the ambient equational theory;
- **(Subst2d)** if u is of the form[2] $y(\ldots, z \leftarrow a(b \leftarrow r), \ldots)$, then

  $$u[\underline{b}/a] := y(\ldots, z \leftarrow r, \ldots),$$

  as in equation (6.1.20), and per equation (6.1.18), but we also add the equality $b = z$ to the ambient equational theory;
- **(Subst2e)** otherwise, if u is of the form $y(\overrightarrow{z_i \leftarrow v_i})$, and if the previous cases do not apply (i.e. a is not the front variable of u or v_i for all i), then we recursively evaluate the substitution:

  $$u[\underline{b}/a] := y(\overrightarrow{z_i \leftarrow v_i[\underline{b}/a]}).$$

Remark 6.1.21. From the formulation of system $\text{Opt}^!$, it is clear that a sequent that is equivalent to a derivable one is itself derivable. Let us now turn our attention to rule `shift` above. It takes a term t, thought of as a pasting diagram, and creates a new variable having t as source. One may thus think of it as a rule creating "fillers", akin to Kan filler condition on simplicial sets.

Example 6.1.22. Consider the term $t = \alpha(g \leftarrow \beta)$ in a suitable context Γ:

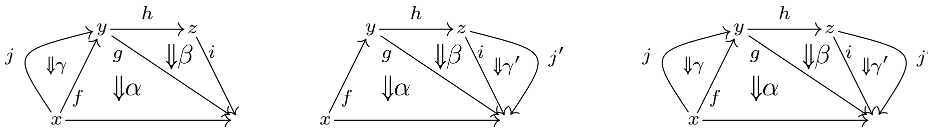
$$\begin{tikzcd} & y \arrow[r, "h"] \arrow[dr, "g"'] \arrow[d, phantom, "\Downarrow\alpha"] & z \arrow[d, "i"] \arrow[dl, phantom, "\Downarrow\beta"] \\ x \arrow[ur, "f"] \arrow[rr] & {} & {} \end{tikzcd}$$

[2] The case in which u is of the form $y(\ldots, z \leftarrow a(b_1 \leftarrow r_1, \ldots, b_k \leftarrow r_k), \ldots)$ with $k > 1$ does not happen in valid derivations.

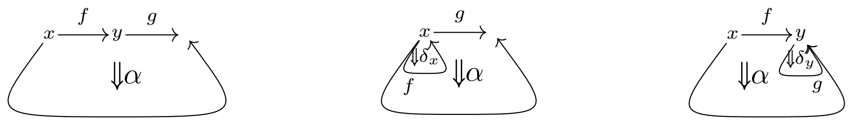
and two variables $\gamma : j \multimap x \multimap \varnothing$ and $\gamma' : j' \multimap z \multimap \varnothing$. We evaluate the following simple graftings:

$$\begin{aligned}
t(f \leftarrow \gamma) &= \alpha(g \leftarrow \beta)(f \leftarrow \gamma) && \text{well-def. since } \mathsf{s}\mathsf{s}\gamma = x = \mathsf{s} f \\
&= \alpha(f \leftarrow \gamma, g \leftarrow \beta) && \text{since } f \in (\mathsf{s}\alpha)^\bullet, \\
t(i \leftarrow \gamma') &= \alpha(g \leftarrow \beta)(i \leftarrow \gamma') && \text{well-def. since } \mathsf{s}\mathsf{s}\gamma' = z = \mathsf{s} i \\
&= \alpha(g \leftarrow \beta(i \leftarrow \gamma')) && \text{since } i \notin (\mathsf{s}\alpha)^\bullet, \\
t(f \leftarrow \gamma)(i \leftarrow \gamma') &= \alpha(f \leftarrow \gamma, g \leftarrow \beta)(i \leftarrow \gamma') && \text{as seen previously} \\
&= \alpha(f \leftarrow \gamma(i \leftarrow \gamma'), g \leftarrow \beta(i \leftarrow \gamma')) && \text{since } i \notin (\mathsf{s}\alpha)^\bullet \\
&= \alpha(f \leftarrow \gamma, g \leftarrow \beta(i \leftarrow \gamma')) && \text{since } i \notin (\mathsf{s}\gamma)^\bullet.
\end{aligned}$$

They can respectively be represented as:



Example 6.1.23. Consider variables $\alpha : g(y \leftarrow f) \multimap x \multimap \varnothing$, $\delta_x : \underline{x} \multimap x \multimap \varnothing$, and $\delta_y : \underline{y} \multimap y \multimap \varnothing$. Then α, $\alpha(f \leftarrow \delta_x)$, and $\alpha(g \leftarrow \delta_y)$ can respectively be represented as:



Then the sources $\alpha(f \leftarrow \delta_x)$ and $\alpha(g \leftarrow \delta_y)$ are respectively

$$\begin{aligned}
g(y \leftarrow f)[\underline{x}/f] &= g && \text{by } \textbf{(Subst2c)}, \\
g(y \leftarrow f)[\underline{y}/g] &= f && \text{by } \textbf{(Subst2b)},
\end{aligned}$$

and in the first case, the equation $x = y$ is added to the ambient equational theory.

Remark 6.1.24. The `degen` rule may be replaced by the following `degen-shift` rule without changing the set of derivable sequents of the form $(E \triangleright \Gamma \vdash y : T)$ with $y \in \mathbb{V}$: if $x \in \mathbb{V}_n$ and $d \in \mathbb{V}_{n+2}$ such that $d \notin \mathbb{V}_{\Gamma,n+2}$, then

$$\frac{E \triangleright \Gamma \vdash_n x : T}{E \triangleright \Gamma, d : \underline{x} \multimap x \multimap T \vdash_{n+2} d : \underline{x} \multimap x \multimap T}\ \texttt{degen-shift}$$

However, note that sequents of the form $\left(E \triangleright \Gamma \vdash \underline{y} : T\right)$ are then no longer derivable.

PROPERTIES OF DERIVABLE SEQUENTS

Let $(E \triangleright \Gamma \vdash x : X)$ be a derivable sequent. We prove in theorem 6.1.26 that the type $X = (\mathsf{s}x \multimap \mathsf{s}\mathsf{s}x \multimap \cdots)$ is completely determined by $\mathsf{s}x$ and Γ. We proceed by extending the source map $\mathsf{s} : \mathbb{V}_{\Gamma,n} \longrightarrow \mathbb{T}_{\Gamma,n-1}$ to a map $\bar{\mathsf{s}} : \mathbb{T}_{\Gamma,n-1} \longrightarrow \mathbb{T}_{\Gamma,n-1}$.

Definition 6.1.25. Define the function $\bar{\mathsf{s}}$ as follows:

$$\begin{aligned} \bar{\mathsf{s}} : \mathbb{T}_\Gamma &\longrightarrow \mathbb{T}_\Gamma \\ x &\longmapsto \mathsf{s}\,x && x \in \mathbb{V}_\Gamma, \\ \underline{x} &\longmapsto x && x \in \mathbb{V}_\Gamma, \\ x(\overrightarrow{y_i \leftarrow u_i}) &\longmapsto (\mathsf{s}\,x)[\overrightarrow{\bar{\mathsf{s}}\,u_i/y_i}] && x, \vec{y_i} \in \mathbb{V}_\Gamma, \vec{u_i} \in \mathbb{T}_\Gamma. \end{aligned}$$

Note that by definition, it agrees with s on variables.

Theorem 6.1.26. *Let $(E \triangleright \Gamma \vdash t : s_1 \multimapdotboth s_2 \multimapdotboth \cdots \multimapdotboth s_n \multimapdotboth \varnothing)$ be a derivable sequent. Then for $1 \leq k \leq n$ we have $s_k = \bar{\mathsf{s}}^k\, t$. Equivalently, for $0 \leq i \leq n$, we have $\bar{\mathsf{s}}\, s_i = s_{i+1}$, where by convention, $s_0 := t$.*

Proof. We proceed by induction on the proof tree of the sequent. For readability, we omit equational theories and contexts.

(1) If the sequent is obtained by the following proof tree:

$$\dfrac{}{x : \varnothing \vdash x : \varnothing}\ \texttt{point}$$

then $\bar{\mathsf{s}}\,x = \mathsf{s}\,x = \varnothing$, since $x \in \mathbb{V}$, and the result trivially holds.

(2) If the last inference of the proof tree is the following instance of `degen`

$$\dfrac{\cdots \vdash s_1 : s_2 \multimapdotboth \cdots \multimapdotboth s_n \multimapdotboth \varnothing}{\cdots \vdash t : s_1 \multimapdotboth s_2 \multimapdotboth \cdots \multimapdotboth s_n \multimapdotboth \varnothing}\ \texttt{degen}$$

then $s_1 \in \mathbb{V}$ and $t = \underline{s_1}$. Thus, $\bar{\mathsf{s}}\,t = s_1$, while for $1 \leq i \leq n$, the equality $\bar{\mathsf{s}}\,s_i = s_{i+1}$ holds by induction.

(3) If the last inference of the proof tree is the following instance of `shift`

$$\dfrac{\cdots \vdash s_1 : s_2 \multimapdotboth \cdots \multimapdotboth s_n \multimapdotboth \varnothing}{\cdots \vdash t : s_1 \multimapdotboth s_2 \multimapdotboth \cdots \multimapdotboth s_n \multimapdotboth \varnothing}\ \texttt{shift}$$

then $t \in \mathbb{V}$, so $\bar{\mathsf{s}}\,t = \mathsf{s}\,t = s_1$, while for $1 \leq i \leq n$, the equality $\bar{\mathsf{s}}\,s_i = s_{i+1}$ holds by induction.

(4) Assume now that the last inference of the proof tree is the following instance of `graft`:

$$\dfrac{\cdots \vdash u : r_1 \multimapdotboth r_2 \multimapdotboth s_3 \multimapdotboth \cdots \multimapdotboth s_n \multimapdotboth \varnothing \qquad \cdots \vdash x : X}{\cdots \vdash t : s_1 \multimapdotboth s_2 \multimapdotboth \cdots \multimapdotboth s_n \multimapdotboth \varnothing}\ \texttt{graft-}a$$

with $a \in r_1^\bullet$ and $x \in \mathbb{V}$ such that $\mathsf{s}\mathsf{s}\,x = \mathsf{s}\,a$. Then $t = u(a \leftarrow x)$, $s_1 = r_1[\mathsf{s}\,x/a]$, and $s_i = r_i$ for $2 \leq i \leq n$. On the one hand, we have

$$\begin{aligned} \bar{\mathsf{s}}\,t &= \bar{\mathsf{s}}\,(u(a \leftarrow x)) \\ &= (\bar{\mathsf{s}}\,u)[\bar{\mathsf{s}}\,x/a] \\ &= (\bar{\mathsf{s}}\,u)[\mathsf{s}\,x/a] && \text{since } x \in \mathbb{V} \\ &= r_1[\mathsf{s}\,x/a] && \text{by induction} \\ &= s_1 && \text{by definition.} \end{aligned}$$

On the other hand, write $r_1 = v(y \leftarrow a(\overrightarrow{z_i \leftarrow w_i}))$, for some $v, \overrightarrow{w_i} \in \mathbb{T}_{n-1}$ and $y \in \mathbb{V}_{n-2}$. This decomposition exhibits r_1 as a grafting (in the sense of definition 6.1.17) of a term $a(\cdots)$ whose head variable is a onto some term v. We then compute:

$$
\begin{aligned}
\bar{\mathsf{s}}\, s_1 &= \bar{\mathsf{s}}\,(r_1[\mathsf{s}\,x/a]) \\
&= \bar{\mathsf{s}}\Big(\quad v(y \leftarrow a(\overrightarrow{z_i \leftarrow w_i}))\quad [\mathsf{s}\,x/a]\quad \Big) \\
&= \bar{\mathsf{s}}\Big(\quad v\left(y \leftarrow (\mathsf{s}\,x)(\overrightarrow{z_i \leftarrow w_i})\right)\quad \Big) \\
&= (\bar{\mathsf{s}}\,v)\Big[\quad (\bar{\mathsf{s}}\,\mathsf{s}\,x)[\overrightarrow{\bar{\mathsf{s}}\,w_i/z_i}]\quad /y\Big] && \text{by definition of } \bar{\mathsf{s}} \\
&= (\bar{\mathsf{s}}\,v)\Big[\quad (\mathsf{s}\,\mathsf{s}\,x)[\overrightarrow{\bar{\mathsf{s}}\,w_i/z_i}]\quad /y\Big] && \text{by induction} \\
&= (\bar{\mathsf{s}}\,v)\Big[\quad (\mathsf{s}\,a)[\overrightarrow{\bar{\mathsf{s}}\,w_i/z_i}]\quad /y\Big] && \text{hypothesis of } \texttt{graft-}a \\
&= \bar{\mathsf{s}}\,r_1 && \text{recall } r_1 = v(y \leftarrow a(\overrightarrow{z_i \leftarrow w_i})) \\
&= r_2 = s_2.
\end{aligned}
$$

Finally, for $1 \le i \le n$, the equality $\bar{\mathsf{s}}\, s_i = s_{i+1}$ holds by induction. ☐

Corollary 6.1.27. *Let $(E \triangleright \Gamma \vdash t : T)$ be a derivable sequent, and $x : s_1 \multimapdotbothA s_2 \multimapdotbothA \cdots \multimapdotbothA s_n \multimapdotbothA \varnothing$ be a typing in Γ. Then for $1 \le k \le n$ we have $s_k = \bar{\mathsf{s}}^k\, t$, or equivalently, for $0 \le i \le n$, we have $\bar{\mathsf{s}}\, s_i = s_{i+1}$, with $s_0 := x$.*

Proof. If $x : s_1 \multimapdotbothA s_2 \multimapdotbothA \cdots \multimapdotbothA s_n \multimapdotbothA \varnothing$ is a typing in Γ, then somewhere in the proof tree of $(E \triangleright \Gamma \vdash t : T)$ appears a sequent of the form $(F \triangleright \Upsilon \vdash x : s_1 \multimapdotbothA s_2 \multimapdotbothA \cdots \multimapdotbothA s_n \multimapdotbothA \varnothing)$, which is necessarily derivable. We conclude by applying theorem 6.1.26 ☐

Remark 6.1.28. A consequence of theorem 6.1.26 and corollary 6.1.27 is that at any stage, a context Γ may be replaced by its "meager form" $\bar{\Gamma}$, obtained by replacing "full typings" $y : Y$ by $y : \mathsf{s}\,y$, i.e. by removing all but the top term of the type Y. Using meager context comes with a cost however: checking the hypothesis of rule `graft` requires to compute the second source $\mathsf{s}\,\mathsf{s}\,x$ of x, which is not contained in $\bar{\Gamma}$. For clarity, we shall not make use of meager forms throughout the rest of this work.

Convention 6.1.29. By definition, $\bar{\mathsf{s}}$ extends s to a function $\mathbb{T}_\Gamma \longrightarrow \mathbb{T}_\Gamma$, and for convenience, we just write it as s in the sequel, and call it the *source* of a term.

Example 6.1.30. Consider the term on the right, representing the pasting diagram on the left:

$$
\begin{tikzcd}
& y \arrow[r, "h"] \arrow[dr, "g"'] \arrow[dr, phantom, "\Downarrow\alpha"', near end] & z \arrow[d, "i"] \arrow[dl, phantom, "\Downarrow\beta"] \\
x \arrow[ur, "f"'] \arrow[ur, bend left=60, "j"] \arrow[ur, phantom, "\Downarrow\gamma", bend left=30] \arrow[rr] & & {}
\end{tikzcd}
\qquad \rightsquigarrow \qquad \alpha(f \leftarrow \gamma, g \leftarrow \beta)
$$

Then its source is computed as follows:

$$
\begin{aligned}
\mathsf{s}\,(\alpha(f \leftarrow \gamma, g \leftarrow \beta)) &= (\mathsf{s}\,\alpha)\,[(\mathsf{s}\,\gamma)/f, (\mathsf{s}\,\beta)/g] && \text{by definition} \\
&= (g(y \leftarrow f))\,[(\mathsf{s}\,\gamma)/f, (\mathsf{s}\,\beta)/g] && \text{since } \mathsf{s}\,\alpha = g(y \leftarrow f) \\
&= (g(y \leftarrow f))\,[j/f, (\mathsf{s}\,\beta)/g] && \text{since } \mathsf{s}\,\gamma = j
\end{aligned}
$$

$$
\begin{aligned}
&= (g(y \leftarrow f))\,[j/f, i(z \leftarrow h)/g] && \text{since } \mathsf{s}\beta = i(z \leftarrow h)\\
&= (g(y \leftarrow j))\,[i(z \leftarrow h)/g] && \text{see equation (6.1.20)}\\
&= (i(z \leftarrow h))\,(y \leftarrow j) && \text{see equation (6.1.20)}\\
&= i(z \leftarrow h(y \leftarrow j)) && \text{since } y \in (\mathsf{s}h)^{\bullet}.
\end{aligned}
$$

The latter term indeed corresponds to the source of the pasting diagram, which is the arrow composition on the top.

Lemma 6.1.31 (Unique occurrence lemma)**.** *Let* $(E \triangleright \Gamma \vdash_n t : s_1 \multimap \cdots)$ *be a derivable sequent, where* $t \in \mathbb{T}_n$ *is not degenerate, say* $t = x(\overrightarrow{a_i \leftarrow u_i})$*,*

(1) Let $y \in t^{\bullet}$*. Then either* $y =_E x$*, or* $y \in u_i^{\bullet}$ *for a unique* i*.*

(2) Let $b \in s_1^{\bullet}$*. Then either* $b \in (\mathsf{s}x)^{\bullet}$*, or* $b \in (\mathsf{s}u_i)^{\bullet}$ *for a unique* i*.*

Proof. (1) By assumption of the `graft` rule, each n-variable of t occurs only once in t.

(2) By the first point, each $(n-1)$-variable of s_1 occurs exactly once. By theorem 6.1.26, $s_1 = (\mathsf{s}x)[\overrightarrow{\bar{\mathsf{s}}u_i/a_i}]$. Thus b either occurs in $\mathsf{s}x$ or on $\mathsf{s}u_i$ for a unique i. □

Proposition 6.1.32. *Let* $(E \triangleright \Gamma \vdash x : X)$ *be a derivable sequent, and* $a \in \mathbb{V}_\Gamma$ *be a variable of type* A*. Then the sequent* $(E|_a \triangleright \Gamma|_a \vdash a : A)$ *is derivable, where* $E|_a$ *(resp.* $\Gamma|_a$*) is the restriction of* E *(resp.* Γ*) to* a *and variables occuring in* A*.*

Proof. (1) If a is 0-dimensional, then $(E|_a \triangleright \Gamma|_a \vdash a : A) = (\triangleright a : \varnothing \vdash a : \varnothing)$ can be obtained by an instance of rule `point`.

(2) If $a = x$, then $(E|_a \triangleright \Gamma|_a \vdash a : A) = (E \triangleright \Gamma \vdash x : X)$ is derivable by assumption.

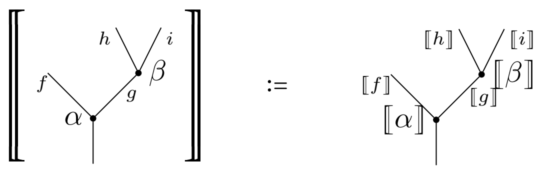
(3) Otherwise, a first appears in the conclusion of an instance of `shift` in the proof tree of $(E \triangleright \Gamma \vdash x : X)$. Then $(E|_a \triangleright \Gamma|_a \vdash a : A)$ is the conclusion of that instance, and is derivable. □

6.2 EQUIVALENCE WITH POLYNOMIAL OPETOPES

In this section, all sequents are assumed derivable in $\textsc{Opt}^!$. We show that sequents typing a variable (up to α-equivalence) are in bijective correspondence with the "polynomial" opetopes of definition 3.1.3. To this end, we define the *polynomial coding* operation $[\![-]\!]_{n+1}$ that maps a sequent $(E \triangleright \Gamma \vdash_n t : T)$ typing an n-term $t \in \mathbb{T}_n$, to an $(n+1)$-opetope $[\![E \triangleright \Gamma \vdash_n t : T]\!]_{n+1} \in \mathbb{O}_{n+1}$, written $[\![t : T]\!]_{n+1}$ or even $[\![t]\!]_{n+1}$ for short, if no ambiguity arises.

The idea of the polynomial coding is to map a pasting diagram described by a term (on the left) to its underlying composition tree, and reapply the coding recursively (on the right):

$[\![\alpha(g \leftarrow \beta)]\!] \quad =$



Convention 6.2.1. A variable $x \in \mathbb{V}_n$ is in particular an n-term, thus the polynomial coding operation gives an $(n+1)$-opetope $[\![x]\!]_{n+1} \in \mathbb{O}_{n+1}$. However, it will be convenient to consider

x as representing an n-opetope, whose tree is described by the $(n-1)$-term $\mathsf{s}\,x$. We thus convene on the notation $[\![x]\!]_n := [\![\mathsf{s}\,x]\!]_n \in \mathbb{O}_n$. Keep in mind that for all $n \in \mathbb{N}$, the function $[\![-]\!]_n$ always yields an n-opetope.

For $t \in \mathbb{T}_n$ and $z \in t^\bullet$, the *address* $\&_t\, z \in \mathbb{A}_n$ of z in t is an n-address (see section 3.3) that, much like in trees (definition 2.2.11), indicates "where z is located in t":

Definition 6.2.2 (Address in a term)**.** Let $(E \triangleright \Gamma \vdash \cdots)$ be a derivable sequent, $t \in \mathbb{T}_\Gamma$ be a non-degenerate term, say $t = x(\overrightarrow{y_i \leftarrow u_i})$.

(1) Let $z \in t^\bullet$. By lemma 6.1.31, either $z =_E x$, or $z \in u_i^\bullet$ for a unique i. The *address* $\&_t\, z \in \mathbb{A}_n$ of z in t is defined as

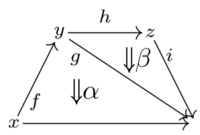
$$\&_t\, z := \begin{cases} [] & \text{if } z =_E x, \\ [\&_{\mathsf{s}\,x}\, y_i] \cdot \&_{u_i}\, z & \text{if } z \in u_i^\bullet. \end{cases}$$

If $[p] = \&_t\, z$, then we write $\mathsf{v}_{[p]}\, t := z$ for the variable of t at address $[p]$. In particular, $\mathsf{v}_{[]}\, t = x$.

(2) Let $a \in (\mathsf{s}\,t)^\bullet$. By lemma 6.1.31, either $a \in (\mathsf{s}\,x)^\bullet$, or $a \in (\mathsf{s}\,u_i)^\bullet$ for a unique i. The *address* $\&_t\, a \in \mathbb{A}_n$ of a in t is defined as

$$\&_t\, a := \begin{cases} [\&_{\mathsf{s}\,x}\, a] & \text{if } a \in (\mathsf{s}\,x)^\bullet, \\ [\&_{\mathsf{s}\,x}\, y_i] \cdot \&_{u_i}\, a & \text{if } a \in (\mathsf{s}\,u_i)^\bullet. \end{cases}$$

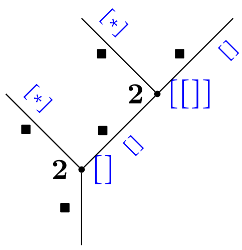
Example 6.2.3. The context describing the pasting diagram:



contains the following typings: $x, y, z : \varnothing$, $f : x \multimap \varnothing$, $g : y \multimap \varnothing$, $h : y \multimap \varnothing$, $i : z \multimap \varnothing$, $\alpha : g(y \leftarrow f) \multimap a \multimap \varnothing$, and $\beta : i(z \leftarrow h) \multimap b \multimap \varnothing$. Write $t := \alpha(g \leftarrow \beta)$. Then,

$$\begin{aligned}
\&_t\, \alpha &= [], \\
\&_t\, \beta &= [\&_{\mathsf{s}\,\alpha}\, g] \cdot \&_\beta\, \beta \;=\; [[]] \cdot [] = [[]], \\
\&_t\, i &= [\&_{\mathsf{s}\,\alpha}\, g] \cdot \&_\beta\, i \;=\; [[]] \cdot [\&_{\mathsf{s}\,\beta}\, i] = [[]] \cdot [[]] = [[][]], \\
\&_t\, h &= [\&_{\mathsf{s}\,\alpha}\, g] \cdot \&_\beta\, h \;=\; [[]] \cdot [\&_{\mathsf{s}\,\beta}\, h] \;=\; [[]] \cdot [[\&_{\mathsf{s}\,i}\, z] \cdot \&_h\, h] \;=\; [[][[]]] = [[][*]], \\
\&_t\, f &= [\&_{\mathsf{s}\,\alpha}\, f] = [[\&_{\mathsf{s}\,g}\, y] \cdot \&_f\, f] \;=\; [[*]].
\end{aligned}$$

Those addresses indeed match with those of the (intuitively) corresponding opetope:



Definition 6.2.4 (Polynomial coding)**.** The polynomial coding operation $\llbracket - \rrbracket_n$ is defined inductively by:

$$\llbracket E \triangleright \Gamma \vdash_0 x : \varnothing \rrbracket_0 := \blacklozenge, \tag{6.2.5}$$

$$\llbracket E \triangleright \Gamma \vdash_1 x : a \multimapdotbothA \varnothing \rrbracket_1 := \blacksquare, \tag{6.2.6}$$

$$\llbracket E \triangleright \Gamma \vdash_{n+1} \underline{x} : x \multimapdotbothA \cdots \rrbracket_{n+2} := \mathsf{I}_{\llbracket x \rrbracket_n}, \tag{6.2.7}$$

$$\left\llbracket E \triangleright \Gamma \vdash_n x(\overrightarrow{y_i \leftarrow u_i}) : \cdots \right\rrbracket_{n+1} := \mathsf{Y}_{\llbracket x \rrbracket_n} \bigcirc_{[\&_{\mathsf{s}x}\, y_i]} \llbracket u_i \rrbracket_{n+1}. \tag{6.2.8}$$

Note that as an immediate consequence of equation (6.2.8), if $x \in \mathbb{V}_n$ is seen as an n-term, then $\llbracket x \rrbracket_{n+1} = \mathsf{Y}_{\llbracket n \rrbracket_n}$. If $(E \triangleright \Gamma \vdash \cdots)$ is a derivable sequent, and $a \in \mathbb{V}_\Gamma$, then the restricted sequent $(E|_a \triangleright \Gamma|_a \vdash a : A)$ is also derivable by proposition 6.1.32, and we may construct an opetope $\llbracket a \rrbracket = \llbracket E|_a \triangleright \Gamma|_a \vdash a : A \rrbracket$.

It is clear that the coding function is well-defined in equations (6.2.5) to (6.2.7). We now establish a series of inductive results to prove proposition 6.2.18 stating that the graftings of equation (6.2.8) are well-defined too, i.e. that for all i, writing $[p_i] := \&_{\mathsf{s}x}\, y_i$, we have

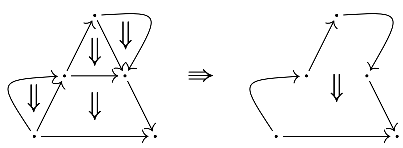
$$[p_i] \in \llbracket x \rrbracket_n^\bullet \qquad \text{and} \qquad \mathsf{s}_{[p_i]} \llbracket x \rrbracket_n = \mathsf{e}_{[]} \llbracket u_i \rrbracket_{n+1}. \tag{6.2.9}$$

The strategy is as follows.

(1) First, we investigate the set of addresses (in the sense of definition 6.2.2) of a term $t \in \mathbb{T}_n$. In lemma 6.2.10, we show that they are the same as its coding $\llbracket t \rrbracket_{n+1}$, i.e. that $(\llbracket t \rrbracket_{n+1})^\bullet = \{\&_t\, x \mid x \in t^\bullet\}$, and further, in proposition 6.2.11, we show that the coding operation is compatible with $\mathsf{s}_{[p]}$ and $\mathsf{v}_{[p]}$. This already proves the left side of (6.2.9).

(2) This compatibility is further investigated in lemmas 6.2.13 and 6.2.15, where we show that $\llbracket - \rrbracket$ commutes with graftings and substitutions.

(3) Then, we show that for an n-term r, $\mathsf{t} \llbracket r \rrbracket_{n+1} = \llbracket \mathsf{s}\, r \rrbracket_n$. This reflects the intuition that the target of an $(n+1)$-opetope ω is just the pasting diagram of $(n-1)$-cells described by the source of ω. In the example below, the composition of 1-cells on top of both sides are indeed the same.



(4) Lastly, we prove the right side of (6.2.9) in proposition 6.2.18, and as an immediate corollary, all the graftings in equation (6.2.8) are well-defined.

If $n = 1$, then all graftings of $(n+1)$-opetopes are possible as there is a unique 1-opetope, and equation (6.2.8) is well-defined in this case.

Lemma 6.2.10. *Let $(E \triangleright \Gamma \vdash_n t : T)$ be a sequent such that $\llbracket t \rrbracket_{n+1}$ is well-defined. Then $\llbracket t \rrbracket_{n+1}^\bullet = \{\&_t\, x \mid x \in t^\bullet\}$, i.e. the node addresses of the opetope $\llbracket t \rrbracket_{n+1}$ are exactly the addresses of the n-variables of t.*

Proof. If $n = 0$, or if t is degenerate, then the result trivially holds. If $t = x(\overrightarrow{y_i \leftarrow u_i})$ as in equation (6.2.8), we have

$$\llbracket t \rrbracket_{n+1}^\bullet = \left\llbracket x(\overrightarrow{y_i \leftarrow u_i}) \right\rrbracket_{n+1}^\bullet$$

$$
\begin{aligned}
&= \left(\mathsf{Y}_{[\![x]\!]_n} \bigcirc_{[\&_{\mathsf{s}x}\, y_i]} [\![u_i]\!]_{n+1} \right)^{\bullet} && \text{by equation (6.2.8)} \\
&= \{[]\} \bigcup_i \{[\&_{\mathsf{s}x}\, y_i] \cdot [p] \mid [p] \in ([\![u_i]\!]_{n+1})^{\bullet}\} \\
&= \{[]\} \bigcup_i \{[\&_{\mathsf{s}x}\, y_i] \cdot \&_t\, x \mid x \in u_i^{\bullet}\} && \text{by induction} \\
&= \{\&_t\, x \mid x \in t^{\bullet}\} && \text{see definition 6.2.2.}
\end{aligned}
$$

□

Proposition 6.2.11. *Let $(E \triangleright \Gamma \vdash_n t : T)$ be a derivable sequent where t is not degenerate, say $t = x(\overrightarrow{y_i \leftarrow u_i})$. Assume that $[\![t]\!]_{n+1}$ is well-defined. For $[p] \in [\![t]\!]_{n+1}^{\bullet}$ we have $\mathsf{s}_{[p]}\, [\![t]\!]_{n+1} = [\![\mathsf{v}_{[p]}\, t]\!]_n$.*

Proof. If $n = 0$, then t is necessarily a 0-variable, so the only possible address in t is $[]$. Then, $\mathsf{s}_{[]}\, [\![t]\!]_1 = \mathsf{s}_{[]}\, \blacksquare = \blacklozenge = [\![t]\!]_0 = [\![\mathsf{v}_{[]}\, t]\!]_0$. Assume that $n \geq 1$. By definition,

$$
[\![t]\!]_{n+1} = \mathsf{Y}_{[\![x]\!]_n} \bigcirc_{[\&_{\mathsf{s}x}\, y_i]} [\![u_i]\!]_{n+1} ,
$$

and we distinguish two cases. If $z = x$, then $[p] = []$, and the result clearly holds. Otherwise, $[p] = [\&_{\mathsf{s}x}\, y_j] \cdot \&_{u_j}\, z$, where j is the unique index such that $z \in u_j^{\bullet}$ (see lemma 6.1.31). Then,

$$
\begin{aligned}
\mathsf{s}_{[p]}\, [\![t]\!]_{n+1} &= \mathsf{s}_{[\&_{\mathsf{s}x}\, y_j] \cdot \&_{u_j}\, z} \left(\mathsf{Y}_{[\![x]\!]_n} \bigcirc_{[\&_{\mathsf{s}x}\, y_i]} [\![u_i]\!]_{n+1} \right) \\
&= \mathsf{s}_{\&_{u_j}\, z}\, [\![u_j]\!]_{n+1} \\
&= [\![z]\!]_n && \text{by induction}
\end{aligned}
$$

□

Corollary 6.2.12. *For x as in equation (6.2.8), and $[p_i] := \&_{\mathsf{s}x}\, y_i$ we have $\mathsf{s}_{[p_i]}\, [\![x]\!]_n = [\![y_i]\!]_{n-1}$.*

Proof. We simply have

$$
\begin{aligned}
\mathsf{s}_{[p_i]}\, [\![x]\!]_n &= \mathsf{s}_{[p_i]}\, [\![\mathsf{s}\, x]\!]_n && \text{see convention 6.2.1} \\
&= [\![y_i]\!]_{n-1} && \text{by proposition 6.2.11.}
\end{aligned}
$$

□

Lemma 6.2.13. *Let $n \geq 1$, and consider the following instance of the `graft` rule:*

$$
\frac{\cdots \vdash_n t : T \qquad \cdots \vdash_n x : X}{\cdots \vdash_n t(a \leftarrow x) : R}\ \texttt{graft-}a
$$

where $(\cdots \vdash_n t : T)$ and $(\cdots \vdash_n x : X)$ are derivable. Writing $r := t(a \leftarrow x)$, and assuming that $[\![r]\!]_{n+1}$ is well-defined, we have $[\![r]\!]_{n+1} = [\![t]\!]_{n+1} \circ_{\&_t\, a} \mathsf{Y}_{[\![x]\!]_n}$.

Proof. By equation (6.2.8), $[\![r]\!]_{n+1} = [\![t]\!]_{n+1} \circ_{[l]} \mathsf{Y}_{[\![x]\!]_n}$, for some leaf address $[l]$. By assumption of the `graft` rule, t is non degenerate, and write it as $t = z(\overrightarrow{y_i \leftarrow u_i})$. According to definition 6.1.17,

$$r = z(\overrightarrow{y_i \leftarrow u_i})(a \leftarrow x) = \begin{cases} z(\overrightarrow{y_i \leftarrow u_i}, a \leftarrow x) & \text{if } a \in (\mathsf{s}\, z)^\bullet, \\ z(\overrightarrow{y_i \leftarrow u_i(a \leftarrow x)}) & \text{if } a \notin (\mathsf{s}\, z)^\bullet, \end{cases}$$

so there are two cases.

(1) If $a \in (\mathsf{s}\, z)^\bullet$, then $[l] = [\&_{\mathsf{s}\, z}\, a] = \&_t\, a$.

(2) If $a \notin (\mathsf{s}\, z)^\bullet$, then

$$[\![r]\!]_{n+1} = \mathsf{Y}_{[\![z]\!]_n} \bigcirc_{[\&_{\mathsf{s}\, z}\, y_i]} [\![u_i(a \leftarrow x)]\!]_{n+1}\,.$$

Let j be the unique index such that $a \in (\mathsf{s}\, u_j)^\bullet$. By induction,

$$[\![u_j(a \leftarrow x)]\!]_{n+1} = [\![u_j]\!]_{n+1} \underset{\&_{u_j}\, a}{\circ} \mathsf{Y}_{[\![x]\!]_n},$$

thus $[l] = [\&_{\mathsf{s}\, z}\, y_j] \cdot \&_{u_j}\, a = \&_t\, a$. □

Lemma 6.2.14 (Named readdressing lemma)**.** *Let $n \geq 1$ and $(E \triangleright \Gamma \vdash_n r : R)$ be a derivable sequent such that $[\![r]\!]_{n+1}$ is well-defined. For $b \in (\mathsf{s}\, r)^\bullet$, we have $\&_{\mathsf{s}\, r}\, b = \wp_{[\![r]\!]_{n+1}} \&_r\, b$ (recall the readdressing map $\wp$ from definition 2.3.11).*

Proof. If r is a variable, then by equation (6.2.8), $[\![r]\!]_{n+1} = \mathsf{Y}_{[\![\mathsf{s}\, r]\!]_n}$, and

$$\begin{aligned} \&_{\mathsf{s}\, r}\, b &= \wp_{\mathsf{Y}_{[\![r]\!]_n}} [\&_{\mathsf{s}\, r}\, b] && \text{by theorem 2.3.6} \\ &= \wp_{[\![r]\!]_{n+1}} \&_r\, b && \text{see definition 6.2.2.} \end{aligned}$$

If r is degenerate, say $r = \underline{x}$, then the result trivially holds as $\mathsf{s}\, r = x$ only has one variable address. Otherwise, the sequent follows from an instance of the `graft` rule, say

$$\frac{\cdots \vdash_n t : T \qquad \cdots \vdash_n x : X}{\cdots \vdash_n r : R}\ \texttt{graft-}a$$

where by induction, $\&_{\mathsf{s}\, t}\, a = \wp_{[\![t]\!]_{n+1}} \&_t\, a$ for all $a \in (\mathsf{s}\, t)^\bullet$. Since $\mathsf{s}\, r = (\mathsf{s}\, t)[\mathsf{s}\, x / a]$, we have

$$\begin{aligned} \&_{\mathsf{s}\, r}\, b &= \begin{cases} \&_{\mathsf{s}\, t}\, a \cdot \&_{\mathsf{s}\, x}\, b & \text{if } b \in (\mathsf{s}\, x)^\bullet, \\ \&_{\mathsf{s}\, t}\, a \cdot \&_{\mathsf{s}\, x}\, c \cdot [p] & \text{if } b \in (\mathsf{s}\, t)^\bullet, \&_{\mathsf{s}\, t}\, a \sqsubseteq \&_{\mathsf{s}\, t}\, b, \\ & \quad \text{say } \&_{\mathsf{s}\, t}\, b = \&_{\mathsf{s}\, t}\, a \cdot [\&_{\mathsf{s}\, a}\, c] \cdot [p], \\ \&_{\mathsf{s}\, t}\, b & \text{if } b \in (\mathsf{s}\, t)^\bullet, \&_{\mathsf{s}\, t}\, a \not\sqsubseteq \&_{\mathsf{s}\, t}\, b. \end{cases} \\ &= \wp_{\left([\![t]\!]_{n+1} \circ_{\&_t\, a} \mathsf{Y}_{[\![x]\!]_n}\right)} \&_r\, b && \text{see theorem 2.4.6} \\ &= \wp_{[\![r]\!]_{n+1}} \&_r\, b && \text{by equation (6.2.8).} \end{aligned}$$

□

Lemma 6.2.15. *Let $(E \triangleright \Gamma \vdash \cdots)$ be a derivable sequent, $u, v \in \mathbb{T}_{\Gamma,n}$ be n-terms, where u is non-degenerate, say $u = y(\overrightarrow{a_i \leftarrow w_i})$. Let $x \in u^\bullet$ be a variable such that $\mathsf{s}\, x = \mathsf{s}\, v$. In particular, the substitution $u[v/x]$ is well-defined. Further, assume that $\mathsf{t}\,[\![v]\!]_{n+1} = [\![x]\!]_n$. Then $[\![u[v/x]]\!]_{n+1} = [\![u]\!]_{n+1} \square_{\&_u\, x} [\![v]\!]_{n+1}$.*

Proof. (1) If $x =_E y$, then

$$\begin{aligned}
&[\![u[v/x]]\!]_{n+1} \\
&= \left[\!\!\left[v(\overrightarrow{a_i \leftarrow w_i})\right]\!\!\right]_{n+1} \\
&= [\![v]\!]_{n+1} \bigcirc_{\&_v a_i} [\![w_i]\!]_{n+1} && \spadesuit \\
&= \left(\mathsf{Y}_{[\![x]\!]_n} \bigcirc_{[\wp_{[\![v]\!]_{n+1}} \&_v a_i]} [\![w_i]\!]_{n+1} \right) \underset{[]}{\square} [\![v]\!]_{n+1} && \text{since } \mathsf{t}\, [\![v]\!]_{n+1} = [\![x]\!]_n \\
&= \left(\mathsf{Y}_{[\![x]\!]_n} \bigcirc_{[\&_{\mathsf{s}\, v}\, a_i]} [\![w_i]\!]_{n+1} \right) \underset{[]}{\square} [\![v]\!]_{n+1} && \diamondsuit \\
&= \left(\mathsf{Y}_{[\![x]\!]_n} \bigcirc_{[\&_{\mathsf{s}\, x}\, a_i]} [\![w_i]\!]_{n+1} \right) \underset{[]}{\square} [\![v]\!]_{n+1} && \text{by assumption} \\
&= [\![u]\!]_{n+1} \underset{[]}{\square} [\![v]\!]_{n+1} \\
&= [\![u]\!]_{n+1} \underset{\&_u x}{\square} [\![v]\!]_{n+1} && \clubsuit
\end{aligned}$$

where $\spadesuit$ is by equation (6.2.8), $\diamondsuit$ follows from lemma 6.2.14, and $\clubsuit$ follows from the fact that x is the head variable of u (see definition 6.2.2).

(2) If $x \neq_E y$, then by lemma 6.1.31, there is a unique index j such that $x \in w_j^\bullet$. We have

$$\begin{aligned}
&[\![u[v/x]]\!]_{n+1} \\
&= [\![y(\ldots, a_j \leftarrow w_j[v/x], \ldots)]\!]_{n+1} \\
&= \left(\mathsf{Y}_{[\![y]\!]_n} \bigcirc_{[\&_{\mathsf{s}\, y}\, a_i], i \neq j} [\![w_i]\!]_{n+1} \right) \underset{[\&_{\mathsf{s}\, y}\, a_j]}{\circ} [\![w_j[v/x]]\!]_{n+1} && \spadesuit \\
&= \left(\mathsf{Y}_{[\![y]\!]_n} \bigcirc_{[\&_{\mathsf{s}\, y}\, a_i], i \neq j} [\![w_i]\!]_{n+1} \right) \\
&\qquad \underset{[\&_{\mathsf{s}\, y}\, a_j]}{\circ} \left([\![w_j]\!]_{n+1} \underset{\&_{w_j} x}{\square} [\![v]\!]_{n+1} \right) && \text{by induction} \\
&= [\![u]\!]_{n+1} \underset{[\&_{\mathsf{s}\, y}\, a_j] \cdot \&_{w_j} x}{\square} [\![v]\!]_{n+1} && \diamondsuit \\
&= [\![u]\!]_{n+1} \underset{\&_u x}{\square} [\![v]\!]_{n+1} && \clubsuit
\end{aligned}$$

where $\spadesuit$ is by equation (6.2.8), $\diamondsuit$ is just a rearrangement of terms, and $\clubsuit$ is definition 6.2.2. $\square$

Proposition 6.2.16. *Let $n \geq 1$ and $(E \triangleright \Gamma \vdash_n r : R)$ be a derivable sequent such that $[\![r]\!]_{n+1}$ is well-defined. Then $\mathsf{t}\, [\![r]\!]_{n+1} = [\![\mathsf{s}\, r]\!]_n$.*

Proof. (1) If $r \in \mathbb{V}_n$ is a variable, then

$$\begin{aligned}
\mathsf{t}\, [\![r]\!]_{n+1} &= \mathsf{t}\, \mathsf{Y}_{[\![r]\!]_n} && \text{by equation (6.2.8)} \\
&= [\![r]\!]_n .
\end{aligned}$$

(2) If r is a degenerate term, say $r = \underline{x}$ for $x \in \mathbb{V}_{n-1}$, then

$$\begin{aligned}
\mathsf{t}\,[\![r]\!]_{n+1} &= \mathsf{t}\,\mathsf{I}_{[\![x]\!]_{n-1}} && \text{by equation (6.2.7)} \\
&= \mathsf{Y}_{[\![x]\!]_{n-1}} \\
&= [\![x]\!]_n && \text{by equation (6.2.8)} \\
&= [\![\mathsf{s}\,r]\!]_n && \text{see rule } \mathtt{degen}.
\end{aligned}$$

(3) Otherwise, the sequent follows from an instance of the `graft` rule, say

$$\dfrac{\cdots \vdash_n t : T \qquad \cdots \vdash_n x : X}{\cdots \vdash_n r : R}\ \mathtt{graft}\text{-}a$$

Let $[p] \coloneqq \wp_{[\![t]\!]_{n+1}} \&_t\, a$. We have

$$\begin{aligned}
\mathsf{t}\,[\![r]\!]_{n+1} &= \mathsf{t}\,[\![t(a \leftarrow x)]\!]_{n+1} && \text{by definition of } \mathtt{graft}\text{-}a \\
&= \mathsf{t}\left([\![t]\!]_{n+1} \underset{\&_{\mathsf{s}\,t}\, a}{\circ} \mathsf{Y}_{[\![x]\!]_n}\right) && \text{by lemma 6.2.13} \\
&= \mathsf{t}\,[\![t]\!]_{n+1} \underset{[p]}{\square} [\![x]\!]_n && \text{by proposition 3.1.6} \\
&= [\![\mathsf{s}\,t]\!]_n \underset{[p]}{\square} [\![x]\!]_n && \text{by induction} \\
&= [\![\mathsf{s}\,t]\!]_n \underset{\&_{\mathsf{s}\,t}\, a}{\square} [\![x]\!]_n && \text{by lemma 6.2.14} \\
&= [\![(\mathsf{s}\,t)[x/a]]\!]_n && \text{by lemma 6.2.15} \\
&= [\![\mathsf{s}\,r]\!]_n && \text{by definition of } \mathtt{graft}\text{-}a.
\end{aligned}$$

$\square$

Corollary 6.2.17. *Let $n \geq 1$ and $(E \triangleright \Gamma \vdash_n t : T)$ be a derivable sequent such that $[\![t]\!]_{n+1}$ is well-defined. Then $\&_t$ exhibits a bijection*

$$(\mathsf{s}\,t)^\bullet \xrightarrow{\cong} [\![t]\!]^{\shortmid}_{n+1}\,,$$

Proof. Since the readdressing map $\wp_{[\![t]\!]_{n+1}}$ is a bijection, $\&_t$ can be expressed as the following composite:

$$\begin{aligned}
(\mathsf{s}\,t)^\bullet &\xrightarrow{\&_t} \{\&_t\, a \mid a \in (\mathsf{s}\,t)^\bullet\} \\
&\xrightarrow{\wp_{[\![t]\!]_{n+1}}} \{\&_{\mathsf{s}\,t}\, a \mid a \in (\mathsf{s}\,t)^\bullet\} && \text{by lemma 6.2.14} \\
&= [\![\mathsf{s}\,t]\!]^\bullet_n && \text{by lemma 6.2.10} \\
&= (\mathsf{t}\,[\![t]\!]_{n+1})^\bullet && \text{by proposition 6.2.16} \\
&\xrightarrow{\wp^{-1}_{[\![t]\!]_{n+1}}} [\![t]\!]^{\shortmid}_{n+1}\,.
\end{aligned}$$

$\square$

Proposition 6.2.18. *With variables as in equation (6.2.8), we have that for all i*

$$\mathsf{e}_{[]}\, [\![u_i]\!]_{n+1} = \mathsf{s}_{\&_{\mathsf{s}\,x}\, y_i}\, [\![x]\!]_n\,,$$

and the graftings are well-defined.

Proof. Write u_i as $a(\overrightarrow{b_j \leftarrow v_j})$, and consider

$$\begin{aligned}
\mathsf{e}_{[]} [\![u_i]\!]_{n+1} &= \mathsf{t}\,\mathsf{s}_{[]} [\![u_i]\!]_{n+1} && \\
&= \mathsf{t} [\![a]\!]_n && \text{by proposition 6.2.11} \\
&= \mathsf{t} [\![\mathsf{s}\, a]\!]_n && \text{see convention 6.2.1} \\
&= [\![\mathsf{s}\,\mathsf{s}\, a]\!]_{n-1} && \text{by proposition 6.2.16} \\
&= [\![\mathsf{s}\, y_i]\!]_{n-1} && \text{by the conditions of } \texttt{graft} \\
&= [\![y_i]\!]_{n-1} && \text{see convention 6.2.1} \\
&= \mathsf{s}_{\&_{\mathsf{s}\,x}\, y_i} [\![x]\!]_n && \text{by corollary 6.2.12.}
\end{aligned}$$

□

This result concludes the proof that equations (6.2.5) to (6.2.8) of definition 6.2.4 are well-defined. The rest of this section is dedicated to prove theorem 6.2.27 stating that $[\![-]\!]_n$ is a bijection modulo α-equivalence. We first prove surjectivity, by defining a sequent $C^!(\omega)$ in $\mathrm{OPT}^!$ such that $[\![C^!(\omega)]\!]_n = \omega$, for any opetope $\omega \in \mathbb{O}_n$.

Definition 6.2.19. We define the *named coding function* $C^!$ as follows.

(1) Trivially, $C^!(\blacklozenge)$ is obtained by the following proof tree:

$$\frac{}{C^!(\blacklozenge)}\ \texttt{point} \tag{6.2.20}$$

with an arbitrary choice of variable (different choices lead to equivalent sequents).

(2) For $\phi \in \mathbb{O}_{n-2}$ the sequent $C^!(\mathsf{I}_\phi)$ is obtained by the following proof tree:

$$\dfrac{\dfrac{\vdots}{C^!(\phi)}}{C^!(\mathsf{I}_\phi)}\ \texttt{degen} \tag{6.2.21}$$

(3) For $\psi \in \mathbb{O}_{n-1}$, the sequent $C^!(\mathsf{Y}_\psi)$ is obtained by the following proof tree:

$$\dfrac{\dfrac{\vdots}{C^!(\psi)}}{C^!(\mathsf{Y}_\psi)}\ \texttt{shift} \tag{6.2.22}$$

with an arbitrary choice of fresh variable (different choices lead to equivalent sequents).

(4) Let $\nu \in \mathbb{O}_n$ having at least one node, $[l] \in \nu^{|}$, and $\psi \in \mathbb{O}_{n-1}$ be such that the grafting $\nu \circ_{[l]} \mathsf{Y}_\psi$ is well-defined. Then the sequent $C^!(\nu \circ_{[l]} \mathsf{Y}_\psi)$ is obtained by the following proof tree:

$$\dfrac{\dfrac{\vdots}{C^!(\nu)} \qquad \dfrac{\dfrac{\vdots}{C^!(\psi)}}{C^!(\mathsf{Y}_\psi)}\ \texttt{shift}}{C^!(\nu \circ_{[l]} \mathsf{Y}_\psi)}\ \texttt{graft-}a \tag{6.2.23}$$

where $C^!(\nu) = (\cdots \vdash_n u : U)$, where the variable $a \in (\mathsf{s}\, u)^\bullet$ is an $(n-1)$-variable such that $\&_{\mathsf{s}\,u}\, a = \wp_\nu[l]$ (see corollary 6.2.17), and where the adequate α-conversions have been performed to fulfill the side conditions of `graft`.

Proposition 6.2.24. *In proof tree* (6.2.23)*, the instance of* `graft` *is well-defined.*

Proof. If $n = 2$, then all graftings are well-defined, as there exists only one 1-opetope. Assume that $n > 2$, write $C^{!}\left(\mathsf{Y}_{\psi}\right) = (\cdots \vdash_{n-1} p : P)$, where $p \in \mathbb{V}_n$, and let $[q] := \wp_{\nu}[l]$. We have

$$\begin{aligned}
[\![\mathsf{s}\, a]\!]_{n-1} &= [\![a]\!]_{n-1} && \text{see convention 6.2.1} \\
&= [\![\mathsf{v}_{[q]}\, \mathsf{s}\, u]\!]_{n-1} && \text{by definition of } a \\
&= \mathsf{s}_{[q]}\, [\![\mathsf{s}\, u]\!]_{n} && \text{by proposition 6.2.11} \\
&= \mathsf{s}_{[q]}\, \mathsf{t}\, [\![u]\!]_{n+1} && \text{by proposition 6.2.16} \\
&= \mathsf{s}_{[q]}\, \mathsf{t}\, \nu && \text{by definition} \\
&= \mathsf{e}_{[l]}\, \nu && \text{by } \textbf{(Glob2)} \\
&= \mathsf{t}\, \psi && \text{by assumption} \\
&= \mathsf{t}\,\mathsf{t}\, \mathsf{Y}_{\psi} && \\
&= \mathsf{t}\,\mathsf{t}\, [\![p]\!]_{n+1} && \text{by definition} \\
&= [\![\mathsf{s}\,\mathsf{s}\, p]\!]_{n-1} && \text{by proposition 6.2.16 twice.}
\end{aligned}$$

By induction on n, the polynomial coding $[\![-]\!]_{n-1}$ is injective modulo α-equivalence. Hence without loss of generality, we can assume $\mathsf{s}\, a = \mathsf{s}\,\mathsf{s}\, p$, and finally, the instance of the `graft` rule is well-defined. □

Proposition 6.2.25. *Let $n \geq 2$ and $\omega \in \mathbb{O}_n$ have at least three nodes. The sequent $C^{!}(\omega)$ does not depend on the decomposition of ω in corollas. Explicitly, for any two decompositions of ω, say*

$$\begin{aligned}
\omega &= \left(\cdots \left(\mathsf{Y}_{\mathsf{s}_{[p_1]}\omega} \underset{[p_2]}{\circ} \mathsf{Y}_{\mathsf{s}_{[p_2]}\omega} \right) \underset{[p_3]}{\circ} \mathsf{Y}_{\mathsf{s}_{[p_3]}\omega} \cdots \right) \underset{[p_k]}{\circ} \mathsf{Y}_{\mathsf{s}_{[p_k]}\omega} \\
&= \left(\cdots \left(\mathsf{Y}_{\mathsf{s}_{[q_1]}\omega} \underset{[q_2]}{\circ} \mathsf{Y}_{\mathsf{s}_{[q_2]}\omega} \right) \underset{[q_3]}{\circ} \mathsf{Y}_{\mathsf{s}_{[q_3]}\omega} \cdots \right) \underset{[q_k]}{\circ} \mathsf{Y}_{\mathsf{s}_{[q_k]}\omega},
\end{aligned}$$

we have

$$C^{!}\left(\left(\mathsf{Y}_{\mathsf{s}_{[p_1]}\omega} \underset{[p_2]}{\circ} \mathsf{Y}_{\mathsf{s}_{[p_2]}\omega} \right) \cdots \underset{[p_k]}{\circ} \mathsf{Y}_{\mathsf{s}_{[p_k]}\omega} \right) = C^{!}\left(\left(\mathsf{Y}_{\mathsf{s}_{[q_1]}\omega} \underset{[q_2]}{\circ} \mathsf{Y}_{\mathsf{s}_{[q_2]}\omega} \right) \cdots \underset{[q_k]}{\circ} \mathsf{Y}_{\mathsf{s}_{[q_k]}\omega} \right).$$

Proof. By definition, the sequence $[p_1], \ldots, [p_k]$ (and likewise for $[q_1], \ldots, [q_k]$) has the following property: for $1 \leq i \leq j \leq k$, either $[p_i] \sqsubseteq [p_j]$ or $[p_i]$ and $[p_j]$ are $\sqsubseteq$-incomparable (recall that $\sqsubseteq$ is the prefix order on $\mathbb{A}_{n-1}$, see definition 3.3.4). Further, $\{[p_1], \ldots, [p_k]\} = \omega^{\bullet} = \{[q_1], \ldots, [q_k]\}$, i.e. the two sequences have the same elements. Consequently, the sequence $[q_1], \ldots, [q_k]$ can be obtained from $[p_1], \ldots, [p_k]$ by a series of transpositions of consecutive $\sqsubseteq$-incomparable addresses.

It is thus enough to check the following: for $\nu \in \mathbb{O}_n$, two different leaf addresses $[l], [l'] \in \nu^{|}$ (which are necessarily $\sqsubseteq$-incomparable), and $\psi, \psi' \in \mathbb{O}_{n-1}$ such that $\mathsf{t}\, \psi = \mathsf{e}_{[l]}\, \nu$ and $\mathsf{t}\, \psi' = \mathsf{e}_{[l']}\, \nu$, we have

$$C^{!}\left((\nu \underset{[l]}{\circ} \mathsf{Y}_{\psi}) \underset{[l']}{\circ} \mathsf{Y}_{\psi'} \right) = C^{!}\left((\nu \underset{[l']}{\circ} \mathsf{Y}_{\psi'}) \underset{[l]}{\circ} \mathsf{Y}_{\psi} \right).$$

Write

$$\begin{aligned} C^!(\nu) &= (E_\nu \triangleright \Gamma_\nu \vdash_n t_\nu : s_\nu \multimapdotbothA X_\nu), \\ C^!\left(\mathsf{Y}_\psi\right) &= \left(E_\psi \triangleright \Gamma_\psi \vdash_n x_\psi : s_\psi \multimapdotbothA X_\psi\right), \\ C^!\left(\mathsf{Y}'_\psi\right) &= \left(E_{\psi'} \triangleright \Gamma_{\psi'} \vdash_n x_{\psi'} : s_{\psi'} \multimapdotbothA X_{\psi'}\right), \end{aligned}$$

with $t_\nu \in \mathbb{T}_n$ and $x_\psi, x_{\psi'} \in \mathbb{V}_n$. Let $a, a' \in (\mathsf{s}\nu)^\bullet$ be such that $\&_{\mathsf{s}\nu}\, a = [l]$ and $\&_{\mathsf{s}\nu}\, a' = [l']$ (see corollary 6.2.17). The sequents above are respectively obtained by the following proof trees:

$$\dfrac{\dfrac{\cdots \vdash t_\nu : s_\nu \multimapdotbothA X_\nu \qquad \cdots \vdash x_\psi : s_\psi \multimapdotbothA X_\psi}{F \triangleright \Gamma_\nu \cup \Gamma_\psi \vdash t_\nu(a \leftarrow x_\psi) : s_\nu[s_\psi/a] \multimapdotbothA X_\nu}\,\texttt{graft-}a \qquad \cdots \vdash x_{\psi'} : s_{\psi'} \multimapdotbothA X_{\psi'}}{G \triangleright \Gamma_\nu \cup \Gamma_\psi \cup \Gamma_{\psi'} \vdash t_\nu(a \leftarrow x_\psi)(a' \leftarrow x_{\psi'}) : s_\nu[s_\psi/a][s_{\psi'}/a'] \multimapdotbothA X_\nu}\,\texttt{graft-}a'$$

$$\dfrac{\dfrac{\cdots \vdash t_\nu : s_\nu \multimapdotbothA X_\nu \qquad \cdots \vdash x_{\psi'} : s_{\psi'} \multimapdotbothA X_{\psi'}}{F' \triangleright \Gamma_\nu \cup \Gamma_{\psi'} \vdash t_\nu(a' \leftarrow x_{\psi'}) : s_\nu[s_{\psi'}/a'] \multimapdotbothA X_\nu}\,\texttt{graft-}a' \qquad \cdots \vdash x_\psi : s_\psi \multimapdotbothA X_\psi}{G' \triangleright \Gamma_\nu \cup \Gamma_{\psi'} \cup \Gamma_\psi \vdash t_\nu(a' \leftarrow x_{\psi'})(a \leftarrow x_\psi) : s_\nu[s_{\psi'}/a'][s_\psi/a] \multimapdotbothA X_\nu}\,\texttt{graft-}a$$

It remains to prove that both those conclusions are α-equivalent.

(1) By assumption on the `graft` rule, $a \notin s_{\psi'}^\bullet$ and $a' \notin s_\psi^\bullet$, and clearly,

$$t_\nu(a \leftarrow x_\psi)(a' \leftarrow x_{\psi'}) = t_\nu(a' \leftarrow x_{\psi'})(a \leftarrow x_\psi).$$

(2) Again, since $a \notin s_{\psi'}^\bullet$ and $a' \notin s_\psi^\bullet$, we have $s_\nu[s_\psi/a][s_{\psi'}/a'] = s_\nu[s_{\psi'}/a'][s_\psi/a]$.

(3) Lastly, the equational theories G and G' are the union of E_ν, E_ψ, and $E_{\psi'}$, and the potential additional equalities incurred by the independent substitutions s_ψ/a and $s_{\psi'}/a'$. Hence $G = G'$. □

Corollary 6.2.26. *For any opetope $\omega \in \mathbb{O}$, the sequent $C^!(\omega)$ is uniquely defined up to α-equivalence.*

Proof. Clearly, proof trees (6.2.20), (6.2.21), and (6.2.22) are well-defined. In proposition 6.2.24, we have shown that the same holds for proof tree equation (6.2.23). Finally, in proposition 6.2.25, we have shown that for a non degenerate opetope $\omega \in \mathbb{O}_n$, the sequent $C^!(\omega)$ does not depend on the decomposition of ω. □

Theorem 6.2.27. *The polynomial coding $[\![-]\!]_n$ is a bijection up to α-equivalence, whose inverse is $C^!(-)$ restricted to $\mathbb{O}_n$.*

Proof. The result is trivial if $n = 0, 1$, so we assume $n \geq 2$. We first show that for $\omega \in \mathbb{O}_n$ we have $[\![C^!(\omega)]\!]_n = \omega$.

(1) By definition of $[\![-]\!]$, $[\![C^!(\blacklozenge)]\!]_0 = \blacklozenge$.

(2) With the same notations as in (6.2.21), and by induction, we have

$$[\![C^!(\mathsf{I}_\phi)]\!]_n = \mathsf{I}_{[\![C^!(\phi)]\!]_{n-2}} = \mathsf{I}_\phi.$$

(3) With the same notations as in (6.2.22), and by induction, we have,

$$[\![C^!(\mathsf{Y}_\psi)]\!]_n = \mathsf{Y}_{[\![C^!(\psi)]\!]_{n-1}} = \mathsf{Y}_\psi.$$

(4) With the same notations as in (6.2.23), and by induction, we have

$$\begin{aligned} \left[\!\!\left[C^! \left(\nu \underset{[l]}{\circ} \mathsf{Y}_\psi \right) \right]\!\!\right]_n &= [\![C^! (\nu)]\!]_n \underset{[\&_{\mathsf{s}\mathsf{s}u}\, a]}{\circ} [\![C^! (\mathsf{Y}_\psi)]\!]_n \\ &= [\![C^! (\nu)]\!]_n \underset{[l]}{\circ} [\![C^! (\mathsf{Y}_\psi)]\!]_n \\ &= \nu \underset{[l]}{\circ} \mathsf{Y}_\psi . \end{aligned}$$

Conversely, we now show that for a derivable sequent $(E \triangleright \Gamma \vdash \alpha : T)$, we have an isomorphism $(E \triangleright \Gamma \vdash \alpha : T) \simeq C^! \left([\![E \triangleright \Gamma \vdash \alpha : T]\!]_n \right)$.

(1) We have that $C^! \left([\![x : \varnothing]\!]_0 \right) = C^! (\blacklozenge) \simeq (\triangleright x : \varnothing \vdash x : \varnothing)$.

(2) With the same notations as in equation (6.2.7), we have

$$C^! \left([\![\cdots \vdash \delta : \underline{x} \mathrel{\bullet\!\!-\!\!\circ} x \mathrel{\bullet\!\!-\!\!\circ} X]\!]_n \right) = C^! \left(\mathsf{I}_{[\![x : X]\!]_n} \right)$$

and both sequents $C^! \left(\mathsf{I}_{[\![x : X]\!]_n} \right)$ and $(\cdots \vdash \delta : \underline{x} \mathrel{\bullet\!\!-\!\!\circ} x \mathrel{\bullet\!\!-\!\!\circ} X)$ are obtained by applying `degen` to $(\cdots \vdash x : X)$. Thus $C^! \left([\![\cdots \vdash \delta : \underline{x} \mathrel{\bullet\!\!-\!\!\circ} x \mathrel{\bullet\!\!-\!\!\circ} X]\!]_n \right) \simeq (\cdots \vdash \delta : \underline{x} \mathrel{\bullet\!\!-\!\!\circ} x \mathrel{\bullet\!\!-\!\!\circ} X)$.

(3) Lastly, consider the sequent $\left(\cdots \vdash \alpha : x(\overrightarrow{y_i \leftarrow u_i}) \mathrel{\bullet\!\!-\!\!\circ} T \right)$ as in equation (6.2.8). Then

$$\begin{aligned} C^! \left(\left[\!\!\left[\cdots \vdash \alpha : x(\overrightarrow{y_i \leftarrow u_i}) \mathrel{\bullet\!\!-\!\!\circ} T \right]\!\!\right]_{n+1} \right) &= C^! \left(\mathsf{Y}_{[\![x]\!]_n} \underset{[\&_{\mathsf{s}x}\, y_i]}{\bigcirc} [\![u_i]\!]_{n+1} \right) \\ &\simeq \left(\cdots \vdash \alpha : x(\overrightarrow{y_i \leftarrow u_i}) \mathrel{\bullet\!\!-\!\!\circ} T' \right) . \end{aligned}$$

Since T and T' are completely determined by $x(\overrightarrow{y_i \leftarrow u_i})$ (see theorem 6.1.26), we have that $T = T'$, whence

$$C^! \left(\left[\!\!\left[\cdots \vdash \alpha : x(\overrightarrow{y_i \leftarrow u_i}) \mathrel{\bullet\!\!-\!\!\circ} T \right]\!\!\right]_{n+1} \right) \simeq \left(\cdots \vdash \alpha : x(\overrightarrow{y_i \leftarrow u_i}) \mathrel{\bullet\!\!-\!\!\circ} T \right) .$$

□

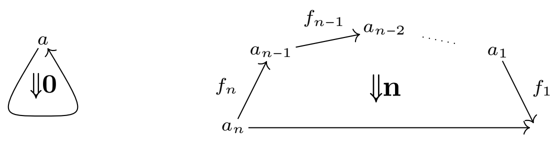
6.3 EXAMPLES

In this section, we showcase the derivation of some low dimensional opetopes. On a scale of a proof tree, specifying the context at every step is redundant. Hence we allow omitting it, only having the equational theory on the left of $\vdash$.

Example 6.3.1 (The arrow)**.** The unique 1-opetope, the *arrow*, is given by the following simple derivation:

$$\dfrac{\dfrac{}{\vdash_0 a : \varnothing}\ \texttt{point}}{\vdash_1 f : a \mathrel{\bullet\!\!-\!\!\circ} \varnothing}\ \texttt{shift}$$

Example 6.3.2 (Opetopic integers)**.** The opetopic integer $\mathbf{n}$ (example 3.1.4) is represented on the left in the case $n = 0$, and on the right if $n \geq 1$:



The derivation of $\mathbf{0}$ is

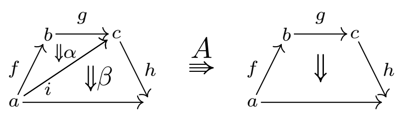
$$\dfrac{\dfrac{\dfrac{}{\vdash_0 a : \varnothing}\,\mathtt{point}}{\vdash_1 \underline{a} : a \multimap \varnothing}\,\mathtt{degen}}{\vdash_1 \mathbf{0} : \underline{a} \multimap a \multimap \varnothing}\,\mathtt{shift}$$

alternatively, we could have used the `degen-shift` rule (remark 6.1.24). For $n \geq 1$, the opetope $\mathbf{n}$ is derived as follows, where `g` is a shorthand for `graft`

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash f_1 : a_1 \multimap \varnothing\end{array} \quad \begin{array}{c}\vdots\\ \vdash f_2 : a_2 \multimap \varnothing\end{array}}{\vdash f_1(a_1 \leftarrow f_2) : a_2 \multimap \varnothing}\,\mathtt{g}\text{-}a_1 \qquad \begin{array}{c}\vdots\\ \vdash f_3 : a_3 \multimap \varnothing\end{array}}{\vdash f_1(a_1 \leftarrow f_2(a_2 \leftarrow f_3)) : a_3 \multimap \varnothing}\,\mathtt{g}\text{-}a_2$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash f_1(a_1 \leftarrow f_2(\cdots a_{n-2} \leftarrow f_{n-1})) : a_{n-1} \multimap \varnothing\end{array} \qquad \begin{array}{c}\vdots\\ \vdash f_n : a_n \multimap \varnothing\end{array}}{\vdash f_1(a_1 \leftarrow f_2(\cdots a_{n-1} \leftarrow f_n)) : a_n \multimap \varnothing}\,\mathtt{g}\text{-}a_{n-1}}{\vdash \mathbf{n} : f_1(a_1 \leftarrow f_2(\cdots a_{n-1} \leftarrow f_n)) \multimap a_n \multimap \varnothing}\,\mathtt{shift}$$

Example 6.3.3. The 3-opetope



is derived as follows.

$$\dfrac{\dfrac{\dfrac{}{\vdash_0 c : \varnothing}\,\mathtt{point}}{\vdash_1 h : c \multimap \varnothing}\,\mathtt{shift} \qquad \dfrac{\dfrac{}{\vdash_0 a : \varnothing}\,\mathtt{point}}{\vdash_1 i : a \multimap \varnothing}\,\mathtt{shift}}{\vdash_1 h(c \leftarrow i) : c[a/c] \multimap \varnothing}\,\mathtt{graft}\text{-}c$$

and $c[a/c] = a$. Then,

$$\dfrac{\begin{array}{c}\vdots\\ \vdash_1 h(c \leftarrow i) : a \multimap \varnothing\end{array}}{\vdash_2 \beta : h(c \leftarrow i) \multimap a \multimap \varnothing}\,\mathtt{shift}$$

On the other hand we have

$$\dfrac{\dfrac{\dfrac{}{\vdash_0 b : \varnothing}\,\mathtt{point}}{\vdash_1 g : b \multimap \varnothing}\,\mathtt{shift} \qquad \dfrac{\dfrac{}{\vdash_0 a : \varnothing}\,\mathtt{point}}{\vdash_1 f : a \multimap \varnothing}\,\mathtt{shift}}{\vdash_1 g(b \leftarrow f) : b[a/b] \multimap \varnothing}\,\mathtt{graft}\text{-}b$$

and $b[a/b] = a$. Then,

$$\dfrac{\begin{array}{c}\vdots\\ \vdash_2 \beta : h(c \leftarrow i) \multimap a \multimap \varnothing\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \vdash_1 g(b \leftarrow f) : \multimap \varnothing\end{array}}{\vdash_2 \alpha : g(b \leftarrow f) \multimap a \multimap \varnothing}\,\mathtt{shift}}{\vdash_2 \beta(i \leftarrow \alpha) : h(c \leftarrow i)[g(b \leftarrow f)/i] \multimap a \multimap \varnothing}\,\mathtt{graft}\text{-}i$$

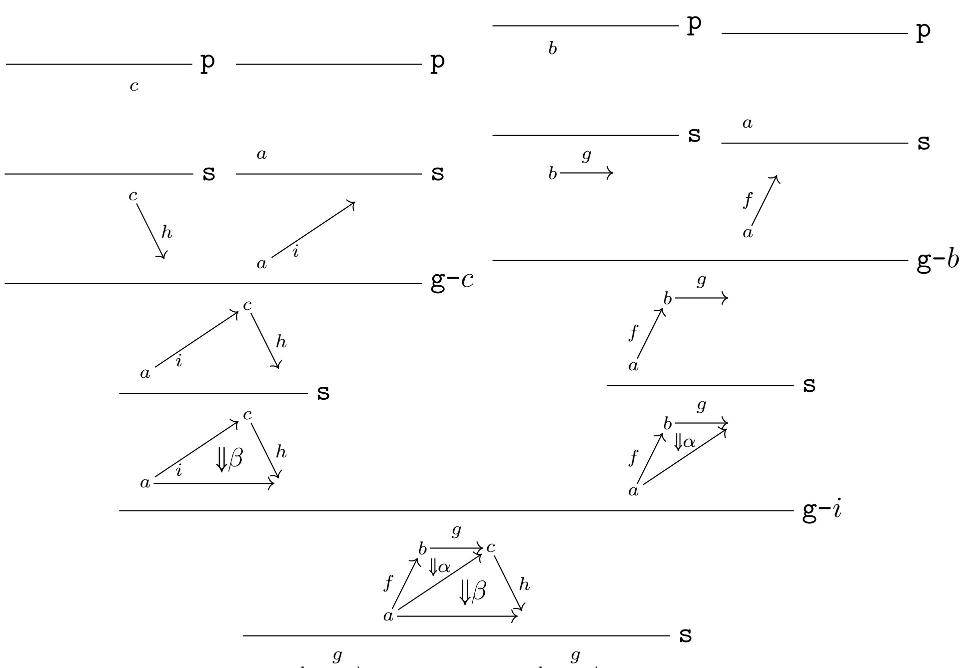
The last grafting is well-defined as $\mathsf{s}\,i = a = \mathsf{s}\mathsf{s}\,\alpha$, and $h(c \leftarrow i)[g(b \leftarrow f)/i] = h(c \leftarrow g(b \leftarrow f))$. Finally

$$\dfrac{\begin{array}{c}\vdots\\ \vdash_2 \beta(i \leftarrow \alpha) : h(c \leftarrow g(b \leftarrow f)) \multimapdotbothA a \multimapdotbothA \emptyset\end{array}}{\vdash_3 A : \beta(i \leftarrow \alpha) \multimapdotbothA h(c \leftarrow g(b \leftarrow f)) \multimapdotbothA a \multimapdotbothA \emptyset}\,\texttt{shift}$$

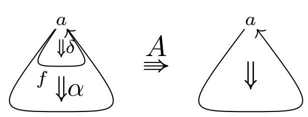
The complete proof tree is as follows, where `p`, `s`, and `g` are abbreviations for `point`, `shift`, and `graft`, respectively

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash c : \emptyset}\,\texttt{p}}{\vdash h : c \multimapdotbothA \emptyset}\,\texttt{s} \quad \dfrac{\dfrac{}{\vdash a : \emptyset}\,\texttt{p}}{\vdash i : a \multimapdotbothA \emptyset}\,\texttt{s}}{\vdash h(c \leftarrow i) : a \multimapdotbothA \emptyset}\,\texttt{g-}c}{\vdash \beta : h(c \leftarrow i) \multimapdotbothA a \multimapdotbothA \emptyset}\,\texttt{s} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash b : \emptyset}\,\texttt{p}}{\vdash g : b \multimapdotbothA \emptyset}\,\texttt{s} \quad \dfrac{\dfrac{}{\vdash a : \emptyset}\,\texttt{p}}{\vdash f : a \multimapdotbothA \emptyset}\,\texttt{s}}{\vdash g(b \leftarrow f) : a \multimapdotbothA \emptyset}\,\texttt{g-}b}{\vdash \alpha : g(b \leftarrow f) \multimapdotbothA a \multimapdotbothA \emptyset}\,\texttt{s}}{\dfrac{\vdash \beta(i \leftarrow \alpha) : h(c \leftarrow g(b \leftarrow f)) \multimapdotbothA a \multimapdotbothA \emptyset}{\vdash A : \beta(i \leftarrow \alpha) \multimapdotbothA h(c \leftarrow g(b \leftarrow f)) \multimapdotbothA a \multimapdotbothA \emptyset}\,\texttt{s}}\,\texttt{g-}i$$

This proof tree can be graphically represented as follows:



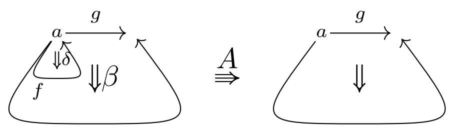
Example 6.3.4 (A degenerate case)**.** The 3-opetope



is derived as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash a : \emptyset}\,\texttt{point}}{\vdash f : a \multimapdotbothA \emptyset}\,\texttt{shift}}{\vdash \alpha : f \multimapdotbothA a \multimapdotbothA \emptyset}\,\texttt{shift} \quad \dfrac{\dfrac{}{\vdash a : \emptyset}\,\texttt{point}}{\vdash \delta : \underline{a} \multimapdotbothA a \multimapdotbothA \emptyset}\,\texttt{degen}}{\dfrac{\vdash \alpha(f \leftarrow \delta) : \underline{a} \multimapdotbothA a \multimapdotbothA \emptyset}{\vdash A : \alpha(f \leftarrow \delta) \multimapdotbothA \underline{a} \multimapdotbothA a \multimapdotbothA \emptyset}\,\texttt{shift}}\,\texttt{graft-}f$$

Example 6.3.5 (Another degenerate case)**.** The 3-opetope



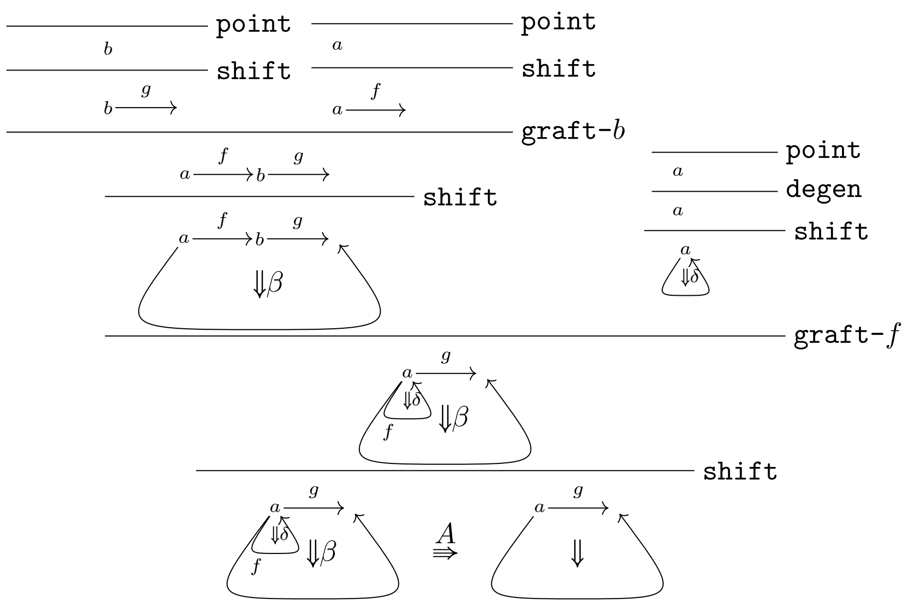
is derived as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash b:\varnothing}\,\texttt{point}}{\vdash g : b \bullet\!\!\multimap \varnothing}\,\texttt{shift} \qquad \dfrac{\dfrac{}{\vdash a:\varnothing}\,\texttt{point}}{\vdash f : a \bullet\!\!\multimap \varnothing}\,\texttt{shift}}{\vdash g(b \leftarrow f) : a \bullet\!\!\multimap \varnothing}\,\texttt{graft-}b}{\vdash \beta : g(b\leftarrow f) \bullet\!\!\multimap a \bullet\!\!\multimap \varnothing}\,\texttt{shift} \qquad \dfrac{\dfrac{\dfrac{}{\vdash a:\varnothing}\,\texttt{point}}{\vdash \underline{a} : a \bullet\!\!\multimap \varnothing}\,\texttt{degen}}{\vdash \delta : \underline{a} \bullet\!\!\multimap a \bullet\!\!\multimap \varnothing}\,\texttt{shift}}{a = b \vdash \beta(f \leftarrow \delta) : g(b\leftarrow f)[\underline{a}/f] \bullet\!\!\multimap a \bullet\!\!\multimap \varnothing}\,\texttt{graft-}f$$

and $g(b \leftarrow f)[\underline{a}/f] = g$, with the added equality $a = b$.

$$\dfrac{\begin{array}{c}\vdots\\ a = b \vdash \beta(f\leftarrow\delta) : g \bullet\!\!\multimap a \bullet\!\!\multimap\varnothing\end{array}}{a=b\vdash A : \beta(f\leftarrow\delta)\bullet\!\!\multimap g \bullet\!\!\multimap a\bullet\!\!\multimap\varnothing}\,\texttt{shift}$$

This proof tree can be graphically represented as follows:



Chapter Seven

The named systems for opetopic sets

This chapter further develops the syntax of chapter 6 in order to include opetopic sets. Recall that in $\mathrm{OPT}^!$, a derivable sequent typing a variable is an expression of of the form

$$E \triangleright \Gamma \vdash x : X,$$

where by construction, all variables in $\mathbb{V}_\Gamma - \{x\}$ appear in the type X (up to $=_E$) of x. Informally, all the variables of Γ "contribute" to the definition of x, and thus are all "faces" of x. This is analogous to a representable presheaf, where all cells derive from a unique cell (corresponding to the identity of the represented object). To describe a finite opetopic set, one would just need an expression of the form

$$E \triangleright \Gamma.$$

This time, no particular variable is the center of attention, and $(E \triangleright \Gamma)$ would need to be constructed in a more liberal manner to reflect arbitrarily complex "adjacency" relations among variables. This is the purpose of system $\mathrm{OPTSET}^!$, present in this chapter.

7.1 THE OPTSET$^!$ SYSTEM

We now present $\mathrm{OPTSET}^!$, a derivation system for opetopic sets that is based on $\mathrm{OPT}^!$ (definition 6.1.16). We first present the required syntactic constructs and conventions, then the inference rules in definition 7.1.2.

SYNTAX

An interesting aspect of the named approach is that only the source faces are specified in the type of terms:

$$x : \mathtt{s}x \multimapdotinv \mathtt{s}\mathtt{s}x \multimapdotinv \cdots$$

Nonetheless, as proven in proposition 6.2.16, all the information about targets remain. This comes from the intuition that any two opetopes with the same source are equal. In opetopic sets however, two cells with the same source faces need not be equal, nor have the same target. To adapt $\mathrm{OPT}^!$ to opetopic sets, all faces, including targets, need to be explicitly specified. This will be part of rule `repr` of system $\mathrm{OPTSET}^!$, presented in definition 7.1.2. Lastly, recall that a typical sequent in system $\mathrm{OPT}^!$ looks like this:

$$E \triangleright \Gamma \vdash t : T.$$

Here, t represents a pasting diagram who will ultimately serve as the source of a new variable, which will be introduced using rule `shift`. It does not provide any additional information about the variables of Γ and their adjacencies. Thus, when describing opetopic sets, we will drop the right hand side of the sequent, and deal with expressions of the form $(E \triangleright \Gamma)$, called *opetopic contexts modulo theory* (or OCMTs for short).

Example 7.1.1. As a preliminary example, the OCMT describing the opetopic set:

$$\begin{tikzcd}
 & b \arrow[dr, "g"] & & \\
a \arrow[ur, "f"] \arrow[rr, "h"'] & {} \arrow[u, phantom, "\Downarrow\alpha"] & a \arrow[r, bend left, "h"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\beta"] & a
\end{tikzcd}$$

is given by

$$\left(\begin{array}{l} b = \mathtt{t} f, a = \mathtt{t} g = \mathtt{t}\mathtt{t}\alpha = \mathtt{t} h = \mathtt{t}\mathtt{t}\beta \\ h = \mathtt{t}\beta = \mathtt{t}\alpha \end{array} \triangleright \begin{array}{l} a : \varnothing, b : \varnothing, \mathtt{t} f : \varnothing, \mathtt{t} g : \varnothing, \mathtt{t}\mathtt{t}\alpha : \varnothing, \\ \qquad \mathtt{t} h : \varnothing, \mathtt{t}\mathtt{t}\beta : \varnothing \\ f : a \multimapdotbothA \varnothing, g : a \multimapdotbothA \varnothing, \mathtt{t}\alpha : a \multimapdotbothA \varnothing, \\ \qquad h : a \multimapdotbothA \varnothing, \mathtt{t}\beta : a \multimapdotbothA \varnothing \\ \alpha : g(b \leftarrow f) \multimapdotbothA a \multimapdotbothA \varnothing, \beta : h \multimapdotbothA a \multimapdotbothA \varnothing \end{array} \right)$$

On the right of $\triangleright$, we have all the variables and types, much like a context in $\mathrm{O{\scriptstyle PT}}^!$. The novelties are variables of the form $\mathtt{t}x$, which indicate targets. For example, $\mathtt{t}\alpha$, the target of α, has type $a \multimapdotbothA \varnothing$, meaning that it is an arrow of source a. On the left hand side is an equational theory as in $\mathrm{O{\scriptstyle PT}}^!$, identifying variables. Note that $\mathtt{t}\alpha$ is identified with h, as shown by the diagram above. See example 7.3.2 for a full treatment of this case.

INFERENCE RULES

Our derivation system for opetopic sets, presented in definition 7.1.2, has four rules:

(1) `repr` that takes an opetope in our previous system and turns it into the representable opetopic set of that opetope by adding all the target faces;
(2) `zero` that constructs the empty OCMT, corresponding to the empty opetopic set;
(3) `sum` that takes the disjoint union of two opetopic sets;
(4) `glue` that identifies cells of an opetopic set.

By lemma 3.5.5, every finite opetopic set is a quotient of a finite sum of representables. Therefore, those rules should be enough to derive all finite opetopic sets, which is formally demonstrated in theorem 7.2.26.

Definition 7.1.2 (The OptSet$^!$ system)**.** *Introduction of all targets.* This rule takes a sequent $(E \triangleright \Gamma \vdash_n x : X)$, and completes it by adding all the targets cells, and turning it into an OCMT:

$$\frac{E \triangleright \Gamma \vdash_n x : X}{E' \triangleright \Gamma'} \, \texttt{repr}$$

where

$$\Gamma' := \Gamma \cup \left\{ \mathsf{t}^k\, a : \mathsf{s}^{k+1}\, a \multimap \mathsf{s}^{k+2}\, a \multimap \cdots \mid a \in \mathbb{V}_{\Gamma,l}, 1 \le k \le l \le n \right\},$$

and

$$\begin{aligned} E' := {} & E \\ & \cup \{ \mathsf{t}\, a = b \mid \text{for all } b \leftarrow a(\cdots) \text{ occurring in a type in } \Gamma \} && (7.1.3) \\ & \cup \left\{ \mathsf{t}\,\mathsf{t}\, a = \mathsf{t}\,\mathsf{v}_{[]}\,\mathsf{s}\, a \mid a \in \mathbb{V}_{\Gamma',k} \text{ non degen.}, 2 \le k \le n \right\} && (7.1.4) \\ & \cup \left\{ \mathsf{t}^{k+2}\, a = b \mid \text{if } \mathsf{t}^k\, a : \underline{b} \multimap b \multimap \cdots, 0 \le k \le n-2 \right\}. && (7.1.5) \end{aligned}$$

Here, $\mathsf{t}^k\, a = \mathsf{t} \cdots \mathsf{t}\, a$ can be thought of as a "tagging" on the variable $a \in \mathbb{V}_l$, but for simplicity, we consider it as a variable of its own: $\mathsf{t}^k\, a \in \mathbb{V}_{l-k}$. By convention, $\mathsf{t}^0\, a := a$, and if $a = b$, then $\mathsf{t}\, a = \mathsf{t}\, b$, for all $a, b \in \Gamma'$. In line (7.1.4), the source of a is assumed non degenerate, thus $\mathsf{s}\, a$ is a term of the form $x(\overrightarrow{y_i \leftarrow u_i})$, and $\mathsf{v}_{[]}\,\mathsf{s}\, a = x$ (see definition 6.2.2).

Zero. This rules introduces the empty OCMT:

$$\frac{}{\triangleright} \, \texttt{zero}$$

Binary sums. This rule takes two disjoint opetopic sets (i.e. whose cells have different names), and produces their sum. If $\Gamma \cap \Upsilon = \varnothing$ (which implies that $E \cap F = \varnothing$), then

$$\frac{E \triangleright \Gamma \qquad F \triangleright \Upsilon}{E, F \triangleright \Gamma, \Upsilon} \, \texttt{sum}$$

Quotients. This rule identifies two parallel cells in an opetopic set by extending the underlying equational theory. If $a, b \in \mathbb{V}_\Gamma$ are such that $\mathsf{s}\, a =_E \mathsf{s}\, b$ and $\mathsf{t}\, a =_E \mathsf{t}\, b$, then

$$\frac{E \triangleright \Gamma}{E, a = b \triangleright \Gamma} \, \texttt{glue}$$

We also write $\texttt{glue-}(a{=}b)$ to make explicit that we added $\{a = b\}$ to the theory.

Remark 7.1.6. In rule `repr`, the additional equalities of (7.1.3) enforce **(Inner)**, those of (7.1.4) enforce **(Glob1)**, and those of (7.1.5) enforce **(Degen)**. Condition **(Glob2)** is implemented in (7.1.3), by the definition of the type of the target variables $\mathtt{t}a$: the bookkeeping of the readdressing map is completely transparent, as for an n-variable x, the correspondence between the $(n-1)$-variables of $\mathtt{s}x$ and $(n-1)$-variables of $\mathtt{st}x$ is already established by their name! See also remark 8.1.17.

Remark 7.1.7. Akin to $\text{OPT}^!$, in $\text{OPTSET}^!$, an OCMT that is equivalent to a derivable one is itself derivable.

Remark 7.1.8. The `sum` and `zero` rules may be replaced by the following `usum` rule (unbiased sum) without changing the set of derivable OCMTs. For $k \geq 0$, and for $(E_1 \triangleright \Gamma_1), \ldots, (E_k \triangleright \Gamma_k)$ OCMTs such that $\Gamma_i \cap \Gamma_j = \varnothing$ for all $i \neq j$, then

$$\frac{E_1 \triangleright \Gamma_1 \qquad \cdots \qquad E_k \triangleright \Gamma_k}{E_1, \ldots, E_k \triangleright \Gamma_1, \ldots, \Gamma_k}\ \mathtt{usum}$$

7.2 EQUIVALENCE WITH OPETOPIC SETS

OPETOPIC SETS FROM OCMT

Notation 7.2.1. Let $E \triangleright \Gamma$ be an OCMTs. We write Γ/E for the set $\mathbb{V}_\Gamma$ quotiented by the equivalence relation generated by the equational theory E.

Definition 7.2.2 (OCMT of a variable)**.** For $(E \triangleright \Gamma \vdash_n x : X)$ a derivable sequent in system $\text{OPT}^!$, where $x \in \mathbb{V}_n$, let $(T_x \triangleright C_x)$, the *OCMT of* x (leaving the sequent around x implicit), be given by the following instance of `repr`:

$$\frac{E \triangleright \Gamma \vdash_n x : X}{T_x \triangleright C_x}\ \mathtt{repr}$$

We now establish a series of results to prove that C_x/T_x carries a natural structure of representable opetopic set. We proceed in 4 steps:

1. we start by noting that C_x/T_x is naturally a set over $\mathbb{O}$ via the polynomial coding map $[\![-]\!]$ (proposition 7.2.5);
2. then, in proposition 7.2.7, we construct source and target maps in C_x/T_x, i.e. the structure maps of an opetopic set;
3. we show in theorem 7.2.8 that the opetopic identities of definition 3.4.2 are satisfied, and that consequently, C_x/T_x has the structure of an opetopic set;
4. finally, we show in proposition 7.2.15 that C_x/T_x is in fact a representable opetopic set, using a counting argument.

From there, we define a structure of opetopic set on an arbitrary OCMT by induction on its proof tree in equations (7.2.17) and (7.2.18).

Definition 7.2.3. Let $(E \triangleright \Gamma \vdash_n x : X)$ be a derivable sequent on $\text{OPT}^!$, where $x \in \mathbb{V}_n$, and $a \in \mathbb{V}_{C_x,k}$. If $a \in \mathbb{V}_{\Gamma,k}$, recall that by proposition 6.1.32, a is typed by the derivable sequent $(E|_a \triangleright \Gamma|_a \vdash_k a : A)$, where $(-)|_a$ denotes restriction of contexts and theories to a and to the variables occurring in the type A. Thus we have a well-defined opetope $[\![a]\!]_k = [\![E|_a \triangleright \Gamma|_a \vdash_k a : A]\!]_k \in \mathbb{O}_k$. Otherwise, if $a = \mathsf{t}^{n-k}\,x$, then $\mathsf{s}a = \mathsf{s}^{n-k+1}\,x$, and define $[\![a]\!]_k := [\![\mathsf{s}^{n-k+1}\,x]\!]_k$. We thus have a map $[\![-]\!] : \mathbb{V}_{C_x} \longrightarrow \mathbb{O}$.

Lemma 7.2.4. *Let $(E \triangleright \Gamma \vdash_n x : X)$ be a derivable sequent on $\text{O}_{\text{PT}}^{!}$, where $x \in \mathbb{V}_n$, and $a \in \mathbb{V}_{C_x,k}$. Then the opetope $[\![a]\!]_k$ of definition 7.2.3 does not depend on the proof tree of $(E \triangleright \Gamma \vdash_n x : X)$.*

Proof. If $a \in \mathbb{V}_{\Gamma,k}$, then $[\![a]\!]_k = [\![E|_a \triangleright \Gamma|_a \vdash_k a : A]\!]_k$, which does not depend on the proof tree of $(E|_a \triangleright \Gamma|_a \vdash_k a : A)$ which by definition (see definition 6.2.4) does not depend on any proof tree. If $a = \mathsf{t}^{n-k} x$, then $[\![a]\!]_k = [\![\mathsf{s}^{n-k+1} x]\!]_k$ which does not depend on the proof tree of $(E \triangleright \Gamma \vdash_n x : X)$ either. □

Proposition 7.2.5. *The map $[\![-]\!] : \mathbb{V}_{C_x} \longrightarrow \mathbb{O}$ factors through C_x/T_x.*

Proof. By construction, the theory T_x identifies variables $a, b \in \mathbb{V}_{C_x,k}$ only if $\mathsf{s}\, a = \mathsf{s}\, b$, thus $[\![a]\!]_k = [\![\mathsf{s}\, a]\!]_k = [\![\mathsf{s}\, b]\!]_k = [\![b]\!]_k$. □

Definition 7.2.6. For $\psi \in \mathbb{O}_k$, write

$$(C_x/T_x)_\psi \;=\; \left\{ a \in \mathbb{V}_{C_x,k} \mid [\![a]\!]_k = \psi \right\}.$$

We now construct source and target maps between those subsets.

Sources. If $[p] \in [\![a]\!]_k^\bullet$, then by corollary 6.2.17, there is a unique $b \in \mathbb{V}_{C_x|_a,k-1}$ such that $\&_{\mathsf{s}\, a}\, b = [p]$. Let $\mathsf{v}_{[p]}\, a \coloneqq b$.

Target. For $a \in \mathbb{V}_{C_x,k}$, $k > 0$, we of course set $\mathsf{t}(a) \coloneqq \mathsf{t}\, a$, the latter being a variable introduced by the `repr` rule.

Proposition 7.2.7. *Let $a \in \mathbb{V}_{C_x,k}$.*

(1) For $[p] \in [\![a]\!]_k^\bullet$ we have $[\![\mathsf{v}_{[p]}\, \mathsf{s}\, a]\!]_{k-1} = \mathsf{s}_{[p]}\, [\![a]\!]_k$.

(2) We have $[\![\mathsf{t}\, a]\!]_{k-1} = \mathsf{t}\, [\![a]\!]_k$.

Proof. (1) It a is not a target i.e. $a \neq \mathsf{t}\, b$ for any $b \in \mathbb{V}_{C_x,k}$, then this is already proven by proposition 6.2.11. If $a = \mathsf{t}^l\, b$ for some $b \in \mathbb{V}_{C_x,k+l}$ that is not a target, and $l \in \mathbb{N}$, then

$$\begin{aligned}
[\![\mathsf{v}_{[p]}\, \mathsf{s}\, a]\!]_{k-1} &= [\![\mathsf{v}_{[p]}\, \mathsf{s}\, \mathsf{t}^l\, b]\!]_{k-1} && \\
&= [\![\mathsf{v}_{[p]}\, \mathsf{s}^{l+1}\, b]\!]_{k-1} && \text{see definition of } \texttt{repr} \\
&= \mathsf{s}_{[p]}\, [\![\mathsf{s}\, \mathsf{s}^l\, b]\!]_k && \text{by proposition 6.2.11} \\
&= \mathsf{s}_{[p]}\, [\![\mathsf{s}\, a]\!]_k && \\
&= \mathsf{s}_{[p]}\, [\![a]\!]_k && \text{see convention 6.2.1.}
\end{aligned}$$

(2) If a is not a target, then

$$\begin{aligned}
\mathsf{t}\, [\![a]\!]_k &= \mathsf{t}\, [\![\mathsf{s}\, a]\!]_k && \text{see convention 6.2.1} \\
&= [\![\mathsf{s}\, \mathsf{s}\, a]\!]_{k-1} && \text{by proposition 6.2.16} \\
&= [\![\mathsf{s}\, \mathsf{t}\, a]\!]_{k-1} && \text{see definition of } \texttt{repr} \\
&= [\![\mathsf{t}\, a]\!]_{k-1} && \text{see convention 6.2.1.}
\end{aligned}$$

If $a = \mathsf{t}^l\, b$ for some $b \in \mathbb{V}_{C_x,k+l}$ that is not a target, and $l \in \mathbb{N}$, then

$$\mathsf{t}\, [\![a]\!]_k \;=\; \mathsf{t}\, [\![\mathsf{s}\, a]\!]_k \qquad \text{see convention 6.2.1}$$

$$
\begin{aligned}
&= \mathsf{t}\left[\!\left[\mathsf{s}\,\mathsf{t}^{l}\,b\right]\!\right]_k \\
&= \mathsf{t}\left[\!\left[\mathsf{s}^{l+1}\,b\right]\!\right]_k && \text{see definition of } \texttt{repr} \\
&= \left[\!\left[\mathsf{s}^{l+2}\,b\right]\!\right]_{k-1} \\
&= \mathsf{t}\left[\!\left[\mathsf{s}\,\mathsf{t}^{l+1}\,b\right]\!\right]_k && \text{see definition of } \texttt{repr} \\
&= [\![\mathsf{s}\,\mathsf{t}\,a]\!]_{k-1} \\
&= [\![\mathsf{t}\,a]\!]_{k-1} && \text{see convention 6.2.1.}
\end{aligned}
$$

□

Theorem 7.2.8. *With all the structure of definition 7.2.6, C_x/T_x is an opetopic set.*

Proof. We check the opetopic identities of definition 3.4.2. Take $a \in \mathbb{V}_{C_x,k}$,

(Inner) Take $[p[q]] \in [\![a]\!]^{\bullet}_k$, and write $d = \mathsf{v}_{[p[q]]}\,\mathsf{s}\,a$. In $\mathsf{s}\,a$, the variable d occurs as

$$\mathsf{s}\,a \;=\; \cdots, b(c \leftarrow d), \cdots$$

where $b = \mathsf{v}_{[p]}\,\mathsf{s}\,a$ and $c = \mathsf{v}_{[q]}\,\mathsf{s}\,b$. By equation (7.1.3), $\mathsf{v}_{[q]}\,\mathsf{v}_{[p]}\,a = \mathsf{v}_{[q]}\,b = c = \mathsf{t}\,d = \mathsf{t}\,\mathsf{v}_{[p[q]]}\,a$.

(Glob1) Assume that $\mathsf{s}\,a$ is not degenerate. Then, by equation (7.1.4), we have $\mathsf{t}\,\mathsf{t}\,a = \mathsf{t}\,\mathsf{v}_{[]}\,a$.

(Glob2) Assume that $\mathsf{s}\,a$ is not degenerate, take $[p[q]] \in [\![a]\!]^{|}_k$, and let $c := \mathsf{v}_{[q]}\,\mathsf{v}_{[p]}\,a$. Then

$$
\begin{aligned}
\wp_{[\![a]\!]_k}[p[q]] &= \wp_{[\![\mathsf{s}\,a]\!]_k}[p[q]] && \text{see convention 6.2.1} \\
&= \wp_{[\![\mathsf{s}\,a]\!]_k}\,\&_{\mathsf{s}\,a}\,c && \text{by definition} \\
&= \&_{\mathsf{s}\,\mathsf{s}\,a}\,c && \text{by lemma 6.2.14} \\
&= \&_{\mathsf{s}\,\mathsf{t}\,a}\,c && \text{see definition of } \texttt{repr}.
\end{aligned}
$$

and thus $\mathsf{v}_{[q]}\,\mathsf{v}_{[p]}\,a = c = \mathsf{v}_{\&_{\mathsf{s}\,\mathsf{t}\,a}\,c}\,\mathsf{t}\,a = \mathsf{v}_{\wp_{[\![a]\!]_k}[p[q]]}\,\mathsf{t}\,a$.

(Degen) Assume that $\mathsf{s}\,a$ is degenerate, say $\mathsf{s}\,a = \underline{b}$. Then by equation (7.1.5), $\mathsf{v}_{[]}\,\mathsf{t}\,a = b = \mathsf{t}\,\mathsf{t}\,a = \mathsf{t}^0\,a = a$. □

Lemma 7.2.9. *The opetopic set C_x/T_x is a quotient of the representable $O[[\![x]\!]_n]$.*

Proof. The category of elements $\mathbb{O}/(C_x/T_x)$ of C_x/T_x is a poset since $\mathbb{O}$ is a directed category. It has a unique maximal element, namely the equivalence class of variable x itself. Moreover, that element has shape $x^{\natural} = [\![x]\!]_n$. By the Yoneda lemma, there is a map $O[[\![x]\!]_n] \longrightarrow C_x/T_x$ having cell x in its image, and since x the maximum, this map is surjective. □

Let $(E \triangleright \Gamma \vdash_n x : X)$ be a derivable sequent, with $x \in \mathbb{V}_n$. In lemma 7.2.9, we established that C_x/T_x is a quotient of the representable opetopic set $O[[\![x]\!]_n]$. We now aim to show that the two are actually isomorphic (proposition 7.2.15) by showing that they have the same number of cells. Recall the cell counting function $\#$ from definition 0.3.2. Since all the slices of $\mathbb{O}$ are finite, $\#\omega = \#O[\omega]$ is a finite number, for all $\omega \in \mathbb{O}$.

Lemma 7.2.10. *For $\omega \in \mathbb{O}$ an opetope, we have $\#\omega = 2 + \#S[\omega]$ (definition 3.5.1).*

Proof. Follows from remark 3.5.2. □

Proposition 7.2.11. *(1) We have* $\# \blacklozenge = 1$, *and* $\# \blacksquare = 3$.

(2) If ω *is an endotope, say* $\omega = \mathsf{Y}_\psi$, *then* $\#\omega = 2 + \#\psi$.

(3) If ω *is a degenerate opetope, say* $\omega = \mathsf{I}_\phi$, *then* $\#\omega = 2 + \#\phi$.

(4) If $\omega = \nu \circ_{[l]} \mathsf{Y}_\psi$, *for some* $\nu \in \mathbb{O}_n$, $[l] \in \nu^{|}$, *and* $\psi \in \mathbb{O}_{n-1}$, *then* $\#\omega = \#\nu + \#\psi - \# \mathsf{e}_{[l]}\, \nu$.

Proof. The first point is clear. If $\omega = \mathsf{Y}_\psi$, then $S[\omega] = O[\psi]$, thus by lemma 7.2.10, $\#\omega = 2 + \#O[\psi] = 2 + \#\psi$. If $\omega = \mathsf{I}_\phi$, then $S[\omega] = O[\phi]$, thus $\#\mathsf{I}_\phi = 2 + \#\phi$. If $\omega = \nu \circ_{[l]} \mathsf{Y}_\psi$, then by lemma 3.5.8, $S[\omega] = S[\nu] \coprod_{O[\mathsf{e}_{[l]}\, \nu]} O[\psi]$, thus $\#\omega = 2 + (\#\nu - 2) + \#\psi - \# \mathsf{e}_{[l]}\, \nu = \#\nu + \#\psi - \# \mathsf{e}_{[l]}\, \nu$. □

Corollary 7.2.12. *If* $\omega \in \mathbb{O}_{\geq 2}$ *is not degenerate, then*

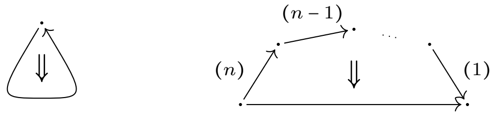
$$\#\omega = 2 + \left(\sum_{[p] \in \omega^\bullet} \# \mathsf{s}_{[p]}\, \omega \right) - \left(\sum_{[p[q]] \in \omega^\bullet} \# \mathsf{s}_{[q]}\, \mathsf{s}_{[p]}\, \omega \right)$$

Proof. If ω is an endotope, the result is already proved in proposition 7.2.11. Otherwise, decompose ω as $\nu \circ_{[l]} \mathsf{Y}_\psi$, and assume that by induction, the result holds for ν. We have

$$\begin{aligned}
\#\omega &= \#\nu + \#\psi - \# \mathsf{e}_{[l]}\, \nu && \spadesuit \\
&= 2 + \left(\#\psi + \sum_{[p] \in \nu^\bullet} \# \mathsf{s}_{[p]}\, \nu \right) - \left(\# \mathsf{e}_{[l]}\, \nu + \sum_{[p[q]] \in \nu^\bullet} \# \mathsf{s}_{[q]}\, \mathsf{s}_{[p]}\, \nu \right) && \diamondsuit \\
&= 2 + \left(\# \mathsf{s}_{[l]}\, \omega + \sum_{\substack{[p] \in \omega^\bullet \\ [p] \neq [l]}} \# \mathsf{s}_{[p]}\, \omega \right) - \left(\# \mathsf{e}_{[l]}\, \omega + \sum_{\substack{[p[q]] \in \omega^\bullet \\ [p[q]] \neq [l]}} \# \mathsf{s}_{[q]}\, \mathsf{s}_{[p]}\, \omega \right) \\
&= 2 + \left(\sum_{[p] \in \omega^\bullet} \# \mathsf{s}_{[p]}\, \omega \right) - \left(\sum_{[p[q]] \in \omega^\bullet} \# \mathsf{s}_{[q]}\, \mathsf{s}_{[p]}\, \omega \right),
\end{aligned}$$

where $\spadesuit$ is by proposition 7.2.11, and $\diamondsuit$ is by induction. □

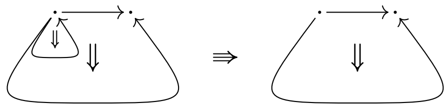
Example 7.2.13. Consider the opetopic integer $\mathbf{n} \in \mathbb{O}_2$ from example 3.1.4:



We show that $\#\mathbf{n} = 2n + 3$. If $n = 0$, then $\#\mathbf{0} = \#\mathsf{I}_\blacklozenge = 2 + \# \blacklozenge = 3$. This can be read on the graphical representation of $\mathbf{0}$, that has one point, one simple arrow, and one double arrow, for a total of 3 cells. If $n = 1$, then $\#\mathbf{1} = \#\mathsf{Y}_\blacksquare = 2 + \# \blacksquare = 5$. Otherwise,

$$\begin{aligned}
\#\mathbf{n} &= \# \left((\mathbf{n} - \mathbf{1}) \underset{[*^{n-1}]}{\circ} \mathsf{Y}_\blacksquare \right) && \text{by definition of } \mathbf{n} \\
&= \#(\mathbf{n} - \mathbf{1}) + \# \blacksquare - \# \mathsf{e}_{[*^{n-1}]}(\mathbf{n} - \mathbf{1}) && \text{by proposition 7.2.11} \\
&= (2n + 1) + 3 - \# \blacklozenge && \text{by induction} \\
&= (2n + 1) + 3 - 1 = 2n + 3.
\end{aligned}$$

Example 7.2.14. Consider the 3-opetope $\omega = \mathsf{Y}_{\mathbf{2}} \circ_{[[*]]} \mathsf{Y}_{\mathbf{0}}$ of example 6.3.5:



Then,

$$\begin{aligned}
\#\omega &= \#\left(\mathsf{Y}_{\mathbf{2}} \underset{[[*]]}{\circ} \mathsf{Y}_{\mathbf{0}}\right) & \\
&= \#\mathsf{Y}_{\mathbf{2}} + \#\mathbf{0} - \#\,\mathsf{e}_{[[*]]}\,\mathsf{Y}_{\mathbf{2}} & \text{by proposition 7.2.11} \\
&= 2 + \#\mathbf{2} + \#\mathbf{0} - \#\,\blacksquare & \text{by proposition 7.2.11} \\
&= 9 & \text{since } \#\mathbf{n} = 2n+3.
\end{aligned}$$

Proposition 7.2.15. *We have* $C_x/T_x \cong O[\llbracket x \rrbracket_n]$.

Proof. (1) If $x = \blacklozenge$, then $\mathbb{V}_{C_x} = \blacklozenge$, while $T_x = \varnothing$. Thus $\#C_x/T_x = 1 = \#\blacklozenge$ by proposition 7.2.11. We know by lemma 7.2.9 that C_x/T_x is a quotient of $O[\llbracket x \rrbracket_n]$, and we just showed that the two have the same number of cells, namely $\#\blacklozenge = 1$. Consequently, $C_x/T_x \cong O[\llbracket x \rrbracket_0]$.

(2) Likewise, if $x = \blacksquare$, then $\mathbb{V}_{C_x} = \{\blacklozenge, \blacksquare, \mathsf{t}\,\blacksquare\}$, while $T_x = \varnothing$. Thus $\#C_x/T_x = 3 = \#\blacksquare$ by proposition 7.2.11, and by the argument above, $C_x/T_x \cong O[\llbracket x \rrbracket_1]$.

(3) Assume now that $x \in \mathbb{V}_n$ for $n \geq 2$. We proceed by case analysis on the form of $\mathsf{s}\,x$.

a) If $\mathsf{s}\,x = y \in \mathbb{V}_{n-1}$, then $\llbracket x \rrbracket_n = \mathsf{Y}_{\llbracket y \rrbracket_{n-1}}$, and by proposition 7.2.11, $\#\llbracket x \rrbracket_n = 2 + \#\llbracket y \rrbracket_{n-1}$. Then $C_x = C_y + \left\{\mathsf{t}^k\,x \mid 0 \leq k \leq n\right\}$, and $\mathsf{t}\,\mathsf{t}\,x =_{T_x} \mathsf{t}\,\mathsf{v}_{[]}\,x = \mathsf{t}\,y$. Consequently, T_x is equivalent to the theory $T_y + \{\mathsf{t}\,\mathsf{t}\,x = \mathsf{t}\,y\}$, and thus

$$C_x/T_x \cong C_y/T_y + \{x, \mathsf{t}\,x\}\,.$$

By induction, $C_y/T_y \cong O[\llbracket y \rrbracket_{n-1}]$, and $\#C_x/T_x = 2 + \#C_y/T_y = 2 + \#\llbracket y \rrbracket_{n-1} = \#\llbracket x \rrbracket_n$, which, by the same argument as above, proves the isomorphism $C_x/T_x \cong O[\llbracket x \rrbracket_n]$.

b) If $\mathsf{s}\,x = \underline{a}$ for some $a \in \mathbb{V}_{n-2}$, then $\llbracket x \rrbracket_n = \mathsf{I}_{\llbracket a \rrbracket_{n-2}}$, and by proposition 7.2.11, $\#\llbracket x \rrbracket_n = 2 + \#\llbracket a \rrbracket_{n-2}$. Then $C_x = C_a + \left\{\mathsf{t}^k\,x \mid 0 \leq k \leq n\right\}$, and $\mathsf{t}\,\mathsf{t}\,x =_{T_x} a$. Therefore T_x is equivalent to the theory $T_a + \{\mathsf{t}\,\mathsf{t}\,x = a\}$, and thus

$$C_x/T_x \cong C_a/T_a + \{x, \mathsf{t}\,x\}\,.$$

Consequently, $\#C_x/T_x = 2 + \#C_a/T_a = 2 + \#\llbracket a \rrbracket_{n-2} = \#\llbracket x \rrbracket_n$.

c) Assume $\mathsf{s}\,x = t(a \leftarrow y)$, for some $t \in \mathbb{T}_{n-1}$, $a \in \mathbb{V}_{n-2}$, and $y \in \mathbb{V}_{n-1}$. Let $z : t \multimap \cdots$ be a fresh n-variable. Clearly, $C_z - \{z, \mathsf{t}\,z\} \subseteq C_x$, thus

$$C_x = C_y \cup \left(C_z - \left\{\mathsf{t}^k\,z \mid 0 \leq k \leq n\right\}\right) + \left\{\mathsf{t}^k\,x \mid 0 \leq k \leq n\right\},$$

while T_x is equivalent to $T_y \cup T_z + \{\mathsf{t}\,y = a, \mathsf{t}\,\mathsf{t}\,x = \mathsf{t}\,\mathsf{v}_{[]}\,x\}$. Since $\mathsf{v}_{[]}\,\mathsf{s}\,x = \mathsf{v}_{[]}\,t = \mathsf{v}_{[]}\,\mathsf{s}\,z$, and $\mathsf{t}\,\mathsf{v}_{[]}\,\mathsf{s}\,z =_{T_z} \mathsf{t}\,\mathsf{t}\,z$ (see equation (7.1.4)), we have

$$C_x/T_x = C_y/T_y \cup C_z/T_z + \{x, \mathsf{t}\,x\} - \{z, \mathsf{t}\,z\}.$$

By hypothesis of the `graft` rule, $C_z/T_z \cap C_y/T_y = C_a/T_a$, and thus we have

$$\begin{aligned}
\#C_x/T_x &= \#C_y/T_y + \#C_z/T_z - \#C_a/T_a \\
&= \#\llbracket y \rrbracket_{n-1} + \#\llbracket z \rrbracket_n - \#\llbracket a \rrbracket_{n-2} \\
&= \#\left(\llbracket t \rrbracket_n \underset{\&_t a}{\circ} \mathsf{Y}_{\llbracket y \rrbracket_{n-1}} \right) && \text{by proposition 7.2.11} \\
&= \#\llbracket x \rrbracket_n .
\end{aligned}$$

□

We now extend the structure of opetopic set defined in definition 7.2.6 to all OCMTs.

Definition 7.2.16. Let $(E \triangleright \Gamma)$ and $(F \triangleright \Upsilon)$ be two OCMTs, and assume by induction that Γ/E and Υ/F have a structure of opetopic set.

(1) If $(G \triangleright \Xi)$ is given by

$$\frac{E \triangleright \Gamma \qquad F \triangleright \Upsilon}{G \triangleright \Xi}\ \texttt{sum}$$

then we have $\Xi = \Gamma + \Upsilon$, and $G = E + F$. We endow the quotient Ξ/G with a structure of opetopic set as follows:

$$\Xi/G := \Gamma/E + \Upsilon/F. \tag{7.2.17}$$

(2) Let $a, b \in \mathbb{V}_{\Gamma,k}$ be such that $\mathsf{s}a =_E \mathsf{s}b$ and $\mathsf{t}a =_E \mathsf{t}b$. Note that $O[\llbracket a \rrbracket_k] = O[\llbracket b \rrbracket_k]$. Then, by definition of rule `glue`, and for $(F \triangleright \Gamma)$ given by

$$\frac{E \triangleright \Gamma}{F \triangleright \Gamma}\ \texttt{glue-}(a{=}b)$$

we have $F = E + \{a = b\}$. We endow the quotient Γ/F with a structure of opetopic set defined by the following coequalizer:

$$O[\llbracket a \rrbracket_k] \overset{a}{\underset{b}{\rightrightarrows}} \Gamma/E \longrightarrow \Gamma/F, \tag{7.2.18}$$

Proposition 7.2.19. *For $(E \triangleright \Gamma)$ a derivable OCMT in $\mathcal{O}\mathrm{PT}\mathcal{S}\mathrm{ET}^!$, the structure of opetopic set on Γ/E does not depend on the proof tree of $(E \triangleright \Gamma)$.*

Proof. If $\Gamma = \varnothing$, i.e. if the OCMT is obtained using the `zero` rule, then the result trivially holds. Otherwise, it is easy to see that the opetopic set Γ/E is given by the following expression that does not depend on the proof tree of $(E \triangleright \Gamma)$:

$$\Gamma/E \cong \frac{\sum_{k \in \mathbb{N}, a \in \mathbb{V}_{\Gamma,k}} O[\llbracket a \rrbracket_k]}{a \sim b, \text{ for all } a, b \in \mathbb{V}_\Gamma \text{ s.t. } a =_E b.}$$

By lemma 7.2.4, for $a \in \mathbb{V}_\Gamma$, the opetope $\llbracket a \rrbracket_k$ does not depend on any proof tree either.

□

Recall that $\mathcal{P}\mathrm{sh}(\mathbb{O})_{\mathrm{fin}}$ is the full subcategory of $\mathcal{P}\mathrm{sh}(\mathbb{O})$ spanned by finite opetopic sets. In this subsection, we provide the last results needed to establish the equivalence between the category of derivable OCMTs and $\mathcal{P}\mathrm{sh}(\mathbb{O})_{\mathrm{fin}}$.

Notation 7.2.20. For $(E \triangleright \Gamma)$ an OCMT, and $a, b \in \mathbb{V}$, the substitution $\Gamma[a/b]$ is defined in the obvious manner, by applying it to all typings in Γ.

Definition 7.2.21 (Morphism of OCMTs)**.** Let $(E \triangleright \Gamma)$ and $(F \triangleright \Upsilon)$ be OCMTs. A morphism $f : (E \triangleright \Gamma) \longrightarrow (F \triangleright \Upsilon)$ is a (non necessarily bijective) map $f : \mathbb{V}_\Gamma \longrightarrow \mathbb{V}_\Upsilon$ compatible with E and F, such that if $x : X$ is a typing in Γ, then $f(x) : f(X)$ is a typing in Υ, where $f(X)$ is the result of applying f to every variable in X. Further, we require that for $n \geq 1$ and $x \in \mathbb{V}_{\Gamma,n}$, we have $f(\mathsf{t}\, x) = \mathsf{t}\, f(x)$. Note that this condition implies that f preserves the dimension of variables. Also, if $f, g : (E \triangleright \Gamma) \longrightarrow (F \triangleright \Upsilon)$, and if for all $x \in \mathbb{V}_\Gamma$ we have $f(x) =_F g(x)$, then we consider f and g to be equivalent, and only consider maps up to equivalence.

Lemma 7.2.22. *Morphisms of OCMTs preserve the shape of variables, i.e. for $f : (E \triangleright \Gamma) \longrightarrow (F \triangleright \Upsilon)$ a morphism and $a \in \mathbb{V}_{\Gamma,k}$, we have $[\![a]\!]_k = [\![f(a)]\!]_k$.*

Proof. Since there is a unique 0-opetope and a unique 1-opetope, the result holds trivially if $k = 0, 1$. If $k \geq 2$, we proceed by induction on $\mathsf{s}\, a$.

(1) If $\mathsf{s}\, a = b \in \mathbb{V}_{k-1}$, then $[\![a]\!]_k = \mathsf{Y}_{[\![b]\!]_{k-1}} = \mathsf{Y}_{[\![f(b)]\!]_{k-1}} = [\![f(a)]\!]_k$.
(2) If $\mathsf{s}\, a = \underline{b}$ for some $b \in \mathbb{V}_{k-2}$, then $[\![a]\!]_k = \mathsf{I}_{[\![b]\!]_{k-2}} = \mathsf{I}_{[\![f(b)]\!]_{k-2}} = [\![f(a)]\!]_{k-2}$.
(3) If $\mathsf{s}\, a = b(\overrightarrow{c_i \leftarrow u_i})$, then by induction, $[\![u_i]\!]_k = [\![f(u_i)]\!]_k$, and

$$[\![a]\!]_k = \mathsf{Y}_{[\![b]\!]_{k-1}} \bigcirc_{[\&_{\mathsf{s}\, b}\, c_i]} [\![u_i]\!]_k = [\![f(a)]\!]_k .$$

□

Definition 7.2.23. Let $\mathcal{C}\mathrm{tx}^!$ for the category of derivable OCMTs and such morphisms. In a sense, it is the syntactic category of system $\textsc{OptSet}^!$.

Definition 7.2.24 (Named stratification functor)**.** The *named stratification functor* $S^! : \mathcal{C}\mathrm{tx}^! \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O})_{\mathrm{fin}}$ is defined as follows:

$$\begin{aligned} S^! : \mathcal{C}\mathrm{tx}^! &\longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O})_{\mathrm{fin}} \\ (E \triangleright \Gamma) &\longmapsto \Gamma/E \\ \Big((E \triangleright \Gamma) \xrightarrow{f} (F \triangleright \Upsilon)\Big) &\longmapsto \Big(\Gamma/E \xrightarrow{f} \Upsilon/F\Big). \end{aligned}$$

Proposition 7.2.25. *Let $f : (E \triangleright \Gamma) \longrightarrow (F \triangleright \Upsilon)$ be a morphism of OCMTs. Then the map $S^! f$ of definition 7.2.24 is indeed a morphism of opetopic sets.*

Proof. By definition, the cells of Γ/E are exactly the variables of $\mathbb{V}_\Gamma$, and likewise for Υ/F. By lemma 7.2.22, $S^! f$ preserves the shapes of the cells. By definition if x is an n-variable, and $x : X$ a typing in Γ, then $f(x) : f(X)$ is a typing in Υ. So for $\omega := [\![x]\!]_n$ and $[p] \in \omega^\bullet$ the following naturality square commutes:

$$\begin{tikzcd}
(\Gamma/E)_\omega \arrow[r, "S^! f_\omega"] \arrow[d, "\mathsf{s}_{[p]}"'] & (\Upsilon/F)_\omega \arrow[d, "\mathsf{s}_{[p]}"] \\
(\Gamma/E)_{\mathsf{s}_{[p]}\,\omega} \arrow[r, "S^! f_{\mathsf{s}_{[p]}\,\omega}"] & (\Upsilon/F)_{\mathsf{s}_{[p]}\,\omega}.
\end{tikzcd}$$

By definition again, if $n \geq 1$, then $f(\mathsf{t}\,x) = \mathsf{t}\,f(x)$, so the analogous naturality square for target embeddings also commutes. Finally, $S^! f$ is a natural transformation. $\square$

Theorem 7.2.26. *The stratification functor $S^! : \mathcal{C}\mathrm{tx}^! \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O})_{\mathrm{fin}}$ is an equivalence of categories.*

Proof. The full subcategory of $\mathcal{P}\mathrm{sh}(\mathbb{O})_{\mathrm{fin}}$ spanned by the essential image of $S^!$ contains all the representables opetopic sets (proposition 7.2.15), the initial object (since $S^!(\triangleright)$ is the opetopic set with no cell), and is closed under finite sums and quotients (definition 7.2.16). Thus it is finitely cocomplete, and equal to the whole category $\mathcal{P}\mathrm{sh}(\mathbb{O})_{\mathrm{fin}}$, so $S^!$ is essentially surjective. By definition, $S^!$ is also faithful, and it remains to show that is it full.

Let $f : \Gamma/E \longrightarrow \Upsilon/F$ be a morphism of opetopic sets. Then, in particular, it is a map between the set of cells of Γ/E and Υ/F. To prove that it is a morphism of OCMT, we show that $\Gamma[f(x)/x \mid x \in \mathbb{V}_\Gamma]$ (see notation 7.2.20) is a subcontext of Υ modulo F, i.e. that for every typing $x : X$ in Γ, for some $x \in \mathbb{V}_k$, the type of $f(x)$ in Υ is $f(X)$ modulo F. If $(E \triangleright \Gamma)$ is the empty OCMT, the result is trivial. Let $x : X$ be a typing in Γ, with $x \in \mathbb{V}_k$. Since f is a morphism of opetopic sets, we have $f(x) \in \mathbb{V}_{\Upsilon,k}$, and by lemma 7.2.22, $[\![x]\!]_k = [\![f(x)]\!]_k$. We show that the type of $f(x)$ in Υ is $f(X)$ by induction on k.

(1) If $k = 0$, then $X = \varnothing$. Since $f(x) \in \mathbb{V}_{\Upsilon,0}$, its type is necessarily $\varnothing = f(X)$, thus $f(x) : f(X)$ is a typing in Υ.
(2) If $k = 1$, then $X = (a \multimap \varnothing)$ where $a = \mathsf{v}_{[]}\,x$ in Γ/E, and since f is a morphism of opetopic sets, $f(\mathsf{v}_{[]}\,x) =_F \mathsf{v}_{[]}\,f(x)$. Thus

$$f(X) \;=\; \big(f(\mathsf{v}_{[]}\,x) \multimap \varnothing\big) \;=_F\; \big(\mathsf{v}_{[]}\,f(x) \multimap \varnothing\big),$$

the latter being the type of $f(x)$ in Υ.
(3) Assume now that $k \geq 2$. The type of x is $X = (\mathsf{s}\,x \multimap \mathsf{s}\mathsf{s}\,x \multimap \cdots \multimap \varnothing)$, and by definition, the type of $\mathsf{t}\,x$ is $Y := (\mathsf{s}\mathsf{s}\,x \multimap \cdots \multimap \varnothing)$ (see equation (7.1.3)). By induction, the type of $f(\mathsf{t}\,x)$ in Υ is $f(Y)$, and since $f(\mathsf{t}\,x) =_F \mathsf{t}\,f(x)$, and the type of the latter is $(\mathsf{s}\mathsf{s}\,f(x) \multimap \cdots \multimap \varnothing)$, we have

$$\big(f(\mathsf{s}\mathsf{s}\,x) \multimap \cdots \multimap \varnothing\big) \;=\; f(Y) \;=_F\; \big(\mathsf{s}\mathsf{s}\,f(x) \multimap \cdots \multimap \varnothing\big),$$

or in other words, $\mathsf{s}^i\,f(x) =_F f(\mathsf{s}^i\,x)$, for $2 \leq i \leq k$. It remains to show that the latter formula holds in the case $i = 1$. Towards a contradiction, assume $\mathsf{s}\,f(x) \neq_F f(\mathsf{s}\,x)$. Then there exists $[p] \in [\![x]\!]_k^\bullet = [\![f(x)]\!]_k^\bullet$ such that $\mathsf{v}_{[p]}\,f(x) \neq_F f(\mathsf{v}_{[p]}\,x)$, which contradicts the fact that f is a morphism of opetopic sets. Consequently, $\mathsf{s}\,f(x) =_F f(\mathsf{s}\,x)$, and $f(X)$ is the type of $f(x)$ in Υ modulo F.

Finally, the underlying map of $f : \Gamma/E \longrightarrow \Upsilon/F$ is a morphism of OCMT, and $S^!$ is full. $\square$

Proposition 7.2.27. *The category* $\mathcal{C}\mathrm{tx}^!$ *has finite colimits, and* $S^!$ *preserves them.*

Proof. The empty OCMT is clearly an initial object of $\mathcal{C}\mathrm{tx}^!$, and the OCMT $(E, F \triangleright \Gamma, \Upsilon)$ obtained by rule `sum` (definition 7.1.2) is clearly a coproduct of $(E \triangleright \Gamma)$ and $(F \triangleright \Upsilon)$. Let now $f, g : (E \triangleright \Gamma) \longrightarrow (F \triangleright \Upsilon)$ be two parallel morphism in $\mathcal{C}\mathrm{tx}^!$. Consider the map

$$(F \triangleright \Upsilon) \xrightarrow{x \mapsto x} (G \triangleright \Upsilon),$$

where $G := F \cup \{f(x) = g(x) \mid x \in \mathbb{V}_\Gamma\}$. Then, $G \triangleright \Upsilon$ is derived from $F \triangleright \Upsilon$ by repeated application of the `glue` rule, and by universal property, it is easy to see that it is a coequalizer of f and g in $\mathcal{C}\mathrm{tx}^!$. So, to summarize, $\mathcal{C}\mathrm{tx}^!$ contains all finite sums and coequalizers, and it is thus finitely cocomplete. The fact that $S^!$ preserves finite colimits is trivial in the case of initial objects, a consequence of equation (7.2.17) for binary sums, and of equation (7.2.18) for coequalizers. □

If $\mathcal{C}$ and $\mathcal{D}$ are small categories with finite limits, recall from theorem 0.5.6 that $\mathcal{L}\mathrm{EX}(\mathcal{C}, \mathcal{D})$ is the category of left exact (i.e. finite limit preserving) functors from $\mathcal{C}$ to $\mathcal{D}$, and natural transformations.

Theorem 7.2.28. *We have an equivalence* $\mathcal{P}\mathrm{sh}(\mathbb{O}) \simeq \mathcal{L}\mathrm{EX}((\mathcal{C}\mathrm{tx}^!)^{\mathrm{op}}, \mathcal{S}\mathrm{et})$.

Proof. This follows directly from theorem 7.2.26 and proposition 7.2.27 and from the Gabriel–Ulmer duality (see corollary 0.5.7). □

7.3 EXAMPLES

In this section, we give example derivations in system $\mathrm{O{\scriptsize PT}S{\scriptsize ET}}^!$. For clarity, we do not repeat the type of previously typed variables in proof trees.

Example 7.3.1. The opetopic set

$$\begin{tikzcd} a \arrow[r, "g"] \arrow[r, bend left=70, "f"] \arrow[r, phantom, bend left=35, "\Downarrow\alpha"] & b \arrow[l, bend left=70, "h"] \end{tikzcd}$$

is derived as follows. First, we derive the cells α, g, and h as opetopes (i.e. in $\mathrm{O{\scriptsize PT}S{\scriptsize ET}}^!$) to obtain the following sequents:

$$\triangleright a : \varnothing, f : a \multimap \varnothing, \alpha : f \multimap a \multimap \varnothing \vdash_2 \alpha : f \multimap a \multimap \varnothing$$
$$\triangleright c : \varnothing, g : c \multimap \varnothing \vdash_1 g : c \multimap \varnothing$$
$$\triangleright b : \varnothing, h : b \multimap \varnothing \vdash_1 h : b \multimap \varnothing$$

and applying the `repr` rule yields respectively:

$$\triangleright\, a : \varnothing, f : a \multimap \varnothing, \alpha : f \multimap a \multimap \varnothing, \mathsf{t} f : \varnothing, \mathsf{t}\alpha : a \multimap \varnothing, \mathsf{t}\mathsf{t}\alpha : \varnothing$$
$$\triangleright\, c : \varnothing, g : c \multimap \varnothing, \mathsf{t}\, g : \varnothing$$
$$\triangleright\, b : \varnothing, h : b \multimap \varnothing, \mathsf{t}\, h : \varnothing.$$

The proof tree then reads:

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{\vdots}{\triangleright a, f, \alpha \vdash_2 \alpha}}{\mathtt{t}\mathtt{t}\alpha = \mathtt{t}f \triangleright a, f, \alpha, \mathtt{t}f, \mathtt{t}\alpha, \mathtt{t}\mathtt{t}\alpha}\,\texttt{repr}
\qquad
\dfrac{\dfrac{\vdots}{\triangleright c, g \vdash_1 g}}{\triangleright c, g, \mathtt{t}g}\,\texttt{repr}
}{\mathtt{t}\mathtt{t}\alpha = \mathtt{t}f \triangleright a, f, \alpha, \mathtt{t}f, \mathtt{t}\alpha, \mathtt{t}\mathtt{t}\alpha, c, g, \mathtt{t}g}\,\texttt{sum}
\qquad
\dfrac{\dfrac{\vdots}{\triangleright b, h \vdash_1 h}}{\triangleright b, h, \mathtt{t}h}\,\texttt{repr}
}{
\mathtt{t}\mathtt{t}\alpha = \mathtt{t}f \quad \triangleright \quad \begin{array}{l} a, b, c, \mathtt{t}f, \mathtt{t}g, \mathtt{t}h, \mathtt{t}\mathtt{t}\alpha \\ f, g, h, \mathtt{t}\alpha \\ \alpha \end{array}
}\,\texttt{sum}
$$

$$
\dfrac{
\mathtt{t}\mathtt{t}\alpha = \mathtt{t}f \quad \triangleright \quad \begin{array}{l} a, b, c, \mathtt{t}f, \mathtt{t}g, \mathtt{t}h, \mathtt{t}\mathtt{t}\alpha \\ f, g, h, \mathtt{t}\alpha \\ \alpha \end{array}
}{
\mathtt{t}\mathtt{t}\alpha = \mathtt{t}f, a = c \quad \triangleright \quad \begin{array}{l} a, b, c, \mathtt{t}f, \mathtt{t}g, \mathtt{t}h, \mathtt{t}\mathtt{t}\alpha \\ f, g, h, \mathtt{t}\alpha \\ \alpha \end{array}
}\,\texttt{glue-}(a = c)
$$

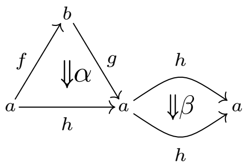
$$
\dfrac{\cdots}{
\mathtt{t}\mathtt{t}\alpha = \mathtt{t}f, a = c, b = \mathtt{t}f \quad \triangleright \quad \begin{array}{l} a, b, c, \mathtt{t}f, \mathtt{t}g, \mathtt{t}h, \mathtt{t}\mathtt{t}\alpha \\ f, g, h, \mathtt{t}\alpha \\ \alpha \end{array}
}\,\texttt{glue-}(b = \mathtt{t}f)
$$

$$
\dfrac{\cdots}{
\mathtt{t}\mathtt{t}\alpha = \mathtt{t}f, a = c, b = \mathtt{t}f, b = \mathtt{t}g \quad \triangleright \quad \begin{array}{l} a, b, c, \mathtt{t}f, \mathtt{t}g, \mathtt{t}h, \mathtt{t}\mathtt{t}\alpha \\ f, g, h, \mathtt{t}\alpha \\ \alpha \end{array}
}\,\texttt{glue-}(b = \mathtt{t}g)
$$

$$
\dfrac{\cdots}{
\mathtt{t}\mathtt{t}\alpha = \mathtt{t}f, a = c, b = \mathtt{t}f, b = \mathtt{t}g, a = \mathtt{t}h \quad \triangleright \quad \begin{array}{l} a, b, c, \mathtt{t}f, \mathtt{t}g, \mathtt{t}h, \mathtt{t}\mathtt{t}\alpha \\ f, g, h, \mathtt{t}\alpha \\ \alpha \end{array}
}\,\texttt{glue-}(a = \mathtt{t}h)
$$

$$
\dfrac{\cdots}{
\begin{array}{r} \mathtt{t}\mathtt{t}\alpha = \mathtt{t}f, a = c, b = \mathtt{t}f, b = \mathtt{t}g, a = \mathtt{t}h \\ g = \mathtt{t}\alpha \end{array} \quad \triangleright \quad \begin{array}{l} a, b, c, \mathtt{t}f, \mathtt{t}g, \mathtt{t}h, \mathtt{t}\mathtt{t}\alpha \\ f, g, h, \mathtt{t}\alpha \\ \alpha \end{array}
}\,\texttt{glue-}(g = \mathtt{t}\alpha)
$$

Example 7.3.2. The opetopic set



is derived as follows. First, we derive the cells α and β as opetopes to obtain the following sequents:

$$
\left(\triangleright \begin{array}{l} a : \varnothing, b : \varnothing \\ f : a \longleftarrow\!\circ\, \varnothing, g : a \longleftarrow\!\circ\, \varnothing \\ \alpha : g(b \leftarrow f) \longleftarrow\!\circ\, a \longleftarrow\!\circ\, \varnothing \end{array} \quad \vdash_2 \quad \alpha : g(b \leftarrow f) \longleftarrow\!\circ\, a \longleftarrow\!\circ\, \varnothing, \right.
$$

$$
\left. \triangleright \begin{array}{l} a' : \varnothing \\ h : a' \longleftarrow\!\circ\, \varnothing \\ \beta : h \longleftarrow\!\circ\, a' \longleftarrow\!\circ\, \varnothing \end{array} \quad \vdash_2 \quad \beta : h \longleftarrow\!\circ\, a' \longleftarrow\!\circ\, \varnothing \right).
$$

Applying the $\texttt{repr}$ rule yields respectively:

$$
\left(b = \mathtt{t}f, \mathtt{t}g = \mathtt{t}\mathtt{t}\alpha \quad \triangleright \quad \begin{array}{l} a : \varnothing, b : \varnothing, \mathtt{t}f : \varnothing, \mathtt{t}g : \varnothing, \mathtt{t}\mathtt{t}\alpha : \varnothing \\ f : a \longleftarrow\!\circ\, \varnothing, g : a \longleftarrow\!\circ\, \varnothing, \mathtt{t}\alpha : a \longleftarrow\!\circ\, \varnothing \\ \alpha : g(b \leftarrow f) \longleftarrow\!\circ\, a \longleftarrow\!\circ\, \varnothing \end{array} , \right.
$$

$$
\left. \mathtt{t}h = \mathtt{t}\mathtt{t}\beta \quad \triangleright \quad \begin{array}{l} a' : \varnothing, \mathtt{t}h : \varnothing, \mathtt{t}\mathtt{t}\beta : \varnothing \\ h : a' \longleftarrow\!\circ\, \varnothing, \mathtt{t}\beta : a' \longleftarrow\!\circ\, \varnothing \\ \beta : h \longleftarrow\!\circ\, a' \longleftarrow\!\circ\, \varnothing \end{array} \right)
$$

The proof tree then reads:

$$
\dfrac{
\dfrac{
\dfrac{
\begin{array}{c}\vdots\\ \triangleright\ \begin{array}{l} a:\varnothing, b:\varnothing\\ f: a \bullet\!\!\multimap \varnothing, g: a\bullet\!\!\multimap\varnothing\\ \alpha: g(b\leftarrow f)\bullet\!\!\multimap a\bullet\!\!\multimap\varnothing\end{array}\quad \vdash_2 \alpha : g(b\leftarrow f)\bullet\!\!\multimap a\bullet\!\!\multimap\varnothing\end{array}
}{
b=\mathtt{t}f,\mathtt{t}g=\mathtt{tt}\alpha\ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f:\varnothing,\mathtt{t}g:\varnothing,\mathtt{tt}\alpha:\varnothing\\ f,g,\mathtt{t}\alpha: a\bullet\!\!\multimap\varnothing\\ \alpha\end{array}
}\ \texttt{repr}
}{
b=\mathtt{t}f, a=\mathtt{t}g=\mathtt{tt}\alpha\ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{tt}\alpha\\ f,g,\mathtt{t}\alpha\\ \alpha\end{array}
}\ \texttt{glue-}(a=\mathtt{t}g)
\qquad
\dfrac{
\dfrac{
\dfrac{
\begin{array}{c}\vdots\\ \triangleright\ \begin{array}{l} a':\varnothing\\ h: a'\bullet\!\!\multimap\varnothing\\ \beta: h\bullet\!\!\multimap a'\bullet\!\!\multimap\varnothing\end{array}\quad \vdash_2 \beta : h\bullet\!\!\multimap a'\bullet\!\!\multimap\varnothing\end{array}
}{
\mathtt{t}h=\mathtt{tt}\beta\ \triangleright\ \begin{array}{l} a',\mathtt{t}h:\varnothing,\mathtt{tt}\beta:\varnothing\\ h,\mathtt{t}\beta: a'\bullet\!\!\multimap\varnothing\\ \beta\end{array}
}\ \texttt{repr}
}{
\begin{array}{l}\mathtt{t}h=\mathtt{tt}\beta\\ h=\mathtt{t}\beta\end{array}\ \triangleright\ \begin{array}{l} a',\mathtt{t}h,\mathtt{tt}\beta\\ h,\mathtt{t}\beta\\ \beta\end{array}
}\ \texttt{glue-}(h=\mathtt{t}\beta)
}{
\begin{array}{l}a'=\mathtt{t}h=\mathtt{tt}\beta\\ h=\mathtt{t}\beta\end{array}\ \triangleright\ \begin{array}{l} a',\mathtt{t}h,\mathtt{tt}\beta\\ h,\mathtt{t}\beta\\ \beta\end{array}
}\ \texttt{glue-}(a'=\mathtt{t}h)
}{
\dfrac{
\dfrac{
\begin{array}{l}b=\mathtt{t}f, a=\mathtt{t}g=\mathtt{tt}\alpha, a'=\mathtt{t}h=\mathtt{tt}\beta\\ h=\mathtt{t}\beta\end{array}\ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{tt}\alpha,a',\mathtt{t}h,\mathtt{tt}\beta\\ f,g,\mathtt{t}\alpha,h,\mathtt{t}\beta\\ \alpha,\beta\end{array}
}{
\begin{array}{l}b=\mathtt{t}f, a=\mathtt{t}g=\mathtt{tt}\alpha=a'=\mathtt{t}h=\mathtt{tt}\beta\\ h=\mathtt{t}\beta\end{array}\ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{tt}\alpha,a',\mathtt{t}h,\mathtt{tt}\beta\\ f,g,\mathtt{t}\alpha,h,\mathtt{t}\beta\\ \alpha,\beta\end{array}
}\ \texttt{glue-}(a=a')
}{
\begin{array}{l}b=\mathtt{t}f, a=\mathtt{t}g=\mathtt{tt}\alpha=a'=\mathtt{t}h=\mathtt{tt}\beta\\ h=\mathtt{t}\beta=\mathtt{t}\alpha\end{array}\ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{tt}\alpha,a',\mathtt{t}h,\mathtt{tt}\beta\\ f,g,\mathtt{t}\alpha,h,\mathtt{t}\beta\\ \alpha,\beta\end{array}
}\ \texttt{glue-}(\mathtt{t}\alpha=h)
}\ \texttt{sum}
$$

Write $(E \triangleright \Gamma)$ for the final OCMT of this proof tree. At the beginning of section 7.1, we gave a seemingly different OCMT for the same opetopic set:

$$
\left(\begin{array}{l}b=\mathtt{t}f, a=\mathtt{t}g=\mathtt{tt}\alpha=\mathtt{t}h=\mathtt{tt}\beta\\ h=\mathtt{t}\beta=\mathtt{t}\alpha\end{array}\quad \triangleright\quad \begin{array}{c} a:\varnothing, b:\varnothing, \mathtt{t}f:\varnothing, \mathtt{t}g:\varnothing, \mathtt{tt}\alpha:\varnothing,\\ \mathtt{t}h:\varnothing, \mathtt{tt}\beta:\varnothing\\ f: a\bullet\!\!\multimap\varnothing, g: a\bullet\!\!\multimap\varnothing, \mathtt{t}\alpha: a\bullet\!\!\multimap\varnothing,\\ h: a\bullet\!\!\multimap\varnothing, \mathtt{t}\beta: a\bullet\!\!\multimap\varnothing\\ \alpha: g(b\leftarrow f)\bullet\!\!\multimap a\bullet\!\!\multimap\varnothing, \beta: h\bullet\!\!\multimap a\bullet\!\!\multimap\varnothing\end{array}\right).
$$

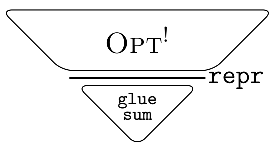
But this α-equivalent to $(E \triangleright \Gamma)$, as instances of a' have been replaced by a, which are both equal according to the equational theory E.

7.4 THE MIXED SYSTEM FOR OPETOPIC SETS

The $\textsc{OptSet}^!$ system, presented in section 7.1, suffers from the following drawback: derivations of opetopic sets start with instances of rules `zero` or `repr`, the latter requiring a full opetope derivation in system $\textsc{Opt}^!$ (presented in section 6.1). This makes derivations somewhat unintuitive, since for an opetopic set $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$ written as

$$X = \frac{\sum_i O[\omega_i]}{\sim}$$

where $\sim$ represents some quotient, the opetopes ω_i have to be derived in $\textsc{Opt}^!$ first, then the `repr` rule has to be used on each one to produce the corresponding representables $O[\omega_i]$, and only then can the sums and gluing be performed:



In this section, we present system $\textsc{OptSet}^!_{\textsc{m}}$ (the M standing for "mixed") for opetopic sets, which does not depend on $\textsc{Opt}^!$, and allows to perform introductions of new cells, sums, and gluings in any sound order. This is done by introducing new cells along with all their targets, effectively rendering $\mathrm{OPTSET}^!$'s `repr` rule superfluous, and removing the "barrier" between $\mathrm{OPT}^!$ and $\mathrm{OPTSET}^!$ in the schema above.

SYNTAX

The syntax of system $\mathrm{OPTSET}^!_{\mathrm{M}}$ uses sequents from $\mathrm{OPT}^!$ (see section 6.1) and OCMTs from $\mathrm{OPTSET}^!$. Specifically, we use two types of judgments.

(1) "$E \triangleright \Gamma$", stating that $E \triangleright \Gamma$ is a well formed OCMT.

(2) "$E \triangleright \Gamma \vdash t : T$", stating that in the OCMT $E \triangleright \Gamma$, the term t is well formed, and has type T. We may also write "$E \triangleright \Gamma \vdash_n t : T$" to emphasize that $t \in \mathbb{T}_n$.

INFERENCE RULES

We present the inference rules of system $\mathrm{OPTSET}^!_{\mathrm{M}}$ in definition 7.4.1. It uses rules `point`, and `graft` from system $\mathrm{OPT}^!$, and rules `zero`, `sum`, and `glue` from system $\mathrm{OPTSET}^!$. As we will see, the system deals with two types of judgments:

(1) "$E \triangleright \Gamma$", stating that $(E \triangleright \Gamma)$ is a well-formed OCMT;

(2) "$E \triangleright \Gamma \vdash t : T$", stating that in $(E \triangleright \Gamma)$, the term t is well-defined, and has type T.

Two new rules, `degen` and `pd`, will go from the first type of judgment to the second, by introducing degenerate terms and single variable terms respectively. In the other direction, rule `shift` is a variant of that of system $\mathrm{OPT}^!$, and from $(E \triangleright \Gamma \vdash t : T)$, introduces a new cell having t as source, along with all the necessary targets. It can be viewed as a fusion of $\mathrm{OPT}^!$'s `shift` rule and $\mathrm{OPTSET}^!$'s `repr` rule.

Definition 7.4.1 (The $\mathrm{OPTSET}^!_{\mathrm{M}}$ system). *Introduction of points.* This rule introduces 0-cells, also called points. If $x \in \mathbb{V}_0$, then

$$\frac{}{\triangleright x : \varnothing}\ \texttt{point}$$

Introduction of degenerate pasting diagrams. This rule creates a new degenerate pasting diagram. If $x \in \mathbb{V}_{\Gamma,k}$, then

$$\frac{E \triangleright \Gamma, x : X}{E \triangleright \Gamma, x : X \vdash_{k+1} \underline{x} : x \bullet\!\!-\!\!\circ\, X}\ \texttt{degen}$$

Introduction of non degenerate pasting diagrams. This rule creates a new non-degenerate pasting diagram consisting of a single cell. It can then be extended using the `graft` rule. If $x \in \mathbb{V}_{\Gamma,k}$, then

$$\frac{E \triangleright \Gamma, x : X}{E \triangleright \Gamma, x : X \vdash_k x : X}\ \texttt{pd}$$

Grafting. This rule extends a previously derived non degenerate pasting diagram by grafting a cell. With the same conditions as rule `graft` of system $\mathrm{OPT}^!$ (see section 6.1), if $x \in \mathbb{V}_n$, $t \in \mathbb{T}_n$ is not degenerate, $a \in (\mathsf{s}\,t)^\bullet$ is such that $\mathsf{s}\,a = \mathsf{s}\mathsf{s}\,x$, then

$$\frac{E \triangleright \Gamma \vdash_n t : s_1 \leftarrow\!\!\circ\; s_2 \leftarrow\!\!\circ \cdots \qquad F \triangleright \Upsilon, x : X}{G \triangleright \Gamma \cup \Upsilon \vdash_n t(a \leftarrow x) : s_1[\mathsf{s}\, x/a] \leftarrow\!\!\circ\; s_2 \leftarrow\!\!\circ \cdots}\ \texttt{graft}$$

where G is the union of E, F, and potentially a set of additional equalities incurred by the substitution $s_1[\mathsf{s}\, x/a]$ (definition 6.1.19). We also write `graft-`a to make explicit the fact that we grafted onto a.

Shifting of pasting diagrams. This rule takes a previously derived pasting diagram (degenerate or not), and introduces a new cell having this pasting diagram as source. It also introduces the targets of all its iterated sources, and extends the ambient equational theory with the required identities, in the same fashion as rule `repr` of definition 6.1.16. If $x \in \mathbb{V}_{n+1}$ is such that $x \notin \mathbb{V}_\Gamma$, then

$$\frac{E \triangleright \Gamma \vdash_n t : T}{F \triangleright \Upsilon}\ \texttt{shift}$$

where

$$\Upsilon := \Gamma \cup \{x : t \leftarrow\!\!\circ\; T\} \cup \left\{\mathsf{t}^i\, x : \mathsf{s}^{i+1}\, x \leftarrow\!\!\circ\; \mathsf{s}^{i+2}\, x \leftarrow\!\!\circ \cdots \mid 0 < i \leq n\right\}$$

by convention, we let $\mathsf{t}^0\, x = x$, and where F is defined as follows:

(1) if t is a degenerate term, say $t = \underline{a}$, then

$$F := E \cup \left\{\mathsf{t}^{i+2}\, x = \mathsf{t}^i\, a \mid 0 \leq i \leq n-1\right\} \tag{7.4.2}$$

(2) if t is not degenerate, say $t = y(\overrightarrow{z_i \leftarrow u_i})$, for some $y \in \mathbb{V}_n$, $\overrightarrow{z_i} \in \mathbb{V}_{n-1}$, and $\overrightarrow{u_i} \in \mathbb{T}_n$, then

$$\begin{aligned} F := E& \\ &\cup \left\{\mathsf{t}^2\, x = \mathsf{t}\, y \mid \text{if } n \geq 1\right\} \\ &\cup \left\{\mathsf{t}\, a = b \mid \text{for all } b \leftarrow a(\cdots) \text{ occurring in } t\right\}. \end{aligned}$$

Zero. This rules introduces the empty OCMT.

$$\frac{}{\triangleright}\ \texttt{zero}$$

Binary sums. This rule takes two disjoint OCMTs (i.e. whose cells have different names), and produces their sum. If $\Gamma \cap \Upsilon = \varnothing$, then

$$\frac{E \triangleright \Gamma \qquad F \triangleright \Upsilon}{E, F \triangleright \Gamma, \Upsilon}\ \texttt{sum}$$

Quotients. This rule identifies two parallel cells in an opetopic set by extending the underlying equational theory. If $a, b \in \mathbb{V}_\Gamma$ are such that $\mathsf{s}\, a =_E \mathsf{s}\, b$ and $\mathsf{t}\, a =_E \mathsf{t}\, b$, then

$$\frac{E \triangleright \Gamma}{E, a = b \triangleright \Gamma}\ \texttt{glue}$$

We also write `glue-(`a=b`)` to make explicit that we added $a = b$ to the theory.

Remark 7.4.3. The `sum` and `zero` rules may be replaced by the following `usum` rule (unbiased sum) without changing the set of derivable OCMTs. For $k \geq 0$, and for $(E_1 \triangleright \Gamma_1), \ldots, (E_k \triangleright \Gamma_k)$ OCMTs such that $\Gamma_i \cap \Gamma_j = \varnothing$ for all $i \neq j$, then

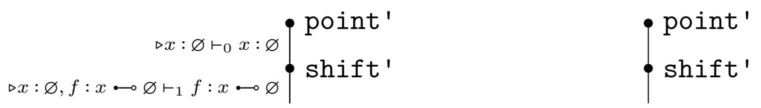
$$\frac{E_1 \triangleright \Gamma_1 \qquad \cdots \qquad E_k \triangleright \Gamma_k}{E_1, \ldots, E_k \triangleright \Gamma_1, \ldots, \Gamma_k} \, \texttt{usum}$$

Remark 7.4.4. Akin to $\mathrm{O{\scriptstyle PT}}^!$ and $\mathrm{O{\scriptstyle PT}S{\scriptstyle ET}}^!$, in $\mathrm{O{\scriptstyle PT}S{\scriptstyle ET}}^!_{\mathrm{M}}$ a sequent or an OCMT that is equivalent to a derivable one is itself derivable.

7.5 EQUIVALENCE WITH OPETOPIC SETS

The aim of this section is to prove theorem 7.5.5, stating that system $\mathrm{O{\scriptstyle PT}S{\scriptstyle ET}}^!_{\mathrm{M}}$ precisely derives opetopic sets, in the sense of theorems 7.2.26 and 7.2.28. In other words, we prove that the set of derivable OCMTs of systems $\mathrm{O{\scriptstyle PT}S{\scriptstyle ET}}^!_{\mathrm{M}}$ and $\mathrm{O{\scriptstyle PT}S{\scriptstyle ET}}^!$ are the same. This is done by rewriting proof trees in $\mathrm{O{\scriptstyle PT}S{\scriptstyle ET}}^!$ to proof trees in $\mathrm{O{\scriptstyle PT}S{\scriptstyle ET}}^!_{\mathrm{M}}$ (see proposition 7.5.2) and conversely (see proposition 7.5.4).

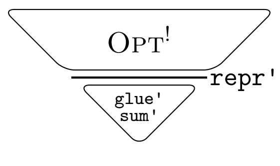
Convention 7.5.1. Throughout this section, the rules of systems $\mathrm{O{\scriptstyle PT}}^!$ and $\mathrm{O{\scriptstyle PT}S{\scriptstyle ET}}^!$ will be decorated by a prime, e.g. `shift'`, in order to differentiate them from the rules of system $\mathrm{O{\scriptstyle PT}S{\scriptstyle ET}}^!_{\mathrm{M}}$. Further, to make notations lighter and the demonstrations more graphical, we write proof trees as actual trees, whose nodes are decorated by rules, and edges by sequents or OCMTs. For instance, derivation of the arrow ▪ (see example 6.3.1) in system $\mathrm{O{\scriptstyle PT}}^!$ is represented as on the left, and more concisely as on the right:



If no uncertainty arise, we leave the decoration of the edges implicit, as on the right.

Proposition 7.5.2. *Every OCMT derivable in system* $\mathrm{O{\scriptstyle PT}S{\scriptstyle ET}}^!$ *is also derivable in system* $\mathrm{O{\scriptstyle PT}S{\scriptstyle ET}}^!_{\mathrm{M}}$.

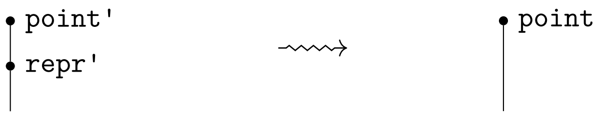
Proof. Recall that a proof tree in system $\mathrm{O{\scriptstyle PT}S{\scriptstyle ET}}^!$ has the following structure:



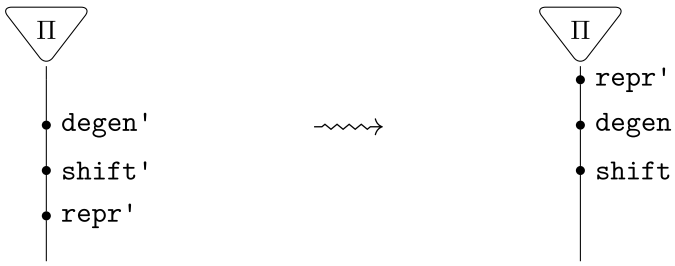
meaning that it begins with derivations in system $\mathrm{O{\scriptstyle PT}}^!$, followed by instances of the `repr'` rule, followed by a derivation in system $\mathrm{O{\scriptstyle PT}S{\scriptstyle ET}}^!$. Remark that rule `glue'` is exactly `glue`, and likewise for `sum`, so that the bottom part of the proof tree is already a derivation in system $\mathrm{O{\scriptstyle PT}S{\scriptstyle ET}}^!_{\mathrm{M}}$.

We now show that we can rewrite the top part to a proof in system $\mathrm{O{\scriptstyle PT}S{\scriptstyle ET}}^!_{\mathrm{M}}$ by "moving up " the instances of rule `repr`, and replacing the other rule instances by those of $\mathrm{O{\scriptstyle PT}S{\scriptstyle ET}}^!_{\mathrm{M}}$. This rewriting procedure is defined by the following cases.

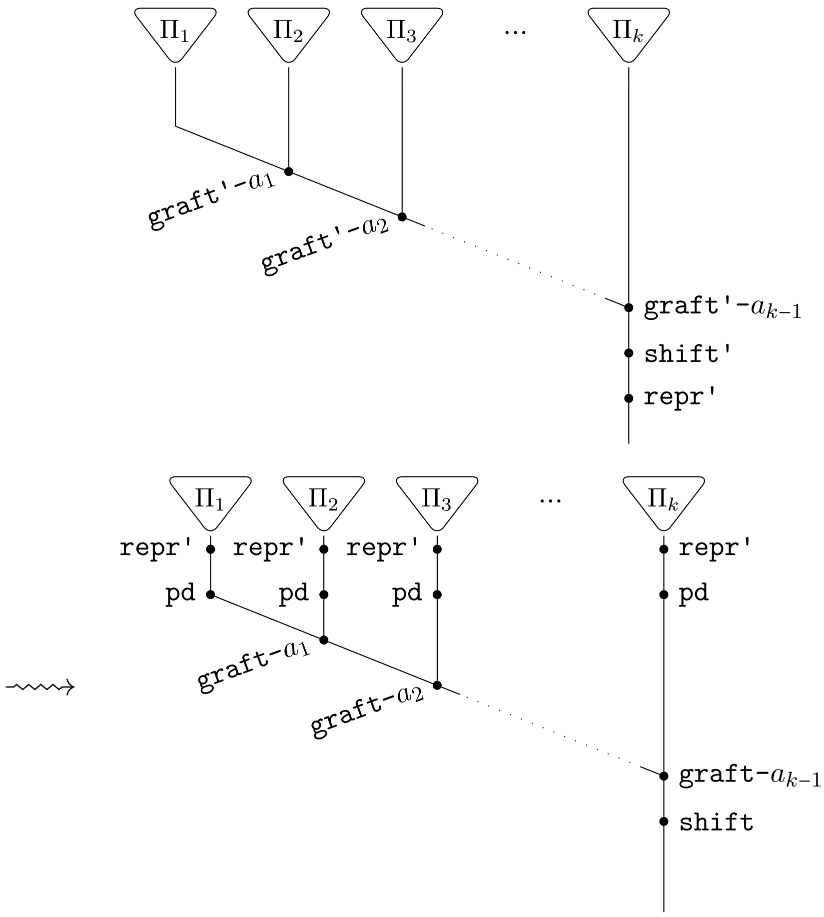
(1) If we have a proof tree as on the left, we rewrite it as on the right:



(2) If we have a derivation as on the left, where Π is a proof tree in system $\textsc{Opt}^!$ or $\textsc{OptSet}^!$, then we rewrite it as on the right:



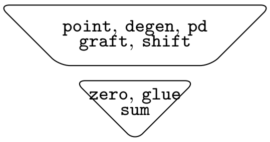
(3) If we have a derivation as on the top, where $\Pi_1, \ldots, \Pi_k$ are proof trees in system $\textsc{Opt}^!$ or $\textsc{OptSet}^!$, then we rewrite it as below:



Here, the new instances of `pd` picks the adequate variables from each sequent, so that they can be used by the instances of `graft`. Once the grafting process is complete, rule `shift` adds all necessary targets, which was previously done by `repr'`.

It is routine verification to check that the conclusion OCMT on the left and the right of any of those cases are the same. This rewriting procedure is convergent (i.e. confluent and terminating), and the normal form of a proof tree in system $\textsc{OptSet}^!$ is a proof tree in system $\textsc{OptSet}^!_{\mathrm{M}}$ that derives the same OCMT. $\square$

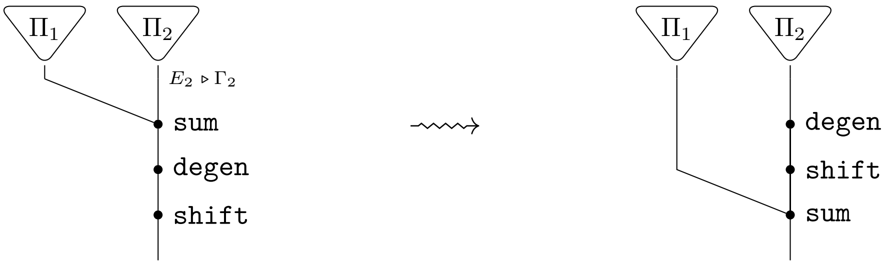
Lemma 7.5.3. *Let $(E \triangleright \Gamma)$ be a derivable OCMT in system $\textsc{OptSet}^!_{\mathrm{M}}$. Then it admits a proof tree of the following form*



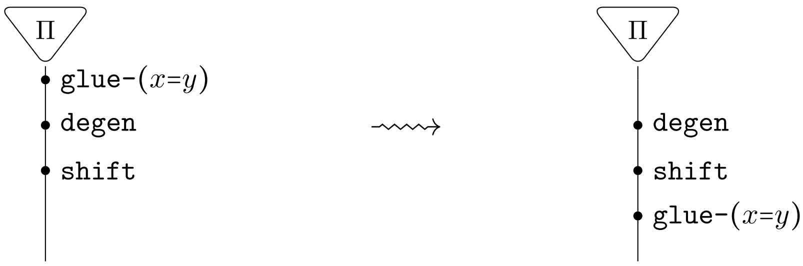
meaning a proof tree starting with derivation in the fragment of system $\textsc{OptSet}^!_{M}$ containing only rules `point`*,* `degen`*,* `pd`*,* `graft`*, and* `shift`*, followed by a derivation in the complementary fragment.*

Proof. If we have a proof tree consisting only of an instance of rule `zero`, then the result trivially holds. Otherwise, we proceed by stating rewriting steps of proof trees in system $\textsc{OptSet}^!_{\mathrm{M}}$, as in the proof of proposition 7.5.2.

(1) If we have a proof tree as on the left, and assuming the instance of `degen` degenerates a variable in Γ_2, we rewrite it as on the right:

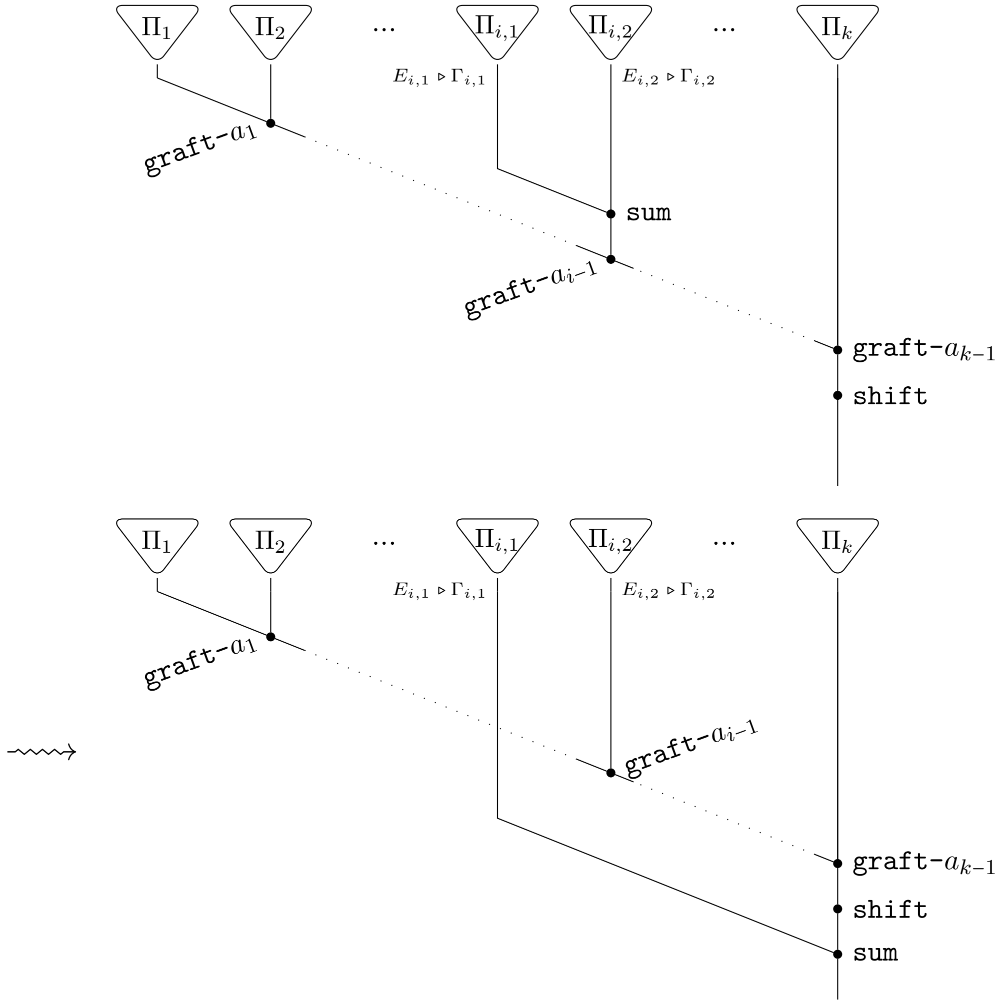


(2) A proof tree as on the left is rewritten as on the right:

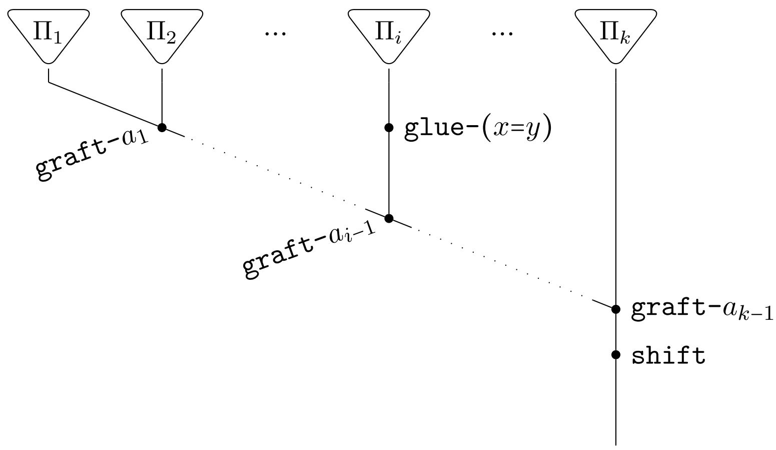


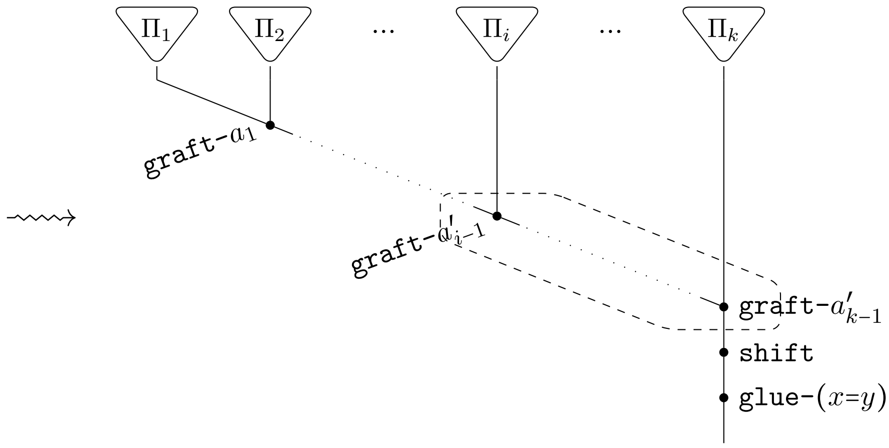
(3) Consider a proof tree as on the top. Then, by assumption on rule `sum`, either $a_{i-1} \in \mathbb{V}_{\Gamma_1}$ or $a_{i-1} \in \mathbb{V}_{\Gamma_2}$. Without loss of generality, assume the latter holds. Then

we rewrite the proof tree as below:



(4) Consider a proof tree as on the top, and rewrite it as below, x.





where in the circled area, for $i - 1 \leq j \leq k - 1$,

$$a'_j = \begin{cases} x & \text{if } a_j = y \\ a_j & \text{otherwise.} \end{cases}$$

In other words, the uses of y in the circled area have been replaced by uses of x. $\square$

Proposition 7.5.4. *Every OCMT derivable in system* $\text{OPTSET}^!_{\text{M}}$ *is also derivable in system* $\text{OPTSET}^!$.

Proof. Consider a proof tree in system $\text{OPTSET}^!_{\text{M}}$. Then it can be rewritten so as to have structure described in lemma 7.5.3. Applying the rewriting steps of proposition 7.5.2 in reverse direction yields a proof tree in systems $\text{OPT}^!$ and $\text{OPTSET}^!$ that derives the same OCMT. $\square$

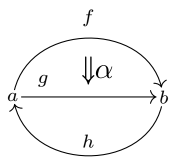
Theorem 7.5.5. *System* $\text{OPTSET}^!_{\text{M}}$ *derives opetopic sets, in the sense of theorems 7.2.26 and 7.2.28.*

Proof. By propositions 7.5.2 and 7.5.4, the OCMTs derived by system $\text{OPTSET}^!_{\text{M}}$ and $\text{OPTSET}^!$ are the same. $\square$

7.6 EXAMPLES

In this section, we give example derivations in system $\text{OPTSET}^!_{\text{M}}$. For clarity, we do not repeat the type of previously typed variables in proof trees.

Example 7.6.1. The opetopic set



of example 7.3.1 can be derived as follows. The first half of the proof tree is on the left, and the second half on the right. Moreover, for clarity, we do not repeat the typing of previously typed variables

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$\texttt{point}$}
\UnaryInfC{$\triangleright\ a : \emptyset$}
\RightLabel{$\texttt{pd}$}
\UnaryInfC{$\triangleright\ a \ \vdash_0 \ a$}
\RightLabel{$\texttt{shift}$}
\UnaryInfC{$\triangleright \begin{array}{l} a, \mathsf{t} f : \emptyset \\ f : a \bullet\!\!-\!\!\circ\, \emptyset \end{array}$}
\RightLabel{$\texttt{pd}$}
\UnaryInfC{$\triangleright \begin{array}{l} a, \mathsf{t} f \\ f \end{array} \ \vdash_0 \ a$}
\RightLabel{$\texttt{shift}$}
\UnaryInfC{$\triangleright \begin{array}{l} a, \mathsf{t} f \\ f, g : a \bullet\!\!-\!\!\circ\, \emptyset \end{array}$}
\AxiomC{}
\RightLabel{$\texttt{point}$}
\UnaryInfC{$\triangleright\ b : \emptyset$}
\RightLabel{$\texttt{pd}$}
\UnaryInfC{$\triangleright\ b \ \vdash_0 \ b$}
\RightLabel{$\texttt{shift}$}
\UnaryInfC{$\triangleright \begin{array}{l} b, \mathsf{t} h : \emptyset \\ h : b \bullet\!\!-\!\!\circ\, \emptyset \end{array}$}
\RightLabel{$\texttt{sum}$}
\BinaryInfC{$\triangleright \begin{array}{l} a, b, \mathsf{t} f, \mathsf{t} g, \mathsf{t} h \\ f, g, h \end{array}$}
\RightLabel{$(b{=}\mathsf{t} f)$}
\UnaryInfC{$b = \mathsf{t} f \ \triangleright \begin{array}{l} a, b, \mathsf{t} f, \mathsf{t} g, \mathsf{t} h \\ f, g, h \end{array}$}
\RightLabel{$(b{=}\mathsf{t} g)$}
\UnaryInfC{$b = \mathsf{t} f = \mathsf{t} g \ \triangleright \begin{array}{l} a, b, \mathsf{t} f, \mathsf{t} g, \mathsf{t} h \\ f, g, h \end{array}$}
\RightLabel{$\texttt{pd}$}
\UnaryInfC{$b = \mathsf{t} f = \mathsf{t} g \ \triangleright \begin{array}{l} a, b, \mathsf{t} f, \mathsf{t} g, \mathsf{t} h \\ f, g, h \end{array} \ \vdash_1 \ f$}
\RightLabel{$\texttt{shift}$}
\UnaryInfC{$b = \mathsf{t} f = \mathsf{t} g = \mathsf{t}^2 \alpha \ \triangleright \begin{array}{l} a, b, \mathsf{t} f, \mathsf{t} g, \mathsf{t} h, \mathsf{t}^2 \alpha : \emptyset \\ f, g, h, \mathsf{t} \alpha : a \bullet\!\!-\!\!\circ\, \emptyset \\ \alpha : f \bullet\!\!-\!\!\circ\, a \bullet\!\!-\!\!\circ\, \emptyset \end{array}$}
\RightLabel{$(g{=}\mathsf{t} \alpha)$}
\UnaryInfC{$\begin{array}{l} b = \mathsf{t} f = \mathsf{t} g = \mathsf{t}^2 \alpha \\ g = \mathsf{t} \alpha \end{array} \ \triangleright \begin{array}{l} a, b, \mathsf{t} f, \mathsf{t} g, \mathsf{t} h, \mathsf{t}^2 \alpha : \emptyset \\ f, g, h, \mathsf{t} \alpha : a \bullet\!\!-\!\!\circ\, \emptyset \\ \alpha : f \bullet\!\!-\!\!\circ\, a \bullet\!\!-\!\!\circ\, \emptyset \end{array}$}
\RightLabel{$(a{=}\mathsf{t} h)$}
\UnaryInfC{$\begin{array}{l} b = \mathsf{t} f = \mathsf{t} g = \mathsf{t}^2 \alpha, a = \mathsf{t} h \\ g = \mathsf{t} \alpha \end{array} \ \triangleright \begin{array}{l} a, b, \mathsf{t} f, \mathsf{t} g, \mathsf{t} h, \mathsf{t}^2 \alpha : \emptyset \\ f, g, h, \mathsf{t} \alpha : a \bullet\!\!-\!\!\circ\, \emptyset \\ \alpha : f \bullet\!\!-\!\!\circ\, a \bullet\!\!-\!\!\circ\, \emptyset \end{array}$}
\end{prooftree}
$$

where rules of the form $(x = y)$ are shorthands for $\texttt{glue-}(x = y)$.

Example 7.6.2. The opetopic set

$$
\begin{tikzcd}
 & b \arrow[dr, "g"] & & \\
a \arrow[ur, "f"] \arrow[rr, "h"'] & \arrow[d, phantom, "\Downarrow\alpha"] & a \arrow[r, bend left, "h"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\beta"] & a
\end{tikzcd}
$$

of example 7.3.2 can be derived as follows.

$$\dfrac{\dfrac{}{\triangleright\ a:\varnothing}\,\texttt{point} \qquad \dfrac{}{\triangleright\ b:\varnothing}\,\texttt{point}}{\triangleright\ a,b}\,\texttt{sum}$$

$$\dfrac{\triangleright\ a,b}{\triangleright\ a,b \vdash_0 a:\varnothing}\,\texttt{pd}$$

$$\dfrac{\triangleright\ a,b \vdash_0 a:\varnothing}{\triangleright\ \begin{array}{l} a,b,\mathtt{t}f:\varnothing \\ f:a \bullet\!\!\multimap \varnothing \end{array}}\,\texttt{shift}$$

$$\dfrac{\triangleright\ \begin{array}{l} a,b,\mathtt{t}f:\varnothing \\ f:a \bullet\!\!\multimap \varnothing \end{array}}{b=\mathtt{t}f \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f \\ f \end{array}}\,\texttt{glue-}(b=\mathtt{t}f)$$

$$\dfrac{b=\mathtt{t}f \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f \\ f \end{array}}{b=\mathtt{t}f \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f \\ f \end{array} \ \vdash_0 \mathtt{t}f:\varnothing}\,\texttt{pd}$$

$$\dfrac{b=\mathtt{t}f \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f \\ f \end{array} \ \vdash_0 \mathtt{t}f:\varnothing}{b=\mathtt{t}f \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g:\varnothing \\ f,g:\mathtt{t}f \bullet\!\!\multimap \varnothing \end{array}}\,\texttt{shift}$$

$$\dfrac{b=\mathtt{t}f \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g:\varnothing \\ f,g:\mathtt{t}f \bullet\!\!\multimap \varnothing \end{array}}{b=\mathtt{t}f \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g \\ f,g \end{array} \ \vdash_0 a:\varnothing}\,\texttt{pd}$$

$$\dfrac{b=\mathtt{t}f \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g \\ f,g \end{array} \ \vdash_0 a:\varnothing}{b=\mathtt{t}f \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{t}h:\varnothing \\ f,g,h:a \bullet\!\!\multimap \varnothing \end{array}}\,\texttt{shift}$$

$$\dfrac{b=\mathtt{t}f \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{t}h:\varnothing \\ f,g,h:a \bullet\!\!\multimap \varnothing \end{array}}{b=\mathtt{t}f \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{t}h \\ f,g,h \end{array} \ \vdash_1 g:\mathtt{t}f \bullet\!\!\multimap \varnothing}\,\texttt{pd}$$

$$\dfrac{b=\mathtt{t}f \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{t}h \\ f,g,h \end{array} \ \vdash_1 g:\mathtt{t}f \bullet\!\!\multimap \varnothing}{b=\mathtt{t}f \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{t}h \\ f,g,h \end{array} \ \vdash_1 g(\mathtt{t}f \leftarrow f):a \bullet\!\!\multimap \varnothing}\,\texttt{graft}$$

$$\dfrac{b=\mathtt{t}f \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{t}h \\ f,g,h \end{array} \ \vdash_1 g(\mathtt{t}f \leftarrow f):a \bullet\!\!\multimap \varnothing}{b=\mathtt{t}f=\mathtt{t}\mathtt{t}\alpha \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{t}h,\mathtt{t}\mathtt{t}\alpha:\varnothing \\ f,g,h,\mathtt{t}\alpha:a \bullet\!\!\multimap \varnothing \\ \alpha:g(\mathtt{t}f \leftarrow f):a \bullet\!\!\multimap \varnothing \end{array}}\,\texttt{shift}$$

$$\dfrac{b=\mathtt{t}f=\mathtt{t}\mathtt{t}\alpha \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{t}h,\mathtt{t}\mathtt{t}\alpha:\varnothing \\ f,g,h,\mathtt{t}\alpha:a \bullet\!\!\multimap \varnothing \\ \alpha:g(\mathtt{t}f \leftarrow f):a \bullet\!\!\multimap \varnothing \end{array}}{\begin{array}{l} b=\mathtt{t}f=\mathtt{t}\mathtt{t}\alpha \\ g=\mathtt{t}\alpha \end{array} \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{t}h,\mathtt{t}\mathtt{t}\alpha \\ f,g,h,\mathtt{t}\alpha \\ \alpha \end{array}}\,\texttt{glue-}(h=\mathtt{t}\alpha)$$

$$\dfrac{\begin{array}{l} b=\mathtt{t}f=\mathtt{t}\mathtt{t}\alpha \\ g=\mathtt{t}\alpha \end{array} \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{t}h,\mathtt{t}\mathtt{t}\alpha \\ f,g,h,\mathtt{t}\alpha \\ \alpha \end{array}}{\begin{array}{l} b=\mathtt{t}f=\mathtt{t}\mathtt{t}\alpha, a=\mathtt{t}g \\ g=\mathtt{t}\alpha \end{array} \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{t}h,\mathtt{t}\mathtt{t}\alpha \\ f,g,h,\mathtt{t}\alpha \\ \alpha \end{array}}\,\texttt{glue-}(a=\mathtt{t}g)$$

$$\dfrac{\begin{array}{l} b=\mathtt{t}f=\mathtt{t}\mathtt{t}\alpha, a=\mathtt{t}g \\ g=\mathtt{t}\alpha \end{array} \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{t}h,\mathtt{t}\mathtt{t}\alpha \\ f,g,h,\mathtt{t}\alpha \\ \alpha \end{array}}{\begin{array}{l} b=\mathtt{t}f=\mathtt{t}\mathtt{t}\alpha, a=\mathtt{t}g=\mathtt{t}h \\ g=\mathtt{t}\alpha \end{array} \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{t}h,\mathtt{t}\mathtt{t}\alpha \\ f,g,h,\mathtt{t}\alpha \\ \alpha \end{array}}\,\texttt{glue-}(a=\mathtt{t}h)$$

$$\dfrac{\begin{array}{l} b=\mathtt{t}f=\mathtt{t}\mathtt{t}\alpha, a=\mathtt{t}g=\mathtt{t}h \\ g=\mathtt{t}\alpha \end{array} \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{t}h,\mathtt{t}\mathtt{t}\alpha \\ f,g,h,\mathtt{t}\alpha \\ \alpha \end{array}}{\begin{array}{l} b=\mathtt{t}f=\mathtt{t}\mathtt{t}\alpha, a=\mathtt{t}g=\mathtt{t}h \\ g=\mathtt{t}\alpha \end{array} \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{t}h,\mathtt{t}\mathtt{t}\alpha \\ f,g,h,\mathtt{t}\alpha \\ \alpha \end{array} \ \vdash_1 h:a \bullet\!\!\multimap \varnothing}\,\texttt{pd}$$

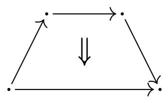
$$\dfrac{\begin{array}{l} b=\mathtt{t}f=\mathtt{t}\mathtt{t}\alpha, a=\mathtt{t}g=\mathtt{t}h \\ g=\mathtt{t}\alpha \end{array} \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{t}h,\mathtt{t}\mathtt{t}\alpha \\ f,g,h,\mathtt{t}\alpha \\ \alpha \end{array} \ \vdash_1 h:a \bullet\!\!\multimap \varnothing}{\begin{array}{l} b=\mathtt{t}f=\mathtt{t}\mathtt{t}\alpha, a=\mathtt{t}g=\mathtt{t}h=\mathtt{t}\mathtt{t}\beta \\ g=\mathtt{t}\alpha \end{array} \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{t}h,\mathtt{t}\mathtt{t}\alpha,\mathtt{t}\mathtt{t}\beta:\varnothing \\ f,g,h,\mathtt{t}\alpha,\mathtt{t}\beta:a \bullet\!\!\multimap \varnothing \\ \alpha,\beta:h \bullet\!\!\multimap a \bullet\!\!\multimap \varnothing \end{array}}\,\texttt{shift}$$

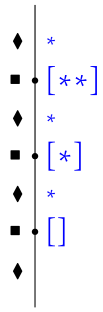
$$\dfrac{\begin{array}{l} b=\mathtt{t}f=\mathtt{t}\mathtt{t}\alpha, a=\mathtt{t}g=\mathtt{t}h=\mathtt{t}\mathtt{t}\beta \\ g=\mathtt{t}\alpha \end{array} \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{t}h,\mathtt{t}\mathtt{t}\alpha,\mathtt{t}\mathtt{t}\beta:\varnothing \\ f,g,h,\mathtt{t}\alpha,\mathtt{t}\beta:a \bullet\!\!\multimap \varnothing \\ \alpha,\beta:h \bullet\!\!\multimap a \bullet\!\!\multimap \varnothing \end{array}}{\begin{array}{l} b=\mathtt{t}f=\mathtt{t}\mathtt{t}\alpha, a=\mathtt{t}g=\mathtt{t}h=\mathtt{t}\mathtt{t}\beta \\ g=\mathtt{t}\alpha, f=\mathtt{t}\beta=h \end{array} \ \triangleright\ \begin{array}{l} a,b,\mathtt{t}f,\mathtt{t}g,\mathtt{t}h,\mathtt{t}\mathtt{t}\alpha,\mathtt{t}\mathtt{t}\beta \\ f,g,h,\mathtt{t}\alpha,\mathtt{t}\beta \\ \alpha,\beta \end{array}}\,\texttt{glue-}(h=\mathtt{t}\beta)$$

Chapter Eight

The unnamed approach for opetopes

The unnamed approach for opetopes relies on the calculus of higher addresses presented in definition 3.3.1 to identify cells, rather than on variables as used in the named approach in chapter 6. For example, recall the opetopic integer $\mathbf{3} \in \mathbb{O}_2$ from example 3.1.4, drawn on the left, with its underlying $\mathfrak{Z}^0$-tree represented on the right:





In this chapter, $\mathbf{3}$ will be encoded as a mapping from its set of node addresses $\mathbf{3}^\bullet = \{[], [*], [**]\}$ to the set of 1-opetopes $\mathbb{O}_1$ as follows:

$$\mathbf{3} \quad \rightsquigarrow \quad \begin{cases} [] \leftarrow \blacksquare \\ [*] \leftarrow \blacksquare \\ [**] \leftarrow \blacksquare \end{cases}$$

The 1-opetope $\blacksquare$ can in turn be encoded as $\{* \leftarrow \blacklozenge$, which gives a complete expression for $\mathbf{3}$:

$$\begin{cases} [] \leftarrow \{* \leftarrow \blacklozenge \\ [*] \leftarrow \{* \leftarrow \blacklozenge \\ [**] \leftarrow \{* \leftarrow \blacklozenge \end{cases}$$

This example will be treated in depth in example 8.3.2.

8.1 THE SYSTEM

PREOPETOPES

Definition 8.1.1 (Preopetope)**.** The sets $\mathbb{P}_n$ of *n-preopetopes* are defined by the following grammar:

$$\mathbb{P}_0 \quad ::= \quad \blacklozenge$$
$$\mathbb{P}_n \quad ::= \quad \begin{cases} \mathbb{A}_{n-1} \leftarrow \mathbb{P}_{n-1} \\ \vdots \\ \mathbb{A}_{n-1} \leftarrow \mathbb{P}_{n-1} \end{cases} \qquad n \geq 1 \tag{8.1.2}$$

$$\mid \quad \{\!\{\mathbb{P}_{n-2} \qquad\qquad n \geq 2 \tag{8.1.3}$$

where the set $\mathbb{A}_n$ of n-addresses is defined in definition 3.3.1. In line (8.1.2), we require further that there is at least one $(n-1)$-address, and that all addresses are distinct.

An n-preopetope $\mathbf{p}$ is *degenerate* if it is of the form of line (8.1.3), it is *non-degenerate* otherwise. We write $\dim \mathbf{p} := n$ for its *dimension*.

Convention 8.1.4. An n-preopetope as in equation (8.1.2) is considered as a *set* of expressions $\mathbb{A}_{n-1} \leftarrow \mathbb{P}_{n-1}$ rather than a list. For instance, the following two n-preopetopes are equal

$$\begin{cases} [p_1] \leftarrow \mathbf{p}_1 \\ [p_2] \leftarrow \mathbf{p}_2 \end{cases} = \begin{cases} [p_2] \leftarrow \mathbf{p}_2 \\ [p_1] \leftarrow \mathbf{p}_1 \end{cases}$$

for any distinct $(n-1)$-addresses $[p_1], [p_2] \in \mathbb{A}_{n-1}$, and any $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{P}_{n-1}$.

Example 8.1.5. (1) There is a unique 1-preopetope $\{* \leftarrow \blacklozenge$, which we simply write $\blacksquare$.

(2) The following are examples of a 2 and 3-preopetope, respectively:

$$\begin{cases} [] \leftarrow \{* \leftarrow \blacklozenge \\ [*\,*\,*\,*\,*] \leftarrow \{* \leftarrow \blacklozenge \end{cases} \qquad\qquad \begin{cases} [[*]] \leftarrow \{\!\{\blacklozenge \\ [[**][*][]] \leftarrow \{[] \leftarrow \{* \leftarrow \blacklozenge \end{cases}$$

We will see that the first does not correspond to an actual opetope, as it is impossible for a 2-opetope to only contain addresses $[]$ and $[*\,*\,*\,*\,*]$ (it would at least need addresses $[*]$, $[**]$, $[*\,*\,*]$, and $[*\,*\,*\,*]$). The second does not correspond to an opetope either, as it does not have a root node (corresponding to address $[]$).

(3) The following is a 4-preopetope $\{\!\{\{\!\{\blacklozenge$. We will see that it corresponds to $\mathsf{I}_{\mathsf{I}_{\blacklozenge}} \in \mathbb{O}_4$.

(4) The following is not a valid preopetope:

$$\begin{cases} [[*]] \leftarrow \blacklozenge \\ [[*][*]] \leftarrow \{* \leftarrow \blacklozenge \end{cases}$$

as $\blacklozenge$ and $\{* \leftarrow \blacklozenge$ do not have the same dimension.

Definition 8.1.6. If we have a non-degenerate n-preopetope of the form

$$\mathbf{p} = \begin{cases} [p_1] \leftarrow \mathbf{q}_1 \\ \vdots \\ [p_k] \leftarrow \mathbf{q}_k \end{cases} \tag{8.1.7}$$

we call $[p_1], \ldots, [p_k] \in \mathbb{A}_{n-1}$ the *source addresses* of $\mathbf{p}$ (or just *sources*), write $\mathbf{p}^\bullet$ for the set of source addresses of $\mathbf{p}$, and $\mathsf{s}_{[p_i]}\,\mathbf{p} := \mathbf{q}_i$ for the $[p_i]$-*source of* $\mathbf{p}$.

Assume $n \geq 2$. A *leaf address* (or just *leaf*) of $\mathbf{p}$ is an $(n-1)$-address of the form $[p[q]]$ such that $[p] \in \mathbf{p}^\bullet$, $[q] \in (\mathsf{s}_{[p]}\,\mathbf{p})^\bullet$, and $[p[q]] \notin \mathbf{p}^\bullet$. We write $\mathbf{p}^| \subseteq \mathbb{A}_{n-1}$ for the set of leaf addresses of $\mathbf{p}$. By convention, if $\mathbf{p}$ is degenerate, then $\mathbf{p}^\bullet := \varnothing$ and $\mathbf{p}^| := \{[]\}$. Further, $\blacklozenge^\bullet = \blacklozenge^| := \varnothing$.

Example 8.1.8. Consider the following preopetopes

$$\mathbf{p} := \begin{cases} [] \leftarrow \begin{cases} [] \leftarrow \{* \leftarrow \blacklozenge \\ [*] \leftarrow \{* \leftarrow \blacklozenge \end{cases} \\ [[*]] \leftarrow \begin{cases} [] \leftarrow \{* \leftarrow \blacklozenge \\ [*] \leftarrow \{* \leftarrow \blacklozenge \end{cases} \end{cases} \qquad\qquad \mathbf{q} := \begin{cases} [] \leftarrow \{[] \leftarrow \{* \leftarrow \blacklozenge \\ [[]] \leftarrow \{\!\{\blacklozenge \end{cases}$$

Then $\mathbf{p}^\bullet = \{[], [[*]]\}$, $\mathbf{p}^| = \{[[]], [[*][]], [[*][*]]\}$, $\mathbf{q}^\bullet = \{[], [[]]\}$, and $\mathbf{q}^| = \varnothing$.

Definition 8.1.9 (Corolla grafting)**.** Let $n \geq 1$, $\mathbf{p} \in \mathbb{P}_n$ be as in equation (8.1.7), and $\mathbf{q} \in \mathbb{P}_{n-1}$. For $[r] \in \mathbf{p}^{|}$ a leaf address of $\mathbf{p}$ (so in particular $[r] \notin \mathbf{p}^{\bullet}$), write

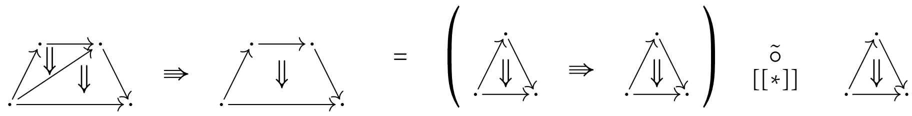
$$\mathbf{p} \underset{[r]}{\tilde{\circ}} \mathbf{q} := \begin{cases} [p_1] \leftarrow \mathbf{q}_1 \\ \vdots \\ [p_k] \leftarrow \mathbf{q}_k \\ [r] \leftarrow \mathbf{q} \end{cases}$$

and call $\mathbf{p} \tilde{\circ}_{[r]} \mathbf{q}$ the *corolla grafting* of $\mathbf{q}$ on $\mathbf{p}$ at address $[r]$. By convention, this operation is associative on the right.

Example 8.1.10. We have

$$\begin{cases} [] \leftarrow \begin{cases} [] \leftarrow \{ * \leftarrow \blacklozenge \\ [*] \leftarrow \{ * \leftarrow \blacklozenge \end{cases} \\ [[*]] \leftarrow \begin{cases} [] \leftarrow \{ * \leftarrow \blacklozenge \\ [*] \leftarrow \{ * \leftarrow \blacklozenge \end{cases} \end{cases} = \left\{ [] \leftarrow \begin{cases} [] \leftarrow \{ * \leftarrow \blacklozenge \\ [*] \leftarrow \{ * \leftarrow \blacklozenge \end{cases} \right. \underset{[[*]]}{\tilde{\circ}} \begin{cases} [] \leftarrow \{ * \leftarrow \blacklozenge \\ [*] \leftarrow \{ * \leftarrow \blacklozenge \end{cases}$$

which, together with the introduction of this chapter, means that graphically,



Remark 8.1.11. The denomination "corolla grafting" is motivated by the fact that $\mathbf{p}$ and $\mathbf{q}$ do not have the same dimension, and thus $\mathbf{q}$ needs to be made into an n-dimensional corolla first. Much like proposition 2.2.22, any preopetope can be obtained by iterated corolla grafting as follows.

Lemma 8.1.12. *Let $n \geq 1$, $\mathbf{p} \in \mathbb{P}_n$ be as in equation* (8.1.7)*, and assume that whenever $1 \leq i < j \leq k$, we have either $[p_i] \sqsubseteq [p_j]$ (definition 3.3.4), or that $[p_i]$ and $[p_j]$ are $\sqsubseteq$-incomparable (in particular, this condition is satisfied if the $[p_i]$'s are lexicographically sorted). Then*

$$\mathbf{p} = \left(\cdots \left(\{ [p_1] \leftarrow \mathbf{q}_1 \right) \underset{[p_2]}{\tilde{\circ}} \mathbf{q}_2 \cdots \right) \underset{[p_k]}{\tilde{\circ}} \mathbf{q}_k.$$

Proof. The condition on the sequence $[p_1], \ldots, [p_k]$ guarantees that the successive corolla graftings are well-defined, i.e. that for $1 \leq i < k$ and

$$\mathbf{p}_i := \left(\cdots \left(\{ [p_1] \leftarrow \mathbf{q}_1 \right) \underset{[p_2]}{\tilde{\circ}} \mathbf{q}_2 \cdots \right) \underset{[p_i]}{\tilde{\circ}} \mathbf{q}_i = \begin{cases} [p_1] \leftarrow \mathbf{q}_1 \\ \vdots \\ [p_i] \leftarrow \mathbf{q}_i \end{cases}$$

we have $[p_{i+1}] \in \mathbf{p}_i^{|}$. □

INFERENCE RULES

We now introduce a typing system for preopetopes in order to characterize those corresponding to opetopes, which is formally shown in theorem 8.2.9. We will deal with sequents of the following form.

Definition 8.1.13 (Sequent)**.** A *sequent* is an expression of the form

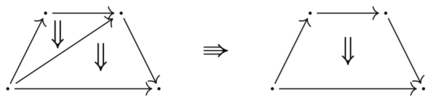
$$\Gamma \vdash \mathbf{p} \longrightarrow \mathbf{t},$$

where $\mathbf{p} \in \mathbb{P}_n$ for some $n \geq 0$, $\mathbf{t} \in \mathbb{P}_{n-1}$, and the *context* Γ is a finite set of pairs consisting of addresses $[l] \in \mathbf{p}^{|}$ and $[q] \in \mathbf{t}^{\bullet}$, denoted by $\frac{[l]}{[q]}$. The preopetope $\mathbf{p}$ is the real object of interest as we will see in subsequent results. We may think of $\mathbf{t}$ as the "target" of $\mathbf{p}$, while Γ establishes a bijection between the leaves of $\mathbf{p}$ and the nodes of its target, playing the role of the readdressing map $\wp$ of definition 2.3.11.

Example 8.1.14. The following is a sequent:

$$\frac{[[]]}{[]}, \frac{[[*][]]}{[*]}, \frac{[[*][*]]}{[**]} \vdash \begin{cases} [] \leftarrow \begin{cases} [] \leftarrow \{ * \leftarrow \blacklozenge \\ [*] \leftarrow \{ * \leftarrow \blacklozenge \end{cases} \\ [[*]] \leftarrow \begin{cases} [] \leftarrow \{ * \leftarrow \blacklozenge \\ [*] \leftarrow \{ * \leftarrow \blacklozenge \end{cases} \end{cases} \longrightarrow \begin{cases} [] \leftarrow \{ * \leftarrow \blacklozenge \\ [*] \leftarrow \{ * \leftarrow \blacklozenge \\ [**] \leftarrow \{ * \leftarrow \blacklozenge \end{cases}$$

As we will see in example 8.3.3, it describes the following 3-opetope:



The operation of *substitution* (definitions 2.2.25 and 6.1.19), which consists in replacing a node by a pasting diagram in an opetope, can be defined as follows in our formalism.

Definition 8.1.15 (Substitution)**.** Let $\mathbf{t}, \mathbf{q} \in \mathbb{P}_n$, $\Upsilon \vdash \mathbf{q} \longrightarrow \mathbf{u}$ be a sequent. Write $\mathbf{t}$ as

$$\mathbf{t} = \begin{cases} [t_1] \leftarrow \mathbf{w}_1 \\ \vdots \\ [t_l] \leftarrow \mathbf{w}_l \end{cases}$$

For $[t_i] \in \mathbf{t}^{\bullet}$, we define $\mathbf{t} \mathbin{\square_{[t_i]}} \mathbf{q}$, the *substitution* by $\mathbf{q}$ in $\mathbf{t}$ at $[t_i]$, as follows:

(1) if $l = 1$ and $\mathbf{q}$ is degenerate, then $\mathbf{t} \mathbin{\square_{[t_1]}} \mathbf{q} := \mathbf{q}$;

(2) if $l \geq 2$ and $\mathbf{q}$ is degenerate, then

$$\mathbf{t} \mathbin{\underset{[t_i]}{\square}} \mathbf{q} := \begin{cases} \rho[t_1] \leftarrow \mathbf{w}_1 \\ \vdots \\ \rho[t_{i-1}] \leftarrow \mathbf{w}_{i-1} \\ \rho[t_{i+1}] \leftarrow \mathbf{w}_{i+1} \\ \vdots \\ \rho[t_l] \leftarrow \mathbf{w}_l \end{cases} \qquad \text{where} \qquad \rho[t_j] := \begin{cases} [t_i r] & \text{if } [t_j] = [t_i[]r], \\ [t_j] & \text{otherwise.} \end{cases}$$

(3) if $l \geq 2$, and $\mathbf{q}$ is not degenerate, write it as

$$\mathbf{q} = \begin{cases} [q_1] \leftarrow \mathbf{v}_1 \\ \vdots \\ [q_k] \leftarrow \mathbf{v}_k \end{cases}$$

and define

$$\mathbf{t} \underset{[t_i]}{\square} \mathbf{q} \coloneqq \begin{cases} \rho[t_1] \leftarrow \mathbf{w}_1 \\ \vdots \\ \rho[t_{i-1}] \leftarrow \mathbf{w}_{i-1} \\ [t_i q_1] \leftarrow \mathbf{v}_1 \\ \vdots \\ [t_i q_k] \leftarrow \mathbf{v}_k \\ \rho[t_{i+1}] \leftarrow \mathbf{w}_{i+1} \\ \vdots \\ \rho[t_l] \leftarrow \mathbf{w}_l \end{cases} \qquad \text{where} \qquad \rho[t_j] \coloneqq \begin{cases} [t_i a r] & \text{if } [t_j] = [t_i[b]r] \\ & \text{for some } \frac{[a]}{[b]} \in \Upsilon, \\ [t_j] & \text{otherwise.} \end{cases}$$

This operation relies on the context Υ, which we leave implicit. By convention, $\square$ is associative on the left.

We refer to section 8.3 for examples of applications of this construction. We now state the inference rules of our unnamed system $\textsc{Opt}^?$ in definition 8.1.16.

Definition 8.1.16 (The $\textsc{Opt}^?$ system)**.** *Introduction of points.*

$$\frac{}{\vdash \blacklozenge \longrightarrow \varnothing}\ \texttt{point}$$

Introduction of degeneracies.

$$\frac{\Gamma \vdash \mathbf{p} \longrightarrow \mathbf{t}}{\frac{[]}{[]} \vdash \left\{\!\!\left\{ \mathbf{p} \right.\right. \longrightarrow \left\{ [] \leftarrow \mathbf{p} \right.}\ \texttt{degen}$$

Note that $\dim(\left\{\!\!\left\{ \mathbf{p} \right.\right.) = 2 + \dim \mathbf{p}$, and that $\dim(\left\{ [] \leftarrow \mathbf{p} \right.) = 1 + \dim \mathbf{p}$.
Shift to the next dimension. Write $\mathbf{p}^\bullet = \{[p_1], \ldots, [p_k]\}$.

$$\frac{\Gamma \vdash \mathbf{p} \longrightarrow \mathbf{t}}{\frac{[[p_1]]}{[p_1]}, \ldots, \frac{[[p_k]]}{[p_k]} \vdash \left\{ [] \leftarrow \mathbf{p} \right. \longrightarrow \mathbf{p}}\ \texttt{shift}$$

As in the previous rule, $\dim(\left\{ [] \leftarrow \mathbf{p} \right.) = 1 + \dim \mathbf{p}$.
Grafting. Assume $\dim \mathbf{p} = n \geq 2$, $[p[q]] \in \mathbf{p}^{\shortmid}$, $\dim \mathbf{q} = n - 1$, write $\mathbf{u} \coloneqq \mathsf{s}_{[q]}\, \mathsf{s}_{[p]}\, \mathbf{p}$ and $\mathbf{q}^\bullet = \{[s_1], \ldots, [s_l]\}$.

$$\frac{\Gamma, \frac{[p[q]]}{[r]} \vdash \mathbf{p} \longrightarrow \mathbf{t} \qquad \Upsilon \vdash \mathbf{q} \longrightarrow \mathbf{u}}{\Gamma', \frac{[p[q][s_1]]}{[rs_1]}, \ldots, \frac{[p[q][s_l]]}{[rs_l]} \vdash \mathbf{p} \underset{[p[q]]}{\tilde{\circ}} \mathbf{q} \longrightarrow \mathbf{t} \underset{[r]}{\square} \mathbf{q}}\ \texttt{graft}$$

where Γ' is given by pairs of the form

(1) $\frac{[a]}{[rxr']}$, where $\frac{[a]}{[r[y]r']} \in \Gamma$ and $\frac{[x]}{[y]} \in \Upsilon$,

(2) $\frac{[a]}{[b]}$, where $\frac{[a]}{[b]} \in \Gamma$ is not as above (i.e. $[b]$ not of the form $[r[y]r']$ for some $\frac{[x]}{[y]} \in \Upsilon$).

In large derivation trees, we will sometimes refer to this rule as $\texttt{graft-}[p[q]]$ for clarity, or simply as $[p[q]]$ in order to make notations lighter.

Remark 8.1.17. Let us explain the transformation of context defined in rule `graft` in definition 8.1.16. Take a derivable sequent in $\mathrm{O{\scriptstyle PT}}^?$, say

$$\Gamma \vdash \mathbf{p} \longrightarrow \mathbf{t},$$

with $\mathbf{p} \in \mathbb{P}_n$. It will be proved in lemma 8.1.19 that Γ exhibits a bijection between $\mathbf{p}^{|}$ and $\mathbf{t}^{\bullet}$. Further, in theorem 8.2.9, we will see that $\mathbf{p}$ corresponds uniquely to an n-opetope $\omega = [\![\mathbf{p}]\!]$, that $\mathsf{t}\omega = [\![\mathbf{t}]\!]$, and that Γ corresponds to the readdressing function $\wp_\omega : \omega^{|} \longrightarrow (\mathsf{t}\omega)^{\bullet}$ or definition 2.3.11.

But where is the readdressing map $\wp_\omega$ implemented in $\mathrm{O{\scriptstyle PT}}^!$? Applying theorem 6.2.27, we know that ω corresponds to a unique sequent (modulo α-equivalence), say

$$E \triangleright \Upsilon \vdash x : s_1 \multimapdotboth s_2 \multimapdotboth \cdots \multimapdotboth \varnothing$$

where $x \in \mathbb{V}_n$. More precisely, considered as a tree, ω is encoded by the term s_1, and by proposition 6.2.16, $\mathsf{t}\omega$ is encoded by s_2. In lemma 6.2.14, we show that $\wp_\omega$ exhibits a bijection

$$\{\&_{s_1} b \mid b \in s_2^{\bullet}\} \xrightarrow{\wp_\omega} \{\&_{s_2} b \mid b \in s_2^{\bullet}\} .$$

Say that a *node* of the term s_2 is an $(n-2)$-variable $b \in s_2^{\bullet}$, while a *leaf* of s_1 is a variable that can be used for grafting (see rule `graft` in definition 6.1.16), i.e. also an $(n-2)$-variable $b \in s_2^{\bullet}$. Then the left hand side can be considered as the set of leaf addresses of s_1, while the right hand side is its set of node addresses of s_2, and $\wp_\omega$ maps the address of $b \in \mathbb{V}_{n-2}$ as a leaf of s_1 to the address of b as a node of s_2. But here, the function $\wp_\omega$ is unnecessary, as this correspondence is already established by the name of the variables.

In $\mathrm{O{\scriptstyle PT}}^?$ however, such bookkeeping is necessary since there are no names, and Γ is designed to precisely be the desired correspondence.

We now prove basic properties of derivable sequents in $\mathrm{O{\scriptstyle PT}}^?$. In proof trees, we may sometimes omit irrelevant information. For instance, if contexts and targets are not important, the `shift` rule may be written as

$$\frac{\mathbf{p}}{\{[] \leftarrow \mathbf{p}}\ \texttt{shift}$$

Lemma 8.1.18. *If* $\Gamma \vdash \mathbf{p} \longrightarrow \mathbf{t}$ *is a derivable sequent, then* $\dim \mathbf{p} = 1 + \dim \mathbf{t}$.

Proof. Easy induction on proof trees. □

Lemma 8.1.19. *Let* $\Gamma \vdash \mathbf{p} \longrightarrow \mathbf{t}$ *be a derivable sequent with* $\dim \mathbf{p} \geq 2$. *Then* Γ *establishes a bijection between* $\mathbf{p}^{|}$ *and* $\mathbf{t}^{\bullet}$ *(i.e. as a set of pairs,* Γ *is the graph of a bijective function).*

Proof. The fact that Γ is a relation from $\mathbf{p}^{|}$ to $\mathbf{t}^{\bullet}$ (i.e. that whenever $\frac{[a]}{[b]} \in \Gamma$ we have $[a] \in \mathbf{p}^{|}$ and $[b] \in \mathbf{t}^{\bullet}$) is clear from the inference rules. It is also clear that Γ is a function (i.e. that whenever $\frac{[a]}{[b]}, \frac{[a']}{[b']} \in \Gamma$ if $[b] \neq [b']$, then $[a] \neq [a']$). Finally, the fact that it is a bijection is clear in the case of `degen` and `shift`, and is a routine verification in the case of `graft`. □

Lemma 8.1.20. *Let* $\Gamma \vdash \mathbf{p} \longrightarrow \mathbf{t}$ *be a derivable sequent with* $\dim \mathbf{p} \geq 2$ *non degenerate. For* $\frac{[p[q]]}{[r]} \in \Gamma$, *we have* $\mathsf{s}_{[q]}\, \mathsf{s}_{[p]}\, \mathbf{p} = \mathsf{s}_{[r]}\, \mathbf{t}$.

Proof. The sequent necessarily follows from an instance of `shift` or `graft`. The result is clear for the former, and follows from inspection for the latter. □

Proposition 8.1.21. *If* $\Gamma \vdash \mathbf{p} \longrightarrow \mathbf{t}$ *is derivable, then so is* $\mathbf{t}$*, i.e. there exists a derivable sequent of the form* $\Upsilon \vdash \mathbf{t} \longrightarrow \mathbf{u}$.

Proof. The only non obvious case is (as always) `graft`, where we have to show that $\mathbf{t} \square_{[r]} \mathbf{q}$ is derivable. Since the sequent $\Gamma, \frac{[p[q]]}{[r]} \vdash \mathbf{p} \longrightarrow \mathbf{t}$ has a nonempty context, $\mathbf{p}$ and $\mathbf{t}$ are non-degenerate. Write $\mathbf{t}$ and $\mathbf{q}$ as in definition 8.1.15. Up to reindexing, we may assume that $\rho[t_j] = [t_j]$ if and only if $j < i$. Assume moreover that the sequences $[t_1], \ldots, [t_{i-1}]$ and $[t_{i+1}], \ldots, [t_l]$ are both lexicographically sorted. In particular, $[t_1] = []$. For $j > i$ write $[t_j] = [t_i[b_j]x_j]$ and $\wp[t_j] = [t_i a_j x_j]$, so that $\Upsilon = \left\{ \frac{[a_j]}{[b_j]} \mid i < j \leq l \right\}$. Then the proof tree of $\mathbf{t} \square_{[t_i]} \mathbf{q}$ is sketched as follows.

(1) If $[t_i] = []$, then necessarily $i = 1$, and $\mathbf{t} \square_{[t_i]} \mathbf{q}$ can be derived as

$$
\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots \\ \mathbf{q}\end{matrix} \quad \begin{matrix}\vdots \\ \mathbf{v}_2\end{matrix}}{\mathbf{q} \mathbin{\tilde{\circ}_{[a_2x_2]}} \mathbf{v}_2} \texttt{graft-}[a_2x_2] \quad \begin{matrix}\vdots \\ \mathbf{v}_3\end{matrix}}{\left(\mathbf{q} \mathbin{\tilde{\circ}_{[a_2x_2]}} \mathbf{v}_2\right) \mathbin{\tilde{\circ}_{[a_3x_3]}} \mathbf{v}_3} \texttt{graft-}[a_3x_3] \\ \vdots \\ \left(\cdots \left(\mathbf{q} \mathbin{\tilde{\circ}_{[a_2x_2]}} \mathbf{v}_2\right) \cdots\right) \mathbin{\tilde{\circ}_{[a_{l-1}x_{l-1}]}} \mathbf{v}_{l-1} \qquad \begin{matrix}\vdots \\ \mathbf{v}_l\end{matrix}}{\left(\cdots \left(\mathbf{q} \mathbin{\tilde{\circ}_{[a_2x_2]}} \mathbf{v}_2\right) \cdots\right) \mathbin{\tilde{\circ}_{[a_lx_l]}} \mathbf{v}_l} \texttt{graft-}[a_lx_l]
$$

and by definition, $\left(\cdots \left(\mathbf{q} \mathbin{\tilde{\circ}_{[a_2x_2]}} \mathbf{v}_2\right) \cdots\right) \mathbin{\tilde{\circ}_{[a_lx_l]}} \mathbf{v}_l = \mathbf{t} \square_{[t_i]} \mathbf{q}$.

(2) If $[t_i] \neq []$, then necessarily $i > 1$, and the process goes similarly. We first derive

$$
\begin{cases} [t_1] \leftarrow \mathbf{v}_1 \\ \vdots \\ [t_{i-1}] \leftarrow \mathbf{v}_{i-1} \end{cases}
$$

then graft the sources of $\mathbf{q}$, and lastly graft the remaining $\mathbf{v}_j$'s, where $j > i$. □

Proposition 8.1.22. *If* $\Gamma_1 \vdash \mathbf{p} \longrightarrow \mathbf{t}_1$ *and* $\Gamma_2 \vdash \mathbf{p} \longrightarrow \mathbf{t}_2$ *are two derivable sequents, then* $\Gamma_1 = \Gamma_2$ *(as sets) and* $\mathbf{t}_1 = \mathbf{t}_2$.

Proof. By inspection of the rules, Γ_i is completely determined by $\mathbf{p}^{\shortmid}$, thus $\Gamma_1 = \Gamma_2$. By lemma 8.1.19, Γ_i exhibits a bijection between $\mathbf{p}^{\shortmid}$ and $\mathbf{t}_i^\bullet$, and in particular, $\mathbf{t}_1^\bullet = \mathbf{t}_2^\bullet$. By lemma 8.1.20, for any $[p] \in \mathbf{t}_1^\bullet = \mathbf{t}_2^\bullet$, we have $\mathsf{s}_{[p]} \mathbf{t}_1 = \mathsf{s}_{[p]} \mathbf{t}_2$. Therefore, $\mathbf{t}_1 = \mathbf{t}_2$. □

Notation 8.1.23. We denote by $\mathbb{P}_n^{\checkmark}$ the set of derivable n-preopetopes, i.e. those $\mathbf{p}$ such that there exists a derivable sequent of the form $\Gamma \vdash \mathbf{p} \longrightarrow \mathbf{t}$. By proposition 8.1.22, this sequent is uniquely determined by $\mathbf{p}$, so let $\mathsf{t}\,\mathbf{p} \coloneqq \mathbf{t}$ be the *target* of $\mathbf{p}$, and $\wp_{\mathbf{p}} : \mathbf{p}^{\shortmid} \longrightarrow \mathbf{t}^\bullet$ be the bijection described by Γ. As such, the sequent around a derivable opetope $\mathbf{p}$ can be reconstructed as $\wp_{\mathbf{p}} \vdash \mathbf{p} \longrightarrow \mathsf{t}\,\mathbf{p}$.

Remark 8.1.24. Our syntax is closely related to the one given for multitopes [HMP02, section 3], called here Hmp. Briefly, in Hmp, the unique 0 and 1-opetopes are respectively denoted ★ and # and, given an n-opetope $\mathbf{p}$, the notation $[\mathbf{p}]$ (resp. $\ulcorner \mathbf{p} \urcorner$) is used for the corresponding degenerate (resp. shifted) $(n+2)$- (resp. $(n+1)$-) opetope. The nodes of an opetope come equipped with a canonical order (just as in our system we could require preopetopes to be always sorted according to the lexicographical order $\preceq$), which apparently dispenses from using addresses. In HMP, an inductive definition of opetopes is given, in the same spirit as our sequent calculus: in particular, typing conditions involving targets when grafting opetopes (grafting is simply application in HMP) are involved. However, no explicit definition at the level of the syntax is given for computing targets (the description given resorts to multicategorical composition), and it is not clear to us how to define it without considering addresses and maintaining more information, as we do with our sequent calculus.

8.2 EQUIVALENCE WITH POLYNOMIAL OPETOPES

We now establish a series of definitions and results to show theorem 8.2.9, stating that the elements of $\mathbb{P}^{\checkmark}_n$ are in bijective correspondence with the set $\mathbb{O}_n$ of polynomial n-opetopes.

Definition 8.2.1 (Unnamed coding)**.** Define the *unnamed coding function* $C^? : \mathbb{O}_n \longrightarrow \mathbb{P}^{\checkmark}_n$ by induction on $n \in \mathbb{N}$. If $n = 0, 1$, then $\mathbb{O}_n$ and $\mathbb{P}_n$ are singletons, so $C^?$ is trivially defined:

$$C^?(\blacklozenge) := \blacklozenge, \qquad C^?(\blacksquare) := \blacksquare = \{* \leftarrow \blacklozenge$$

Assume $n \geq 2$, that $C^?$ is defined for all $k < n$, and take $\omega \in \mathbb{O}_n$.

(1) If ω is degenerate, say $\omega = \mathsf{I}_\phi$ for some $\phi \in \mathbb{O}_{n-2}$, then

$$C^?\left(\mathsf{I}_\phi\right) := \{\!\{ C^?(\phi) \ .$$

(2) If ω is an endotope, say $\omega = \mathsf{Y}_\psi$ for some $\psi \in \mathbb{O}_{n-1}$, then

$$C^?\left(\mathsf{Y}_\psi\right) := \{[] \leftarrow C^?(\psi) \ .$$

(3) Otherwise, decompose ω as $\omega = \nu \circ_{[l]} \mathsf{Y}_\psi$, for $\nu \in \mathbb{O}_n$, $\psi \in \mathbb{O}_{n-1}$, and $[l] \in \nu^{|}$, and let

$$C^?(\omega) := C^?(\nu) \mathbin{\underset{[l]}{\tilde{\circ}}} C^?(\psi) .$$

Proposition 8.2.2. *Let $n \geq 2$ and $\omega \in \mathbb{O}_n$ have at least three nodes. Then the preopetope $C^?(\omega)$ does not depend in the decomposition of ω in corollas.*

Proof. Akin to proposition 6.2.25, it is enough to check that for $\nu \in \mathbb{O}_n$ non degenerate, two different leaf addresses $[l], [l'] \in \nu^{|}$ (which are necessarily $\sqsubseteq$-incomparable), and $\psi, \psi' \in \mathbb{O}_{n-1}$ such that $\mathsf{t}\,\psi = \mathsf{e}_{[l]}\,\nu$ and $\mathsf{t}\,\psi' = \mathsf{e}_{[l']}\,\nu$, we have

$$C^?\left((\nu \mathbin{\underset{[l]}{\circ}} \mathsf{Y}_\psi) \mathbin{\underset{[l']}{\circ}} \mathsf{Y}_{\psi'} \right) = C^?\left((\nu \mathbin{\underset{[l']}{\circ}} \mathsf{Y}_{\psi'}) \mathbin{\underset{[l]}{\circ}} \mathsf{Y}_\psi \right).$$

To unclutter notations, write $C^?(\nu) = \{\vdots$. We have

$$C^?\left((\nu \mathbin{\underset{[l]}{\circ}} \mathsf{Y}_\psi) \mathbin{\underset{[l']}{\circ}} \mathsf{Y}_{\psi'} \right)$$

$$
\begin{aligned}
&= C^{?}\left(\nu \underset{[l]}{\circ} \mathsf{Y}_{\psi}\right) \underset{[l']}{\tilde{\circ}} C^{?}\left(\psi'\right) && \text{by definition} \\
&= \left(C^{?}(\nu) \underset{[l]}{\tilde{\circ}} C^{?}(\psi)\right) \underset{[l']}{\tilde{\circ}} C^{?}\left(\psi'\right) && \text{by definition} \\
&= \left(\begin{cases} \vdots \\ [l] \leftarrow C^{?}(\psi) \end{cases}\right) \underset{[l']}{\tilde{\circ}} C^{?}\left(\psi'\right) && \text{see definition 8.1.9} \\
&= \begin{cases} \vdots \\ [l] \leftarrow C^{?}(\psi) \\ [l'] \leftarrow C^{?}(\psi') \end{cases} && \text{see definition 8.1.9} \\
&= \begin{cases} \vdots \\ [l'] \leftarrow C^{?}(\psi') \\ [l] \leftarrow C^{?}(\psi) \end{cases} && \text{see convention 8.1.4} \\
&= \left(\begin{cases} \vdots \\ [l'] \leftarrow C^{?}(\psi') \end{cases}\right) \underset{[l]}{\tilde{\circ}} C^{?}(\psi) && \text{since } [l] \not\sqsubseteq [l'] \\
&= \left(C^{?}(\nu) \underset{[l']}{\tilde{\circ}} C^{?}\left(\psi'\right)\right) \underset{[l]}{\tilde{\circ}} C^{?}(\psi) \\
&= C^{?}\left(\left(\nu \underset{[l']}{\circ} \mathsf{Y}_{\psi'}\right) \underset{[l]}{\circ} \mathsf{Y}_{\psi}\right).
\end{aligned}
$$

□

We now establish a series of results to prove proposition 8.2.6 stating that $C^{?}(\omega)$ is always a derivable preopetope.

Lemma 8.2.3. *For $\omega \in \mathbb{O}_n$, we have $\omega^{\bullet} = C^{?}(\omega)^{\bullet}$, and $\omega^{|} = C^{?}(\omega)^{|}$.*

Proof. We proceed by induction.

(1) If $n \leq 1$, then the claims trivially hold.

(2) Assume $\omega = \mathsf{I}_{\phi}$, for some $\phi \in \mathbb{O}_{n-2}$. Then $\omega^{\bullet} = \varnothing = \left(\lbrace\!\lbrace C^{?}(\phi)\right)^{\bullet}$. For leaves, $\omega^{|} = \{[]\} = \left(\lbrace\!\lbrace C^{?}(\phi)\right)^{|} = C^{?}(\omega)^{|}$ (see definition 8.1.6).

(3) Assume $\omega = \mathsf{Y}_{\psi}$, for some $\psi \in \mathbb{O}_{n-1}$, so that we have $C^{?}(\omega) = \left\{[] \leftarrow C^{?}(\psi)\right.$. Then $\omega^{\bullet} = \{[]\} = \left(\left\{[] \leftarrow C^{?}(\psi)\right.\right)^{\bullet} = C^{?}(\omega)^{\bullet}$. By induction, $\psi^{\bullet} = C^{?}(\psi)^{\bullet}$, so the leaf addresses of ω and $\left\{[] \leftarrow C^{?}(\psi)\right.$ are both of the form $[[q]]$, where $[q]$ ranges over $\psi^{\bullet}$, hence $\omega^{|} = C^{?}(\omega)^{|}$.

(4) Assume $\omega = \nu \circ_{[l]} \mathsf{Y}_{\psi}$, for some $\nu \in \mathbb{O}_n$, $\psi \in \mathbb{O}_{n-1}$, and $[l] \in \nu^{|}$. Then, by induction,

$$
\begin{aligned}
\omega^{\bullet} &= \nu^{\bullet} + \{[l]\} && \text{by definition} \\
&= C^{?}(\nu)^{\bullet} + \{[l]\} && \text{by induction} \\
&= \left(C^{?}(\nu) \underset{[l]}{\tilde{\circ}} C^{?}(\psi)\right)^{\bullet} && \text{see definition 8.1.9} \\
&= C^{?}(\omega)^{\bullet} && \text{see definition 8.2.1,}
\end{aligned}
$$

and

$$
\begin{aligned}
\omega^{|} &= \nu^{|} - \{[l]\} + \{[l[q]] \mid [q] \in \psi^{\bullet}\} && \text{by definition} \\
&= C^{?}(\nu)^{|} - \{[l]\} + \left\{[l[q]] \mid [q] \in C^{?}(\psi)^{\bullet}\right\} && \text{by induction}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(C^? (\nu) \underset{[l]}{\tilde{\circ}} C^? (\psi) \right)^{|} && \text{see definition 8.1.9} \\
&= C^? (\omega)^{|} && \text{see definition 8.2.1.}
\end{aligned}
$$

□

Lemma 8.2.4. *For $\omega \in \mathbb{O}_n$ non degenerate and $[p] \in \omega^\bullet$, we have $C^? \left(\mathsf{s}_{[p]}\, \omega\right) = \mathsf{s}_{[p]}\, C^? (\omega)$.*

Proof. We proceed by induction. Since $\omega^\bullet \neq \varnothing$, ω is either $\blacksquare$, an endotope, or a grafting.

(1) If $\omega = \blacksquare$, then $[p] = *$, and trivially, $C^? (\mathsf{s}_* \blacksquare) = C^? (\blacklozenge) = \blacklozenge = \mathsf{s}_* \left\{ * \leftarrow \blacklozenge \right. = \mathsf{s}_* C^? (\blacksquare)$.

(2) Assume that ω is an endotope, say $\omega = \mathsf{Y}_\psi$, for some $\psi \in \mathbb{O}_{n-1}$. Necessarily, $[p] = []$, and $C^? \left(\mathsf{s}_{[]}\, \omega\right) = C^? (\psi) = \mathsf{s}_{[]} \left\{ [] \leftarrow C^? (\psi) \right. = \mathsf{s}_{[]}\, C^? (\omega)$.

(3) Assume $\omega = \nu \circ_{[l]} \mathsf{Y}_\psi$, for some $\nu \in \mathbb{O}_n$, $\psi \in \mathbb{O}_{n-1}$, and $[l] \in \nu^{|}$. Let $[p] \in \omega^\bullet$. If $[p] = [l]$, then

$$
\begin{aligned}
C^? \left(\mathsf{s}_{[l]}\, \omega\right) &= C^? (\psi) \\
&= \mathsf{s}_{[l]} \left(C^? (\nu) \underset{[l]}{\tilde{\circ}} C^? (\psi) \right) && \text{see definition 8.1.9} \\
&= \mathsf{s}_{[l]}\, C^? (\omega) && \text{see definition 8.2.1.}
\end{aligned}
$$

Otherwise, we have

$$
\begin{aligned}
C^? \left(\mathsf{s}_{[p]}\, \omega\right) &= C^? \left(\mathsf{s}_{[p]}\, \nu\right) \\
&= \mathsf{s}_{[p]}\, C^? (\nu) && \text{by induction} \\
&= \mathsf{s}_{[p]} \left(C^? (\nu) \underset{[l]}{\tilde{\circ}} C^? (\psi) \right) && \text{see definition 8.1.9} \\
&= \mathsf{s}_{[p]}\, C^? (\omega) && \text{see definition 8.2.1.}
\end{aligned}
$$

□

Lemma 8.2.5. *For $\omega \in \mathbb{O}_n$, we have $C^? (\mathsf{t}\, \omega) = \mathsf{t}\, C^? (\omega)$, and $\wp_\omega = \wp_{C^?(\omega)}$ (see notation 8.1.23).*

Proof. We proceed by induction.

(1) Assume $\omega = \mathsf{I}_\phi$, for some $\phi \in \mathbb{O}_{n-2}$. Then

$$
\begin{aligned}
C^? (\mathsf{t}\, \omega) &= C^? \left(\mathsf{Y}_\phi\right) \\
&= \left\{ [] \leftarrow C^? (\phi) \right. && \text{see definition 8.2.1} \\
&= \mathsf{t} \left\{\!\!\left\{ C^? (\phi) \right.\right. && \text{see } \texttt{degen} \text{ rule} \\
&= \mathsf{t}\, C^? (\omega) && \text{see definition 8.2.1.}
\end{aligned}
$$

Since ω and $C^? (\omega)$ are both degenerate (as opetope and preopetope, respectively), $\wp_\omega$ and $\wp_{C^?(\omega)}$ both map $[] \in \omega^{|} = C^? (\omega)^{|}$ to $[] \in (\mathsf{t}\, \omega)^\bullet = \mathsf{t}\, C^? (\omega)^\bullet$ (see the `degen` rule).

(2) Assume $\omega = \mathsf{Y}_\psi$, for some $\psi \in \mathbb{O}_{n-1}$, so that we have $C^? (\omega) = \left\{ [] \leftarrow C^? (\psi) \right.$. Then

$$
C^? (\mathsf{t}\, \omega) = C^? (\psi)
$$

$$
\begin{aligned}
&= \mathsf{t}\left\{[] \leftarrow C^?(\psi)\right. && \text{see } \texttt{shift} \text{ rule} \\
&= \mathsf{t}\, C^?(\omega) && \text{see definition 8.2.1.}
\end{aligned}
$$

Moreover, we have $\omega^{\shortmid} = C^?(\omega)^{\shortmid} = \{[[p]] \mid [p] \in \psi^\bullet\}$, and by definition, $\wp_\omega[[p]] = [p] = \wp_{C^?(\omega)}[p]$ (see the `shift` rule).

(3) Assume $\omega = \nu \circ_{[l]} \mathsf{Y}_\psi$, for some $\nu \in \mathbb{O}_n$, $\psi \in \mathbb{O}_{n-1}$, and $[l] \in \nu^{\shortmid}$. Then,

$$
\begin{aligned}
C^?(\mathsf{t}\omega) &= C^?\left((\mathsf{t}\nu) \underset{\wp_\nu[l]}{\square} \psi\right) && \text{by proposition 3.1.6} \\
&= C^?(\mathsf{t}\nu) \underset{\wp_{C^?(\nu)}[l]}{\square} C^?(\psi) && \text{by induction, } \wp_\nu = \wp_{C^?(\nu)} \\
&= \mathsf{t}\, C^?(\nu) \underset{\wp_{C^?(\nu)}[l]}{\square} C^?(\psi) && \text{by induction} \\
&= \mathsf{t}\left(C^?(\nu) \underset{[l]}{\tilde{\circ}} C^?(\psi)\right) && \text{see } \texttt{graft} \text{ rule} \\
&= \mathsf{t}\, C^?(\omega)
\end{aligned}
$$

The equality $\wp_\omega = \wp_{C^?(\omega)}$ follows by inspection of rule `graft`. □

Proposition 8.2.6. *If $\omega \in \mathbb{O}_n$, then $C^?(\omega)$ is a derivable n-preopetope.*

Proof. If $n \leq 1$, then the result is trivial, so assume that $n \geq 2$.

(1) If $\omega = \mathsf{I}_\phi$ for some $\phi \in \mathbb{O}_{n-2}$, then $C^?(\omega)$ is simply obtained by an instance of `degen`:

$$
\dfrac{\begin{array}{c}\vdots \\ C^?(\phi)\end{array}}{C^?(\omega)}\ \texttt{degen}
$$

(2) Likewise, if $\omega = \mathsf{Y}_\psi$ for some $\psi \in \mathbb{O}_{n-1}$, then $C^?(\omega)$ can be obtained using an instance of `shift`.

(3) Assume that $\omega = \nu \circ_{[l]} \mathsf{Y}_\psi$, for $\nu\mathbb{O}_n$ non degenerate, $\psi \in \mathbb{O}_{n-1}$, and $[l] \in \nu^{\shortmid}$. By lemma 8.2.3, $\nu^{\shortmid} = C^?(\nu)^{\shortmid}$, thus $[l] \in C^?(\nu)^{\shortmid}$. Since ν is not degenerate, the leaf address $[l]$ can be decomposed as $[l] = [p[q]]$, where $[p] \in \nu^\bullet$ and $[q] \in (\mathsf{s}_{[p]}\nu)^\bullet$. We have

$$
\begin{aligned}
\mathsf{s}_{[q]}\mathsf{s}_{[p]}C^?(\nu) &= \mathsf{s}_{[q]}C^?\left(\mathsf{s}_{[p]}\nu\right) && \text{by lemma 8.2.3} \\
&= C^?\left(\mathsf{s}_{[q]}\mathsf{s}_{[p]}\nu\right) && \text{by lemma 8.2.3} \\
&= C^?(\mathsf{t}\psi) && \text{by assumption} \\
&= \mathsf{t}\, C^?(\psi) && \text{by lemma 8.2.5}
\end{aligned}
$$

Finally, rule `graft` can be used to derive $C^?(\omega)$:

$$
\dfrac{\begin{array}{c}\vdots \\ C^?(\nu)\end{array} \quad \begin{array}{c}\vdots \\ C^?(\psi)\end{array}}{C^?(\omega)}\ \texttt{graft-}[l]
$$

□

We finally prove that $C^?$ is a bijection by constructing its inverse.

Definition 8.2.7. Define the *polynomial coding function* $[\![-]\!] : \mathbb{P}^{\checkmark}_n \longrightarrow \mathbb{O}_n$ by induction on $n \in \mathbb{N}$. If $n = 0, 1$, then both sets are singletons, and $[\![-]\!]$ is trivially defined. Assume $n \geq 2$, and that $[\![-]\!]$ is defined for all $k < n$. We distinguish three cases.

(1) If $\mathbf{q} \in \mathbb{P}^{\checkmark}_{n-2}$, then $\left[\!\!\left[\{\!\{ \mathbf{q} \right]\!\!\right] := \mathsf{I}_{[\![\mathbf{q}]\!]}$.
(2) If $\mathbf{q} \in \mathbb{P}^{\checkmark}_{n-1}$, then $\left[\!\!\left[\{ [] \leftarrow \mathbf{q} \right]\!\!\right] := \mathsf{Y}_{[\![\mathbf{q}]\!]}$.
(3) If $\mathbf{p} \in \mathbb{P}^{\checkmark}_n$, $\mathbf{q} \in \mathbb{P}^{\checkmark}_{n-1}$, and $[l] \in \mathbf{p}^{\mid}$ are such that the corresponding instance of rule `graft` is well-defined, then let

$$\left[\!\!\left[\mathbf{p} \underset{[l]}{\tilde{\circ}} \mathbf{q} \right]\!\!\right] := [\![\mathbf{p}]\!] \underset{[l]}{\circ} \mathsf{Y}_{[\![\mathbf{q}]\!]},$$

which is well-defined by lemmas 8.2.3 to 8.2.5.

Lemma 8.2.8. *Let $n \geq 1$ and $\mathbf{p} \in \mathbb{P}^{\checkmark}_n$ have at least three node addresses. Then $[\![\mathbf{p}]\!]$ does not depend on the decomposition of $\mathbf{p}$ into corolla graftings.*

Proof. Akin to propositions 6.2.25 and 8.2.2, it is enough to check the that for $\mathbf{p} \in \mathbb{P}^{\checkmark}_n$ non degenerate, $[l], [l'] \in \mathbf{p}^{\mid}$ different leaf addresses (in particular, they are $\sqsubseteq$-incomparable), $\mathbf{q}, \mathbf{q}' \in \mathbb{P}^{\checkmark}_{n-1}$ such that $\mathsf{t}\,\mathbf{q} = \mathsf{e}_{[l]}\,\mathbf{p}$ and $\mathsf{t}\,\mathbf{q}' = \mathsf{e}_{[l']}\,\mathbf{p}$, we have

$$\left[\!\!\left[\left(\mathbf{p} \underset{[[l]]}{\tilde{\circ}} \mathbf{q} \right) \underset{[[l']]}{\tilde{\circ}} \mathbf{q}' \right]\!\!\right] = \left[\!\!\left[\left(\mathbf{p} \underset{[[l']]}{\tilde{\circ}} \mathbf{q}' \right) \underset{[[l]]}{\tilde{\circ}} \mathbf{q} \right]\!\!\right].$$

This is straightforward:

$$\begin{aligned}
\left[\!\!\left[\left(\mathbf{p} \underset{[[l]]}{\tilde{\circ}} \mathbf{q} \right) \underset{[[l']]}{\tilde{\circ}} \mathbf{q}' \right]\!\!\right] &= \left([\![\mathbf{p}]\!] \underset{[[l]]}{\circ} \mathsf{Y}_{[\![\mathbf{q}]\!]} \right) \underset{[[l']]}{\circ} \mathsf{Y}_{[\![\mathbf{q}']\!]} && \text{by definition} \\
&= \left([\![\mathbf{p}]\!] \underset{[[l']]}{\circ} \mathsf{Y}_{[\![\mathbf{q}']\!]} \right) \underset{[[l]]}{\circ} \mathsf{Y}_{[\![\mathbf{q}]\!]} && \text{since } [[l]] \not\sqsubseteq [[l']] \\
&= \left[\!\!\left[\left(\mathbf{p} \underset{[[l']]}{\tilde{\circ}} \mathbf{q}' \right) \underset{[[l]]}{\tilde{\circ}} \mathbf{q} \right]\!\!\right] && \text{by definition.}
\end{aligned}$$

$\square$

Theorem 8.2.9. *The functions $C^?$ and $[\![-]\!]$ are mutually inverse.*

Proof. Straightforward verifications. $\square$

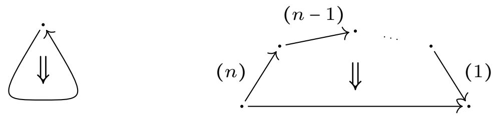
8.3 EXAMPLES

In this section, we give example derivations in system $\textsc{Opt}^?$.

Example 8.3.1 (The arrow)**.** The unique 1-opetope $\blacksquare = \{* \leftarrow \blacklozenge$ is derived by

$$\cfrac{\cfrac{}{\vdash \blacklozenge \longrightarrow \varnothing}\ \texttt{point}}{\vdash \{* \leftarrow \blacklozenge \longrightarrow \blacklozenge}\ \texttt{shift}$$

Example 8.3.2 (Opetopic integers)**.** The opetopic integer **n** (example 3.1.4) is represented on the left in the case $n = 0$, and on the right if $n \geq 1$:



The derivation of **0** is simply

$$\dfrac{\dfrac{}{\vdash \blacklozenge}\,\texttt{point}}{\frac{[]}{[]} \vdash \{\!\{\blacklozenge \longrightarrow \blacksquare}\,\texttt{degen}$$

whereas for $n \geq 1$, the opetope **n** is derived as

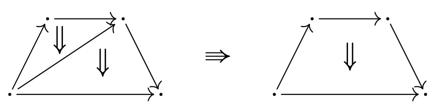
$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash \blacksquare \longrightarrow \blacklozenge\end{array}}{\frac{[*]}{*} \vdash \{[] \leftarrow \blacksquare \longrightarrow \blacksquare}\,\texttt{shift} \qquad \begin{array}{c}\vdots\\ \vdash \blacksquare \longrightarrow \blacklozenge\end{array}}{\frac{[**]}{*} \vdash \begin{cases}[] \leftarrow \blacksquare\\ [*] \leftarrow \blacksquare\end{cases} \longrightarrow \blacksquare}\,[*] \qquad \begin{array}{c}\vdots\\ \vdash \blacksquare \longrightarrow \blacklozenge\end{array}}{\frac{[***]}{*} \vdash \begin{cases}[] \leftarrow \blacksquare\\ [*] \leftarrow \blacksquare\\ [**] \leftarrow \blacksquare\end{cases} \longrightarrow \blacksquare}\,[**]$$

$$\vdots$$

$$\dfrac{\frac{[*^{n-1}]}{*} \vdash \begin{cases}[] \leftarrow \blacksquare\\ \vdots\\ [*^{n-2}] \leftarrow \blacksquare\end{cases} \longrightarrow \blacksquare \qquad \begin{array}{c}\vdots\\ \vdash \blacksquare \longrightarrow \blacklozenge\end{array}}{\frac{[*^{n}]}{*} \vdash \begin{cases}[] \leftarrow \blacksquare\\ \vdots\\ [*^{n-1}] \leftarrow \blacksquare\end{cases} \longrightarrow \blacksquare}\,[*^{n-1}]$$

where there is a total of $n - 1$ instances of the `graft` rule.

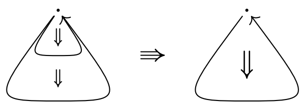
Example 8.3.3 (A classic)**.** The 3-opetope



can be derived as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \frac{[**]}{*} \vdash \mathbf{2} \longrightarrow \blacksquare\end{array}}{\frac{[[*]]}{[*]} \vdash \{[] \leftarrow \mathbf{2} \longrightarrow \mathbf{2}}\,\texttt{shift} \qquad \begin{array}{c}\vdots\\ \frac{[**]}{*} \vdash \mathbf{2} \longrightarrow \blacksquare\end{array}}{\frac{[[]]}{[]}, \frac{[[*][]]}{[*]}, \frac{[[*][*]]}{[**]} \vdash \begin{cases}[] \leftarrow \mathbf{2}\\ [[*]] \leftarrow \mathbf{2}\end{cases} \longrightarrow \mathbf{3}}\,\texttt{graft-}[[*]]$$

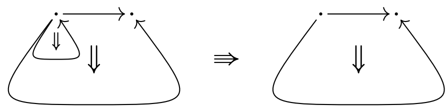
Example 8.3.4 (A degenerate case)**.** The 3-opetope



can be derived as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \frac{[*]}{*} \vdash \mathbf{1} \longrightarrow \blacksquare\end{array}}{\frac{[[]]}{[]} \vdash \left\{[] \leftarrow \mathbf{1}\right. \longrightarrow \mathbf{1}}\,\texttt{shift} \qquad \begin{array}{c}\vdots\\ \frac{[]}{[]} \vdash \mathbf{0} \longrightarrow \blacksquare\end{array}}{\vdash \begin{cases}[] \leftarrow \mathbf{1}\\ [[]] \leftarrow \mathbf{0}\end{cases} \longrightarrow \mathbf{0}}\,\texttt{graft-}[[]]$$

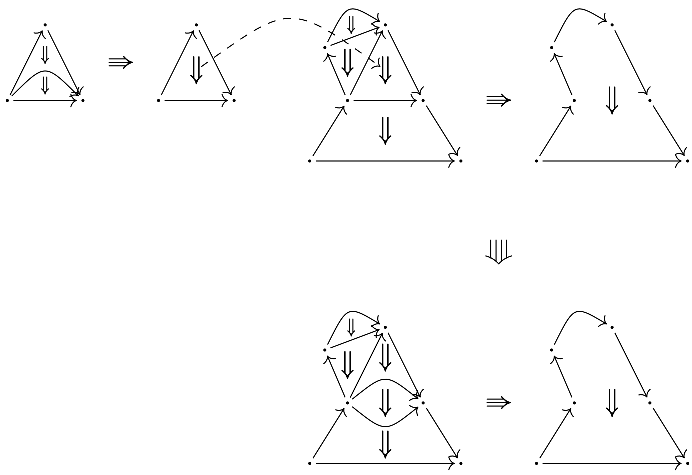
Example 8.3.5 (Another degenerate case)**.** The 3-opetope



can be derived as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \frac{[**]}{*} \vdash \mathbf{2} \longrightarrow \blacksquare\end{array}}{\frac{[[]]}{[]}, \frac{[[*]]}{[*]} \vdash \left\{[] : \mathbf{2}\right. \longrightarrow \mathbf{2}}\,\texttt{shift} \qquad \begin{array}{c}\vdots\\ \vdash \mathbf{0} \longrightarrow \blacksquare\end{array}}{\frac{[[]]}{[]} \vdash \begin{cases}[] \leftarrow \mathbf{2}\\ [[*]] \leftarrow \mathbf{0}\end{cases} \longrightarrow \mathbf{1}}\,\texttt{graft-}[[*]]$$

Example 8.3.6 (A 4-opetope)**.** The 4-opetope



is derived by

$$\dfrac{\dfrac{\vdots}{\frac{[***]}{*} \vdash \mathbf{3} \longrightarrow \blacksquare}}{\frac{[[]]}{[]}, \frac{[[*]]}{[*]}, \frac{[[**]]}{[**]} \vdash \big\{[] \leftarrow \mathbf{3} \longrightarrow \mathbf{3}}\ \texttt{shift}$$

$$\dfrac{\dfrac{\vdots}{\frac{[***]}{*} \vdash \mathbf{3} \longrightarrow \blacksquare}\ \texttt{shift} \qquad \dfrac{\vdots}{\frac{[**]}{*} \vdash \mathbf{2} \longrightarrow \blacksquare}}{\frac{[[]]}{[]}, \frac{[[**]]}{[***]}, \frac{[[*][]]}{[*]}, \frac{[[*][*]]}{[**]} \vdash \begin{cases} [] \leftarrow \mathbf{3} \\ [[*]] \leftarrow \mathbf{2} \end{cases} \longrightarrow \mathbf{4}}\ [[*]]$$

$$\dfrac{\cdots \qquad \dfrac{\vdots}{\frac{[**]}{*} \vdash \mathbf{2} \longrightarrow \blacksquare}}{\frac{[[]]}{[]}, \frac{[[**]]}{[****]}, \frac{[[*][]]}{[*]}, \frac{[[*][*][]]}{[**]}, \frac{[[*][*][*]]}{[***]} \vdash \begin{cases} [] \leftarrow \mathbf{3} \\ [[*]] \leftarrow \mathbf{2} \\ [[*][*]] \leftarrow \mathbf{2} \end{cases} \longrightarrow \mathbf{5}}\ [[*][*]]$$

$$\dfrac{\cdots \qquad \dfrac{\vdots}{\frac{[*]}{*} \vdash \mathbf{1} \longrightarrow \blacksquare}}{\frac{[[]]}{[]}, \frac{[[**]]}{[****]}, \frac{[[*][]]}{[*]}, \frac{[[*][*][*]]}{[***]}, \frac{[[*][*][][]]}{[**]} \vdash \begin{cases} [] \leftarrow \mathbf{3} \\ [[*]] \leftarrow \mathbf{2} \\ [[*][*]] \leftarrow \mathbf{2} \\ [[*][*][]] \leftarrow \mathbf{1} \end{cases} \longrightarrow \mathbf{5}}\ [[*][*][]]$$

$$\dfrac{\cdots}{\frac{[[]]}{[]}, \frac{[[[*]]]}{[[*]]}, \frac{[[[*][*]]]}{[[*][*]]}, \frac{[[[*][*][]]]}{[[*][*][]]} \vdash \Bigg\{ [] \leftarrow \begin{cases} [] \leftarrow \mathbf{3} \\ [[*]] \leftarrow \mathbf{2} \\ [[*][*]] \leftarrow \mathbf{2} \\ [[*][*][]] \leftarrow \mathbf{1} \end{cases} \longrightarrow \begin{cases} [] \leftarrow \mathbf{3} \\ [[*]] \leftarrow \mathbf{2} \\ [[*][*]] \leftarrow \mathbf{2} \\ [[*][*][]] \leftarrow \mathbf{1} \end{cases}}\ \texttt{shift}$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\frac{[*]}{*} \vdash \mathbf{1} \longrightarrow \blacksquare}}{\frac{[[]]}{[]} \vdash \big\{[] \leftarrow \mathbf{1} \longrightarrow \mathbf{1}}\ \texttt{shift} \qquad \dfrac{\vdots}{\frac{[**]}{*} \vdash \mathbf{2} \longrightarrow \blacksquare}}{\frac{[[][]]}{[]}, \frac{[[][*]]}{[*]} \vdash \begin{cases} [] \leftarrow \mathbf{1} \\ [[]] \leftarrow \mathbf{2} \end{cases} \longrightarrow \mathbf{2}}\ [[]]$$

$$\dfrac{\cdots \qquad \cdots}{\frac{[[]]}{[]}, \frac{[[[*][*]]]}{[[*][][*]]}, \frac{[[[*][*][]]]}{[[*][][*][]]}, \frac{[[[*]][[]]}{[[*]]}, \frac{[[[*]][[[]]]]}{[[*][]]} \vdash \begin{cases} [] \leftarrow \begin{cases} [] \leftarrow \mathbf{3} \\ [[*]] \leftarrow \mathbf{2} \\ [[*][*]] \leftarrow \mathbf{2} \\ [[*][*][]] \leftarrow \mathbf{1} \end{cases} \\ [[[*]]] \leftarrow \begin{cases} [] \leftarrow \mathbf{1} \\ [[]] \leftarrow \mathbf{2} \end{cases} \end{cases} \longrightarrow \begin{cases} [] \leftarrow \mathbf{3} \\ [[*]] \leftarrow \mathbf{1} \\ [[*][]] \leftarrow \mathbf{2} \\ [[*][][*]] \leftarrow \mathbf{2} \\ [[*][][*][]] \leftarrow \mathbf{1} \end{cases}}\ [[[*]]]$$

8.4 DECIDING OPETOPES

We now present in algorithm 1 the isOpetope function that, given a preopetope $\mathbf{p} \in \mathbb{P}$, decides if $\mathbf{p} \in \mathbb{P}^{\checkmark}$, as proved in proposition 8.4.1. This algorithm tries to deconstruct $\mathbf{p}$ by finding the last rule instance in a potential proof tree for it, and recursively checking the validity of the premises. We emphasize that this algorithm, while straightforward, is extremely inefficient.

Algorithm 1 Well formation algorithm

1: **procedure** isOpetope($\mathbf{p} \in \mathbb{P}$) ▷ Returns a boolean
2:     **if** $\mathbf{p} = \blacklozenge$ **then**
3:         **return true**
4:     **else if** $\mathbf{p} = \{\!\{ \mathbf{q}$ **then**
5:         **return** isOpetope($\mathbf{q}$)
6:     **else**
7:         **while** $\mathbf{p}$ has an address of the form $[p[q]]$ **do**
8:             Let $[p[q]]$ be the maximal such address
9:             **if** $[p] \notin \mathbf{p}^\bullet$ **or** $[q] \notin (\mathsf{s}_{[p]}\, \mathbf{p})^\bullet$ **then**
10:                 **return false**
11:             **else if not** isOpetope($\mathsf{s}_{[p]}\, \mathbf{p}$) **then**
12:                 **return false**
13:             **else if not** isOpetope($\mathsf{s}_{[p[q]]}\, \mathbf{p}$) **then**
14:                 **return false**
15:             **else if** $\mathsf{t}\,\mathsf{s}_{[p]}\, \mathbf{p} \neq \mathsf{s}_{[p[q]]}\, \mathbf{p}$ **then**
16:                 **return false**
17:             **else**
18:                 Remove address $[p[q]]$ from $\mathbf{p}$
19:             **end if**
20:         **end while**
21:         **if** $\mathbf{p}$ is of the form $\{[] \leftarrow \mathbf{q}$ **then**
22:             **return** isOpetope($\mathbf{q}$)
23:         **else**
24:             **return false**
25:         **end if**
26:     **end if**
27: **end procedure**

Proposition 8.4.1. *For* $\mathbf{p} \in \mathbb{P}$*, the execution* isOpetope($\mathbf{p}$) *returns* ***true*** *if and only if* $\mathbf{p} \in \mathbb{P}^{\checkmark}$.

Proof. This algorithm tries to deconstruct the potential proof tree of $\mathbf{p}$ in system $\mathrm{Opt}^?$. Condition at line (2) removes an instance of the `point` rule. Condition at line (4) removes an instance of `degen`. Each iteration of the **while** loop at line (7) removes an instance of `graft`. Finally the condition at line (21) removes an instance of `shift`. If the algorithm encounters an expression that is not the conclusion of any instance of any rule of $\mathrm{Opt}^?$, it returns **false**. Otherwise, if all branches of the proof tree lead to $\blacklozenge$, it returns **true**. □

Chapter Nine

The unnamed approach for opetopic sets

We now present $\mathrm{OptSet}^?$, a derivation system for opetopic set that is *controlled* by $\mathrm{Opt}^?$, unlike system $\mathrm{OptSet}^!$ that is *based* on $\mathrm{Opt}^!$.

9.1 THE OPTSET$^?$ SYSTEM

As always, contexts are considered as sets, so that the order in which the typings are written is irrelevant, even though those typings might be interdependent. We rely on two types of judgment that can be understood as follows:

(1) "Γ", meaning that Γ is a well-formed context,

(2) "$\Gamma \vdash \mathbf{P}$", meaning that $\mathbf{P}$ is a well-formed *pasting diagram* in context Γ.

We now state the inference rules in definition 9.1.1, and simultaneously assign a *shape* $x^\natural \in \mathbb{P}^\checkmark$ (see notation 8.1.23) to any variable x in a derivable context.

Definition 9.1.1 (The $\textsc{OptSet}^?$ system)**.** *Introduction of points.*

$$\frac{\Gamma}{\Gamma, x : \blacklozenge}\,\texttt{point}$$

for x a fresh variable name. Such a cell x has no source, no target, and its shape is given by $x^\natural := \blacklozenge$.

Introduction of degenerate pasting diagrams.

$$\frac{\Gamma, x : T}{\Gamma, x : T \vdash \{\!\{ x}\,\texttt{degen}$$

The shape of this pasting diagram is $(\{\!\{ x\,)^\natural := \{\!\{ x^\natural$.

Introduction of non degenerate pasting diagrams. If there exists $\mathbf{p} \in \mathbb{P}^{\checkmark}$ a non degenerate opetope, say

$$\mathbf{p} = \begin{cases} [p_1] \leftarrow \psi_1 \\ \vdots \\ [p_k] \leftarrow \psi_k \end{cases}$$

and typings $x_1 : T_1, \ldots, x_k : T_k$ in the current context such that

(1) $x_i^\natural = \psi_i$,

(2) **(Inner)** whenever $[p_j] = [p_i[q]]$ we have $\mathsf{t}\, x_j = \mathsf{s}_{[q]}\, x_i$ (the latter notation is defined in rule `shift` below),

then:

$$\frac{\Gamma, x_1 : T_1, \ldots, x_k : T_k}{\Gamma, x_1 : T_1, \ldots, x_k : T_k \vdash \begin{cases} [p_1] \leftarrow x_1 \\ \vdots \\ [p_k] \leftarrow x_k \end{cases}}\,\texttt{graft}$$

Denote this pasting diagram (the expression on the right of $\vdash$) by $\mathbf{P}$. Its shape is given by $\mathbf{P}^\natural := \mathbf{p}$, and for $1 \leq i \leq k$, let $\mathsf{s}_{[p_i]}\, \mathbf{P} := x_i$. Forming a pasting diagram in this manner is essentially an unbiased (or non binary) grafting, whence the name of the rule.

Shift to the next dimension. If we have a pasting diagram $\mathbf{P}$ of shape $\mathbf{P}^\natural = \mathbf{p}$, a cell $x : \mathbf{Q} \longrightarrow a$ in the current context, such that

(1) $x^\natural = \mathsf{t}\, \mathbf{p}$,

(2) **(Glob1)** if $\mathbf{p}$ is non degenerate, then $\mathsf{t}\, \mathsf{s}_{[]}\, \mathbf{P} = \mathsf{t}\, x$,

(3) **(Glob2)** if $\mathbf{p}$ is non degenerate, then for a leaf $[p[q]] \in \mathbf{p}^{|}$, we have $\mathsf{s}_{[q]}\, \mathsf{s}_{[p]}\, \mathbf{P} = \mathsf{s}_{\wp_{\mathbf{p}}[p[q]]}\, x$,

(4) **(Degen)** if $\mathbf{p}$ is degenerate, then $\mathbf{Q} = \{[] \leftarrow a$,

then:

$$\frac{\Gamma, x : \mathbf{Q} \longrightarrow a \vdash \mathbf{P}}{\Gamma, x : \mathbf{Q} \longrightarrow a, y : \mathbf{P} \longrightarrow x}\,\texttt{shift}$$

for y a fresh name. The shape of y is given by $y^\natural := \mathbf{P}^\natural$, its source is $\mathsf{s}\, y := \mathbf{P}$, for $[p] \in \mathbf{p}^\bullet$, its $[p]$-source is $\mathsf{s}_{[p]}\, y := \mathsf{s}_{[p]}\, \mathbf{P}$, and its target is $\mathsf{t}\, y := x$.

Definition 9.2.1 (Substitution)**.** Let Υ and $\Gamma = (x_1 : T_1, \ldots, x_k : T_k)$ be two derivable contexts in $\text{OptSet}^?$. Akin to classical type theory (see e.g. [Hof97, definition 2.11]), a substitution $\sigma : \Upsilon \longrightarrow \Gamma$ is a sequence of variables $(\sigma_1, \ldots, \sigma_k)$ such that for $1 \leq i \leq k$, the typing $\sigma_i : T_i[\sigma_1/x_1]\cdots[\sigma_{i-1}/x_{i-1}]$ is in Υ.

Let $\mathcal{C}\text{tx}^?$ be the syntactic category of our type theory, i.e. the category whose objects are derivable contexts, and morphisms are substitutions as defined above.

Lemma 9.2.2. *In the setting above, we have $\sigma_i^\natural = x_i^\natural$.*

Proof. The shape of a variable, i.e. the shape of its source pasting diagram, does not depend on the variables present in it, only on its underlying preopetope. □

Recall from definition 8.2.1 the *unnamed coding function* $C^? : \mathbb{O}_n \longrightarrow \mathbb{P}_n^{\checkmark}$.

Definition 9.2.3 (Unnamed stratification)**.** We now construct the *unnamed stratification functor* $S^? : (\mathcal{C}\text{tx}^?)^{\text{op}} \longrightarrow \mathcal{P}\text{sh}(\mathbb{O})_{\text{fin}}$. For $\Gamma \in \mathcal{C}\text{tx}^?$ and $\omega \in \mathbb{O}$, let

$$S^?\Gamma_\omega := \left\{ x \in \Gamma \mid x^\natural = C^? (\omega) \right\}.$$

If $x^\natural \neq \blacklozenge$, then the type X of x is of the form $\mathbf{P} \longrightarrow z$, and we let $\mathsf{t}\, x := z$. This is well defined as by construction of Γ we have $z^\natural = \mathsf{t}(x^\natural)$. For $[p] \in \omega^\bullet$, we let $\mathsf{s}_{[p]}\, x := \mathsf{s}_{[p]}\, \mathbf{P}$. Again, this is well-defined as $(\mathsf{s}_{[p]}\, \mathbf{P})^\natural = \mathsf{s}_{[p]}(\mathbf{P}^\natural) = \mathsf{s}_{[p]}(x^\natural)$. From there, the opetopic identities clearly hold, and $S^?\Gamma$ is a finite opetopic set.

On morphisms, write $\Gamma = (x_1 : T_1, \ldots, x_k : T_k)$, let $\sigma = (\sigma_1, \ldots, \sigma_k) : \Upsilon \longrightarrow \Gamma$ be a substitution, and define a morphism $S^?\sigma : S^?\Gamma \longrightarrow S^?\Upsilon$ as follows. For x_i a variable of Γ, and $\omega \in \mathbb{O}$ such that $C^? (\omega) = x_i^\natural$, there is a corresponding cell $x_i \in S^?\Gamma_\omega$, and we let $(S^?\sigma)(x_i) := \sigma_i$. This is well-defined since by lemma lemma 9.2.2, we have $\sigma_i^\natural = x^\natural = \omega$, thus $\sigma_i \in S^?\Upsilon_\omega$.

Lemma 9.2.4. *The map $S^?\sigma$ of definition 9.2.3 is a morphism of opetopic sets.*

Proof. Assume $\omega \neq \blacklozenge$, so that the type of x_i is $\mathbf{P} \longrightarrow x_j$ for some $j < i$, and the type of σ_i is $\mathbf{P}[\sigma_1/x_1]\cdots[\sigma_{i-1}/x_{i-1}] \longrightarrow \sigma_j$. Then $(S^?\sigma)(\mathsf{t}\, x_i) = \sigma_j = \mathsf{t}(S^?\sigma)(x_i)$. If $[p] \in \omega^\bullet$, then $\mathsf{s}_{[p]}\, x_i = x_l$, for some $l < i$, and

$$\begin{aligned}
(S^?\sigma)(\mathsf{s}_{[p]}\, x_i) &= (S^?\sigma)(x_l) \\
&= \sigma_l && \text{see definition 9.2.1} \\
&= \mathsf{s}_{[p]}\, (\mathbf{P}[\sigma_1/x_1]\cdots[\sigma_l/x_l]\cdots[\sigma_{i-1}/x_{i-1}]) \\
&= \mathsf{s}_{[p]}(S^?\sigma(x_i)) && \text{see definition 9.2.1.}
\end{aligned}$$

In conclusion, $S^?\sigma$ is compatible with the source and target maps, and thus is a morphism of opetopic sets $S^?\Gamma \longrightarrow S^?\Upsilon$. □

Theorem 9.2.5. *The stratification functor $S^? : (\mathcal{C}\text{tx}^?)^{\text{op}} \longrightarrow \mathcal{P}\text{sh}(\mathbb{O})_{\text{fin}}$ is an equivalence of categories.*

Proof. It is clear from the definition that $S^?$ is faithful. Let $\Gamma, \Upsilon \in \mathcal{C}\mathrm{tx}^?$, with $\Gamma = (x_1 : T_1, \ldots, x_k : T_k)$, and $f : S^?\Gamma \longrightarrow S^?\Upsilon$. For σ_f the substitution $(f(x_1), \ldots, f(x_k)) : \Gamma \longrightarrow \Upsilon$, we clearly have $f = S^?\sigma_f$, showing that $S^?$ is fully faithful.

We now show that $S^?$ is essentially surjective. Take $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})_{\mathrm{fin}}$, and enumerate its cells as $x_1 \in X_{\omega_1}, \ldots, x_k \in X_{\omega_k}$, such that whenever $i < j$ we have $\dim \omega_i \leq \dim \omega_j$. In other words, they are sorted by dimension. We produce a sequence of derivable contexts $\Gamma^{(0)} \subseteq \Gamma^{(1)} \subseteq \cdots \subseteq \Gamma^{(k)}$, where $\Gamma^{(i)} = (\overline{x_1} : T_1, \ldots, \overline{x_i} : T_i)$ is such that $\overline{x_i}^\natural = C^?(\omega_i)$. For $i = 0$, let $\Gamma^{(0)} = ()$. Assume $1 \leq i \leq k$, and that $\Gamma^{(i-1)}$ is defined and derivable.

(1) If $\omega_i = \blacklozenge$, let $\Gamma^{(i)}$ be given by the following proof tree:

$$\dfrac{\begin{matrix}\vdots \\ \Gamma^{(i-1)}\end{matrix}}{\Gamma^{(i-1)}, \overline{x_i} : \blacklozenge}\ \texttt{point}$$

Note that in this case, $T_i = \blacklozenge$.

(2) Assume $\omega_i \neq \blacklozenge$ is not degenerate. Then, by induction, we have $\overline{\mathsf{t}\, x_i}^\natural = \mathsf{t}\,\omega_i$, and for every address $[p] \in \omega_i^\bullet$, we have $\overline{\mathsf{s}_{[p]}\, x_i}^\natural = \mathsf{s}_{[p]}\, C^?(\omega_i)$. From this, $\Gamma^{(i)}$ is given by the following proof tree:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma^{(i-1)}\end{matrix}}{\Gamma^{(i-1)} \vdash \begin{cases} [p_1] \leftarrow \overline{\mathsf{s}_{[p_1]}\, x_i} \\ \vdots \end{cases}}\ \texttt{graft}}{\Gamma^{(i-1)}, \overline{x_i} : \begin{cases} [p_1] \leftarrow \overline{\mathsf{s}_{[p_1]}\, x_i} \\ \vdots \end{cases} \longrightarrow \overline{\mathsf{t}\, x_i}}\ \texttt{shift}$$

where the pasting diagram has shape $C^?(\omega)$, and $\{[p_1], \ldots\} := \omega^\bullet$.

(3) If ω_i is degenerate, then $\Gamma^{(i)}$ is given by the following proof tree:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma^{(i-1)}\end{matrix}}{\Gamma^{(i-1)} \vdash \{\!\{\overline{\mathsf{t}\,\mathsf{t}\, x_i}}\ \texttt{degen}}{\Gamma^{(i-1)}, \overline{x_i} : \{\!\{\overline{\mathsf{t}\,\mathsf{t}\, x_i} \longrightarrow \overline{\mathsf{t}\, x_i}}\ \texttt{shift}$$

Finally, the mapping $x_i \longmapsto \overline{x_i}$ exhibits an isomorphism $X \longrightarrow S^?\Gamma^{(k)}$, and $S^?$ is essentially surjective. □

The category $\mathcal{C}\mathrm{tx}^?$ has finite limits, induced from finite colimits in $\mathcal{P}\mathrm{sh}(\mathbb{O})_{\mathrm{fin}}$ through $S^?$. We conclude this section with a result similar to theorem 7.2.28, stating that opetopic sets essentially are "models of the algebraic theory $\mathcal{C}\mathrm{tx}^?$".

Theorem 9.2.6. *We have an equivalence* $\mathcal{P}\mathrm{sh}(\mathbb{O}) \simeq \mathcal{L}\mathrm{EX}(\mathcal{C}\mathrm{tx}^?, \mathcal{S}\mathrm{et})$.

Proof. This follows directly from theorem 9.2.5 and from the Gabriel–Ulmer duality (see corollary 0.5.7). □

9.3 EXAMPLES

In this section, we give example derivations in system $\text{OptSet}^?$. For clarity, we do not repeat the type of previously typed variables in proof trees.

Example 9.3.1. We show how to derive the following opetopic set, which is not representable:

$$\begin{tikzcd} a \arrow[r, "g"] \arrow[r, bend left=70, "f"] \arrow[r, phantom, bend left=35, "\Downarrow\alpha"] & b \arrow[l, bend left=70, "h"] \end{tikzcd}$$

First, we introduce all the points:

$$\dfrac{\dfrac{}{a : \blacklozenge}\,\texttt{point}}{a, b : \blacklozenge}\,\texttt{point}$$

Then we introduce f, by first specifying its source pasting diagram with the `graft` rule, parameterized by opetope $\blacksquare = \left\{ * \leftarrow \blacklozenge \right.$, and then applying the `shift` rule:

$$\dfrac{\dfrac{\vdots \atop a, b}{a, b \vdash \left\{ * \leftarrow a \right.}\,\texttt{graft}}{a, b, f : \left\{ * \leftarrow a \longrightarrow b \right.}\,\texttt{shift}$$

We proceed similarly for g and h:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots \atop a, b, f}{a, b, f \vdash \left\{ * \leftarrow a \right.}\,\texttt{graft}}{a, b, f, g : \left\{ * \leftarrow a \longrightarrow b \right.}\,\texttt{shift}}{a, b, f, g \vdash \left\{ * \leftarrow b \right.}\,\texttt{graft}}{a, b, f, g, h : \left\{ * \leftarrow b \longrightarrow a \right.}\,\texttt{shift}$$

Lastly, we introduce α, first by specifying its source with the `graft` rule, parameterized by opetope $\mathbf{1} = \left\{ [] \leftarrow \blacksquare \right.$ (see the opetopic integers defined in example 8.3.2), and applying the `shift` rule:

$$\dfrac{\dfrac{\vdots \atop a, b, f, g, h}{a, b, f, g, h \vdash \left\{ [] \leftarrow f \right.}\,\texttt{graft}}{a, b, f, g, h, \alpha : \left\{ [] \leftarrow f \longrightarrow g \right.}\,\texttt{shift}$$

Example 9.3.2. The opetopic set

$$\begin{tikzcd} & b \arrow[dr, "g"] & & \\ a \arrow[ur, "f"] \arrow[rr, "h"'] & \Downarrow\alpha & a \arrow[r, bend left, "h"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\beta"] & a \end{tikzcd}$$

is straightforwardly derived as

$$
\begin{array}{c}
\hline a : \blacklozenge \quad \texttt{point} \\
\hline a, b : \blacklozenge \quad \texttt{point} \\
\hline a, b \vdash \left\{ * \leftarrow a \right. \quad \texttt{graft} \\
\hline a, b, f : \left\{ * \leftarrow a \right. \longrightarrow b \quad \texttt{shift} \\
\hline a, b, f \vdash \left\{ * \leftarrow b \right. \quad \texttt{graft} \\
\hline a, b, f, g : \left\{ * \leftarrow b \right. \longrightarrow a \quad \texttt{shift} \\
\hline a, b, f, g \vdash \left\{ * \leftarrow a \right. \quad \texttt{graft} \\
\hline a, b, f, g, h : \left\{ * \leftarrow a \right. \longrightarrow a \quad \texttt{shift} \\
\hline a, b, f, g, h \vdash \begin{cases} [] \leftarrow g \\ [*] \leftarrow f \end{cases} \quad \texttt{graft} \\
\hline a, b, f, g, h, \alpha : \begin{cases} [] \leftarrow g \\ [*] \leftarrow f \end{cases} \longrightarrow h \quad \texttt{shift} \\
\hline a, b, f, g, h, \alpha \vdash \left\{ [] \leftarrow h \right. \quad \texttt{graft} \\
\hline a, b, f, g, h, \alpha, \beta : \left\{ [] \leftarrow h \right. \longrightarrow h \quad \texttt{shift}
\end{array}
$$

9.4 APPLICATION TO OPETOPIC CATEGORIES

In this section, we present system $\text{OptCat}^?$, an extension of $\text{OptSet}^?$ (definition 9.1.1). It is a direct implementation of the axioms of opetopic categories [BD98] [CL04] [Fin16].

OPERATIONS ON PASTING DIAGRAMS

Recall that in system $\text{OptSet}^?$, a non degenerate pasting diagram is a syntactical construct that reads

$$
\mathbf{P} = \begin{cases} [p_1] \leftarrow f_1 \\ \vdots \\ [p_k] \leftarrow f_k \end{cases} \tag{9.4.1}
$$

where there is an opetope $\omega \in \mathbb{O}$ such that $\omega^\bullet = \{[p_1], \ldots, [p_k]\}$.

Proposition 9.4.2. *Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$ be an opetopic set, and Γ be a derivable context in $\text{OptSet}^?$ such that $S^?\Gamma \cong X$. Then a map $S[\omega] \longrightarrow X$ (recall the definition of the spine $S[\omega]$ of ω from definition 3.5.1) corresponds exactly to a derivable pasting diagram $\mathbf{P}$ in Γ such that $\mathbf{P}^\natural = C^?(\omega)$, or equivalently, to an instance of the `graft` or `degen` rule on the context Γ.*

Proof. This follows by straightforward derivation of $S[\omega]$ in $\text{OptSet}^?$. □

Definition 9.4.3 (Substitution in a pasting diagram)**.** Let Γ be a derivable context, and $\mathbf{P}$ a non-degenerate pasting diagram in Γ as in equation (9.4.1). If h is a cell in Γ *parallel* to f_i (written $h \parallel f_i$), i.e. having the same type, then write

$$\mathbf{P} \underset{[p_i]}{\square} h := \mathbf{P}[h/f_i] = \begin{cases} [p_1] \leftarrow f_1 \\ \vdots \\ [p_i] \leftarrow h \\ \vdots \\ [p_k] \leftarrow f_k \end{cases}$$

Since f_i and h have the same type, they can be used interchangeably, and $\Gamma \vdash \mathbf{P} \square_{[p_i]} h$ is a derivable sequent.

Definition 9.4.4 (Source and target of a pasting diagram)**.** Let Γ be a derivable context, and $\mathbf{P}$ a pasting diagram in Γ (degenerate or not), of shape $\mathbf{P}^\natural = \mathbf{p}$. The *target* $\mathsf{t}\,\mathbf{P}$ of $\mathbf{P}$ is defined as follows:

(1) if $\mathbf{P}$ is degenerate, say $\mathbf{P} = \{\!\!\{ a$, then $\mathsf{t}\,\mathbf{P} := a$;
(2) otherwise, for $\mathbf{P}$ as in equation (9.4.1), let $\mathsf{t}\,\mathbf{P} := \mathsf{t}\,\mathsf{s}_{[]}\,\mathbf{P}$.

By definition, $\mathsf{t}\,\mathbf{P}$ is a cell in Γ of shape $(\mathsf{t}\,\mathbf{P})^\natural = \mathsf{t}\,\mathbf{p}$. By the same disjunction, define the *source* $\mathsf{s}\,\mathbf{P}$ of $\mathbf{P}$ as follows:

(1) if $\mathbf{P}$ is degenerate, say $\mathbf{P} = \{\!\!\{ a$, then $\mathsf{s}\,\mathbf{P} := a$;
(2) otherwise, for $\mathbf{P}$ as in equation (9.4.1), and $[p_1[q_1]], \ldots, [p_l[q_l]]$ the leaf addresses of $\mathbf{p}$, let

$$\mathsf{s}\,\mathbf{P} := \begin{cases} \wp_{\mathbf{p}}[p_1[q_1]] \leftarrow \mathsf{s}_{[q_1]}\,\mathsf{s}_{[p_1]}\,\mathbf{P} \\ \vdots \\ \wp_{\mathbf{p}}[p_l[q_l]] \leftarrow \mathsf{s}_{[q_l]}\,\mathsf{s}_{[p_l]}\,\mathbf{P} \end{cases}$$

In the first case, $\mathsf{s}\,\mathbf{P}$ is a cell of Γ, whereas in the second case, $\Gamma \vdash \mathsf{s}\,\mathbf{P}$ is derivable by the assumptions of the `graft` rule. In both cases, $(\mathsf{s}\,\mathbf{P})^\natural = \mathsf{s}_{[p]}\,\mathbf{p}$.

Lemma 9.4.5. *For Γ a derivable sequent, and $\alpha : \mathbf{P} \longrightarrow u$ a cell of Γ, we have $\mathsf{s}\,\mathbf{P} = \mathsf{s}\,u$, and $\mathsf{t}\,\mathbf{P} = \mathsf{t}\,u$.*

Proof. By assumption on the `fill` rule. ☐

INFERENCE RULES

In the definition of opetopic categories [BD98] [CL04] [Fin16], faces of a given cell can be annotated as "universal", and if so, certain lifting properties hold (see definition 9.4.7).

Notation 9.4.6. Recall that in system OptSet$^?$, a cell is typed as follows:

$$\alpha : \begin{cases} [p_1] \leftarrow f_1 \\ \vdots \\ [p_k] \leftarrow f_k \end{cases} \longrightarrow g.$$

We can also write $\alpha : \mathbf{P} \longrightarrow g$ for short. If α is *source universal* at address $[p_i]$ (or equivalently, at source f_i), for $1 \leq i \leq k$, we write

$$\alpha : \begin{cases} \vdots \\ [p_i] \leftarrow \forall f_i \\ \vdots \end{cases} \longrightarrow g,$$

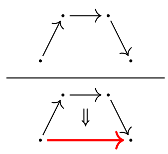
or $\alpha : \forall_{[p_i]}\mathbf{P} \longrightarrow g$ for short. Likewise, if α is *target universal*, we write

$$\alpha : \begin{cases} [p_1] \leftarrow f_1 \\ \vdots \\ [p_k] \leftarrow f_k \end{cases} \longrightarrow \forall g$$

or $\alpha : \mathbf{P} \longrightarrow \forall g$ for short.

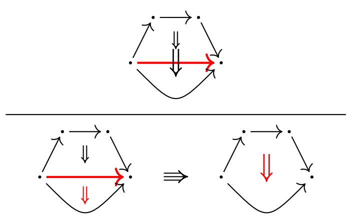
In definition 9.4.7, we present the four inference rules of system OptCat$^?$, implementing the axioms of opetopic categories of [BD98] [CL04] [Fin16] in our syntax. We specifically rely on the formulation of [Fin16]. But first, let us sketch them informally.

Compositions. Of course, there is a notion of composition, which in this case is *unbiased*, meaning the composition operation takes as input pasting diagrams, rather than pairs of cells satisfying some condition. Graphically,


`tfill`

Moreover, the filler 2-cell above is target universal, which is hinted by the thickened red arrow[1].

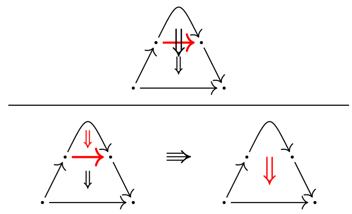
Target universality. Assume that we have a target universal cell, here the upper trapezoidal 2-cell, and another cell which has the same source, represented by the bigger double arrow. Then that cell factorizes through the target universal one, and further, the "factorizator" 3-cell is target universal and source universal at the factorization (red double arrow on the bottom left).


`tuniv`

This rule suggests that target universal cells should really be considered as compositors, even if they are not unique for a given pasting diagram.

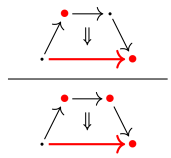
Source universality. Akin to target universality, assume that we have a source universal cell, here the lower trapezoidal 2-cell, and another cell which has the same target and sources, except for that universal one, represented by the bigger double arrow. Then that cell factorizes through the source universal one, and further, the "factorizator" 3-cell is target universal and source universal at the factorization (red double arrow on the top left of the conclusion).

[1]Note that being target universal is a property of the filler and not of the target. Thus strictly speaking, the filler should be highlighted rather than the target cell, but doing so results in less clear graphical representations.



suniv

Closure of target universal cells[2]*.* Lastly, the target universal cells enjoy the following closure property. Assume we have a target universal cell whose faces are also target universal, except for one. Then we may infer that one to be target universal too. Here, target universality of the 1-cells is represented by the red dots (the bottom right dot depicts the target universality of *both* arrows that have it as a target), and note that the top horizontal arrow is not target universal in the premise.



tclose

We now formulate those rules in the syntax of $\textsc{OptSet}^?$:

Definition 9.4.7 (The $\textsc{OptCat}^?$ system)**.** *Filling of pasting diagrams.* From a pasting diagram, this rule creates the "composite" cell u, and a target universal "compositor" γ.

$$\frac{\Gamma \vdash \mathbf{P}}{\Gamma, u : \mathsf{s}\,\mathbf{P} \longrightarrow \mathsf{t}\,\mathbf{P}, \gamma : \mathbf{P} \longrightarrow \forall u}\ \texttt{tfill}$$

Target universality.

$$\frac{\Gamma, \alpha : \mathbf{P} \longrightarrow \forall u, \beta : \mathbf{P} \longrightarrow v}{\Gamma, \alpha, \beta, \xi : \{[] \leftarrow u \longrightarrow v, A : \begin{cases} [] \leftarrow \forall \xi \\ [[]] \leftarrow \alpha \end{cases} \longrightarrow \forall \beta}\ \texttt{tuniv-}\alpha/\beta$$

Source universality. For s a cell of Γ, parallel to $\mathsf{s}_{[p]}\,\mathbf{P}$:

$$\frac{\Gamma, \alpha : \forall_{[p]}\mathbf{P} \longrightarrow u, \beta : \mathbf{P} \circ_{[p]} s \longrightarrow u}{\Gamma, \alpha, \beta, \xi : \{[] \leftarrow s \longrightarrow \mathsf{s}_{[p]}\,\mathbf{P}, A : \begin{cases} [] \leftarrow \alpha \\ [[p]] \leftarrow \forall \xi \end{cases} \longrightarrow \forall \beta}\ \texttt{suniv-}\alpha/\beta$$

Closure of target universal cells. Let A be a target universal cell such that its faces (sources and target) $\alpha_i : \mathbf{Q}_i \longrightarrow u_i$ are also target universal, for all $1 \leq i \leq k+1$, except for a given j. Then we may infer that α_j is target universal as well.

$$\frac{\Gamma, \alpha_j : \mathbf{Q}_j \longrightarrow u_j, A : \mathbf{P} \longrightarrow \forall \alpha_{k+1}}{\Gamma, \alpha_j : \mathbf{Q}_j \longrightarrow \forall u_j, A : \mathbf{P} \longrightarrow \forall \alpha_{k+1}}\ \texttt{tclose-}A$$

We derive some of Finster's examples [Fin16] in our system.

Example 9.4.8 (id, refl)**.** Given a variable x, we derive its *identity* id_x and *reflexivity* refl_x as follows.

$$\dfrac{\dfrac{\dfrac{\Gamma, x : \mathbf{P} \longrightarrow a}{\Gamma, x \vdash \{\!\!\{ x}\ \texttt{degen}}{\Gamma, x, \mathsf{id}_x : \{[] \leftarrow x \longrightarrow x, \mathsf{refl}_x : \{\!\!\{ x \longrightarrow \forall \mathsf{id}_x}\ \texttt{tfill}}{\Gamma, x, \mathsf{id}_x : \{[] \leftarrow x \longrightarrow \forall x, \mathsf{refl}_x}\ \texttt{tclose-}\mathsf{refl}_x$$

The fact that id_x indeed behaves like an identity is argued by the upcoming derivations of runit_f (example 9.4.10) and lunit_f (example 9.4.11), where $f : \{[] : x \longrightarrow y$ is a unary cell (i.e. an arrow).

Convention 9.4.9*.* For clarity, we do not mention the ambient context, or repeat previously derived variables. For instance, the proof tree of example 9.4.8 will be more concisely written as:

$$\dfrac{\dfrac{\dfrac{x : \mathbf{P} \longrightarrow a}{\cdots \vdash \{\!\!\{ x}\ \texttt{degen}}{\mathsf{id}_x : \{[] \leftarrow x \longrightarrow x, \quad \mathsf{refl}_x : \{\!\!\{ x \longrightarrow \forall \mathsf{id}_x}\ \texttt{tfill}}{\mathsf{id}_x : \{[] \leftarrow x \longrightarrow \forall x}\ \texttt{tclose-}\mathsf{refl}_x$$

Example 9.4.10 (runit)**.** For id_x and refl_x derived as in example 9.4.8,

$$\dfrac{\dfrac{\dfrac{\dfrac{x, y, \mathsf{id}_x, \mathsf{refl}_x, f : \{[] : x \longrightarrow y}{\cdots \vdash \begin{cases} [] \leftarrow f \\ [[]] \leftarrow \mathsf{id}_x \end{cases}}\ \texttt{graft}}{f\mathsf{id}_x : \{[] \leftarrow x \longrightarrow y, \quad \alpha : \begin{cases} [] \leftarrow f \\ [[]] \leftarrow \mathsf{id}_x \end{cases} \longrightarrow \forall f\mathsf{id}_x}\ \texttt{tfill}}{\cdots \vdash \begin{cases} [] \leftarrow \alpha \\ [[[]]] \leftarrow \mathsf{refl}_x \end{cases}}\ \texttt{graft}}{\dfrac{\mathsf{runit}_f : \{[] \leftarrow f \longrightarrow f\mathsf{id}_x, \quad \beta : \begin{cases} [] \leftarrow \alpha \\ [[[]]] \leftarrow \mathsf{refl}_x \end{cases} \longrightarrow \forall \mathsf{runit}_f}{\mathsf{runit}_f : \{[] \leftarrow f \longrightarrow \forall f\mathsf{id}_x}\ \texttt{tclose-}\beta}\ \texttt{tfill}$$

Example 9.4.11 (lunit)**.** For id_x and id_y derived as in example 9.4.8,

$$
\dfrac{\dfrac{\dfrac{\dfrac{x, y, \mathsf{id}_x, \mathsf{id}_y, f : \big\{[\,] : x \longrightarrow y}{\cdots \vdash \begin{cases}[\,] \leftarrow \mathsf{id}_y \\ [[\,]] \leftarrow f\end{cases}}\,\texttt{graft}}{\mathsf{id}_y f : \big\{[\,] \leftarrow x \longrightarrow y, \quad \alpha : \begin{cases}[\,] \leftarrow \mathsf{id}_y \\ [[\,]] \leftarrow f\end{cases} \longrightarrow \forall \mathsf{id}_y f}\,\texttt{tfill}}{\cdots \vdash \begin{cases}[\,] \leftarrow \alpha \\ [[\,]] \leftarrow \mathsf{refl}_y\end{cases}}\,\texttt{graft}}{\mathsf{lunit}_f : \big\{[\,] \leftarrow f \longrightarrow \mathsf{id}_y f, \quad \beta : \begin{cases}[\,] \leftarrow \alpha \\ [[\,]] \leftarrow \mathsf{refl}_y\end{cases} \longrightarrow \forall \mathsf{lunit}_f}\,\texttt{tfill}
$$
$$
\dfrac{\mathsf{lunit}_f : \big\{[\,] \leftarrow f \longrightarrow \mathsf{id}_y f, \quad \beta : \begin{cases}[\,] \leftarrow \alpha \\ [[\,]] \leftarrow \mathsf{refl}_y\end{cases} \longrightarrow \forall \mathsf{lunit}_f}{\mathsf{lunit}_f : \big\{[\,] \leftarrow f \longrightarrow \forall \mathsf{id}_y f}\,\texttt{tclose-}\beta
$$

Example 9.4.12 (f^{-1})**.** A unary target universal cell $f : \big\{[\,] \leftarrow x \longrightarrow \forall y$ (for x and y of arbitrary dimension) admits an antiparallel target universal cell f^{-1} constructed as follows. For id_x derived as in example 9.4.8,

$$
\dfrac{\dfrac{x, y, \mathsf{id}_x : \big\{[\,] \leftarrow x \longrightarrow \forall x, f : \big\{[\,] \leftarrow x \longrightarrow \forall y}{f^{-1} : \big\{[\,] \leftarrow y \longrightarrow x, \quad \mathsf{linv}_f : \begin{cases}\forall [\,] \leftarrow f \\ [[\,]] \leftarrow f^{-1}\end{cases} \longrightarrow \forall \mathsf{id}_x}\,\texttt{tuniv-}f/\mathsf{id}_x}{f^{-1} : \big\{[\,] \leftarrow y \longrightarrow \forall x,}\,\texttt{tclose-}\mathsf{linv}_f
$$

Example 9.4.13 (Associativity)**.** Consider three composable unary cells f, g and h. We show that there is a universal "coherence" cell between the unbiased composite fgh and the iterated binary composite $(fg)h$. The dual case $f(gh)$ is derived similarly. First, we derive fgh.

$$
\dfrac{\dfrac{f : \big\{[\,] \leftarrow a \longrightarrow b, \quad g : \big\{[\,] \leftarrow b \longrightarrow c, \quad h : \big\{[\,] \leftarrow c \longrightarrow d}{\cdots \vdash \begin{cases}[\,] \leftarrow h \\ [[\,]] \leftarrow g \\ [[\,][\,]] \leftarrow f\end{cases}}\,\texttt{ps}}{fgh : \big\{[\,] \leftarrow a \longrightarrow d, \quad \mathsf{cmp}_{fgh} : \begin{cases}[\,] \leftarrow h \\ [[\,]] \leftarrow g \\ [[\,][\,]] \leftarrow f\end{cases} \longrightarrow \forall fgh}\,\texttt{tfill}
$$

Next, we derive $(fg)h$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\cdots \vdash \begin{cases}[\,] \leftarrow g \\ [[\,]] \leftarrow f\end{cases}}\,\texttt{ps}}{fg : \big\{[\,] \leftarrow a \longrightarrow c, \quad \mathsf{cmp}_{fg} : \begin{cases}[\,] \leftarrow g \\ [[\,]] \leftarrow f\end{cases} \longrightarrow \forall fg}\,\texttt{tfill}}{\cdots \vdash \begin{cases}[\,] \leftarrow h \\ [[\,]] \leftarrow fg\end{cases}}\,\texttt{ps}}{(fg)h : \big\{[\,] \leftarrow a \longrightarrow d, \quad \mathsf{cmp}_{(fg)h} : \begin{cases}[\,] \leftarrow h \\ [[\,]] \leftarrow fg\end{cases} \longrightarrow \forall (fg)h}\,\texttt{tfill}
$$

Then, we compose cmp_{fg} and $\mathsf{cmp}_{(fg)h}$ to obtain a new cell parallel to cmp_{fgh}.

$$\dfrac{\dfrac{\vdots}{\cdots \vdash \begin{cases} [] \leftarrow \mathsf{cmp}_{(fg)h} \\ [[[]]] \leftarrow \mathsf{cmp}_{fg} \end{cases}}\ \texttt{ps}}{\mathsf{lcmp}_{fgh} : \begin{cases} [] \leftarrow h \\ [[]] \leftarrow g \\ [[][]] \leftarrow f \end{cases} \longrightarrow (fg)h, \quad A : \begin{cases} [] \leftarrow \mathsf{cmp}_{(fg)h} \\ [[[]]] \leftarrow \mathsf{cmp}_{fg} \end{cases} \longrightarrow \forall \mathsf{lcmp}_{fgh}}\ \texttt{tfill}$$

Finally, we establish a coherence cell between fgh and $(fg)h$.

$$\dfrac{\dfrac{\vdots}{\mathsf{lassoc}_{fgh} : \{[] \leftarrow fgh \longrightarrow (fg)h, \quad B : \begin{cases} [] \leftarrow \forall \mathsf{lassoc}_{fgh} \\ [[]] \leftarrow \mathsf{cmp}_{fgh} \end{cases} \longrightarrow \forall \mathsf{lcmp}_{fgh}}\ \spadesuit}{\mathsf{lassoc}_{fgh} : \{[] \leftarrow fgh \longrightarrow \forall (fg)h}\ \texttt{tclose-}B$$

where $\spadesuit$ is $\texttt{tuniv-}\mathsf{cmp}_{fgh}/\mathsf{lcmp}_{fgh}$. From here, example 9.4.12 transposes to lassoc_{fgh} to show that it is weakly invertible.

Part III

Algebras

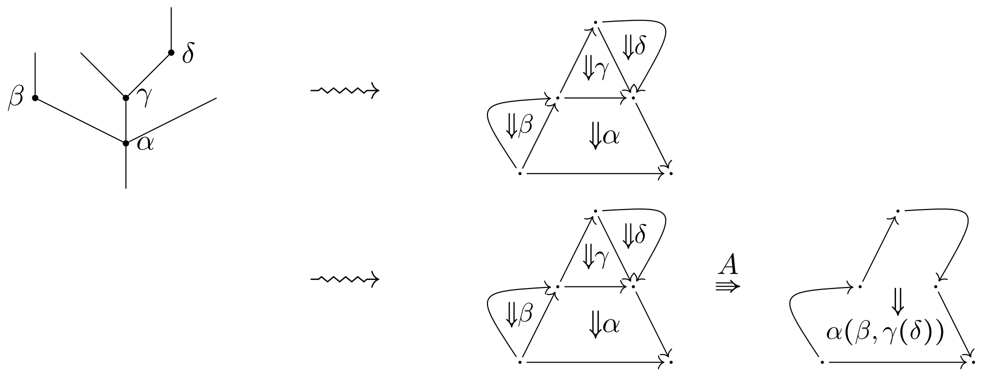
Chapter Ten

Introduction

THIS part of the thesis is devoted to studying a family of structures called *opetopic algebras*, which are algebraic structures whose operations have *higher dimensional tree-like arities*. As an example in lieu of a definition, a category is an algebraic structure where *operations* (a.k.a. morphisms) are shaped like arrows, in that their input and output consist of a single *color* (a.k.a. object). Consequently, the action of composing those operations takes as input sequences of (composable) arrows, which can be seen as linear trees of operations. Thus, in categories, the *shapes of compositions* are linear trees, i.e. 2-opetopes.

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d \xrightarrow{i} e \quad \rightsquigarrow \quad \begin{tikzcd} & b \arrow[r, "g"] & c \arrow[r, "h"] \arrow[d, phantom, "\Downarrow\mu"] & d \arrow[dr, "i"] & \\ a \arrow[ur, "f"] \arrow[rrrr, "i \cdot h \cdot g \cdot f"'] & & {} & & e \end{tikzcd}$$

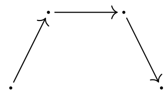
A second example, one dimension above, is that of a planar colored $\mathcal{S}$et-operad (also called nonsymmetric multicategory). Here, operations are shaped like 2-opetopes, since their inputs are sequences of colors. As before, composition takes as input a pasting diagram of operations, and since they are shaped like 2-opetopes, composition itself is shaped like a 3-opetope:



Heuristically extending this pattern leads one to presume that an algebra one dimension above planar operads should have a composition whose arities are planar trees of operations whose arities are also planar trees. Indeed, such algebraic structures are precisely the $\mathbb{PT}$-*combinads in* $\mathcal{S}$et (combinads over the combinatorial pattern of planar trees) of Loday [Lod12]. These examples fit in the following table:

Algebraic structure	Sets =0-algebras	Categories =1-algebras	Operads =2-algebras	$\mathbb{PT}$–combinads =3-algebras	… ?
Arity of compositions	Trivial =1-opetopes	Lists =2-opetopes	Trees =3-opetopes	Trees of trees =4-opetopes	… ?

The polynomial definition of opetopes and the category $\mathbb{O}$ of section 3.1 provide a very natural and uniform framework to deal with these examples and generalizations thereof. Recall that if $\omega \in \mathbb{O}$ and $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$, then a morphism $f : S[\omega] \longrightarrow X$ amounts to pasting diagram of shape ω of elements of X. For example, if $\omega = \mathbf{3}$, then its spine is

$S[\mathbf{3}] \quad =$ 

and thus f corresponds to a sequence of 3 “composable arrows” of X. Factoring f through the spine inclusion s_ω requires to find a *compositor*, i.e. a cell in $x \in X_\omega$ whose source is the pasting diagram defined by f. In other words, x is a *coherence cell* corresponding to f, while $\mathsf{t}x$ can be thought of as a *composition*

$$\begin{tikzcd} S[\omega] \arrow[r, "f"] \arrow[d, "\mathsf{s}_\omega"'] & X \\ O[\omega]. \arrow[ur, "x"'] & \end{tikzcd}$$

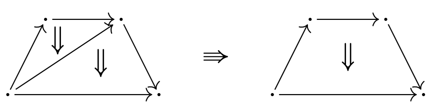
In particular, if $\mathsf{S}_{n+1} \perp X$, i.e. if all liftings against spine inclusions are unique, then we have a map

$$\mu : \{n\text{-pasting diagrams of } X\} \longrightarrow X_n.$$

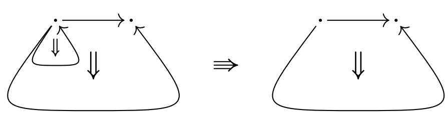
For example, if $n = 1$, then μ maps chains of arrows of X to arrows, which looks just like a category (ignoring higher-dimensional cells for now). If $Y \in \mathcal{P}\mathrm{sh}(\mathbb{O})$ is such that $\mathsf{S}_3 \perp Y$, then trees of elements of Y_2 can be composed into 2-cells, which is reminiscent of operads. A natural definition thus arise:

Definition 10.0.1 (Opetopic algebra (tentative))**.** An *n-opetopic algebra* is an opetopic set X such that $\mathsf{S}_{n+1} \perp X$.

Unfortunately, S_{n+1} cannot enforce suitable coherence conditions (i.e. associativity and unitality) by itself. The fundamental issue is the inability to “merge” multiple pasting diagrams into one. This can be addressed by requiring that $\mathsf{S}_{n+2} \perp X$. For example, in the case of categories ($n = 1$), if ω is the following 3-opetope



then the fact that $\mathsf{s}_\omega \perp X$ guarantees that for all composable arrows f, g, and h, we have $(fg)h = fgh$. A similar opetope would enforce $f(gh) = fgh$. If

$\nu \quad =$ 

then the fact that $\mathsf{s}_\nu \perp X$ guarantees that for all arrow $f : a \longrightarrow b$, $\mathrm{id}_a f = f$, where id_a is the composition of the empty pasting diagram at point a.

It turns out that $(n+2)$-opetopes are enough to retrieve all suitable coherence laws, as being orthogonal to S_{n+k} for greater values of k does not bring anything new (see corollary 3.5.11). We arrive at this new definition:

Definition 10.0.2 (Opetopic algebra (tentative))**.** An *n-opetopic algebra* is an opetopic set X such that $\mathsf{S}_{n+1,n+2} \perp X$.

It remains to deal with cells of dimension $> n+2$ and $< n$. For the former, it is enough to impose $\mathsf{B}_{>n+2} \perp X$, and for the former, $\mathsf{O}_{<n} \perp X$ (see definition 3.5.15). That last condition can be replaced by $\mathsf{O}_{<n-k} \perp X$ to allow for "colors" (akin to that of colored operads).

This shall be treated in more details in chapter 11, where we give two equivalent definitions for the notion of *k-colored n-dimensional opetopic algebras* (or *(k,n)-algebras* for short). The second one follows the intuitive approach above, while the main one presents opetopic algebras as algebras over an extension of the $\mathfrak{Z}^n$ monad (definition 3.1.1) over $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$. As advertised, we recover categories in the case $(k,n) = (1,1)$, colored operads in the case $(1,2)$, as well as $\mathbb{PT}$-combinads in the case $(0,3)$. The category $\mathcal{A}\mathrm{lg}_{k,n}$ of (k,n)-algebras comes with an adjunction

$$h : \mathcal{P}\mathrm{sh}(\mathbb{O}) \rightleftarrows \mathcal{A}\mathrm{lg}_{k,n} : M,$$

where M is the *opetopic nerve*, and where h maps an opetopic set X to the algebra whose operations are the n-cells of X, and where the $(n+1)$-cell are interpreted as relations[1]. In a result we call the "nerve theorem for $\mathbb{O}$" (theorem 11.2.33), we show that this adjunction is reflective, i.e. that being an algebra is a *property* of an opetopic set, rather than a structure. Therefore, opetopes and opetopic sets encode the underlying geometry of algebras with "higher arity". This adjunction shows that the monadic approach and the lifting approach presented above are equivalent.

As it turns out, the monad $\mathfrak{Z}^n$ over $\mathcal{P}\mathrm{sh}(\mathbb{O})$ is a *parametric right adjoint monad* [Web07] [BMW12]. The standard theory for such monads calls for another shape category, $\mathbb{\Lambda}_{k,n}$, and another nerve theorem in the form of a reflective adjunction

$$\tau : \mathcal{P}\mathrm{sh}(\mathbb{\Lambda}_{k,n}) \rightleftarrows \mathcal{A}\mathrm{lg}_{k,n} : N.$$

It is this adjunction, rather than $h \dashv M$ above, that allows us to recover the well-known examples

$$\mathcal{P}\mathrm{sh}(\mathbb{\Delta}) \rightleftarrows \mathcal{C}\mathrm{at}, \qquad \mathcal{P}\mathrm{sh}(\mathbb{\Omega}) \rightleftarrows \mathcal{O}\mathrm{p}_{\mathrm{col}},$$

where $\mathbb{\Omega}$ is the planar version of Moerdijk and Weiss's category of *dendrices* [MW07]. The category $\mathbb{\Lambda}_{k,n}$ of *opetopic shapes* is explicitly defined as the full subcategory of $\mathcal{A}\mathrm{lg}_{k,n}$ spanned by free algebras over $\mathbb{O}_{n-k,n+2}$. In the categorical case ($k = n = 1$), this simply says that $\mathbb{\Delta}$ is the category of finite ordinals, and in the operadic case ($k = 1$, $n = 2$), that $\mathbb{\Omega}$ is the category of operads spanned by trees. Thus, the functor $h : \mathbb{O}_{n-k,n+2} \longrightarrow \mathbb{\Lambda}_{k,n}$ gives rise to an adjunction that sits at the bottom of the following triangle

$$\begin{tikzcd}
 & \mathcal{A}\mathrm{lg}_{k,n} \arrow[dl, shift left, "M"] \arrow[dr, shift left, "N"] & \\
\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \arrow[ur, shift left, "h"] \arrow[rr, shift left, "h_!"] \arrow[rr, phantom, "\perp"] & & \mathcal{P}\mathrm{sh}(\mathbb{\Lambda}_{k,n}), \arrow[ul, shift left, "\tau"'] \arrow[ll, shift left, "h^*"]
\end{tikzcd}$$

[1] This is analogous to the homotopy category functor $\mathcal{P}\mathrm{sh}(\mathbb{\Delta}) \longrightarrow \mathcal{C}\mathrm{at}$, which interprets 0-cells as objects, 1-cells as arrows, and 2-cells as relations.

and gives a direct comparison between the opetopic nerve $M : \mathcal{A}\mathrm{lg}_{k,n} \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O})$ and the classical nerve $N : \mathcal{A}\mathrm{lg}_{k,n} \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{A}_{k,n})$, which is better-known in low-dimensional examples. The adjunction $h \dashv M$ that we presented first is in fact constructed last, as the composite of the lower and right adjunctions of the triangle above.

With the notion of opetopic algebras now established, we move on to the study of their homotopy theory, with the objective of generalizing existing results that apply to low-dimensional opetopic algebras and presheaves over $\mathbb{A} = \mathbb{A}_{k,n}$. In chapter 13, we introduce a model structure "à la Cisinski" on $\mathcal{P}\mathrm{sh}(\mathbb{A})$, denoted by $\mathcal{P}\mathrm{sh}(\mathbb{A})_\infty$. It is cofibrantly generated, the cofibrations are the monomorphisms, and the fibrant objects are those satisfying some horn filling conditions. This construction generalizes Joyal's structure on $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ for quasi-categories [JT07, theorem 1.9], and Cisinski–Moerdijk's structure on $\mathcal{P}\mathrm{sh}(\mathbb{\Omega})$ for ∞-operads in the planar case [CM13, theorem 1.1].

Then, in chapter 14, we define the *folk model structure* $\mathcal{A}\mathrm{lg}_{\mathrm{folk}}$ (we omit k and n to unclutter notations), which is a direct generalization of that for categories [Cis19, theorem 3.3.10], and operads [Wei07, theorem 1.6.2]. As expected, the weak equivalences are an adequate notion of "equivalence of algebras", i.e. invertible up to "natural isomorphism", and every object is fibrant and cofibrant. As in the familiar cases, we have a Quillen adjunction

$$\tau : \mathcal{P}\mathrm{sh}(\mathbb{A})_\infty \underset{\perp}{\rightleftarrows} \mathcal{A}\mathrm{lg}_{\mathrm{folk}} : M,$$

and in particular, nerves of algebras are "∞-algebras", i.e. fibrant in $\mathcal{P}\mathrm{sh}(\mathbb{A})_\infty$.

Next, in chapter 15, we move to the level of simplicial presheaves over $\mathbb{A}$. Much in the spirit of Rezk [Rez01], Joyal–Tierney [JT07], and Cisinski–Moerdijk [CM13], we show that there is an adequate notion of *Segal space* and *complete Segal space* in the opetopic setting, and that their homotopy theory can be studied by the means of model structures on $\mathcal{S}\mathrm{p}(\mathbb{A}) := \mathcal{P}\mathrm{sh}(\mathbb{\Delta})^{\mathbb{A}^{\mathrm{op}}}$, denoted by $\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{Segal}}$ and $\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{Rezk}}$, respectively. Further, the discrete space functor $(-)^{\mathrm{disc}} : \mathcal{P}\mathrm{sh}(\mathbb{A}) \longrightarrow \mathcal{S}\mathrm{p}(\mathbb{A})$ gives rise to a Quillen equivalence

$$\mathcal{P}\mathrm{sh}(\mathbb{A})_\infty \underset{\perp}{\rightleftarrows} \mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{Rezk}},$$

showing that complete Segal spaces are a model for ∞-algebras. Again, in low dimensions, those results are already known, see [JT07, theorem 4.11] for the simplicial case, and [CM13, corollary 6.7] for the planar dendroidal case.

Lastly, in chapter 16, we provide another simplicial model for ∞-algebras. Instead of considering simplicial presheaves like in chapter 15, we consider simplicial algebras. Recall that $\mathcal{A}\mathrm{lg}$ is the category of ($\mathcal{S}\mathrm{et}$-)models over a projective sketch, and let $\mathrm{I}\mathcal{A}\mathrm{lg}$ be the category of models in $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ of that same sketch. Generalizing the constructions of Horel [Hor15], we first endow $\mathrm{I}\mathcal{A}\mathrm{lg}$ with a model structure equivalent to $\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{proj}}$, the projective structure on $\mathcal{S}\mathrm{p}(\mathbb{A})$. Then, by successive localizations, we arrive to the desired structure $\mathrm{I}\mathcal{A}\mathrm{lg}_{\mathrm{Rezk}}$, which is related to $\mathcal{P}\mathrm{sh}(\mathbb{A})_\infty$ via a zig-zag of Quillen equivalences.

Most of the material of this part originates from a series of paper in collaboration with Chaitanya Leena Subramaniam [HL19, HL20a, HL20b].

Chapter Eleven

Opetopic algebras

LET $k \leq n \in \mathbb{N}$, and recall from notation 3.4.5 that $\mathbb{O}_{n-k,n} \hookrightarrow \mathbb{O}$ is the full subcategory of those opetopes ω such that $n - k \leq \dim \omega \leq n$. A *k-colored, n-dimensional opetopic algebra*, or (k, n)-opetopic algebra, will be an algebraic structure on a presheaf over $\mathbb{O}_{n-k,n}$, whose cells of dimension n are "operations" that can be "composed" in ways encoded by $(n+1)$-cells[1]. As we will see, the fact that the operations and relations of a (k, n)-opetopic algebra are encoded by opetopes of dimension $\geq n$ results in the category $\mathcal{A}\mathrm{lg}_{k,n}$ of (k, n)-opetopic algebras always having a canonical fully faithful *nerve functor* to the category $\mathcal{P}\mathrm{sh}(\mathbb{O})$ of opetopic sets (theorem 11.2.33).

We begin this chapter by surveying elements of the theory of *parametric right adjoint (p.r.a.) monads*. This will be essential to the definition of the *colored* $\mathfrak{Z}^n$ monad, which is an extension of $\mathfrak{Z}^n$ (in the sense of definition 3.1.1) to $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$. The algebras of this new monad will be the (k, n)-opetopic algebras. Then, we introduce the category $\mathbb{A}$ of *opetopic shapes*, which is the category of free algebras over $\mathbb{O}_{n-k,n}$. We investigate ways to construct algebras from presheaves over $\mathbb{A}$ and $\mathbb{O}$. Specifically, we obtain two adjunctions

$$h : \mathcal{P}\mathrm{sh}(\mathbb{O}) \rightleftarrows \mathcal{A}\mathrm{lg} : M, \qquad \tau : \mathcal{P}\mathrm{sh}(\mathbb{A}) \rightleftarrows \mathcal{A}\mathrm{lg} : N,$$

where the left adjoints are called *algebraic realizations*, and where the right adjoints are their respective *nerve functors*. The theory of p.r.a., which we review in section 11.1, provides remarkable information about the nerves, which we state in theorems 11.1.39 and 11.2.33. Finally, in section 11.3, we investigate a phenomenon we call *algebraic trompe-l'œil*, whereby, in a sense we make precise, any opetopic algebra can be reduced to a $(1, 3)$-algebra.

11.1 MONADIC APPROACH

PARAMETRIC RIGHT ADJOINTS

The goal of this section is to present elements of the theory of p.r.a. monads, and state the *nerve theorem for p.r.a. monads* (theorem 11.1.13). Informally, it gives a geometrical characterization of algebras over a p.r.a. monad. This is further investigated in corollary 11.1.14.

Definition 11.1.1 (Parametric right adjoint [Web07, definition 2.3] [Str00, section 5])**.** If $T : \mathcal{C} \longrightarrow \mathcal{D}$ is functor, and $\mathcal{C}$ has a terminal object 1, then T naturally factors as

$$\mathcal{C} \xrightarrow{\cong} \mathcal{C}/1 \xrightarrow{T_1} \mathcal{D}/T1 \longrightarrow \mathcal{D}, \tag{11.1.2}$$

[1] Recall that an $(n+1)$-opetope is precisely a pasting diagram of n-opetopes.

where the second functor is the dependent sum along the terminal morphism $! : T1 \longrightarrow 1$. We say that T is a *parametric right adjoint* (abbreviated p.r.a.) if T_1 has a left adjoint E.

Definition 11.1.3 (P.r.a monad)**.** A *p.r.a. monad* T is a monad whose endofunctor is a p.r.a. and whose unit $\mathrm{id} \longrightarrow T$ and multiplication $TT \longrightarrow T$ are cartesian natural transformations.

Notation 11.1.4. With a slight (but standard) abuse of notations, let $T : \mathcal{P}\mathrm{sh}(\mathcal{A}) \longrightarrow \mathcal{A}\mathrm{lg}(T)$ be the free T-algebra functor. The (identity-on-objects, fully faithful) factorization of the composite $\Theta_0 \hookrightarrow \mathcal{P}\mathrm{sh}(\mathcal{A}) \xrightarrow{T} \mathcal{A}\mathrm{lg}(T)$ will be denoted by

$$\Theta_0 \xrightarrow{j_T} \Theta_T \xrightarrow{i_T} \mathcal{A}\mathrm{lg}(T) \tag{11.1.5}$$

In other words, Θ_T is the full subcategory of $\mathcal{A}\mathrm{lg}(T)$ spanned by free algebras over elements of Θ_0.

Remark 11.1.6. We shall immediately restrict definition 11.1.1 to the case where $\mathcal{C} = \mathcal{D} = \mathcal{P}\mathrm{sh}(\mathcal{A})$ for a small category $\mathcal{A}$. Recall that $\mathcal{A}/T1$ is the category of elements of $T1 \in \mathcal{P}\mathrm{sh}(\mathcal{A})$, and using proposition 0.3.6, the factorization of (11.1.2) becomes

$$\mathcal{P}\mathrm{sh}(\mathcal{A}) \xrightarrow{T_1} \mathcal{P}\mathrm{sh}(\mathcal{A}/T1) \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{A}). \tag{11.1.7}$$

Let E be the left adjoint of T_1. Then T_1 is the nerve of the restriction $E : \mathcal{A}/T1 \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{A})$ of E to the representable presheaves, and the usual formula (see definition 0.4.9 and proposition 0.4.11) gives

$$(T_1 X)_x = \mathcal{P}\mathrm{sh}(\mathcal{A})(Ex, X),$$

where $X \in \mathcal{P}\mathrm{sh}(\mathcal{A})$ and $x \in \mathcal{A}/T1$. Therefore, for $a \in \mathcal{A}$, we have

$$(TX)_a = \sum_{x \in (T1)_a} \mathcal{P}\mathrm{sh}(\mathcal{A})(Ex, X) \tag{11.1.8}$$

In fact, it is clear that the data of the object $T1 \in \mathcal{P}\mathrm{sh}(\mathcal{A})$ and of the functor $E : \mathcal{A}/T1 \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{A})$ completely describe (via equation (11.1.8)) the functor T up to isomorphism. Let Θ_0 (leaving T implicit) be the full subcategory of $\mathcal{P}\mathrm{sh}(\mathcal{A})$ spanned by the image of the restriction of the left adjoint $E : \mathcal{A}/T1 \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{A})$ of T_1. Objects of Θ_0 are called *T-cardinals.*

Remark 11.1.9. A p.r.a. monad T on a presheaf category is an example of a *monad with arities* [BMW12]. The theory of monads with arities provides a remarkable amount of information about the free-forgetful adjunction $\mathcal{P}\mathrm{sh}(\mathcal{A}) \rightleftarrows \mathcal{A}\mathrm{lg}(T)$ and about the category of algebras $\mathcal{A}\mathrm{lg}(T)$. In particular, T has arities in Θ_0 [Web07, proposition 4.22].

Proposition 11.1.10 ([Web07, proposition 4.20])**.** *Let $T : \mathcal{P}\mathrm{sh}(\mathcal{A}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{A})$ be a p.r.a. monad, and Θ_0 be as in definition 11.1.3. Then the Yoneda embedding $\mathsf{y}_{\mathcal{A}}$ factors as*

$$\mathcal{A} \xrightarrow{i} \Theta_0 \xrightarrow{i_0} \mathcal{P}\mathrm{sh}(\mathcal{A})$$

or in other words, representable presheaves are T-cardinals. Since i_0 and $\mathsf{y}_{\mathcal{A}}$ are embeddings of categories, so is i.

Lemma 11.1.11. *Let $\mathcal{C}$ be a small category, and let*

$$\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{j} \mathcal{P}\mathrm{sh}(\mathcal{C})$$

be a factorization of the Yoneda embedding $\mathsf{y}_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{C})$, where i and j are also embeddings. In other words, $\mathcal{D}$ is a full subcategory of $\mathcal{P}\mathrm{sh}(\mathcal{C})$ containing all representable presheaves. Let $N_j : \mathcal{P}\mathrm{sh}(\mathcal{C}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{D})$ be the nerve of j. Then $N_j \cong i_*$, and j is dense.

Proof. Recall that N_j is right adjoint to $L := \mathrm{Lan}_{\mathsf{y}_{\mathcal{D}}} j$. For $X \in \mathcal{P}\mathrm{sh}(\mathcal{D})$ and $a \in \mathcal{C}$,

$$\begin{aligned}
(LX)_a &= \int^{b \in \mathcal{D}} X_b \times j(b)_a && \text{by definition of } L \\
&= \int^{b \in \mathcal{D}} X_b \times \mathcal{P}\mathrm{sh}(\mathcal{C})(a, j(b)) && \text{by the Yoneda lemma} \\
&= \int^{b \in \mathcal{D}} X_b \times \mathcal{P}\mathrm{sh}(\mathcal{C})(ji(a), j(b)) && \text{since } \mathsf{y}_{\mathcal{C}} = ji \\
&\cong \int^{b \in \mathcal{D}} X_b \times \mathcal{P}\mathrm{sh}(\mathcal{D})(i(a), b) && \text{since } j \text{ is fully faithful} \\
&\cong X_{i(a)} && \text{by the density formula} \\
&= (i^* X)_a && \text{by definition of } i^*,
\end{aligned}$$

naturally in X and a. Therefore, $L \cong i^*$, and by adjunction, $N_j \cong i_*$. By assumption, i is fully faithful, so by lemma 0.4.14, $i_* \cong N_j$ is fully faithful, and by proposition 0.4.11, j is dense. $\square$

Corollary 11.1.12. *Let i and j be as in lemma 11.1.11, and*

$$\mathsf{J}_{\mathcal{C}} := \{\varepsilon_\theta : i_! i^* \theta \longrightarrow \theta \mid \theta \in \mathcal{D} - \mathrm{im}\, i\},$$

where $\varepsilon : i_! i^* \longrightarrow \mathrm{id}_{\mathcal{P}\mathrm{sh}(\mathcal{D})}$ is the counit of the adjunction $i_! : \mathcal{P}\mathrm{sh}(\mathcal{C}) \rightleftarrows \mathcal{P}\mathrm{sh}(\mathcal{D}) : i^*$ (see notation 0.4.12). Then a presheaf $X \in \mathcal{P}\mathrm{sh}(\mathcal{D})$ is in the essential image of N_j if and only if $\mathsf{J}_{\mathcal{C}} \perp X$ (it is not hard to see that ε_θ is an isomorphism if $\theta \in \mathrm{im}\, i$, so $\mathsf{J}_{\mathcal{C}} \perp X$ if and only if $\{\varepsilon_\theta \mid \theta \in \mathcal{D}\} \perp X$).

Proof. Let $Y \in \mathcal{P}\mathrm{sh}(\mathcal{C})$ and $b \in \mathcal{D}$. Since i_* is fully faithful, the counit $i^* i_* Y \longrightarrow Y$ of the adjunction $i^* \dashv i_*$ is an isomorphism. We have

$$\begin{aligned}
\mathcal{P}\mathrm{sh}(\mathcal{D})(b, N_j Y) &\cong \mathcal{P}\mathrm{sh}(\mathcal{D})(b, i_* Y) && \text{by lemma 11.1.11} \\
&\cong \mathcal{P}\mathrm{sh}(\mathcal{C})(i^* b, i^* i_* Y) && \text{since } i_* \text{ is fully faithful} \\
&\cong \mathcal{P}\mathrm{sh}(\mathcal{D})(i_! i^* b, i_* Y) && \text{since } i_! \dashv i^* \\
&\cong \mathcal{P}\mathrm{sh}(\mathcal{D})(i_! i^* b, N_j Y) && \text{by lemma 11.1.11.}
\end{aligned}$$

It is easy to check that one direction of the previous isomorphism is pre-composition by ε_b, thus $\varepsilon_b \perp N_j Y$ for all $b \in \mathcal{D}$. Conversely, take $X \in \mathcal{P}\mathrm{sh}(\mathcal{D})$ such that $\mathsf{J}_{\mathcal{C}} \perp X$. As previously mentionned, $\varepsilon_b \perp X$ for all $b \in \mathcal{D}$, so

$$\begin{aligned}
(N_j i^* X)_b &\cong (i_* i^* X)_b && \text{by lemma 11.1.11} \\
&\cong \mathcal{P}\mathrm{sh}(\mathcal{D})(b, i_* i^* X) && \text{by the Yoneda lemma} \\
&\cong \mathcal{P}\mathrm{sh}(\mathcal{D})(i_! i^* b, X) && \text{since } i_! \dashv i^* \dashv i_*
\end{aligned}$$

$$
\begin{aligned}
&\cong \mathcal{P}\mathrm{sh}(\mathcal{D})(b, X) && \text{since } \varepsilon_b \perp X \\
&\cong X_b && \text{by the Yoneda lemma,}
\end{aligned}
$$

and thus X is in the essential image of N_j. □

Theorem 11.1.13. *(1) The functors $i_0 : \Theta_0 \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{A})$ and $i_T : \Theta_T \longrightarrow \mathcal{A}\mathrm{lg}(T)$ are dense. Equivalently, their nerve $N_{i_0} : \mathcal{P}\mathrm{sh}(\mathcal{A}) \longrightarrow \mathcal{P}\mathrm{sh}(\Theta_0)$ and $N_{i_T} : \mathcal{A}\mathrm{lg}(T) \longrightarrow \mathcal{P}\mathrm{sh}(\Theta_T)$ are fully faithful.*

(2) The square of the left is an exact adjoint square *[BMW12, section 1.4], i.e. there exists a natural isomorphism $N_{i_0} U \longrightarrow j_T^* N_{i_T}$ whose mate $t_! N_{i_0} \longrightarrow N_{i_T} T$ is invertible[2]:*

$$
\begin{tikzcd}
\mathcal{P}\mathrm{sh}(\mathcal{A}) \arrow[d, "N_{i_0}"'] & \mathcal{A}\mathrm{lg}(T) \arrow[l, "U"'] \arrow[d, "N_{i_T}"] \\
\mathcal{P}\mathrm{sh}(\Theta_0) & \mathcal{P}\mathrm{sh}(\Theta_T) \arrow[l, "j_T^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}\mathrm{sh}(\mathcal{A}) \arrow[r, "T"] \arrow[d, "N_{i_0}"'] & \mathcal{A}\mathrm{lg}(T) \arrow[d, "N_{i_T}"] \\
\mathcal{P}\mathrm{sh}(\Theta_0) \arrow[r, "t_!"] & \mathcal{P}\mathrm{sh}(\Theta_T)
\end{tikzcd}
$$

In particular, both squares commute up to natural isomorphism.

(3) (Segal condition) A presheaf $X \in \mathcal{P}\mathrm{sh}(\Theta_T)$ is in the essential image of N_{i_T} if and only if $j_T^* X$ is in the essential image of N_{i_0}.

Proof. Density of i_0 is a direct consequence of lemma 11.1.11 and proposition 11.1.10, and density of i_T is [BMW12, theorem 1.10]. Point (2) is [BMW12, proposition 1.9], and the Segal condition is [Web07, theorem 4.10 (2)]. □

Corollary 11.1.14. *Let*

$$
\mathsf{J}_T \coloneqq t_! \mathsf{J}_{\mathcal{A}} = \{ t_! \varepsilon_\theta : t_! i_! i^* \theta \longrightarrow t_! \theta \mid \theta \in \Theta_0 - \operatorname{im} i \},
$$

where ε is the counit of the adjunction $i_! \dashv i^*$. Then a presheaf $X \in \mathcal{P}\mathrm{sh}(\Theta_T)$ is in the essential image of N_{i_T} if and only if $\mathsf{J}_T \perp X$. As a consequence, the left adjoint $\mathcal{P}\mathrm{sh}(\Theta_T) \longrightarrow \mathcal{A}\mathrm{lg}(T)$ of N_{i_T} (i.e. the left Kan extension of i_T along the Yoneda embedding) exhibits an equivalence of categories

$$
\mathsf{J}_T^{-1} \mathcal{P}\mathrm{sh}(\Theta_T) \xrightarrow{\simeq} \mathcal{A}\mathrm{lg}(T).
$$

Proof. The first claim follows from corollary 11.1.12 and theorem 11.1.13. For the second, see corollary 0.5.12. □

COLORED $\mathfrak{Z}^n$-ALGEBRAS

Remark 11.1.15. Recall the definition of the polynomial monad $\mathfrak{Z}^n$ from definition 3.1.1. If $X = (X_\psi \mid \psi \in \mathbb{O}_n)$ is a set over $\mathbb{O}_n$, and if $\omega \in \mathbb{O}_n$, then

$$
(\mathfrak{Z}^n X)_\omega = \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathsf{t}\,\nu = \omega}} \prod_{[p] \in \nu^\bullet} X_{\mathsf{s}_{[p]}\,\nu}.
$$

[2] This definition is a slight generalization of the *Beck–Chevalley condition* [MP00, remark 2.6].

Under the equivalence $\mathcal{S}\mathrm{et}/\mathbb{O}_n \simeq \mathcal{P}\mathrm{sh}(\mathbb{O}_n)$, this formula can be rewritten as

$$(\mathfrak{Z}^n X)_\omega = \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathsf{t}\,\nu = \omega}} \mathcal{P}\mathrm{sh}(\mathbb{O}_n)(S[\nu], X),$$

where $S[\nu]$ is the truncated spine of ν (see definition 3.5.1).

In this section, we extend the polynomial monad $\mathfrak{Z}^n$ over $\mathcal{S}\mathrm{et}/\mathbb{O}_n = \mathcal{P}\mathrm{sh}(\mathbb{O}_n)$ to a p.r.a. over $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$, where $k \leq n$. This new setup will encompass more known examples than the uncolored case (see proposition 11.1.29). For instance, recall that the polynomial monad $\mathfrak{Z}^2$ on $\mathcal{S}\mathrm{et}/\mathbb{O}_2 \cong \mathcal{S}\mathrm{et}/\mathbb{N}$ is exactly the monad of planar operads. The extension of $\mathfrak{Z}^2$ to $\mathcal{P}\mathrm{sh}(\mathbb{O}_{1,2})$ will retrieve *colored* planar operads as algebras. Similarly, the polynomial monad $\mathfrak{Z}^1$ on $\mathcal{S}\mathrm{et}$ is the free-monoid monad, which we would like to vary to obtain “colored monoids”, i.e. small categories.

The first step of this construction is to define $\mathfrak{Z}^n$ as a p.r.a. functor, i.e. such that $\mathfrak{Z}^n_1$ below is a right adjoint:

$$\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) \xrightarrow{\mathfrak{Z}^n_1} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}/\mathfrak{Z}^n 1) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}).$$

Following remark 11.1.6, it suffices to define its value $\mathfrak{Z}^n 1$ on the terminal presheaf, and to specify a functor $E : \mathbb{O}_{n-k,n}/\mathfrak{Z}^n 1 \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$.

Definition 11.1.16. Define $\mathfrak{Z}^n 1 \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$ as

$$(\mathfrak{Z}^n 1)_\psi := \{*\}, \qquad\qquad (\mathfrak{Z}^n 1)_\omega := \{\nu \in \mathbb{O}_{n+1} \mid \mathsf{t}\,\nu = \omega\},$$

where $\psi \in \mathbb{O}_{n-k,n-1}$ and $\omega \in \mathbb{O}_n$. We now define a functor $E : \mathbb{O}_{n-k,n}/\mathfrak{Z}^n 1 \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$. On objects, for $* \in (\mathfrak{Z}^n 1)_\psi$ and $\nu \in (\mathfrak{Z}^n 1)_\omega$, let[3]

$$E(*) := O[\psi], \qquad\qquad E(\nu) := S[\nu]. \tag{11.1.17}$$

On morphisms, E maps face embeddings to the canonical inclusions. The functor $\mathfrak{Z}^n_1 : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}/\mathfrak{Z}^n 1)$ is defined as the right adjoint to the left Kan extension of E along the Yoneda embedding, i.e. $\mathfrak{Z}^n_1 = N_E$ (see definition 0.4.9). We now recover the endofunctor $\mathfrak{Z}^n$ explicitly using equation (11.1.8): for $\psi \in \mathbb{O}_{n-k,n-1}$ we have $(\mathfrak{Z}^n X)_\psi \cong X_\psi$, and for $\omega \in \mathbb{O}_n$, we end up with a formula similar to remark 11.1.15

$$(\mathfrak{Z}^n X)_\omega \cong \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathsf{t}\,\nu = \omega}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})(S[\nu], X).$$

Example 11.1.18. Let us unfold definition 11.1.16 in the case $n = 1$ and $k = 1$. Here, $\mathcal{P}\mathrm{sh}(\mathbb{O}_{0,1})$ is the category of directed graphs, whose terminal object 1 is the graph with one vertex and a loop. The graph $\mathfrak{Z}^1 1$ also has one vertex, but this time, it has an many loops as there are 2-opetopes, i.e. one loop per element in $\mathbb{N}$. The category of elements $\mathbb{O}_{0,1}/\mathfrak{Z}^1 1$ looks like this:

$$\begin{tikzcd}
* \arrow[d, "s_0{,}t_0"'] \arrow[dr, bend left, "s_1{,}t_1"] \arrow[drr, bend left, "s_2{,}t_2"] \arrow[drrrr, bend left, "s_m{,}t_m"] \\
0 & 1 & 2 & \cdots & m & \cdots
\end{tikzcd}$$

[3]Note that in equation (11.1.17), the presheaves $O[\psi]$ and $S[\nu]$ are considered in $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$, but as per convention 3.5.20, the truncations are left implicit.

where $*$ corresponds to the vertex of $\mathfrak{Z}^1 1$, the numbers on the second row correspond to its vertices, and the morphisms are the inclusions of $*$ as the source or target of these vertices. The functor $E : \mathbb{O}_{0,1}/\mathfrak{Z}^1 1 \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{0,1})$ maps $*$ to the graph with one vertex and no edges, and maps m to the linear graph with m consecutive edges:

$$E(*) = (\bullet), \qquad E(m) = (\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet).$$

On morphisms, $E(s_n)$ (resp. $E(t_n)$) is the inclusion of $\bullet$ as the first (resp. as the last) vertex of $E(m)$. Then, for $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{0,1})$, the graph $\mathfrak{Z}^1 X$ has the same vertices as X, but its edges are paths in X. In other words, $\mathfrak{Z}^1 : \mathcal{P}\mathrm{sh}(\mathbb{O}_{0,1}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{0,1})$ is the free category monad.

Recall from definition 11.1.3 that a p.r.a. monad is a monad T whose unit $\mathrm{id} \longrightarrow T$ and multiplication $TT \longrightarrow T$ are cartesian, and such that its underlying functor is a p.r.a. We now endow $\mathfrak{Z}^n$ with the structure of a p.r.a. monad over $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$. We first specify the unit and multiplication $\eta_1 : 1 \longrightarrow \mathfrak{Z}^n 1$ and $\mu_1 : \mathfrak{Z}^n \mathfrak{Z}^n 1 \longrightarrow \mathfrak{Z}^n 1$ on the terminal object 1, and extend them to cartesian natural transformations (lemma 11.1.23). Next, we check that the required monad identities hold for 1 (lemma 11.1.25), which automatically gives us the desired monad structure on $\mathfrak{Z}^n$.

Proposition 11.1.19. *Recall from definition 3.1.7 that $\mathbb{O}_{n+2}^{(2)}$ is the set of $(n+2)$-opetopes of* of uniform height 2, *and let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$. Then as usual, $(\mathfrak{Z}^n \mathfrak{Z}^n X)_{<n} = X_{<n}$, and if $\omega \in \mathbb{O}_n$, then*

$$(\mathfrak{Z}^n \mathfrak{Z}^n X)_\omega \cong \sum_{\substack{\xi \in \mathbb{O}_{n+2}^{(2)} \\ \mathsf{t}\mathsf{t}\xi = \omega}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})(S[\mathsf{t}\xi], X).$$

Proof. Take $\omega \in \mathbb{O}_n$, and $x \in (\mathfrak{Z}^n \mathfrak{Z}^n X)_\omega$, say $x : S[\nu] \longrightarrow \mathfrak{Z}^n X$, where $\mathsf{t}\nu = \omega$. For $[p_i] \in \nu^\bullet$, write $x_i := x[p_i] : S[\nu_i] \longrightarrow X$, where $\mathsf{t}\nu_i = \mathsf{s}_{[p_i]}\nu$. Informally, x is a "pasting diagram of pasting diagrams" of X, i.e. a pasting diagram of the x_i's, which are themselves pasting diagrams in X. The goal is to assemble the x_i's in a single pasting diagram $\Phi(x)$. Let

$$\xi := \mathsf{Y}_\nu \bigcirc_{[[p_i]]} \mathsf{Y}_{\nu_i},$$

and note that $\mathsf{t}\mathsf{t}\xi = \mathsf{t}\mathsf{s}_{[]}\xi = \mathsf{t}\nu = \omega$ by **(Glob1)**. We now define a map $\Phi(x) : S[\mathsf{t}\xi] \longrightarrow X$. Note that leaf addresses of ξ are of the form $[[p_i][l]]$, where $[l] \in \nu_i^{|}$, thus node addresses of $\mathsf{t}\xi$ are of the form $\wp_\xi[[p_i][l]]$. Let

$$\Phi(x)\left(\wp_\xi[[p_i][l]]\right) := x_i\left(\wp_{\nu_i}[l]\right).$$

The construction of $\Phi(x)$ provides a map

$$\Phi : (\mathfrak{Z}^n \mathfrak{Z}^n X)_\omega \longrightarrow \sum_{\substack{\xi \in \mathbb{O}_{n+2}^{(2)} \\ \mathsf{t}\mathsf{t}\xi = \omega}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})(S[\mathsf{t}\xi], X)$$

whose inverse we now construct. Let $\xi \in \mathbb{O}_{n+2}^{(2)}$, say

$$\xi = \mathsf{Y}_\alpha \bigcirc_{[[p]]} \mathsf{Y}_{\beta_{[p]}},$$

be such that $\mathsf{t}\mathsf{t}\xi = \omega$, and take $y : S[\mathsf{t}\xi] \longrightarrow X$. Write $\nu := \mathsf{t}\xi$. As noted in definition 3.1.7, ξ exhibits a partition of ν into subtrees, and let $\Psi(y) : S[\alpha] \longrightarrow \mathfrak{Z}^n X$ map $[p]$ to the restriction of y to the subtree $\beta_{[p]}$ of ν. It is routine verification to check that Φ and Ψ are mutually inverse. $\square$

Definition 11.1.20. We now define $\eta_1 : 1 \longrightarrow \mathfrak{Z}^n 1$ and $\mu_1 : \mathfrak{Z}^n \mathfrak{Z}^n 1 \longrightarrow \mathfrak{Z}^n 1$, the monads laws of $\mathfrak{Z}^n$, on the terminal presheaf $1 \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$. In dimension $< n$, they are the identity. Let $\omega \in \mathbb{O}_n$. Recall from definition 11.1.16 that $(\mathfrak{Z}^n 1)_\omega = \{\nu \in \mathbb{O}_{n+1} \mid \mathsf{t}\nu = \omega\}$, and by proposition 11.1.19,

$$(\mathfrak{Z}^n \mathfrak{Z}^n 1)_\omega = \left\{\xi \in \mathbb{O}_{n+2}^{(2)} \mid \mathsf{t}\mathsf{t}\xi = \omega\right\}. \tag{11.1.21}$$

Now, let $(\eta_1)_\omega$ map the unique element of 1_ω to $\mathsf{Y}_\omega \in (\mathfrak{Z}^n 1)_\omega$, and let $(\mu_1)_\omega$ map $\xi \in (\mathfrak{Z}^n \mathfrak{Z}^n 1)_\omega$ to $\mathsf{t}\xi \in (\mathfrak{Z}^n 1)_\omega$.

Remark 11.1.22. Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$, and consider the terminal map $! : X \longrightarrow 1$. The map $\mathfrak{Z}^n ! : (\mathfrak{Z}^n X)_\omega \longrightarrow (\mathfrak{Z}^n 1)_\omega$ simply maps a pasting diagram $f : S[\nu] \longrightarrow X$ (where $\mathsf{t}\nu = \omega$) to its shape ν.

Lemma 11.1.23. *Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$, and consider the terminal map $! : X \longrightarrow 1$. To alleviate notations, write $p := \mathfrak{Z}^n ! : \mathfrak{Z}^n X \longrightarrow \mathfrak{Z}^n 1$. there exists maps $\eta_X : X \longrightarrow \mathfrak{Z}^n X$ and $\mu_X : \mathfrak{Z}^n \mathfrak{Z}^n X \longrightarrow \mathfrak{Z}^n$ such that the following squares are cartesian:*

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "!"'] & \mathfrak{Z}^n X \arrow[d, "p"] \\ 1 \arrow[r, "\eta_1"] & \mathfrak{Z}^n 1, \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{Z}^n \mathfrak{Z}^n X \arrow[r, "\mu_X"] \arrow[d, "\mathfrak{Z}^n p"'] & \mathfrak{Z}^n X \arrow[d, "p"] \\ \mathfrak{Z}^n \mathfrak{Z}^n 1 \arrow[r, "\mu_1"] & \mathfrak{Z}^n 1. \end{tikzcd} \tag{11.1.24}$$

In particular, the maps η_X and μ_X assemble into cartesian natural transformations $\eta : \mathrm{id} \longrightarrow \mathfrak{Z}^n$ and $\mu : \mathfrak{Z}^n \mathfrak{Z}^n \longrightarrow \mathfrak{Z}^n$.

Proof. Both squares trivially cartesian in dimension $< n$, so it suffices to check in dimension n.

(1) If P is the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{Z}^n X \arrow[d, "p"] \\ 1 \arrow[r, "\eta_1"] & \mathfrak{Z}^n 1, \end{tikzcd}$$

then for $\omega \in \mathbb{O}_n$ we have

$$P_\omega = \{x \in \mathfrak{Z}^n X \mid p(x) = \mathsf{Y}_\omega\} = \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})(S[\mathsf{Y}_\omega], X) = X_\omega,$$

as $S[\mathsf{Y}_\omega] = O[\omega]$.

(2) Let P be the bullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{Z}^n X \arrow[d, "p"] \\ \mathfrak{Z}^n \mathfrak{Z}^n 1 \arrow[r, "\mu_1"] & \mathfrak{Z}^n 1, \end{tikzcd}$$

and let $\omega \in \mathbb{O}_n$. By definition, and with equation (11.1.21), P_ω is the set of all pairs (ξ, x), where $\xi \in \mathbb{O}_{n+2}^{(2)}$ is such that $\mathsf{t}\mathsf{t}\xi = \omega$, $x : S[\nu] \longrightarrow X$ is such that $\mathsf{t}\nu = \omega$, and subject to the constraint that $\mathsf{t}\xi = \nu$. By proposition 11.1.19, it is clear that $P_\omega \cong (\mathfrak{Z}^n \mathfrak{Z}^n X)_\omega$. ☐

Lemma 11.1.25. *The following diagrams commute:*

$$\begin{tikzcd}
\mathfrak{Z}^n 1 \arrow[r, "\eta_{\mathfrak{Z}^n 1}"] \arrow[dr, equal] & \mathfrak{Z}^n \mathfrak{Z}^n 1 \arrow[d, "\mu_1"] & \mathfrak{Z}^n 1 \arrow[l, "\mathfrak{Z}^n \eta_1"'] \arrow[dl, equal] \\
& \mathfrak{Z}^n 1, &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{Z}^n \mathfrak{Z}^n \mathfrak{Z}^n 1 \arrow[r, "\mathfrak{Z}^n \mu_1"] \arrow[d, "\mu_{\mathfrak{Z}^n 1}"'] & \mathfrak{Z}^n \mathfrak{Z}^n 1 \arrow[d, "\mu_1"] \\
\mathfrak{Z}^n \mathfrak{Z}^n 1 \arrow[r, "\mu_1"] & \mathfrak{Z}^n 1.
\end{tikzcd}$$

Proof. Recall from definition 11.1.16 that for $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$, $(\mathfrak{Z}^n X)_{<n} = X_{<n}$. Thus all diagrams commute trivially in dimension $< n$.

(1) Let $\omega \in \mathbb{O}_n$ and $\nu \in \mathfrak{Z}^n 1_\omega$, i.e. $\nu \in \mathbb{O}_{n+1}$ such that $\mathsf{t}\nu = \omega$. Then

$$\begin{aligned}
\mu_1 \eta_{\mathfrak{Z}^n 1}(\nu) &= \mu_1 \left(\mathsf{Y}_{\mathsf{Y}_{\mathsf{t}\nu}} \underset{[[]]}{\circ} \mathsf{Y}_\nu \right) && \text{see definition 11.1.20} \\
&= \mathsf{t} \left(\mathsf{Y}_{\mathsf{Y}_{\mathsf{t}\nu}} \underset{[[]]}{\circ} \mathsf{Y}_\nu \right) && \text{see definition 11.1.20} \\
&= \mathsf{Y}_{\mathsf{t}\nu} \underset{[]}{\square} \nu && \text{by proposition 3.1.6} \\
&= \nu,
\end{aligned}$$

and similarly, if $\{[p_1], \ldots\} = \nu^\bullet$,

$$\begin{aligned}
\mu_1 (\mathfrak{Z}^n \eta_1)(\nu) &= \mu_1 \left(\mathsf{Y}_\nu \underset{[[p_i]]}{\bigcirc} \mathsf{Y}_{\mathsf{Y}_{\mathsf{s}_{[p_i]}\nu}} \right) && \spadesuit \\
&= \mathsf{t} \left(\mathsf{Y}_\nu \underset{[[p_i]]}{\bigcirc} \mathsf{Y}_{\mathsf{Y}_{\mathsf{s}_{[p_i]}\nu}} \right) && \spadesuit \\
&= \left(\nu \underset{[p_1]}{\square} \mathsf{Y}_{\mathsf{s}_{[p_1]}\nu} \right) \underset{[p_2]}{\square} \mathsf{Y}_{\mathsf{s}_{[p_2]}\nu} \cdots && \text{by proposition 3.1.6} \\
&= \nu,
\end{aligned}$$

where $\spadesuit$ follows from definition 11.1.20.

(2) Akin to proposition 11.1.19, one can show that elements of $\mathfrak{Z}^n \mathfrak{Z}^n \mathfrak{Z}^n 1_\omega$ are $(n+2)$-opetopes ξ of uniform height 3 such that $\mathsf{t}\mathsf{t}\xi = \omega$. Let ξ be such an opetope, and write it as

$$\xi = \mathsf{Y}_\alpha \underset{[[p_i]]}{\bigcirc} \underbrace{\left(\mathsf{Y}_{\beta_i} \underset{[[q_{i,j}]]}{\bigcirc} \mathsf{Y}_{\gamma_{i,j}} \right)}_{A_i :=} = \underbrace{\left(\mathsf{Y}_\alpha \underset{[[p_i]]}{\bigcirc} \mathsf{Y}_{\beta_i} \right)}_{B :=} \underset{[[p_i][q_{i,j}]]}{\bigcirc} \mathsf{Y}_{\gamma_{i,j}}$$

where $\alpha, \beta_i, \gamma_{i,j} \in \mathbb{O}_n$, $[p_i]$ ranges over $\alpha^\bullet$ and $[q_{i,j}]$ over $\beta_i^\bullet$. Then

$$\begin{aligned}
\mu_1 (\mathfrak{Z}^n \mu_1)(\xi) &= \mu_1 (\mathfrak{Z}^n \mu_1) \left(\mathsf{Y}_\alpha \underset{[[p_i]]}{\bigcirc} A_i \right) \\
&= \mu_1 \left(\mathsf{Y}_\alpha \underset{[[p_i]]}{\bigcirc} \mathsf{Y}_{\mathsf{t} A_i} \right)
\end{aligned}$$

$$= \mathsf{t}\left(\mathsf{Y}_\alpha \bigcirc_{[[p_i]]} \mathsf{Y}_{\mathsf{t}A_i}\right)$$

$$= \mathsf{t}\left(\mathsf{Y}_\alpha \bigcirc_{[[p_i]]} A_i\right) \qquad \text{by proposition 3.1.6}$$

$$= \mathsf{t}\left(B \bigcirc_{[[p_i][q_{i,j}]]} \mathsf{Y}_{\gamma_{i,j}}\right) \qquad \text{by definition}$$

$$= \mathsf{t}\left(\mathsf{Y}_{\mathsf{t}B} \bigcirc_{[\wp_B[[p_i][q_{i,j}]]]} \mathsf{Y}_{\gamma_{i,j}}\right) \qquad \text{by proposition 3.1.6}$$

$$= \mu_1 \mu_{\mathfrak{Z}^n 1}(\xi).$$

□

Proposition 11.1.26. *The cartesian natural transformations μ and η (whose components are defined in definition 11.1.20 and lemma 11.1.23) give $\mathfrak{Z}^n$ a structure of p.r.a. monad on $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$.*

Proof. This is a direct consequence of lemmas 11.1.23 and 11.1.25. □

Remark 11.1.27. Clearly, when $k = 0$, we recover the usual polynomial monad on $\mathcal{S}\mathrm{et}/\mathbb{O}_n$.

Definition 11.1.28 (Opetopic algebra)**.** A *k-colored n-dimensional opetopic algebras* is an algebra of $\mathfrak{Z}^n$ in $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$. We write $\mathcal{A}\mathrm{lg}_{k,n}$ for the Eilenberg–Moore category of $\mathfrak{Z}^n$.

Proposition 11.1.29. *Up to equivalence, and for small values of k and n with $k \le n$, the category $\mathcal{A}\mathrm{lg}_{k,n}$ is given by the following table:*

$k \backslash n$	0	1	2	3
0	$\mathcal{S}\mathrm{et}$	$\mathcal{M}\mathrm{on}$	$\mathcal{O}\mathrm{p}$	$\mathcal{C}\mathrm{omb}_{\mathbb{PT}}$
1		$\mathcal{C}\mathrm{at}$	$\mathcal{O}\mathrm{p}_{\mathrm{col}}$	$\mathcal{A}\mathrm{lg}_{1,3}$
2			$\mathcal{A}\mathrm{lg}_{2,2}$	$\mathcal{A}\mathrm{lg}_{2,3}$
3				$\mathcal{A}\mathrm{lg}_{3,3}$

where $\mathcal{M}\mathrm{on}$ is the category of monoids, $\mathcal{O}\mathrm{p}$ of non colored planar operads, $\mathcal{O}\mathrm{p}_{\mathrm{col}}$ of colored planar operads, and $\mathcal{C}\mathrm{omb}_{\mathbb{PT}}$ of combinads over the combinatorial pattern of planar trees [Lod12].

Proof (sketch). (1) The case $k = 0$ is treated in proposition 3.1.5.

(2) Assume $k = n = 1$. Then $\mathcal{P}\mathrm{sh}(\mathbb{O}_{0,1})$ is the category of graphs, and a $\mathfrak{Z}^1$ maps a graph to its graph of paths. A $\mathfrak{Z}^1$-algebra is just a graph with an adequate notion of composition of paths, i.e. a category.

(3) Similarly, in the case $k = 1$ and $n = 2$, the category $\mathcal{P}\mathrm{sh}(\mathbb{O}_{1,2})$ is the category of signatures whose inputs and output of functions are typed. Extending the reasoning of the proof of proposition 3.1.5, it is easy to see that a $\mathfrak{Z}^2$-algebra is a colored planar operad. □

Proposition 11.1.30. *Let $\mathcal{C}$ be a complete and cocomplete category, and T be a monad on $\mathcal{C}$.*

(1) *The forgetful functor $U : \mathcal{A}\mathrm{lg}(T) \longrightarrow \mathcal{C}$ creates limits. If T preserves $\mathcal{J}$-indexed colimits, then U creates them.*
(2) *If T is finitary, then $\mathcal{A}\mathrm{lg}(T)$ is cocomplete.*

Proof. The first point is [Bor94b, propositions 4.3.1 and 4.3.2], and the second is [BW05, theorem 3.9 on p. 265]. □

Remark 11.1.31. Unlike completeness, cocompleteness of categories of algebras is a tricky subject, but is thankfully well-studied in the literature, see e.g. [Lin69] [Adá77] [AK80] [BW05]. We would also like to cite the paper of Hermelink [Her19], which is a convenient survey on the matter.

Proposition 11.1.32. *The category $\mathcal{A}\mathrm{lg}_{k,n}$ is complete and cocomplete.*

Proof. Recall that $\mathcal{A}\mathrm{lg}_{k,n}$ is the Eilenberg–Moore category of $\mathfrak{Z}^n$. Since it is a presheaf category, $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$ is complete and cocomplete. In particular, $\mathcal{A}\mathrm{lg}_{k,n}$ has all limits. By definition 11.1.16, $\mathfrak{Z}^n$ preserves all colimits (in particular, it is finitary), and cocompleteness follows from proposition 11.1.30. □

OPETOPIC SHAPES

In this section, we state and prove the nerve theorem for $\mathfrak{Z}^n$. In particular, we show that the category $\mathcal{A}\mathrm{lg}_{k,n}$ is a localization of $\mathcal{P}\mathrm{sh}(\mathbb{A}_{k,n})$, where $\mathbb{A}_{k,n} := \Theta_{\mathfrak{Z}^n}$ (see definition 11.1.33). This serves as an intermediate result to obtain a similar nerve theorem over opetopic sets.

Definition 11.1.33 (Opetopic shape)**.** By definitions 11.1.3 and 11.1.16, the category of $\mathfrak{Z}^n$-cardinals is the full subcategory of $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$ spanned by the representables $O[\omega]$, where $\omega \in \mathbb{O}_{n-k,n-1}$, and the spines $S[\nu]$, where $\nu \in \mathbb{O}_{n+1}$. Analogous to notation 11.1.4, let $\mathbb{A}_{k,n}$, the category of *opetopic shapes*, be the full subcategory of $\mathcal{A}\mathrm{lg}_{k,n}$ spanned by $\mathfrak{Z}^n\Theta_0$.

Convention 11.1.34. Throughout this work, we will frequently fix parameters $k \leq n \in \mathbb{N}$ in an implicit manner, and suppress them in notation whenever it is unambiguous. For example, we write $\mathbb{A}$ instead of $\mathbb{A}_{k,n}$, $\mathfrak{Z}$ instead of $\mathfrak{Z}^n$, $\mathcal{A}\mathrm{lg}$ instead of $\mathcal{A}\mathrm{lg}_{k,n}$, etc.

Definition 11.1.35 (Algebraic realization for $\mathbb{A}$)**.** Recall from proposition 11.1.32 that $\mathcal{A}\mathrm{lg}$ is cocomplete. From $\mathbb{A} \longrightarrow \mathcal{A}\mathrm{lg}$ the inclusion of definition 11.1.33, we derive an adjunction

$$\tau : \mathcal{P}\mathrm{sh}(\mathbb{A}) \underset{\perp}{\rightleftarrows} \mathcal{A}\mathrm{lg} : N,$$

by left Kan extension along the Yoneda embedding. The left adjoint is called the *algebraic realization*, and the right adjoint is the *nerve*.

Example 11.1.36. (1) Take $n = k = 1$. By proposition 11.1.29, $\mathcal{A}\mathrm{lg}_{1,1} = \mathcal{C}\mathrm{at}$, and $\mathbb{A}_{1,1}$ is the full subcategory of $\mathcal{C}\mathrm{at}$ spanned by $[m] = \mathfrak{Z}^1 O[\mathbf{m}]$, where $m \in \mathbb{N}$. Therefore, $\mathbb{A}_{1,1} = \mathbb{\Delta}$. The algebraic realization $\tau_{1,1} : \mathcal{P}\mathrm{sh}(\mathbb{\Delta}) \longrightarrow \mathcal{C}\mathrm{at}$ is just the realization of a simplicial set into a category, and its right adjoint $N_{1,1}$ is the classical nerve.
(2) Likewise, $\mathbb{A}_{1,2}$ is the category of colored operads generated by trees, thus it is the planar version of Moerdijk and Weiss's category of dendrices Ω. The functor $N_{1,2}$ is the *dendroidal nerve* of [MW07, section 4], and $\tau_{1,2}$ is its left adjoint. In that paper, they are respectively denoted by N_d and τ_d.

Akin to $\mathcal{P}\mathrm{sh}(\mathbb{O})$, the category $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ enjoys an adequate notion of *spine*. As we shall see, the set S of spine inclusions in $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ will characterize the nerves of algebras in the sense of corollary 11.1.14.

Definition 11.1.37 (Spine)**.** For $\nu \in \mathbb{O}_{n+1}$, write $\lambda := \mathfrak{z}S[\nu]$, and let $S[\lambda]$, the *spine* of the opetopic shape λ, be the colimit

$$S[\lambda] := h_! S[\nu] = \operatorname{colim}\left(\mathbb{O}_{n-k,n}/S[\nu] \longrightarrow \mathbb{O}_{n-k,n} \xrightarrow{\mathfrak{z}} \mathbb{\Lambda} \xrightarrow{\mathsf{y}} \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})\right).$$

Let $\mathsf{s}_\lambda : S[\lambda] \hookrightarrow \lambda$ be the *spine inclusion of* λ, and let S be the set of spine inclusions in $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$:

$$\mathsf{S} := \left\{ \mathsf{s}_\lambda : S[\lambda] \hookrightarrow \lambda \mid \nu \in \mathbb{O}_{n+1} \right\}.$$

Example 11.1.38. If $k = n = 1$, then $\mathbb{\Lambda}_{1,1} = \mathbb{\Delta}$, and the $(n+1)$-opetopes are the opetopic integers (example 3.1.4). For $m \in \mathbb{N}$, the diagram $\mathbb{O}_{0,1}/S[\mathbf{m}] \longrightarrow \mathbb{O}_{0,1}$ is

$$\begin{tikzcd}[column sep=small]
 & \blacksquare & & \blacksquare & & \blacksquare & \cdots & \blacksquare & & \blacksquare \\
\blacklozenge \arrow[ur, "\mathsf{s}_*"] & & \blacklozenge \arrow[ul, "\mathsf{t}"'] \arrow[ur, "\mathsf{s}_*"] & & \blacklozenge \arrow[ul, "\mathsf{t}"'] \arrow[ur, "\mathsf{s}_*"] & & & & \blacklozenge \arrow[ul, "\mathsf{t}"'] \arrow[ur, "\mathsf{s}_*"] &
\end{tikzcd}$$

where there are m instances of $\blacksquare$. By definition, $\mathfrak{z}\blacklozenge = \Delta[0]$ and $\mathfrak{z}\blacksquare = \Delta[1]$. Further, $\mathfrak{z}\mathsf{s}_* = d_1$ and $\mathfrak{z}\mathsf{t} = d_0$. Thus, if $\lambda := \mathfrak{z}S[\mathbf{m}]$, then $S[\lambda]$ is the colimit of the following diagram in $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$:

$$\begin{tikzcd}[column sep=small]
 & \Delta[1] & & \Delta[1] & & \Delta[1] & \cdots & \Delta[1] & & \Delta[1] \\
\Delta[0] \arrow[ur, "d_1"] & & \Delta[0] \arrow[ul, "d_0"'] \arrow[ur, "d_1"] & & \Delta[0] \arrow[ul, "d_0"'] \arrow[ur, "d_1"] & & & & \Delta[0] \arrow[ul, "d_0"'] \arrow[ur, "d_1"] &
\end{tikzcd}$$

Therefore, $S[\lambda]$ is the simplicial spine $S[m]$.

Theorem 11.1.39 (Nerve theorem for $\mathbb{\Lambda}$)**.** *(1) The functor $\tau : \mathbb{\Lambda} \longrightarrow \mathcal{A}\mathrm{lg}$ is dense, or equivalently, the nerve $N : \mathcal{A}\mathrm{lg} \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ is fully faithful.*

(2) A presheaf $X \in \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ is in the essential image of N if and only if $\mathsf{S} \perp X$. In particular, using point (1), $\mathcal{A}\mathrm{lg}$ is equivalent to the orthogonality class induced by the set S.

(3) (Segal condition) The reflective adjunction $\tau : \mathcal{P}\mathrm{sh}(\mathbb{\Lambda}) \rightleftarrows \mathcal{A}\mathrm{lg} : N$ exhibits $\mathcal{A}\mathrm{lg}$ as the localization of $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ at the spine inclusions, i.e. $\mathcal{A}\mathrm{lg} \simeq \mathsf{S}^{-1}\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$.

Proof. (1) This is theorem 11.1.13.

(2) Consider the inclusions

$$\mathbb{O}_{n-k,n} \xrightarrow{i} \Theta_0 \xrightarrow{j} \mathbb{\Lambda}$$

as in definition 11.1.33, and the counit $\varepsilon : i_! i^* \longrightarrow \mathrm{id}_{\mathcal{P}\mathrm{sh}(\Theta_0)}$ of the adjunction $i_! : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) \rightleftarrows \mathcal{P}\mathrm{sh}(\Theta_0) : i^*$. The category $\Theta_0 - \operatorname{im} i$ is spanned by the spines $S[\nu]$, for $\nu \in \mathbb{O}_{n+1}$ (see definition 11.1.33).

Since i maps an opetope $\omega \in \mathbb{O}_{n-k,n}$ to the associated representable $O[\omega] \in \Theta_0$, we have $i^* S[\nu] = \Theta_0(i-, S[\nu]) = S[\nu]$ as presheaves over $\mathbb{O}_{n-k,n}$. Next, by definition of left Kan extensions, the presheaf $i_! i^* S[\nu] = i_! S[\nu]$ is the colimit

$$\operatorname{colim}\left(\mathbb{O}_{n-k,n}/S[\nu] \longrightarrow \mathbb{O}_{n-k,n} \xrightarrow{i} \Theta_0 \xrightarrow{\mathsf{y}} \mathcal{P}\mathrm{sh}(\Theta_0)\right),$$

thus

$$\begin{aligned}
t_! i_! i^* S[\nu] &= t_! \operatorname{colim} \left(\mathbb{O}_{n-k,n}/S[\nu] \longrightarrow \mathbb{O}_{n-k,n} \xrightarrow{i} \Theta_0 \xrightarrow{\mathsf{y}} \mathcal{P}\mathrm{sh}(\Theta_0) \right) \\
&\cong \operatorname{colim} \left(\mathbb{O}_{n-k,n}/S[\nu] \longrightarrow \mathbb{O}_{n-k,n} \xrightarrow{i} \Theta_0 \xrightarrow{\mathsf{y}} \mathcal{P}\mathrm{sh}(\Theta_0) \xrightarrow{t_!} \mathcal{P}\mathrm{sh}(\mathbb{A}) \right) \\
&\cong \operatorname{colim} \left(\mathbb{O}_{n-k,n}/S[\nu] \longrightarrow \mathbb{O}_{n-k,n} \xrightarrow{i} \Theta_0 \xrightarrow{j} \mathbb{A} \xrightarrow{\mathsf{y}} \mathcal{P}\mathrm{sh}(\mathbb{A}) \right) \\
&= \operatorname{colim} \left(\mathbb{O}_{n-k,n}/S[\nu] \longrightarrow \mathbb{O}_{n-k,n} \xrightarrow{h} \mathbb{A} \xrightarrow{\mathsf{y}} \mathcal{P}\mathrm{sh}(\mathbb{A}) \right) && \spadesuit \\
&= S[h\nu] && \diamondsuit
\end{aligned}$$

where $\spadesuit$ is by definition of h on $\mathbb{O}_{n-k,n}$, and $\diamondsuit$ is by definition of $S[h\nu]$. On the other hand, $t_! S[\nu] = \mathfrak{Z}^n S[\nu] = h\nu$, and the counit $\varepsilon_{S[\nu]}$ is simply the spine inclusion $\mathsf{s}_{h\nu} : S[h\nu] \longrightarrow h\nu$. We apply corollary 11.1.14 to conclude that a presheaf $X \in \mathcal{P}\mathrm{sh}(\mathbb{A})$ is in the essential image of N if and only if $\mathsf{S} \perp X$.

(3) Follows from the previous two points and corollary 0.5.12. □

Remark 11.1.40. Then theorem 11.1.39 generalizes the well-known fact that $\mathcal{C}\mathrm{at}$ (in the case $k = n = 1$) and $\mathcal{O}\mathrm{p}_{\mathrm{col}}$ (in the case $k = 1$ and $n = 2$) have fully faithful nerve functors to $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ and $\mathcal{P}\mathrm{sh}(\Omega)$ [MW07, example 4.2] respectively, exhibiting them as localizations of the respective presheaf categories at a set of *spine inclusions*, sometimes called *Grothendieck–Segal colimits*.

11.2 ALGEBRAIC REALIZATION

In this section, we show how to construct opetopic algebras from opetopic sets, by the means of the *algebraic realization* $h_{k,n} : \mathcal{P}\mathrm{sh}(\mathbb{O}) \longrightarrow \mathcal{A}\mathrm{lg}_{k,n}$, for all $k, n \in \mathbb{N}$ with $k \leq n$. Much in the spirit of the classical realization $\mathcal{P}\mathrm{sh}(\mathbb{\Delta}) \longrightarrow \mathcal{C}\mathrm{at}$, given $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$, we shall interpret its n-cells as "generators", and its $(n+1)$-cells as "relations". The first step to implement this idea is to extend $\mathfrak{Z}^n O[-] : \mathbb{O}_{n-k,n} \longrightarrow \mathbb{A}$ to a functor from $\mathbb{O}_{n-k,n+2}$. Informally, the image of an $(n+1)$-opetope represents an algebra with essentially one relation, and the image of an $(n+2)$-opetope is an algebra, also with essentially a single relation, but which is presented with many smaller composable relations (see example 11.2.2 for an illustration of this intuition). Thus, realizations of $(n+1)$-opetopes implement the idea of "relation" in opetopic algebras, while realizations of $(n+2)$-opetopes enforce "associativity among relations". Then, in definition 11.2.3, the realization $h_{k,n}$ for opetopes is defined as a composite of left adjoints

$$\mathcal{P}\mathrm{sh}(\mathbb{O}) \underset{\perp}{\xrightarrow{(-)_{n-k,n+2}}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \underset{\perp}{\rightleftarrows} \mathcal{P}\mathrm{sh}(\mathbb{A}_{k,n}) \underset{\perp}{\overset{\tau_{k,n}}{\rightleftarrows}} \mathcal{A}\mathrm{lg}_{k,n}.$$

To declutter notations, we shall use convention 11.1.34 and omit parameters k and n in most notations, e.g. $\mathbb{A} = \mathbb{A}_{k,n}$, $\mathcal{A}\mathrm{lg} = \mathcal{A}\mathrm{lg}_{k,n}$, $\mathfrak{Z} = \mathfrak{Z}^n$, etc.

DEFINITION

Definition 11.2.1. There is a natural functor $\mathbb{O}_{n-k,n} \longrightarrow \mathbb{A}$, mapping an opetope ω to $\mathfrak{Z}O[\omega]$, see proposition 11.1.10 and equation (11.1.5). We now extend it to a functor

$h : \mathbb{O}_{n-k,n+2} \longrightarrow \mathbb{A}$. On objects, it is given by

$$\begin{aligned} h : \mathbb{O}_{n-k,n+2} &\longrightarrow \mathbb{A} \\ \omega &\longmapsto \begin{cases} \mathfrak{Z}O[\omega] & \text{if } \dim \omega \leq n, \\ \mathfrak{Z}S[\omega] & \text{if } \dim \omega = n+1, \\ \mathfrak{Z}S[\mathsf{t}\,\omega] & \text{if } \dim \omega = n+2. \end{cases} \end{aligned}$$

We now specify h on morphisms. Since it extends the natural functor $\mathbb{O}_{n-k,n} \longrightarrow \mathbb{A}$, it is enough to consider morphisms in $\mathbb{O}_{n,n+2}$, so take $\nu \in \mathbb{O}_{n+1}$ and $\xi \in \mathbb{O}_{n+2}$.

(1) For $[p] \in \nu^\bullet$, let $h\left(\mathsf{s}_{[p]}\,\nu \xrightarrow{\mathsf{s}_{[p]}} \nu\right) := \mathfrak{Z}\left(O[\mathsf{s}_{[p]}\,\nu] \xrightarrow{\mathsf{s}_{[p]}} S[\nu]\right)$.

(2) In order to define $h\left(\mathsf{t}\,\nu \xrightarrow{\mathsf{t}} \nu\right) = \left(\mathfrak{Z}O[\mathsf{t}\,\nu] \xrightarrow{h\,\mathsf{t}} \mathfrak{Z}S[\nu]\right)$, it is enough to provide a morphism $O[\mathsf{t}\,\nu] \longrightarrow \mathfrak{Z}S[\nu]$, i.e. a cell in $\mathfrak{Z}S[\nu]_{\mathsf{t}\,\nu}$. Let it be $\left(S[\nu] \xrightarrow{\mathrm{id}} S[\nu]\right) \in \mathfrak{Z}S[\nu]_{\mathsf{t}\,\nu}$.

(3) Let $h\left(\mathsf{t}\,\xi \xrightarrow{\mathsf{t}} \xi\right) = \left(\mathfrak{Z}S[\mathsf{t}\,\xi] \xrightarrow{h\,\mathsf{t}} \mathfrak{Z}S[\mathsf{t}\,\xi]\right)$ be the identity map.

(4) Let $[p] \in \xi^\bullet$. In order to define a morphism of $\mathfrak{Z}$-algebras

$$h\left(\mathsf{s}_{[p]}\,\xi \xrightarrow{\mathsf{s}_{[p]}} \xi\right) = \left(\mathfrak{Z}S[\mathsf{s}_{[p]}\,\xi] \xrightarrow{h\,\mathsf{s}_{[p]}} \mathfrak{Z}S[\mathsf{t}\,\xi]\right),$$

it is enough to provide a morphism $h\,\mathsf{s}_{[p]} : S[\mathsf{s}_{[p]}\,\xi] \longrightarrow \mathfrak{Z}S[\mathsf{t}\,\xi]$ in $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$, which we now construct.

a) Using equation (2.2.26), ξ decomposes as

$$\xi = \alpha \underset{[p]}{\circ} \mathsf{Y}_{\mathsf{s}_{[p]}\,\xi} \underset{[[q_i]]}{\bigcirc} \beta_i,$$

for some $\alpha, \beta_i \in \mathbb{O}_{n+2}$, and where $[q_i]$ ranges over $(\mathsf{s}_{[p]}\,\xi)^\bullet$. The leaves of β_i are therefore a subset of the leaves of ξ. More precisely, a leaf address $[l] \in \beta_i^{|}$ corresponds to the leaf $[p[q_i]l]$ of ξ. This defines an inclusion $f_i : S[\mathsf{t}\,\beta_i] \longrightarrow S[\mathsf{t}\,\xi]$ that maps the node $\wp_{\beta_i}[l] \in (\mathsf{t}\,\beta_i)^\bullet$ to $\wp_\xi[p[q_i]l] \in (\mathsf{t}\,\xi)^\bullet$.

b) Note that by definition, the map f_i is an element of

$$\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})(S[\mathsf{t}\,\beta_i], S[\mathsf{t}\,\xi]) \subseteq \mathfrak{Z}S[\mathsf{t}\,\xi]_{\mathsf{t}\,\mathsf{t}\,\beta_i},$$

and since $\mathsf{t}\,\mathsf{t}\,\beta_i = \mathsf{t}\,\mathsf{s}_{[]}\,\beta_i = \mathsf{e}_{[p[q_i]]}\,\xi$ (by **(Glob1)** and **(Inner)**), we have $f_i \in \mathfrak{Z}S[\mathsf{t}\,\xi]_{\mathsf{e}_{[p[q_i]]}\,\xi}$.

c) Together, the f_i assemble into the required morphism $h\,\mathsf{s}_{[p]} : S[\mathsf{s}_{[p]}\,\xi] \longrightarrow \mathfrak{Z}S[\mathsf{t}\,\xi]$, that maps the node $[q_i] \in (\mathsf{s}_{[p]}\,\xi)^\bullet$ to f_i. So in conclusion, we have

$$\begin{aligned} h\,\mathsf{s}_{[p]} : S[\mathsf{s}_{[p]}\,\xi] &\longrightarrow \mathfrak{Z}S[\mathsf{t}\,\xi] \\ (h\,\mathsf{s}_{[p]})[q_i] : S[\mathsf{t}\,\beta_i] &\longrightarrow S[\mathsf{t}\,\xi] \\ \wp_{\beta_i}[l] &\longmapsto \wp_\xi[p[q_i]l], \end{aligned}$$

for $[q_i] \in (\mathsf{s}_{[p]}\,\xi)^\bullet$ and $[l] \in \beta_i^{|}$.

This defines h on object and morphisms, and functoriality is straightforward.

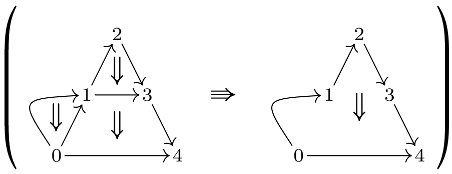
Example 11.2.2. Consider the case $k = n = 1$, so that $h = h_{1,1}$ is a functor $\mathbb{O}_{0,3} \longrightarrow \mathbb{A}_{1,1} \cong \mathbb{\Delta}$. In low dimensions, we have $h\blacklozenge = [0]$, $h\blacksquare = [1]$, and $h\mathbf{m} = [m]$ with $m \in \mathbb{N}$, since h is $\mathfrak{Z}$ in this case. For instance,

$$h\mathbf{3} = h\left(\begin{tikzcd} & 1 \arrow[r] & 3 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rrr] & & \Downarrow & 4 \end{tikzcd}\right) = [3]$$

is the category with 3 generating morphisms, and the 2-cell of $\mathbf{3}$ just witnesses their composition.

Consider now the following 3-opetope ξ:

$$\xi = \mathsf{Y}_{\mathbf{3}} \underset{[[*]]}{\circ} \mathsf{Y}_{\mathbf{2}} \underset{[[**]]}{\circ} \mathsf{Y}_{\mathbf{1}} =$$



Then $h\xi = \mathfrak{Z}S[\mathsf{t}\,\xi] = \mathfrak{Z}S[\mathbf{4}] = [4]$. This result should be understood as the poset of points of ξ (represented as dots in the pasting diagram above) ordered by the topmost arrows. The 2-dimensional faces of ξ provide several relations among the generating arrows, and the 3-cell is a witness of the composition of those relations.

Take the face embedding $\mathsf{s}_{[]} : \mathbf{3} \longrightarrow \xi$, corresponding to the trapezoid at the base of the pasting diagram. Then $h\,\mathsf{s}_{[]}$ maps points 0, 1, 2, 3 of $h\mathbf{3} = [3]$ to points 0, 1, 3, 4 of $h\xi$, respectively. In other words, it "skips" point 2, which is exactly what the pasting diagram above depicts: the $[]$-source of ξ does not touch point 2 (the topmost one). Likewise, the map $h\,\mathsf{s}_{[[**]]} : [1] = h\mathbf{1} \longrightarrow h\xi$ maps 0, 1 to 0, 1, respectively.

Consider now the target embedding $\mathsf{t} : \mathbf{4} \longrightarrow \xi$. Since the target face touches all the points of ξ (this can be checked graphically, but more generally follows from **(Glob2)**), $h\,\mathsf{t}$ should be the identity map on $[4]$, which is precisely what the definition gives.

Definition 11.2.3 (Algebraic realization for $\mathbb{O}$)**.** With a slight abuse of notation, let $h : \mathcal{P}\mathrm{sh}(\mathbb{O}) \rightleftarrows \mathcal{A}\mathrm{lg} : M$ be the composite adjunction

$$\mathcal{P}\mathrm{sh}(\mathbb{O}) \underset{\perp}{\overset{(-)_{n-k,n+2}}{\rightleftarrows}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \underset{\perp}{\overset{h_!}{\rightleftarrows}} \mathcal{P}\mathrm{sh}(\mathbb{A}) \underset{\perp}{\overset{\tau}{\rightleftarrows}} \mathcal{A}\mathrm{lg},$$

where $h_!$ is the extension of $h : \mathbb{O}_{n-k,n+2} \longrightarrow \mathbb{A}$ (definition 11.2.1) to the presheaf categories (see notation 0.4.12).

Remark 11.2.4. The first adjunction of the composite is just a truncation, and does not carry any information; the part between $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$ and $\mathcal{A}\mathrm{lg}$ is actually what implements the n-cells of a presheaf as operations, and $(n+1)$-cells as relations. The $(n+2)$-cells represent relations among relations (e.g. associativity of composition in categories) and cannot be discarded, i.e. one cannot obtain an adequate realization adjunction of the form $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+1}) \rightleftarrows \mathcal{A}\mathrm{lg}$. Formally, the nerve theorem 11.2.33 will not hold if h is defined as the composite

$$\mathcal{P}\mathrm{sh}(\mathbb{O}) \xrightarrow{(-)_{n-k,n+1}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+1}) \xrightarrow{h_!} \mathcal{P}\mathrm{sh}(\mathbb{A}) \xrightarrow{\tau} \mathcal{A}\mathrm{lg}.$$

Remark 11.2.5. We now have a commutative triangle of adjunctions:

$$\begin{tikzcd} & \mathcal{A}\mathrm{lg} \arrow[dl, shift left, "M"] \arrow[dr, shift right, "N"'] & \\ \mathcal{P}\mathrm{sh}(\mathbb{O}) \arrow[ur, shift left, "h"] \arrow[ur, phantom, "\perp"] \arrow[rr, shift left] \arrow[rr, phantom, "\perp"] & & \mathcal{P}\mathrm{sh}(\mathbb{\Lambda}), \arrow[ul, shift right, "\tau"'] \arrow[ul, phantom, "\perp"] \arrow[ll, shift left] \end{tikzcd} \tag{11.2.6}$$

The notation h might seem a bit overloaded, but its meaning is quite simple: it always takes an opetopic set and produces an algebra. If that opetopic set is the representable of an opetope in $\mathbb{O}_{n-k,n+2}$, then it falls within the scope of definition 11.2.1, and the output algebra is in fact an opetopic shape, i.e. in $\mathbb{\Lambda}$.

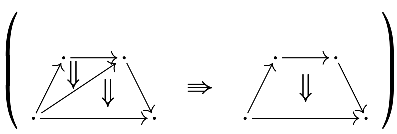
DIAGRAMMATIC MORPHISMS

This section is devoted to proving various (rather technical) facts about the functor $h : \mathbb{O}_{n-k,n+2} \longrightarrow \mathbb{\Lambda}$ of definition 11.2.1, eventually leading to lemma 11.2.16, stating that most morphisms in $\mathbb{\Lambda}$ admit a good "geometrical description" (see definition 11.2.7 and example 11.2.8). This result shall be used when proving the *nerve theorem for* $\mathbb{O}$ (theorem 11.2.33), but as a first application, we show in proposition 11.2.20 that h it is essentially surjective on morphisms[4].

Definition 11.2.7 (Diagrammatic morphism)**.** Let $\nu_1, \nu_2 \in \mathbb{O}_{n+1}$. A morphism $f : h\nu_1 \longrightarrow h\nu_2$ in $\mathbb{\Lambda}$ is *diagrammatic* if there exists an opetope $\xi \in \mathbb{O}_{n+2}$ and a node address $[p] \in \xi^\bullet$ such that $\mathsf{s}_{[p]}\,\xi = \nu_1$, $\mathsf{t}\,\xi = \nu_2$, and $f = (h\,\mathsf{t})^{-1} \cdot (h\,\mathsf{s}_{[p]})$. This situation is summarized by the following diagram, called a *diagram of* f:

$$\frac{\begin{tikzcd} & \xi \\ \nu_1 \arrow[ur, "\mathsf{s}_{[p]}"] & \nu_2 \arrow[u, "\mathsf{t}"'] \end{tikzcd}}{h\nu_1 \xrightarrow{f} h\nu_2.}$$

Example 11.2.8. Consider the case $k = n = 1$ again, and recall from example 11.1.36 that in this case, $\mathbb{\Lambda} = \mathbb{\Delta}$. Consider the map $f : [2] \longrightarrow [3]$ in $\mathbb{\Delta}$, where $f(0) = 0$, $f(1) = 1$, and $f(2) = 2$. In other words, $f = d^3$ is the 3^{rd} coface map. Taking ξ as on the left, we obtain a diagram of f on the right:

$\xi = \mathsf{Y}_{\mathbf{2}} \underset{[[*]]}{\circ} \mathsf{Y}_{\mathbf{2}} =$  ,

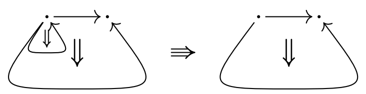
$$\frac{\begin{tikzcd} & \xi \\ \mathbf{2} \arrow[ur, "\mathsf{s}_{[[*]]}"] & \mathbf{3} \arrow[u, "\mathsf{t}"'] \end{tikzcd}}{[2] \xrightarrow{f} [3]}$$

Consider now a non injective map $g : [2] \longrightarrow [1]$ where $g(0) = g(1) = 0$ and $g(2) = 1$. In other words, $g = s^0$ is the 0^{th} codegeneracy map. Taking ξ' as on the left, we obtain a

[4] While pleasant, this proposition is not put to use in the present work

diagram of g on the right:

$$\xi' = \mathsf{Y}_{\mathbf{2}} \underset{[[*]]}{\circ} \mathsf{Y}_{\mathbf{0}} = \left(\right.$$



$$\left. \right), \qquad \begin{array}{c} \begin{tikzcd} & \xi' \\ \mathbf{2} \arrow[ur, "\mathsf{s}_{[]}"] & \mathbf{1} \arrow[u, "\mathsf{t}"'] \end{tikzcd} \\ \hline [2] \xrightarrow{g} [1] \end{array}$$

On the one hand, lemma 11.2.9 below states that diagrammatic morphisms are stable under composition, and on the other hand, those two examples seem to indicate that all simplicial cofaces and codegeneracies are diagrammatic. One might thus expect all morphisms of $\mathbb{\Delta}$ to be in the essential image of $h_{1,1} : \mathbb{O}_{0,3} \longrightarrow \mathbb{\Delta}$. This is indeed true, and a more general statement is proved in proposition 11.2.20.

Lemma 11.2.9. *If f_1 and f_2 are diagrammatic as on the left, the diagram on the right is well-defined, and is a diagram of $f_2 f_1$.*

$$\begin{array}{c} \begin{tikzcd} & \xi_1 & & \xi_2 \\ \nu_1 \arrow[ur, "\mathsf{s}_{[p_1]}"] & \nu_2 \arrow[u, "\mathsf{t}"'] \arrow[ur, "\mathsf{s}_{[p_2]}"] & & \nu_3 \arrow[u, "\mathsf{t}"'] \end{tikzcd} \\ \hline h\nu_1 \xrightarrow{f_1} h\nu_2 \xrightarrow{f_2} h\nu_3, \end{array} \qquad \begin{array}{c} \begin{tikzcd} & \xi_2 \underset{[p_2]}{\square} \xi_1 \\ \nu_1 \arrow[ur, "\mathsf{s}_{[p_2 p_1]}"] & \nu_3 \arrow[u, "\mathsf{t}"'] \end{tikzcd} \\ \hline h\nu_1 \xrightarrow{f_2 f_1} h\nu_3 \end{array}$$

Proof. It is a simple but lengthy matter of unfolding the definition of h. First, note that

$$\begin{aligned} \mathsf{t}(\xi_2 \underset{[p_2]}{\square} \xi_1) &= \mathsf{t}\,\mathsf{t}(\mathsf{Y}_{\xi_2} \underset{[[p_2]]}{\circ} \mathsf{Y}_{\xi_1}) && \text{by proposition 3.1.6} \\ &= \mathsf{t}\,\mathsf{s}_{[]}(\mathsf{Y}_{\xi_2} \underset{[[p_2]]}{\circ} \mathsf{Y}_{\xi_1}) && \text{by } \textbf{(Glob2)} \\ &= \mathsf{t}\,\xi_2 = \nu_3. \end{aligned}$$

Using equation (2.2.26), we decompose ξ_1 as

$$\xi_1 = \alpha_1 \underset{[p_1]}{\circ} \mathsf{Y}_{\nu_1} \bigcirc_{[[q_i]]} \beta_i, \tag{11.2.10}$$

where $[q_i]$ ranges over $\nu_1^\bullet$. If $\beta_i^{\shortmid} = \{[l_{i,j}] \mid j\}$, then $\xi_1^{\shortmid} = \{[p_1[q_i]l_{i,j}] \mid i, j\}$, and so $\nu_2^\bullet = (\mathsf{t}\,\xi_1)^\bullet = \left\{\wp_{\xi_1}[p_1[q_i]l_{i,j}] \mid i, j\right\}$. Using equation (2.2.26) again, we decompose ξ_2 as

$$\xi_2 = \alpha_2 \underset{[p_2]}{\circ} \mathsf{Y}_{\nu_2} \bigcirc_{[\wp_{\xi_1}[p_1[q_i]l_{i,j}]]} \gamma_{i,j} \tag{11.2.11}$$

and write

$$\begin{aligned} \xi_2 \underset{[p_2]}{\square} \xi_1 &= \left(\alpha_2 \underset{[p_2]}{\circ} \mathsf{Y}_{\nu_2} \bigcirc_{[\wp_{\xi_1}[p_1[q_i]l_{i,j}]]} \gamma_{i,j} \right) \underset{[p_2]}{\square} \xi_1 && \text{see (11.2.10)} \\ &= \alpha_2 \underset{[p_2]}{\circ} \xi_1 \bigcirc_{[p_1[q_i]l_{i,j}]} \gamma_{i,j} && \text{see definition 2.2.25} \\ &= \alpha_2 \underset{[p_2]}{\circ} \left(\alpha_1 \underset{[p_1]}{\circ} \mathsf{Y}_{\nu_1} \bigcirc_{[[q_i]]} \beta_i \right) \bigcirc_{[[q_i]l_{i,j}]} \gamma_{i,j} && \text{see (11.2.11)} \end{aligned}$$

$$= \left(\alpha_2 \underset{[p_2]}{\circ} \alpha_1\right) \underset{[p_2 p_1]}{\circ} \mathsf{Y}_{\nu_1} \underset{[[q_i]]}{\bigcirc} \underbrace{\left(\beta_i \underset{[l_{i,j}]}{\bigcirc} \gamma_{i,j}\right)}_{\delta_i} \qquad \text{rearranging terms.}$$

Applying the definition of h we have, for $[q_i] \in \nu_1^\bullet$, $[l_{i,j}] \in \beta_i^{|}$, and $[r] \in \gamma_{i,j}^{|}$,

$$\begin{aligned}
h\,\mathsf{s}_{[p_2 p_1]} : S[\nu_1] &\longrightarrow \mathfrak{Z} S[\nu_3] \\
(h\,\mathsf{s}_{[p_2 p_1]})[q_i] : S[\mathsf{t}\,\delta_i] &\longrightarrow S[\nu_3] \\
\wp_{\delta_i}[l_{i,j} r] &\longmapsto \wp_\zeta[p_2 p_1 [q_i] l_{i,j} r]; && (11.2.12) \\
h\,\mathsf{s}_{[p_1]} : S[\nu_1] &\longrightarrow \mathfrak{Z} S[\nu_2] \\
(h\,\mathsf{s}_{[p_1]})[q_i] : S[\mathsf{t}\,\beta_i] &\longrightarrow S[\nu_2] \\
\wp_{\beta_i}[l_{i,j}] &\longmapsto \wp_{\xi_1}[p_1[q_i] l_{i,j}]; && (11.2.13) \\
h\,\mathsf{s}_{[p_2]} : S[\nu_2] &\longrightarrow \mathfrak{Z} S[\nu_3] \\
(h\,\mathsf{s}_{[p_2]})(\wp_{\xi_1}[p_1[q_i] l_{i,j}]) : S[\mathsf{t}\,\gamma_{i,j}] &\longrightarrow S[\nu_3] \\
\wp_{\gamma_{i,j}}[r] &\longmapsto \wp_{\xi_2}[p_2 \;\; \wp_{\xi_1}[p_1[q_i] l_{i,j}] \;\; r]. && (11.2.14)
\end{aligned}$$

Thus,

$$\begin{aligned}
&(h\,\mathsf{s}_{[p_2 p_1]})([q_i])(\wp_{\delta_i}[l_{i,j} r]) \\
&= \wp_\zeta[p_2 p_1 [q_i] l_{i,j} r] && \text{by (11.2.12)} \\
&= \wp_{\xi_2}[p_2 \;\; \wp_{\xi_1}[p_1[q_i] l_{i,j}] \;\; r] && \spadesuit \\
&= (h\,\mathsf{s}_{[p_2]})(\wp_{\xi_1}[p_1[q_i] l_{i,j}])(\wp_{\gamma_{i,j}}[r]) && \text{by (11.2.14)} \\
&= (h\,\mathsf{s}_{[p_2]})\left((h\,\mathsf{s}_{[p_1]})([q_i])(\wp_{\beta_i}[l_{i,j}])\right)(\wp_{\gamma_{i,j}}[r]) && \text{by (11.2.13)} \\
&= \left(h\,\mathsf{s}_{[p_2]} \cdot h\,\mathsf{s}_{[p_1]}\right)([q_i])(\wp_{\delta_i}[l_{i,j} r]), && \diamond
\end{aligned}$$

where equality $\spadesuit$ comes from the monad structure on $\mathfrak{Z}$, and $\diamond$ from the definition of the composition in $\mathbb{A}$ when considered as the Kleisli category of $\mathfrak{Z}$. $\square$

Lemma 11.2.15. *(1) Let $\nu \in \mathbb{O}_{n+1}$, $\omega := \mathsf{t}\,\nu$, and $\xi := \mathsf{Y}_{\mathsf{Y}_\omega} \circ_{[[]]} \mathsf{Y}_\nu$. Note that $\nu = \mathsf{t}\,\xi$. The following is a diagram of $h\,\mathsf{t} : h\omega \longrightarrow h\nu$:*

$$\frac{\begin{tikzcd} & \xi \\ \mathsf{Y}_\omega \arrow[ur, "\mathsf{s}_{[]}"] & \nu \arrow[u, "\mathsf{t}"'] \end{tikzcd}}{h\omega \xrightarrow{h\,\mathsf{t}} h\nu.}$$

(2) Let $\beta, \nu \in \mathbb{O}_{n+1} = \operatorname{tr} \mathfrak{Z}^{n-1}$, and $i : S[\beta] \longrightarrow S[\nu]$ a morphism of presheaves. Then i corresponds to an inclusion $\beta \longhookrightarrow \nu$ of $\mathfrak{Z}^{n-1}$ trees, mapping node at address $[q]$ to $[pq]$, where $[p] := i[] \in \nu^\bullet$ is the address of the image of the root node. Write $\nu = \bar{\beta} \,\square_{[p]}\, \beta$, for an adequate $\bar{\beta} \in \mathbb{O}_{n+1}$, and let $\xi := \mathsf{Y}_{\bar{\beta}} \circ_{[[p]]} \mathsf{Y}_\beta$. Note that $\nu = \mathsf{t}\,\xi$ by proposition 3.1.6. The following is a diagram of hi:

$$\frac{\begin{tikzcd} & \xi \\ \beta \arrow[ur, "\mathsf{s}_{[p]}"] & \nu \arrow[u, "\mathsf{t}"'] \end{tikzcd}}{h\beta \xrightarrow{hi} h\nu.}$$

Proof. Tedious but straightforward matter of unfolding definition 11.2.1. □

Lemma 11.2.16 (Diagrammatic lemma)**.** *Let $\nu, \nu' \in \mathbb{O}_{n+1}$ with ν non degenerate, and $f : h\nu \longrightarrow h\nu'$ be a morphism in $\mathbb{A}$. Then f is diagrammatic.*

Proof. Let us first sketch the proof. The idea is to proceed by induction on ν. The case $\nu = \mathsf{Y}_\psi$ for some $\psi \in \mathbb{O}_n$ is fairly simple. In the inductive case, we essentially show that f exhibits an inclusion $\nu \hookrightarrow \nu'$ of $\mathfrak{Z}^{n-1}$-trees by constructing an $(n+1)$-opetope $\bar{\nu}$ such that $\nu' = \bar{\nu} \square_{[q]} \nu$. Thus by lemma 11.2.15, the following is a diagram of hf:

$$
\begin{array}{c}
\begin{tikzcd} & \xi \\ \nu \arrow[ur, "\mathsf{s}_{[[q_1]]}"] & \nu' \arrow[u, "\mathsf{t}"'] \end{tikzcd} \\ \hline h\nu \xrightarrow{f} h\nu',
\end{array}
$$

where $\xi := \mathsf{Y}_{\bar{\nu}} \circ_{[[q_1]]} \mathsf{Y}_\nu$.

Let us now dive into the details. As advertised, the proof proceeds by induction on ν, which by assumption is not degenerate.

(1) Assume $\nu = \mathsf{Y}_\psi$ for some $\psi \in \mathbb{O}_n$. Then

$$\mathbb{A}(h\mathsf{Y}_\psi, h\nu') = \mathbb{A}(\mathfrak{Z}S[\mathsf{Y}_\psi], \mathfrak{Z}S[\nu']) \cong (\mathfrak{Z}S[\nu])_\psi.$$

Thus f corresponds to a unique morphism $\tilde{f} : S[\nu''] \longrightarrow S[\nu']$, for some $\nu'' \in \mathbb{O}_{n+1}$ such that $\mathsf{t}\,\nu'' = \psi$, and f is the composite

$$h\mathsf{Y}_\psi = h\psi \xrightarrow{h\,\mathsf{t}} h\nu'' \xrightarrow{\mathfrak{Z}\tilde{f}} h\nu'.$$

Those two arrows are diagrammatic by lemma 11.2.15, and by lemma 11.2.9, so is f.

(2) By induction, write $\nu = \nu_1 \circ_{[l]} \mathsf{Y}_{\psi_2}$ for some $\nu_1 \in \mathbb{O}_{n+1}$, $[l] \in \nu_1^{\shortmid}$, and $\psi_2 \in \mathbb{O}_n$. Write $\psi_1 := \mathsf{t}\,\nu_1$, and $\nu_2 := \mathsf{Y}_{\psi_2}$. Then f restricts as f_i, $i = 1, 2$, given by the composite $h\nu_i \longrightarrow h\nu \xrightarrow{f} h\nu'$.

Let $[l']$ be the edge address of ν' (or equivalently, the $(n-1)$-cell of $S[\nu'] \subseteq h\nu'$) such that $\mathsf{e}_{[l']}\,\nu' = f(\mathsf{e}_{[l]}\,\nu)$. Then ν' decomposes as $\nu' = \beta_1 \circ_{[l']} \beta_2$, for some $\beta_1, \beta_2 \in \mathbb{O}_{n+1}$ (in particular, β_1 and β_2 are sub $\mathfrak{Z}^{n-1}$-trees of ν'), and f_1 and f_2 factor as

$$
\begin{tikzcd} h\nu_i \arrow[r, "\bar{f}_i"] \arrow[dr, "f_i"'] & h\beta_i \arrow[d, "b_i"] \\ & h\nu', \end{tikzcd}
$$

where b_i correspond to the subtree inclusion $\beta_i \hookrightarrow \nu'$. By induction, $\bar{f}_i$ is diagrammatic, say with the following diagram:

$$
\begin{array}{c}
\begin{tikzcd} & \xi_i \\ \nu_i \arrow[ur, "\mathsf{s}_{[p_i]}"] & \beta_i \arrow[u, "\mathsf{t}"'] \end{tikzcd} \\ \hline h\nu_i \xrightarrow{\bar{f}_i} h\beta_i,
\end{array}
$$

and thus β_i can be written as $\beta_i = \bar{\nu}_i \,\square_{[q_i]}\, \nu_i$, for some $\bar{\nu}_i \in \mathbb{O}_{n+1}$ and $[q_i] \in \bar{\nu}_i^\bullet$. In the case $i = 2$, note that $\beta_2 = \bar{\nu}_2 \,\square_{[q_2]}\, \nu_2 = \bar{\nu}_2 \,\square_{[q_2]}\, \mathsf{Y}_{\psi_2} = \bar{\nu}_2$.
On the one hand we have

$$\begin{aligned}
\mathsf{e}_{[l']}\, \nu' &= f(\mathsf{e}_{[l]}\, \nu) && \text{by definition of } [l'] \\
&= f_1(\mathsf{e}_{[l]}\, \nu_1) && \text{since } \nu = \nu_1 \underset{[l]}{\circ} \mathsf{Y}_{\psi_2} \\
&= b_1 \bar{f}_1(\mathsf{e}_{[l]}\, \nu_1) && \text{since } f_1 = b_1 \bar{f}_1 \\
&= b_1(\mathsf{e}_{[q_1 l]}\, \beta_1) && \text{since } \beta_1 = \bar{\nu}_1 \underset{[q_1]}{\square} \nu_1 \\
&= \mathsf{e}_{[q_1 l]}\, \nu,
\end{aligned}$$

showing $[l'] = [q_1 l]$, and thus that $\bar{\nu}_1$ is of the form

$$\bar{\nu}_1 = \mu_1 \underset{[q_1]}{\circ} \mathsf{Y}_{\psi_1} \bigcirc_{[[r_{1,j}]]} \mu_{1,j}, \tag{11.2.17}$$

where $[r_{1,j}]$ ranges over $\psi_1^\bullet - \{\wp_{\nu_1}[l]\}$, and $\mu_1, \mu_{1,j} \in \mathbb{O}_{n+1}$. On the other hand,

$$\begin{aligned}
\mathsf{e}_{[l']}\, \nu' &= f(\mathsf{e}_{[l]}\, \nu) && \text{by definition of } [l'] \\
&= f_2(\mathsf{e}_{[]}\, \nu_2) && \text{since } \nu = \nu_1 \underset{[l]}{\circ} \mathsf{Y}_{\psi_2} \\
&= b_2 \bar{f}_2(\mathsf{e}_{[]}\, \nu_2) && \text{since } f_1 = b_2 \bar{f}_2 \\
&= b_2(\mathsf{e}_{[q_2]}\, \beta_2) && \text{since } \beta_2 = \bar{\nu}_2 \underset{[q_2]}{\square} \nu_2 \\
&= \mathsf{e}_{[l']}\, \nu',
\end{aligned}$$

showing $[q_2] = []$, and so $\mathsf{s}_{[]}\, \beta_2 = \mathsf{s}_{[]}\, \bar{\nu}_2 = \psi_2$, and we can write β_2 as

$$\beta_2 = \mathsf{Y}_{\psi_2} \bigcirc_{[[r_{2,j}]]} \mu_{2,j}, \tag{11.2.18}$$

where $[r_{2,j}]$ ranges over $\psi_2^\bullet$, and $\mu_{2,j} \in \mathbb{O}_{n+1}$. Finally, we have

$$\begin{aligned}
\nu' &= \beta_1 \underset{[l']}{\circ} \beta_2 = (\bar{\nu}_1 \underset{[q_1]}{\square} \nu_1) \underset{[l']}{\circ} \beta_2 \\
&= \left(\mu_1 \underset{[q_1]}{\circ} \nu_1 \bigcirc_{\wp_{\nu_1}^{-1}[r_{1,j}]} \mu_{1,j} \right) \underset{[l']}{\circ} \left(\mathsf{Y}_{\psi_2} \bigcirc_{[[r_{2,j}]]} \mu_{2,j} \right) && \text{by (11.2.17) and (11.2.18)} \\
&= \left(\left(\mu_1 \underset{[q_1]}{\circ} \underbrace{\nu_1 \underset{[l]}{\circ} \mathsf{Y}_{\psi_2}}_{=\nu} \right) \bigcirc_{[q_1]\cdot\wp_{\nu_1}^{-1}[r_{1,j}]} \mu_{1,j} \right) \bigcirc_{[l'[r_{2,j}]]} \mu_{2,j} && \text{rearranging terms} \\
&= \bar{\nu} \underset{[q_1]}{\square} \nu,
\end{aligned}$$

for some $\bar{\nu}' \in \mathbb{O}_{n+1}$. Finally, by lemma 11.2.15, the following is a diagram of hf, where $\xi := \mathsf{Y}_{\bar{\nu}} \circ_{[[q_1]]} \mathsf{Y}_{\nu}$:

$$\dfrac{\begin{tikzcd} & \xi \\ \nu \arrow[ur, "\mathsf{s}_{[[q_1]]}"] & \nu' \arrow[u, "\mathsf{t}"'] \end{tikzcd}}{h\nu \xrightarrow{f} h\nu'}.$$

□

Lemma 11.2.19. *(1) If $\omega \in \mathbb{O}_{n-1}$, then h maps $\mathsf{tt} : \omega \longrightarrow \mathsf{I}_\omega$ to an identity.*

(2) If $\omega \in \mathbb{O}_n$, then h maps $\mathsf{s}_{[]} : \omega \longrightarrow \mathsf{Y}_\omega$ to an identity.

(3) If $\omega \in \mathbb{O}_{n+2}$, then h maps $\mathsf{t} : \mathsf{t}\omega \longrightarrow \omega$ to an identity.

Proof. By inspection of definition 11.2.1. □

Proposition 11.2.20. *The functor $h : \mathbb{O}_{n-k,n+2} \longrightarrow \mathbb{A}$ is essentially surjective on morphisms.*

Proof. Let $\omega, \omega' \in \mathbb{O}_{n-k,n+2}$.

(1) If $\dim \omega, \dim \omega' < n-1$, then by definition 11.2.1, $h\omega = \omega$ and $h\omega' = \omega'$ as presheaves over $\mathbb{O}_{n-k,n+2}$, and thus

$$\mathbb{A}(h\omega, h\omega') = \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})(\omega, \omega') = \mathbb{O}(\omega, \omega').$$

(2) Assume that $\dim \omega < n-1$ and $\dim \omega' \geq n-1$. We first show that $O[\omega']_{<n-1} = (h\omega')_{<n-1}$ by inspection of definition 11.2.1. If $\dim \omega' \leq n$, then the claim trivially holds. If $\dim \omega' = n+1$, then $h\omega' = \mathfrak{Z} S[\omega']$, and

$$\begin{aligned} (h\omega')_{<n-1} &= (\mathfrak{Z} S[\omega'])_{<n-1} \\ &= S[\omega']_{<n-1} && \text{see definition 11.1.16} \\ &= O[\omega']_{<n-1} && \text{see remark 3.5.2.} \end{aligned}$$

The case where $\dim \omega' = n+2$ is proved similarly. Thus, $O[\omega']_{<n-1} = (h\omega')_{<n-1}$, and in particular, $O[\omega']_\omega = (h\omega')_\omega$. Finally,

$$\begin{aligned} \mathbb{A}(h\omega, h\omega') &\cong \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})(\omega, h\omega') \\ &= \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})(\omega, \omega') && \spadesuit \\ &= \mathbb{O}(\omega, \omega'), \end{aligned}$$

where ♠ results from the observation above.

(3) If $\dim \omega \geq n-1$ and $\dim \omega' < n-1$, then $\mathbb{A}(h\omega, h\omega') = \varnothing$.

(4) Lastly, assume $\dim \omega, \dim \omega' \geq n-1$. By lemma 11.2.19, we may assume that $\dim \omega = \dim \omega' = n+1$. If ω is non degenerate, then by lemma 11.2.16, every morphism in $\mathbb{A}(h\omega, h\omega')$ is diagrammatic, thus in the essential image of h. Assume that ω is degenerate, say $\omega = \mathsf{I}_\phi$ for some $\phi \in \mathbb{O}_{n-1}$. Akin to point (2), by inspection of definition 11.2.1, one can prove that $O[\omega']_\phi = (h\omega')_\phi$. Finally,

$$\begin{aligned} \mathbb{A}(h\omega, h\omega') &\cong \mathbb{A}(h\phi, h\omega') && \text{by corollary 3.5.14} \\ &\cong \mathbb{O}(\phi, \omega') && \spadesuit, \end{aligned}$$

where ♠ results from the observation above. □

Remark 11.2.21. It is worthwhile to note that $h : \mathbb{O}_{n-k,n+2} \longrightarrow \mathbb{A}$ is *not* full. Take for example $n = k = 1$, so that h is a functor $\mathbb{O}_{0,3} \longrightarrow \mathbb{\Delta}$. Let $a, b \in \mathbb{N}$, $a \neq b$, and consider the corresponding opetopic integers $\mathbf{a}, \mathbf{b} \in \mathbb{O}_2$. Since they are different but have the same dimension, $\mathbb{O}(\mathbf{a}, \mathbf{b}) = \varnothing$, but of course, $\mathbb{\Delta}(h\mathbf{a}, h\mathbf{b}) = \mathbb{\Delta}([a], [b])$ is not empty. The diagrammatic lemma says that if $a \neq 0$, then a morphism in $\mathbb{\Delta}([a], [b])$ can be recovered as the image of a face embedding of $\mathbf{a}$ in some 3-opetope whose target is $\mathbf{b}$.

Recall from corollary 11.1.14 that we have a reflective adjunction

$$\tau : \mathcal{P}\mathrm{sh}(\mathbb{A}) \overset{\longrightarrow}{\underset{\perp}{\longleftarrow}} \mathcal{A}\mathrm{lg} : N$$

that exhibits $\mathcal{A}\mathrm{lg}$ as the localization of $\mathcal{P}\mathrm{sh}(\mathbb{A})$ at the set S of spine inclusions. This result is part of what we call the *nerve theorem for* $\mathbb{A}$. In this section, we prove a similar result in $\mathcal{P}\mathrm{sh}(\mathbb{O})$. The strategy is to study the adjunction $h_! : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \rightleftarrows \mathcal{P}\mathrm{sh}(\mathbb{A}) : h^*$, and to show that it preserves the orthogonality classes of spine inclusions. It follows that it restricts and corestricts as an adjunction $\mathsf{S}_{n+1,n+2}^{-1}\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \rightleftarrows \mathsf{S}^{-1}\mathcal{P}\mathrm{sh}(\mathbb{A}) \simeq \mathcal{A}\mathrm{lg}$, and it remains to prove that it is an equivalence. More formally, we make use of the following observation:

Lemma 11.2.22. *Let $F : \mathcal{A} \rightleftarrows \mathcal{B} : U$ be an adjunction, and write $\eta : \mathrm{id} \longrightarrow UF$ for the unit and $\varepsilon : FU \longrightarrow \mathrm{id}$ for the counit. If $\mathcal{A}'$ (resp. $\mathcal{B}'$) is a full subcategory of $\mathcal{A}$ (resp. $\mathcal{B}$) such that*

(1) $F\mathcal{A}' \subseteq \mathcal{B}'$ and $U\mathcal{B}' \subseteq \mathcal{A}'$,

(2) for all $a \in \mathcal{A}'$, the unit $\eta_a : a \longrightarrow UFa$ is an isomorphism, and dually, for all $b \in \mathcal{B}'$, ε_b is an isomorphism,

then the adjunction restricts and corestricts to an adjoint equivalence $F : \mathcal{A}' \rightleftarrows \mathcal{B}' : U$. In particular, if $\mathcal{A}'$ (resp. $\mathcal{B}'$) is an orthogonality class induced by a class of morphism K (resp. K'), then condition (1) above translates as follows:

(1') for all $a \in \mathcal{A}$, if $\mathsf{K} \perp a$, then $\mathsf{K}' \perp Fa$, and dually, for all $b \in \mathcal{B}$, if $\mathsf{K}' \perp b$, then $\mathsf{K} \perp Ub$.

Proposition 11.2.23. *The functor $h_! : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{A})$ (see definition 11.2.3) takes the set $\mathsf{S}_{n+1} \subseteq \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})^{[1]}$ (of definition 3.5.1) to $\mathsf{S} \subseteq \mathcal{P}\mathrm{sh}(\mathbb{A})^{[1]}$ (of definition 11.1.37), and takes morphisms in S_{n+2} to S-local isomorphisms.*

Proof. (1) Let $\nu \in \mathbb{O}_{n+1}$, and recall from definition 11.1.37 that $\mathbb{O}_{n-k,n}/S[\nu]$ is the category of elements of $S[\nu]$. We have

$$\begin{aligned} h_! S[\nu] &= h_! \operatorname*{colim}_{\psi \in \mathbb{O}_{n-k,n}/S[\nu]} O[\psi] \\ &\cong \operatorname*{colim}_{\psi \in \mathbb{O}_{n-k,n}/S[\nu]} h_! O[\psi] \\ &= \operatorname*{colim}_{\psi \in \mathbb{O}_{n-k,n}/S[\nu]} \mathsf{y}_{\mathbb{A}}(h\psi) \\ &= S[h\nu] && \text{see definition 11.1.37.} \end{aligned}$$

(2) For $\xi \in \mathbb{O}_{n+2}$, the inclusion $S[\mathsf{t}\xi] \longrightarrow S[\xi]$ is a relative S_{n+1}-cell complex by lemma 3.5.12. Since $h_!$ preserves colimits, and since $h_! \mathsf{S}_{n+1} = \mathsf{S}$, we have that $h_!(S[\mathsf{t}\xi] \longrightarrow S[\xi])$ is a relative S-cell complex, and thus an S-local isomorphism. In the square below

$$\begin{tikzcd} h_! S[\mathsf{t}\xi] \arrow[r] \arrow[d, "h_! \mathsf{s}_{\mathsf{t}\xi}"'] & h_! S[\xi] \arrow[d, "h_! \mathsf{s}_{\xi}"] \\ h_! O[\mathsf{t}\xi] \arrow[r, "h_! \mathsf{t}"] & h_! O[\xi] \end{tikzcd}$$

the top arrow is an S-local isomorphism, the right arrow is in S by the previous point, and the bottom arrow is an isomorphism by definition. By 3-for-2, we conclude that $h_! \mathsf{s}_\xi$ is an S-local isomorphism. □

Lemma 11.2.24. *Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$ be such that $\mathsf{S}_{n+1,n+2} \perp X$, and take $\omega \in \mathbb{O}_{n-k,n+2}$. The following are spans of isomorphisms:*

(1) for $\psi \in \mathbb{O}_{n-1}$,

$$\mathbb{A}(h\omega, h\psi) \times X_\psi \xleftarrow{\mathrm{id} \times \mathsf{t}\,\mathsf{t}} \mathbb{A}(h\omega, h\psi) \times X_{\mathsf{I}_\psi} \xrightarrow{\mathbb{A}(h\omega, h\,\mathsf{t}\,\mathsf{t}) \times \mathrm{id}} \mathbb{A}(h\omega, h\mathsf{I}_\psi) \times X_{\mathsf{I}_\psi};$$

(2) for $\psi \in \mathbb{O}_n$,

$$\mathbb{A}(h\omega, h\psi) \times X_\psi \xleftarrow{\mathrm{id} \times \mathsf{s}_{[]}} \mathbb{A}(h\omega, h\psi) \times X_{\mathsf{Y}_\psi} \xrightarrow{\mathbb{A}(h\omega, h\,\mathsf{s}_{[]}) \times \mathrm{id}} \mathbb{A}(h\omega, h\mathsf{Y}_\psi) \times X_{\mathsf{Y}_\psi};$$

(3) for $\psi \in \mathbb{O}_{n+2}$,

$$\mathbb{A}(h\omega, h\,\mathsf{t}\,\psi) \times X_{\mathsf{t}\,\psi} \xleftarrow{\mathrm{id} \times \mathsf{t}} \mathbb{A}(h\omega, h\,\mathsf{t}\,\psi) \times X_\psi \xrightarrow{\mathbb{A}(h\omega, h\,\mathsf{t}) \times \mathrm{id}} \mathbb{A}(h\omega, h\psi) \times X_\psi.$$

Proof. Follows from lemma 11.2.19. □

Lemma 11.2.25. *Let $\omega \in \mathbb{O}_{n-k,n+2}$. If $\psi \in \mathbb{O}_{n-k,n-1}$, then $\mathbb{A}(h\omega, h\psi) \cong \mathbb{O}_{n-k,n+2}(\omega, \psi)$.*

Proof. Easy verification. □

Proposition 11.2.26. *Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$. If $\mathsf{S}_{n+1,n+2} \perp X$, then the unit $\eta_X : X \longrightarrow h^* h_! X$ is an isomorphism.*

Proof. It suffices to show that for each $\omega \in \mathbb{O}_{n-k,n+2}$, the following map is a bijection:

$$X_\omega \xrightarrow{\eta_X} h^* h_! X_\omega = \int^{\psi \in \mathbb{O}_{n-k,n+2}} \mathbb{A}(h\omega, h\psi) \times X_\psi.$$

If $\omega \in \mathbb{O}_{n-k,n-1}$, then $h\omega = O[\omega]$, and $\mathbb{A}(h\omega, h-) \cong \mathbb{O}_{n-k,n+2}(\omega, -)$. Thus,

$$\begin{aligned} h^* h_! X_\omega &= \int^{\psi \in \mathbb{O}_{n-k,n+2}} \mathbb{A}(h\omega, h\psi) \times X_\psi && \text{by definition} \\ &\cong \int^{\psi \in \mathbb{O}_{n-k,n+2}} \mathbb{O}_{n-k,n+2}(\omega, \psi) \times X_\psi && \text{since } \dim \omega \leq n-1 \\ &\cong X_\omega && \text{by the density formula.} \end{aligned}$$

Assume how that $\dim \omega \geq n$. We construct an inverse of η_X via a cowedge $\mathbb{A}(h\omega, h-) \times X_- \xrightarrow{\cdot\cdot} X_\omega$.

(1) Assume $\omega \in \mathbb{O}_n$. By lemma 11.2.24, it suffices to consider the case $\psi \in \mathbb{O}_{n+1}$. To unclutter notations, write $\mathcal{P} := \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$. We have the sequence of morphisms

$$\begin{aligned} \mathbb{A}(h\omega, h\psi) \times X_\psi &\xrightarrow{\cong} \Big(\sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathsf{t}\,\nu = \omega}} \mathcal{P}(S[\nu], S[\psi]) \Big) \times \mathcal{P}(S[\psi], X) && \spadesuit \\ &\xrightarrow{\text{comp.}} \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathsf{t}\,\nu = \omega}} \mathcal{P}(S[\nu], X) \end{aligned}$$

$$\xrightarrow{\cong} \sum_{\substack{\nu\in\mathbb{O}_{n+1}\\ \mathsf{t}\,\nu=\omega}} X_\nu \qquad\qquad \spadesuit$$
$$\xrightarrow{\mathsf{t}} X_\omega,$$

where $\spadesuit$ follow from the assumption that $\mathsf{S}_{n+1} \perp X$. It is straightforward to verify that this defines a cowedge whose induced map is the required inverse.

(2) Assume $\omega \in \mathbb{O}_{n+1}$. If ω is degenerate, say $\omega = \mathsf{I}_\phi$ for some $\phi \in \mathbb{O}_{n-1}$, then $\mathbb{A}(h\omega, h-) \cong \mathbb{A}(h\phi, h-)$ and we are in a case we have treated before. So let ω be non-degenerate. By lemmas 11.2.24 and 11.2.25, we may suppose $\psi \in \mathbb{O}_{n,n+1}$. Recall that for every $f \in \mathbb{A}(h\omega, h\psi)$, the diagrammatic lemma 11.2.16 computes a $\xi \in \mathbb{O}_{n+2}$ and $[p] \in \xi^\bullet$ such that $\mathsf{s}_{[p]}\,\xi = \omega$, $\mathsf{t}\,\xi = \psi$ and $h\,\mathsf{s}_{[p]} \cong f$. By corollary 3.5.13, the target embedding $\mathsf{t} : \psi \longrightarrow \xi$ is an $\mathsf{S}_{n+1,n+2}$-local isomorphism, and by assumption, $\mathsf{S}_{n+1,n+2} \perp X$. Therefore, we have an isomorphism $\mathsf{t} : X_\xi \longrightarrow X_\psi$, which gives rise to a map

$$\begin{aligned} \mathbb{A}(h\omega, h\psi) \times X_\psi &\longrightarrow X_\omega \\ (f, x) &\longmapsto \mathsf{s}_{[p]}\,\mathsf{t}^{-1}\,x. \end{aligned}$$

It is straightforward to verify that this assignment defines a cowedge, whose associated map is the required inverse.

(3) Assume $\omega \in \mathbb{O}_{n+2}$. Then by definition of h, $\mathbb{A}(h\omega, h-) \cong \mathbb{A}(h\,\mathsf{t}\,\omega, h-)$, and this is the case we have just treated. □

Corollary 11.2.27. *Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$. If $\mathsf{S}_{n+1,n+2} \perp X$, then $\mathsf{S} \perp h_!X$.*

Proof. Recall from proposition 11.2.23 that $\mathsf{S} = h_!\mathsf{S}_{n+1}$. Let $\nu \in \mathbb{O}_{n+1}$. To unclutter notations, write $\mathcal{P} := \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$. We have

$$\begin{aligned} \mathcal{P}\mathrm{sh}(\mathbb{A})(h_!\nu, h_!X) &\cong \mathcal{P}(\nu, h^*h_!X) && \text{since } h_! \dashv h^* \\ &\cong \mathcal{P}(\nu, X) && \text{by proposition 11.2.26} \\ &\cong \mathcal{P}(S[\nu], X) && \text{since } \mathsf{s}_\nu \perp X \\ &\cong \mathcal{P}(S[\nu], h^*h_!X) && \text{by proposition 11.2.26,} \\ &\cong \mathcal{P}\mathrm{sh}(\mathbb{A})(h_!S[\nu], h_!X) && \text{since } h_! \dashv h^* \end{aligned}$$

and by construction, this isomorphism is the precomposition by $h_!\mathsf{s}_\nu$. Therefore, $h_!\mathsf{s}_\nu \perp X$. □

This second proposition will provide the other half of the equivalence between $\mathcal{A}\mathrm{lg}$ and the localization $\mathsf{S}_{n+1,n+2}^{-1}\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$.

Proposition 11.2.28. *Let $Y \in \mathcal{P}\mathrm{sh}(\mathbb{A})$. If $\mathsf{S} \perp Y$, then the counit map $\varepsilon_Y : h_!h^*Y \longrightarrow Y$ is an isomorphism.*

Proof. We have to prove that for each $\lambda \in \mathbb{A}$, the map

$$h_!Y_\lambda = \int^{\psi\in\mathbb{O}_{n-k,n+2}} \mathbb{A}(\lambda, h\psi) \times Y_{h\psi} \xrightarrow{(\varepsilon_Y)_\lambda} Y_\lambda \tag{11.2.29}$$

is a bijection. Recall notation 0.4.2, and consider, the map

$$s : Y_\lambda \longrightarrow \int^{\psi\in\mathbb{O}_{n-k,n+2}} \mathbb{A}(\lambda, h\psi) \times Y_{h\psi}$$

mapping $y \in Y_\lambda$ to $\mathrm{id}_\lambda \otimes y$. It is well-defined, as h is surjective on objects, and it is easy to verify that $s(y)$ it is independent of the choice of an antecedent $h\nu = \lambda$. Note that s is a section of $(\varepsilon_Y)_\lambda$, and we proceed to prove that s is surjective. In other words, we show that that every element $f \otimes y$, with $f \in \mathbb{A}(\lambda, h\psi)$ for some $\psi \in \mathbb{O}_{n-k,n+2}$ and $y \in Y_\lambda$, is equal to an element of the form $\mathrm{id}_\lambda \otimes y'$, for some $y' \in Y_\lambda$.

(1) Assume $\lambda = h\phi$ for some $\phi \in \mathbb{O}_{n-k,n-1}$. Then $\mathbb{A}(\lambda, h\psi) = \mathbb{O}_{n-k,n+2}(\phi, \psi)$, and $f \otimes y = \mathrm{id}_\phi \otimes f(y)$ has the required form.

(2) Assume $\lambda = h\nu = hS[\nu]$ for some $\nu \in \mathbb{O}_{n+1}$. If ν is degenerate, say $\nu = \mathsf{I}_\phi$, then by lemma 11.2.19, $h\nu = h\phi$, so we fall in the previous case. Thus, we may assume that ν is not degenerate. Further, by lemmas 11.2.24 and 11.2.25, we may consider only the case where $\psi \in \mathbb{O}_{n+1}$. By lemma 11.2.16, f admits a diagram, say

$$\frac{\begin{tikzcd} & \xi \\ \nu \arrow[ur, "\mathsf{s}_{[p]}"] & \psi \arrow[u, "\mathsf{t}"'] \end{tikzcd}}{h\nu \xrightarrow{f} h\psi,}$$

i.e. $f \cong h\,\mathsf{s}_{[p]}$. We then have $f \otimes y = \mathrm{id}_{\mathsf{Y}_\omega} \otimes (h\,\mathsf{s}_{[p]})(y)$.

(3) Assume $\lambda = h\omega$ for some $\omega \in \mathbb{O}_n$. By lemma 11.2.19, $h\omega = h\mathsf{Y}_\omega$, and we fall in the previous case. ☐

Definition 11.2.30. Let the adjunction induced by the localization of $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$ at the set of spine inclusions $\mathsf{S}_{n+1,n+2}$ be denoted by

$$u : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \underset{\perp}{\rightleftarrows} \mathsf{S}^{-1}_{n+1,n+2}\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) : N_u$$

On the other hand, recall from theorem 11.1.39 that we have an adjunction $\tau \dashv N$ that exhibits $\mathcal{A}\mathrm{lg}$ as the localization $\mathsf{S}^{-1}\mathcal{P}\mathrm{sh}(\mathbb{A})$. We are now well-equipped to prove that $\mathcal{A}\mathrm{lg}$ is equivalent to the localized category $\mathsf{S}^{-1}_{n+1,n+2}\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$.

Lemma 11.2.31. *The adjunction* $h_! : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \underset{\perp}{\rightleftarrows} \mathcal{P}\mathrm{sh}(\mathbb{A}) : h^*$ *restricts to an adjoint equivalence* $\tilde{h}_! \dashv \tilde{h}^*$*, as shown below.*

$$\begin{tikzcd} \mathsf{S}^{-1}_{n+1,n+2}\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \arrow[r, shift left, "\tilde{h}_!"] \arrow[r, phantom, "\perp"] \arrow[d, "N_u"'] & \mathsf{S}^{-1}\mathcal{P}\mathrm{sh}(\mathbb{A}) \simeq \mathcal{A}\mathrm{lg} \arrow[l, shift left, "\tilde{h}^*"] \arrow[d, "N"] \\ \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \arrow[r, shift left, "h_!"] \arrow[r, phantom, "\perp"] & \mathcal{P}\mathrm{sh}(\mathbb{A}). \arrow[l, shift left, "h^*"] \end{tikzcd}$$

Proof. We check the conditions of lemma 11.2.22.

(1) By proposition 11.2.23, for all $Y \in \mathcal{A}\mathrm{lg} \simeq \mathsf{S}^{-1}\mathcal{P}\mathrm{sh}(\mathbb{A})$, we have that $h_!\mathsf{S}_{n+1,n+2} \perp N_u Y$, or equivalently, that $\mathsf{S}_{n+1,n+2} \perp h^* N_u Y$. Thus $h^* N_u$ factors through the localization $\mathsf{S}^{-1}_{n+1,n+2}\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$. Next, by corollary 11.2.27, $h_! N_u$ factors through $\mathcal{A}\mathrm{lg}$.

(2) By proposition 11.2.26, if $X \in \mathsf{S}^{-1}_{n+1,n+2}\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$, then the unit map $\eta_X : X \longrightarrow h^* h_! X$ is an isomorphism, and dually, by proposition 11.2.28, if $Y \in \mathsf{S}^{-1}\mathcal{P}\mathrm{sh}(\mathbb{A})$, then the counit map ε_Y is an isomorphism. ☐

Definition 11.2.32. Recall the definition of O and S from definitions 3.5.1 and 3.5.15, and let $\mathsf{A} = \mathsf{A}_{k,n} := \mathsf{O}_{<n-k} \cup \mathsf{S}_{\geq n+1}$.

Theorem 11.2.33 (Nerve theorem for $\mathbb{O}$)**.** *The reflective adjunction* $h : \mathcal{P}\mathrm{sh}(\mathbb{O}) \rightleftarrows \mathcal{A}\mathrm{lg} : M$ *exhibits* $\mathcal{A}\mathrm{lg}$ *as the localization* $\mathsf{A}^{-1}\mathcal{P}\mathrm{sh}(\mathbb{O})$*, or equivalently, as the orthogonality class induced by* A *in* $\mathcal{P}\mathrm{sh}(\mathbb{O})$*.*

Proof. Recall from definition 11.2.3 that h is the composite

$$\mathcal{P}\mathrm{sh}(\mathbb{O}) \xrightarrow{(-)_{n-k,n+2}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \xrightarrow{h_!} \mathcal{P}\mathrm{sh}(\mathbb{\Lambda}) \xrightarrow{\tau} \mathcal{A}\mathrm{lg},$$

and by lemma 11.2.31, it is isomorphic to the composite

$$\mathcal{P}\mathrm{sh}(\mathbb{O}) \xrightarrow{(-)_{n-k,n+2}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \xrightarrow{u} \mathsf{S}^{-1}_{n+1,n+2}\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \xrightarrow{\simeq} \mathcal{A}\mathrm{lg}.$$

By proposition 3.5.19, the truncation $(-)_{n-k,n+2}$ is the localization at $\mathsf{O}_{<n-k} \cup \mathsf{B}_{>n+2}$. By corollary 0.5.12, u is the localization at $\mathsf{S}_{n+1,n+2}$. Therefore, h is the localization at $\mathsf{O}_{<n-k} \cup \mathsf{S}_{n+1,n+2} \cup \mathsf{B}_{>n+2}$, which by lemma 3.5.10 is the localization at A. By corollary 0.5.12, $\mathcal{A}\mathrm{lg}$ is equivalent to the orthogonality class induced by A. $\square$

Corollary 11.2.34. *Assume* $k = 1$*. Any opetopic shape* $\lambda \in \mathbb{\Lambda} = \mathbb{\Lambda}_{1,n}$ *is isomorphic to one of the form* $h\omega$ *for* $\omega \in \mathbb{O}_{n+1}$*.*

Proof. By theorem 11.2.33, $h : \mathcal{P}\mathrm{sh}(\mathbb{O}) \longrightarrow \mathcal{A}\mathrm{lg}$ maps spine inclusions in $\mathsf{S}_{\geq n+1}$ to isomorphism. Thus, it maps $\mathsf{S}_{\geq n+1}$-local isomorphisms to isomorphisms. The result then follows from corollary 3.5.14. $\square$

We now present a notation that shows how solutions of lifting problems $S[\omega] \longrightarrow X$ can really be understood as compositions of "tree-shaped arities".

Notation 11.2.35. For $\phi \in \mathbb{O}_{n-1}$ and $x \in X_\phi$, write $\mathrm{id}_x \in X_{\mathsf{Y}_\phi}$ for the target of the unique solution of the lifting problem

$$\begin{tikzcd} S[\mathsf{I}_\phi] = O[\phi] \arrow[r, "x"] \arrow[d, "\mathsf{tt}"'] & X \\ O[\mathsf{I}_\phi]. \arrow[ur, dashed, "\exists ! f"'] & \end{tikzcd}$$

Explicitly, if f is the solution, then id_x is the cell of X_{Y_ϕ} selected by

$$O[\mathsf{Y}_\phi] \xrightarrow{\mathsf{t}} O[\mathsf{I}_\phi] \xrightarrow{f} X.$$

Note that by **(Degen)**, we have $\mathsf{t}\,\mathrm{id}_x = \mathsf{s}_{[]}\,\mathrm{id}_x = x$. Let now $\psi, \psi' \in \mathbb{O}_n$, and $[p] \in \psi^\bullet$ such that $\mathsf{s}_{[p]}\,\psi = \mathsf{t}\,\psi'$. In particular, $\omega := \mathsf{Y}_\psi \circ_{[[p]]} \mathsf{Y}_{\psi'}$ is a well-defined $(n+1)$-opetope. For $x \in X_\psi$, $x' \in X_{\psi'}$ such that $\mathsf{s}_{[p]}\,x = \mathsf{t}\,x'$, let $x \circ_{[p]} x'$ be the target of the unique solution of the following lifting problem:

$$\begin{tikzcd} S[\omega] \arrow[r, "{[] \mapsto x,\ [[p]] \mapsto x'}"] \arrow[d] & X \\ O[\omega]. \arrow[ur, dashed, "\exists ! g"'] & \end{tikzcd}$$

Note that $(x \circ_{[p]} x')^\natural = \mathsf{t}\omega = \psi \square_{[p]} \psi'$. An iterated composition as on the right can be concisely written as on the left:

$$x \bigcirc_{[p_i]} y_i := (\cdots(x \underset{[p_1]}{\circ} y_1) \underset{[p_2]}{\circ} y_2 \cdots) \underset{[p_k]}{\circ} y_k,$$

where $x \in X_\psi$, $\psi^\bullet = \{[p_1], \ldots, [p_k]\}$, $y_i \in X_{\psi_i}$, and $\mathsf{t}\psi_i = \mathsf{s}_{[p_i]} \psi$. Equivalently, $x \bigcirc_{[p_i]} y_i$ is the target of the unique solution of

$$\begin{tikzcd} S[\omega'] \arrow[r, "{[] \mapsto x,\ [[p_i]] \mapsto y_i}"] \arrow[d] & X \\ O[\omega']. \arrow[ur, dashed] & \end{tikzcd}$$

where $\omega' := \mathsf{Y}_\psi \bigcirc_{[[p_i]]} \mathsf{Y}_{\psi_i}$.

Proposition 11.2.36. *Let U be the following composite*

$$\mathcal{A}\mathrm{lg} \xrightarrow{M} \mathcal{P}\mathrm{sh}(\mathbb{O}) \xrightarrow{(-)_{n-k,n}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}).$$

In other words, it considers an algebra as an opetopic set (via the fully faithful nerve functor M) and truncates it. For $A, B \in \mathcal{A}\mathrm{lg}$, a map $f : UA \longrightarrow UB$ is in the image of U if and only if

(1) for all $a \in UA_{n-1}$, we have $f(\mathrm{id}_a) = \mathrm{id}_{f(a)}$;

(2) for a well defined composition $x \circ_{[p]} y$, where $x, y \in A_n$ (see notation 11.2.35), we have $f(x \circ_{[p]} y) = f(x) \circ_{[p]} f(y)$.

Note that if condition (1) holds, then condition (2) is equivalent to the following

(3) for a well defined composition $x \bigcirc_{[p_i]} y_i$, where $x, y_1, \ldots \in A_n$, and $[p_i]$ ranges over the set of node addresses of the shape of x (see notation 11.2.35), we have

$$f\left(x \bigcirc_{[p_i]} y_i\right) = f(x) \bigcirc_{[p_i]} f(y_i)$$

Proof. The direct implication is clear. For the converse, note that the conditions simply state that f can be extended as a map $MA \longrightarrow MB$, as $\mathsf{S} \perp MA, MB$. Since M is fully faithful, $f : MA \longrightarrow MB$ is the nerve a (unique) morphism of algebras. □

11.3 THE ALGEBRAIC TROMPE-L'ŒIL

As we saw in section 11.1, for all $k, n \in \mathbb{N}$ with $k \leq n$, we have a notion of *k-colored n-opetopic algebra*. For such an algebra $B \in \mathcal{A}\mathrm{lg}_{k,n}$, operations are n-cells (so that their shapes are n-opetopes), and colors are cells of dimension $n-k$ to $n-1$, thus the "color space" is stratified over k dimensions. Notable examples include

$$\mathcal{C}\mathrm{at} \simeq \mathcal{A}\mathrm{lg}_{1,1}, \qquad \mathcal{O}\mathrm{p}_{\mathrm{col}} \simeq \mathcal{A}\mathrm{lg}_{1,2}.$$

(see proposition 11.1.29). But are all $\mathcal{A}\mathrm{lg}_{k,n}$ fundamentally different?

In this section, we answer this question negatively: in a sense that we make precise, the most "algebraically rich" notion of opetopic algebra is given in the case $k = 1$ and $n = 3$. Although opetopes can be arbitrarily complex, the algebraic data they carry can be expressed by 3-opetopes, a.k.a. trees. We call this phenomenon *algebraic trompe-l'œil*, a French expression that literally translates as "fools-the-eye". And indeed, the eye is fooled in two ways: by color (proposition 11.3.4) and shape (proposition 11.3.15). In the former, we argue that the color space of an algebra $B \in \mathcal{A}\mathrm{lg}_{k,n}$, expressing how operations may or may not be composed, only needs 1 dimension, and thus that cells of dimension less than $n-1$ do not bring new algebraic data, only geometrical one. For the latter, recall from definition 3.1.3 that opetopes are trees of opetopes. In particular, 3-opetopes are just plain trees, and $\mathbb{O}_3$ already contains all the possible underlying tree shapes of all opetopes. Consequently, operations of B, which are its n-cells, may be considered as 3-cells in a very similar 3-algebra $B^\vee$. Finally, we combine those two results in theorem 11.3.16, which states that an algebra $B \in \mathcal{A}\mathrm{lg}_{k,n}$ is exactly a presheaf $B \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$ with a 1-colored 3-algebra structure on $B^\vee_{n-1,n}$ (see definition 3.5.16).

COLOR

For $B \in \mathcal{A}\mathrm{lg}_{k,n}$, recall that the colors of B are its cells of dimension $n-k$ to $n-1$. They express which operations (n-cells) of B may or may not be composed. However, since that criterion only depends on $(n-1)$-cells, constraints expressed by lower dimensional cells should be redundant. In proposition 11.3.4, we show that this is indeed the case, in that the algebra structure on B is completely determined by a 1-colored n-algebra structure on the truncation $B_{n-1,n}$.

Lemma 11.3.1. *Let $k, n \geq 1$, and $\nu \in \mathbb{O}_{n+1}$. Then*

$$S[\nu]_{n-k,n} \cong \iota_!(S[\nu]_{n-1,n}),$$

where $\iota_!$ is the left adjoint to the truncation $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$.

Proof. It follows from the fact that $S[\nu]$ is completely determined by the incidence relation of the n- and $(n-1)$-faces of ν (see lemma 3.5.8). □

Proposition 11.3.2. *For $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$ we have $\mathfrak{Z}^n(X_{n-1,n}) \cong (\mathfrak{Z}^n X)_{n-1,n}$. Consequently, the truncation functor $(-)_{n-1,n} : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$ lifts as*

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}_{k,n} \arrow[r, "(-)_{n-1,n}"] \arrow[d] & \mathcal{A}\mathrm{lg}_{1,n} \arrow[d] \\
\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) \arrow[r, "(-)_{n-1,n}"] & \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n}).
\end{tikzcd} \tag{11.3.3}$$

Proof. To unclutter notations, write $\mathcal{P} := \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$. First, $\mathfrak{Z}^n(X_{n-1,n})_{n-1} = X_{n-1} = (\mathfrak{Z}^n X)_{n-1}$. Then, for $\omega \in \mathbb{O}_n$, we have

$$\begin{aligned}
\mathfrak{Z}^n(X_{n-1,n})_\omega &= \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathsf{t}\,\nu = \omega}} \mathcal{P}(S[\nu]_{n-1,n}, X_{n-1,n}) && \text{see definition 11.1.16} \\
&\cong \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathsf{t}\,\nu = \omega}} \mathcal{P}(\iota_! S[\nu], X) && \text{since } \iota_! \dashv (-)_{n-1,n}
\end{aligned}$$

$$\begin{aligned}
&\cong \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathsf{t}\,\nu = \omega}} \mathcal{P}(S[\nu]_{n-k,n}, X) && \text{by lemma 11.3.1} \\
&= \mathfrak{Z}^n X_\omega.
\end{aligned}$$

□

Proposition 11.3.4. *The square* (11.3.3) *is a pullback. That is, a $\mathfrak{Z}^n$-algebra structure on $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$ is completely determined by a $\mathfrak{Z}^n$-algebra structure on $X_{n-1,n}$.*

Proof. Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$. By proposition 11.3.2, a $\mathfrak{Z}^n$-algebra structure on X restricts to one on $X_{n-1,n}$. Since the truncation functor $(-)_{n-1,n} : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$ is faithful, its lift $\mathcal{A}\mathrm{lg}_{k,n} \longrightarrow \mathcal{A}\mathrm{lg}_{1,n}$ is injective on objects. In particular, different algebra structures on X truncate to different algebra structures on $X_{n-1,n}$. Conversely, since $(\mathfrak{Z}^n X)_{<n} = X_{<n}$, a $\mathfrak{Z}^n$-algebra structure on $X_{n-1,n}$ extends to one on X. Therefore, the truncation functor establishes a bijective correspondence between the algebra structures on X and on $X_{n-1,n}$. □

SHAPE

We start by defining the *flattening operator* $(-)^\vee : \mathbb{O}_{n-1,n} \longrightarrow \mathbb{O}_{2,3}$, for $n \geq 1$, mapping an n-opetope ω to a 3-opetope $\omega^\vee$ having the same underlying polynomial tree, i.e. $\langle \omega^\vee \rangle \cong \langle \omega \rangle$ (see notation 2.2.9).

Definition 11.3.5. (1) If $n = 1$, then $(-)^\vee$ simply maps $\mathbb{O}_{0,1} = \left(\blacklozenge \xrightarrow{\mathsf{s}_*, \mathsf{t}} \blacksquare \right)$ to the diagram $\left(\mathbf{0} \xrightarrow{\mathsf{s}_{[]}, \mathsf{t}} \mathsf{Y}_{\mathbf{0}} \right)$.

(2) Assume now that $n \geq 2$. Recall from definition 3.1.3 that a 3-opetope is a $\mathfrak{Z}^1$-tree, where $\mathfrak{Z}^1$ is given by

$$\{\blacksquare\} \xleftarrow{\mathsf{s}} E_2 \xrightarrow{p} \mathbb{O}_2 \xrightarrow{\mathsf{t}} \{\blacksquare\},$$

where $\mathbb{O}_2 = \{\mathbf{m} \mid m \in \mathbb{N}\}$, and where $E_2(\mathbf{m}) = \mathbf{m}^\bullet = \left\{ [*^i] \mid 0 \leq i < m \right\}$. Let $f : \mathfrak{Z}^{n-2} \longrightarrow \mathfrak{Z}^1$ be the morphism of polynomial functors given by

$$\begin{tikzcd}
\mathbb{O}_{n-2} \arrow[d, "f_0"'] & E_{n-2} \arrow[l, "\mathsf{s}"'] \arrow[r, "p"] \arrow[d, "f_2"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{O}_{n-1} \arrow[r, "\mathsf{t}"] \arrow[d, "f_1"] & \mathbb{O}_{n-2} \arrow[d, "f_0"] \\
\{\blacksquare\} & E_2 \arrow[l, "\mathsf{s}"'] \arrow[r, "p"] & \mathbb{O}_2 \arrow[r, "\mathsf{t}"] & \{\blacksquare\},
\end{tikzcd}$$

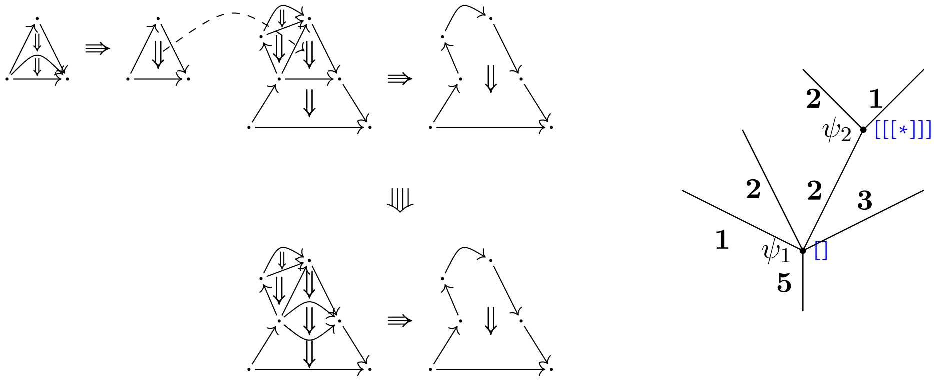
where $f_1(\psi) = \mathbf{m}$, $m = \#\psi^\bullet$, and where f_2 is fiberwise increasing with respect to the lexicographical order $\leq$ on addresses. This morphism induces a functor $f_* : \mathbb{O}_n = \operatorname{tr} \mathfrak{Z}^{n-2} \longrightarrow \operatorname{tr} \mathfrak{Z}^1 = \mathbb{O}_3$ (see definition 2.2.7) mapping an n-opetope to its underlying tree, seen as a 3-opetope. Explicitly,

$$f_*(\mathsf{I}_\phi) = \mathsf{I}_\blacksquare, \qquad f_*\left(\mathsf{Y}_\psi \bigcirc_{[[p_i]]} \omega_i \right) = \mathbf{m} \bigcirc_{[[*^i]]} f_*(\omega_i),$$

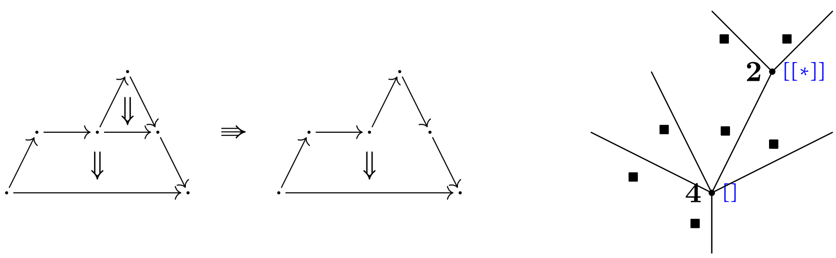
where $\phi \in \mathbb{O}_{n-2}$, $\psi \in \mathbb{O}_{n-1}$, $\psi^\bullet = \{[p_0] < [p_1] < \cdots\}$, and $\omega_0, \ldots, \omega_{m-1} \in \mathbb{O}_n$. For $\omega \in \mathbb{O}_n$, since ω and $\omega^\vee$ have the same underlying tree, they have the same number of source faces, i.e. $\#\omega^\bullet = \#(\omega^\vee)^\bullet$, and we write $a_\omega : \omega^\bullet \longrightarrow (\omega^\vee)^\bullet$ for the unique increasing map with respect to the lexicographical order. Intuitively, a_ω maps a node of the underlying tree $\langle\omega\rangle$ of ω to that same node in $\langle\omega^\vee\rangle$. However, since the source faces of ω and $\omega^\vee$ are not the same, a_ω is not strictly speaking an identity, but rather a conversion of a "walking instruction in the tree ω" (which is what an address is) to one in $\omega^\vee$. Explicitly, a node address $[[q_1]\cdots[q_k]] \in \omega^\bullet$ (with $[q_{i+1}] \in \mathsf{s}_{[[q_1]\cdots[q_i]]}\,\omega$) is mapped to $[f_{2,\mathsf{s}_{[]}\,\omega}[q_1]\ \cdots\ f_{2,\mathsf{s}_{[[q_1]\cdots[q_{k-1}]]}\,\omega}[q_k]]$.

(3) Define now the *flattening operator* $(-)^\vee : \mathbb{O}_{n-1,n} \longrightarrow \mathbb{O}_{2,3}$ as follows: for $\psi \in \mathbb{O}_{n-1}$ and $\omega \in \mathbb{O}_n$,
   a) $\psi^\vee := f_1(\psi)$ and $\omega^\vee := f_*(\omega)$ as above;
   b) clearly, $(\mathsf{t}\,\omega)^\vee = \mathbf{m} = \mathsf{t}\,\omega^\vee$, where $m := \#(\mathsf{t}\,\omega)^\bullet$, so let $\left(\mathsf{t}\,\omega \xrightarrow{\mathsf{t}} \omega\right)^\vee$ simply be $\left((\mathsf{t}\,\omega)^\vee \xrightarrow{\mathsf{t}} \omega^\vee\right)$;
   c) likewise, for $[p] \in \omega^\bullet$, we have $\left(\mathsf{s}_{[p]}\,\omega\right)^\vee = \mathsf{s}_{a_\omega[p]}\,\omega^\vee$, and let $\left(\mathsf{s}_{[p]}\,\omega \xrightarrow{\mathsf{s}_{[p]}} \omega\right)^\vee$ simply be $\left(\mathsf{s}_{[p]}\,\omega\right)^\vee \xrightarrow{\mathsf{s}_{a_\omega[p]}} \omega^\vee$.

Example 11.3.6. Consider the 4-opetope ω, represented graphically and in tree form below:



where ψ_1 and ψ_2 are the 3-opetopes on the top right and top left hand corner respectively. Then its flattening $\omega^\vee$ is as follows:



Although the graphical representations of ω and $\omega^\vee$ look nothing alike, their underlying (undecorated) trees are identical.

Remark 11.3.7. Clearly, $(-)^\vee : \mathbb{O}_{n-1,n} \longrightarrow \mathbb{O}_{2,3}$ is faithful, and if $n \leq 3$, then $(-)^\vee$ is also injective on objects. Note that this is no longer the case if $n \geq 4$, as distinct n-opetopes may have the same underlying tree. For example, the underlying tree of the any degenerate is just a single edge, and for all $n \geq 4$, there exists infinitely many degenerate n-opetopes.

Definition 11.3.8 (Flattening)**.** With a slight abuse of notations, let

$$(-)^{\vee} : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{2,3}),$$

the *flattening operation*, be the left Kan extension of $\mathbb{O}_{n-1,n} \xrightarrow{(-)^{\vee}} \mathbb{O}_{2,3} \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{2,3})$ along the Yoneda embedding.

Lemma 11.3.9. *Explicitly, for $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$, we have*

$$X^{\vee}_{\mathbf{m}} \cong \sum_{\substack{\psi \in \mathbb{O}_{n-1} \\ \#\psi^{\bullet} = m}} X_{\psi}, \qquad X^{\vee}_{\gamma} \cong \sum_{\substack{\omega \in \mathbb{O}_{n} \\ \omega^{\vee} = \gamma}} X_{\omega},$$

with $m \in \mathbb{N}$ and $\gamma \in \mathbb{O}_3$.

Proof. (1) Assume that $X = O[\omega]$ for some $\omega \in \mathbb{O}_{n-1}$, and let $d := \#\omega^{\bullet}$. If $m \in \mathbb{N}$, then by definition

$$O[\omega]^{\vee}_{\mathbf{m}} = O[\mathbf{d}]_{\mathbf{m}} = \begin{cases} \{\mathrm{id}_{\mathbf{d}}\} & \text{if } d = m \\ \varnothing & \text{otherwise.} \end{cases}$$

On the other hand, if $\psi \in \mathbb{O}_{n-1}$,

$$O[\omega]_{\psi} = \mathbb{O}(\psi, \omega) = \begin{cases} \{\mathrm{id}_{\psi}\} & \text{if } \omega = \psi \\ \varnothing & \text{otherwise.} \end{cases}$$

Thus,

$$\sum_{\substack{\psi \in \mathbb{O}_{n-1} \\ \#\psi^{\bullet} = m}} O[\omega]_{\psi} \cong \begin{cases} \{\mathrm{id}_{\mathbf{d}}\} & \text{if } d = m \\ \varnothing & \text{otherwise} \end{cases} = O[\omega]^{\vee}_{\mathbf{m}}.$$

On the other hand, if $\gamma \in \mathbb{O}_3$, then

$$O[\omega]^{\vee}_{\gamma} = O[\mathbf{d}]_{\gamma} = \varnothing = \sum_{\substack{\omega' \in \mathbb{O}_{n} \\ \omega'^{\vee} = \gamma}} X_{\omega'}.$$

(2) With the same reasoning, one can prove the lemma in the case $X = O[\omega]$ for some $\omega \in \mathbb{O}_n$.

(3) Let us now consider the general case. If $m \in \mathbb{N}$, then

$$\begin{aligned} X^{\vee}_{\mathbf{m}} &= \int^{\omega \in \mathbb{O}_{n-1,n}} X_{\omega} \times O[\omega^{\vee}]_{\mathbf{m}} && \text{see equation (0.4.8)} \\ &\cong \int^{\omega \in \mathbb{O}_{n-1,n}} X_{\omega} \times \sum_{\substack{\psi \in \mathbb{O}_{n-1} \\ \#\psi^{\bullet} = m}} O[\omega]_{\psi} && \text{by the previous points} \\ &\cong \sum_{\substack{\psi \in \mathbb{O}_{n-1} \\ \#\psi^{\bullet} = m}} \int^{\omega \in \mathbb{O}_{n-1,n}} X_{\omega} \times O[\omega]_{\psi} \\ &\cong \sum_{\substack{\psi \in \mathbb{O}_{n-1} \\ \#\psi^{\bullet} = m}} X_{\psi} && \text{by theorem 0.4.1.} \end{aligned}$$

We prove the second isomorphism of the lemma in a similar manner. □

Remark 11.3.10. Take $n \geq 1$ and $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$. By lemma 11.3.9, X and $X^\vee$ essentially have the same cells and the same incidence relations among them. Formally, there is a canonical isomorphism $\mathbb{O}_{n-1,n}/X \longrightarrow \mathbb{O}_{2,3}/X^\vee$ between the categories of elements of X and $X^\vee$, which maps source (resp. target) maps to source (resp. target) maps. Further, if $f : X \longrightarrow Y$ is a morphism in $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$, then we have a commutative square

$$\begin{tikzcd} \mathbb{O}_{n-1,n}/X \arrow[r, "f"] \arrow[d, "\cong"'] & \mathbb{O}_{n-1,n}/Y \arrow[d, "\cong"] \\ \mathbb{O}_{2,3}/X^\vee \arrow[r, "f^\vee"] & \mathbb{O}_{2,3}/Y^\vee \end{tikzcd}$$

In particular, $(-)^\vee : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{2,3})$ is faithful.

Lemma 11.3.11. *Let $n \geq 1$, and consider the flattening operator $(-)^\vee : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{2,3})$.*

(1) For $\nu \in \mathbb{O}_{n+1}$, there exists a unique 4-opetope $\nu' \in \mathbb{O}_4$ such that $S[\nu]^\vee_{n-1,n} \cong S[\nu']_{2,3}$.

(2) Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$, $\nu \in \mathbb{O}_4$, and $f : S[\nu] \longrightarrow X^\vee$. Then there exists a unique $\nu' \in \mathbb{O}_{n+1}$ and $g : S[\nu']_{2,3} \longrightarrow X$ such that $S[\nu']^\vee_{n-1,n} = S[\nu]_{2,3}$, and $g^\vee = f$.

Proof. (1) If $\nu = \mathsf{I}_\phi$ for $\phi \in \mathbb{O}_{n-1}$, let $\nu' = \mathsf{I}_{\phi^\vee}$. If $\nu = \mathsf{Y}_\omega \bigcirc_{[[p_i]]} \nu_i$, let

$$\nu' := \mathsf{Y}_{\omega^\vee} \bigcirc_{[a_\omega[p_i]]} \nu'_i,$$

where the ν'_i are given by induction. The graftings are well defined since

$$\mathsf{t}\,\mathsf{s}_{[]}\,\nu'_i = \mathsf{t}\big(\mathsf{s}_{[]}\,\nu_i\big)^\vee = \big(\mathsf{t}\,\mathsf{s}_{[]}\,\nu_i\big)^\vee = \big(\mathsf{s}_{[p_i]}\,\omega\big)^\vee = \mathsf{s}_{a_\omega[p_i]}\,\omega^\vee.$$

The isomorphism $S[\nu]^\vee_{n-1,n} \cong S[\nu']_{2,3}$ can easily be shown by induction on the structure of ν and using lemma 3.5.8.

(2) For $\nu^\bullet = \{[p_1], \ldots, [p_m]\}$, f maps $[p_i]$ to a cell $x_i \in X^\vee = X_{n-1}$, and let $\psi_i \in \mathbb{O}_{n-1}$ be the shape of x_i as a cell of X. If $[p_i] = [p_j[q]]$ for some j and $[q]$, then $\mathsf{s}_{[q]}\,x_j = \mathsf{t}\,x_i$ in $X^\vee$, so $\mathsf{s}_{a^{-1}_{\psi_j}[q]}\,x_j = \mathsf{t}\,x_i$ in X, and in particular, $\mathsf{s}_{a^{-1}_{\psi_j}[q]}\,\psi_j = \mathsf{t}\,\psi_i$. Consequently, the ψ_is may be grafted together into a $(n+1)$-opetope ν' such that $\nu'^\vee = \nu$, and $\mathsf{s}_{a^{-1}_{\nu'}[p_i]} = \psi_i$. Define $g : S[\nu']_{2,3} \longrightarrow X$ mapping $\mathsf{s}_{a^{-1}_{\nu'}[p_i]}\,\nu'$ to x_i, and observe that $g^\vee = f$. □

Proposition 11.3.12. *For $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$ we have $\mathfrak{Z}^3(X^\vee) \cong (\mathfrak{Z}^n X)^\vee$. Consequently, the functor $(-)^\vee$ lifts as*

$$\begin{tikzcd} \mathcal{A}\mathrm{lg}_{1,n} \arrow[r, "(-)^\vee"] \arrow[d] & \mathcal{A}\mathrm{lg}_{1,3} \arrow[d] \\ \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n}) \arrow[r, "(-)^\vee"] & \mathcal{P}\mathrm{sh}(\mathbb{O}_{2,3}). \end{tikzcd} \tag{11.3.13}$$

Proof. First, $\mathfrak{Z}^3(X^\vee)_2 = X^\vee_2 \cong X_{n-1} = (\mathfrak{Z}^n X)_{n-1} = (\mathfrak{Z}^n X)^\vee_2$. Then,

$$\mathfrak{Z}^3(X^\vee)_3 = \sum_{\nu \in \mathbb{O}_4} \mathcal{P}\mathrm{sh}(\mathbb{O}_{2,3})(S[\nu], X^\vee)$$

$$
\begin{aligned}
&\cong \sum_{\nu \in \mathbb{O}_{n+1}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})(S[\nu], X) && \text{by lemma 11.3.11} \\
&= (\mathfrak{Z}^n X)_n \\
&= (\mathfrak{Z}^n X)^\vee_3 .
\end{aligned}
$$

□

Lemma 11.3.14. *Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$ and $m : \mathfrak{Z}^n X \longrightarrow X$. Then m is a $\mathfrak{Z}^n$-algebra structure on X if and only if $m^\vee : \mathfrak{Z}^3 X^\vee \longrightarrow X^\vee$ is a $\mathfrak{Z}^3$-algebra structure on $X^\vee$.*

Proof. Clearly, $(-)^\vee$ maps the multiplication $\mu^n : \mathfrak{Z}^n\mathfrak{Z}^n \longrightarrow \mathfrak{Z}^n$ to μ^3, and the unit $\eta^n : \mathrm{id} \longrightarrow \mathfrak{Z}^n$ to η^3. By remark 11.3.10, $(-)^\vee$ is faithful, and the square on the left commutes if and only if the square on the right commutes

$$
\begin{tikzcd}
\mathfrak{Z}^n\mathfrak{Z}^n X \arrow[r, "\mathfrak{Z}^n m"] \arrow[d, "\mu^n"'] & \mathfrak{Z}^n X \arrow[d, "m"] \\
\mathfrak{Z}^n X \arrow[r, "m"] & X,
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{Z}^3\mathfrak{Z}^3 X^\vee \arrow[r, "\mathfrak{Z}^3 m^\vee"] \arrow[d, "\mu^3"'] & \mathfrak{Z}^3 X^\vee \arrow[d, "m^\vee"] \\
\mathfrak{Z}^3 X^\vee \arrow[r, "m^\vee"] & X^\vee,
\end{tikzcd}
$$

and likewise for the diagram involving η^n and η^3. □

Proposition 11.3.15. *The square* (11.3.13) *is a pullback. That is, a $\mathfrak{Z}^n$-algebra structure on $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$ is completely determined by a $\mathfrak{Z}^3$-algebra structure on $X^\vee$.*

Proof. Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$. By proposition 11.3.12, a $\mathfrak{Z}^n$-algebra structure on X induces a $\mathfrak{Z}^3$-algebra structure on $X^\vee$. By remark 11.3.10, $(-)^\vee : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{2,3})$ is faithful, and thus its lift $\mathcal{A}\mathrm{lg}_{1,n} \longrightarrow \mathcal{A}\mathrm{lg}_{1,3}$ is injective on object. In particular, different algebra structures on X result in different algebra structures on $X^\vee$.

Conversely, let $m : \mathfrak{Z}^3 X^\vee \longrightarrow X^\vee$ be a $\mathfrak{Z}^3$-algebra structure on $X^\vee$, and define $m' : \mathfrak{Z}^n X \longrightarrow X$ as the identity in dimension $n-1$, and mapping $f : S[\nu] \longrightarrow X$ to $m(f^\vee) \in X^\vee_2 \cong X_{n-1}$. Recall that $f^\vee$ is a map of the form $S[\nu'] \longrightarrow X^\vee$, for some ν' such that $\mathsf{t}\nu' = (\mathsf{t}\nu)^\vee$, and thus m' is a map of opetopic sets. By lemma 11.3.14, it is a $\mathfrak{Z}^n$-algebra structure on X.

Finally, the flattening operation establishes a bijective correspondence between the $\mathfrak{Z}^n$-algebra structures on X and the $\mathfrak{Z}^3$-algebra structures on $X^\vee$. □

Theorem 11.3.16 (Algebraic trompe-l'œil)**.** *The following square is a pullback:*

$$
\begin{tikzcd}
\mathcal{A}\mathrm{lg}_{k,n} \arrow[r, "(-)^\vee_{n-1,n}"] \arrow[d] & \mathcal{A}\mathrm{lg}_{1,3} \arrow[d] \\
\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) \arrow[r, "(-)^\vee_{n-1,n}"] & \mathcal{P}\mathrm{sh}(\mathbb{O}_{2,3}).
\end{tikzcd}
\tag{11.3.17}
$$

In other words, a $\mathfrak{Z}^n$-algebra structure on $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$ is completely determined by a $\mathfrak{Z}^3$-algebra structure on $X^\vee_{n-1,n}$.

Proof. This is a direct consequence of propositions 11.3.4 and 11.3.15, and the pasting lemma for pullbacks. □

Chapter Twelve

Preliminaries in homotopy theory

We review elements of abstract homotopy theory in preparation for the upcoming chapters, with an emphasis on the homotopy theory of presheaves. For more complete references, see [Hov99] [Hir09] [Lur09, section A.2] as well as [Cis06, chapter 1].

12.1 MODEL CATEGORIES

DEFINITIONS

Definition 12.1.1 (Model category [Hir09, definition 7.1.3])**.** A *model category* is a category $\mathcal{M}$ endowed with three classes of maps $\mathsf{Cof}, \mathsf{Fib}, \mathsf{Weq} \subseteq \mathcal{M}^{[1]}$, whose elements are respectively called *cofibrations*, *fibrations*, and *weak equivalences*, subject to the following conditions.

- **(M0)** *Limit axiom.* The category $\mathcal{M}$ is complete and cocomplete.
- **(M1)** *3-for-2 axiom.* The class Weq of weak equivalences has the 3-for-2 property.
- **(M2)** *Retract axiom.* The classes Cof, Fib, and Weq are closed under retracts (in $\mathcal{M}^{[1]}$).
- **(M3)** *Lifting axiom.* For $\mathsf{ACof} := \mathsf{Cof} \cap \mathsf{Weq}$ and $\mathsf{AFib} := \mathsf{Fib} \cap \mathsf{Weq}$ the classes of *acyclic cofibrations* and *acyclic fibrations*[1], respectively, we have $\mathsf{ACof} \pitchfork \mathsf{Fib}$ and $\mathsf{Cof} \pitchfork \mathsf{AFib}$.
- **(M4)** *Factorization axiom.* For any morphism $f \in \mathcal{M}$, there exists a factorization $f = pj$, where $j \in \mathsf{ACof}$ and $p \in \mathsf{Fib}$. Dually, there exists a factorization $f = qi$, where $i \in \mathsf{Cof}$ and $q \in \mathsf{AFib}$. Further, those factorizations are functorial[2].

Definition 12.1.2. Let $\mathcal{M}$ be a model category. An object $x \in \mathcal{M}$ is *cofibrant* (resp. *fibrant*) if the initial map $\mathsf{i} : \varnothing \longrightarrow x$ (resp. the terminal map $! : x \longrightarrow 1$) is a cofibration (resp. a fibration). Consider the factorizations of i and $!$ of **(M4)**:

$$\begin{tikzcd} \varnothing \arrow[rr, "\mathsf{i}"] \arrow[dr, "\in \mathsf{Cof}"'] & & x \\ & Qx, \arrow[ur, "q_x \in \mathsf{AFib}"'] & \end{tikzcd} \qquad \begin{tikzcd} x \arrow[rr, "!"] \arrow[dr, "r_x \in \mathsf{ACof}"'] & & 1 \\ & Rx. \arrow[ur, "\in \mathsf{Fib}"'] & \end{tikzcd}$$

The object Qx (resp. Rx) is called a *cofibrant replacement* (resp. a *fibrant replacement*) of x. Since the factorizations of **(M4)** are functorial, these constructions give rise to two functors $Q, R : \mathcal{M} \longrightarrow \mathcal{M}$.

---

[1] We resort to the term "acyclic fibration" rather than the most common "trivial fibration", reserved for definition 12.3.15.

[2] In the sense that they define functors $\mathcal{M}^{[1]} \longrightarrow \mathcal{M}^{[2]}$ that are sections of the composition map.

Definition 12.1.3 (Proper model category [Hir09, definition 13.1.1])**.** A model category $\mathcal{M}$ is *left proper* if the cobase change of a weak equivalence along a cofibration is a weak equivalence. Explicitly, if $w \in \mathsf{Weq}$ and $c \in \mathsf{Cof}$ share the same domain, then the morphism $\bar{w}$ below is a weak equivalence

$$\begin{tikzcd} \cdot \arrow[r, "c"] \arrow[d, "w"'] & \cdot \arrow[d, "\bar{w}"] \\ \cdot \arrow[r] & \cdot \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Dually, $\mathcal{M}$ is *right proper* if the base change of a weak equivalence along a fibration is a weak equivalence. Finally, $\mathcal{M}$ is *proper* if it is left and right proper.

Definition 12.1.4 (Homotopy category)**.** Let $\mathcal{M}$ be a model category, and let $\operatorname{ho}\mathcal{M}$, the *homotopy category of* $\mathcal{M}$ be the localization $\operatorname{ho}\mathcal{M} := \mathsf{Weq}^{-1}\mathcal{M}$. It is a category in the same universe as $\mathcal{M}$ [Hov99, corollary 1.2.9], e.g. locally small if $\mathcal{M}$ is.

Proposition 12.1.5 ([Hov99, proposition 1.2.8])**.** *Let $\mathcal{M}$ be a model category, A be cofibrant, and P be fibrant. Then a morphism $f : A \longrightarrow P$ is a weak equivalence if and only if it is a* homotopy equivalence*, i.e. there exists $g : P \longrightarrow A$ (in $\mathcal{M}$) such that f and g are mutually inverse in* $\operatorname{ho}\mathcal{M}$.

Definition 12.1.6 (Small object)**.** Let $\mathcal{C}$ be a category with small colimits.

(1) [Hir09, definition 10.2.1] Let λ be an ordinal. A λ*-sequence* is a functor $F : \lambda \longrightarrow \mathcal{C}$ such that for every limit ordinal $\beta < \lambda$, the canonical map $\operatorname{colim}_{\alpha<\beta} F\alpha \longrightarrow F\beta$ is an isomorphism.

(2) [Hir09, definition 10.4.1] [Hov99, definition 2.1.3] Let K be a class of morphisms of $\mathcal{C}$ and κ be a cardinal. An object $c \in \mathcal{C}$ is κ*-small relative to* K if $\mathcal{C}(c, -)$ preserves colimits of λ-sequences, for every regular cardinal $\lambda \geq \kappa$. It is *small relative to* K if it is λ-small relative to K for some regular cardinal λ. It is *small* if it is small relative to the class of all morphisms of $\mathcal{C}$.

(3) We say that K *admits the small object argument* if domains of morphisms of K are small relative to $\mathsf{Cell}_{\mathsf{K}}$.

Remark 12.1.7. Note that small objects are presentable. Conversely, finitely presentable objects are $\aleph_0$-small [AR94, corollary 1.7].

Lemma 12.1.8 ([Cis06, theorem 1.2.23])**.** *Let K be a set of morphisms of $\mathcal{P}\mathrm{sh}(\mathcal{C})$ allowing the small object argument. Then its saturation ${}^{\pitchfork}(\mathsf{K}^{\pitchfork})$ is the closure of $\mathsf{Cell}_{\mathsf{K}}$ under retracts (in $\mathcal{P}\mathrm{sh}(\mathcal{C})^{[1]}$).*

Definition 12.1.9 (Cofibrantly generated model category)**.** We say that a model category $\mathcal{M}$ is *cofibrantly generated* if there exists two *sets* $\mathsf{I}, \mathsf{J} \subseteq \mathcal{M}^{[1]}$ that admit the small object argument, such that $\mathsf{Cof} = {}^{\pitchfork}(\mathsf{I}^{\pitchfork})$ and $\mathsf{ACof} = {}^{\pitchfork}(\mathsf{J}^{\pitchfork})$. We say that $\mathcal{M}$ is *combinatorial* if it is locally presentable and cofibrantly generated.

Example 12.1.10. There is a model structure $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})_{\mathrm{Quillen}}$ on simplicial sets, where the cofibrations are the monomorphisms, and the weak equivalence are those maps f such that their geometric realization $|f| \in \mathcal{T}\mathrm{op}^{[1]}$ is a weak homotopy equivalence. In this structure, fibrations are called *Kan fibrations*, and fibrant object are called *Kan complexes*.

It is combinatorial, the set of boundary inclusions B (see example 0.3.10) is a set of generating cofibrations, and the set of horn inclusions H is a set of generating acyclic cofibrations. In particular, map p is a Kan fibration if and only if $\mathsf{H} \pitchfork p$.

Definition 12.1.11 (Simplicial model category)**.** A model category $\mathcal{M}$ is *simplicial* if the following additional axioms hold.

(SM0) *Enrichment.* The category $\mathcal{M}$ is simplicially enriched, and we write $\operatorname{map}(-,-)$ for the simplicial hom-space. It is furthermore *tensored*, and *cotensored* over simplicial sets, meaning that we have two functors

$$- \otimes - : \mathcal{P}\mathrm{sh}(\mathbb{\Delta}) \times \mathcal{M} \longrightarrow \mathcal{M}, \qquad (-)^{(-)} : \mathcal{M} \times \mathcal{P}\mathrm{sh}(\mathbb{\Delta})^{\mathrm{op}} \longrightarrow \mathcal{M},$$

such that for $K \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ and $x, y \in \mathcal{M}$, the following isomorphisms hold

$$\operatorname{map}(K \otimes x, y) \cong \operatorname{map}(K, \operatorname{map}(x, y)) \cong \operatorname{map}(x, y^K)$$

naturally in K, x, and y.

(SM7) *HELP*[3]. Let $i : a \longrightarrow b$ be a cofibration and $p : x \longrightarrow y$ be a fibration. Then the cartesian gap map $\widehat{\operatorname{map}}(i,p)$ below is a Kan fibration (see example 12.1.10), and a weak equivalence if either i or p is:

$$\begin{tikzcd}
\operatorname{map}(b,x) \arrow[dr, dashed, "\widehat{\operatorname{map}}(i,p)"] \arrow[drr, bend left, "i^*"] \arrow[ddr, bend right, "p_*"'] & & \\
& \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \operatorname{map}(a,x) \arrow[d, "p_*"] \\
& \operatorname{map}(b,y) \arrow[r, "i^*"] & \operatorname{map}(a,y).
\end{tikzcd}$$

Remark 12.1.12. If $\mathcal{M}$ is a model category that satisfies **(SM0)**, then **(SM7)** has two equivalent formulations involving the simplicial tensor and cotensor.

(SM7') Let $i : a \longrightarrow b$ be a cofibration in $\mathcal{M}$ and $j : K \longrightarrow L$ be a cofibration in $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$. Then the cocartesian gap map $j \mathbin{\hat{\otimes}} i$ below is a cofibration, and a weak equivalence if either i or j is:

$$\begin{tikzcd}
K \otimes a \arrow[r, "K \otimes i"] \arrow[d, "j \otimes a"'] & K \otimes b \arrow[d] \arrow[ddr, bend left, "j \otimes b"] & \\
L \otimes a \arrow[r] \arrow[drr, bend right, "L \otimes i"'] & \cdot \arrow[ul, phantom, very near start, "\ulcorner"] \arrow[dr, dashed, "j \hat{\otimes} i"] & \\
& & L \otimes b.
\end{tikzcd}$$

(SM7") Let $p : a \longrightarrow b$ be a fibration in $\mathcal{M}$ and $j : K \longrightarrow L$ be a cofibration in $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$. Then the cartesian gap map $\widehat{\hom}(j,p)$ below is a fibration, and a weak equivalence

[3] This is an acronym for "Homotopy Extension Lifting Property" and not a covert distress call.

if either p or j is:

$$\begin{tikzcd}
a^L \arrow[rrd, bend left, "j^*"] \arrow[rd, dashed, "\widehat{\hom}(j,p)"] \arrow[rdd, bend right, "p_*"'] & & \\
& \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & a^K \arrow[d, "p_*"] \\
& b^L \arrow[r, "j^*"] & b^K .
\end{tikzcd}$$

QUILLEN FUNCTORS

Definition 12.1.13 (Quillen functor)**.** Let $\mathcal{M}$ and $\mathcal{N}$ be two model categories. An adjunction $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$ is a *Quillen adjunction* if F preserves cofibrations and acyclic cofibrations, or equivalently (see lemma 0.2.7), if U preserves fibrations and acyclic fibrations. In this case, we say that F is a *left Quillen functor*, and that U is a *right Quillen functor*.

Let Q be a cofibrant replacement functor in $\mathcal{M}$, and R be functorial fibrant replacement in $\mathcal{N}$. The *left derived functor of* F is defined as $\mathrm{d}F := FQ$, and dually, the *right derived functor of* U is $\mathrm{d}U := UR$. In the sequel, we never deal with functors that are both left and right Quillen, so the notation d is not ambiguous.

Lemma 12.1.14 (Ken Brown's lemma [Hov99, lemma 1.1.12])**.** *Let $F : \mathcal{M} \longrightarrow \mathcal{N}$ be a functor between model categories*[4] *that maps acyclic cofibrations between cofibrant objects (resp. acyclic fibrations between fibrant objects) to weak equivalences. Then F preserves all weak equivalences between cofibrant objects (resp. between fibrant objects).*

Corollary 12.1.15. *Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$ be a Quillen adjunction. Then F (resp. U) preserves weak equivalences between cofibrant (resp. between fibrant) objects.*

Lemma 12.1.16 ([Dug01, corollary A.2] [Hir09, proposition 8.5.4])**.** *Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$ be an adjunction between model categories. The following are equivalent:*

(1) the adjunction is Quillen;

(2) F preserves cofibrations and U preserves fibrations between fibrant objects;

(3) F preserves cofibrations between cofibrant objects and U preserves fibrations.

Definition 12.1.17 (Quillen equivalence)**.** A Quillen adjunction $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$ is a *Quillen equivalence* if for all $x \in \mathcal{M}$ and $y \in \mathcal{N}$, the following composites are weak equivalences:

$$x \xrightarrow{\eta} UFx \xrightarrow{Uj} URFx, \qquad FQUy \xrightarrow{Fq} FUy \xrightarrow{\varepsilon} y,$$

where η and ε are the unit and counit of the adjunction, $j : Fx \longrightarrow RFx$ is a fibrant replacement of Fx, and $q : QUy \longrightarrow Uy$ is a cofibrant replacement of Uy.

Proposition 12.1.18. *If $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$ is a Quillen adjunction (resp. a Quillen equivalence), then the derived functors* $\mathrm{d}F$ *and* $\mathrm{d}U$ *form an adjunction (resp. an adjoint equivalence)* $\mathrm{ho}\,\mathcal{M} \rightleftarrows \mathrm{ho}\,\mathcal{N}$.

[4] In fact, $\mathcal{N}$ only needs to be equipped with a class of "weak equivalences" that satisfies 3-for-2.

Definition 12.1.19 ([Lur09, definition A.2.8.1]). Let $\mathcal{M}$ be a model category, and $\mathcal{C}$ be a small category.

(1) A morphism in $\mathcal{M}^{\mathcal{C}}$ is an *injective cofibration* (resp. an *injective weak equivalence*) if it is pointwise a cofibration in $\mathcal{M}$ (resp. a weak equivalence in $\mathcal{M}$). An *injective fibration* is a morphism that has the right lifting property against all injective acyclic cofibrations.

(2) Dually, a morphism in $\mathcal{M}^{\mathcal{C}}$ is an *projective fibration* (resp. a *projective weak equivalence*) if it is pointwise a fibration in $\mathcal{M}$ (resp. a weak equivalence in $\mathcal{M}$). A *projective cofibration* is a morphism that has the left lifting property against all projective acyclic fibrations.

Theorem 12.1.20 ([Lur09, proposition A.2.8.2, A.3.3.2] [Hir09, theorem 11.6.1]). *Let* $\mathcal{M}$ *be a combinatorial model category with* $\mathsf{I}_{\mathcal{M}}$ *and* $\mathsf{J}_{\mathcal{M}}$ *as sets of generating cofibrations and generating acyclic cofibrations respectively, and* $\mathcal{C}$ *be a small category.*

(1) If for every set $X \in \mathcal{S}\mathrm{et}$*, the box product* $X \boxtimes - : \mathcal{M} \longrightarrow \mathcal{M}$ *(see definition 0.3.9) preserves acyclic cofibrations, then* $\mathcal{M}^{\mathcal{C}}$ *admits a combinatorial model structure whose cofibrations, weak equivalences, and fibrations are the projective ones. We write this structure* $\mathcal{M}^{\mathcal{C}}_{\mathrm{proj}}$ *and call it the* projective model structure *on* $\mathcal{M}^{\mathcal{C}}$*. The following are sets of generating projective cofibrations and generating projective acyclic cofibrations, respectively:*

$$\{\mathcal{C}(c,-) \boxtimes i \mid c \in \mathcal{C}, i \in \mathsf{I}_{\mathcal{M}}\}, \qquad \{\mathcal{C}(c,-) \boxtimes j \mid c \in \mathcal{C}, j \in \mathsf{J}_{\mathcal{M}}\}.$$

(2) If for every set $X \in \mathcal{S}\mathrm{et}$*, the exponentiation* $(-)^X : \mathcal{M} \longrightarrow \mathcal{M}$ *(see definition 0.3.9) preserves acyclic fibrations, then* $\mathcal{M}^{\mathcal{C}}$ *admits a combinatorial model structure whose cofibrations, weak equivalences, and fibrations are the injective ones. We write this structure* $\mathcal{M}^{\mathcal{C}}_{\mathrm{inj}}$ *and call it the* injective model structure *on* $\mathcal{M}^{\mathcal{C}}$*. The following are sets of generating injective cofibrations and generating injective acyclic cofibrations, respectively:*

$$\left\{i^{\mathcal{C}(-,c)} \mid c \in \mathcal{C}, i \in \mathsf{I}_{\mathcal{M}}\right\}, \qquad \left\{j^{\mathcal{C}(-,c)} \mid c \in \mathcal{C}, j \in \mathsf{J}_{\mathcal{M}}\right\}.$$

Furthermore, the projective and injective structures (provided that they exist) are left proper, right proper, or simplicial, whenever $\mathcal{M}$ *is. Lastly, the identity functor induces a Quillen equivalence*

$$\mathrm{id} : \mathcal{M}^{\mathcal{C}}_{\mathrm{proj}} \underset{\perp}{\overset{\sim}{\rightleftarrows}} \mathcal{M}^{\mathcal{C}}_{\mathrm{inj}} : \mathrm{id}.$$

Definition 12.1.21 (Reedy category [RV14, definition 2.1]). A Reedy category is a small category $\mathcal{R}$ with two *wide* subcategories (i.e. containing all objects of $\mathcal{R}$) $\mathcal{R}_+$ and $\mathcal{R}_-$ containing all the isomorphisms, and a map $\deg : \mathrm{ob}\,\mathcal{R} \longrightarrow \mathbb{N}$ such that for every morphism $f : a \longrightarrow b$ that is not an isomorphism, if $f \in \mathcal{R}_+$ (resp. $f \in \mathcal{R}_-$), then $\deg a < \deg b$ (resp. $\deg a > \deg b$). Further, we require that every morphism factors uniquely as one from $\mathcal{R}_-$ followed by one in $\mathcal{R}_+$. Morphisms in $\mathcal{R}_+$ (resp. $\mathcal{R}_-$) are called *increasing* (resp. *decreasing*).

If $\mathcal{R}_-$ is a discrete category (i.e. there is no non-trivial decreasing morphism), then $\mathcal{R}$ is said to be a *direct category.*

Remark 12.1.22. In [RV14, definition 2.1] and [Hir09, definition 15.1.2], the subcategories $\mathcal{R}_+$ and $\mathcal{R}_-$ are denoted by $\overrightarrow{\mathcal{R}}$ and $\overleftarrow{\mathcal{R}}$ respectively. It is possible to use a more general degree map $\deg : \operatorname{ob} \mathcal{R} \longrightarrow \kappa$, where κ is a regular ordinal, and most of the theory of Reedy categories still hold in this setting [Hir09, remark 15.1.4]. In this work, we shall only consider the case $\kappa = \mathbb{N}$.

Definition 12.1.23 (Latching and matching objects)**.** Let $\mathcal{R}$ be a Reedy category, $\mathcal{C}$ be a complete and cocomplete category, $X \in \mathcal{C}^{\mathcal{R}}$, and $a \in \mathcal{R}$. The *latching object* $\mathrm{L}_a X$ and *matching object* $\mathrm{M}_a X$ of X at a are defined as

$$\mathrm{L}_a X := \operatorname*{colim}_{\substack{f : b \to a \\ f \in \mathcal{R}_+ - \{\mathrm{id}_a\}}} X_b, \qquad\qquad \mathrm{M}_a X := \lim_{\substack{f : a \to b \\ f \in \mathcal{R}_- - \{\mathrm{id}_a\}}} X_b.$$

For $f : X \longrightarrow Y$ a morphism in $\mathcal{C}^{\mathcal{R}}$ and $a \in \mathsf{R}$, the *relative latching map* $\hat{\mathrm{L}}_a f$ and *relative matching map* $\hat{\mathrm{M}}_a f$ of f at a are defined as the gap maps below:

$$\begin{tikzcd}
\mathrm{L}_a X \arrow[r] \arrow[d, "\mathrm{L}_a f"'] & X \arrow[d] \arrow[ddr, bend left, "f"] & \\
\mathrm{L}_a Y \arrow[r] \arrow[drr, bend right] & \cdot \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\hat{\mathrm{L}}_a f"] & \\
& & Y,
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[dr, dashed, "\hat{\mathrm{M}}_a f"] \arrow[drr, bend left] \arrow[ddr, bend right, "f"'] & & \\
& \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{M}_a X \arrow[d, "\mathrm{M}_a f"] \\
& Y \arrow[r] & \mathrm{M}_a Y.
\end{tikzcd}$$

Example 12.1.24. The category $\mathbb{\Delta}$ is naturally a Reedy category, where $\deg[n] := n$, and where a map f is in $\mathbb{\Delta}_+$ (resp. in $\mathbb{\Delta}_-$) if it is injective (resp. surjective). The opposite category $\mathbb{\Delta}^{\mathrm{op}}$ is also Reedy, with the same the degree map, $(\mathbb{\Delta}^{\mathrm{op}})_+ := \mathbb{\Delta}_-$, and $(\mathbb{\Delta}^{\mathrm{op}})_- := \mathbb{\Delta}_+$. If $X \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ and $n \in \mathbb{N}$, then $\mathrm{L}_n X$ is the smallest subobject of X such that $(\mathrm{L}_n X)_{<n} = X_{<n}$, i.e. it contains of the k-cells of X, where $0 \leq k < n$ and their degeneracies. On the other hand, $\mathrm{M}_n X = \iota_* X$, where ι is the inclusion $\mathbb{\Delta}_{<n} \longrightarrow \mathbb{\Delta}$. Note that $\mathrm{L}_n X$ (resp. $\mathrm{M}_n X$) is initial (resp. terminal) among all the simplicial sets Y such that $X_{<n} = Y_{<n}$.

Theorem 12.1.25 ([Lur09, proposition A.2.9.19] [Hir09, theorems 15.3.4 and 15.6.27])**.** *Let $\mathcal{R}$ be a Reedy category, and $\mathcal{M}$ be a model category. There exists a model structure on $\mathcal{M}^{\mathcal{R}}$ where a morphism f is*

1. *a cofibration (resp. a fibration) if all its relative latching maps (resp. matching maps) are cofibrations (resp. fibrations) in $\mathcal{M}$,*
2. *a weak equivalence if it is a pointwise weak equivalence.*

Further, a morphism is an acyclic cofibration (resp. an acyclic fibration) if and only if all its relative latching maps (resp. relative matching maps) are. We call this structure the Reedy model structure *and denote it by $\mathcal{M}^{\mathcal{R}}_{\mathrm{Reedy}}$. If $\mathcal{M}$ is left proper, right proper, cofibrantly generated, or simplicial, then so is $\mathcal{M}^{\mathcal{R}}_{\mathrm{Reedy}}$. Lastly, the identity functor induces two Quillen equivalences*

$$\mathcal{M}^{\mathcal{R}}_{\mathrm{proj}} \underset{\perp}{\overset{\sim}{\rightleftarrows}} \mathcal{M}^{\mathcal{R}}_{\mathrm{Reedy}} \underset{\perp}{\overset{\sim}{\rightleftarrows}} \mathcal{M}^{\mathcal{R}}_{\mathrm{inj}}.$$

LEFT BOUSFIELD LOCALIZATIONS

Similar to the 1-categorical setting, where a localization of a category formally inverts morphisms, a localization of a model category $\mathcal{M}$ formally turns some morphisms into weak equivalences, thereby localizing (in the 1-categorical sense) its homotopy category $\operatorname{ho}\mathcal{M}$. While declaring arbitrary morphisms to be weak equivalence is always possible, there is no guarantee that the resulting model structure has an adequate universal property, let alone that we have any control over the new cofibrations, fibrations, or even weak equivalences.

In this section, we recall the notion of *left Bousfield localization*, which is a type of localization that keeps the class of cofibrations constant. Under mild conditions, it is guaranteed to exist, and all the enjoyable properties of the model structure (e.g. being proper) carry over. The point of such a construction is to enforce new lifting properties on fibrant objects. Indeed, by increasing the amount of weak equivalences while keeping the same cofibrations, the left Bousfield localization mechanically increases the constraints on the fibrant objects.

Definition 12.1.26 (Local object)**.** Let $\mathcal{M}$ be a simplicial model category, and $\mathsf{K} \subseteq \mathcal{M}^{[1]}$ be a class of morphisms. We say that an object $x \in \mathcal{M}$ is K-*local* if it is fibrant, and if for all $f : a \longrightarrow b$ in K, the precomposition map $f^* : \operatorname{map}(b, x) \longrightarrow \operatorname{map}(a, x)$ is a weak equivalence. A morphism g is a K-*local equivalence* if every K-local object is also a $\{g\}$-local object.

Remark 12.1.27. Note the similarity with the definition of local isomorphisms (see definition 0.2.8), where instead of weak equivalences between mapping spaces, we require bijections between hom-sets.

Definition 12.1.28 (Left Bousfield localization [Hir09, section 3.3])**.** Let $\mathcal{M}$ be a simplicial model category, and $\mathsf{K} \subseteq \mathcal{M}^{[1]}$ be a class of morphism. The *left Bousfield localization* $\mathsf{K}^{-1}\mathcal{M}$ *of* $\mathcal{M}$ *at* K, if it exists, the model structure on $\mathcal{M}$ where $\mathsf{Cof}_{\mathsf{K}^{-1}\mathcal{M}} = \mathsf{Cof}_{\mathcal{M}}$, and $\mathsf{Weq}_{\mathsf{K}^{-1}\mathcal{M}}$ is the class of K-local equivalences. In this case, fibrant objects in $\mathsf{K}^{-1}\mathcal{M}$ are exactly the K-local objects, and $\mathsf{AFib}_{\mathsf{K}^{-1}\mathcal{M}} = \mathsf{AFib}_{\mathcal{M}}$. Furthermore, we have a Quillen adjunction $\mathrm{id} : \mathcal{M} \rightleftarrows \mathsf{K}^{-1}\mathcal{M} : \mathrm{id}$, and the left Quillen functor $\mathrm{id} : \mathcal{M} \longrightarrow \mathsf{K}^{-1}\mathcal{M}$ is essentially initial among all left Quillen functors from $\mathcal{M}$ mapping elements of K to weak equivalences.

The next two results involve the technical notion of *cellular model category* [Hir09, definition 12.1.1]. We shall skip its definition, for in this work, results that depend on cellular model structures either produce or request them in a “black box” manner.

Theorem 12.1.29 ([Hir09, theorem 4.1.1])**.** *If* $\mathcal{M}$ *is a left proper cellular model category, and* K *is a set of morphisms, then the left Bousfield localization* $\mathsf{K}^{-1}\mathcal{M}$ *exists, and is left proper and cellular, with the same simplicial enrichment, tensor, and cotensor as* $\mathcal{M}$.

Theorem 12.1.30 ([Hir09, theorem 3.3.20])**.** *Let* $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$ *be a Quillen adjunction,* $\mathsf{K} \subseteq \mathcal{M}^{[1]}$ *a class of morphism, and* $\mathrm{d}F$ *the left derived functor of* F*. If the left Bousfield localizations* $\mathsf{K}^{-1}\mathcal{M}$ *and* $(\mathrm{d}F\mathsf{K})^{-1}\mathcal{N}$ *exist, then* $F \dashv U$ *descends to a Quillen adjunction* $F : \mathsf{K}^{-1}\mathcal{M} \rightleftarrows (\mathrm{d}F\mathsf{K})^{-1}\mathcal{N} : U$*. Furthermore, if the original adjunction was a Quillen equivalence, then so is the localized one.*

12.2 PRESHEAVES AS MODELS FOR HOMOTOPY TYPES

In this section, we survey some of the results and constructions of [Cis06] that lead to the powerful theorem 12.2.8. Informally, given $\mathcal{C}$ a small category, and a good notion of “cylinder object”, this result allows us to construct a model structure on $\mathcal{P}\mathrm{sh}(\mathcal{C})$. In this structure, the cofibrations are exactly the monomorphisms, while the homotopy equivalences are determined by those cylinders. In the second part of this section, we present the notion of *normal skeletal category*, which is a stronger form of Reedy category. Much like in the Reedy setting, a skeletal category $\mathcal{C}$ has a notion of “dimension” for its objects, but here, the elementary maps that “jump dimension” must satisfy some additional requirements. From this, we can define a set of *boundary inclusions* B such that the class of monomorphisms of $\mathcal{P}\mathrm{sh}(\mathcal{C})$ is exactly $\mathsf{Cell}_{\mathsf{B}}$, see proposition 12.2.13. Combined together, those two theories produce model category structures on presheaf categories where we have a very fine control over the weak equivalences and cofibrations. By lifting arguments, it thus makes reasoning with fibrations a manageable endeavor.

THE CISINSKI MODEL STRUCTURE

Definition 12.2.1 (Cylinder object [Cis06, definition 1.3.1])**.** Let $\mathcal{C}$ be a small category, and $X \in \mathcal{P}\mathrm{sh}(\mathcal{C})$ be a presheaf over $\mathcal{C}$. A *cylinder* of X is a factorization of the fold map

$$\begin{tikzcd} X + X \arrow[rr, "\nabla"] \arrow[dr, "(i_0, i_1)"'] & & X \\ & \mathrm{I}X, \arrow[ur, "\nabla"'] & \end{tikzcd}$$

such that $(i_0, i_1) : X + X \longrightarrow \mathrm{I}X$ is a monomorphism. We write $X^{(e)}$ for the image of $i_e : X \longrightarrow \mathrm{I}X$.

Definition 12.2.2 (I-homotopy [Cis06, definition 1.3.3, remark 1.3.4])**.** Let $\mathcal{C}$ be a small category, $f, g : X \longrightarrow Y$ be two parallel maps in $\mathcal{P}\mathrm{sh}(\mathcal{C})$, and $\mathrm{I}X$ be a cylinder of X (definition 12.2.1). An *elementary* I-*homotopy* from f to g is a morphism $H : \mathrm{I}A \longrightarrow B$ such that the following triangle commutes:

$$\begin{tikzcd} A + A \arrow[d, "(i_0, i_1)"'] \arrow[dr, "(f, g)"] & \\ \mathrm{I}A \arrow[r, "H"'] & B. \end{tikzcd}$$

Let $\simeq_{\mathrm{I}}$ (or just $\simeq$ is the context is clear), the I-*homotopy relation*, be the equivalence relation spanned by this relation on $\mathcal{P}\mathrm{sh}(\mathcal{C})(A, B)$.

One readily checks that $\simeq$ is a congruence on the category $\mathcal{P}\mathrm{sh}(\mathcal{C})$, and let $\mathrm{ho}\,\mathcal{P}\mathrm{sh}(\mathcal{C})$ be the quotient category. A morphism $f : X \longrightarrow Y$ is a I-*homotopy equivalence* (or just *homotopy equivalence* if the context is clear) if it is invertible in $\mathrm{ho}\,\mathcal{P}\mathrm{sh}(\mathcal{C})$.

Definition 12.2.3 (Elementary homotopical data [Cis06, definition 1.3.6])**.** For $\mathcal{C}$ a small category, an *elementary homotopical data* on $\mathcal{P}\mathrm{sh}(\mathcal{C})$ is a functorial cylinder $\mathrm{I} : \mathcal{P}\mathrm{sh}(\mathcal{C}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{C})$ (see definition 12.2.1) such that

(DH1) I preserves small colimits and monomorphisms;

(DH2) for every monomorphism $f : X \longrightarrow Y$ in $\mathcal{P}\mathrm{sh}(\mathcal{C})$, and for $e = 0, 1$, the following square is a pullback:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_e"'] & Y \arrow[d, "i_e"] \\ \mathrm{I}X \arrow[r, "\mathrm{I}f"] & \mathrm{I}Y. \end{tikzcd}$$

Definition 12.2.4 (Relative anodyne extension)**.** Let $\mathcal{C}$ be a small category, and I be an elementary homotopical data on $\mathcal{P}\mathrm{sh}(\mathcal{C})$ (definition 12.2.3). A *class of anodyne extensions* relative to I is a class $\mathsf{K} \subseteq \mathcal{P}\mathrm{sh}(\mathcal{C})^{[1]}$ such that

(An0) there exists a *set* $A \subseteq \mathcal{P}\mathrm{sh}(\mathcal{C})^{[1]}$ of monomorphisms such that $\mathsf{K} = {}^{\pitchfork}(A^{\pitchfork})$ (in particular, elements of K are monomorphisms);

(An1) for every monomorphism $m : X \longrightarrow Y$ in $\mathcal{P}\mathrm{sh}(\mathcal{C})$, and $e = 0, 1$, the cocartesian gap map g is in K:

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "i_e"'] & Y \arrow[d, "i_e"] \arrow[ddr, bend left, "i_e"] & \\ \mathrm{I}X \arrow[r] \arrow[drr, bend right, "\mathrm{I}m"'] & \mathrm{I}X \cup Y^{(e)} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "g"] & \\ & & \mathrm{I}Y; \end{tikzcd}$$

(An2) for all $m : X \longrightarrow Y$ in K, the cocartesian gap map g is in K:

$$\begin{tikzcd} X + X \arrow[r, "m+m"] \arrow[d, "{(i_0,i_1)}"'] & Y + Y \arrow[d] \arrow[ddr, bend left, "{(i_0,i_1)}"] & \\ \mathrm{I}X \arrow[r] \arrow[drr, bend right, "\mathrm{I}m"'] & \mathrm{I}X \cup (Y + Y) \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "g"] & \\ & & \mathrm{I}Y. \end{tikzcd}$$

Lemma 12.2.5. *Let* I *be an elementary homotopical data, and* K *be a class of anodyne extensions relative to* I*. Then* I *preserves* K*.*

Proof. By axiom **(An0)**, K is closed under compositions and cobase change. The claim then follows from axiom **(An1)**. $\square$

Definition 12.2.6 (Homotopical structure [Cis06, definition 1.3.14])**.** Let $\mathcal{C}$ be a small category. A *homotopical structure* on $\mathcal{P}\mathrm{sh}(\mathcal{C})$ is a pair (I, K), where I is an elementary homotopical data on $\mathcal{P}\mathrm{sh}(\mathcal{C})$ (definition 12.2.3), and K is a class of anodyne extension relative to I (definition 12.2.4).

Definition 12.2.7 (Cisinski model category)**.** A *Cisinski model category* is a model structure on a presheaf category over a small category, that is cofibrantly generated, and where Cof is the class of monomorphisms.

A notable source of such structures is the following theorem:

Theorem 12.2.8 ([Cis06, definition 1.3.21, theorem 1.3.22]). *Let $\mathcal{C}$ be a small category, and (I, K) be a homotopical structure on $\mathcal{P}\mathrm{sh}(\mathcal{C})$ (definition 12.2.6). There is a model structure on $\mathcal{P}\mathrm{sh}(\mathcal{C})$ where:*

(1) a morphism f is said to be a naive fibration *if $\mathsf{K} \pitchfork f$; a presheaf $X \in \mathcal{P}\mathrm{sh}(\mathbb{A})$ is fibrant if the terminal morphism $X \longrightarrow 1$ is a naive fibration;*

(2) a morphism $f : X \longrightarrow Y$ is a weak equivalence if for all fibrant object $P \in \mathcal{P}\mathrm{sh}(\mathbb{A})$, the induced map $f^* : \mathrm{ho}\,\mathcal{P}\mathrm{sh}(\mathcal{C})(Y, P) \longrightarrow \mathrm{ho}\,\mathcal{P}\mathrm{sh}(\mathcal{C})(X, P)$ is a bijection (definition 12.2.2);

(3) the cofibrations are the monomorphism.

This model structure is of Cisinski type (definition 12.2.7), cellular, and proper (definition 12.1.3).

Lemma 12.2.9. *Let $\mathcal{C}$ be a small category, and $\mathcal{P}\mathrm{sh}(\mathcal{C})$ is endowed with a model structure as in theorem 12.2.8. Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{P}\mathrm{sh}(\mathcal{C})$, where Y is fibrant. Then f is a fibration if and only if it is a naive fibration.*

Proof. *($\Longrightarrow$)* By construction, anodyne extensions are monomorphisms (hence cofibrations), and by [Cis06, proposition 1.3.31], they are also weak equivalences. In particular, if f is a fibration, then $\mathsf{K} \pitchfork f$, and f is a naive fibration.

($\Longleftarrow$) This is [Cis06, proposition 1.3.36]. □

SKELETAL CATEGORIES

Definition 12.2.10 (Skeletal category). A *skeletal category* [Cis06, definition 8.1.1] is a small category $\mathcal{C}$ endowed with a map $\deg : \mathrm{ob}\,\mathcal{C} \longrightarrow \mathbb{N}$ and two wide subcategories $\mathcal{C}_+$ and $\mathcal{C}_-$ such that the following axioms are satisfied.

(Sq0) *Invariance.* Isomorphisms are in $\mathcal{C}_+$ and $\mathcal{C}_-$, and if $c, c' \in \mathcal{C}$ are isomorphic, then $\deg c = \deg c'$.

(Sq1) *Dimension.* If $f : c \longrightarrow c'$ is an arrow in $\mathcal{C}_+$ (resp. $\mathcal{C}_-$) that is not an isomorphism, then $\deg c < \deg c'$ (resp. $\deg c > \deg c'$).

(Sq2) *Factorization.* Every arrow f of $\mathcal{C}$ can be factored as $f = f_+ f_-$, with $f_+ \in \mathcal{C}_+$ and $f_- \in \mathcal{C}_-$, in an essentially unique way.

(Sq3) *Section.* Every arrow in $\mathcal{C}_-$ admits a section. Two arrows $f, f' \in \mathcal{C}_-$ are equal if and only if they have the same sections.

The skeletal category $\mathcal{C}$ is *normal* [Cis06, definition 8.1.36, proposition 8.1.37] if $\mathcal{C}$ is rigid, i.e. does not have non-trivial automorphisms.

Example 12.2.11. The usual Reedy structure on Δ is in fact a normal skeletal structure.

Definition 12.2.12 (Boundary). Let $\mathcal{C}$ be a skeletal category. The *boundary* [Cis06, paragraph 8.1.30] $\partial c \in \mathcal{P}\mathrm{sh}(\mathcal{C})$ of an object $c \in \mathcal{C}$ is defined as

$$\partial c := \mathrm{L}_c \mathsf{y}_{\mathcal{C}} = \operatorname*{colim}_{\substack{f : d \to c \\ f \in \mathcal{C}_+ - \{\mathrm{id}_c\}}} \mathsf{y}_{\mathcal{C}} d.$$

It is the maximal subpresheaf of $c = \mathsf{y}_{\mathcal{C}} c$ not containing id_c. The canonical map $\mathsf{b}_c : \partial c \longrightarrow c$ is a monomorphism, and is called the *boundary inclusion* of c. Let $\mathsf{B}_{\mathcal{C}}$ (or just B if the context is clear) be the set of boundary inclusions.

Proposition 12.2.13 ([Cis06, propositions 8.1.35 and 8.1.37])**.** *If $\mathcal{C}$ is a normal skeletal category, then the class of monomorphisms of $\mathcal{P}\mathrm{sh}(\mathcal{C})$ is $\mathsf{Cell}_{\mathsf{B}}$.*

12.3 JOYAL–TIERNEY CALCULUS

In this section, we present the so-called *Joyal–Tierney calculus*, which is a symbolic calculus introduced in [JT07] in order to facilitate the use of lifting arguments. At its core is the *box product* (introduced in definition 0.3.9)

$$- \boxtimes - : \mathcal{P}\mathrm{sh}(\mathbb{\Delta}) \times \mathcal{P}\mathrm{sh}(\mathbb{\Delta}) \longrightarrow \mathcal{S}\mathrm{p}(\mathbb{\Delta}),$$

where $\mathcal{S}\mathrm{p}(\mathbb{\Delta}) := \mathcal{P}\mathrm{sh}(\mathbb{\Delta})^{\mathbb{\Delta}^{\mathrm{op}}}$ is the category of bisimplicial sets, that is left adjoint in both variables. If $u : A \longrightarrow B$ and $v : K \longrightarrow L$ are maps between simplicial sets, then the *Leibniz box product* $u \mathbin{\hat{\boxtimes}} v$ (also called *pushout-product*) is the cocartesian gap map

$$\begin{tikzcd}
A \boxtimes K \arrow[r, "A \boxtimes v"] \arrow[d, "u \boxtimes K"'] & A \boxtimes L \arrow[d] \arrow[ddr, bend left, "u \boxtimes L"] & \\
B \boxtimes K \arrow[r] \arrow[drr, bend right, "B \boxtimes v"'] & \cdot \arrow[ul, phantom, very near start, "\ulcorner"] \arrow[dr, dashed, "u \hat{\boxtimes} v"] & \\
 & & B \boxtimes L.
\end{tikzcd}$$

and the functor $- \mathbin{\hat{\boxtimes}} - : \mathcal{P}\mathrm{sh}(\mathbb{\Delta})^{[1]} \times \mathcal{P}\mathrm{sh}(\mathbb{\Delta})^{[1]} \longrightarrow \mathcal{S}\mathrm{p}(\mathbb{\Delta})^{[1]}$ is also left adjoint in both variables. If $\langle u \backslash - \rangle$ (resp. $\langle -/v \rangle$) is the right adjoint of $u \mathbin{\hat{\boxtimes}} -$ (resp. $- \mathbin{\hat{\boxtimes}} v$), and if $f \in \mathcal{S}\mathrm{p}(\mathbb{\Delta})^{[1]}$, then by adjunction we have

$$v \pitchfork \langle u \backslash f \rangle \iff (u \mathbin{\hat{\boxtimes}} v) \pitchfork f \iff u \pitchfork \langle f / v \rangle, \tag{12.3.1}$$

which allows to seamlessly translate lifting problems between $\mathcal{S}\mathrm{p}(\mathbb{\Delta})$ and $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$. This formalism has many applications in Reedy theory [RV14], and as we shall see, in the study of presheaves over a skeletal category. In preparation for upcoming results, we need to generalize the Joyal–Tierney calculus slightly, by considering a box product of the form

$$- \boxtimes - : \mathcal{P}\mathrm{sh}(\mathcal{C}) \times \mathcal{P}\mathrm{sh}(\mathbb{\Delta}) \longrightarrow \mathcal{S}\mathrm{p}(\mathcal{C}),$$

where $\mathcal{S}\mathrm{p}(\mathcal{C}) := \mathcal{P}\mathrm{sh}(\mathbb{\Delta})^{\mathcal{C}^{\mathrm{op}}}$ (see definition 12.3.2). The equivalences of (12.3.1) then become a convenient "Rosetta stone" for lifting problems between $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$, $\mathcal{P}\mathrm{sh}(\mathcal{C})$, and $\mathcal{S}\mathrm{p}(\mathcal{C})$.

Definition 12.3.2. Let $\mathcal{S}\mathrm{p}(\mathcal{C}) := \mathcal{P}\mathrm{sh}(\mathbb{\Delta})^{\mathcal{C}^{\mathrm{op}}}$ be the category of *simplicial presheaves* over $\mathcal{C}$. If no ambiguity arise, elements of $\mathcal{S}\mathrm{p}(\mathcal{C})$ are also called *spaces*, leaving the shape category $\mathcal{C}$ implicit. There are obvious equivalences $\mathcal{S}\mathrm{p}(\mathcal{C}) \simeq \mathcal{P}\mathrm{sh}(\mathcal{C})^{\mathbb{\Delta}^{\mathrm{op}}} \simeq \mathcal{P}\mathrm{sh}(\mathcal{C} \times \mathbb{\Delta})$.

Example 12.3.3. The category of *bisimplicial sets* [JT07, section 2] [Jar06, chapter 4] is simply $\mathcal{S}\mathrm{p}(\mathbb{\Delta})$.

Definition 12.3.4 (Box product [JT07, section 2])**.** Specifying definition 0.3.9 to the case $\mathcal{D} = \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$, the *box product*[5] is a functor $- \boxtimes - : \mathcal{P}\mathrm{sh}(\mathcal{C}) \times \mathcal{P}\mathrm{sh}(\mathbb{\Delta}) \longrightarrow \mathcal{S}\mathrm{p}(\mathcal{C})$, where for $X \in \mathcal{P}\mathrm{sh}(\mathcal{C})$, $K \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$, $c \in \mathcal{C}$, and $n \in \mathbb{N}$,

$$(X \boxtimes K)_{c,n} = X_c \times K_n.$$

Clearly, it preserves colimits in both variables, and therefore, it is left adjoint in both variables[6] since its domains and codomain are presheaf categories. The right adjoints of $X \boxtimes - : \mathcal{P}\mathrm{sh}(\mathbb{\Delta}) \longrightarrow \mathcal{S}\mathrm{p}(\mathcal{C})$ and $- \boxtimes K : \mathcal{P}\mathrm{sh}(\mathcal{C}) \longrightarrow \mathcal{S}\mathrm{p}(\mathcal{C})$ will be denoted by

$$X\backslash - : \mathcal{S}\mathrm{p}(\mathcal{C}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{\Delta}), \qquad\qquad -/K : \mathcal{S}\mathrm{p}(\mathcal{C}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{C}),$$

respectively. Note that $X\backslash -$ and $-/K$ are contravariant in X and K respectively. Consequently, for $W \in \mathcal{S}\mathrm{p}(\mathcal{C})$, the functors $-\backslash W : \mathcal{P}\mathrm{sh}(\mathcal{C})^{\mathrm{op}} \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ and $W/- : \mathcal{P}\mathrm{sh}(\mathbb{\Delta})^{\mathrm{op}} \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{C})$ are mutually right adjoint.

Lemma 12.3.5. *Let $W \in \mathcal{S}\mathrm{p}(\mathcal{C})$. For $c \in \mathcal{C}$, we have $c\backslash W = W_c$. Dually, for $n \in \mathbb{N}$, $W/\Delta[n] = W_{-,n}$.*

Proof. Straightforward computations. □

Lemma 12.3.6 (Generalization of [JT07, lemma 4.8])**.** *Let $\mathcal{C}$ be a small category, and $F : \mathcal{P}\mathrm{sh}(\mathbb{\Delta})^{\mathrm{op}} \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{C})$ be a continuous functor. Then $F \cong G/-$, where $G \in \mathcal{S}\mathrm{p}(\mathcal{C})$ is the restriction of F to $\mathbb{\Delta}$, i.e. $G_{-,k} := F\Delta[k]$, for $k \in \mathbb{N}$.*

Proof. For $k \in \mathbb{N}$, we have $G/\Delta[k] = G_{-,k} = F\Delta[k]$, thus $G/-$ and F coincide on $\mathbb{\Delta}^{\mathrm{op}}$. Since $\mathbb{\Delta}^{\mathrm{op}}$ freely generates $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})^{\mathrm{op}}$ under small limits, we are done. □

Definition 12.3.7 (Leibniz construction [RV14, definition 4.4])**.** Consider a functor of two variables $- \otimes - : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$, where $\mathcal{C}$ has pushouts. Its *Leibniz construction* is the functor $- \hat{\otimes} - : \mathcal{A}^{[1]} \times \mathcal{B}^{[1]} \longrightarrow \mathcal{C}^{[1]}$ which maps an arrow $f : a_1 \longrightarrow a_2$ in $\mathcal{A}$ and $g : b_1 \longrightarrow b_2$ in $\mathcal{B}$ to the cocartesian gap map below:

$$\begin{tikzcd}
a_1 \otimes b_1 \arrow[r, "a_1 \otimes g"] \arrow[d, "f \otimes b_1"'] & a_1 \otimes b_2 \arrow[d, "f'"] \arrow[ddr, bend left, "f \otimes b_2"] & \\
a_2 \otimes b_1 \arrow[r, "g'"] \arrow[drr, bend right, "a_2 \otimes g"'] & \cdot \arrow[dr, dashed, "f \hat{\otimes} g"] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & a_2 \otimes b_2.
\end{tikzcd}$$

The Leibniz construction $\hat{\otimes}$ essentially has the same properties as the original functor $\otimes$, see [RV14, section 4].

Definition 12.3.8 (Leibniz box product)**.** The *Leibniz box product*[7]

$$- \hat{\boxtimes} - : \mathcal{P}\mathrm{sh}(\mathcal{C})^{[1]} \times \mathcal{P}\mathrm{sh}(\mathbb{\Delta})^{[1]} \longrightarrow \mathcal{S}\mathrm{p}(\mathcal{C})^{[1]}$$

[5] The box product is denoted by □ in [JT07, section 2]. In [RV14, notation 4.1], it is called the *exterior product* and written $\underline{\times}$.

[6] Such functors are also called *divisible on both sides*.

[7] In [JT07, section 2], it is denoted by □′.

is simply the Leibniz construction of definition 12.3.7 applied to the box product of definition 12.3.4. If K and L are classes of morphisms of $\mathcal{P}\mathrm{sh}(\mathcal{C})$ and $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ respectively, let $\mathsf{K} \hat{\boxtimes} \mathsf{L} := \{k \hat{\boxtimes} l \mid k \in \mathsf{K}, l \in \mathsf{L}\}$.

Akin to the box product, the Leibniz box product is divisible on both sides. Specifically, if $h : W_1 \longrightarrow W_2$ is a morphism in $\mathcal{S}\mathrm{p}(\mathcal{C})$, let $\langle f \backslash h \rangle$ be the cartesian gap map on the left:

$$\begin{tikzcd}
X_2\backslash W_1 \arrow[dr, dashed, "\langle f\backslash h\rangle"] \arrow[drr, bend left, "f\backslash W_1"] \arrow[ddr, bend right, "X_2\backslash h"'] & & \\
& \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_1\backslash W_1 \arrow[d, "X_1\backslash h"] \\
& X_2\backslash W_2 \arrow[r, "f\backslash W_2"] & X_1\backslash W_2.
\end{tikzcd}
\qquad
\begin{tikzcd}
W_1/Y_2 \arrow[dr, dashed, "\langle h/g\rangle"] \arrow[drr, bend left, "h/Y_2"] \arrow[ddr, bend right, "W_1/g"'] & & \\
& \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & W_2/Y_2 \arrow[d, "W_2/g"] \\
& W_1/Y_1 \arrow[r, "h/Y_1"] & W_2/Y_1.
\end{tikzcd}$$

Then the functor $\langle f\backslash - \rangle$ is right adjoint to $f \hat{\boxtimes} -$. Dually, if $\langle h/g \rangle$ be the cartesian gap map on the right above, then the functor $\langle -/g \rangle$ is right adjoint to $- \hat{\boxtimes} g$.

Lemma 12.3.9. *(1) Let h be a morphism in $\mathcal{S}\mathrm{p}(\mathcal{C})$, and $K \in \mathcal{P}\mathrm{sh}(\mathcal{C})$. Then $K\backslash h = \langle (\varnothing \hookrightarrow K)\backslash h \rangle$. Similarly, if $L \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$, then $h/L = \langle h/(\varnothing \hookrightarrow L) \rangle$.*

(2) Let $X \in \mathcal{S}\mathrm{p}(\mathcal{C})$, and f be a morphism in $\mathcal{P}\mathrm{sh}(\mathcal{C})$. Then $f\backslash X = \langle f \backslash (X \twoheadrightarrow 1) \rangle$. Likewise, if g is a morphism in $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$, then $X/g = \langle (X \twoheadrightarrow 1)/g \rangle$.

Proof. Straightforward computations. □

We now lift some technical results of [JT07, section 2] from the setting of bisimplicial sets to simplicial presheaves over a normal skeletal category $\mathcal{C}$ (definition 12.2.10).

Definition 12.3.10. (1) The *discrete space functor* $(-)^{\mathrm{disc}} : \mathcal{P}\mathrm{sh}(\mathcal{C}) \longrightarrow \mathcal{S}\mathrm{p}(\mathcal{C})$ maps a presheaf X to $X^{\mathrm{disc}} := X \boxtimes \Delta[0]$. Explicitly, X^{disc} is the space such that for all $c \in \mathcal{C}$, $(X^{\mathrm{disc}})_c$ is the discrete simplicial set at X_c. Recall that the $- \boxtimes \Delta[0]$ is left adjoint to the "evaluation at 0" $(-)_{-,0} := -/\Delta[0]$:

$$(-)^{\mathrm{disc}} : \mathcal{P}\mathrm{sh}(\mathcal{C}) \rightleftarrows \mathcal{S}\mathrm{p}(\mathcal{C}) : (-)_{-,0}.$$

Note that if $i : \mathcal{C} \hookrightarrow \mathcal{C} \times \mathbb{\Delta}$ is the embedding mapping $c \in \mathcal{C}$ to the tuple $(c, [0])$, then the discrete space functor is $i_!$, the left Kan extension of $\mathcal{C} \xrightarrow{i} \mathcal{C} \times \mathbb{\Delta} \xrightarrow{\mathrm{y}} \mathcal{S}\mathrm{p}(\mathcal{C})$ along the Yoneda embedding, and the evaluation at 0 is i^*, the precomposition by i.

(2) Dual to the discrete space functor, let $1 \in \mathcal{P}\mathrm{sh}(\mathcal{C})$ be the terminal presheaf, and $(-)^{\mathrm{const}} : \mathcal{P}\mathrm{sh}(\mathbb{\Delta}) \longrightarrow \mathcal{S}\mathrm{p}(\mathcal{C})$, the *constant space functor*, map a simplicial set K to $K^{\mathrm{const}} := 1 \boxtimes K$. Explicitly, the functor $K^{\mathrm{const}} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ is constant at K. Recall that $(-)^{\mathrm{const}} = 1 \boxtimes -$ is left adjoint to $r := 1\backslash -$:

$$(-)^{\mathrm{const}} : \mathcal{P}\mathrm{sh}(\mathbb{\Delta}) \rightleftarrows \mathcal{S}\mathrm{p}(\mathcal{C}) : r,$$

Note that if $p : \mathcal{C} \times \mathbb{\Delta} \longrightarrow \mathbb{\Delta}$ is the projection onto the second component, then $(-)^{\mathrm{const}}$ is simply p^*, the precomposition by p, whereas $r := p_*$ is the right Kan extension of $\mathcal{C} \times \mathbb{\Delta} \xrightarrow{p} \mathbb{\Delta} \xrightarrow{\mathrm{y}} \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ along the Yoneda embedding.

Proposition 12.3.11. *The functor* $r : \mathcal{S}\mathrm{p}(\mathcal{C}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ *provides a simplicial enrichment on* $\mathcal{S}\mathrm{p}(\mathcal{C})$ *with* $\mathrm{map}(X,Y) := r(Y^X)$. *Note that* $\mathrm{map}(X,Y)_n = \mathcal{S}\mathrm{p}(\mathcal{C})(\Delta[n]^{\mathrm{const}} \times X, Y)$.

Proof. Straightforward verifications. □

Lemma 12.3.12. *For* $X, Y \in \mathcal{S}\mathrm{p}(\mathcal{C})$ *we have*

$$\mathrm{map}(X,Y) \cong \int_{c\in\mathcal{C}} \mathrm{map}(X_c, Y_c),$$

where the map *on the right is the mapping space in* $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$.

Proof. For $n \in \mathbb{N}$ we have

$$\begin{aligned}
\mathrm{map}(X,Y)_n &\cong \mathcal{S}\mathrm{p}(\mathcal{C})(\Delta[n]^{\mathrm{const}} \times X, Y) && \text{by definition} \\
&\cong \int_{c\in\mathcal{C}} \mathcal{P}\mathrm{sh}(\mathbb{\Delta})(\Delta[n] \times X_c, Y_c) && \text{by theorem 0.4.1} \\
&= \int_{c\in\mathcal{C}} \mathrm{map}(X_c, Y_c)_n && \text{by definition.}
\end{aligned}$$

□

Proposition 12.3.13 (Generalization of [JT07, proposition 2.4])**.** *The category* $\mathcal{S}\mathrm{p}(\mathcal{C})$ *endowed with the simplicial enrichment of proposition 12.3.11 is simplicially tensored and cotensored, where for for* $K \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ *and* $X \in \mathcal{S}\mathrm{p}(\mathcal{C})$, *we let* $K \otimes X := K^{\mathrm{const}} \times X$ *and* $X^K := X^{K^{\mathrm{const}}}$.

Proof. Let $Y \in \mathcal{S}\mathrm{p}(\mathcal{C})$.

(1) We have

$$\begin{aligned}
\mathrm{map}(K \otimes X, Y) &= \mathrm{map}(K^{\mathrm{const}} \times X, Y) && \text{by definition} \\
&= \int_{c\in\mathcal{C}} \mathrm{map}(K \times X_c, Y_c) && \text{by lemma 12.3.12} \\
&\cong \int_{c\in\mathcal{C}} \mathrm{map}\left(K, \mathrm{map}(X_c, Y_c)\right) && \spadesuit \\
&\cong \mathrm{map}(K, \int_{c\in\mathcal{C}} \mathrm{map}(X_c, Y_c)) && \spadesuit \\
&= \mathrm{map}\left(K, \mathrm{map}(X,Y)\right) && \text{by lemma 12.3.12,}
\end{aligned}$$

naturally in all variables, where $\spadesuit$ comes from the fact that $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ is cartesian closed. Thus $- \otimes X$ is an enriched left adjoint to $\mathrm{map}(X, -)$.

(2) We have

$$\begin{aligned}
\mathrm{map}(X, Y^K) &= \mathrm{map}(X, Y^{K^{\mathrm{const}}}) && \text{by definition} \\
&\cong \mathrm{map}(K^{\mathrm{const}} \times X, Y) && \text{by def. of exponentials} \\
&= \mathrm{map}(K \otimes X, Y) && \text{by definition}
\end{aligned}$$

naturally in all variables, and thus $(-)^K$ is an enriched right adjoint to the left tensor $K \otimes -$. □

Proposition 12.3.14. *If* $\mathcal{C}$ *is a normal skeletal category, then the class of monomorphisms of* $\mathcal{S}\mathrm{p}(\mathcal{C})$ *is* $\mathsf{Cell}_{\mathsf{B}_{\mathcal{C}} \hat{\boxtimes} \mathsf{B}_{\mathbb{\Delta}}}$.

Proof. Observe that $\mathbb{\Delta}$ is normal skeletal (with $\deg[n] := n$), and thus so is the product $\mathcal{C} \times \mathbb{\Delta}$ in an evident way [Cis06, remark 8.1.7]. In particular, for $(c, [n]) \in \mathcal{C} \times \mathbb{\Delta}$, maps $f \in (\mathcal{C} \times \mathbb{\Delta})_+$ with codomain $(c, [n])$ are pairs or morphisms $f = (f_{\mathcal{C}}, f_{\mathbb{\Delta}}) \in \mathcal{C}_+ \times \mathbb{\Delta}_+$, and f is an isomorphism if and only if both $f_{\mathcal{C}}$ and $f_{\mathbb{\Delta}}$ are[8]. Thus it is easy to see that the boundary and boundary inclusion of $(c, [n])$ are given by

$$(c \boxtimes \partial\Delta[n]) \coprod_{\partial c \boxtimes \partial\Delta[n]} (\partial c \boxtimes \Delta[n]) \xrightarrow{\mathsf{b}_c \hat{\boxtimes} \mathsf{b}_n} c \boxtimes \Delta[n].$$

We apply proposition 12.2.13 to conclude. □

Definition 12.3.15 (Trivial fibration)**.** We say that a morphism f in some category $\mathcal{A}$ is a *trivial fibration* if it has the right lifting property with respect to all monomorphisms.

Remark 12.3.16. In a model category, the classes of acyclic and trivial fibrations are in general different. They only coincide when the cofibrations are the monomorphisms, e.g. in Cisinski model categories.

Proposition 12.3.17 (Generalization of [JT07, proposition 2.3])**.** *Let $\mathcal{C}$ be a normal skeletal category, and $f : X \longrightarrow Y$ be a morphism in $\mathcal{S}\mathrm{p}(\mathcal{C})$. The following are equivalent:*

(1) f is a trivial fibration;
(2) the map $\langle \mathsf{b}_c \backslash f \rangle$ is a trivial fibration, for all $c \in \mathcal{C}$;
(3) the map $\langle u \backslash f \rangle$ is a trivial fibration, for all monomorphisms u in $\mathcal{P}\mathrm{sh}(\mathcal{C})$;
(4) the map $\langle f / \mathsf{b}_n \rangle$ is a trivial fibration, for all $n \in \mathbb{N}$;
(5) the map $\langle f / v \rangle$ is a trivial fibration, for all monomorphisms v in $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$.

Proof. Simple consequence of proposition 12.3.14 and the adjunctions $u \mathbin{\hat{\boxtimes}} - \dashv \langle u \backslash - \rangle$ and $- \mathbin{\hat{\boxtimes}} v \dashv \langle - / v \rangle$ of section 12.3. □

Lemma 12.3.18. *Let $\mathcal{C}$ be a normal skeletal category (in particular, it is Reedy), and $\mathcal{A}$ be a small category.*

(1) For $X \in \mathcal{P}\mathrm{sh}(\mathcal{A})^{\mathcal{C}}$ and $c \in \mathcal{C}$, the matching object of X at c is $\partial c \backslash X$. For $f : X \longrightarrow Y$ in $\mathcal{P}\mathrm{sh}(\mathcal{A})^{\mathcal{C}}$ and $c \in \mathcal{C}$, the relative matching map of f at c is $\langle \mathsf{b}_c \backslash f \rangle$.
(2) Assume that $\mathcal{P}\mathrm{sh}(\mathcal{A})$ be endowed with some model structure, and consider the Reedy model structure on $\mathcal{P}\mathrm{sh}(\mathcal{A})^{\mathcal{C}}$. Then a morphism f is a Reedy fibration (resp. acyclic fibration) if and only if for all $c \in \mathcal{C}$, the map $\langle \mathsf{b}_c \backslash f \rangle$ is a fibration (resp. acyclic fibration) in $\mathcal{P}\mathrm{sh}(\mathcal{A})$.

Proof. For the fist claim, observe that

$$\partial c \backslash X = \Big(\operatorname*{colim}_{\substack{d \to c \\ \text{in } \mathcal{C}_+, \text{not iso}}} d \Big) \backslash X \cong \lim_{\substack{d \to c \\ \text{in } \mathcal{C}_+, \text{not iso}}} (d \backslash X) \cong \lim_{\substack{d \to c \\ \text{in } \mathcal{C}_+, \text{not iso}}} X_d,$$

which is the matching object of X at c. The rest is by definition. □

For the the rest of this section, we assume that $\mathcal{C}$ is a normal skeletal category, and that $\mathcal{P}\mathrm{sh}(\mathcal{C})$ is endowed with a Cisinski model structure (definition 12.2.7). Speaking of the Reedy model structure on $\mathcal{S}\mathrm{p}(\mathcal{C}) \simeq \mathcal{P}\mathrm{sh}(\mathcal{C} \times \mathbb{\Delta})$ can lead to confusions as to which Reedy category is used as shape category. We introduce the *vertical* and *horizontal model structures* to remedy this.

[8] This observation is called *Leibniz's formula* in [RV14, observation 4.2].

Definition 12.3.19. (1) Let $\mathcal{S}\mathrm{p}(\mathcal{C})_v$, the *vertical model structure* on the category of spaces $\mathcal{S}\mathrm{p}(\mathcal{C}) = \mathcal{S}\mathrm{p}(\mathbb{\Delta})^{\mathcal{C}^{\mathrm{op}}}$, be the Reedy structure induced by the Quillen model structure on $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$. In this structure, a map $f : X \longrightarrow Y$ is a weak equivalence (also called *column-wise weak equivalence*) if for all $c \in \mathcal{C}$, the map of simplicial sets $c\backslash f = f_c : X_c \longrightarrow Y_c$ is a weak equivalence. It is a fibration (also called *vertical fibration*) if for all $c \in \mathcal{C}$, the relative matching map $\langle \mathsf{b}_c\backslash f\rangle$ is a Kan fibration, where $\mathsf{b}_c : \partial c \longrightarrow c$ is the boundary inclusion of c. Fibrant spaces in $\mathcal{S}\mathrm{p}(\mathcal{C})_v$ are also called *vertically fibrant*.

(2) Dually, let $\mathcal{S}\mathrm{p}(\mathcal{C})_h$, the *horizontal model structure* on $\mathcal{S}\mathrm{p}(\mathcal{C}) \simeq \mathcal{P}\mathrm{sh}(\mathcal{C})^{\mathbb{\Delta}^{\mathrm{op}}}$, be the Reedy structure induced by the model structure on $\mathcal{P}\mathrm{sh}(\mathcal{C})$. The description of weak equivalence and fibrations transpose from the vertical model structure *mutadis mutandis*.

Proposition 12.3.20. *The model structures $\mathcal{S}\mathrm{p}(\mathcal{C})_v$ and $\mathcal{S}\mathrm{p}(\mathcal{C})_h$ are of Cisinski type (definition 12.2.7).*

Proof. By [Hir09, theorem 15.6.27], both are cofibrantly generated. A map $f \in \mathcal{S}\mathrm{p}(\mathcal{C})$ is a vertical (resp. horizontal) acyclic fibration if and only if for all $c \in \mathcal{C}$ (resp. for all $n \in \mathbb{N}$), the matching map $\langle \mathsf{b}_c\backslash f\rangle$ (resp. $\langle f/\mathsf{b}_n\rangle$) is an acyclic fibration in $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})_{\mathrm{Quillen}}$ (resp. in the model structure on $\mathcal{P}\mathrm{sh}(\mathcal{C})$, which is assumed to be of Cisinski type), i.e. a trivial fibration. By proposition 12.3.17, f is a trivial fibration. Finally, f is a vertical or a horizontal acyclic fibration if and only if it is a trivial fibration. Therefore, vertical and horizontal cofibrations are the monomorphisms. □

Proposition 12.3.21 (Generalization of [JT07, proposition 2.5])**.** *Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{S}\mathrm{p}(\mathcal{C})$. The following are equivalent:*

(1) f is a vertical fibration, i.e. the map $\langle \mathsf{b}_c\backslash f\rangle$ is a Kan fibration (see example 12.1.10), for $c \in \mathcal{C}$;
(2) the map $\langle u\backslash f\rangle$ is a Kan fibration, for all monomorphism $u \in \mathcal{P}\mathrm{sh}(\mathcal{C})$;
(3) the map $\langle f/\mathsf{h}^k_n\rangle$ is a trivial fibration, where $\mathsf{h}^k_n : \Lambda^k[n] \longrightarrow \Delta[n]$ is the k-th horn inclusion of $[n]$, for all $n \in \mathbb{N}$ and $0 \leq k \leq n$;
(4) the map $\langle f/v\rangle$ is a trivial fibration, for all anodyne extension $v \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$.

Definition 12.3.22 (Homotopically constant space [RSS01, section 3])**.** A space $X \in \mathcal{S}\mathrm{p}(\mathcal{C})$ is *homotopically constant* if for every map $f : [k] \longrightarrow [l]$ in $\mathbb{\Delta}$, the structure map $X/f : X_{-,l} \longrightarrow X_{-,k}$ is a weak equivalence $\mathcal{P}\mathrm{sh}(\mathcal{C})$.

Lemma 12.3.23. *Let $X \in \mathcal{S}\mathrm{p}(\mathcal{C})$. The following are equivalent:*

(1) X is homotopically constant;
(2) for all $k \in \mathbb{N}$, writing $s : [k] \longrightarrow [0]$ the terminal map in $\mathbb{\Delta}$, the structure map $X/s : X_{-,0} \longrightarrow X_{-,k}$ a weak equivalence;
(3) for every codegeneracy $s^i : [k] \longrightarrow [k-1]$ in $\mathbb{\Delta}$, the structure map $X/s^i : X_{-,k-1} \longrightarrow X_{-,k}$ is a weak equivalence;
(4) for all $k \in \mathbb{N}$ and all map $d : [0] \longrightarrow [k]$ in $\mathbb{\Delta}$, the structure map $X/d : X_{-,k} \longrightarrow X_{-,0}$ is a weak equivalence;
(5) for every coface map $d^i : [k] \longrightarrow [k+1]$ in $\mathbb{\Delta}$, the structure map $X/d^i : X_{-,k+1} \longrightarrow X_{-,k}$ is a weak equivalence.

Proof. The implications (1) $\implies$ (2) $\implies$ (3) and (1) $\implies$ (4) $\implies$ (5) are trivial.

(2) $\implies$ *(1)* Take a map $f : [k] \longrightarrow [l]$ in $\mathbb{\Delta}$. Clearly, $s = sf$, so $X/s = (X/f)(X/s)$. By 3-for-2, X/f is a weak equivalence.

(3) $\implies$ *(2)* Note that the terminal map $s : [k] \longrightarrow [0]$ is a composite of codegeneracies $[k] \longrightarrow [k-1] \longrightarrow \cdots \longrightarrow [1] \longrightarrow [0]$.

(4) $\implies$ *(2)* The terminal map $s : [k] \longrightarrow [0]$ is a retraction of any map $d : [0] \longrightarrow [k]$, thus X/s is a section of X/d. By 3-for-2, X/s is a weak equivalence.

(5) $\implies$ *(4)* Note that all map $d : [0] \longrightarrow [k]$ is a composite of coface maps. □

Proposition 12.3.24 (Generalization of [JT07, proposition 2.8])**.** *If a space is vertically fibrant (definition 12.3.19), then it is homotopically constant.*

Proof. Let $X \in \mathcal{S}\mathrm{p}(\mathcal{C})$ be vertically fibrant. Take $d : [0] \longrightarrow [n]$ in $\mathbb{\Delta}$. Since $d : \Delta[0] \longrightarrow \Delta[n]$ is a trivial cofibration in $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})_{\mathrm{Quillen}}$, by proposition 12.3.21, the map $X/d = \langle (X \to 1)/d \rangle$ is a trivial fibration. Apply lemma 12.3.23 to conclude. □

Proposition 12.3.25 (Generalization of [JT07, proposition 2.9])**.** *A map $f : X \longrightarrow Y$ between vertically fibrant spaces is a weak equivalence in $\mathcal{S}\mathrm{p}(\mathcal{C})_h$ if and only if $f_{-,0} : X_{-,0} \longrightarrow Y_{-,0}$ is a weak equivalence.*

Proof. Let $n \in \mathbb{N}$ and $s : [n] \longrightarrow [0]$ be the terminal map in $\mathbb{\Delta}$. We have a commutative square

$$\begin{tikzcd} X_{-,0} \arrow[r, "f_{-,0}"] \arrow[d, "X/s"'] & Y_{-,0} \arrow[d, "Y/s"] \\ X_{-,n} \arrow[r, "f_{-,n}"] & Y_{-,n} \end{tikzcd}$$

where by proposition 12.3.24, the vertical morphisms are weak equivalences. The result follows by 3-for-2. □

Chapter Thirteen

The homotopy theory of ∞-opetopic algebra

We now take interest in the notion of "weak opetopic algebra", or ∞-algebras. In dimension 1, ∞-categories are modeled by the *weak Kan complexes* (or *quasi-categories*) of Boardman–Vogt [BV73], further studied by Joyal [Joy08], Lurie [Lur09], and others. In particular, the fibrant objects of Joyal's model structure $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})_{\mathrm{Joyal}}$ on simplicial sets are exactly quasi-categories. One dimension above, at the level of operads, similar results have been achieved by Cisinski and Moerdijk [CM11b] [CM13], using the category $\mathbb{\Omega}$ of dendrices as shape theory, rather than $\mathbb{\Delta}$. The idea is similar: ∞-operads are presheaves over $\mathbb{\Omega}$ satisfying some non strict "horn filling condition". In particular, operads are recovered as presheaves which satisfy said conditions strictly. In this chapter, we generalize those ideas by considering the category $\mathbb{\Lambda}$ of opetopic shapes (definition 11.1.33). More precisely, by applying Cisinski's homotopy theory (recalled in section 12.2) we show that $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ can be endowed with a model structure whose fibrant objects are exactly the presheaves satisfying some inner Kan filling condition.

13.1 A SKELETAL STRUCTURE ON $\mathbb{\Lambda}$

Let $n \geq 1$. In this section, we endow $\mathbb{\Lambda} = \mathbb{\Lambda}_{1,n}$ with the structure of a skeletal category (cf. definition 12.2.10). To that end, we need to assign a notion of *degree* to objects $\lambda \in \mathbb{\Lambda}$, which shall be denoted by $\deg \lambda \in \mathbb{N}$, and specify two wide subcategories $\mathbb{\Lambda}_+$ and $\mathbb{\Lambda}_-$ satisfying axioms **(Sq0)** to **(Sq3)**.

Definition 13.1.1. Recall from corollary 11.2.34 that the objects of $\mathbb{\Lambda}$ are (up to isomorphisms) the free algebras on $(n+1)$-opetopes. For $\omega \in \mathbb{O}_{n+1}$, let $\deg h\omega := \#\omega^\bullet$ be the number of nodes of ω.

Definition 13.1.2. Let $f : h\omega \longrightarrow h\omega'$ be a morphism in $\mathbb{\Lambda}$, with $\omega, \omega' \in \mathbb{O}_{n+1}$. In particular, it induces a set map between $(n-1)$-cells: $(Mf)_{n-1} : (Mh\omega)_{n-1} \longrightarrow (Mh\omega')_{n-1}$. Let $\mathbb{\Lambda}_+$ (resp. $\mathbb{\Lambda}_-$) be the wide subcategory spanned by those morphisms f such that $(Mf)_{n-1}$ is a monomorphism (resp. epimorphism).

The rest of this section is dedicated to prove the following result.

Theorem 13.1.3. *With the data of definitions 13.1.1 and 13.1.2, $\mathbb{\Lambda}$ is a skeletal category.*

Proof of theorem 13.1.3, axiom **(Sq0)**. Since $\mathbb{\Lambda}$ has no isomorphisms beside the identities, this axiom holds trivially. □

Lemma 13.1.4. *Let $\omega \in \mathbb{O}_{n+1} = \operatorname{tr} \mathfrak{Z}^{n-1}$ and $\psi \in \mathbb{O}_n$. Then $(Mh\omega)_\psi$ is the set of sub-$\mathfrak{Z}^{n-1}$-trees ν of ω such that $\mathsf{t}\nu = \psi$.*

Proof. By definitions 11.1.16 and 11.2.1,

$$(Mh\omega)_\psi = \mathfrak{Z}S[\omega]_\psi = \sum_{\substack{\nu\in\mathbb{O}_{n+1}\\ \mathsf{t}\,\nu=\psi}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})(S[\nu], S[\omega]),$$

and morphisms $S[\nu] \longrightarrow S[\omega]$ are precisely the $\mathfrak{Z}^{n-1}$-tree embeddings of ν in ω. □

Lemma 13.1.5. *Let $f : h\omega \longrightarrow h\omega'$ be a morphism in $\mathbb{\Lambda}$, with $\omega, \omega' \in \mathbb{O}_{n+1}$. The following are equivalent:*

(1) $f \in \mathbb{\Lambda}_+$ (resp. $\mathbb{\Lambda}_-$);

(2) the map between n-cells $Mf_n : (Mh\omega)_n \longrightarrow (Mh\omega')_n$ is a monomorphism (resp. an epimorphism);

(3) Mf is a monomorphism (resp. an epimorphism).

Proof. Recall from theorem 11.2.33 that the image of $M : \mathcal{A}\mathrm{lg} \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{\geq n-1})$ is the orthogonality class induced by $\mathsf{S}_{\geq n+1}$, thus $Mf_{\geq n}$ is a monomorphism (resp. an epimorphism) if and only if Mf_n is. It remains to show that Mf_{n-1} is a monomorphism (resp. epimorphism) if and only if Mf_n is. But this is a direct consequence of lemma 13.1.4, stating that $(Mh\omega)_n$ is the set of sub-$\mathfrak{Z}^{n-1}$-trees of ω. □

Corollary 13.1.6. *Let $f : h\omega \longrightarrow h\omega'$ be a morphism in $\mathbb{\Lambda}$, with $\omega, \omega' \in \mathbb{O}_{n+1}$. The following are equivalent:*

(1) $f \in \mathbb{\Lambda}_+ \cap \mathbb{\Lambda}_-$, i.e. f_{n-1} is an isomorphism;

(2) Mf_n is an isomorphism;

(3) Mf is an isomorphism;

(4) f is an isomorphism.

Proof. Follows from lemma 13.1.5 and the fact that M reflects isomorphisms (since it is equivalent to the inclusion of a small orthogonality class). □

Proof of theorem 13.1.3, axiom **(Sq1)**. Let $f : h\omega \longrightarrow h\omega'$ be a morphism in $\mathbb{\Lambda}$ that is not an isomorphism, with $\omega, \omega' \in \mathbb{O}_{n+1}$. If $f \in \mathbb{\Lambda}_+$, then by lemma 13.1.5 and corollary 13.1.6, Mf_n is a monomorphism that is not an isomorphism, thus $\#(Mh\omega)_n < \#(Mh\omega')_n$, i.e. the number of subtrees of ω is strictly less than that of ω'. In particular, the number of nodes of ω is strictly less than that of ω', i.e. $\deg h\omega < \deg h\omega'$. The case where $f \in \mathbb{\Lambda}_-$ instead of $\mathbb{\Lambda}_+$ is treated similarly. □

Proof of theorem 13.1.3, axiom **(Sq2)**. Let $f : h\omega \longrightarrow h\omega'$ be a morphism in $\mathbb{\Lambda}$, with $\omega, \omega' \in \mathbb{O}_{n+1}$. It maps the maximal subtree $\omega \subseteq \omega$ to a subtree of ω', say $\nu \subseteq \omega'$. Then the following is the desired factorization:

$$\begin{tikzcd} h\omega \arrow[rr, "f"] \arrow[dr, two heads, "f"'] & & h\omega' \\ & h\nu. \arrow[ur, hook] & \end{tikzcd}$$

By lemma 13.1.5, $\mathbb{\Lambda}_+$ only contains inclusions (seen in $\mathcal{P}\mathrm{sh}(\mathbb{O}_{\geq n-1})$), and uniqueness of the factorization follows. □

Lemma 13.1.7. *Let $f_1, f_2 : h\omega \longrightarrow h\omega'$ be two morphisms in $\mathbb{\Lambda}$, with $\omega, \omega' \in \mathbb{O}_{n+1}$. Recall from lemma 13.1.4 that $(Mh\omega)_n$ is the set of sub 3^{n-1} trees of ω. Then, $f_1 = f_2$ if and only if Mf_1 and Mf_2 agree on all one-node subtrees of ω.*

Proof. Under the adjunction $\mathcal{P}\mathrm{sh}(\mathbb{O}_{\geq n-1}) \rightleftarrows \mathcal{A}\mathrm{lg}$, f_i corresponds to a morphism $\bar{f}_i : O[\omega] \longrightarrow Mh\omega'$, and since $\mathsf{S}_{\geq n+1} \perp Mh\omega'$, it is uniquely determined by its restriction $f_i : S[h\omega] \longrightarrow h\omega'$. □

Notation 13.1.8. As a consequence of lemma 13.1.7, if $\omega \in \mathbb{O}_{n+1}$ is not degenerate, the algebra $h\omega$ is freely generated by its one-node subtrees. We denote these generators by $c_{\omega,[p]}$, where $[p]$ ranges over $\omega^\bullet$.

Proof of theorem 13.1.3, axiom **(Sq3)***.* Let $f : h\omega \longrightarrow h\omega'$ be a morphism in $\mathbb{\Lambda}_-$. We define a section g of f. By adjointness, defining a map $g : h\omega' \longrightarrow h\omega$ is equivalent to specifying a map $\bar{g} : O[\omega'] \longrightarrow Mh\omega$ in $\mathcal{P}\mathrm{sh}(\mathbb{O}_{\geq n-1})$. Since $\mathsf{S}_{\geq n+1} \perp Mh\omega$, it is enough to define $\bar{g}$ on the spine $S[\omega']$ of ω'. For $[p] \in (\omega')^\bullet$, let $\bar{g}(\mathsf{s}_{[p]}\,\omega') := \mathsf{s}_{[q]}\,\omega$, where

$$[q] := \min_{\text{wrt. } \lessdot} \left\{ [r] \in \omega^\bullet \mid Mf(\mathsf{s}_{[r]}\,\omega) = \mathsf{s}_{[p]}\,\omega' \right\}.$$

In other words, g maps the node $\mathsf{s}_{[p]}\,\omega'$ to the lexicographically minimal node in the fiber $f^{-1}(\mathsf{s}_{[p]}\,\omega')$. The composite

$$S[\omega'] \xrightarrow{\bar{g}} Mh\omega \xrightarrow{Mf} Mh\omega'$$

maps a node $\mathsf{s}_{[p]}\,\omega'$ to $\mathsf{s}_{[p]}\,\omega'$, thus g is a section of f.

Let now $f_1, f_2 : h\omega \longrightarrow h\omega'$ be a morphism in $\mathbb{\Lambda}_-$ having the same sections. In particular, for g_i the section of f_i constructed as above, where $i = 1, 2$, we have $f_1 g_2 = \mathrm{id}_{h\omega'}$. Thus for $[p] \in (\omega')^\bullet$, we have $Mg_2(\mathsf{s}_{[p]}\,\omega') \leq Mg_1(\mathsf{s}_{[p]}\,\omega')$, meaning that the node $Mg_2(\mathsf{s}_{[p]}\,\omega')$ is "below" $Mg_1(\mathsf{s}_{[p]}\,\omega')$ in the tree ω. Conversely, since $f_2 g_1 = \mathrm{id}_{h\omega'}$, we have $Mg_1(\mathsf{s}_{[p]}\,\omega') \leq Mg_2(\mathsf{s}_{[p]}\,\omega')$, and finally, $Mg_1(\mathsf{s}_{[p]}\,\omega') = Mg_2(\mathsf{s}_{[p]}\,\omega')$. By lemma 13.1.7, $g_1 = g_2$, so clearly, $f_1 = f_2$. □

Example 13.1.9. Take $n = 1$, so that $\mathbb{\Lambda} = \mathbb{\Delta}$, and consider the map $f : [4] \longrightarrow [1]$ in $\mathbb{\Delta}_-$ defined by $f(0) = f(1) = 0$ and $f(2) = f(3) = f(4) = 1$. Then f admits several sections (6 in total), but the proof of **(Sq3)** constructs that which maps each element $i \in [1]$ to the minimal element of $f^{-1}(i)$. Explicitly, $g(0) := 0$ and $g(1) := 2$.

13.2 ANODYNE EXTENSIONS

Proposition 13.2.1. *The class of monomorphisms of $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ is exactly the class of relative B-cell complexes Cell_B.*

Proof. By theorem 13.1.3, $\mathbb{\Lambda}$ is a normal skeletal category, and proposition 12.2.13 applies. □

Definition 13.2.2 (Elementary face)**.** Let $\lambda \in \mathbb{\Lambda}$.

(1) A *elementary face* of λ is a morphism $f : \lambda' \longrightarrow \lambda$ in $\mathbb{\Lambda}_+$, where $\deg \lambda' = \deg \lambda - 1$.

(2) Let $e : \lambda' \longrightarrow \lambda$ be an elementary face of λ, and write $\lambda = h\omega$ and $\lambda' = h\omega'$, with $\omega, \omega' \in \mathbb{O}_{n+1}$. The face e is *inner* if Me_{n-1} exhibits a bijection between the leaves of ω' (seen as $(n-1)$-cells of $Mh\omega'$) and the leaves of ω, and if it maps the root edge $\mathsf{e}_{[]}\,\omega$ to $\mathsf{e}_{[]}\,\omega'$. In other words, e induces an isomorphism $h\partial O[\mathsf{t}\,\omega] \longrightarrow h\partial O[\mathsf{t}\,\omega']$.

Remark 13.2.3. If $e : h\omega' \longrightarrow h\omega$ is an inner face of $h\omega$, then, by a counting argument on the number of nodes of ω' and ω, there exists a unique $[p] \in (\omega')^\bullet$ such that $e(c_{\omega',[p]})$ (see notation 13.1.8) is a subtree of ω with two nodes. If $[p] \neq [p'] \in (\omega')^\bullet$, then $e(c_{\omega',[p']})$ is a generator of $h\omega$, i.e. a one-node subtree. Thus f exhibits a subtree $\nu := e(c_{\omega,[p]})$ of ω with two nodes, or equivalently, an inner edge.

This remark motivates the following terminology:

Definition 13.2.4 (Inner horn)**.** (1) Let $\omega \in \mathbb{O}_{n+1}$. If ω is degenerate (resp. an endotope), say $\omega = \mathsf{I}_\phi$ for some $\phi \in \mathbb{O}_{n-1}$ (resp. Y_ϕ for some $\phi \in \mathbb{O}_n$), then $h\omega$ does not have any inner face, but by convention, let its inner horn simply be $\Lambda[h\omega] := h\phi$. Write $\mathsf{h}_{h\omega} : \Lambda[h\omega] \longrightarrow h\omega$ for the unique inner horn inclusion of $h\omega$.

(2) Otherwise, for e an inner face of λ, define $\Lambda^e[\lambda]$, the *inner horn* of λ at e, as the colimit in $\mathcal{P}\mathrm{sh}(\mathbb{A})$

$$\Lambda^e[\lambda] := \operatorname*{colim}_{\substack{g:\lambda'\to\lambda \text{ elem. face} \\ g\neq e}} \lambda'.$$

Let $\mathsf{h}^e_\lambda : \Lambda^e[\lambda] \longrightarrow \lambda$ be the *inner horn inclusion* λ at e.

Let $\mathsf{H}_{\text{inner}} \subseteq \mathcal{P}\mathrm{sh}(\mathbb{A})^{[1]}$ be the set of inner horn inclusions.

Definition 13.2.5 (Inner anodyne extension)**.** Let the class $\mathsf{An}_{\text{inner}}$ of *inner anodyne extensions* (also called *mid anodyne extensions*) be the saturation of $\mathsf{H}_{\text{inner}}$. An *inner fibration* (also called *mid fibration*) is a morphism $f \in \mathcal{P}\mathrm{sh}(\mathbb{A})$ such that $\mathsf{H}_{\text{inner}} \pitchfork f$, or equivalently, such that $\mathsf{An}_{\text{inner}} \pitchfork f$.

Remark 13.2.6. Note that definitions 13.2.4 and 13.2.5 are compatible with the classical notions of inner horn and inner anodyne extension in the simplicial setting ($n = 1$, see example 0.3.10) and in the dendroidal setting ($n = 2$, see [CM13, section 1]).

Definition 13.2.7. Generalizing definition 13.2.4, if $\omega \in \mathbb{O}_{n+1}$, and I is a set of (not necessarily inner) elementary faces of $h\omega$, let $\Lambda^I[\lambda]$ be the following colimit in $\mathcal{P}\mathrm{sh}(\mathbb{A})$:

$$\Lambda^I[\lambda] := \operatorname*{colim}_{\substack{g:\lambda'\to\lambda \text{ elem. face} \\ g\notin I}} \lambda'$$

and write $\mathsf{h}^I_{h\omega} : \Lambda^I[h\omega] \longrightarrow h\omega$ for the canonical inclusion.

Lemma 13.2.8. *Let $\omega \in \mathbb{O}_{n+1}$ have $d \geq 2$ nodes, and $\varnothing \neq I \subseteq J$ be two nonempty sets of inner faces of $h\omega$. Then the inclusion $\mathsf{h}^J_{h\omega}$ factors as*

$$\Lambda^J[h\omega] \xrightarrow{u} \Lambda^I[h\omega] \xrightarrow{\mathsf{h}^I_{h\omega}} h\omega,$$

and u is a relative cell complex of inner horn inclusions of opetopes with at most $d-1$ nodes[1]*.*

[1] As we see in the proof, it is even a *finite* composite of pushouts of such maps.

Proof. We proceed by induction on $m := \#(J - I)$. If $m = 0$, then u is an identity, and the result holds trivially. Assume $m \geq 1$, and that the holds up to $m - 1$. Take $e \in J - I$, and let $J' := J - \{e\}$. The inclusion u decomposes as

$$\Lambda^J[h\omega] \xrightarrow{v} \Lambda^{J'}[h\omega] \xrightarrow{w} \Lambda^I[h\omega].$$

By induction, since $\mathsf{h}^{J'}_{h\omega} = \mathsf{h}^I_{h\omega} \cdot w$, the inclusion w is a cell complex of inner horn inclusions of opetopes with at most $d - 1$ nodes. To conclude, it remains to show that v is too.

The inner face e of $h\omega$ exhibits a subtree $\nu \subseteq \omega$ with two nodes (see remark 13.2.3), or equivalently, an inner edge of ω, say at address $[p]$. Let ω' be ω where this inner edge has been contracted. Formally, decomposing ω on the left so as to exhibits the subtree ν, the opetope ω' is defined on the right:

$$\omega = \alpha \underset{[p]}{\circ} \nu \underset{[l_i]}{\bigcirc} \beta_i, \qquad\qquad \omega' := \alpha \underset{[p]}{\circ} \mathsf{Y}_{\mathsf{t}\nu} \underset{[\wp_\nu[l_i]]}{\bigcirc} \beta_i,$$

where $[l_i]$ ranges over $\nu^{|}$. Equivalently, ω' is the only opetope such that $\omega' \square_{[p]} \nu = \omega$. Then the inclusion $w : \Lambda^J[h\omega] \longrightarrow \Lambda^{J'}[h\omega]$ above is obtained as the following pushout

$$\begin{tikzcd}
\Lambda^{J'}[h\omega'] \arrow[r] \arrow[d, "\mathsf{h}^{J'}_{h\omega}"'] & \Lambda^J[h\omega] \arrow[d, "w"] \\
h\omega' \arrow[r] & \Lambda^{J'}[h\omega] \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Hence w is a pushout of $\mathsf{h}^{J'}_{h\omega}$, and ω' has $d - 1$ nodes. $\square$

Lemma 13.2.9 (Generalization of [MW09, lemma 5.1])**.** *Let $\omega \in \mathbb{O}_{n+1}$, and I be a nonempty set of inner faces of $h\omega$. Then the inclusion $\mathsf{h}^I_{h\omega} : \Lambda^I[h\omega] \hookrightarrow h\omega$ is an inner anodyne extension.*

Proof. We proceed by induction on $d := \deg h\omega$ and on $m := \#I$. Since I is nonempty, $d \geq 2$ and $m \geq 1$. If $d = 2$, then $h\omega$ has a unique inner face e, and $\mathsf{h}^I_{h\omega} = \mathsf{h}^e_{h\omega}$.

Assume now that $d \geq 3$. If $m = 1$, then $\mathsf{h}^I_{h\omega}$ is an inner horn inclusion, and the claim trivially holds. If $m \geq 2$, take $e \in I$, and let $J := I - \{e\}$. The inclusion $\mathsf{h}^I_{h\omega}$ decomposes as

$$\Lambda^I[h\omega] \xrightarrow{u} \Lambda^J[h\omega] \xrightarrow{\mathsf{h}^J_{h\omega}} h\omega.$$

By induction on m, $\mathsf{h}^J_{h\omega}$ is an inner anodyne extension. By lemma 13.2.8, u is a cell complex of inner horn inclusions of opetopes of at most $d - 1$ nodes, and so by induction on d, u is an inner anodyne extension as well. $\square$

Lemma 13.2.10. *Let $\omega \in \mathbb{O}_{n+1}$ have $d \geq 2$ nodes, and I be the set of all inner faces of ω. Note that $\Lambda^I[h\omega]$ contains all the generators $c_{\omega,[p]}$ (see notation 13.1.8), and thus the spine inclusion $\mathsf{s}_{h\omega}$ decomposes as*

$$S[h\omega] \xrightarrow{u} \Lambda^I[h\omega] \xrightarrow{\mathsf{h}^I_{h\omega}} h\omega.$$

Then the inclusion u is a relative cell complex of spine inclusions of opetopes of at most $d - 1$ nodes.

Proof. Note that an outer face (i.e. an elementary faces that are not inner) of $h\omega$ is an inclusion $\omega' \subseteq \omega$ of a subtree of ω of $d-1$ nodes. Hence $\Lambda^I[h\omega]$ is the colimit on the left, while $S[h\omega]$ can be expressed as on the right

$$\Lambda^I[h\omega] = \operatorname*{colim}_{\substack{\omega'\subseteq\omega \\ \deg h\omega' = d-1}} h\omega', \qquad S[h\omega] = \operatorname*{colim}_{\substack{\omega'\subseteq\omega \\ \deg h\omega' = d-1}} S[h\omega'].$$

Thus

$$u : \operatorname*{colim}_{\substack{\omega'\subseteq\omega \\ \deg h\omega' = d-1}} S[h\omega'] \hookrightarrow \operatorname*{colim}_{\substack{\omega'\subseteq\omega \\ \deg h\omega' = d-1}} h\omega'$$

can be obtained as a composition of pushouts of the $\mathsf{s}_{\omega'}$'s. Explicitly, let $\omega_1, \ldots, \omega_m \subseteq \omega$ be the subtrees of ω with $d-1$ nodes, and let $g_1, \ldots, g_m$ be the associated outer face inclusions of $h\omega$. Let $X^{(0)} := S[\omega]$, let ι_0 be the identity on $X^{(0)}$, and for $1 \leq i \leq m$, let $X^{(i)}$ be the pushout

$$\begin{tikzcd} S[\omega_i] \arrow[r, "\iota_{i-1} g_i \mathsf{s}_{\omega_i}"] \arrow[d, hook] & X^{(i-1)} \arrow[d, hook, "v"] \\ h\omega_i \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & X^{(i)}, \end{tikzcd}$$

and $\iota_i := v\iota_{i-1}$. Then $\Lambda^I[h\omega] = X^{(m)}$. □

Proposition 13.2.11 (Generalization of [JT07, lemma 1.21]). *Spine inclusions are inner anodyne extensions, and consequently, $^{\pitchfork}(\mathsf{S}^{\pitchfork}) \subseteq \mathsf{An}_{\text{inner}}$.*

Proof. Let $\omega \in \mathbb{O}_{n+1}$, and consider the spine inclusion $\mathsf{s}_{h\omega} : S[h\omega] \hookrightarrow h\omega$.

(1) If $\deg h\omega = \#\omega^\bullet = 0, 1$, then $\mathsf{s}_{h\omega}$ is an identity, thus an inner anodyne extension.
(2) If $\deg h\omega = 2$, then $h\omega$ admits a unique inner face $e : h\mathsf{Y}_{\mathsf{t}\omega} \longrightarrow h\omega$, and note that $\Lambda^e[h\omega] = S[h\omega]$. Thus in this case, the spine inclusion is an inner horn inclusion.
(3) Assume $d = \deg h\omega \geq 3$, and let I be the set of all inner faces of $h\omega$. Note that since $\Lambda^I[h\omega]$ contains all generators of $h\omega$ (i.e. subtrees of one node), the spine inclusion $\mathsf{s}_{h\omega}$ decomposes as

$$S[h\omega] \xrightarrow{u} \Lambda^I[h\omega] \xrightarrow{\mathsf{h}^e_{h\omega}} h\omega,$$

and in order to show that $\mathsf{s}_{h\omega}$ is an inner anodyne extension, it suffices to show that u is. By lemma 13.2.10, it is a cell complex of spine inclusions of opetopes of at most $d-1$ nodes, thus by induction, it is an inner anodyne extension. □

Proposition 13.2.12. *Inner anodyne extensions are* S*-local isomorphisms.*

Proof. It is enough to show that inner horn inclusions are S-local isomorphisms. Let $\omega \in \mathbb{O}_{n+1}$ have $d \geq 2$ nodes. If $d = 2$, then it has a unique inner face, and the corresponding inner horn inclusion is simply the spine inclusion $S[h\omega] \hookrightarrow h\omega$.

Assume $d \geq 3$, let I be the set of all inner faces of $h\omega$, and $e \in I$. Then the spine inclusion $\mathsf{s}_{h\omega}$ decomposes as

$$S[h\omega] \xrightarrow{u} \Lambda^I[h\omega] \xrightarrow{v} \Lambda^e[h\omega] \xrightarrow{\mathsf{h}^e_{h\omega}} h\omega.$$

By lemma 13.2.10, u is a spine complex, thus an S local-isomorphism. By lemma 13.2.8, v is a cell complex of inner horn inclusions of opetopes with at most $d-1$ nodes. By induction, v is an S local-isomorphism. Thus vu and $\mathsf{s}_{h\omega} = \mathsf{h}^e_{h\omega} vu$ are S local-isomorphisms, and by 3-for-2, so is $\mathsf{h}^e_{h\omega}$. □

Definition 13.3.1 (Parallel cells)**.** Let $X \in \mathcal{A}\mathrm{lg}$ be an opetopic algebra, $\omega \in \mathbb{O}_n$, and $x, y \in X_\omega$. We say that x and y are *parallel* if the following two composites are equal:

$$\partial O[\omega] \xrightarrow{\mathsf{b}_\omega} O[\omega] \underset{y}{\overset{x}{\rightrightarrows}} X.$$

Explicitly, $\mathsf{t}\, x = \mathsf{t}\, x'$, and for all $[p] \in \omega^\bullet$, $\mathsf{s}_{[p]}\, x = \mathsf{s}_{[p]}\, x'$.

Definition 13.3.2 (Quotient of an opetopic algebra)**.** Recall that by proposition 11.1.30, the category $\mathcal{A}\mathrm{lg}$ is cocomplete. Let $X \in \mathcal{A}\mathrm{lg}$ be an opetopic algebra, $\omega \in \mathbb{O}_n$, and $x, y \in X_\omega$ be parallel cells. Let the *quotient* of X by the equality $x = y$ be the following coequalizer in $\mathcal{A}\mathrm{lg}$:

$$h\omega \underset{y}{\overset{x}{\rightrightarrows}} X \longrightarrow X / \{x = y\}.$$

More generally, consider X_n as a set over $\mathbb{O}_n$, and let $K \subseteq X_n \times X_n$ be a set of pairs of parallel cells. Then the quotient of X by K is the following coequalizer in $\mathcal{A}\mathrm{lg}$:

$$\textstyle\sum_{\omega \in \mathbb{O}_n} \sum_{(x,y) \in K_\omega} h\omega \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \longrightarrow X/K,$$

where $p_1(x, y) \coloneqq x$ and $p_2(x, y) \coloneqq y$.

Example 13.3.3. Assume $k = n = 1$, so that $\mathcal{A}\mathrm{lg} = \mathcal{C}\mathrm{at}$. Let $\mathcal{C}$ be a category. Then definition 13.3.1 corresponds to the classical definition of parallel morphisms. If K is a set of pairs of parallel morphisms, then $\mathcal{C}/K$ is the usual quotient of $\mathcal{C}$ by the congruence relation generated by K, see e.g. [Awo10, definition 4.6].

Definition 13.3.4 (Rezk interval in $\mathcal{A}\mathrm{lg}$)**.** Let $\phi \in \mathbb{O}_{n-1}$. The *Rezk interval of shape* ϕ is the algebra $\mathfrak{J}_\phi \in \mathcal{A}\mathrm{lg}$ generated by one invertible operation j_ϕ of shape Y_ϕ. Explicitly, $\mathfrak{J}_\phi$ has two $(n-1)$-cells $0_\phi, 1_\phi \in (\mathfrak{J}_\phi)_\phi$, and four n-cells $j_\phi, j_\phi^{-1}, \mathrm{id}_0, \mathrm{id}_1 \in (\mathfrak{J}_\phi)_{\mathsf{Y}_\phi}$, satisfying the following equalities

$$\mathsf{s}_{[]}\, j_\phi = \mathsf{t}\, j_\phi^{-1} = 0_\phi, \quad \mathsf{t}\, j_\phi = \mathsf{s}_{[]}\, j_\phi^{-1} = 1_\phi, \quad j_\phi \underset{[]}{\circ} j_\phi^{-1} = \mathrm{id}_1, \quad j_\phi^{-1} \underset{[]}{\circ} j_\phi = \mathrm{id}_0 .$$

The interval $\mathfrak{J}_\phi$ admits an involutive automorphism that swaps 0_ϕ and 1_ϕ, and j_ϕ and j_ϕ^{-1}. In particular, up to isomorphism, there is a unique endpoint inclusion $\mathsf{j}_\phi : h\phi \longrightarrow \mathfrak{J}_\phi$, and let $\mathsf{E}_{\mathfrak{J}} \coloneqq \{\mathsf{j}_\phi \mid \phi \in \mathbb{O}_{n-1}\}$.

If $X = (X_\phi \mid \phi \in \mathbb{O}_{n-1})$ is a set over $\mathbb{O}_{n-1}$, let $\mathfrak{J}_X \coloneqq \sum_{\phi \in \mathbb{O}_{n-1}} \sum_{x \in X_\phi} \mathfrak{J}_\phi$. In the $x \in X_\phi$ component, we write j_x instead of j_ϕ, and similarly, j_x^{-1}, 0_x, and 1_x. The *Rezk interval* (without any mention of shape) is the sum $\mathfrak{J} = \mathfrak{J}_{\mathbb{O}_{n-1}} \coloneqq \sum_{\phi \in \mathbb{O}_{n-1}} \mathfrak{J}_\phi$ in $\mathcal{A}\mathrm{lg}$.

Notation 13.3.5. Let $A \in \mathcal{A}\mathrm{lg}$ be an algebra, $\phi \in \mathbb{O}_{n-1}$, and $a \in MA_{\mathsf{Y}_\phi}$ (recall that $M : \mathcal{A}\mathrm{lg} \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O})$ is the opetopic nerve of algebras, see definition 11.2.3). If $x = \mathsf{s}_{[]}\, a$ and $y = \mathsf{t}\, a$, we write $a : x \longrightarrow y$. For example, in the Rezk interval $\mathfrak{J}_\phi$, the cell j_ϕ is of the form $0_\phi \longrightarrow 1_\phi$.

Definition 13.3.6 (Rezk cylinder of an algebra)**.** Let $A \in \mathcal{A}\mathrm{lg}$, let hA_{n-1} be the subalgebra of A spanned by all identity operations, and consider the following pushout in $\mathcal{A}\mathrm{lg}$:

$$\begin{tikzcd} hA_{n-1} + hA_{n-1} \arrow[r] \arrow[d] & A + A \arrow[d, "(i_0, i_1)"] \\ \mathfrak{J}_{A_{n-1}} \arrow[r] & A' \arrow[ul, phantom, "\ulcorner", very near start] , \end{tikzcd}$$

where $\mathfrak{J}_{A_{n-1}}$ is defined in definition 13.3.4. For a a cell of A and $e \in \{0, 1\}$, write $a^{(e)} := i_e(a)$. Explicitly, A' is generated by two copies of A, and for each $a \in A_\psi$, $\psi \in \mathbb{O}_{n-1}$, an additional isomorphism $j_a : a^{(0)} \longrightarrow a^{(1)}$. The *Rezk cylinder* of A is the quotient $\mathfrak{J}A := A'/K$, where

$$K := \left\{ a^{(1)} \bigcirc_{[p_i]} j_{\mathsf{s}_{[p_i]} a} = j_{\mathsf{t}\, a} \circ_{[]} a^{(0)} \mid \omega \in \mathbb{O}_n, \omega^\bullet = \{[p_1], \ldots\}, a \in A_\omega \right\}. \tag{13.3.7}$$

See notation 11.2.35 for the $\circ$ notation. We write $A^{(e)}$ for the image of i_e, and, by abuse of notation, $i_e : A^{(e)} \hookrightarrow \mathfrak{J}A$ the obvious inclusion. It is a cylinder object in the sense of definition 12.2.1, i.e. we have a canonical factorization of the codiagonal map

$$\begin{tikzcd} A + A \arrow[rr, "\nabla"] \arrow[dr, hook, "(i_0, i_1)"'] & & A \\ & \mathfrak{J}A. \arrow[ur, "\nabla"'] & \end{tikzcd}$$

Explicitly, $\nabla : \mathfrak{J}A \longrightarrow A$ maps $a^{(e)}$ to a, for a cell $a \in A$, and j_a to id_a, if a is $(n-1)$-dimensional. Note that $(i_0, i_1) : A + A \longrightarrow \mathfrak{J}A$ is a monomorphism, since the relation K of equation (13.3.7) does not identify cells of $A^{(e)}$, for $e = 0, 1$.

Example 13.3.8. Let $\psi \in \mathbb{O}_n$. Then $\mathfrak{J}h\psi$ is generated by

(1) $\psi^{(0)}, \psi^{(1)} \in (\mathfrak{J}h\psi)_\psi$, i.e. two copies of $h\psi$,

(2) for each $[p] \in \psi^\bullet$, writing $\phi := \mathsf{s}_{[p]}\, \psi$, two cells $j_\phi, j_\phi^{-1} \in (\mathfrak{J}h\psi)_{\mathsf{Y}_{\mathsf{s}_{[p]}\,\psi}}$, i.e. one copy of $\mathfrak{J}_\phi$,

(3) $j_{\mathsf{t}\,\psi}$, $j_{\mathsf{t}\,\psi}^{-1}$, i.e. one copy of $\mathfrak{J}_{\mathsf{t}\,\psi}$,

subject to the equation (13.3.7). Likewise, for $\omega \in \mathbb{O}_{n+1}$, the cylinder $\mathfrak{J}h\omega$ is generated by two copies of $h\omega$, and one copy $\mathfrak{J}_{\mathsf{e}_{[q]}\,\omega}$ for each edge address $[q]$ of ω.

Remark 13.3.9. The cylinder $\mathfrak{J}A$ of an algebra A can be thought of the *Boardman–Vogt tensor product* [Wei11] [BV73] [May72] $\mathfrak{J} \otimes_{\mathrm{BV}} A$. Unfortunately, in the planar case, a general construction is not possible as there is no way to "shuffle the inputs". In fact, this is the only obstruction, so one could still define $A \otimes_{\mathrm{BV}} B$ as long as either A of B only have unary (i.e. endotopic) operations.

The Rezk cylinder construction of definition 13.3.6 extends to $\mathcal{P}\mathrm{sh}(\mathbb{A})$:

Definition 13.3.10 (Rezk interval in $\mathcal{P}\mathrm{sh}(\mathbb{A})$)**.** For $\phi \in \mathbb{O}_{n-1}$, recall the definition of the Rezk interval $\mathfrak{J}_\phi \in \mathcal{A}\mathrm{lg}$ from definition 13.3.4. Let $\mathfrak{I}_\phi := N\mathfrak{J}_\phi$ be the *Rezk interval* of shape $h\phi$. Write $\mathsf{i}_\phi = N\mathsf{j}_\phi : h\phi \longrightarrow \mathfrak{I}_\phi$ for the endpoint inclusion (which is unique up to isomorphism), and $\mathsf{E}_{\mathfrak{J}} := \left\{ \mathsf{i}_\phi \mid \phi \in \mathbb{O}_{n-1} \right\}$.

Definition 13.3.11 (Rezk cylinder of a presheaf over $\mathbb{A}$)**.** Let $\omega \in \mathbb{O}_{\geq n-1}$. The *Rezk cylinder* $\mathfrak{I}h\omega$ of the representable $h\omega$ is defined as the nerve $N(\mathfrak{J}h\omega)$. Extend $\mathfrak{I}$ by colimits to obtain a functor $\mathfrak{I} : \mathcal{P}\mathrm{sh}(\mathbb{A}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{A})$. The *Rezk cylinder* $\mathfrak{I}X$ of a presheaf $X \in \mathcal{P}\mathrm{sh}(\mathbb{A})$ is a cylinder object in the sense of definition 12.2.1, i.e. we have a factorization of the codiagonal map

$$\begin{tikzcd} X + X \arrow[rr, "\nabla"] \arrow[dr, hook, "{(i_0,i_1)}"'] & & X \\ & \mathfrak{I}X. \arrow[ur, "\nabla"'] & \end{tikzcd}$$

Remark 13.3.12. Explicitly, for $X \in \mathcal{P}\mathrm{sh}(\mathbb{A})$, we have

$$\mathfrak{I}X := \operatorname*{colim}_{h\omega \to X} N(\mathfrak{J}h\omega).$$

In dimension $(n-1)$ and n, $\mathfrak{I}X$ is the following pushout

$$\begin{tikzcd} X_{n-1} + X_{n-1} \arrow[r] \arrow[d, "b"'] & X_{n-1,n} + X_{n-1,n} \arrow[d] \\ N\mathfrak{J}_{X_{n-1}} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & (\mathfrak{I}X)_{n-1,n}, \end{tikzcd}$$

where b maps $x \in X_\psi$ in the first (resp. the second) component to 0_x (resp. 1_x) in $N\mathfrak{J}_x \subseteq N\mathfrak{J}_{X_{n-1}}$. The $(n+1)$-cells of $\mathfrak{I}X$ are so that in $\tau\mathfrak{I}X$, the following relation (analogous to equation (13.3.7)) holds:

$$x^{(1)} \bigcirc_{[p_i]} j_{\mathsf{s}_{[p_i]} x} = j_{\mathsf{t} x} \circ_{[]} x^{(0)}$$

for a cell $x \in X_\omega$, $\omega \in \mathbb{O}_n$, and $\omega^\bullet = \{[p_1], \ldots\}$.

With that in mind, we readily deduce the following:

Lemma 13.3.13. *For $X \in \mathcal{P}\mathrm{sh}(\mathbb{A})$ we have a canonical isomorphism $\mathfrak{J}\tau X \cong \tau\mathfrak{I}X$.*

Proposition 13.3.14. *The functorial cylinder $\mathfrak{I}$ is an elementary homotopical data (definition 12.2.3).*

Proof. Straightforward unpacking of the definition of $\mathfrak{I}$. □

Definition 13.3.15 ($\mathfrak{I}$-anodyne extension)**.** Recall from definition 13.3.10 the set $\mathsf{E}_\mathfrak{J}$ of endpoint inclusions of the Rezk intervals in $\mathcal{P}\mathrm{sh}(\mathbb{A})$. Let the class $\mathsf{An}_\mathfrak{J}$ of $\mathfrak{I}$-*anodyne extensions*, be the saturation of $\mathsf{H}_{\mathrm{inner}} \cup \mathsf{E}_\mathfrak{J}$.

Remark 13.3.16. At this stage, the term "anodyne" is a bit overloaded, so to summarize:

1. an *anodyne extension* is an element of $\mathsf{An} \subseteq \mathcal{P}\mathrm{sh}(\mathbb{\Delta})^{[1]}$, which is the saturation of the set $\mathsf{H} = \mathsf{H}_{1,1}$ of simplicial horn inclusions (example 0.3.10); we do not need to generalize this notion to the opetopic setting;
2. an *inner anodyne extension* is an element of $\mathsf{An}_{\mathrm{inner}}$, (definition 13.2.5) which is the saturation of the set $\mathsf{H}_{\mathrm{inner}}$ of inner horn inclusions (definition 13.2.5), and this terminology is compatible with the classical one in the simplicial setting;
3. an $\mathfrak{I}$-*anodyne extension* is an element of $\mathsf{An}_\mathfrak{J}$ (definition 13.3.15), which is the saturation of the set $\mathsf{H}_{\mathrm{inner}} \cup \mathsf{E}_\mathfrak{J}$; by definition, $\mathsf{An}_{\mathrm{inner}} \subseteq \mathsf{An}_\mathfrak{J}$.

Lemma 13.3.17. *Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{A})$, and consider the unit map $\eta_X : X \longrightarrow N\tau X$. If η_X is a monomorphism, then it is an inner anodyne extension.*

Proof. Let $X^{(0)} := X$ and $\iota_0 := \eta_X$. We construct a sequence $X = X^{(0)} \longrightarrow X^{(1)} \longrightarrow \cdots \longrightarrow X^{(\beta)} \longrightarrow \cdots$. Assume by induction that for an ordinal β and all $\alpha < \beta$, we have a factorization

$$X \xrightarrow{\in \mathsf{An}_{\text{inner}}} X^{(\alpha)} \xrightarrow{\iota_\alpha} N\tau X$$

of η_X, and that ι_α is injective. If β is a limit ordinal, simply set $X^{(\beta)} := \operatorname{colim}_{\alpha<\beta} X^{(\alpha)}$, and ι_β to be the induced inclusion.

Assume that β is a successor ordinal, say $\beta = \alpha+1$, choose an unsolved inner horn lifting problem $l : \Lambda^e[h\omega] \longrightarrow X^{(\alpha)}$ with $\omega \in \mathbb{O}_{n+1}$, and write $e : h\nu \longrightarrow h\omega$ for the corresponding inner face. If such a lifting problem does not exist, simply set $X^{(\alpha+1)} := X^{(\alpha)}$ and $\iota_{\alpha+1} := \iota_\alpha$. Otherwise, freely adjoin a solution to l by the means of the following pushout

$$\begin{tikzcd}
\Lambda^e[h\omega] \arrow[r, "l"] \arrow[d, "\mathsf{h}^e_{h\omega}"] & X^{(\alpha)} \arrow[d, "i"] \arrow[ddr, bend left, "\iota_\alpha"] & \\
h\omega \arrow[r, "\bar{l}"] \arrow[drr, bend right, "\exists!"] & Y \arrow[dr, dashed, "j"] \arrow[ul, phantom, "\ulcorner"] & \\
& & N\tau X,
\end{tikzcd}$$

and let $j : Y \longrightarrow N\tau X$ be the map induced by ι_α, and the unique lift of $\iota_\alpha l$ along $\mathsf{h}^e_{h\omega}$ (recall that by proposition 13.2.12, $\mathsf{H}_{\text{inner}} \perp N\tau X$). One of the following cases occurs.

(1) If j is a monomorphism, simply set $X^{(\alpha+1)} := Y$ and $\iota_{\alpha+1} := j$.

(2) Assume that j is not injective. Since j is injective on $X^{(\alpha)}$ (as j extends ι_α which is injective by induction), there must exist $x \in X^{(\alpha)}$ and $y \in Y - X^{(\alpha)}$ such that $j(x) = j(y)$. Because l was not solved in $X^{(\alpha)}$, the offending cell y cannot be $\bar{l} \in Y_{h\omega}$, thus it is necessarily $y := e(\bar{l})$. Consider the coequalizer

$$h\nu \underset{y}{\overset{x}{\rightrightarrows}} Y \xrightarrow{r} X^{(\alpha+1)}$$

that collapses y onto x. Then r has a section s that maps the class $[x] = [y]$ to y. Thus r exhibits $X^{(\alpha+1)}$ as a retract of Y, and the map $ri : X^{(\alpha)} \longrightarrow X^{(\alpha+1)}$ as a retract of i:

$$\begin{tikzcd}
X^{(\alpha)} \arrow[r, equal] \arrow[d, "ri"'] & X^{(\alpha)} \arrow[d, "i"] \\
X^{(\alpha+1)} \arrow[r, "s"] & Y, \arrow[l, bend left, "r"]
\end{tikzcd}$$

where s is a section of r. In particular, the composite $X \longrightarrow X^{(\alpha)} \xrightarrow{ri} X^{(\alpha+1)}$ is an inner anodyne extension. Note that $\iota_{\alpha+1} := js$ is injective and extends ι_α.

In all cases, we constructed a factorization

$$X \xrightarrow{\in \mathsf{An}_{\text{inner}}} X^{(\alpha+1)} \xrightarrow{\iota_{\alpha+1}} N\tau X$$

of η_X, where $\iota_{\alpha+1}$ is injective. If κ is the cardinal of the set of horn lifting problems of $N\tau X$ (solved or unsolved), then the sequence X_α stabilizes after κ steps, as all lifting problems have been solved. Clearly, $X^{(\kappa)} = N\tau X$, and by construction, η_X is an inner anodyne extension. $\square$

Corollary 13.3.18. *Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$, $A \in \mathcal{A}\mathrm{lg}$, and $m : X \longrightarrow NA$ be a monomorphism such that its transpose $\bar{m} : \tau X \longrightarrow A$ is an isomorphism. Then m is an inner anodyne extension.*

Proof. The condition states that up to isomorphism, m is the unit map $\eta_X : X \longrightarrow N\tau X$. We can apply lemma 13.3.17 to conclude. $\square$

Proposition 13.3.19. *The pair $(\mathfrak{J}, \mathsf{An}_{\mathfrak{J}})$ satisfies condition* **(An1)** *of definition 12.2.4, i.e. for every monomorphism $m : X \longrightarrow Y$ in $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$, and $e = 0, 1$, the cocartesian gap map g is $\mathfrak{J}$-anodyne:*

$$\begin{tikzcd}
X \arrow[r, "m"] \arrow[d, "i_e"'] & Y \arrow[d, "i_e"] \arrow[ddr, bend left, "i_e"] & \\
\mathfrak{J}X \arrow[r] \arrow[drr, bend right, "\mathfrak{J}m"'] & \mathfrak{J}X \cup Y^{(e)} \arrow[dr, dashed, "g"] \arrow[ul, phantom, "\ulcorner"] & \\
& & \mathfrak{J}Y.
\end{tikzcd}$$

Proof. By proposition 13.2.1, it is enough to check the claim when m is a boundary inclusion, say $m = \mathsf{b}_{h\omega} : \partial h\omega \longrightarrow h\omega$ with $\omega \in \mathbb{O}_{n+1}$.

(1) Assume $\deg h\omega = 0$, i.e. ω is degenerate, say $\omega = \mathsf{I}_\phi$. Then $\partial h\omega = \varnothing$, and since $\mathfrak{J}$ preserves colimits, $\mathfrak{J}\partial h\omega = \varnothing$ as well. Thus g is the inclusion $h\omega^{(e)} = h\mathsf{I}_\phi = h\phi \longrightarrow \mathfrak{J}h\omega = \mathfrak{J}_\phi$, i.e. an endpoint inclusion of the Rezk interval $\mathfrak{J}_\phi$, which by definition is $\mathfrak{J}$-anodyne

(2) Assume $\deg h\omega \geq 1$. We only treat the case $e = 0$, the other one being similar. Let $[p] \in \omega^\bullet$, and consider the generator $c_{\omega,[p]}$ of $h\omega$ (see notation 13.1.8). By equation (13.3.7), in $\mathfrak{J}h\omega$, we have

$$c^{(1)}_{\omega,[p]} = j_{\mathsf{t}\,a} \underset{[]}{\circ} c^{(0)}_{\omega,[p]} \bigcirc_{[q_i]} j^{-1}_{\mathsf{s}_{[q_i]}\,a}$$

where $[q_i]$ ranges over $(\mathsf{s}_{[p]}\,\omega)^\bullet$. Therefore, $\mathfrak{J}h\omega$ is freely generated by $\mathfrak{J}\partial h\omega \cup h\omega^{(0)}$. By corollary 13.3.18, g is an inner anodyne extension. $\square$

Proposition 13.3.20. *The pair $(\mathfrak{J}, \mathsf{An}_{\mathfrak{J}})$ satisfies condition* **(An2)** *of definition 12.2.4, i.e. if $m : X \longrightarrow Y$ is an $\mathfrak{J}$-anodyne extension, then so is the cocartesian gap map g:*

$$\begin{tikzcd}
X + X \arrow[r, "m+m"] \arrow[d, "{(i_0,i_1)}"'] & Y + Y \arrow[d] \arrow[ddr, bend left, "{(i_0,i_1)}"] & \\
\mathfrak{J}X \arrow[r] \arrow[drr, bend right, "\mathfrak{J}m"'] & \mathfrak{J}X \cup (Y + Y) \arrow[dr, dashed, "g"] \arrow[ul, phantom, "\ulcorner"] & \\
& & \mathfrak{J}Y.
\end{tikzcd}$$

Proof. By proposition 13.2.1, it is enough to check the claim when m is an inner horn inclusion or an endpoint inclusion of a Rezk interval.

(1) Assume m is a inner horn inclusion, say $m = \mathsf{h}^e_{h\omega} : \Lambda^e[h\omega] \longrightarrow h\omega$, where $\omega \in \mathbb{O}_{n+1}$. Recall that by proposition 13.2.12, $\tau(\Lambda^e[h\omega]) = h\omega$. Thus applying τ to the diagram of definition 12.2.3 **(An2)** yields

$$\begin{tikzcd}
h\omega + h\omega \arrow[r, equal] \arrow[d] & h\omega + h\omega \arrow[d] \arrow[ddr, bend left] & \\
\mathfrak{J}h\omega \arrow[r, equal] \arrow[rrd, equal, bend right] & \mathfrak{J}h\omega \arrow[ul, phantom, "\ulcorner" very near start] \arrow[dr, dashed, "\tau g"] & \\
 & & \mathfrak{J}h\omega .
\end{tikzcd}$$

Therefore $g : \mathfrak{J}(\Lambda^e[h\omega]) \cup (h\omega + h\omega) \longrightarrow \mathfrak{J}h\omega = N\mathfrak{J}h\omega$ is such that its transpose

$$\mathfrak{J}h\omega \xrightarrow{\tau g} \tau N(\mathfrak{J}h\omega) \xrightarrow{\cong} \mathfrak{J}h\omega$$

is an isomorphism. By corollary 13.3.18, g is an inner anodyne extension.

(2) Assume m is an endpoint inclusion of a Rezk interval, say $m = \mathsf{i}_\phi : Nh\phi \hookrightarrow N\mathfrak{J}_\phi$, for a $\phi \in \mathbb{O}_{n-1}$. Write $X := Nh\phi$ and $Y := N\mathfrak{J}_\phi$. Note that

$$\mathfrak{J}X \cup (Y + Y) \;=\; \mathfrak{J}(Nh\phi) \cup (N\mathfrak{J}_\phi + N\mathfrak{J}_\phi) \;=\; \mathfrak{J}X \cup (Y + Y).$$

Further, $\mathfrak{J}Y = \mathfrak{J}(N\mathfrak{J}_\phi) = N(\mathfrak{J}\tau(N\mathfrak{J}_\phi)) \cong N(\mathfrak{J}\mathfrak{J}_\phi)$.
The algebra $\mathfrak{J}\mathfrak{J}_\phi$ contains four $(n-1)$-cells 00_ϕ, 01_ϕ, 10_ϕ, and 11_ϕ of shape ϕ, and is generated by the n-cells $j^{(0*)}_\phi : 00_\phi \longrightarrow 01_\phi$, $j^{(1*)}_\phi : 10_\phi \longrightarrow 11_\phi$, $j^{(*0)}_\phi : 00_\phi \longrightarrow 10_\phi$, $j^{(1*)}_\phi : 10_\phi \longrightarrow 11_\phi$ and their inverses. Further, the equality on the left holds, which can be depicted as a commutative square of invertible arrows on the right:

$$j^{(*1)}_\phi \underset{[]}{\circ} j^{(0*)}_\phi \;=\; j^{(1*)}_\phi \underset{[]}{\circ} j^{(*0)}_\phi , \qquad \begin{tikzcd}
00_\phi \arrow[r, "j^{(0*)}_\phi"] \arrow[d, "j^{(*0)}_\phi"'] & 01_\phi \arrow[d, "j^{(*1)}_\phi"] \\
10_\phi \arrow[r, "j^{(1*)}_\phi"] & 11_\phi .
\end{tikzcd}$$

On the other hand, the pushout $\mathfrak{J}X \cup (Y + Y)$ contains $j^{(0*)}_\phi$, $j^{(*0)}_\phi$, $j^{(*1)}_\phi$, and their inverses. Thus it freely generates $\mathfrak{J}\mathfrak{J}_\phi$, i.e. g satisfies the conditions of corollary 13.3.18. Consequently, it is an inner anodyne extension. □

Theorem 13.3.21. *The category $\mathbb{A}$ with the functorial cylinder $\mathfrak{J}$ (definition 13.3.11) and the class $\mathsf{An}_\mathfrak{J}$ of $\mathfrak{J}$-anodyne extensions (definition 13.3.15) forms a homotopical structure (definition 12.2.6). Using theorem 12.2.8, we obtain the following* model structure for ∞-algebras $\mathcal{P}\mathrm{sh}(\mathbb{A})_\infty$ *on $\mathcal{P}\mathrm{sh}(\mathbb{A})$:*

1. *a morphism f is a* naive fibration *if $\mathsf{An}_\mathfrak{J} \pitchfork f$ (definition 13.3.15); a presheaf $X \in \mathcal{P}\mathrm{sh}(\mathbb{A})$ is fibrant if the terminal morphism $X \longrightarrow 1$ is a naive fibration;*
2. *a morphism $f : X \longrightarrow Y$ is a weak equivalence if for all fibrant object $P \in \mathcal{P}\mathrm{sh}(\mathbb{A})$, the induced map $f^* : \mathrm{ho}\,\mathcal{P}\mathrm{sh}(\mathbb{A})(Y, P) \longrightarrow \mathrm{ho}\,\mathcal{P}\mathrm{sh}(\mathbb{A})(X, P)$ is a bijection;*
3. *a morphism f is a cofibrations if it is a monomorphism.*

In particular, $\mathcal{P}\mathrm{sh}(\mathbb{A})_\infty$ *is of Cisinski type (definition 12.2.7), cellular, and proper. Fibrant objects in* $\mathcal{P}\mathrm{sh}(\mathbb{A})_\infty$ *are called* ∞-algebras *(or* inner Kan complexes*).*

Proof. By definition, $\mathsf{An}_{\mathfrak{J}}$ is the class of relative cell complexes over a set of monomorphisms, thus it satisfies axiom **(An0)**. Axioms **(An1)** and **(An2)** have been established by propositions 13.3.19 and 13.3.20 respectively. ◻

This construction is a direct generalization of the *Joyal model structure* $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})_{\mathrm{Joyal}}$ for quasi-categories [JT07, theorem 1.9] [Ber18, theorem 7.1.7 and section 7.3] [Cis19, definition 3.3.7], and of the *Cisinski–Moerdijk model structure* $\mathcal{P}\mathrm{sh}(\mathbb{\Omega})_{\mathrm{CM}}$ for planar ∞-operads [CM11b, theorem 2.4]. The following results come as “sanity checks” for $\mathcal{P}\mathrm{sh}(\mathbb{A})_\infty$.

Lemma 13.3.22. *A presheaf* $X \in \mathcal{P}\mathrm{sh}(\mathbb{A})$ *is an* ∞*-algebra is and only if* $\mathsf{H}_{\mathrm{inner}} \pitchfork X$.

Proof. It is enough to show that if $\mathsf{H}_{\mathrm{inner}} \pitchfork X$, then $\mathsf{E}_{\mathfrak{J}} \pitchfork X$. So assume that $\mathsf{H}_{\mathrm{inner}} \pitchfork X$ let $\phi \in \mathbb{O}_{n-1}$, and consider a lifting problem $x : h\phi \longrightarrow X$. Recall from definition 13.2.4 that $h\mathsf{I}_\phi$ has a unique inner horn, and that the associated horn inclusion $\mathsf{h} : \Lambda[h\omega] \longrightarrow h\omega$, and let $\bar{x}$ be the solution on the left:

$$
\begin{tikzcd}
h\phi \arrow[r, "x"] \arrow[d, "\mathsf{h}_{\mathsf{I}_\phi}"'] & X \\
h\mathsf{I}_\phi . \arrow[ur, "\bar{x}"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
h\phi \arrow[r, "x"] \arrow[d, "\mathsf{i}_\phi"'] & X \\
\mathfrak{J}_\phi . \arrow[ur, "l"'] &
\end{tikzcd}
$$

Then the map $l : \mathfrak{J}_\phi \longrightarrow X$ mapping j_ϕ and j_ϕ^{-1} (see definition 13.3.4 for notations) to $\mathsf{t}\,\bar{x}$ is the desired solution as on the right[2]. ◻

Proposition 13.3.23. *The nerve* NA *of an algebra* $A \in \mathcal{A}\mathrm{lg}$ *is an* ∞*-algebra.*

Proof. By theorem 11.1.39, $\mathsf{S} \perp NA$, and thus by proposition 13.2.12, $\mathsf{H}_{\mathrm{inner}} \perp NA$. Apply lemma 13.3.22 to conclude. ◻

13.4 SIMPLICIAL ACTIONS

The category $\mathcal{P}\mathrm{sh}(\mathbb{A})$ admits a natural simplicial tensor and cotensor that behave well with the model structure of theorem 13.3.21. In this section, we establish some technical results that shall be of use in subsequent proofs.

Definition 13.4.1. For $k \in \mathbb{N}$ and $\lambda \in \mathbb{A}$, let $\Delta[k] \otimes \lambda$ be the nerve

$$\Delta[k] \otimes \lambda := N\left(\mathfrak{J}\lambda \coprod_\lambda \mathfrak{J}\lambda \coprod_\lambda \mathfrak{J}\lambda \coprod_\lambda \cdots \coprod_\lambda \mathfrak{J}\lambda \right),$$

where there is k instances of $\mathfrak{J}\lambda$. In other words, it is the nerve of k instances of the (algebraic) cylinder $\mathfrak{J}\lambda$ “glued end-to-end”. Extending in both variables by colimits yields a tensor product $- \otimes - : \mathcal{P}\mathrm{sh}(\mathbb{\Delta}) \times \mathcal{P}\mathrm{sh}(\mathbb{A}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{A})$.

[2] In fact, $\mathsf{t}\,\bar{x}$ is the identity cell id_x of notation 11.2.35, as the horn inclusion $\mathsf{h}_{\mathsf{I}_\phi} : h\phi \longrightarrow h\mathsf{I}_\phi$ is the spine inclusion $\mathsf{s}_{\mathsf{I}_\phi}$.

Remark 13.4.2. Let us unfold definition 13.4.1 a little bit. Take $X \in \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ and $K \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$. Then for each $x \in X_{h\phi}$ with $\phi \in \mathbb{O}_{n-1}$, and $k \in K_0$, there is a cell $k \otimes x \in (K \otimes X)_{h\phi}$. For every edge $e \in K_1$, there is an *internal isomorphism* (an operation that is invertible, see definition 14.1.6 for a precise definition) $e \otimes x : d_1(e) \otimes x \longrightarrow d_0(e) \otimes x$. More generally, for every m-cell $k \in K_m$, write $k_0, \ldots, k_m \in K_0$ for its vertices, and $k_{i,j}$ for its edge from k_i to k_j, where $0 \leq i < j \leq m$. We have a cell

$$k \otimes x \in (K \otimes X)_{h\omega}, \qquad \omega := \mathsf{Y}_{\mathsf{Y}_\phi} \underset{[]}{\circ} \mathsf{Y}_{\mathsf{Y}_\phi} \underset{[]}{\circ} \cdots \underset{[]}{\circ} \mathsf{Y}_{\mathsf{Y}_\phi},$$

such that $\mathsf{s}_{[*^i]}(k \otimes x) = k_{i,i+1} \otimes x$ is an internal isomorphism $k_i \otimes x \longrightarrow k_j \otimes x$.

With this description, it is clear that for $\mathfrak{I}_\blacksquare \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ (which is the nerve of the groupoid generated by one isomorphism), we have $\mathfrak{I}X \cong \mathfrak{I}_\blacksquare \otimes X$ for all $X \in \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$.

Definition 13.4.3. A mapping space and cotensor can be constructed from the tensor product $- \otimes -$ of definition 13.4.1 so as to make $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ enriched, tensored and cotensored over $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$:

$$\mathrm{map}(X, Y)_k := \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})(\Delta[k] \otimes X, Y), \qquad (Y^K)_\lambda := \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})(K \otimes \lambda, Y),$$

where $X, Y \in \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$, $K \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$, $\lambda \in \mathbb{\Lambda}$, and $k \in \mathbb{N}$.

Lemma 13.4.4. *For $X, Y \in \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ and $K \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$, consider the natural* hom-*set isomorphism*

$$\Phi : \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})(K \otimes X, Y) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})(X, Y^K)$$

of the adjunction $K \otimes - : \mathcal{P}\mathrm{sh}(\mathbb{\Lambda}) \overset{\perp}{\rightleftarrows} \mathcal{P}\mathrm{sh}(\mathbb{\Lambda}) : (-)^K$. The map Φ preserves and reflects the $\mathfrak{I}$-homotopy relation (definitions 12.2.2 and 13.3.11), i.e. it induces an isomorphism

$$\Phi : \mathrm{ho}\,\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})(K \otimes X, Y) \longrightarrow \mathrm{ho}\,\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})(X, Y^K).$$

Proof. It is enough to show that Φ preserves and reflects the elementary $\mathfrak{I}$-homotopy relation. Let $f, g : K \otimes X \longrightarrow Y$ be elementary homotopic maps, i.e. such that there exists a homotopy $H : \mathfrak{I}(K \otimes X) \longrightarrow Y$ making the following triangle commute:

$$\begin{tikzcd}
(K \otimes X) + (K \otimes X) \arrow[d, "{(i_0, i_1)}"'] \arrow[dr, "f+g"] & \\
\mathfrak{I}(K \otimes X) \arrow[r, "H"'] & Y.
\end{tikzcd}$$

Note that $(K \otimes X) + (K \otimes X) \cong K \otimes (X + X)$, and $\mathfrak{I}(K \otimes X) \cong \mathfrak{I}_\blacksquare \otimes K \otimes X \cong (\mathfrak{I}_\blacksquare \times K) \otimes X \cong (K \times \mathfrak{I}_\blacksquare) \otimes X \cong K \otimes \mathfrak{I}X$. Under the adjunction, the triangle above transposes as

$$\begin{tikzcd}
X + X \arrow[d, "{(i_0, i_1)}"'] \arrow[dr, "\Phi f + \Phi g"] & \\
\mathfrak{I}X \arrow[r, "\Phi H"'] & Y^K,
\end{tikzcd}$$

exhibiting a homotopy from Φf to Φg. Reflection of homotopies is proved similarly. □

Lemma 13.4.5. *Let $K \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$.*

(1) *For $\omega \in \mathbb{O}_{\geq n-1}$, and e an inner face of $h\omega$, the map $K \otimes \mathsf{h}^e_{h\omega} : K \otimes \Lambda^e[h\omega] \longrightarrow K \otimes h\omega$ is an inner anodyne extension.*

(2) *For $\phi \in \mathbb{O}_{n-1}$, the map $K \otimes \mathsf{i}_\phi : K \otimes h\phi \longrightarrow K \otimes \mathfrak{J}_\phi$ is a $\mathfrak{J}$-anodyne extension.*

Proof. Since $\mathsf{An}_{\text{inner}}$ and $\mathsf{An}_{\mathfrak{J}}$ are closed under colimits (in $\mathcal{P}\text{sh}(\mathbb{A})^{[1]}$), it is enough to check the claims when K is representable, say $K = \Delta[m]$, where $m \in \mathbb{N}$. But since $\Delta[m] \otimes X$ is a $(\mathbb{\Delta}/S[m])$-indexed colimit involving $\Delta[0] \otimes X$ and $\Delta[1] \otimes X$, it is enough to check the claims in the case $m = 0, 1$. If $m = 0$, then $\Delta[0] \otimes -$ is the identity functor, and both claims are tautological. If $m = 1$, then $\Delta[1] \otimes - \cong \mathfrak{J}-$, and the claims follows from lemma 12.2.5. $\square$

Corollary 13.4.6. *For $K \in \mathcal{P}\text{sh}(\mathbb{\Delta})$, the functor $(-)^K$ preserves naive fibrations (definition 12.2.7), and in particular, ∞-algebras.*

Proof. Let $\mathsf{L} := \mathsf{H}_{\text{inner}} \cup \mathsf{E}_{\mathfrak{J}}$, so that by definition $\mathsf{An}_{\mathfrak{J}} = {}^{\pitchfork}(\mathsf{L}^{\pitchfork})$. Let p be a naive fibration, i.e. a map such that $\mathsf{An}_{\mathfrak{J}} \pitchfork p$, or equivalently, such that $\mathsf{L} \pitchfork p$. By lemma 13.4.5, we have $K \otimes \mathsf{L} \subseteq \mathsf{An}_{\mathfrak{J}}$, so $K \otimes \mathsf{L} \pitchfork p$, and by adjunction, $\mathsf{L} \pitchfork p^K$. Consequently, p^K is a naive fibration. $\square$

Lemma 13.4.7. (1) *For $K \in \mathcal{P}\text{sh}(\mathbb{\Delta})$, the tensor $K \otimes - : \mathcal{P}\text{sh}(\mathbb{A})_\infty \longrightarrow \mathcal{P}\text{sh}(\mathbb{A})_\infty$ preserves cofibrations and weak equivalences.*

(2) *Let $X \in \mathcal{P}\text{sh}(\mathbb{A})$ be an ∞-algebra. Then $- \otimes X : \mathcal{P}\text{sh}(\mathbb{\Delta})_{\text{Quillen}} \longrightarrow \mathcal{P}\text{sh}(\mathbb{A})_\infty$ preserves cofibrations, and weak equivalences between Kan complexes.*

Proof. (1) Clearly, $K \otimes -$ preserves monomorphisms. Let $u : X \longrightarrow Y$ be a weak equivalence, and $P \in \mathcal{P}\text{sh}(\mathbb{A})$ be an ∞-algebra. Recall from lemma 13.4.4 that we have a natural isomorphism

$$\Phi : \text{ho}\,\mathcal{P}\text{sh}(\mathbb{A})(K \otimes X, Y) \longrightarrow \text{ho}\,\mathcal{P}\text{sh}(\mathbb{A})(X, Y^K),$$

where Φ is the natural hom-set isomorphism of the adjunction $K \otimes - \dashv (-)^K$. Therefore, we have the following naturality square:

$$\begin{tikzcd}
\text{ho}\,\mathcal{P}\text{sh}(\mathbb{A})(K \otimes Y, P) \arrow[r, "(K\otimes u)^*"] \arrow[d, "\Phi"'] & \text{ho}\,\mathcal{P}\text{sh}(\mathbb{A})(K \otimes X, P) \arrow[d, "\Phi"] \\
\text{ho}\,\mathcal{P}\text{sh}(\mathbb{A})(Y, P^K) \arrow[r, "u^*"] & \text{ho}\,\mathcal{P}\text{sh}(\mathbb{A})(X, P^K).
\end{tikzcd}$$

The vertical maps are bijections. By corollary 13.4.6, P^K is an ∞-algebra, thus u^* is a bijection as well. Therefore, $(K \otimes u)^*$ is a bijection for all ∞-algebras P. By definition, $K \otimes u$ is a weak equivalence.

(2) Clearly, $- \otimes X$ preserves monomorphisms. Let $w : K \longrightarrow L$ be a weak equivalence between Kan complexes. By proposition 12.1.5, it is a homotopy equivalence, meaning that it admits a homotopy inverse $w^{-1} : L \longrightarrow K$ up to homotopy, and $w^{-1} \otimes X$ is a homotopy inverse of $w \otimes X$. $\square$

Corollary 13.4.8. *Let $u : K \longrightarrow L$ be a cofibration between Kan complexes, $v : X \longrightarrow Y$ be a cofibration in $\mathcal{P}\text{sh}(\mathbb{A})$, and consider the Leibniz tensor $u \hat{\otimes} v$ (definition 12.3.7). Then $u \hat{\otimes} v$ is a cofibration. If either u or v is an acyclic cofibration, then so is $u \hat{\otimes} v$.*

Proof. Surely, since u and v are monomorphisms, $u \hat{\otimes} v$ is too. Assume that u is an acyclic cofibration. By lemma 13.4.7, $u \otimes X$ and $u \otimes Y$ are too, and so is the pushout u' of $u \otimes X$ along $K \otimes v$. By 3-for-2, $u \hat{\otimes} v$ is a weak equivalence. The case where v is an acyclic cofibration instead of u is proved similarly. □

Proposition 13.4.9. *Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{A})$ be an ∞-algebra, and $v : K \longrightarrow L$ be a cofibration (resp. an acyclic cofibration) between Kan complexes. Then $X^v : X^L \longrightarrow X^K$ is a fibration (resp. an acyclic fibration).*

Proof. Assume that v is a cofibration (resp. an acyclic cofibration). In order to show that X^v is a fibration (resp. an acyclic fibration), we must show that $u \pitchfork X^v$ for every acyclic cofibration (resp. all cofibration) u in $\mathcal{P}\mathrm{sh}(\mathbb{A})_\infty$. This is equivalent to $u \hat{\otimes} v \pitchfork X$. By corollary 13.4.8, $u \hat{\otimes} v$ is an acyclic cofibration, and since X is an ∞-algebra, the result holds. □

Remark 13.4.10. Unfortunately, the model structure $\mathcal{P}\mathrm{sh}(\mathbb{A})_\infty$, together with the tensor and cotensor of definitions 13.4.1 and 13.4.3 cannot be promoted into a simplicial model category. In fact, this already fails if $n = 1$ [JT07, section 6]. However, $\mathcal{P}\mathrm{sh}(\mathbb{A})_\infty$ is Quillen equivalent to a model structure that is simplicial, see theorems 15.2.8 and 15.2.11.

13.5 WHAT ABOUT $\mathbb{O}$?

Recall from definition 3.4.2 that in $\mathbb{O}$, morphisms are (iterated) face embeddings of opetopes. Therefore, the geometry of opetopic sets is fairly intuitive, as opposed to presheaves over $\mathbb{A}$, and it is natural to ask wether they enjoy a model structure similar to $\mathcal{P}\mathrm{sh}(\mathbb{A})_\infty$, while still modeling ∞-algebras. Unfortunately, severe obstructions are in the way. First, note that $\mathbb{O}$ is skeletal (definition 12.2.10), where the degree map $\mathbb{O} \longrightarrow \mathbb{N}$ maps an opetopic to its dimension, where $\mathbb{O}_+ := \mathbb{O}$, and where $\mathbb{O}_-$ only contains the identity maps. That last point means that $\mathbb{O}$ does not have "degeneracy maps", so in general, there is no way to "create identity cells". More formally we shall see that because of this limitation, there is no good notion of cylinder object in $\mathcal{P}\mathrm{sh}(\mathbb{O})$. This argument was first presented in notes of Van den Berg [VdB13] regarding semi-simplicial sets, which are presheaves over $\mathbb{\Delta}_+$ (for the usual Reedy structure on $\mathbb{\Delta}$).

Lemma 13.5.1 (Van den Berg argument)**.** *Recall from definition 12.1.21 that a Reedy category $\mathcal{R}$ is direct if its subcategory $\mathcal{R}_-$ of decreasing morphisms is discrete. It is* connected *if it has a unique object of degree 0, say z. If $\mathcal{R}$ is direct and connected, then there is no model structure on $\mathcal{P}\mathrm{sh}(\mathcal{R})$ satisfying all of the following three conditions:*

- **(VdB1)** *the representable presheaf $\mathsf{y}z$ is cofibrant;*
- **(VdB2)** *there exists a fibrant presheaf P with $\# P_z \geq 2$, such that every cofibrant object A, there is only one map $A \longrightarrow P$ up to homotopy;*
- **(VdB3)** *there exists an object with more than one connected component, i.e. a fibrant object P such that for every cofibrant object A with $\# A_z > 0$, there are at least two non-homotopic maps $A \longrightarrow P$.*

Proof. Towards a contradiction, assume there is such a model structure. Consider a cylinder object of $Z := \mathrm{y}z$ as in

$$\begin{tikzcd} Z + Z \arrow[rr, "\nabla"] \arrow[dr, "(i_0, i_1)"'] & & Z \\ & \mathrm{I}Z, \arrow[ur, "\nabla"'] & \end{tikzcd}$$

where ∇ is a weak equivalence. Recall that since $\mathcal{R}$ is direct, $Z_r = \varnothing$ for all $r \neq z$. Hence the mere existence of ∇ as above forces $\mathrm{I}Z$ to only have cells of shape z, i.e. $(\mathrm{I}Z)_r = \varnothing$ for all $r \neq z$. We now wonder if i_0 and i_1 map the cell id_z to the same cell of $\mathrm{I}Z$.

(1) Assume they do, and take P a presheaf as in **(VdB2)**. By **(VdB1)**, Z is cofibrant, and the assumption on P means that there is two distinct but left-homotopic maps $f, g : Z \longrightarrow P$. However, the homotopy cannot be witnessed by any map $\mathrm{I}Z \longrightarrow P$.

(2) Assume now $i_0(\mathrm{id}_z) \neq i_1(\mathrm{id}_z)$, and take P a presheaf as in **(VdB3)**. Again, Z is cofibrant, and the disconnectedness condition entails that there exists two non-homotopic maps $f, g : Z \longrightarrow P$. Consider $H : \mathrm{I}Z \longrightarrow P$ mapping $i_0(\mathrm{id}_z)$ to $f(\mathrm{id}_z)$, $i_1(\mathrm{id}_z)$ to $g(\mathrm{id}_z)$, and the other cells of $\mathrm{I}Z$ (necessarily all of shape z by the remark above) arbitrarily. Then $Hi_0 = f$ and $Hi_1 = g$, thus H is a homotopy between f and g, a contradiction. □

Lemma 13.5.2. *Let $\mathcal{M}$ be a model category. A fibrant object P is* contractible *if the terminal map $P \longrightarrow 1$ is a homotopy equivalence. If P is contractible, then for every cofibrant object A, there is only one map $A \longrightarrow P$ up to homotopy.*

Proof. Let $r : 1 \longrightarrow P$ be a homotopy inverse to $! : P \longrightarrow 1$. For A a cofibrant object and $f, g : A \longrightarrow P$, we have $f \simeq r\,!\,f = r\,!\,g \simeq g$. □

Proposition 13.5.3. *Let $k = n = 1$, so that $\mathcal{P}\mathrm{sh}(\mathbb{A}_{1,1})_\infty = \mathcal{P}\mathrm{sh}(\Delta)_{\mathrm{Joyal}}$. There is no Cisinski model structure on $\mathcal{P}\mathrm{sh}(\mathbb{O})$ such that the adjunction $h_! : \mathcal{P}\mathrm{sh}(\mathbb{O}) \rightleftarrows \mathcal{P}\mathrm{sh}(\Delta)_{\mathrm{Joyal}} : h^*$ is Quillen.*

Proof. Clearly, $\mathbb{O}$ is a connected and direct Reedy category. Towards a contradiction with lemma 13.5.1, assume that such a model structure on $\mathcal{P}\mathrm{sh}(\mathbb{O})$ exists. Composing with the Quillen adjunction $\tau \dashv N$ (see proposition 14.2.4) gives a Quillen adjunction $h : \mathcal{P}\mathrm{sh}(\mathbb{O}) \rightleftarrows \mathcal{C}\mathrm{at}_{\mathrm{folk}} : M$, where $\mathcal{C}\mathrm{at}_{\mathrm{folk}}$ is the usual folk (or canonical) model structure on $\mathcal{C}\mathrm{at}$ (see theorem 14.2.1 for a general construction). In particular, for a (necessarily cofibrant) $A \in \mathcal{P}\mathrm{sh}(\mathbb{O})$ and a (necessarily fibrant) $P \in \mathcal{C}\mathrm{at}_{\mathrm{folk}}$, there is a natural isomorphism

$$\mathrm{ho}\,\mathcal{C}\mathrm{at}_{\mathrm{folk}}(hA, P) \xrightarrow{\cong} \mathrm{ho}\,\mathcal{P}\mathrm{sh}(\mathbb{O})(A, MP).$$

Since the model structure on $\mathcal{P}\mathrm{sh}(\mathbb{O})$ is of Cisinski type, it satisfies **(VdB1)**. The terminal object in $\mathcal{C}\mathrm{at}$ is $[0]$, and for $\mathfrak{J}$ the Rezk interval of definition 13.3.4 (which in this case is the groupoid $\cdot \leftrightarrow \cdot$), the inclusion $i_0 : [0] \longrightarrow \mathfrak{J}$ is clearly a retract of $\mathfrak{J} \longrightarrow [0]$ up to homotopy (i.e. up to equivalence of algebras). Consequently, $\mathfrak{J}$ is contractible, but not trivial in the sense that it has two objects. By lemma 13.5.2, $\mathfrak{J}$ is an object satisfying the condition of **(VdB2)**, and so does $M\mathfrak{J}$. Likewise, $[0] + [0]$ satisfies condition **(VdB3)**, and so does $M([0] + [0])$. Finally, the model structure on $\mathcal{P}\mathrm{sh}(\mathbb{O})$ satisfies all the requirements of lemma 13.5.1, which is not possible. □

Chapter Fourteen

The homotopy theory of opetopic algebras

THE goal of this chapter is to define the *folk model structure* on $\mathcal{A}\mathrm{lg}_{k,n}$, which is a direct generalization of that for categories [Cis19, theorem 3.3.10], and operads [Wei07, theorem 1.6.2]. Unlike most model structures that appear in the literature (folk or otherwise), this one can be constructed directly and by elementary means. Throughout this chapter, we shall use convention 11.1.34: we fix parameters $k := 1$ and $n \geq 1$, and omit them from various notations, e.g. $\mathcal{A}\mathrm{lg} = \mathcal{A}\mathrm{lg}_{1,n}$, $\mathfrak{Z} = \mathfrak{Z}^n$, etc.

14.1 PRELIMINARIES

Convention 14.1.1. Recall from theorem 11.1.39 that the adjunction $\tau : \mathcal{P}\mathrm{sh}(\mathbb{\Lambda}) \rightleftarrows \mathcal{A}\mathrm{lg} : N$ exhibits $\mathcal{A}\mathrm{lg}$ as a small orthogonality class in $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$. To simplify notation, we will shall consider $\mathcal{A}\mathrm{lg}$ as a *subcategory* of $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ via the nerve functor, i.e. an algebra $A \in \mathcal{A}\mathrm{lg}$ will ubiquitously be a presheaf over $\mathbb{\Lambda}$. Similarly, by theorem 11.2.33, $\mathcal{A}\mathrm{lg}$ is a small orthogonality class of $\mathcal{P}\mathrm{sh}(\mathbb{O})$, and we shall consider $\mathcal{A}\mathrm{lg}$ as a subcategory of $\mathcal{P}\mathrm{sh}(\mathbb{O})$ via the fully faithful functor $M : \mathcal{A}\mathrm{lg} \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O})$. In particular, if $A \in \mathcal{A}\mathrm{lg}$ and $m \in \mathbb{N}$, then

$$A_m = MA_m = \sum_{\omega \in \mathbb{O}_m} MA_\omega .$$

Definition 14.1.2 (Boundary, see definition 12.2.12)**.** Let $\lambda \in \mathbb{\Lambda}$ be an opetopic shape. The *boundary* $\partial\lambda \in \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ of λ is the colimit (in $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$)

$$\partial\lambda := \operatorname*{colim}_{\substack{f:\lambda'\to\lambda \text{ in } \mathbb{\Lambda}_+ \\ f \text{ not an iso.}}} \lambda' .$$

We write $\mathsf{b}_\lambda : \partial\lambda \longrightarrow \lambda$ for the *boundary inclusion* of λ, and $\mathsf{B} := \{\mathsf{b}_\lambda \mid \lambda \in \mathbb{\Lambda}\}$ the set of boundary inclusions of $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$.

Example 14.1.3. (1) If $n = 1$, then $\mathcal{A}\mathrm{lg} = \mathcal{C}\mathrm{at}$ is the category of small categories. The only 1-opetope is $\blacksquare$, and $\partial h \blacksquare = h \blacklozenge + h \blacklozenge$, the discrete category with two objects.

(2) If $n = 2$, then $\mathcal{A}\mathrm{lg} = \mathcal{O}\mathrm{p}_{\mathrm{col}}$ is the category of colored operads. If $\omega \in \mathbb{O}_3$ is a tree, then $\partial h\omega$ is the maximal subpresheaf of $Nh\omega$ not containing the ω, i.e. the cell witnessing the composition of all its source faces.

Notation 14.1.4. Let $A \in \mathcal{A}\mathrm{lg}$ be an algebra, $\psi \in \mathbb{O}_n$, and $d : \partial h\psi \longrightarrow A$, and consider the following pullback

$$\begin{tikzcd} A_d \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A_{h\psi} = \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})(h\psi, A) \arrow[d, "\mathsf{b}^*_{h\psi}"] \\ * \arrow[r, "d"] & \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})(\partial h\psi, A). \end{tikzcd}$$

Explicitly, A_d is the subset of $A_{h\psi}$ of all cells a such that $\mathsf{t}\,a = d(\mathsf{t}\,\psi)$, and such that for all $[q] \in \psi^\bullet$, $\mathsf{s}_{[q]}\,a = d(\mathsf{s}_{[q]}\,\psi)$.

Example 14.1.5. (1) Take $n = 1$, and recall from example 14.1.3 that $\partial h \blacksquare = h\blacklozenge + h\blacklozenge$ is the discrete category with two objects. Hence, if $\mathcal{C} \in \mathcal{C}\mathrm{at}$, a morphism $d : \partial h \blacksquare \longrightarrow \mathcal{C}$ is simply a pair of objects $(c, c') := (d\,\mathsf{s}_* \blacksquare, d\,\mathsf{t}\,\blacksquare)$ of objects of $\mathcal{C}$, and $\mathcal{C}_d = \mathcal{C}(c, c')$.

(2) Take $n = 2$, so that $\mathcal{A}\mathrm{lg}$ is the category of planar colored operads. For $P \in \mathcal{A}\mathrm{lg}$ and $\mathbf{k} \in \mathbb{O}_2$, a morphism $d : \partial h\mathbf{k} \longrightarrow P$ is just a $(k+1)$-tuple of colors $(c_0, \ldots, c_{k-1}; c_k)$ of P, where $c_i := d\,\mathsf{s}_{[*^i]}\,\mathbf{k}$ for $0 \leq i < k$, and $c_k := d\,\mathsf{t}\,\mathbf{k}$. The set P_d is simply the set of operations $P(c_0, \ldots, c_{k-1}; c_k)$.

Definition 14.1.6 (Internal isomorphism)**.** Let $A \in \mathcal{A}\mathrm{lg}$, $\phi \in \mathbb{O}_{n-1}$, $x, y \in A_{h\phi}$, and $a : x \longrightarrow y$ (see notation 13.3.5). We say that a is an *internal isomorphism* (or just *isomorphism*) if it is invertible, i.e. if there exists a cell $a^{-1} : y \longrightarrow x$ such that, using notation 11.2.35, we have

$$a \underset{[]}{\circ} a^{-1} = \mathrm{id}_y, \qquad\qquad a^{-1} \underset{[]}{\circ} a = \mathrm{id}_x .$$

In this case, we also say that x and y are *isomorphic*, and write $x \cong y$. Equivalently, a is an internal isomorphism if the following lifting problem has a solution:

$$\begin{tikzcd} h\mathsf{Y}_\phi \arrow[r, "a"] \arrow[d] & A \\ \mathfrak{I}_\phi . \arrow[ur] & \end{tikzcd}$$

Definition 14.1.7 (Natural transformation)**.** Let $f, g : A \longrightarrow B$ be two morphisms of algebras. A *natural transformation* $\alpha : f \longrightarrow g$ is the datum of a cell $\alpha_a : f(a) \longrightarrow g(a)$ for each $a \in A_{n-1}$, called a *component* of α, such that for all $\psi \in \mathbb{O}_n$ and $x \in A_{h\psi}$, the following relation holds:

$$g(x) \underset{[q]}{\bigcirc} \alpha_{\mathsf{s}_{[q]}\,a} = \alpha_{\mathsf{t}\,a} \underset{[]}{\circ} f(x).$$

Natural transformations can be composed in the obvious way, and the usual exchange law holds. A *natural isomorphism* is an invertible natural transformation, or equivalently, one whose components are all isomorphisms.

Definition 14.1.8 (Algebraic equivalence)**.** Let $f : A \longrightarrow B$ be a morphism of algebras.

(1) We say that f is *fully faithful* if for all $\psi \in \mathbb{O}_n$ and for every morphism $d : \partial h\psi \longrightarrow A$, the induced map $f_d : A_d \longrightarrow B_{fd}$ is a bijection.

(2) We say that f is *essentially surjective* if for all $b \in B_{n-1}$, there exists $a \in A_{n-1}$ such that $f(a) \cong b$.

(3) We say that f is an *algebraic equivalence* (or *equivalence of algebras*, or simply *equivalence*) if it is invertible up to natural isomorphism, i.e. there exists $g : B \longrightarrow A$ and natural isomorphisms $\varepsilon : gf \longrightarrow \mathrm{id}_A$ and $\eta : \mathrm{id}_B \longrightarrow fg$. In this case, g is called a *weak inverse* of f.

Proposition 14.1.9 (Generalization of [ML98, theorem IV.4.1] [Wei07, lemma 1.1.19])**.** *A morphism $f : A \longrightarrow B$ of algebras is an equivalence if and only if it is fully faithful and essentially surjective.*

Proof. This is similar to [ML98, theorem IV.4.1]. The direct implication is easy. For the converse, assume that f is fully faithful and essentially surjective. For $b \in B_{n-1}$, choose a cell $g(b) \in A_{n-1}$ and an isomorphism $\eta_b : b \longrightarrow f(g(b))$. Let $\psi \in \mathbb{O}_n$ and $y \in B_{h\psi}$ be such that $\mathsf{t}y$ is in the image of f, say $\mathsf{t}y = f(a)$ for some $a \in A_{n-1}$. Then the cell

$$z := y \bigcirc_{[q]} \eta^{-1}_{\mathsf{s}_{[q]} y}$$

where $[q]$ ranges over $\psi^\bullet$, has all its faces in the image of f, as $\mathsf{t}z = \mathsf{t}y$, and $\mathsf{s}_{[q]} z = \mathsf{s}_{[]} \eta^{-1}_{\mathsf{s}_{[q]} y} = f(g(\mathsf{s}_{[q]} y))$. Since f is fully faithful, there exists a unique cell in $A_{h\psi}$, which we call $g(y)$, such that $f(g(y)) = z$. Using proposition 11.2.36, this defines a morphism $g : B \longrightarrow A$, and a natural isomorphism $\eta : \mathrm{id}_B \longrightarrow fg$.

Lastly, let $a \in A_{n-1}$, and consider the isomorphism $e := \eta^{-1}_{f(a)} : fgf(a) \longrightarrow f(a)$. Since f is fully faithful, there exists a unique $\varepsilon_a : gf(a) \longrightarrow a$ such that $e = f(\varepsilon_a)$. It is straightforward to check that the ε_a are the components of a natural isomorphism $\varepsilon : gf \longrightarrow \mathrm{id}_A$. □

Lemma 14.1.10. *Let $f : A \longrightarrow B$ be a fully faithful essentially surjective morphism that is injective on $(n-1)$-cells. Then f admits a retract up to isomorphism, i.e. a weak inverse $g : B \longrightarrow A$ together with natural isomorphisms $\varepsilon : gf \longrightarrow \mathrm{id}_A$ and $\eta : \mathrm{id}_B \longrightarrow fg$ as in proposition 14.1.9, but where $gf = \mathrm{id}_A$ and ε is an identity (i.e. all its components are identities), where $f = fgf$, and where for $a \in A_{n-1}$, $\eta_{f(a)} = \mathrm{id}_{f(a)}$*

Proof. It suffices to amend the proof of proposition 14.1.9 so that g, ε and η have the desired properties. For $a \in A_{n-1}$, since f is injective on objects, we may choose $g(f(a))$ to be a, and $\eta_{f(a)}$ to be $\mathrm{id}_{f(a)}$. It follows that, after extending g to a morphism $B \longrightarrow A$, we have $gf = \mathrm{id}_A$. Further, for $a \in A_{n-1}$, $\varepsilon_a : gf(a) \longrightarrow a$ is the only n-cell of A such that $f(\varepsilon_a) = \eta^{-1}_{f(a)} = \mathrm{id}_{f(a)}$, whence ε is the identity. □

14.2 THE FOLK MODEL STRUCTURE

Theorem 14.2.1 (Generalization of [Cis19, theorem 3.3.10] and [Wei07, theorem 1.6.2])**.** *The category $\mathcal{A}\mathrm{lg} = \mathcal{A}\mathrm{lg}_{1,n}$ of 1-colored n-opetopic algebras admits a model structure where*

(1) the weak equivalences are the algebraic equivalences (definition 14.1.8);
(2) the cofibrations are the morphisms that are injective on $(n-1)$-cells;
(3) the fibrations are the morphisms $f : A \longrightarrow B$ such that $\mathsf{E}_\mathfrak{J} \pitchfork f$ (also called isofibrations*), i.e. such that for every isomorphism in B of the form $x : f(a) \longrightarrow b'$, there exist an isomorphism $y : a \longrightarrow a'$ in A such that $f(y) = x$.*

We call this structure the folk model structure *on $\mathcal{A}\mathrm{lg}$, and denote it by $\mathcal{A}\mathrm{lg}_{\mathrm{folk}}$. Furthermore, acyclic fibrations are the algebraic equivalences that are surjective on $(n-1)$-cells, and every object is both fibrant and cofibrant.*

The second claim can easily be checked once the model structure is established. The rest of this section is dedicated to proving this theorem. To that end, we verify each of Quillen's axioms, recalled in definition 12.1.1.

Proof of theorem 14.2.1, **(M0)***: limit axiom.* This is proposition 11.1.32. □

Proof of theorem 14.2.1, **(M1)***: 3-for-2 axiom.* It is easy to check that the class of morphisms that are weakly invertible satisfies 3-for-2. ☐

Proof of theorem 14.2.1, **(M2)***: retract axiom.* Let f be a retract of g as in

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[l, bend left, "r"] \arrow[d, "g"] \\ C \arrow[r, "i'"] & D \arrow[l, bend left, "r'"] \end{tikzcd}$$

where $ri = \mathrm{id}_A$ and $r'i' = \mathrm{id}_C$.

(1) If g is a cofibration, i.e. injective on $(n-1)$-cells, then clearly, so is f.

(2) Assume that g is a fibration, and $x : f(a) \longrightarrow c$ an isomorphism in C. Since $i'f(a) = gi(a)$, there exists an isomorphism $y : i(a) \longrightarrow b$ in B such that $g(y) = i'(x)$, and $r(y)$ is an isomorphism in A such that $fr(y) = x$. Therefore, f is a fibration.

(3) Assume that g is a weak equivalence. By proposition 14.1.9, this is equivalent to g being fully faithful and essentially surjective. Since f is a retract of g, it is clearly fully faithful as well. Let $c \in C_{n-1}$. Since g is essentially surjective, there exists $b \in B_{n-1}$ and an isomorphism $x : g(b) \longrightarrow i'(c)$. This induces an isomorphism $r'(x) : fr(b) \longrightarrow c$, and consequently, f is essentially surjective. ☐

Proof of theorem 14.2.1, **(M3)***: lifting axiom.* Consider a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "p"] \\ C \arrow[r, "g"] & D, \end{tikzcd}$$

where i is a cofibration and p is a fibration. We show that a lift exists whenever i or p is a weak equivalence.

(1) Assume that i is a weak equivalence, and let $r : C \longrightarrow A$ be a weak retract of i as in lemma 14.1.10, together with the natural isomorphism $\eta : \mathrm{id}_C \longrightarrow ir$. Let $c \in C_{n-1}$, and consider $gir(c) = pfr(c) \in D_{n-1}$. Since p is an isofibration, there exists an isomorphism $\beta_c : fr(c) \longrightarrow b$ in B such that $p(\beta_c) = g(\eta_c^{-1}) : gir(c) = pfr(c) \longrightarrow g(c)$. In particular, $p(b) = g(c)$, and define $l(c) \coloneqq b$. This defines a lift $l : C_{n-1} \longrightarrow B_{n-1}$.
The construction above also provides a natural isomorphism $\beta : fr \longrightarrow l$. Without loss of generality, we choose l and the components of β such that for all $a \in A_{n-1}$, $li(a) = f(a)$, and $\beta_{i(a)} = \mathrm{id}_{f(a)}$. For $\psi \in \mathbb{O}_n$ and $x \in C_{h\psi}$, let

$$l(x) \coloneqq \beta_{\mathsf{t}x} \underset{[]}{\circ} fr(x) \underset{[q]}{\bigcirc} \beta^{-1}_{\mathsf{s}_{[q]}x}.$$

It can easily be checked that l is then a morphism of algebras $C \longrightarrow B$, and finally that it provides the desired lift.

(2) Assume that p is a weak equivalence. In particular, on $(n-1)$-cells, i is an injection, and p a surjection. Thus a lift $l : C_{n-1} \longrightarrow B_{n-1}$ can be found. We now extend l to a morphism of algebras $l : C \longrightarrow B$.

Let $\psi \in \mathbb{O}_n$, $x \in C_\psi$, and d be the composite

$$\partial h\psi \xrightarrow{\mathsf{b}_{h\psi}} h\psi \xrightarrow{x} C.$$

Since p is fully faithful, it induces a bijection $p_d : C_{ld} \longrightarrow D_{gd}$. By proposition 11.2.36, letting $l(x) := p_d^{-1} g(x)$ extends l to an algebra morphism which is the desired lift. $\square$

Proof of theorem 14.2.1, **(M4)***: factorization axiom.* Let $f : A \longrightarrow B$ be a morphism of algebras.

(1) We decompose f as $f = pi$, where i is an acyclic cofibration, and where p is fibration. Define $P \in \mathcal{P}\mathrm{sh}(\mathbb{A})$ as follows. For $\phi \in \mathbb{O}_{n-1}$, let

$$P_{h\phi} := \left\{ (a, v, b) \mid a \in A_{h\phi}, b \in B_{h\phi}, v : f(a) \xrightarrow{\cong} b \right\}.$$

There is an obvious projection $p_1 : P_{n-1} \longrightarrow A_{n-1}$ mapping a tuple (a, v, b) to a. If $\psi \in \mathbb{O}_n$ and $d : \partial h\psi \longrightarrow P_{n-1}$, let $P_d := A_{p_1 d}$. At this stage, P clearly extends as an algebra, essentially inheriting the same law as A.

Let $i : A \longrightarrow P$ map an $(n-1)$-cell a to $(a, \mathrm{id}_{f(a)}, f(a))$. This completely determines i, as indeed, for an n-cell $x \in A_{\geq n}$ we necessarily have $i(x) = x$. Clearly, i is a fully faithful cofibration. It remains to show that it is essentially surjective. If $(a, v, b) \in P_{n-1}$, note that id_a exhibits an isomorphism $i(a) = (a, \mathrm{id}_{f(a)}, f(a)) \longrightarrow (a, v, b)$ in P. Therefore, i is an acyclic cofibration.

Let $p_3 : P \longrightarrow B$ be the projection mapping (a, v, b) to b. Let $\psi \in \mathbb{O}_n$ and $x \in P_{h\psi}$. Write $\mathsf{t}x = (a_\mathsf{t}, v_\mathsf{t}, b_\mathsf{t})$, for a source address $[q] \in \psi^\bullet$, write $\mathsf{s}_{[q]} x = (a_{[q]}, v_{[q]}, b_{[q]})$, and define

$$p_3(x) := v_\mathsf{t} \underset{[]}{\circ} f(x) \underset{[q]}{\bigcirc} v_{[q]}^{-1}.$$

This defines a morphism $p_3 : P \longrightarrow B$ which we claim to be a fibration. Indeed, if $(a, v, b) \in P_{n-1}$ and $w : b \longrightarrow b'$ is an isomorphism in P, we have an isomorphism $\mathrm{id}_a : (a, v, b) \longrightarrow (a, w \circ_{[]} v, b')$ in P, and

$$p_3(\mathrm{id}_{f(a)}) = (w \underset{[]}{\circ} v) \underset{[]}{\circ} f(\mathrm{id}_a) \underset{[]}{\circ} v = w.$$

Lastly, it is clear that $f = p_3 i$, and thus f decomposes as an acyclic cofibration followed by a fibration.

(2) We decompose f as $f = pi$, where i is an cofibration, and where p is an acyclic fibration. Define $C \in \mathcal{P}\mathrm{sh}(\mathbb{A})$ as follows. On $(n-1)$-cells, it is given as on the left, and let $\bar{f}$ be defined as on the right:

$$C_{n-1} := A_{n-1} + B_{n-1}, \qquad \bar{f} := (f, \mathrm{id}_{B_{n-1}}) : C_{n-1} \longrightarrow B_{n-1}.$$

Explicitly, $\bar{f}$ maps $a \in A_{n-1}$ to $f(a)$, and $b \in B_{n-1}$ to b. For $\psi \in \mathbb{O}_n$ and $d : \partial h\psi \longrightarrow C_{n-1}$, let the fiber C_d be simply $B_{\bar{f}d}$. At this stage, C clearly extends as an algebra, essentially inheriting the same law as B.

Let $i : A \longrightarrow C$ map an $(n-1)$-cell a to a, and an n-cell x to $f(x)$. Obviously, this is a cofibration. Let $p : C \longrightarrow A$ map an $(n-1)$-cell d to $\bar{f}(d)$, and an n-cell x to x. This can easily be seen to be an acyclic fibration. Lastly, $f = pi$, so that f can be decomposed into a cofibration followed by an acyclic fibration. $\square$

Remark 14.2.2. The algebras P and C above can be thought of as *mapping path space* [May99, section 7.2] and the *mapping cylinder* [May99, section 6.2] of f, respectively. In the classical model structure on topological spaces [Qui67, section II.3], those constructions are used to provide the factorizations of axiom **(M5)**.

Theorem 14.2.3. *The model structure* $\mathcal{A}\mathrm{lg}_{\mathrm{folk}}$ *is cofibrantly generated, and* $\mathsf{E}_{\mathfrak{J}}$ *can be taken as a set of generating acyclic cofibrations.*

Proof. By definition, a morphism f is an fibration if and only if $\mathsf{E}_{\mathfrak{J}} \pitchfork f$. It is straightforward to check that f is an acyclic fibration of and only if $\mathsf{J} \pitchfork f$, where J is

$$\left\{ \varnothing \longrightarrow h\phi \mid \phi \in \mathbb{O}_{n-1} \right\} + v\mathsf{B} + \left\{ h\psi \coprod_{\partial h\psi} h\psi \longrightarrow h\psi \mid \psi \in \mathbb{O}_n \right\},$$

where B is introduced in definition 14.1.2. Finally, domains of maps in $\mathsf{E}_{\mathfrak{J}}$ and J are small, since they are finite colimits of representable presheaves, which are small. In particular, $\mathsf{E}_{\mathfrak{J}}$ and J admit the small object argument. □

Proposition 14.2.4. *We have a Quillen adjunction* $\tau : \mathcal{P}\mathrm{sh}(\mathbb{A})_\infty \rightleftarrows \mathcal{A}\mathrm{lg}_{\mathrm{folk}} : N$.

Proof. Clearly, if $f : X \longrightarrow Y$ is a monomorphism in $\mathcal{P}\mathrm{sh}(\mathbb{A})$, then τf is injective on $(n-1)$-cells. Therefore, τ preserves cofibrations. To conclude, it suffices to show that N preserves fibrations. First, note that τ maps $\mathsf{H}_{\mathrm{inner}}$ (see definition 13.2.4) to isomorphisms, and since the adjunction is reflective, it maps $\mathsf{E}_{\mathfrak{J}} = N\mathsf{E}_{\mathfrak{J}}$ (see definition 13.3.10) to $\mathsf{E}_{\mathfrak{J}}$ up to isomorphism. Therefore, τ maps $\mathsf{An}_{\mathfrak{J}}$ to acyclic cofibrations. Let now f be a fibration in $\mathcal{A}\mathrm{lg}_{\mathrm{folk}}$. Then f has the right lifting property against all acyclic cofibrations, and in particular, $\tau\mathsf{An}_{\mathfrak{J}} \pitchfork f$. By adjointness, $\mathsf{An}_{\mathfrak{J}} \pitchfork Nf$, and thus Mf is a naive fibration. By proposition 13.3.23, the codomain of Nf is an ∞-algebra, and we apply lemma 12.2.9 to conclude that Nf is a fibration. □

Lemma 14.2.5. *Let* $f, g : A \longrightarrow B$ *be two parallel morphisms of algebras. The following are equivalent:*

(1) $f \simeq_{\mathfrak{J}} g$ *(see definition 12.2.2);*
(2) there is an elementary $\mathfrak{J}$*-homotopy from* f *to* g*;*
(3) there exists a natural isomorphism $f \longrightarrow g$.

Proof. *(1)* $\implies$ *(3)* A homotopy H from f to g as in definition 12.2.2 induces a natural isomorphism α with components $\alpha_a = H(j_a)$ (see definition 13.3.6 for notations), for $a \in A_{n-1}$.

(3) $\implies$ *(2)* A natural isomorphism $\alpha : f \longrightarrow g$ induces a homotopy $H : \mathfrak{J}A \longrightarrow B$ from f to g, where for $a \in A$, $H(a^{(0)}) := f(a)$ (see definition 13.3.6 for notations), $H(a^{(1)}) := g(a)$, $H(j_a) = \alpha_a$, and $H(j_a^{-1}) := \alpha_a^{-1}$.

(2) $\implies$ *(1)* By definition. □

Proposition 14.2.6. *A morphism* $f : A \longrightarrow B$ *is a weak equivalence (for the folk structure of theorem 14.2.1) if and only if it is an isomorphism in* $\mathrm{ho}\,\mathcal{A}\mathrm{lg}$ *(with respect to* $\mathfrak{J}$*, see definition 12.2.2). Therefore, the category* $\mathrm{ho}\,\mathcal{A}\mathrm{lg}$ *is the localization of* $\mathcal{A}\mathrm{lg}$ *at the class of algebraic equivalences.*

Proof. Follows from proposition 14.1.9 and lemma 14.2.5. □

Chapter Fifteen

∞-*opetopic algebras vs. complete Segal spaces*

Much in the spirit of [JT07] [CM13], we show that an adequate notion of *opetopic complete Segal space* provides a model for the homotopy theory of ∞-algebras. Formally, we show in theorem 15.2.11 that there is a Quillen equivalence between $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})_\infty$, the model structure for ∞-algebras of theorem 13.3.21, and $\mathcal{S}\mathrm{p}(\mathbb{\Lambda})_{\mathrm{Rezk}}$, the *Rezk structure* on simplicial presheaves over $\mathbb{\Lambda}$, which we construct in this chapter.

As usual, and following convention 11.1.34, we fix parameters $k := 1$ and $n \geq 1$, and omit them from various notations, e.g. $\mathbb{\Lambda} = \mathbb{\Lambda}_{1,n}$, $\mathcal{A}\mathrm{lg} = \mathcal{A}\mathrm{lg}_{1,n}$, $\tau = \tau_{1,n}$, etc.

15.1 SEGAL SPACES

Definition 15.1.1. Recall from definition 12.3.19 that $\mathcal{S}\mathrm{p}(\mathbb{\Lambda})_v$ is the Reedy model structure on the category $\mathcal{S}\mathrm{p}(\mathbb{\Lambda}) = \mathcal{P}\mathrm{sh}(\mathbb{\Delta})^{\mathbb{\Lambda}^{\mathrm{op}}}$ induced by $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})_{\mathrm{Quillen}}$. Let $\mathcal{S}\mathrm{p}(\mathbb{\Lambda})_{\mathrm{Segal}}$, the *Segal model structure* on $\mathcal{S}\mathrm{p}(\mathbb{\Lambda})$, be the left Bousfield localization

$$\mathcal{S}\mathrm{p}(\mathbb{\Lambda})_{\mathrm{Segal}} := \mathsf{S}^{-1}\mathcal{S}\mathrm{p}(\mathbb{\Lambda})_v,$$

where S is the set of spine inclusions (definition 11.1.37). This localization exists by theorem 12.1.29. Fibrant objects (resp. weak equivalences) in $\mathcal{S}\mathrm{p}(\mathbb{\Lambda})_{\mathrm{Segal}}$ are called *Segal spaces* (resp. *Segal weak equivalences*). Unfolding the definition, a Segal space $X \in \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ is a vertically fibrant space such that for all $\nu \in \mathbb{O}_{n+1}$, the map

$$\mathsf{s}_{h\nu}\backslash X : \underbrace{h\nu\backslash X}_{=X_{h\nu}} \longrightarrow S[h\nu]\backslash X$$

is a weak equivalence.

Definition 15.1.2 (Right cancellation property [JT07, section 3])**.** Let $\mathcal{C}$ be a category. A class of monomorphisms $\mathsf{K} \subseteq \mathcal{C}^{[1]}$ has the *right cancellation property* if for any composable pair or monomorphisms $f, g \in \mathcal{C}^{[1]}$, if $gf, f \in \mathsf{K}$, then $g \in \mathsf{K}$.

$$\begin{tikzcd} \cdot \arrow[rr, "gf \in \mathsf{K}"] \arrow[dr, "f \in \mathsf{K}"'] & & \cdot \\ & \cdot \arrow[ur, "\therefore g \in \mathsf{K}"'] & \end{tikzcd}$$

Lemma 15.1.3 (Generalization of [JT07, lemma 3.5] and [CM13, proposition 2.5])**.** *If* K *is a saturated class of monomorphism of* $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ *having the right cancellation property, and if* $\mathsf{S} \subseteq \mathsf{K}$*, then* $\mathsf{An}_{\mathrm{inner}} \subseteq \mathsf{K}$*.*

Proof. Since K is saturated, it is enough to show that $\mathsf{H}_{\text{inner}} \subseteq \mathsf{K}$. Let $\omega \in \mathbb{O}_{n+1}$ and $\mathsf{h}^e_{h\omega} : \Lambda^e[h\omega] \longrightarrow h\omega$ be an inner horn inclusion. By lemma 13.2.10, the spine inclusion $\mathsf{s}_\omega : S[h\omega] \longrightarrow h\omega$ decomposes as

$$S[h\omega] \xrightarrow{u} \Lambda^e[h\omega] \xrightarrow{\mathsf{h}^e_{h\omega}} h\omega$$

where u is a spine complex. By assumption, $\mathsf{s}_\omega \in \mathsf{K}$, and since K is saturated, $u \in \mathsf{K}$. By right cancellation, $\mathsf{h}^e_{h\omega} \in \mathsf{K}$. □

Proposition 15.1.4 (Generalization of [JT07, proposition 3.4]). *Let $X \in \mathcal{S}\mathrm{p}(\mathbb{A})$ be vertically fibrant. The following are equivalent:*

(1) X is a Segal space;

(2) the map $\mathsf{h}^e_{h\omega} \backslash X$ is an acyclic Kan fibration, for all $\omega \in \mathbb{O}_{n+1}$ and all inner horn inclusions $\mathsf{h}^e_{h\omega} : \Lambda^e[h\omega] \longrightarrow h\omega$;

(3) the map $u \backslash X$ is an acyclic Kan fibration (see example 12.1.10), for every inner anodyne extension $u \in \mathcal{P}\mathrm{sh}(\mathbb{A})$;

(4) the map X / b_n is an inner fibration (definition 13.2.5), for all $n \in \mathbb{N}$;

(5) the map X/v is an inner fibration, for every monomorphism $v \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$.

Proof. The equivalences (2) $\iff$ (3) $\iff$ (4) $\iff$ (5) are clear. We now show (1) $\iff$ (3).

(1) $\implies$ (3) Recall from proposition 12.3.21 that since X is vertically fibrant, $u \backslash X$ is a Kan fibration for every monomorphism u. In particular, $u \backslash X$ is an acyclic Kan fibration if and only if it is a weak equivalence.

Let K be the class of monomorphisms $u \in \mathcal{P}\mathrm{sh}(\mathbb{A})$ such that $u \backslash X$ is a weak equivalence. We show that K satisfies the conditions of lemma 15.1.3.

a) Since X is a Segal space, $\mathsf{S} \subseteq \mathsf{K}$.

b) The class of weak equivalences has 3-for-2, thus, so does K. In particular, K has the right cancellation property.

c) We now show that K is saturated. By the remark above, $u \in \mathsf{K}$ if and only if $u \backslash X$ is an acyclic Kan fibration, i.e. $\mathsf{b}_n \pitchfork u \backslash X$ for all $n \in \mathbb{N}$. But this is equivalent to $u \pitchfork X / \mathsf{b}_n$, and thus K is the intersection of the class of monomorphisms of $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ and ${}^{\pitchfork}\{X/\mathsf{b}_n \mid n \in \mathbb{N}\}$, both of which are saturated. Therefore, K is saturated.

Finally, by lemma 15.1.3, K contains all inner anodyne extensions.

(3) $\implies$ (1) By proposition 13.2.11, spine inclusions are inner anodyne extensions. □

Corollary 15.1.5 (Generalization of [JT07, corollary 3.6]). *Let $K \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ be a simplicial set, and $X \in \mathcal{S}\mathrm{p}(\mathbb{A})$ be a Segal space. Then $\mathsf{H}_{\text{inner}} \pitchfork X/K$. In particular, $\mathsf{H}_{\text{inner}} \pitchfork X/\Delta[n] = X_{-,n}$ for all $n \in \mathbb{N}$.*

Proof. For $\mathsf{i} : \varnothing \hookrightarrow K$ the initial map, the map $X/\mathsf{i} : X/K \longrightarrow X/\varnothing = 1$ is an inner fibration by proposition 15.1.4. □

Lemma 15.1.6 ([JT07, lemma 3.7]). *Let $\mathsf{K} \subseteq \mathcal{P}\mathrm{sh}(\mathbb{\Delta})^{[1]}$ be a saturated class of monomorphisms having the right cancellation property (see definition 15.1.2). If K contains all simplicial coface maps $d^i : \Delta[m-1] \longrightarrow \Delta[m]$, then it contains all simplicial horn inclusions $\mathsf{h}^k_m : \Lambda^k[m] \longrightarrow \Delta[m]$.*

Lemma 15.1.7 (Generalization of [JT07, lemma 3.8]). *Let $f : X \longrightarrow Y$ be an inner fibration between ∞-algebras. It is an acyclic fibration if and only if it is a weak equivalence surjective on $(n-1)$-cells.*

Proof. *($\Longrightarrow$)* Surely, if f is an acyclic fibration, it is a weak equivalence. Moreover, f has the right lifting property against all monomorphisms (since $\mathcal{P}\mathrm{sh}(\mathbb{A})_\infty$ is of Cisinski type, see theorem 13.3.21), and in particular, against the inclusions of the form $\varnothing \hookrightarrow h\phi$, for $\phi \in \mathbb{O}_{n-1}$. Therefore, f is surjective on $(n-1)$-cells[1].

($\Longleftarrow$) By lemma 12.2.9 f is a fibration if and only if it is a naive fibration, i.e. if $(\mathsf{H}_{\mathrm{inner}} \cup \mathsf{E}_{\mathfrak{J}}) \pitchfork f$. By assumption, $\mathsf{H}_{\mathrm{inner}} \pitchfork f$. By adjunction, in order to show that $\mathsf{E}_{\mathfrak{J}} \pitchfork f$, it suffices to show that $\mathsf{E}_{\mathfrak{J}} \pitchfork vf$. Note that f is a weak equivalence between cofibrant objects. Recall from proposition 14.2.4 that $\tau : \mathcal{P}\mathrm{sh}(\mathbb{A})_\infty \longrightarrow \mathcal{A}\mathrm{lg}_{\mathrm{folk}}$ is a left Quillen functor, thus as a consequence of Ken Brown's lemma (see corollary 12.1.15), τf is a weak equivalence. Besides, it is also surjective on $(n-1)$-cells, so by definition of the folk structure (theorem 14.2.1), it is an acyclic fibration. In particular, it is a fibration, thus $\mathsf{E}_{\mathfrak{J}} \pitchfork vf$. □

Lemma 15.1.8. *Let $X \in \mathcal{S}\mathrm{p}(\mathbb{A})$ be a space such that*

(1) for all $n \in \mathbb{N}$, the map X/b_n is an inner fibration;

(2) $X_{h\phi}$ is a Kan complex for all $\phi \in \mathbb{O}_{n-1}$.

Let $u : K \longrightarrow L$ be an anodyne extension in $\mathcal{P}\mathrm{sh}(\Delta)$. Then X/u is an acyclic fibration if and only if it is a weak equivalence.

Proof. Necessity is tautological, so suppose that X/u is a weak equivalence. By assumption, X/u is an inner fibration whenever u is a monomorphism. In particular, $X/u : X/L \longrightarrow X/K$ is a weak equivalence between ∞-algebras. By lemma 15.1.7, in order to show that X/u is an acyclic fibration, it is enough to show that it is surjective on $(n-1)$-cells. Note that for $\phi \in \mathbb{O}_{n-1}$,

$$\begin{aligned}
(X/L)_{h\phi} &\cong \mathcal{P}\mathrm{sh}(\mathbb{A})(h\phi, X/L) && \text{by the Yoneda lemma} \\
&\cong \mathcal{S}\mathrm{p}(\mathbb{A})(h\phi \boxtimes L, X) && \text{since } - \boxtimes L \dashv -/L \\
&\cong \mathcal{P}\mathrm{sh}(\Delta)(L, h\phi \backslash X) && \text{since } h\phi \boxtimes - \dashv h\phi\backslash - \\
&\cong \mathcal{P}\mathrm{sh}(\Delta)(L, X_{h\phi}) && \text{by lemma 12.3.5,}
\end{aligned}$$

and likewise, $(X/K)_{h\phi} \cong \mathcal{P}\mathrm{sh}(\Delta)(K, X_{h\phi})$. By assumption, $X_{h\phi}$ is a Kan complex, and since u is an anodyne extension, the precomposition map on top is surjective

$$\begin{tikzcd}
\mathcal{P}\mathrm{sh}(\Delta)(L, X_{h\phi}) \arrow[r, "u^*"] \arrow[d, "\cong"'] & \mathcal{P}\mathrm{sh}(\Delta)(K, X_{h\phi}) \arrow[d, "\cong"] \\
(X/L)_{h\phi} \arrow[r, "(X/u)_{h\phi}"] & (X/K)_{h\phi}.
\end{tikzcd}$$

Therefore, X/u is surjective on $(n-1)$-cells, and thus an acyclic fibration as claimed. □

Proposition 15.1.9 (Generalization of [JT07, proposition 3.9]). *A space $X \in \mathcal{S}\mathrm{p}(\mathbb{A})$ is a Segal space if and only if the following conditions are satisfied:*

[1] In fact, the same argument shows that f is surjective in all dimension.

(1) for all $n \in \mathbb{N}$, the map X/b_n is an inner fibration;
(2) $X_{h\phi}$ is a Kan complex for all $\phi \in \mathbb{O}_{n-1}$;
(3) X is homotopically constant (see definition 12.3.22).

Proof. $(\Longrightarrow)$ Assume that $X \in \mathcal{S}\mathrm{p}(\mathbb{\Lambda})$ is a Segal space. In particular, it is vertically fibrant.

(1) This is proposition 15.1.4 (4).
(2) The terminal map $! : X \longrightarrow 1$ is a vertical fibration, so by definition 12.3.19, for $\phi \in \mathbb{O}_{n-1}$, the map $\langle \mathsf{b}_{h\phi} \backslash ! \rangle : X_{h\phi} \longrightarrow 1$ is a Kan fibration.
(3) This is proposition 12.3.24.

$(\Longleftarrow)$ We first show that X is vertically fibrant. By proposition 12.3.21, this is equivalent to X/u being an acyclic fibration for every anodyne extension $u \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})^{[1]}$. Let K be the class of anodyne extensions u such that X/u is an acyclic fibration. By lemma 15.1.8, this is equivalent to X/u being a weak equivalence. By proposition 12.3.21, to show that X is vertically fibrant, we must check that K contains all simplicial horn inclusions. Using lemma 15.1.6, it suffices to show that K has the right cancellation property, is saturated, and that it contains all simplicial face maps.

(1) We show that K has the right cancellation property. Let u and w be two composable monomorphisms such that $u, wu \in \mathsf{K}$. In particular, u and wu are acyclic cofibrations. By 3-for-2, w is a weak equivalence, and since it is a monomorphism, it is an acyclic cofibration as well. Next, X/u and $X/(wu) = (X/u) \cdot (X/w)$ are weak equivalences, so by 3-for-2 again, so is X/w. Finally, $w \in \mathsf{K}$, and K has the right cancellation property.
(2) We now show that K is saturated. By definition, an anodyne extension u is in K if and only if $\mathsf{b}_{h\omega} \pitchfork X/u$ for all $\omega \in \mathbb{O}_{\geq n-1}$. By adjunction, this is equivalent to $u \pitchfork \mathsf{b}_{h\omega} \backslash X$, thus K is the intersection of the class of acyclic cofibrations and ${}^{\pitchfork}\{\mathsf{b}_{h\omega} \backslash X \mid \omega \in \mathbb{O}_{\geq n-1}\}$, both of which are saturated.
(3) Lastly, let us show that the simplicial face maps $d^i : \Delta[m-1] \longrightarrow \Delta[m]$ belong to K. It is well known that d^i is an anodyne extension. By assumption, X is homotopically constant, so by lemma 12.3.23, $d^i \in \mathsf{K}$.

Therefore, by proposition 12.3.21, K contains all simplicial horn inclusions, and by proposition 12.3.21, X is vertically fibrant.

Lastly, we show that X satisfies the Segal condition of definition 15.1.1. It suffices to show that for every $\omega \in \mathbb{O}_{\geq n-1}$, the map $\mathsf{s}_{h\omega} \backslash X$ is an acyclic fibration, i.e. that for all $n \in \mathbb{N}$, we have $\mathsf{b}_n \pitchfork \mathsf{s}_{h\omega} \backslash X$. This is equivalent to $\mathsf{s}_{h\omega} \pitchfork X/\mathsf{b}_n$, which holds since X/b_n is an inner fibration by assumption, and $\mathsf{s}_{h\omega}$ is an inner anodyne extension by proposition 13.2.11. □

Proposition 15.1.10 (Generalization of [JT07, proposition 3.10])**.** *Let $f : X \longrightarrow Y$ be a vertical fibration between two Segal spaces.*

(1) If $u : A \longrightarrow B$ is an inner anodyne extension in $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ (definition 13.2.5), then $\langle u \backslash f \rangle$ is an acyclic fibration.
(2) If $w : K \longrightarrow L$ is a monomorphism in $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$, then $\langle f/w \rangle$ is an inner fibration between ∞-algebras.

Proof. (1) In particular, u is a monomorphism, so by proposition 12.3.21, the map $\langle w\backslash f\rangle$ is a Kan fibration. It remains to show that it is a weak equivalence. Consider the following diagram:

$$\begin{tikzcd}
B\backslash X \arrow[dr, "\langle u\backslash f\rangle"] \arrow[drr, bend left, "u\backslash X"] \arrow[ddr, bend right, "B\backslash f"'] & & \\
& \cdot \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A\backslash X \arrow[d, "A\backslash f"] \\
& B\backslash Y \arrow[r, "u\backslash Y"] & A\backslash Y.
\end{tikzcd}$$

By proposition 15.1.4, $u\backslash X$ and $u\backslash Y$ are acyclic fibrations, and so is the pullback map p. By 3-for-2, $\langle u\backslash f\rangle$ is a weak equivalence.

(2) By proposition 15.1.4, X/K and X/L are ∞-algebras. In the pullback square

$$\begin{tikzcd}
X/L \arrow[dr, "\langle f/w\rangle"] \arrow[drr, bend left, "X/w"] \arrow[ddr, bend right, "f/Y"'] & & \\
& P \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X/K \arrow[d, "f/K"] \\
& Y/L \arrow[r, "Y/w"] & Y/K,
\end{tikzcd}$$

the bottom map Y/w is an inner fibration by proposition 15.1.4, and thus p is too. Since X/K is an ∞-algebra, so is P. We now show that $\langle f/w\rangle$ is an inner fibration, i.e. that $u \pitchfork \langle f/w\rangle$ for all $u \in \mathsf{An}_{\text{inner}}$. By (1), $\langle u\backslash f\rangle$ is an acyclic fibration, so $w \pitchfork \langle u\backslash f\rangle$, and by adjunction, $u \pitchfork \langle f/w\rangle$ as desired. □

15.2 COMPLETE SEGAL SPACES

Definition 15.2.1 (Rezk map)**.** For $\phi \in \mathbb{O}_{n-1}$, recall from definition 13.3.10 the definition of the Rezk interval $\mathfrak{J}_\phi$, and the endpoint inclusion $\mathsf{i}_\phi : h\phi \longrightarrow \mathfrak{J}_\phi$. There is a canonical morphism, called the *Rezk map* :

$$\mathsf{r}_\phi : \mathfrak{J}_\phi \longrightarrow h\phi,$$

mapping j_ϕ and j_ϕ^{-1} to id_ϕ (see definition 13.3.4 for notations), and let R be the set of Rezk maps.

Remark 15.2.2. Let $\phi \in \mathbb{O}_{n-1}$. The endpoint inclusion $\mathsf{i}_\phi : h\phi \longrightarrow \mathfrak{J}_\phi$ (definition 13.3.11) and the Rezk map r_ϕ displays $h\phi$ as a *deformation retract* of $\mathfrak{J}_\phi$: $\mathsf{r}_\phi \mathsf{i}_\phi = \mathrm{id}_{h\phi}$, and $\mathsf{i}_\phi \mathsf{r}_\phi \simeq \mathrm{id}_{\mathfrak{J}_\phi}$.

Definition 15.2.3. Let $\mathcal{S}\mathrm{p}(\mathbb{A})_{\text{Rezk}}$, the *Rezk model structure* on $\mathcal{S}\mathrm{p}(\mathbb{A})$, be the left Bousfield localization

$$\mathcal{S}\mathrm{p}(\mathbb{A})_{\text{Rezk}} := \mathsf{R}^{-1}\mathcal{S}\mathrm{p}(\mathbb{A})_{\text{Segal}},$$

which exists by theorem 12.1.29. Fibrant objects (resp. weak equivalences) in $\mathcal{S}\mathrm{p}(\mathbb{A})_{\text{Rezk}}$ are called *complete Segal spaces* (resp. *Rezk weak equivalences*). Unfolding the definition, a Segal space $X \in \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ is complete if and only if for all $\phi \in \mathbb{O}_{n-1}$, the map

$$\mathsf{r}_\phi \backslash X : \underbrace{h\phi \backslash X}_{= X_{h\phi}} \longrightarrow \mathfrak{J}_\phi \backslash X$$

is a weak equivalence.

Lemma 15.2.4 (Generalization of [JT07, lemma 4.2])**.** *A Segal space $X \in \mathcal{S}\mathrm{p}(\mathbb{\Lambda})$ is complete if and only if for all $\phi \in \mathbb{O}_{n-1}$, the map $\mathsf{i}_\phi \backslash X$ is a trivial fibration, where $\mathsf{i}_\phi : h\phi \longrightarrow \mathfrak{J}_\phi$ is the endpoint inclusion of the Rezk interval $\mathfrak{J}_\phi$ (definition 13.3.10).*

Proof. By definition, X is complete if and only if for all $\phi \in \mathbb{O}_{n-1}$, the map $\mathsf{r}_\phi \backslash X$ is a weak equivalence. Since $\mathsf{r}_\phi \mathsf{i}_\phi = \mathrm{id}_{h\phi}$, we have $(\mathsf{i}_\phi \backslash X)(\mathsf{r}_\phi \backslash X) = \mathrm{id}_{X_{h\phi}}$, and by 3-for-2, $\mathsf{i}_\phi \backslash X$ is a weak equivalence if and only if $\mathsf{r}_\phi \backslash X$ is. On the other hand, $\mathsf{i}_\phi \backslash X$ is always a Kan fibration by proposition 12.3.21. Hence it is a trivial fibration if and only if it is a weak equivalence. □

Lemma 15.2.5 (Generalization of [JT07, lemma 4.3])**.** *Let $f : X \longrightarrow Y$ be a Rezk fibration (i.e. a fibration in $\mathcal{S}\mathrm{p}(\mathbb{\Lambda})_{\mathrm{Rezk}}$) between two complete Segal spaces, and $u : K \hookrightarrow L$ be a monomorphism in $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$. Then the map $\langle f/u \rangle$ is a fibration.*

Proof. By proposition 15.1.10, $\langle f/u \rangle$ is an inner fibration between ∞-algebras. Therefore, by lemma 12.2.9, $\langle f/u \rangle$ is a fibration if and only if it is a naive fibration, so it remains to show that $\mathsf{E}_{\mathfrak{J}} \pitchfork \langle f/u \rangle$. By adjunction, for $\phi \in \mathbb{O}_{n-1}$, we have $\mathsf{i}_\phi \pitchfork \langle f/u \rangle$ if and only if $u \pitchfork \langle \mathsf{i}_\phi \backslash f \rangle$. Thus, we must show that $\langle \mathsf{i}_\phi \backslash f \rangle$ is an acyclic fibration. By proposition 12.3.21, it is a fibration. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{J}_\phi \backslash X \arrow[dr, "\langle \mathsf{i}_\phi \backslash f \rangle"] \arrow[drr, bend left, "\mathsf{i}_\phi \backslash X"] \arrow[ddr, bend right, "\mathfrak{J}_\phi \backslash f"'] & & \\
& \cdot \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & h\phi \backslash Y \arrow[d, "h\phi \backslash f"] \\
& \mathfrak{J}_\phi \backslash Y \arrow[r, "\mathsf{i}_\phi \backslash Y"] & h\phi \backslash Y .
\end{tikzcd}$$

By lemma 15.2.4, $\mathsf{i}_\phi \backslash X$ and $\mathsf{i}_\phi \backslash Y$ are acyclic fibrations. Since f is a Rezk fibration, it is a vertical fibration, and its matching map $\langle (\partial h\phi \hookrightarrow h\phi) \backslash f \rangle$ is a fibration by proposition 12.3.21. Since $\phi \in \mathbb{O}_{n-1}$, $\partial h\phi = \varnothing$, thus $h\phi \backslash f = \langle (\partial h\phi \hookrightarrow h\phi) \backslash f \rangle$ is a fibration. The pullback map p is an acyclic fibration, and by 3-for-2, so is $\langle \mathsf{i}_\phi \backslash f \rangle$. □

Proposition 15.2.6 (Generalization of [JT07, proposition 4.4])**.** *A space $X \in \mathcal{S}\mathrm{p}(\mathbb{\Lambda})$ is a complete Segal space if and only if it is horizontally fibrant (i.e. X/b_m is a fibration for all $m \in \mathbb{N}$, see definition 12.3.19) and homotopically constant. In particular, complete Segal spaces are exactly the* simplicial resolutions *of [Dug01, definition 4.7].*

Proof. *($\Longrightarrow$)* By proposition 15.1.9, X is homotopically constant. By lemma 15.2.5, $X/\mathsf{b}_m = \langle (X \to 1)/\mathsf{b}_m \rangle$ is a fibration as $X \longrightarrow 1$ is a Rezk fibration.

$(\Longleftarrow)$ We first show that X is vertically fibrant. By proposition 12.3.21, this is equivalent to X/u being an acyclic fibration for every anodyne extension $u \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})^{[1]}$. Let K be the class of monomorphisms u such that X/u is an acyclic fibration. We need to show that K contains all simplicial horn inclusions. Using lemma 15.1.6, it suffices to show that K has the right cancellation property, is saturated, and that it contains all simplicial face maps.

(1) Since X is horizontally fibrant, X/u is a fibration for all monomorphism $u \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$. Thus, X/u is an acyclic fibration if and if it is a weak equivalence. The right cancellation property of K then follows from 3-for-2.

(2) For saturation, note that by definition, $u \in \mathsf{K}$ if and only if $\mathsf{b}_{h\omega} \pitchfork X/u$ for all $\omega \in \mathbb{O}_{\geq n-1}$. This is equivalent to $v \pitchfork \mathsf{b}_{h\omega} \backslash X$, thus K is the intersection of the class of monomorphisms of $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ and ${}^{\pitchfork}\{\mathsf{b}_{h\omega} \mid \omega \in \mathbb{O}_{\geq n-1}\}$, both of which are saturated.

(3) By assumption, X is homotopically constant. By lemma 12.3.23, K contains all simplicial face maps.

Therefore, by proposition 12.3.21, K contains all simplicial horn inclusions, and by proposition 12.3.21, X is vertically fibrant.

Since X is vertically fibrant, the terminal map $! : X \longrightarrow 1$ is a vertical fibration, and by definition 12.3.19, for $\phi \in \mathbb{O}_{n-1}$, the relative matching map $\langle \mathsf{b}_{h\phi} \backslash ! \rangle : X_{h\phi} \longrightarrow 1$ is a Kan fibration. Therefore, $X_{h\phi}$ is a Kan complex, and by proposition 15.1.9, X is a Segal space.

Lastly, let us show that X is complete. By lemma 15.2.4, it suffices to show that $\mathsf{i}_\phi \backslash X$ is an acyclic fibration, for all $\phi \in \mathbb{O}_n$. The map i_ϕ is is an acyclic cofibration since it is a monomorphism and a homotopy equivalence by remark 15.2.2. Since X is horizontally fibrant, $\mathsf{i}_\phi \pitchfork X/\mathsf{b}_m$, for all $m \in \mathbb{N}$. By adjunction, $\mathsf{b}_n \pitchfork \mathsf{i}_\phi \backslash X$, and $\mathsf{i}_\phi \backslash X$ is an acyclic fibration. □

Theorem 15.2.7 (Generalization of [JT07, theorem 4.5]). *The Rezk model category structure $\mathcal{S}\mathrm{p}(\mathbb{\Lambda})_{\mathrm{Rezk}}$ is a left Bousfield localization of the horizontal model structure $\mathcal{S}\mathrm{p}(\mathbb{\Lambda})_h$ (definition 12.3.19). In particular, a weak equivalence in $\mathcal{S}\mathrm{p}(\mathbb{\Lambda})_h$ is a Rezk weak equivalence.*

Proof. By proposition 12.3.20, $\mathcal{S}\mathrm{p}(\mathbb{\Lambda})_v$ and $\mathcal{S}\mathrm{p}(\mathbb{\Lambda})_h$ are both of Cisinski type. Since the Rezk model structure $\mathcal{S}\mathrm{p}(\mathbb{\Lambda})_{\mathrm{Rezk}}$ is a left Bousfield localization of $\mathcal{S}\mathrm{p}(\mathbb{\Lambda})_v$, it is also of Cisinski type. In particular, $\mathcal{S}\mathrm{p}(\mathbb{\Lambda})_{\mathrm{Rezk}}$ and $\mathcal{S}\mathrm{p}(\mathbb{\Lambda})_h$ have the same cofibrations, namely the monomorphisms. Thus, in order to prove the claim, it is enough to show that the identity functor induces a Quillen adjunction $\mathrm{id} : \mathcal{S}\mathrm{p}(\mathbb{\Lambda})_h \rightleftarrows \mathcal{S}\mathrm{p}(\mathbb{\Lambda})_{\mathrm{Rezk}} : \mathrm{id}$. By lemma 12.1.16, it suffices to show that id preserves cofibrations, and Rezk fibrations between complete Segal spaces. In both structures, cofibrations are the monomorphisms. By lemma 15.2.5, for $f : X \longrightarrow Y$ a Rezk fibration between complete Segal spaces, the matching map $\langle f/\mathsf{b}_m \rangle$ is a fibration for all $m \in \mathbb{N}$, thus f is a horizontal fibration. □

Theorem 15.2.8. *With the simplicial action of proposition 12.3.13, $\mathcal{S}\mathrm{p}(\mathbb{\Lambda})_{\mathrm{Rezk}}$ is a simplicial model category.*

Proof. Recall from [Dug01, section 6] the model structure $s\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})_{hc}$, which is a left Bousfield localization of $\mathcal{S}\mathrm{p}(\mathbb{\Lambda})_h$ whose fibrant objects are the simplicial resolutions. It is simplicial by [Dug01, theorem 6.1], and by [RSS01, theorem 3.1], it coincides with $\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{Rezk}}$. □

Definition 15.2.9. Recall from [JT07, proposition 1.16] that we have a coreflective adjunction (meaning that the left adjoint is an embedding)

$$\mathcal{K}\mathrm{an} \rightleftarrows q\mathcal{C}\mathrm{at} : J$$

between Kan complexes and quasi-categories. The right adjoint J maps a quasi-category to its maximal sub Kan complex, so it can be thought of as the "groupoidal core" functor. It preserves cofibrations, fibrations, and weak equivalences from $\mathcal{P}\mathrm{sh}(\Delta)_{\mathrm{Joyal}}$. For $X \in \mathcal{P}\mathrm{sh}(\mathbb{A})$, let $\Gamma X \in \mathcal{S}\mathrm{p}(\mathbb{A})$ be the space given by

$$\Gamma X_{-,m} := X^{J(\Delta[m])},$$

for $m \in \mathbb{N}$, and where the simplicial cotensor is defined in definition 13.4.3.

Proposition 15.2.10 (Generalization of [JT07, proposition 4.10])**.** *Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{A})$ be an ∞-algebra. Then ΓX is a complete Segal space, and there is a canonical acyclic cofibration $X^{\mathrm{disc}} \longrightarrow \Gamma X$ in $\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{Rezk}}$. In particular, ΓX is a fibrant replacement of X^{disc} in the Rezk structure.*

Proof. (1) Note that $X^{J(-)} : \mathcal{P}\mathrm{sh}(\Delta)^{\mathrm{op}} \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{A})$ is a continuous functor. Using lemma 12.3.6 we readily deduce that $X^{J(-)} \cong \Gamma X/-$.

(2) For u a map in $\mathcal{P}\mathrm{sh}(\Delta)$, we have $\Gamma X/u = X^{J(u)}$, and recall that J preserves cofibrations, fibrations, and weak equivalences of $\mathcal{P}\mathrm{sh}(\Delta)_{\mathrm{Joyal}}$. In particular, by proposition 13.4.9, $\Gamma X/\mathsf{b}_k = X^{J(\mathsf{b}_k)}$ is a fibration for all $k \in \mathbb{N}$. Likewise, since any map $d : \Delta[0] \longrightarrow \Delta[n]$ is an acyclic cofibration in $\mathcal{P}\mathrm{sh}(\Delta)_{\mathrm{Quillen}}$, $\Gamma X/d = X^{J(d)} : \Gamma X_{-,n} \longrightarrow \Gamma X_{-,0}$ is an acyclic fibration, and by lemma 12.3.23, ΓX is homotopically constant. By proposition 15.2.6, we conclude that ΓX is a complete Segal space.

(3) Note that $\Gamma X_{-,0} \cong \Gamma X/\Delta[0] \cong X^{J(\Delta[0])} = X$, and let $\gamma_0 : X \longrightarrow \Gamma X_{-,0}$ be the identity. We extend γ_0 to a morphism $\gamma : X^{\mathrm{disc}} \longrightarrow \Gamma X$. Note that in Δ, the terminal map $s^k : [k] \longrightarrow [0]$ is a retraction of any iterated coface map $d : [0] \longrightarrow [k]$. Thus, in addition to being a weak equivalence (as ΓX is homotopically constant), the structure map $s^k : \Gamma X_{-,0} \longrightarrow \Gamma X_{-,k}$ is a section of $d : \Gamma X_{-,k} \longrightarrow \Gamma X_{-,0}$, and in particular injective. Letting $\gamma_k := s^k \gamma_0 : X^{\mathrm{disc}}_{-,k} = X \longrightarrow \Gamma X_{-,k}$ gives rise to the desired map $\gamma : X^{\mathrm{disc}} \longrightarrow \Gamma X$ which by construction is a horizontal acyclic cofibration. By theorem 15.2.7, it is a Rezk acyclic cofibration. □

Theorem 15.2.11 (Generalization of [JT07, proposition 4.7 and theorem 4.11])**.** *The adjunction*

$$(-)^{\mathrm{disc}} : \mathcal{P}\mathrm{sh}(\mathbb{A})_{\infty} \rightleftarrows \mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{Rezk}} : (-)_{-,0}$$

of definition 12.3.10 is a Quillen equivalence. Thus, complete Segal spaces model ∞-algebras.

Proof. (1) We first show that the adjunction is Quillen. Clearly, $(-)^{\mathrm{disc}}$ preserves monomorphisms and maps weak equivalences to horizontal weak equivalences, which by theorem 15.2.7 are Rezk weak equivalences. Therefore, $(-)^{\mathrm{disc}}$ is a left Quillen functor.

(2) We show that for a complete Segal space $X \in \mathcal{S}\mathrm{p}(\mathbb{A})$, the map

$$(QX_{-,0})^{\mathrm{disc}} \xrightarrow{q^{\mathrm{disc}}} (X_{-,0})^{\mathrm{disc}} \xrightarrow{\varepsilon} X$$

is a Rezk weak equivalence, where $q : QX_{-,0} \longrightarrow X_{-,0}$ is a cofibrant replacement of $X_{-,0}$. First, since all objects are cofibrant in $\mathcal{P}\mathrm{sh}(\mathbb{A})_\infty$, we choose q to be the identity. Next, by theorem 15.2.7, it suffices to show that ε is a weak equivalence in $\mathcal{S}\mathrm{p}(\mathbb{A})_h$, i.e. that $\varepsilon_{-,n} : (X_{-,0})^{\mathrm{disc}}_{-,n} = X_{-,0} \longrightarrow X_{-,n}$ is a weak equivalence. Clearly, $\varepsilon_{-,n}$ is induced by the terminal map $[n] \longrightarrow [0]$ in $\mathbb{\Delta}$, and thus is a weak equivalence since X is homotopically constant by proposition 15.2.6.

(3) We show that for an ∞-algebra $X \in \mathcal{P}\mathrm{sh}(\mathbb{A})_\infty$, the map

$$X \xrightarrow{\cong} (X^{\mathrm{disc}})_{-,0} \xrightarrow{r_{-,0}} (RX^{\mathrm{disc}})_{-,0}$$

is a Rezk weak equivalence, where $r : X^{\mathrm{disc}} \longrightarrow RX^{\mathrm{disc}}$ is a fibrant replacement of X^{disc}. Choosing it to be $\gamma : X^{\mathrm{disc}} \longrightarrow \Gamma X$ (proposition 15.2.10) concludes the proof. □

Chapter Sixteen

Homotopy coherent opetopic algebras

The goal of this chapter is to present an alternative model for weak opetopic algebras, a.k.a. ∞-algebras. In section 16.1, we consider the category $\mathbb{I}\mathcal{A}\mathrm{lg} = \mathbb{I}\mathcal{A}\mathrm{lg}_{1,n}$ of opetopic algebras internal to simplicial sets (or equivalently, simplicial objects in $\mathcal{A}\mathrm{lg}_{k,n}$), and in section 16.2, we transfer (in the sense of theorem 16.2.2) the projective model structure on $\mathcal{S}\mathrm{p}(\mathbb{A})$ to one on $\mathbb{I}\mathcal{A}\mathrm{lg}$, which we call the *Horel model structure*. The strategy used here is a direct adaptation of [Hor15, section 3 and 4], where this result is proved in the case $k = n = 1$ (i.e. $\mathbb{A} = \mathbb{\Delta}$ and $\mathcal{A}\mathrm{lg} = \mathcal{C}\mathrm{at}$). Finally, by successive localizations of the Horel structure, we arrive at the notion of complete Segal algebra, and show in theorem 16.3.5 that they are a model for ∞-algebras.

Throughout this chapter, we shall make use of convention 11.1.34, and fix parameters $k := 1$ and $n \geq 1$, and omit them from various notations, e.g. $\mathbb{A} = \mathbb{A}_{1,n}$, $\mathcal{A}\mathrm{lg} = \mathcal{A}\mathrm{lg}_{1,n}$, etc.

16.1 INTERNAL ALGEBRAS

Remark 16.1.1. Recall from theorem 11.1.39 that $\mathcal{A}\mathrm{lg} = \mathcal{A}\mathrm{lg}_{k,n}$ is the small orthogonality class induced by $\mathsf{A}_{k,n}$ (see definition 11.2.32) in $\mathcal{P}\mathrm{sh}(\mathbb{O})$. By theorem 0.5.15, it is thus equivalent to the category of models of a small projective sketch whose underlying category is $\mathbb{O}^{\mathrm{op}}$. The distinguished cones are:

(1) for all $\phi \in \mathbb{O}_{<n-k}$, the object ϕ over the empty diagram $\varnothing \longrightarrow \mathbb{O}^{\mathrm{op}}$;

(2) for all $\omega \in \mathbb{O}_{\geq n+1}$, the object ω over the diagram

$$(\mathbb{O}/S[\omega])^{\mathrm{op}} \longrightarrow \mathbb{O}^{\mathrm{op}}$$

via the morphisms $\omega \longrightarrow \mathsf{s}_{[p]}\,\omega$ (where $[p]$ ranges over $\omega^\bullet$) and $\omega \longrightarrow \mathsf{t}\,\omega$ in $\mathbb{O}^{\mathrm{op}}$.

Definition 16.1.2 (Algebras internal to simplicial sets)**.** Let $\mathbb{I}\mathcal{A}\mathrm{lg}$ be the category of simplicial models of the sketch described in remark 16.1.1. Explicitly, it is the full subcategory of $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})^{\mathbb{O}^{\mathrm{op}}}$ spanned by the simplicial presheaves X such that

(1) for all $\phi \in \mathbb{O}_{<n-k}$, $X_\phi = \Delta[0]$;

(2) for all $\omega \in \mathbb{O}_{\geq n+1}$, we have

$$X_\omega \cong \lim\Big((\mathbb{O}/S[\omega])^{\mathrm{op}} \longrightarrow \mathbb{O}^{\mathrm{op}} \xrightarrow{X} \mathcal{P}\mathrm{sh}(\mathbb{\Delta})\Big).$$

We have an isomorphism $\mathbb{I}\mathcal{A}\mathrm{lg} \cong \mathcal{A}\mathrm{lg}^{\mathbb{\Delta}^{\mathrm{op}}}$ that simply swaps the indices (considering $\mathcal{A}\mathrm{lg}$ as a subcategory of $\mathcal{P}\mathrm{sh}(\mathbb{O})$, see theorem 11.2.33). By proposition 11.1.32, $\mathcal{A}\mathrm{lg}$ is complete and cocomplete, thus so is $\mathbb{I}\mathcal{A}\mathrm{lg}$.

Definition 16.1.3 (Internal realization-nerve adjunction)**.** The realization-nerve adjunction $\tau : \mathcal{P}\mathrm{sh}(\mathbb{A}) \rightleftarrows \mathcal{A}\mathrm{lg} : N$ of definition 11.1.35 induces an adjunction

$$\bar{\tau} : \mathcal{S}\mathrm{p}(\mathbb{A}) \rightleftarrows \mathrm{I}\mathcal{A}\mathrm{lg} : \bar{N},$$

where $\bar{\tau}$ is the composite

$$\mathcal{S}\mathrm{p}(\mathbb{A}) \xrightarrow{\cong} \mathcal{P}\mathrm{sh}(\mathbb{A})^{\mathbb{\Delta}^{\mathrm{op}}} \xrightarrow{\tau^{\mathbb{\Delta}^{\mathrm{op}}}} \mathcal{A}\mathrm{lg}^{\mathbb{\Delta}^{\mathrm{op}}} \xrightarrow{\cong} \mathrm{I}\mathcal{A}\mathrm{lg}.$$

Explicitly, for $X \in \mathcal{S}\mathrm{p}(\mathbb{A})$ and $m \in \mathbb{N}$, we have $\bar{\tau} X_m = \tau(X_{-,m}) = \tau(X/\Delta[m])$, and for $A \in \mathrm{I}\mathcal{A}\mathrm{lg}$, we have $(\bar{N}A)_{-,m} = N(A_m)$. Since $\tau \dashv N$ is a reflective adjunction (see theorem 11.1.39), so is $\bar{\tau} \dashv \bar{N}$.

Proposition 16.1.4. *The category* $\mathrm{I}\mathcal{A}\mathrm{lg}$ *is simplicially enriched. The mapping spaces are given by*

$$\mathrm{map}_{\mathrm{I}\mathcal{A}\mathrm{lg}}(A, B) \coloneqq \mathrm{map}_{\mathcal{S}\mathrm{p}(\mathbb{A})}(\bar{N}A, \bar{N}B),$$

for $A, B \in \mathrm{I}\mathcal{A}\mathrm{lg}$*. We shall write* map *for* $\mathrm{map}_{\mathrm{I}\mathcal{A}\mathrm{lg}}$ *or* $\mathrm{map}_{\mathcal{S}\mathrm{p}(\mathbb{A})}$ *if no ambiguity arises. Moreover,* $\mathrm{I}\mathcal{A}\mathrm{lg}$ *is simplicially tensored, with, for* $K \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$*,*

$$(K \otimes A)_m \coloneqq \bar{\tau}(K \otimes \bar{N}A) = \bar{\tau}(K^{\mathrm{const}} \times \bar{N}A).$$

In the middle, $\otimes$ *is the simplicial tensor of proposition 12.3.13.*

Proof. The first claim follows from the fact that $\bar{N}$ is fully faithful. We now check that $- \otimes -$ above indeed defines a simplicial tensor.

$$\begin{aligned}
&\mathrm{I}\mathcal{A}\mathrm{lg}(K \otimes A, B) \\
&= \mathrm{I}\mathcal{A}\mathrm{lg}(\bar{\tau}(K \otimes \bar{N}A), B) && \text{by definition} \\
&= \mathcal{S}\mathrm{p}(\mathbb{A})(K \otimes \bar{N}A, \bar{N}B) && \text{since } \bar{\tau} \dashv \bar{N} \\
&\cong \mathcal{S}\mathrm{p}(\mathbb{A})((\operatorname*{colim}_{\Delta[m]\to K} \Delta[m]) \otimes \bar{N}A, \bar{N}B) \\
&\cong \lim_{\Delta[m]\to K} \mathcal{S}\mathrm{p}(\mathbb{A})(\Delta[m] \otimes \bar{N}A, \bar{N}B) && \text{since } - \otimes \bar{N}A \text{ cocont.} \\
&= \lim_{\Delta[m]\to K} \mathcal{P}\mathrm{sh}(\mathbb{\Delta})(\Delta[m], \mathrm{map}_{\mathcal{S}\mathrm{p}(\mathbb{A})}(\bar{N}A, \bar{N}B)) && \text{see proposition 12.3.13} \\
&\cong \mathcal{P}\mathrm{sh}(\mathbb{\Delta})(K, \mathrm{map}_{\mathrm{I}\mathcal{A}\mathrm{lg}}(A, B)) && \text{by definition.}
\end{aligned}$$

□

Lemma 16.1.5. *For* $K \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$*,* $A \in \mathrm{I}\mathcal{A}\mathrm{lg}$*, and* $m \in \mathbb{N}$*, we have*

$$(K \otimes A)_m \cong K_m \boxtimes A_m = \sum_{K_m} A_m.$$

Proof. Straightforward computations:

$$\begin{aligned}
(K \otimes A)_m &= \bar{\tau}(K^{\mathrm{const}} \times \bar{N}A)_m && \text{by definition} \\
&= \tau\left((K^{\mathrm{const}} \times \bar{N}A)_{-,m}\right) && \text{see definition 16.1.3} \\
&= \tau\left(K_m \boxtimes (\bar{N}A)_{-,m}\right) \\
&= \tau(K_m \boxtimes N(A_m)) && \text{see definition 16.1.3}
\end{aligned}$$

$$
\begin{aligned}
&\cong K_m \boxtimes \tau N(A_m) && \text{since } \tau \text{ cocont.} \\
&\cong K_m \boxtimes A_m && \text{since } \tau \dashv N \text{ reflective.}
\end{aligned}
$$

□

Definition 16.1.6 (Connected object)**.** An object $c \in \mathcal{C}$ in some category $\mathcal{C}$ is *connected* if $\mathcal{C}(c, -) : \mathcal{C} \longrightarrow \mathcal{S}\mathrm{et}$ preserves sums.

Lemma 16.1.7. *(1) A connected colimit (i.e. a colimit whose scheme is a connected category) of connected objects is connected.*
(2) Let $\mathcal{C}$ be a small category. Representable presheaves in $\mathcal{P}\mathrm{sh}(\mathcal{C})$ are connected.

Proof. The first claim follows from the fact that in $\mathcal{S}\mathrm{et}$, sums commute with connected limits. The second is by the Yoneda lemma. □

Lemma 16.1.8. *The nerve $N : \mathcal{A}\mathrm{lg} \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{A})$ preserves sums.*

Proof. By proposition 11.1.30, it is enough to show that the monad $\mathfrak{Z}^n$ on $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$ preserves sums. Let $\sum_i X_i$ be a sum in $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$ and $\omega \in \mathbb{O}_{n-k,n}$. If $\dim \omega < n$, then by definition of $\mathfrak{Z}^n$ (see definition 11.1.16), we have $(\mathfrak{Z}^n \sum_i X_i)_\omega = (\sum_i X_i)_\omega = \sum_i X_{i,\omega}$. If $\dim \omega = n$, then

$$
(\mathfrak{Z}^n \sum_i X_i)_\omega = \sum_{\mathsf{t}\,\nu = \omega} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})(S[\nu], \sum_i X_i).
$$

Since $k \geq 1$, and by lemma 3.5.8, the spine $S[\nu]$ is a connected colimit of representables, hence connected by lemma 16.1.7. □

Lemma 16.1.9. *Let $K \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ and $X \in \mathcal{P}\mathrm{sh}(\mathbb{A})$, and recall the box product $K \boxtimes X \in \mathcal{S}\mathrm{p}(\mathbb{A})$ from definition 12.3.4.*

(1) We have

$$
\bar{\tau}(K \boxtimes X) \cong K \otimes (\tau X)^{\mathrm{const}}.
$$

In particular, if $\lambda \in \mathbb{A}$, then $\bar{\tau}(K \boxtimes \lambda) \cong K \otimes \lambda^{\mathrm{const}}$, where on the left, λ is considered as a representable presheaf, and on the right, as an algebra.

(2) We have

$$
\bar{N}\left(K \otimes (\tau X)^{\mathrm{const}}\right) \cong K \boxtimes N\tau X.
$$

In particular, if $\lambda \in \mathbb{A}$, then $\bar{N}(K \otimes \lambda^{\mathrm{const}}) = K \boxtimes \lambda$, where on the left, λ is considered as an algebra, and on the right, as a representable presheaf.

Proof. For $m \in \mathbb{N}$, we have

$$
\begin{aligned}
\bar{\tau}(K \boxtimes X)_m &= \bar{\tau}\left((K \boxtimes X)_{-,m}\right) && \text{see definition 16.1.3} \\
&= \tau(K_m \boxtimes X) && \text{see definition 12.3.4} \\
&\cong K_m \boxtimes \tau X && \text{since } \tau \text{ cocont.} \\
&= K_m \boxtimes (\tau X)^{\mathrm{const}}_m && \\
&\cong (K \otimes (\tau X)^{\mathrm{const}})_m && \text{by lemma 16.1.5,}
\end{aligned}
$$

and for the second point,

$$
\bar{N}(K \otimes (\tau X)^{\mathrm{const}})_{-,m} = N\left((K \boxtimes (\tau X)^{\mathrm{const}})_m\right) \qquad \text{see definition 16.1.3}
$$

$$\cong N(K_m \boxtimes \tau X) \qquad \text{by lemma 16.1.5}$$
$$\cong K_m \boxtimes N\tau X \qquad \text{by lemma 16.1.8.}$$

□

16.2 THE HOREL MODEL STRUCTURE

Definition 16.2.1 (Projective structure on $\mathcal{S}\mathrm{p}(\mathbb{A})$)**.** By theorem 12.1.20, the category $\mathcal{S}\mathrm{p}(\mathbb{A}) \simeq \mathcal{P}\mathrm{sh}(\mathbb{\Delta})^{\mathbb{A}^{\mathrm{op}}}$ admits the *projective model structure*, whereby a map f is a fibration (resp. weak equivalence) if for all $\lambda \in \mathbb{A}$, f_λ is a fibration (resp. a weak equivalence) in the classical model structure $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})_{\mathrm{Quillen}}$. We shall denote this structure by $\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{proj}}$. It is combinatorial, and we write

$$\mathsf{I}_{\mathbb{A}} := \left\{ \partial\Delta[m] \boxtimes \lambda \xrightarrow{\mathsf{b}_n \boxtimes \lambda} \Delta[m] \boxtimes \lambda \mid m \in \mathbb{N}, \lambda \in \mathbb{A} \right\}$$
$$\mathsf{J}_{\mathbb{A}} := \left\{ \Lambda^l[m] \boxtimes \lambda \xrightarrow{\mathsf{h}_n^l \boxtimes \lambda} \Delta[m] \boxtimes \lambda \mid m \in \mathbb{N}, 0 \leq l \leq m, \lambda \in \mathbb{A} \right\}$$

for the sets of generating cofibrations and generating acyclic cofibrations respectively (see theorem 12.1.20). Lastly, recall that the identity functor induces a Quillen equivalence

$$\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{proj}} \rightleftarrows \mathcal{S}\mathrm{p}(\mathbb{A})_v .$$

The goal of this section is to use the following result to induce a model structure on $\mathbb{I}\mathcal{A}\mathrm{lg}$ along the adjunction of definition 16.1.3:

Theorem 16.2.2 ([Fre09, proposition 11.1.14])**.** *Let $F : \mathcal{M} \rightleftarrows \mathcal{C} : U$ be an adjunction between complete and cocomplete categories, and where $\mathcal{M}$ is a model category cofibrantly generated by $\mathsf{I}_{\mathcal{M}}$ and $\mathsf{J}_{\mathcal{M}}$ as sets of generating cofibrations and acyclic cofibrations respectively. Assume that*

(1) U preserves filtered colimits,
(2) U sends pushouts of maps in $F\mathsf{I}_{\mathcal{M}}$ (resp. $F\mathsf{J}_{\mathcal{M}}$) to cofibrations (resp. trivial cofibrations).

Then the right-induced model structure *on $\mathcal{C}$ exists, i.e. that in which a morphism f is a fibration (resp. a weak equivalence) if and only if Uf is a fibration (resp. a weak equivalence). By definition, the adjunction $F \dashv U$ is a Quillen adjunction, and furthermore, U preserves cofibrations.*

Proposition 16.2.3 (Generalization of [Hor15, proposition 3.5])**.** *The nerve $\bar{N}$ preserves filtered colimits.*

Proof. Let $\lambda \in \mathbb{A}$. By definition, for $A \in \mathbb{I}\mathcal{A}\mathrm{lg}$ and $m \in \mathbb{N}$ we have $\bar{N}A_{\lambda,m} = (NA_m)_\lambda$, and by theorems 0.5.11 and 11.1.39, the nerve N preserves filtered colimits. Thus, so does $\bar{N}$. □

Lemma 16.2.4. *For $\omega \in \mathbb{O}_{\geq n-k}$, the algebra $h\omega$ is a connected object in $\mathcal{A}\mathrm{lg}$.*

Proof. For $A, B \in \mathcal{A}\mathrm{lg}$ we have

$$\begin{aligned}
&\mathcal{A}\mathrm{lg}(h\omega, A + B) \\
&\cong \mathcal{P}\mathrm{sh}(\mathbb{A})(h\omega, N(A + B)) && \text{since } \tau \dashv N \\
&\cong \mathcal{P}\mathrm{sh}(\mathbb{A})(h\omega, NA + NB) && \text{by lemma 16.1.8} \\
&\cong \mathcal{P}\mathrm{sh}(\mathbb{A})(h\omega, NA) + \mathcal{P}\mathrm{sh}(\mathbb{A})(h\omega, NB) && \text{since } h\omega \text{ is repres.} \\
&\cong \mathcal{A}\mathrm{lg}(h\omega, A) + \mathcal{A}\mathrm{lg}(h\omega, B) && \text{since } \tau \dashv N,
\end{aligned}$$

and therefore, $h\omega$ is connected in $\mathcal{A}\mathrm{lg}$. ◻

Lemma 16.2.5 (Generalization of [Hor15, lemma 4.1])**.** *Let $A, B \in \mathcal{A}\mathrm{lg}$, $C, D \in \mathrm{I}\mathcal{A}\mathrm{lg}$, $i : K \longrightarrow L$ a monomorphism in $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$, and consider a pushout square as on the left:*

$$\begin{tikzcd}
K \boxtimes A \arrow[r] \arrow[d, "i \boxtimes A"'] & C \arrow[d] \\
L \boxtimes A \arrow[r] & D, \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{map}(B, K \boxtimes A) \arrow[r] \arrow[d, "\mathrm{map}(B, i \boxtimes A)"'] & \mathrm{map}(B, C) \arrow[d] \\
\mathrm{map}(B, L \boxtimes A) \arrow[r] & \mathrm{map}(B, D).
\end{tikzcd}$$

If B is a connected algebra, then the square on the right is a pushout too.

Proof. Since colimits in $\mathrm{I}\mathcal{A}\mathrm{lg}$ are calculated pointwise, for all $m \in \mathbb{N}$, the following square is a pushout in $\mathcal{A}\mathrm{lg}$:

$$\begin{tikzcd}
K_m \boxtimes A \arrow[r] \arrow[d, "i_m \boxtimes A"'] & C_m \arrow[d] \\
L_m \boxtimes A \arrow[r] & D_m. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Defining $M_m := L_m - i_m(K_m)$, we clearly have $D_m \cong (M_m \boxtimes A) + C_m$. Next,

$$\begin{aligned}
&\mathrm{colim}\begin{pmatrix} \mathcal{A}\mathrm{lg}(B, K_m \boxtimes A) \longrightarrow \mathcal{A}\mathrm{lg}(B, C_m) \\ \downarrow \\ \mathcal{A}\mathrm{lg}(B, L_m \boxtimes A) \end{pmatrix} \\
&\cong \mathrm{colim}\begin{pmatrix} K_m \times \mathcal{A}\mathrm{lg}(B, A) \longrightarrow \mathcal{A}\mathrm{lg}(B, C_m) \\ \downarrow \\ L_m \times \mathcal{A}\mathrm{lg}(B, A) \end{pmatrix} && \spadesuit \\
&\cong M_m \times \mathcal{A}\mathrm{lg}(B, A) + \mathcal{A}\mathrm{lg}(B, C_m) \\
&\cong \mathcal{A}\mathrm{lg}(B, M_m \boxtimes A) + \mathcal{A}\mathrm{lg}(B, C_m) && \spadesuit \\
&\cong \mathcal{A}\mathrm{lg}(B, (M_m \boxtimes A) + C_m) \cong \mathcal{A}\mathrm{lg}(B, D_m), && \spadesuit
\end{aligned}$$

where in steps $\spadesuit$, we used the fact that $\mathcal{A}\mathrm{lg}(B, -)$ preserves sums, as B is connected. Therefore, the following square is a pushout:

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}(B, K_m \boxtimes A) \arrow[r] \arrow[d] & \mathcal{A}\mathrm{lg}(B, C_m) \arrow[d] \\
\mathcal{A}\mathrm{lg}(B, L_m \boxtimes A) \arrow[r] & \mathcal{A}\mathrm{lg}(B, D_m),
\end{tikzcd}$$

but this is just the desired square evaluated at m. ◻

Corollary 16.2.6 (Generalization of [Hor15, corollary 4.2]). *With the same hypotheses as lemma 16.2.5, the following square is a pushout:*

$$\begin{tikzcd} \bar{N}(K \boxtimes A) \arrow[r] \arrow[d, "\bar{N}(i \boxtimes A)"'] & \bar{N}C \arrow[d] \\ \bar{N}(L \boxtimes A) \arrow[r] & \bar{N}D. \end{tikzcd}$$

Proof. This follows from lemmas 16.2.4 and 16.2.5. □

Theorem 16.2.7 (Generalization of [Hor15, theorem 5.1]). *The right induced model structure on* $\mathbb{I}\mathcal{A}\mathrm{lg}$ *along* $\bar{N}$ *(in the sense of theorem 16.2.2) exists. We shall denote it by* $\mathbb{I}\mathcal{A}\mathrm{lg}_{\mathrm{Horel}}$ *and call it the* Horel model structure[1]. *Recall that a morphism* f *of* $\mathbb{I}\mathcal{A}\mathrm{lg}$ *is a fibration (resp. a weak equivalence) if and only if* $\bar{N}f$ *is a projective fibration (resp. a projective weak equivalence). Furthermore, we have a Quillen adjunction*

$$\bar{\tau} : \mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{proj}} \rightleftarrows \mathbb{I}\mathcal{A}\mathrm{lg}_{\mathrm{Horel}} : \bar{N},$$

and $\bar{N}$ *preserves cofibrations.*

Proof. We apply theorem 16.2.2. Recall that $\mathbb{I}\mathcal{A}\mathrm{lg}$ is complete and cocomplete, and by proposition 16.2.3, $\bar{N}$ preserves filtered colimits. Let $f \in \mathsf{I}_{\mathbb{A}}$ be a generating projective cofibration, say $f = \mathsf{b}_m \boxtimes \lambda : \partial\Delta[m] \boxtimes \lambda \longrightarrow \Delta[m] \boxtimes \lambda$, where $m \in \mathbb{N}$ and $\lambda \in \mathbb{A}$. By lemma 16.1.9, $\bar{\tau}(K \boxtimes \lambda) \cong K \otimes \lambda^{\mathrm{const}}$ for all $K \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$, and in particular, $\bar{\tau}f = \mathsf{b}_m \otimes \lambda^{\mathrm{const}} : \partial\Delta[m] \otimes \lambda^{\mathrm{const}} \longrightarrow \Delta[m] \otimes \lambda^{\mathrm{const}}$, Consider a pushout of $\bar{\tau}f$, say

$$\begin{tikzcd} \partial\Delta[n] \otimes \lambda^{\mathrm{const}} \arrow[r] \arrow[d, "\bar{\tau}f"'] & C \arrow[d, "g"] \\ \Delta[n] \otimes \lambda^{\mathrm{const}} \arrow[r] & D. \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

By lemma 16.1.9, $\bar{N}\bar{\tau}f \cong f$, and by corollary 16.2.6, $\bar{N}g$ is a pushout of $\bar{N}\bar{\tau}f \cong f$, thus a cofibration. Consequently, $\bar{N}$ maps pushouts of maps in $\bar{\tau}\mathsf{I}_{\mathbb{A}}$ to cofibrations. A similar reasoning shows that $\bar{N}$ maps pushouts of maps in $\bar{\tau}\mathsf{J}_{\mathbb{A}}$ to trivial cofibrations. □

Example 16.2.8. If $k = n = 1$, then $\mathbb{I}\mathcal{A}\mathrm{lg}$ is the category $\mathbb{I}\mathcal{C}\mathrm{at}$ of internal categories in $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$, and one can check that $\mathbb{I}\mathcal{A}\mathrm{lg}_{\mathrm{Horel}}$ matches the model structure on $\mathbb{I}\mathcal{C}\mathrm{at}$ given in [Hor15].

Proposition 16.2.9 (Generalization of [Hor15, proposition 4.3]). *If* $X \in \mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{proj}}$ *is cofibrant, then the unit map* $\eta_X : X \longrightarrow \bar{N}\bar{\tau}X$ *is an isomorphism.*

Proof. Since the adjunction $\bar{\tau} \dashv \bar{N}$ is reflective, the unit map η_X is an isomorphism if and only if X is a nerve, i.e. of the form $\bar{N}B$ for some $B \in \mathbb{I}\mathcal{A}\mathrm{lg}$. We now show that all cofibrant objects (i.e. retracts of $\mathsf{I}_{\mathbb{A}}$-cell complexes) are nerves.

(1) The initial object $\varnothing$ is a nerve, as $\varnothing = \bar{N}\bar{\tau}\varnothing$.
(2) Clearly, if Y is a retract of X, then η_Y is the retract of η_X. Therefore, if X is a nerve, so is Y.

[1] In [Hor15, section 5.1], it is called the *levelwise model structure*

(3) We now show that pushouts of nerves along maps in $\mathsf{I}_{\mathbb{A}}$ are nerves. Let $n \in \mathbb{N}$, $\lambda \in \mathbb{A}$, X be a nerve, and Y be the pushout on the right.

$$\begin{tikzcd} \partial\Delta[n] \otimes \lambda^{\mathrm{const}} \arrow[r] \arrow[d] & \bar{\tau}X \arrow[d] \\ \Delta[n] \otimes \lambda^{\mathrm{const}} \arrow[r] & C, \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} \partial\Delta[n] \boxtimes \lambda \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta[n] \boxtimes \lambda \arrow[r] & Y, \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

By assumption, $X \cong \bar{N}\bar{\tau}X$, and by lemma 16.1.9 we have $\bar{N}(K \otimes \lambda^{\mathrm{const}}) \cong K \boxtimes \lambda$ for all $K \in \mathcal{P}\mathrm{sh}(\mathbb{\Delta})$. Therefore, the cospan in the right square is the nerve of the cospan in the left square, and let C be its pushout. By corollary 16.2.6, $Y \cong \bar{N}C$.

(4) Let $X = \operatorname{colim}_i X_i$ be a filtered colimit of nerves. As a left adjoint, $\bar{\tau}$ preserves all colimits, and by proposition 16.2.3, the nerve $\bar{N}$ preserve filtered colimits. In particular, the unit map η_X of X is the filtered colimit $\operatorname{colim}_i \eta_{X_i}$ (in $\mathcal{S}\mathrm{p}(\mathbb{A})^{[1]}$), and thus an isomorphism.

(5) Let $X_\beta = \operatorname{colim}_{\alpha<\beta} X_\alpha$ be a relative $\mathsf{I}_{\mathbb{A}}$-cell complex, where X_0 is a nerve. We show that each X_γ is a nerve by transfinite induction, for $\gamma \leq \beta$. The case $\gamma = 0$ is an assumption. If X_γ is a nerve, then so is $X_{\gamma+1}$ by point 3. The limit case is handled by point 4.

Finally, all cofibrant objects of $\mathcal{S}\mathrm{p}(\mathbb{A})$ are nerves. □

Theorem 16.2.10 (Generalization of [Hor15, proposition 5.4]). *The Quillen adjunction*

$$\bar{\tau} : \mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{proj}} \rightleftarrows \mathbb{I}\mathcal{A}\mathrm{lg}_{\mathrm{Horel}} : \bar{N}$$

of theorem 16.2.7 is a Quillen equivalence.

Proof. Let $C \in \mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{proj}}$ be cofibrant, $P \in \mathbb{I}\mathcal{A}\mathrm{lg}_{\mathrm{Horel}}$ be fibrant, and $f : \bar{\tau}C \longrightarrow P$ be a morphism. Then its adjoint $\tilde{f} : C \longrightarrow \bar{N}P$ is given by the composite

$$C \xrightarrow{\eta_C} \bar{N}\bar{\tau}C \xrightarrow{\bar{N}f} \bar{N}P.$$

By proposition 16.2.9, and since C is cofibrant, η_C is an isomorphism. By definition of the right induced model structure in $\mathbb{I}\mathcal{A}\mathrm{lg}$ (see theorem 16.2.2), $\bar{N}$ preserves and reflects weak equivalences. Finally, f is a weak equivalence if and only its adjoint $\tilde{f}$ is, and the adjunction is a Quillen equivalence. □

16.3 ANOTHER MODEL FOR ∞-ALGEBRAS

The Horel model structure of theorem 16.2.7 serves as a basis for an alternative model for ∞-algebras, namely *complete Segal algebras* (definition 16.3.4), which we introduce in this section. More We show that they are the fibrant objects of $\mathbb{I}\mathcal{A}\mathrm{lg}_{\mathrm{Rezk}}$, the *Rezk model structure* for internal algebras, and in theorem 16.3.5, we establish a span of Quillen equivalences

$$\mathbb{I}\mathcal{A}\mathrm{lg}_{\mathrm{Rezk}} \overset{\bar{\tau}}{\leftrightarrows} \mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{pRezk}} \overset{\mathrm{id}}{\rightleftarrows} \mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{Rezk}},$$

where $\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{pRezk}}$ is analogous to $\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{Rezk}}$ but built from the projective model structure instead of the vertical one.

Definition 16.3.1. Akin to definitions 15.1.1 and 15.2.3, the *projective Segal model structure* (resp. the *projective Rezk model structure*) on $\mathcal{S}\mathrm{p}(\mathbb{A})$ is the left Bousfield localization $\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{pSegal}} := \mathsf{S}^{-1}\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{proj}}$ (resp. $\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{pRezk}} := \mathsf{R}^{-1}\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{pSegal}}$).

Proposition 16.3.2. *Localizing the Quillen equivalence between the projective and vertical (see definition 12.3.19) model structures on* $\mathcal{S}\mathrm{p}(\mathbb{A})$ *we obtain a Quillen equivalence*

$$\mathrm{id} : \mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{pSegal}} \rightleftarrows \mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{Segal}} : \mathrm{id},$$

and likewise for the Rezk structures.

Proof. Since $\mathrm{id} : \mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{proj}} \longrightarrow \mathcal{S}\mathrm{p}(\mathbb{A})_v$ preserves and reflects weak equivalences, it maps a cofibrant approximation of a spine inclusions $\mathsf{s}_{h\omega}$ (in $\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{proj}}$) to a map that is weakly equivalent to $\mathsf{s}_{h\omega}$ (in $\mathcal{S}\mathrm{p}(\mathbb{A})_v$). Thus, we have a chain of Quillen equivalences

$$\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{pSegal}} \rightleftarrows (\mathrm{d}\,\mathrm{id}\,\mathsf{S})^{-1}\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{proj}} \rightleftarrows \mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{Segal}},$$

where the first one derives from theorem 12.1.30. The proof for the equivalence between the Rezk structures is similar. □

Proposition 16.3.3. *The identity adjunction* $\mathrm{id} : \mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{proj}} \rightleftarrows \mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{pSegal}} : \mathrm{id}$ *is a Quillen equivalence. Consequently,* $\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{pRezk}}$ *is equivalent to* $\mathsf{R}^{-1}\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{proj}}$.

Proof. Localize the Quillen equivalence of theorem 16.2.10 as follows:

$$\begin{tikzcd}
\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{proj}} \arrow[r, shift left, "\bar{\tau}"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathrm{I}\mathcal{A}\mathrm{lg}_{\mathrm{Horel}} \arrow[l, shift left, "\bar{N}"] \arrow[d] \\
\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{pSegal}} \arrow[r, shift left, "\bar{\tau}"] \arrow[r, phantom, "\perp"] & (\mathrm{d}\bar{\tau}\mathsf{S})^{-1}\mathrm{I}\mathcal{A}\mathrm{lg}_{\mathrm{Horel}}. \arrow[l, shift left, "\bar{N}"]
\end{tikzcd}$$

By theorem 11.1.39, $\tau : \mathcal{P}\mathrm{sh}(\mathbb{A}) \longrightarrow \mathcal{A}\mathrm{lg}$ maps spine inclusions to isomorphisms. Therefore, $\bar{\tau}$ maps spine inclusions to isomorphisms, and the left derived functor $\mathrm{d}\bar{\tau}$ maps spine inclusions to weak equivalences. Consequently, the left Bousfield localization $(\mathrm{d}\bar{\tau}\mathsf{S})^{-1}\mathrm{I}\mathcal{A}\mathrm{lg}_{\mathrm{Horel}}$ is equivalent to $\mathrm{I}\mathcal{A}\mathrm{lg}_{\mathrm{Horel}}$. On the other hand, the lower adjunction is a Quillen equivalence by definition 12.1.28. Quillen equivalences satisfy the 3-for-2 property, and the result follows. □

Definition 16.3.4 (Rezk structure for $\mathrm{I}\mathcal{A}\mathrm{lg}$)**.** Let $\mathrm{I}\mathcal{A}\mathrm{lg}_{\mathrm{Rezk}}$, the *Rezk model structure* on $\mathrm{I}\mathcal{A}\mathrm{lg}$, be the left Bousfield localization $\mathrm{I}\mathcal{A}\mathrm{lg}_{\mathrm{Rezk}} := (\mathrm{d}\bar{\tau}\mathsf{R})^{-1}\mathrm{I}\mathcal{A}\mathrm{lg}_{\mathrm{Horel}}$. Fibrant objects are called *complete Segal algebras*.

Theorem 16.3.5. *The Quillen equivalence of theorem 16.2.10 descends to the Rezk structures, and we have a span of Quillen equivalences:*

$$\mathrm{I}\mathcal{A}\mathrm{lg}_{\mathrm{Rezk}} \underset{\longrightarrow}{\overset{\bar{\tau}}{\longleftarrow}} \mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{pRezk}} \underset{\longleftarrow}{\overset{\mathrm{id}}{\longrightarrow}} \mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{Rezk}}.$$

Proof. Follows from propositions 16.3.2 and 16.3.3 and theorem 12.1.30. □

Chapter Seventeen

Conclusion

In this thesis, we explored the theory and applications of opetopes. Starting from the polynomial approach of Kock et. al., we presented developments of this field, with an emphasis on syntactical and algebraic aspects. We now briefly summarize the main results of this work.

The polynomial approach to opetopes. The very first undertaking of this thesis is to carefully review the polynomial definition of opetopes of Kock et. al. [KJBM10]. Special attention is given to the structure of polynomial monads, and we show that, as algebras over $(-)^\star$ (definition 2.3.3 and theorem 2.3.5), they essentially are tree calculi (theorem 2.3.6 and remark 2.3.13). From there, a completely formal definition of the Baez–Dolan construction in the polynomial setting is given: if M is a polynomial monad, then M^+ is the calculus of M-tree substitutions (theorem 2.4.6).

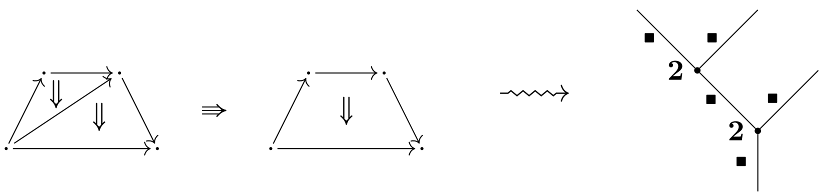
Then, following Kock et. al. [KJBM10], the set $\mathbb{O}_n$ of n-opetopes is defined as the set of colors of $\mathfrak{Z}^n := \mathrm{id}_{\mathcal{S}\mathrm{et}}^{++\cdots+}$. Very informally, this means that an n-opetope is a

$$\underbrace{\text{tree of trees of trees of trees of } \ldots \text{ of trees of}}_{n \text{ times}} \text{ points.}$$

Ultimately, manipulating opetopes amounts to reasoning about their nodes and adjacencies. Applying the detailed review of the theory of polynomial functors and trees provides the formalism of higher-dimensional addresses (section 3.3), a compact notation that gives "simultaneous walking instructions" through all of the trees that make up a given opetope.

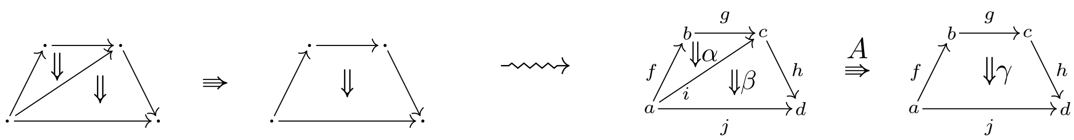
Opetopic sets and many-to-one polygraphs. In the opetopic setting, trees and pasting diagrams are the two faces of the same coin. Using the latter point of view, it is apparent that opetopes assemble into a category $\mathbb{O}$ whose morphisms are geometrical face embeddings (section 3.4). Presheaves over $\mathbb{O}$ are called opetopic sets, and we show that they are equivalent to many-to-one polygraphs. In more details, if $\mathcal{P} \in \mathcal{P}\mathrm{ol}^{\mathrm{mto}}$, then its opetopic nerve $N\mathcal{P} \in \mathcal{P}\mathrm{sh}(\mathbb{O})$ is $\mathcal{P}$ stripped of its algebraic structure. The source $\mathsf{s}\,x$ of a generator $x \in \mathcal{P}_n$ can no longer be a composition of $(n-1)$-generators, but the information about the adjacency of the various occurrences of those $(n-1)$-generators is enough to fully reconstruct $\mathsf{s}\,x$. The occurrence and adjacency data is precisely what $N\mathcal{P}$ stores. We recall that this correspondence was already known from [HMP00] [HMZ02] [HMZ08], but our approach is much more concise.

Syntax. In an effort to facilitate the use of opetopes, we turn our attention to syntactical systems that fully encode them. A formalization which is adapted for computerized manipulation would indeed be desirable. The tree and pasting diagram dichotomy mentioned earlier gives rise to two very different systems, which we respectively call the unnamed and named approach. In the unnamed approach, the recursive tree nature of opetopes is represented in a construct called preopetope. For example,



$$\rightsquigarrow \quad \begin{cases} [] \leftarrow \begin{cases} [] \leftarrow \{ * \leftarrow \blacklozenge \\ [*] \leftarrow \{ * \leftarrow \blacklozenge \end{cases} \\ [[*]] \leftarrow \begin{cases} [] \leftarrow \{ * \leftarrow \blacklozenge \\ [*] \leftarrow \{ * \leftarrow \blacklozenge \end{cases} \end{cases}$$

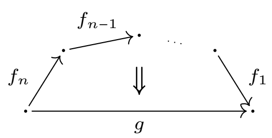
The named approach, on the other hand, gives names to the (source) faces of an opetope, and encodes the adjacency data in terms:



$$\rightsquigarrow \qquad A : \beta(i \leftarrow \alpha) \multimap h(c \leftarrow g(b \leftarrow f)) \multimap a \multimap \varnothing.$$

We present the $\mathrm{O{\scriptsize PT}}^?$ derivation system, which characterizes opetopes among all pre-opetopes, and similarly in the named approach, where system $\mathrm{O{\scriptsize PT}}^!$ determines which terms are well-formed, i.e. correspond to opetopes. We also present variants of these systems, respectively $\mathrm{O{\scriptsize PT}S{\scriptsize ET}}^?$ and $\mathrm{O{\scriptsize PT}S{\scriptsize ET}}^!$, that derive finite opetopic sets.

Opetopic algebras. A category $\mathcal{C}$ has a very natural opetopic nerve $M_{1,1}\mathcal{C} \in \mathcal{P}\mathrm{sh}(\mathbb{O})$ whose points are the objects of $\mathcal{C}$, arrows are the morphisms, and the 2-cells witness the relations of $\mathcal{C}$: an equality of the form of the form $f_1 \cdots f_{n-1} f_n = g$ is witnessed by a 2-cell of the form



The same idea works for operads, but this time, relations are encoded at the level of 3-cells. In continuation of this idea, we present the notion of k-colored n-dimensional opetopic algebra, which are algebraic structures that can be presented by opetopic sets. Specifically, we extend the polynomial monad $\mathfrak{Z}^n$ of chapter 3 to a p.r.a. monad over $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$, which can be thought of as a "free pasting diagram" monad (section 11.1). Then, we establish a nerve theorem which exhibits $\mathcal{A}\mathrm{lg}_{k,n}$, the category of (k,n)-algebras, as an orthogonality class in $\mathcal{P}\mathrm{sh}(\mathbb{O})$. Roughly speaking, it shows that an opetopic algebra is an opetopic set X such that $\mathsf{S}_{\geq n+1} \perp X$, where $\mathsf{S}_{\geq n+1}$ is the set of spine inclusions of opetopes of dimension $\geq n+1$. This is very similar to the fact that a category is a simplicial set Y such that $\mathsf{S}_{\geq 2} \perp Y$, where this time $\mathsf{S}_{\geq 2}$ is the set of spine inclusions of simplices of dimension ≥ 2.

The notion of opetopic algebra seems to provide different structures for each choice of parameters $k, n \in \mathbb{N}$ with $k \leq n$, but this is not really the case. In a phenomenon we call algebraic trompe-l'œil, we show that all opetopic algebra can essentially be reduced to a "colored combinad", i.e. a $(1,3)$-algebra. These structures sit 1 dimension above operads, and, at the time of writing, have not been thoroughly studied in the literature.

Models of ∞-algebras. This thesis also studies ∞-algebras (or weak algebras), where associativity and unitality only hold up to coherent homotopy (in this case, coherent higher cell). For example, instead of requiring that $\mathsf{S}_{\geq n+1} \perp X$ as in the strict case, we only have $\mathsf{S}_{\geq n+1} \pitchfork X$. The nerve of a strict opetopic algebra can also be described in terms of a presheaf over the category $\mathbb{A} = \mathbb{A}_{k,n}$ of opetopic shapes (definition 11.1.33). Although the interplay between $\mathcal{P}\mathrm{sh}(\mathbb{O})$ and $\mathcal{P}\mathrm{sh}(\mathbb{A})$ is rather technical, the latter proves to be a much better presheaf model when studying weak algebras (see e.g. proposition 13.5.3).

The homotopy theory of ∞-algebras is encapsulated in the model structure $\mathcal{P}\mathrm{sh}(\mathbb{A})_\infty$, constructed in theorem 13.3.21 using methods of Cisinski's homotopy theory of presheaves [Cis06]. We also provide simplicial models for ∞-algebras. Then, in the style of Rezk [Rez01], Joyal and Tierney [JT07], and Cisinski and Moerdijk [CM13], we consider the category $\mathcal{S}\mathrm{p}(\mathbb{A})$ of simplicial presheaves over $\mathbb{A}$. Starting from the Reedy model structure, and by successive left Bousfield localizations, we construct the Rezk structure $\mathcal{S}\mathrm{p}(\mathbb{A})_{\mathrm{Rezk}}$, and show that it is Quillen-equivalent to $\mathcal{P}\mathrm{sh}(\mathbb{A})_\infty$ (theorem 15.2.11). We also investigate an alternative approach, where in the style of [Hor15], we consider the category $\mathrm{I}\mathcal{A}\mathrm{lg}$ of opetopic algebras internal to simplicial sets. There, we start from the Horel structure $\mathrm{I}\mathcal{A}\mathrm{lg}_{\mathrm{Horel}}$ (theorem 16.2.7), and by a similar process of left Bousfield localization, obtain a new model $\mathrm{I}\mathcal{A}\mathrm{lg}_{\mathrm{Rezk}}$ for ∞-algebras (theorem 16.3.5).

PERSPECTIVES

We now expose some research directions that are in continuation of the themes of this thesis.

Symmetric opetopes. Perhaps the most immediate field left uncharted in this thesis is that of symmetric opetopes. The "original" opetopes of Baez and Dolan [BD98] where built from symmetric operads and the so-called slice construction (analogous to the $(-)^+$ construction presented in definition 2.4.1). Cheng later improved the theory by constructing a category of opetopes [Che03a] [Che04b] by generators and relations, much in the style of definition 3.4.2. However, the geometrical intuition stems from a different formalism for trees, relying on Kelly–Mac Lane graphs [Che03b] [Che06].

The approach using polynomial trees offers the valuable notion of higher address, used ubiquitously throughout this work in order to refer to source faces (or nodes) of opetopes. Therefore, it would be desirable to formulate a definition of symmetric opetopes in this context. It is not clear how the formalism of polynomial monads and trees can be lifted to the symmetric setting, however. For example, monads describing structures with symmetries (e.g. commutativity) tend to not be cartesian, thus not polynomial.

Computational models for ∞-categories. System $\textsc{OptCat}^?$ is an elegant implementation of Hermida's "coherence via universality" principle [Her01], but the correspondence with ∞-categories is not proved in this thesis. More formally, one would like an equivalence between the category of contexts of $\textsc{OptCat}^?$ (or rather, an adequate definition thereof) and the category of finitely presentable ∞-categories (e.g. quasi-categories). Unfortunately, experimental evidences suggest that rule `tclose` is too strong, although the subject is still actively being investigated.

A computer implementation of opetopes leveraging either syntactic approach presented in part II would further promote opetopes as a convenient formalism for higher structures. In particular, a proof assistant based on a correct version of system $\textsc{OptCat}^?$ would be of practical use for the study of higher coherence laws in ∞-categories or ∞-groupoids. An implementation of the other syntactical systems of part II already exists [Ho 18a] but is only a proof of concept at the moment.

Beyond ∞-categories, it seems very plausible that these syntactical methods can be pushed to the setting of ∞-algebras. How much work would be required partially depends on the perspective we present next.

Homotopical trompe-l'œil. The algebraic trompe-l'œil (theorem 11.3.16) states that a (k, n)-opetopic algebra can essentially be reduced to a $(1, 3)$-opetopic algebra, which can be thought of as colored combinad. What about weak opetopic algebras? More formally, if $(-)^\vee : \mathcal{P}\mathrm{sh}(\mathbb{A}_{k,n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{A}_{1,3})$ is the left Kan extension of the composite

$$\mathbb{A}_{k,n} \longrightarrow \mathcal{A}\mathrm{lg}_{k,n} \xrightarrow{(-)^\vee} \mathcal{A}\mathrm{lg}_{1,3} \xrightarrow{N_{1,3}} \mathcal{P}\mathrm{sh}(\mathbb{A}_{1,3})$$

along the Yoneda embedding, and D is its right adjoint, what can be said about the adjunction $(-)^\vee \dashv D$ when considering the model structures of theorem 13.3.21 on $\mathcal{P}\mathrm{sh}(\mathbb{A}_{k,n})$ and $\mathcal{P}\mathrm{sh}(\mathbb{A}_{1,3})$?

Towards oriented types. Homotopy type theory [Uni13] (thereafter HoTT) is a dependent type theory where special attention is given to identity types. In a nutshell, types are thought of as spaces, elements of a type as points, and an identity between two elements as a path in the ambient space. Paths between paths, i.e. homotopies, are then identities between identities, and so on. Naturally, HoTT acts as an internal language for ∞-groupoids, where higher identities are interpreted as higher dimensional cells. One would be interested in types that model ∞-categories instead. Much like opetopes representing compositions of all valid pasting diagrams of lower dimensional cells, "opetopic types" should have unbiased compositions of identities. This would be in contrast to the usual binary composition of homotopies, which defined by induction on identity types. A possible approach is to introduce identity types parametrized by pasting diagrams of lower dimensional identities, along with an adequate induction principle, which should be related to the axioms of system $\textsc{OptCat}^?$ (see section 9.4).

Further generalizations of the methods of Joyal–Tierney and Horel. In chapter 13, we construct the model structure $\mathcal{P}\mathrm{sh}(\mathbb{A})_\infty$, and in chapters 15 and 16, we provide alternative models for ∞-algebras by generalizing the approaches of Joyal and Tierney [JT07] and

Horel [Hor15], respectively. However, at their core, these methods do not seem to fundamentally rely on the combinatorics of opetopes. We ask if it is possible to lift them to a more general setting.

For example, let $\mathcal{C}$ be a normal skeletal category, and (I, K) be a homotopical structure on $\mathcal{P}\mathrm{sh}(\mathcal{C})$. In particular, the Cisinski model structure of theorem 12.2.8 exists, and we denote it by $\mathcal{P}\mathrm{sh}(\mathcal{C})_{\mathrm{Cisinski}}$. By **(An0)**, K is the saturation of a set J of monomorphisms, and assume that J admits the small object argument. In particular, J is a set of generating acyclic cofibrations. Consider now $\mathcal{S}\mathrm{p}(\mathcal{C})_v$, the category of simplicial presheaves over $\mathcal{C}$ endowed with the vertical model structure. Let the *Segal model structure* $\mathcal{S}\mathrm{p}(\mathcal{C})_{\mathrm{Segal}}$ be the left Bousfield localization

$$\mathcal{S}\mathrm{p}(\mathcal{C})_{\mathrm{Segal}} \coloneqq (\mathsf{J}^{\mathrm{disc}})^{-1} \mathcal{S}\mathrm{p}(\mathcal{C})_v .$$

For $c \in \mathcal{C}$, let $i_0, i_1 : c \longrightarrow \mathrm{I}c$ be the endpoint inclusions of c into its corresponding cylinder object, and E be the set of all endpoint inclusions. Define the *Rezk model structure* $\mathcal{S}\mathrm{p}(\mathcal{C})_{\mathrm{Rezk}}$ to be the left Bousfield localization

$$\mathcal{S}\mathrm{p}(\mathcal{C})_{\mathrm{Rezk}} \coloneqq (\mathsf{E}^{\mathrm{disc}})^{-1} \mathcal{S}\mathrm{p}(\mathcal{C})_{\mathrm{Segal}} .$$

Conjecture. *We have a Quillen equivalence*

$$(-)^{\mathrm{disc}} : \mathcal{P}\mathrm{sh}(\mathcal{C})_{\mathrm{Cisinski}} \underset{\perp}{\overset{\sim}{\rightleftarrows}} \mathcal{S}\mathrm{p}(\mathcal{C})_{\mathrm{Rezk}} : (-)_{-,0} .$$

Recall that by theorem 0.5.15, presheaves $X \in \mathcal{P}\mathrm{sh}(\mathcal{C})$ such that $\mathsf{J} \perp X$ correspond to models of a projective sketch over $\mathcal{C}^{\mathrm{op}}$. Let $\mathbb{M}\mathrm{od}$ be the category of simplicial models of that sketch. We have a natural reflective adjunction $\tau : \mathcal{S}\mathrm{p}(\mathcal{C}) \underset{\perp}{\rightleftarrows} \mathbb{M}\mathrm{od} : N$.

Conjecture. *The projective model structure over* $\mathcal{S}\mathrm{p}(\mathcal{C})$ *can be transferred along* N*, i.e. there exists a model structure over* $\mathbb{M}\mathrm{od}$ *where a morphism* f *is a fibration (resp. a weak equivalence) if and only of* Nf *is a projective fibration (resp. a projective weak equivalence).*

Assuming it exists, denote this model structure by $\mathbb{M}\mathrm{od}_{\mathrm{Horel}}$. The localization methods of section 16.3 should produce a Rezk structure $\mathbb{M}\mathrm{od}_{\mathrm{Rezk}}$ which is related to $\mathcal{S}\mathrm{p}(\mathcal{C})_{\mathrm{Rezk}}$ (and thus to $\mathcal{P}\mathrm{sh}(\mathcal{C})_{\mathrm{Cisinski}}$) via a zig-zag of Quillen equivalences.

to be continued...

Part IV

Additional material

Appendix A

Linear opetopic sets

In this chapter, we generalize some results of [LV12] to the opetopic setting. Let R be a commutative ring, $\mathcal{M}\mathrm{od}_R$ be the category of two sided R-modules, and ${}_{\mathrm{dg}}\mathcal{M}\mathrm{od}_R$ be the category of differential graded R-modules. Recall that a (planar) algebraic R-operad P is a sequence of R-modules $(P_0, P_1, P_2, \ldots)$, and for each finite sequence of integers $k_1, \ldots, k_m$, a *composition map*

$$\begin{aligned} \gamma : P_m \bigotimes_{i=1}^{m} P_{k_i} &\longrightarrow P_{k_1 + \cdots + k_m} \\ (a; b_1, \ldots, b_m) &\longmapsto a(b_1, \ldots, b_m), \end{aligned}$$

satisfying adequate unitality and associativity conditions. Intuitively, $a \in P_m$ is an operation with m inputs, and γ grafts all the b_i on these inputs, giving an operation of $k_1 + \cdots + k_m$ inputs. In the opetopic setting, the arity of an operation x of an "n-opetopic operad" would not be a list, but rather an n-opetope. Thus an n-opetopic operad X would be a collection of R-modules $(X_\omega \mid \omega \in \mathbb{O}_n)$, and the composition map would be of the form

$$\gamma : P_\omega \bigotimes_{[p] \in \omega^\bullet} P_{\psi_{[p]}} \longrightarrow P_{\omega'},$$

where $\mathsf{t}\, \psi_{[p]} = \mathsf{s}_{[p]}\, \omega$, and $\omega' := \omega \,\square_{[p]}\, \psi_{[p]}$. Of course, γ is expected to satisfy unitality and associativity conditions analogous to the classical operadic (or 2-opetopic) case. This can be extremely tricky to write down. A homogeneous element of $P_\omega \otimes_{[p] \in \omega^\bullet} P_{\psi_{[p]}}$ is a tensor of an element of P_ω and a *tree* of elements of P, rather than just a list. Rearranging terms to express the associativity equation of γ is an excruciating endeavor. In the differential graded setting, the Koszul sign rule is simply unmanageable.

In this chapter, we present a formalism designed to tackle this problem, and manage elements of an "higher" opetopic operad in a more tractable fashion. Classical operads are recovered in the case $n = 2$, and associative R-algebras are recovered in the case $n = 1$. We reformulate classical constructions and results in this setting, such as the bar-cobar adjunction.

A.1 OPETOPIC MODULES AND SCHUR FUNCTORS

OPERATIONS ON $\mathbb{O}_n$-MODULES

Definition A.1.1 (Opetopic module)**.** Let $\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n} := \prod_{\omega \in \mathbb{O}_n} \mathcal{M}\mathrm{od}_R$ be the category of *n-opetopic modules*, or *$\mathbb{O}_n$-modules* for short. If $X \in \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$ and $\omega \in \mathbb{O}_n$, we write $X_\omega \in \mathcal{M}\mathrm{od}_R$ for its component corresponding to ω.

Notation A.1.2. Let $X, Y_1, \ldots, Y_k \in \mathcal{M}\mathrm{od}_R$. Much in the spirit of the $\bigcirc$ notation (see notation 2.2.23), we write

$$X \bigotimes_i Y_i := X \otimes \bigotimes_{i=1}^{k} Y_i = X \otimes Y_1 \otimes \cdots \otimes Y_k.$$

Definition A.1.3 (Operations on opetopic modules)**.** For $X, Y \in \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$, let their *sum*, *tensor product*, and *composite product* be

$$(X \oplus Y)_\omega := X_\omega \oplus Y_\omega, \tag{A.1.4}$$

$$(X \otimes Y)_\omega := \bigoplus_{\omega_1 \circ_{[l]} \omega_2 = \omega} X_{\omega_1} \otimes Y_{\omega_2}, \tag{A.1.5}$$

$$(X \circ Y)_\omega := \bigoplus_{\nu \,\square_{[p_i]}\, \nu_i = \omega} X_\nu \bigotimes_i Y_{\nu_i}, \tag{A.1.6}$$

where $\omega \in \mathbb{O}_n$. In equation (A.1.5), the sum ranges over all decompositions of ω as a grafting of the form $\omega_1 \circ_{[l]} \omega_2$, for $\omega_1, \omega_2 \in \mathbb{O}_n$ and $[l] \in \omega_1$. In equation (A.1.6), the sum ranges over all decomposition of ω as a simultaneous substitution, and by convention, the node addresses $[p_i]$ of ν are sorted in lexicographical order. Equivalently, the sum of equation (A.1.6) ranges over all $\xi \in \mathbb{O}_{n+1}^{(2)}$ (see definition 3.1.7) such that $\mathsf{t}\,\xi = \omega$, in which case $\nu := \mathsf{s}_{[l]}\,\xi$, and $\nu_i := \mathsf{s}_{[[p_i]]}\,\xi$.

Proposition A.1.7. *The operations $\oplus$, $\otimes$, and $\circ$ of definition A.1.3 are associative.*

Proof. Straightforward computations. □

Examples A.1.8.

(1) If $n = 1$, then $\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n} = \mathcal{M}\mathrm{od}_R$ is just the category of R-modules, and $\circ$ is the usual tensor product.

(2) If $n = 2$, then $\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n} = \mathcal{M}\mathrm{od}_R^{\mathbb{N}}$ is the category of non negatively graded R-modules, which we consider as a planar version of $\mathbb{S}$-modules [LV12, section 5.1]. For $X, Y \in \mathcal{M}\mathrm{od}_R^{\mathbb{O}_2}$ and $k \in \mathbb{N}$, we have that

$$(X \circ Y)_n = \bigoplus_{a_1 + \cdots + a_k = n} X_k \bigotimes_i Y_{a_i}$$

which corresponds to the composite product of [LV12, section 5.1.6].

Notation A.1.9. Let $X, Y \in \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$ be opetopic modules, $\omega \in \mathbb{O}_n$, and consider a homogeneous element $x \otimes y_1 \otimes y_2 \otimes \cdots \in (X \circ Y)_\omega$, say in the $X_\nu \otimes_i Y_{\nu_i}$ component. It will be concisely denoted as $\frac{y}{x}$, where y is the sequence $y_1, y_2, \ldots$. Recall that by convention, the node addresses $[p_i]$ of ν are sorted in lexicographical order. In particular, $y_1 \in Y_{\nu_1}$ "corresponds" to the node address $[] \in \nu^\bullet$. Similarly, if $W \in \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$, then homogeneous elements of $W \circ X \circ Y$ we be denoted by

$$\frac{\dfrac{y}{x}}{w}$$

Definition A.1.10. The operations on n-opetopic modules presented above have neutral elements that we define now. The identically null module 0 is clearly a two sided neutral element for $\oplus$. Define $R, I \in \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$ as follows:

$$R_\omega := \begin{cases} R\omega & \text{if } \omega \text{ is degenerate,} \\ 0 & \text{otherwise,} \end{cases} \qquad I_\omega := \begin{cases} R\omega & \text{if } \omega \text{ is an endotope,} \\ 0 & \text{otherwise,} \end{cases}$$

where $R\omega$ is the free R-module on 1 generator ω. An element of R_ω will be written $r\omega$, for $r \in R$, or simply by r if no ambiguity arise.

Example A.1.11. If $n = 2$, then R is the $\mathbb{N}$-graded module $(R, 0, 0, \ldots)$, while $I = (0, R, 0, 0, \ldots)$.

Lemma A.1.12. *(1) The $\mathbb{O}_n$-module R is a left[1] neutral element for $\otimes$.*
(2) The $\mathbb{O}_n$-module I is a two sided neutral element for $\circ$.

Proof. (1) For $X \in \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$ and $\omega \in \mathbb{O}_n$ we have

$$(R \otimes X)_\omega = \bigoplus_{\omega_1 \circ \omega_2 = \omega} R_{\omega_1} \otimes X_{\omega_2} = R_{\mathsf{I}_{\mathsf{e}_{[]}\omega}} \otimes X_\omega = X_\omega,$$

as indeed, the only decomposition of ω as a grafting $\omega_1 \circ_{[l]} \omega_2$ where ω_1 is degenerate is $\omega = \mathsf{I}_{\mathsf{e}_{[]}\omega} \circ_{[]} \omega$.

(2) For $X \in \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$ and $\omega \in \mathbb{O}_n$ we have

$$(I \circ X)_\omega = \bigoplus_{\nu \,\square_i\, \nu_i = \omega} I_\nu \bigotimes_i X_{\nu_i} = I_{\mathsf{Y}_{\mathsf{t}\omega}} \otimes X_\omega = X_\omega,$$

as indeed, the only decomposition of ω as a simultaneous substitution $\nu \,\square_{[p_i]}\, \nu_i$ where ν is an endotope is $\omega = \mathsf{Y}_{\mathsf{t}\omega} \,\square_{[]}\, \omega$. For the other side, we have

$$(X \circ I)_\omega = \bigoplus_{\nu \,\square_i\, \nu_i = \omega} X_\nu \bigotimes_i I_{\nu_i} = X_\omega \bigotimes_i I_{\mathsf{Y}_{\mathsf{s}_{[p_i]}\omega}} = X_\omega,$$

as indeed, the only decomposition of ω as a simultaneous substitution $\nu \,\square_{[p_i]}\, \nu_i$ where all ν_i's are endotopes is $\omega \,\square_{[p_i]}\, \mathsf{Y}_{\mathsf{s}_{[p_i]}\omega}$. $\square$

Definition A.1.13 (Augmented, coaugmented)**.** An opetopic module $X \in \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$ is *augmented* if it is equipped with an *augmentation map* $\varepsilon : X \longrightarrow I$. In this case, we write $\bar{X} := \ker \varepsilon$ for the *augmentation ideal* of X. Dually, X is *coaugmented* if it is equipped with a *coaugmentation map* $\eta : I \longrightarrow X$, and we let $\bar{X} := \operatorname{coker} \eta$ be the *coaugmentation quotient* of X.

SCHUR FUNCTORS

Definition A.1.14 (Opetopic vector space)**.** Let $\mathcal{V}\mathrm{ect}_R^{\mathbb{O}_n} := \mathcal{M}\mathrm{od}_R^{\mathbb{O}_{n-1}}$ be the category of $\mathbb{O}_n$*-vector spaces*. It has direct sums, given pointwise, akin to equation (A.1.4). We now endow it with a tensor product. For $V, W \in \mathcal{V}\mathrm{ect}_R^{\mathbb{O}_n}$, and $\psi \in \mathbb{O}_{n-1}$, let

$$(V \otimes W)_\psi := \bigoplus_{\psi_1 \,\square_{[p]}\, \psi_2 = \psi} V_{\psi_1} \otimes W_{\psi_2}. \tag{A.1.15}$$

This is *not* the same tensor product as equation (A.1.5) if we consider V and W to be in $\mathcal{M}\mathrm{od}_R^{\mathbb{O}_{n-1}}$.

Example A.1.16. If $n = 2$, then $\mathcal{V}\mathrm{ect}_R^{\mathbb{O}_2}$ is just the category of R-modules, and the tensor product above is just the usual tensor product of R-modules.

[1] Unfortunately this does not hold on the right if $n \geq 2$. Indeed, there are as many ways of decomposing ω as a grafting $\omega_1 \circ_{[l]} \omega_2$ with ω_2 degenerate, as there are leaves in ω. Thus if $k = \#\omega^{|}$, then $(X \otimes R)_\omega = X^{\oplus k}$.

Definition A.1.17 (Schur functor)**.** Let $X \in \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$ be an n-opetopic module. It induces a functor $\tilde{X} : \mathcal{V}\mathrm{ect}_R^{\mathbb{O}_n} \longrightarrow \mathcal{V}\mathrm{ect}_R^{\mathbb{O}_n}$, called its *Schur functor*, defined as follows. For $V \in \mathcal{V}\mathrm{ect}_R^{\mathbb{O}_n}$ and $\psi \in \mathbb{O}_{n-1}$, let

$$\tilde{X}(V)_\psi \coloneqq \bigoplus_{\mathsf{t}\,\omega = \psi} X_\omega \otimes V_\omega,$$

where $V_\omega \coloneqq \bigotimes_{[p] \in \omega} V_{\mathsf{s}_{[p]}\,\omega}$.

Proposition A.1.18. *The operations $\oplus$, $\otimes$, and $\circ$ are compatible with $\widetilde{(-)}$, i.e. for $X, Y \in \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$, we have $\widetilde{X \oplus Y} \cong \tilde{X} \oplus \tilde{Y}$, $\widetilde{X \otimes Y} \cong \tilde{X} \otimes \tilde{Y}$, and $\widetilde{X \circ Y} \cong \tilde{X} \circ \tilde{Y}$.*

Proof. (1) The first claim is clear.

(2) For $V \in \mathcal{V}\mathrm{ect}_R^{\mathbb{O}_n}$ and $\psi \in \mathbb{O}_{n-1}$, we have

$$\begin{aligned}
\widetilde{X \otimes Y}(V)_\psi &= \bigoplus_{\mathsf{t}\,\omega = \psi} \left(\bigoplus_{\omega_1 \circ \omega_2 = \omega} X_{\omega_1} \otimes Y_{\omega_2} \right) \otimes V_\omega \\
&\cong \bigoplus_{\mathsf{t}\,\omega = \psi} \bigoplus_{\omega_1 \circ \omega_2 = \omega} (X_{\omega_1} \otimes V_{\omega_1}) \otimes (X_{\omega_2} \otimes V_{\omega_2}) && \spadesuit \\
&\cong \bigoplus_{\psi_1 \square \psi_2 = \psi} \bigoplus_{\mathsf{t}\,\omega_1 = \psi_1} \bigoplus_{\mathsf{t}\,\omega_2 = \psi_2} (X_{\omega_1} \otimes V_{\omega_1}) \otimes (X_{\omega_2} \otimes V_{\omega_2}) \\
&\cong \left(\tilde{X}(V) \otimes \tilde{Y}(V) \right)_\psi,
\end{aligned}$$

where in $\spadesuit$, the isomorphism $V_\omega \cong V_{\omega_1} \otimes V_{\omega_2}$ is by definition.

(3) For $V \in \mathcal{V}\mathrm{ect}_R^{\mathbb{O}_n}$ and $\psi \in \mathbb{O}_{n-1}$, we have

$$\begin{aligned}
\tilde{X}(\tilde{Y}(V))_\psi &= \bigoplus_{\mathsf{t}\,\nu = \psi} X_\nu \bigotimes_{[p_i] \in \nu^\bullet} \tilde{Y}(V)_{\mathsf{s}_{[p_i]}\,\nu} \\
&= \bigoplus_{\mathsf{t}\,\nu = \psi} X_\nu \bigotimes_{[p_i] \in \nu^\bullet} \left(\bigoplus_{\mathsf{t}\,\nu_i = \mathsf{s}_{[p_i]}\,\nu} Y_{\nu_i} \otimes V_{\nu_i} \right) \\
&\cong \bigoplus_{\mathsf{t}\,\omega = \psi} \bigoplus_{\nu \square_i \nu_i = \omega} \left(X_\nu \bigotimes_i Y_{\nu_i} \right) \otimes V_\omega \\
&= \widetilde{X \circ Y}(V)_\psi.
\end{aligned}$$

□

Proposition A.1.19. *(1) The Schur functor $\tilde{0}$ of 0 is constant at the null $\mathbb{O}_n$-module;*

(2) the Schur functor $\tilde{R}$ of R is constant at R;

(3) the Schur functor $\tilde{I}$ of I is the identity functor.

Proof. (1) Obvious.

(2) For $V \in \mathcal{V}\mathrm{ect}_R^{\mathbb{O}_n}$ and $\psi \in \mathbb{O}_{n-1}$ we have

$$\begin{aligned}
\tilde{R}(V)_\psi &= \bigoplus_{\mathsf{t}\,\omega = \psi} R_\omega \otimes V_\omega = \begin{cases} R_{\mathsf{I}_\phi} \otimes V_{\mathsf{I}_\phi} & \text{if } \psi = \mathsf{Y}_\phi \\ 0 & \text{if } \psi \text{ is not an endotope} \end{cases} \\
&= R,
\end{aligned}$$

as $V_{\mathsf{I}_\phi} = \bigotimes_{[p] \in \mathsf{I}_\phi^\bullet} V_{\mathsf{s}_{[p]}\,\mathsf{I}_\phi} = R$ since $\mathsf{I}_\phi^\bullet = \varnothing$.

(3) For $V \in \mathcal{V}\mathrm{ect}_R^{\mathbb{O}_n}$ and $\psi \in \mathbb{O}_{n-1}$ we have

$$\tilde{I}(V)_\psi = \bigoplus_{\mathsf{t}\,\omega=\psi} I_\omega \otimes V_\omega = I_{\mathsf{Y}_\psi} \otimes V_{\mathsf{Y}_\psi} = V_\psi$$

where the second equality comes from the fact that the only endotope ω such that $\mathsf{t}\,\omega = \psi$ is Y_ψ, and where the last equality is the observation that $V_{\mathsf{Y}_\psi} = \bigotimes_{[p]\in\mathsf{Y}_\psi^\bullet} V_{\mathsf{s}_{[p]}\,\mathsf{Y}_\psi} = V_\psi$ since $\mathsf{Y}_\psi^\bullet = \{[]\}$ and $\mathsf{s}_{[]}\,\mathsf{Y}_\psi = \psi$. □

Proposition A.1.20. *The tuple $(\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}, \circ, I)$ is a monoidal category, and the "Schur functor" functor $\widetilde{(-)} : \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n} \longrightarrow [\mathcal{V}\mathrm{ect}_R^{\mathbb{O}_n}, \mathcal{V}\mathrm{ect}_R^{\mathbb{O}_n}]$ is monoidal. In particular, a monoid (X, m, u) in $\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$ induces a monad $\tilde{X}$ on $\mathcal{V}\mathrm{ect}_R^{\mathbb{O}_n}$.*

Proof. The first claim is a direct consequence of proposition A.1.7 and lemma A.1.12, and the second of propositions A.1.18 and A.1.19. □

Definition A.1.21. An object $C \in \mathcal{C}$ is *finitely generated* [AR94, definition 1.67] if $\mathcal{C}(C, -)$ preserves filtered colimits of monomorphisms. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is *finitely bounded* [AMSW19, definition 3.1] if for every object $C \in \mathcal{C}$ and every finitely generated subobject $d : D \longrightarrow FC$, there exists a finitely generated subobject $c : C' \longrightarrow C$ such that d factors through Fc:

$$\begin{tikzcd} & FC' \arrow[d, "Fc"] \\ D \arrow[ur, dashed] \arrow[r, "d"] & FC. \end{tikzcd}$$

Theorem A.1.22 ([AMSW19, theorem 3.4])**.** *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between locally finitely presentable categories. If*

(1) in $\mathcal{C}$, finitely generated objects are finitely presentable,
(2) F preserves monomorphisms,
(3) F is finitely bounded,

then F is finitary.

Proposition A.1.23. *A R-module M is finitely generated in the sense of definition A.1.21 if and only if it is finitely generated in the classical sense, i.e. there is a short exact sequence*

$$0 \longrightarrow K \longrightarrow R^{\oplus k} \longrightarrow M \longrightarrow 0$$

for some $k \in \mathbb{N}$. Further, M is finitely presentable (in the sense of definition 0.5.2) if K is finitely generated.

Proof. This follows from a more general result regarding algebraic theories, see [AR94, proposition 3.11 and theorem 3.12]. □

Proposition A.1.24. *If R is a Noetherian domain, then for all $X \in \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$, the Schur functor $\tilde{X}$ is finitary.*

Proof. We apply theorem A.1.22. Since R is Noetherian, every finitely generated R-module is finitely presentable [Rot09, corollary 3.19]. It is clear that $\tilde{X}$ preserves monomorphisms, and it remains to check that it is finitely bounded. Let $V \in \mathcal{V}\mathrm{ect}_R^{\mathbb{O}_n}$ and $W \subseteq \tilde{X}(V)$ be finitely generated. Let S be a finite generating family of W. An element $s \in S$ is a finite sums of generator of $\tilde{X}(V)$:

$$s = \sum_{i \in I_s} \frac{v_{s,i}}{x_{s,i}}.$$

The set $T := \{v_{s,i} \mid s \in S, i \in I_s\}$ is thus finite, and consider the subspace V' of V generated by T. Clearly, $W \subseteq \tilde{X}(V')$. □

THE DIFFERENTIAL GRADED CASE

Definition A.1.25 (Differential graded opetopic module)**.** Let the category of *differential graded (dg) n-opetopic modules* be ${}_{\mathrm{dg}}\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n} := \prod_{\omega \in \mathbb{O}_n} {}_{\mathrm{dg}}\mathcal{M}\mathrm{od}_R$.

Definition A.1.26 (Suspension)**.** Recall that if M is a dg R-module, its *suspension* $\blacktriangle M$ (also denoted sM in [LV12], $M(-1)$ in [Wu16], $\uparrow M$ in [Pro11], or even $M[1]$) is defined as $(\blacktriangle M)_k := M_{k-1}$, with differential $\partial_{\blacktriangle M} := -\partial_M$. Equivalently, $\blacktriangle M = (R\blacktriangle) \otimes M$, where $R\blacktriangle$ is the dg R-module concentrated in degree 1, freely generated by an element $\blacktriangle$. We write $\blacktriangle^n M := \blacktriangle \cdots \blacktriangle\, M$, and $\blacktriangledown^n$ the operation that is inverse to $\blacktriangle^n$, also called *desuspension*.

Let now X be a dg $\mathbb{O}_n$-module. The *suspension* $\blacktriangle X$ of X is the $\mathbb{O}_n$-module such that for all $\omega \in \mathbb{O}_n$ we have $(\blacktriangle X)_\omega = \blacktriangle(X_\omega)$ as dg R-modules. Equivalently, $\blacktriangle X = \blacktriangle R \otimes X$. The notations $\blacktriangle^n$ and $\blacktriangledown^n$ transpose to this case.

Remark A.1.27. The constructions and results of definitions A.1.3 and A.1.17 transpose to the differential graded case. Let $X, Y \in {}_{\mathrm{dg}}\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$.

(1) In $X \oplus Y$, for $x \in X$ and $y \in Y$, we have $\partial(x \oplus y) = \partial(x) \oplus \partial(y)$.

(2) Take a homogeneous element $x \otimes y$ in $X \otimes Y$. Then $\partial(x \otimes y) = \partial(x) \otimes y + (-1)^x x \otimes \partial(y)$, where we write $(-1)^x$ as a shorthand for $(-1)^{|x|}$, and where $|x|$ stands for for the degree of x.

(3) Take a homogeneous element $\frac{y}{x} \in X \circ Y$, and recall that by convetion, the sequence $y = y_1, \ldots$ is lexicographically sorted, meaning that if $\frac{y}{x}$ is in the $\nu \,\square_{[p_i]}\, \nu_i = \omega$ component of $(X \circ Y)_\omega = \bigoplus_{\nu \,\square_{[p_i]}\, \nu_i = \omega} X_\nu \otimes_i Y_{\nu_i}$, then $[p_1] < [p_2] < \cdots$. The formula for differentials gives,

$$\partial\left(\frac{y}{x}\right) = \frac{y}{\partial(x)} + (-1)^x \frac{\partial[y]}{x}, \tag{A.1.28}$$

where

$$\partial[y] := \sum_i (-1)^{y_1 + \cdots + y_{i-1}} y_1 \otimes \cdots \otimes y_{i-1} \otimes \partial(y_i) \otimes y_{i+1} \otimes \cdots,$$

and where as before, $(-1)^{y_1 + \cdots + y_{i-1}}$ is a shorthand for $(-1)^{|y_1| + \cdots + |y_{i-1}|}$.

Definition A.1.29 (Homology)**.** If $X \in {}_{\mathrm{dg}}\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$, then its *homology* $H(X)$ is the dg $\mathbb{O}_n$-module given by:

$$H_k(X)_\omega := \frac{\operatorname{im}\left(\partial : X_{\omega,k+1} \to X_{\omega,k}\right)}{\ker\left(\partial : X_{\omega,k} \to X_{\omega,k-1}\right)},$$

and endowed with the trivial differential. We say that X is *acyclic* if as dg $\mathbb{O}_n$-modules, $H(X) = 0$. If X is augmented or coaugmented, then X is acyclic if for $H_k(X) = 0$ as $\mathbb{O}_n$-modules, whenever $k \neq 0$, and $H_0(X) \cong I$. Equivalently, the augmentation or coaugmentation map is a quasi-isomorphism between I (concentrated in degree 0 with trivial differential) and X.

Definition A.1.30. The composite product $-\circ-:\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}\times\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}\longrightarrow\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$ is linear in its first variable, but not in its second one, meaning that in general, $X\circ(Y\oplus Z)\ncong(X\circ Y)\oplus(X\circ Z)$. To remedy this, we define a ternary operator $-\circ(-;-)$ [LV12, section 6.1], which is the maximal subfunctor of $-\circ(-\oplus-)$ that is linear in its third variable. Explicitly, $(X\circ(Y;Z))_\omega$ is the maximal submodule of $(X\circ(Y\oplus Z))_\omega$ where Z appears exactly once in each summand:

$$(X\circ(Y;Z))_\omega := \bigoplus_{\nu\,\square_{[p_i]}\,\nu_i=\omega}\ \bigoplus_{[p]\in\nu^\bullet} X_\nu \bigotimes_i \begin{cases} Z_{\mathsf{s}_{[p]}\,\nu} & \text{if } [p_i]=[p],\\ Y_{\mathsf{s}_{[p_i]}\,\nu} & \text{otherwise.}\end{cases}$$

If $f:Y\longrightarrow Y'$ and $g:Z\longrightarrow Z'$, then the morphism $X\circ(f;g):X\circ(Y;Z)\longrightarrow X\circ(Y';Z')$ maps an element $\frac{w}{x}$ to $\frac{(f;g)(w)}{x}$, where (assuming that Y and Z are disjoint)

$$(f;g)(w_i)=\begin{cases} f(w_i) & \text{if } w_i\in Y,\\ g(w_i) & \text{if } w_i\in Z.\end{cases}$$

Remark A.1.31. By definition, a homogeneous element of $X\circ(Y;Z)$ is an element $\frac{w}{x}\in X\circ(Y\oplus Z)$ with a distinguished element of the sequence w, such that that distinguished element lies in Z while all the others are in Y. With this point of view, there is an obvious morphism $X\circ Y\longrightarrow X\circ(Y;Y)$ which maps a homogeneous element $\frac{y}{x}\in X\circ Y$ to the sum $\sum\frac{y}{x}$ over all possible choices of distinguished element in the sequence y.

Definition A.1.32 (Infinitesimal composition of morphisms)**.** If $f:X\longrightarrow X'$ and $g:Y\longrightarrow Y'$ are morphisms of $\mathbb{O}_n$-modules, then their *infinitesimal composite* [LV12, section 6.1.5] $f\circ' g$ is

$$X\circ Y\longrightarrow X\circ(Y;Y)\xrightarrow{f\circ(\mathrm{id}_Y;g)} X'\circ(Y;Y'),$$

which is given explicitly by

$$(f\circ' g)\left(\frac{y}{x}\right) := (-1)^{gx}\frac{g[y]}{f(x)} \tag{A.1.33}$$

where $\frac{y}{x}\in X\circ Y$ is a homogeneous element, and $g[y]:=\sum_i(-1)^{y_1+\cdots+y_{i-1}}y_1\otimes\cdots\otimes y_{i-1}\otimes g(y_i)\otimes y_{i+1}\otimes\cdots$. For example, the differential on $X\circ Y$ is concisely given by $\partial_{X\circ Y}=\partial_X\circ\mathrm{id}+\mathrm{id}\circ'\partial_Y$.

Definition A.1.34 (Infinitesimal composite product)**.** The *infinitesimal composite product* [LV12, section 6.1.1] $X\circ_{(1)}Y$ of X and Y is defined as

$$(X\circ_{(1)}Y)_\omega := X\circ(I;Y) = \bigoplus_{\omega_1\,\square_{[p]}\,\omega_2=\omega} X_{\omega_1}\otimes Y_{\omega_2}, \tag{A.1.35}$$

which is just the tensor product of $\mathcal{V}\mathrm{ect}_R^{\mathbb{O}_{n+1}}$ (see equation (A.1.15)). If $f:X\longrightarrow X'$ and $g:Y\longrightarrow Y'$, the map $f\circ(I;g):X\circ_{(1)}Y\longrightarrow X'\circ_{(1)}Y'$ will conveniently be denoted by $f\circ_{(1)}g$.

THE LINEAR $\mathfrak{Z}^n$

Definition A.2.1. Let $f : A \longrightarrow B$ be a map between sets. It induces three functors $f_!, f_* : \mathcal{M}\mathrm{od}_R^A \longrightarrow \mathcal{M}\mathrm{od}_R^B$, and $f^* : \mathcal{M}\mathrm{od}_R^B \longrightarrow \mathcal{M}\mathrm{od}_R^A$, given as follow. For $M \in \mathcal{M}\mathrm{od}_R^A$, $N \in \mathcal{M}\mathrm{od}_R^B$, $a \in A$, and $b \in B$

$$f_! M_b := \bigoplus_{f(a)=b} M_a, \qquad f^* N_a := N_{f(a)}, \qquad f_* M_b := \bigotimes_{f(a)=b} M_a.$$

Proposition A.2.2. *We have an adjunction*[2] $f_! \dashv f^*$.

Proof. Straightforward verifications. □

Definition A.2.3 (Linear $\mathfrak{Z}^n$). Using definition A.2.1, and akin to the $\mathcal{S}\mathrm{et}$-theoretical case (see remark 2.1.4), a polynomial functor P in $\mathcal{S}\mathrm{et}$ given by

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$$

induces a functor $P = t_! p_* s^* : \mathcal{M}\mathrm{od}_R^I \longrightarrow \mathcal{M}\mathrm{od}_R^J$. In particular, for $\mathfrak{Z}^n$ the polynomial functor introduced in definition 3.1.1

$$\mathbb{O}_n \xleftarrow{\mathsf{s}} E_{n+1} \xrightarrow{p} \mathbb{O}_{n+1} \xrightarrow{\mathsf{t}} \mathbb{O}_n,$$

we consider the induced functor $\mathfrak{Z}^n : \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n} \longrightarrow \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$. Unfolding the definitions, for $X \in \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$ and $\omega \in \mathbb{O}_n$, we have

$$\mathfrak{Z}^n X_\omega = \bigoplus_{\mathsf{t}\,\xi=\omega} \bigotimes_{[p_i]\in\xi^\bullet} X_{\mathsf{s}_{[p_i]}\,\xi} = \bigoplus_{\mathsf{t}\,\xi=\omega} X_\xi. \tag{A.2.4}$$

Remark that $\mathfrak{Z}^n X_\omega = \widetilde{Z^n}(X)_\omega$, where $Z^n \in \mathcal{M}\mathrm{od}_R^{\mathbb{O}_{n+1}}$ is identically R (i.e. $Z^n_\xi = R\xi$ for all $\xi \in \mathbb{O}_{n+1}$), and X is considered in $\mathcal{V}\mathrm{ect}_R^{\mathbb{O}_{n+1}}$. A homogeneous element of $(\mathfrak{Z}^n X)_\omega$ will be denoted by $\frac{x}{\xi}$ where $\mathsf{t}\,\xi = \omega$, and where x is a sequence indexed by node addresses $[p] \in \xi^\bullet$ (in lexicographical order) such that $x_{[p]} \in X_{\mathsf{s}_{[p]}\,\xi}$.

Proposition A.2.5. *The $\mathbb{O}_n$-module Z^n is naturally a $\circ$-monoid. Therefore, the endofunctor $\mathfrak{Z}^n$ on $\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$ is naturally a monad.*

Proof. We endow Z^n with a monoid structure. The unit morphism $u : I \longrightarrow Z^n$ is simply the inclusion. On the other hand, for $\omega \in \mathbb{O}_n$, we have

$$(Z^n \circ Z^n)_\omega \cong \bigoplus_{\nu \,\square_i\, \nu_i=\omega} R,$$

and we let the multiplication morphism $m : Z^n \circ Z^n \longrightarrow Z^n$ map the $\nu \,\square_i\, \nu_i = \omega$ component to the ω component via the identity map id_R. The fact that (Z^n, m, u) us a monoid follows from the monad structure on $\mathfrak{Z}^n : \mathcal{S}\mathrm{et}/\mathbb{O}_n \longrightarrow \mathcal{S}\mathrm{et}/\mathbb{O}_n$. The second claim follows from proposition A.1.20. □

[2] In general, it is not true that $f^* \dashv f_*$.

Notation A.2.6. We write $\eta : \mathrm{id} \longrightarrow \mathfrak{Z}^n$ and $\mu : \mathfrak{Z}^n\mathfrak{Z}^n \longrightarrow \mathfrak{Z}^n$ for the monad laws of $\mathfrak{Z}^n$ induced by the monoid structure on Z^n of proposition A.2.5. Explicitly, for $X \in \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$,

$$\eta_X(x) = \frac{x}{\mathsf{Y}_\omega}, \qquad \mu_X\left(\begin{array}{c} y \\ \hline \zeta \\ \hline \xi \end{array}\right) = \frac{y}{\xi \,\square_i\, \zeta_i}, \tag{A.2.7}$$

where $x \in X_\omega$.

MONOIDAL APPROACH

Notation A.2.8. Let $X = (X, m, u)$ is a $\circ$-monoid in $\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$. The morphism $u : I \longrightarrow X$ maps $1 \in I_{\mathsf{Y}_\psi}$ to $1 \in X_{\mathsf{Y}_\psi}$. For $\frac{y}{x}$ a typical element of $X \circ X$, let

$$\genfrac{}{}{1.5pt}{}{y}{x} := m\left(\frac{y}{x}\right).$$

The associativity and unit axioms read

$$\genfrac{}{}{1.5pt}{}{z}{\left(\genfrac{}{}{1.5pt}{}{y}{x}\right)} = \genfrac{}{}{1.5pt}{}{\left(\genfrac{}{}{1.5pt}{}{z}{y}\right)}{x} \qquad \genfrac{}{}{1.5pt}{}{x}{1} = x = \genfrac{}{}{1.5pt}{}{1}{x}. \tag{A.2.9}$$

The expression on the left will be more consisely denoted by

$$\begin{array}{c} z \\ \hline\hline y \\ \hline\hline x \end{array}.$$

If $X = (X, m, u)$ is a dg $\circ$-monoid, then the derivation rule reads

$$\partial\left(\genfrac{}{}{1.5pt}{}{y}{x}\right) = \genfrac{}{}{1.5pt}{}{y}{\partial(x)} + (-1)^x \genfrac{}{}{1.5pt}{}{\partial[y]}{x}. \tag{A.2.10}$$

Example A.2.11. For Z^n the $\circ$-monoid defined in proposition A.2.5, we have

$$\genfrac{}{}{1.5pt}{}{\zeta}{\xi} = \xi \mathop{\square}_{[p]\in\xi^\bullet} \zeta_{[p]} \tag{A.2.12}$$

Definition A.2.13. A $\circ$-monoid (X, m, u) is *augmented* if it is endowed with an *augmentation map* $\varepsilon : X \longrightarrow I$ such that $\varepsilon u = \mathrm{id}_I$. In particular, the underlying module X is augmented by ε.

Definition A.2.14 (Infinitesimal multiplication)**.** If $A = (A, m, u)$ is a $\circ$-monoid, we have an *infinitesimal multiplication* $m = m_{(1)} : A \circ_{(1)} A \longrightarrow A$ given by the composite

$$A \circ_{(1)} A = A \circ (I; A) \xrightarrow{\mathrm{id}_A \circ (u; \mathrm{id}_A)} A \circ (A; A) \longrightarrow A \circ A \xrightarrow{m} A.$$

where the map $A \circ (A; A) \longrightarrow A \circ A$ just forgets the distinguished element. Explicitly, for an element of $A \circ_{(1)} A$ of the form $\frac{1,1,\ldots,b,\ldots,1}{a}$, we have

$$m_{(1)}\left(\frac{1, 1, \ldots, b, \ldots, 1}{a}\right) = \genfrac{}{}{1.5pt}{}{u(1), u(1), \ldots, b, \ldots, u(1)}{a}.$$

Remark A.2.15. Let X be an algebra over the monad $\mathfrak{Z}^n$ of proposition A.2.5, with structure map $m : \mathfrak{Z}^n X \longrightarrow X$, and let

$$\frac{x}{\overline{\xi}} := m\left(\frac{x}{\xi}\right).$$

The associativity and unit axioms read

$$\frac{\left(\frac{x}{\overline{\zeta}}\right)}{\overline{\xi}} = \frac{x}{\overline{\left(\frac{\zeta}{\overline{\xi}}\right)}} = \frac{x}{\overline{\xi \square_{[p] \in \xi^\bullet} \zeta_{[p]}}}, \qquad \frac{x}{\overline{\mathsf{Y}_\omega}} = x, \tag{A.2.16}$$

where on the right, we assume $x \in X_\omega$.

Proposition A.2.17. *Let $X \in \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$. Then a $\circ$-monoid structure on X is equivalent to a $\mathfrak{Z}^n$-algebra structure on X.*

Proof. Assume we have a $\circ$-monoid structure (X, m, u) on X. A structure map $m' : \mathfrak{Z}^n X \longrightarrow X$ can be defined inductively as follows:

$$m'\left(\frac{}{\mathsf{I}_\psi}\right) := u(1), \qquad m'\left(\frac{x}{\mathsf{Y}_\omega}\right) := x, \qquad m'\left(\frac{y}{\mathsf{Y}_\omega \bigcirc_{[[q_i]]} \xi_i}\right) := \frac{m'\left(\frac{y^\xi}{\xi}\right)}{\overline{y_{[]}}},$$

where $\psi \in \mathbb{O}_{n-1}$, $\omega \in \mathbb{O}_n$, $x \in X_\omega$, and y^{ξ_i} is the subsequence of y consisting of elements corresponding to nodes in ξ_i. Conversely, given a $\mathfrak{Z}^n$-algebra structure m' on X, define

$$u(1) := m'\left(\frac{}{\mathsf{I}_\psi}\right), \qquad \frac{y}{\overline{x}} := m'\left(\frac{x, y}{\mathsf{Y}_\omega \bigcirc_i \mathsf{Y}_{\omega_i}}\right),$$

where $\psi \in \mathbb{O}_{n-1}$, $x \in X_\omega$, and $y_i \in X_{\omega_i}$ for some $\omega, \omega_1, \ldots \in \mathbb{O}_n$. $\square$

Examples A.2.18. (1) If $n = 1$, then a $\mathfrak{Z}^1$-algebra is just an associative unital R-algebra.
(2) If $n = 2$, then a $\mathfrak{Z}^2$-algebra is a planar algebraic operad [LV12, chapter 5].

Definition A.2.19 (Augmented algebra)**.** A $\mathfrak{Z}^n$-algebra A is *augmented* if it is endowed with a morphism $\varepsilon : A \longrightarrow I$ of degree 0. This morphism is then called the *augmentation morphism* of A, and the sub $\mathbb{O}_n$-module $\bar{A} := \ker \varepsilon$ is the *augmentation ideal.* We write ${}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R^+(\mathfrak{Z}^n)$ for the category of augmented dg $\mathfrak{Z}^n$-algebras, and morphisms preserving the augmentations, i.e. those $f : A \longrightarrow B$ in ${}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R(\mathfrak{Z}^n)$ that lift as $f : \bar{A} \longrightarrow \bar{B}$.

Remark A.2.20. Let $X \in \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$ and $f : X \longrightarrow X$. It can be extended as a morphism $f : \mathfrak{Z}^n X \longrightarrow \mathfrak{Z}^n X$ by letting

$$f\left(\frac{x}{\xi}\right) := \frac{f[x]}{\xi}.$$

Note that this is *not* $\mathfrak{Z}^n f$, as $(\mathfrak{Z}^n f)\left(\frac{x}{\xi}\right) = \frac{f(x)}{\xi}$, i.e. f is applied to each element of the list.

Definition A.2.21 (Derivation)**.** Let A be a $\mathfrak{Z}^n$-algebra with structure map $\gamma : \mathfrak{Z}^n A \longrightarrow A$. A *derivation* on A is a morphism $d : A \longrightarrow A$ such that $d\gamma = \gamma d$, explicitly:

$$d\left(\frac{x}{\overline{\overline{\xi}}}\right) = \frac{d[x]}{\overline{\overline{\xi}}}. \tag{A.2.22}$$

We write $\mathrm{Der}(A)$ for the space of derivations of A. If A is free, say $A = \mathfrak{Z}^n X$, then equation (A.2.22) reads

$$d\left(\frac{x}{\xi \,\square_i\, \zeta_i}\right) = d\left(\begin{array}{c} x \\ \hline \zeta \\ \hline\hline \xi \end{array}\right) = \frac{d\left[\frac{x}{\zeta}\right]}{\overline{\overline{\xi}}}. \tag{A.2.23}$$

Proposition A.2.24 (Analogous to [LV12, proposition 1.1.8])**.** *Let $X \in \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$. The restriction map*

$$\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}(\mathfrak{Z}^n X, \mathfrak{Z}^n X) \longrightarrow \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}(X, \mathfrak{Z}^n X)$$

restricts to an isomorphism $\mathrm{Der}(\mathfrak{Z}^n X) \longrightarrow \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}(X, \mathfrak{Z}^n X)$*. Specifically, let $f : X \longrightarrow \mathfrak{Z}^n X$, and define $d_f : \mathfrak{Z}^n X \longrightarrow \mathfrak{Z}^n X$ as*

$$d_f\left(\frac{x}{\xi}\right) := \frac{(\eta_X ; f)(x)}{\overline{\overline{\xi}}}, \tag{A.2.25}$$

where $\eta_X : X \longrightarrow \mathfrak{Z}^n X$ is the unit of the monad $\mathfrak{Z}^n$ at X, which maps $x \in X_\omega$ to $\frac{x}{\mathsf{Y}_\omega}$. Then d_f is a derivation, and the map $d_{(-)} : \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}(X, \mathfrak{Z}^n X) \longrightarrow \mathrm{Der}(\mathfrak{Z}^n X)$ is an inverse to the restriction map.

Proof. We first check that d_f defined in equation (A.2.25) really is a derivation:

$$\begin{aligned}
d_f\left(\begin{array}{c} x \\ \hline \zeta \\ \hline\hline \xi \end{array}\right) &= d_f\left(\frac{x}{\xi \,\square_i\, \zeta_i}\right) && \text{by equation (A.2.12)} \\
&= \frac{(\eta_X ; f)(x)}{\overline{\overline{\xi \,\square_i\, \zeta_i}}} && \text{see equation (A.2.25)} \\
&= \begin{array}{c} (\eta_X ; f)(x) \\ \hline\hline \zeta \\ \hline\hline \xi \end{array} && \text{by equation (A.2.12)} \\
&= \frac{d_f\left[\frac{x}{\zeta}\right]}{\overline{\overline{\xi}}} && \text{see equation (A.2.25).}
\end{aligned}$$

Next, we check that d_f extends f, i.e. that $d_f \eta_X = f$. For $x \in X_\omega$

$$\begin{aligned}
d_f \eta_X(x) &= d_f\left(\frac{x}{\mathsf{Y}_\omega}\right) && \text{see equation (A.2.7)} \\
&= \frac{(\eta_X ; f)(x)}{\overline{\overline{\mathsf{Y}_\omega}}} && \text{since } \#\mathsf{Y}_\omega^\bullet = 1 \\
&= f(x) && \text{see equation (A.2.16).}
\end{aligned}$$

Thus the map $d_{(-)}$ is injective. Finally we check that it is surjective, specifically that for $d \in \mathrm{Der}(\mathfrak{z}^n X)$, and $f := d\eta_X$, we have $d = d_f$.

$$
\begin{aligned}
d_f\left(\frac{x}{\xi}\right) &= \frac{(\eta_X; d_f)(x)}{\xi} && \text{since } d_f \text{ is a derivation} \\
&= \frac{d\left[\frac{x}{\mathsf{Y}_\omega}\right]}{\xi} && \text{dy definition of } f \text{ and } d_f \\
&= d\left(\frac{\frac{x}{\mathsf{Y}_\omega}}{\xi}\right) && \text{since } d \text{ is a derivation} \\
&= d\left(\frac{x}{\xi \,\square_i\, \mathsf{Y}_{\omega_i}}\right) && \text{by equation (A.2.12)} \\
&= d\left(\frac{x}{\xi}\right).
\end{aligned}
$$

□

Lemma A.2.26. *Let* $X \in {}_{\mathrm{dg}}\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$*,* $f, g : X \longrightarrow \mathfrak{z}^n X$*, and* $d_f, d_g \in \mathrm{Der}(\mathfrak{z}^n X)$ *the unique derivations extending* f *and* g*, as defined in equation* (A.2.25)*.*

(1) If f *has an odd degree, then* $d_f d_f = d_{d_f f}$*, explicitly,*

$$
d_f d_f\left(\frac{x}{\xi}\right) = \frac{(\eta_X; d_f f)(x)}{\xi} = d_{d_f f}\left(\frac{x}{\xi}\right).
$$

(2) If f *and* g *both have (possibly distinct) odd degrees, then* $d_f d_g + d_g d_f = d_{d_f g + d_g f}$*.*

(3) If f *and* g *both have (possibly distinct) even degrees, then* $d_f d_g - d_g d_f = d_{d_f g - d_g f}$*.*

Proof. We have

$$
\begin{aligned}
d_f d_f\left(\frac{x}{\xi}\right) &= d_f\left(\frac{(\eta_X; f)(x)}{\xi}\right) \\
&= \frac{d_f\left[(\eta_X; f)(x)\right]}{\xi} && \spadesuit \\
&= \sum_i (-1)^{x_1 + \cdots + x_{i-1}} \frac{d_f\left[x_1, \ldots, x_{i-1}, f(x_i), x_{i+1}, \ldots\right]}{\xi} \\
&= \sum_i \sum_{j<i} (-1)^{x_j + \cdots + x_{i-1}} \frac{\ldots, f(x_j), \ldots, f(x_i), \ldots}{\xi} \\
&\quad + \sum_i \frac{\ldots, d_f f(x_i), \ldots}{\xi} \\
&\quad + \sum_i \sum_{j>i} (-1)^{f + x_i + \cdots + x_{j-1}} \frac{\ldots, f(x_i), \ldots, f(x_j), \ldots}{\xi} \\
&= \sum_i \frac{\ldots, d_f f(x_i), \ldots}{\xi}
\end{aligned}
$$

$$= \frac{(\eta_X; d_f f)(x)}{\xi}.$$

where ♠ follows from equation (A.2.23). The second and third points can be proved using the same technique. ☐

FREE MONAD

Definition A.2.27 (Algebra over a functor)**.** Let $P : \mathcal{C} \longrightarrow \mathcal{C}$ be a functor. A P*-algebra* is a pair (c, m), where c is an object of $\mathcal{C}$, and m is a morphism $Pc \longrightarrow c$. If (c', m') is another P-algebra, then an algebra morphism $f : (c, m) \longrightarrow (c', m')$ is simply a morphism $f : c \longrightarrow c'$ in $\mathcal{C}$ such that the following square commutes:

$$\begin{tikzcd} Pc \arrow[r, "Pf"] \arrow[d, "m"'] & Pc' \arrow[d, "m'"] \\ c \arrow[r, "f"] & c'. \end{tikzcd}$$

We denote by $\mathcal{A}\mathrm{lg}(P)$ the category of P-algebras and algebra morphisms.

Theorem A.2.28. *Let $\mathcal{C}$ be a category and $P : \mathcal{C} \longrightarrow \mathcal{C}$ be a functor.*

(1) [BW05, theorem 4.4] If the forgetful functor $U^P : \mathcal{A}\mathrm{lg}(P) \longrightarrow \mathcal{C}$ has a left adjoint F^P, then the monad $U^P F^P$ is the free monad on P.

(2) [BW05, theorem 4.5] If $\mathcal{C}$ is complete, and if P has a free monad T, then the forgetful functor U^P above has a left adjoint, and the canonical comparison functor $E : \mathcal{A}\mathrm{lg}(T) \longrightarrow \mathcal{A}\mathrm{lg}(P)$ is an equivalence.

Corollary A.2.29 ([GH04, proposition 17])**.** *Let $\mathcal{C}$ be a complete category and $P : \mathcal{C} \longrightarrow \mathcal{C}$. A monad T over $\mathcal{C}$ is the free monad on P if and only if there is an equivalence $E : \mathcal{A}\mathrm{lg}(T) \longrightarrow \mathcal{A}\mathrm{lg}(P)$ such that the following triangle commutes:*

$$\begin{tikzcd} \mathcal{A}\mathrm{lg}(T) \arrow[rr, "E"] \arrow[dr, "U^T"'] & & \mathcal{A}\mathrm{lg}(P) \arrow[dl, "U^P"] \\ & \mathcal{C}, & \end{tikzcd}$$

where U^T and U^P are the forgetful functors.

Theorem A.2.30. *For $X \in \mathrm{Mod}_R^{\mathbb{O}_n}$, the Schur functor $\widetilde{\mathfrak{Z}^n X}$ is the free monad over $\tilde{X}$.*

Proof. The unit map $\eta_X : X \longrightarrow \mathfrak{Z}^n X$ induces a functor $\eta_X^* : \mathcal{A}\mathrm{lg}(\widetilde{\mathfrak{Z}^n X}) \longrightarrow \mathcal{A}\mathrm{lg}(\tilde{X})$ that commutes with the forgetful functors. We apply corollary A.2.29 by showing that η_X^* is an equivalence of categories.

To that end, we define a functor $S : \mathcal{A}\mathrm{lg}(\tilde{X}) \longrightarrow \mathcal{A}\mathrm{lg}(\widetilde{\mathfrak{Z}^n X})$ that maps a $\tilde{X}$-algebra $m : \tilde{X}(V) \longrightarrow V$ to a $\widetilde{\mathfrak{Z}^n X}$-algebra $\bar{m} : \widetilde{\mathfrak{Z}^n X}(V) \longrightarrow V$ that we now define. Applying definitions A.1.17 and A.2.3, for $\psi \in \mathbb{O}_{n-1}$,

$$\widetilde{\mathfrak{Z}^n X}(V)_\psi = \bigoplus_{\mathtt{t}\,\xi = \psi} X_\xi \otimes V_{\mathtt{t}\,\xi}.$$

A homogeneous element

$$\begin{array}{c} v \\ \hline x \\ \hline \xi \end{array} \in \widetilde{\mathfrak{Z}^n X}(V)_\psi \tag{A.2.31}$$

may thus be considered as a decoration of the opetope ξ by elements of X on its nodes, and elements of V in its leaves. Given a node decorated by $x \in X_\omega$ and whose input edges are decoraged by a sequence $w \in V_\omega$, the algebra structure m gives an element

$$m\left(\frac{w}{x}\right) \in V_{\mathsf{t}\omega}.$$

Thus, to define $\bar{m}$ on a homogeneous element as in equation (A.2.31), we recursively apply m to every node of ξ. Formally, it is given as follows.

(1) If $\psi \in \mathbb{O}_{n-1}$ and $v \in V_\psi$ then

$$\bar{m}\left(\begin{array}{c} v \\ \hline \\ \hline \mathsf{I}_\psi \end{array}\right) := v.$$

(2) If $\omega \in \mathbb{O}_n$ and $\frac{v}{x} \in \tilde{X}(V)_\omega$, then

$$\bar{m}\left(\begin{array}{c} v \\ \hline x \\ \hline \mathsf{Y}_\omega \end{array}\right) := m\left(\frac{v}{x}\right).$$

(3) Consider an element of $\widetilde{\mathfrak{Z}^n X}(V)_\psi$ as in equation (A.2.31), where ξ has at least two nodes. Let $[p]$ be the maximal element of $\xi^\bullet$ (with respect to the lexicographical order). Then ξ decomposes as $\xi = \xi' \circ_{[p]} \mathsf{Y}_\omega$, where $\omega := \mathsf{s}_{[p]} \xi$. Further, the sequence x of elements decorating the nodes of ξ decomposes as $x = x', y$, where $y \in X_\omega$ corresponds to the node at $[p]$. Likewise, the sequence v decomposes as $v = v', w$, where the subsequence w correspond to all the elements of v decorating the input edges (which are necessarily leaves by maximality of $[p]$) of node node at address $[p]$. The idea is to decorate the output edge of that node by $m\left(\frac{w}{x}\right)$. Formally,

$$\bar{m}\left(\begin{array}{c} v \\ \hline x \\ \hline \xi \end{array}\right) = \bar{m}\left(\begin{array}{c} v', w \\ \hline x', y \\ \hline \xi' \circ_{[p]} \mathsf{Y}_\omega \end{array}\right) := \bar{m}\left(\begin{array}{c} v', m\left(\frac{w}{y}\right) \\ \hline x' \\ \hline \xi' \end{array}\right).$$

It is easy to see that $\bar{m} : \widetilde{\mathfrak{Z}^n X}(V) \longrightarrow V$ is indeed a $\widetilde{\mathfrak{Z}^n X}$-algebra. On morphisms, S simply maps the square on the left to the one on the right:

$$\begin{tikzcd} \tilde{X}(V) \arrow[r, "\tilde{X}(f)"] \arrow[d, "m"'] & \tilde{X}(V') \arrow[d, "m'"] \\ V \arrow[r, "f"] & V' \end{tikzcd} \qquad \longmapsto \qquad \begin{tikzcd} \widetilde{\mathfrak{Z}^n X}(V) \arrow[r, "\widetilde{\mathfrak{Z}^n X}(f)"] \arrow[d, "\bar{m}"'] & \widetilde{\mathfrak{Z}^n X}(V') \arrow[d, "\bar{m}'"] \\ V \arrow[r, "f"] & V'. \end{tikzcd}$$

Clearly, S is a section of η_X^*, and in particular, η_X^* is full and surjective on objects. We now show that S is a retraction of η_X^*. Take a $\widetilde{\mathfrak{Z}^n X}$-algebra $p : \widetilde{\mathfrak{Z}^n X}(W) \longrightarrow W$, and consider the algebra structure $\bar{p} : \widetilde{\mathfrak{Z}^n X}(W) \longrightarrow W$ induced by the restriction $p' : \tilde{X}(W) \longrightarrow W$ of p to $\tilde{X}(W)$, as defined above. We show that $p = \bar{p}$. Specifically, we show that p and $\bar{p}$ agree on homogeneous elements as in equation (A.2.31) by induction on $\#\xi^\bullet$.

(1) If $\#\xi^\bullet = 0$, then ξ is degenerate, say $\xi = \mathsf{I}_\psi$. If $w \in W_\psi$, then

$$\bar{p}\begin{pmatrix} w \\ \hline \\ \hline \mathsf{I}_\psi \end{pmatrix} = w = p\begin{pmatrix} w \\ \hline \\ \hline \mathsf{I}_\psi \end{pmatrix}.$$

(2) If $\#\xi^\bullet = 1$, then ξ is an endotope, say $\xi = \mathsf{Y}_\omega$. If $\frac{w}{x} \in \tilde{X}(W)_\omega$, then by definition

$$\bar{p}\begin{pmatrix} w \\ \hline x \\ \hline \mathsf{Y}_\omega \end{pmatrix} = p'\left(\frac{w}{x}\right) = p\begin{pmatrix} w \\ \hline x \\ \hline \mathsf{Y}_\omega \end{pmatrix}.$$

(3) If $\#\xi^\bullet \geq 2$, then ξ can be decomposed as $\xi = \xi' \circ_{[p]} \mathsf{Y}_\omega$, where $[p]$ is the maximal element of $\xi^\bullet$. For an element as in equation (A.2.31), and with the same decomposition as in the definition of $\bar{m}$ above,

$$\bar{p}\begin{pmatrix} w \\ \hline x \\ \hline \xi \end{pmatrix} = \bar{p}\begin{pmatrix} v', w \\ \hline x', y \\ \hline \xi' \circ_{[p]} \mathsf{Y}_\omega \end{pmatrix} = \bar{p}\begin{pmatrix} v', p'\left(\frac{w}{y}\right) \\ \hline x' \\ \hline \xi' \end{pmatrix} \overset{\spadesuit}{=} p\begin{pmatrix} v', p'\left(\frac{w}{y}\right) \\ \hline x' \\ \hline \xi' \end{pmatrix} \overset{\diamond}{=} p\begin{pmatrix} w \\ \hline x \\ \hline \xi \end{pmatrix}$$

where $\spadesuit$ is by induction, and $\diamond$ is by the axioms of $\widetilde{\mathfrak{Z}^n X}$-algebra.

Finally, $p = \bar{p}$, and S is a two sided inverse to η_X^*. In particular, it is an equivalence of categories, and by corollary A.2.29, $\widetilde{\mathfrak{Z}^n X}$ is the free monad over $\tilde{X}$. $\square$

A.3 LINEAR OPETOPIC COALGEBRAS

COMONOIDAL APPROACH

Notation A.3.1. Let (C, Δ, ε) be a comonoid for $\circ$ in $\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$. Akin to Sweedler's notation [LV12, definition 1.2.1], for $c \in C$ we write

$$\Delta(c) = \sum_{(c)} \frac{c_2}{c_1}$$

although we leave the $\sum$ symbol implicit most of the time. The coassociativity and counit axioms read

$$\frac{c_3}{\left(\frac{c_2}{c_1}\right)} = \frac{\left(\frac{c_3}{c_2}\right)}{c_1} = \frac{c_3}{\frac{c_2}{c_1}}, \qquad \frac{\varepsilon(c_2)}{c_1} = \frac{1}{c}, \qquad \frac{c_2}{\varepsilon(c_1)} = \frac{c}{1}. \tag{A.3.2}$$

If $Y = (Y, \Delta, \varepsilon)$ is a dg $\circ$-comonoid, then

$$\frac{\partial(c)_2}{\partial(c)_1} = \Delta\partial(c) = \partial\Delta(c) = \frac{c_2}{\partial(c_1)} + (-1)^{c_1}\frac{\partial[c_2]}{c_1}. \tag{A.3.3}$$

Definition A.3.4 (Conilpotent comonoid)**.** Let (C, Δ, ε) be a $\circ$-comonoid. Its n-fold comultiplication is denoted by $\Delta^n : C \longrightarrow C^{\circ n}$. In particular, $\Delta^0 = \varepsilon$. The comonoid C is *conilpotent* if for every $c \in C$, there exists an $N \in \mathbb{N}$ such that $\Delta^N(c) = 0$.

Remark A.3.5. In [LV12, section 5.7], an algebraic cooperad is defined as a comonoid $\Delta : C \longrightarrow C \bar{\circ} C$, where $\bar{\circ}$ is a special composite product relying on invariants rather than coinvariants [LV12, section 5.1.21]. In the opetopic setting developed here, we do not deal with any action of the symmetric group, and $\bar{\circ}$ coindices with the composite product $\circ$ defined in equation (A.1.6).

Definition A.3.6 (Infinitesimal comultiplication)**.** Let $C = (C, \Delta, \varepsilon)$ be a $\circ$-comonoid. Using the counit map, define a projection $p : C \circ C \longrightarrow C \circ_{(1)} C$ as the composite

$$C \circ C \longrightarrow C \circ (C; C) \xrightarrow{C \circ (\varepsilon; \mathrm{id}_C)} C \circ (I; C) = C \circ_{(1)} C,$$

which is given explicitly by

$$p\left(\frac{d}{c}\right) := \frac{(\varepsilon; \mathrm{id})(d)}{c}, \tag{A.3.7}$$

where $\frac{d}{c} \in C \circ C$ is a homogeneous element. The *infinitesimal comultiplication* of C is the composite $\Delta_{(1)} := p\Delta$. Explicitly, if $c \in C$, then

$$\Delta_{(1)}(c) = \frac{(\varepsilon; \mathrm{id})(c_2)}{c_1}.$$

COMONADIC APPROACH

Recall proposition A.2.5 stating that the linear $\mathfrak{Z}^n : \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n} \longrightarrow \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$ is naturally a monad. The proof proceeds by noting that $\mathfrak{Z}^n$ is the Schur functor of a $\mathbb{O}_{n+1}$-module Z^n (definition A.2.3), which is naturally a $\circ$-monoid. We now show that Z^n can also be endowed with a $\circ$-comonoid structure.

Lemma A.3.8. *Let $\varepsilon : Z^n \longrightarrow I$ be the obvious projection, and for $\xi \in \mathbb{O}_{n+1}$, let*

$$\Delta(\xi) = \frac{\xi_2}{\xi_1} := \sum_{\zeta \,\square_{[p]}\, \zeta'_{[p]} = \xi} \frac{\zeta'}{\zeta}.$$

Then the tuple $(Z^n, \Delta, \varepsilon)$ is a $\circ$-comonoid. In particular, $\mathfrak{Z}^n$ is naturally a comonad on $\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$.

Proof. By definition, $\Delta(\xi)$ is the formal sum of all the possible decompositions of ξ as a simultaneous substitution. Equivalently, it is the formal sum of all $(n+2)$-opetopes $\nu \in \mathbb{O}_{n+2}^{(2)}$ such that $\mathsf{t}\,\nu = \xi$. Therefore, $(Z^n \circ \Delta)\Delta(\xi)$ and $(\Delta \circ Z^n)\Delta(\xi)$ are both the formal sum of all $(n+2)$-opetopes $\nu \in \mathbb{O}_{n+2}^{(3)}$ such that $\mathsf{t}\,\nu = \xi$, and Δ is indeed coassociative. Counit axioms are easily checked. □

Notation A.3.9. Let $\Delta : C \longrightarrow \mathfrak{Z}^n C$ be a $\mathfrak{Z}^n$-coalgebra. Recall that for $\omega \in \mathbb{O}_n$ we have

$$(\mathfrak{Z}^n C)_\omega = \widetilde{Z^n}(C)_\omega = \bigoplus_{\mathsf{t}\,\xi = \omega} R\xi \otimes X_\xi$$

where $X_\xi = \bigotimes_{[p]\in\xi^\bullet} X_{\mathsf{s}_{[p]}\xi}$, and that we write a homogeneous element of $R\xi \otimes X_\xi$ as $\frac{x}{\xi}$, where x is an adequate sequence of elements of X indexed by $\xi^\bullet$. Then, the structure morphism Δ maps $c \in C$ to

$$\Delta(c) = \sum_{(c)} \frac{c_2}{\xi_1^c},$$

although we leave the $\sum$ symbol implicit most of the time. The coassociativity and counit axioms read

$$\frac{\frac{(c_2)_2}{\xi_1^{c_2}}}{\xi_1^c} = \frac{\frac{c_3}{\xi_2^c}}{\xi_1^c} = \frac{\frac{c_2}{(\xi_1^c)_2}}{(\xi_1^c)_1}, \qquad \frac{\varepsilon(c_2)}{\xi_1^c} = c = \frac{c_2}{\varepsilon(\xi_1^c)}. \tag{A.3.10}$$

Definition A.3.11 (Coaugmented coalgebra)**.** In the comonoid structure $(Z^n, \Delta, \varepsilon)$ of lemma A.3.8, recall that $\varepsilon(\xi) = 1$ if ξ is an endotope, and 0 otherwise. A coalgebra C is *coaugmented* if it is endowed with a map $\eta : I \longrightarrow C$, called the *coaugmentation map*. In this case, we write $\bar{C} := C/\operatorname{im}\eta$. Let ${}^{\mathrm{co}}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}^+_R(\mathfrak{Z}^n)$ be the category of dg $\mathfrak{Z}^n$-coalgebras and morphisms preserving the coaugmentation maps, i.e. those $f : C \longrightarrow D$ in ${}^{\mathrm{co}}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R(\mathfrak{Z}^n)$ that descend to a morphism $\bar{C} \longrightarrow \bar{D}$.

Proposition A.3.12. *Let $C \in \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$. Then a conilpotent $\circ$-comonoid structure on C is equivalent to a $\mathfrak{Z}^n$-coalgebra structure on C.*

Proof. Assume we have a $\circ$-comonoid structure (C, Δ, ε) on C. A structure map $\gamma : C \longrightarrow \mathfrak{Z}^n C$ can be defined as

$$\gamma(c) := \sum_{n\in\mathbb{N}} \Delta^n(c).$$

Since C is conilpotent, this sum has finitely many terms, so γ is well-defined. Conversely, let $\gamma : C \longrightarrow \mathfrak{Z}^n C$ be a coalgebra structure on C. Recall that

$$\mathfrak{Z}^n C_\omega = \widetilde{Z^n}(C)_\omega = \bigoplus_{\mathsf{t}\xi=\omega} R\xi \otimes C_\xi.$$

Therefore, the submodule of $\mathfrak{Z}^n C$ spanned by the summands where $\#\xi^\bullet = k$ is exactly $C^{\circ k} = C \circ \cdots \circ C$. Write $\mathrm{proj}^{(k)}$ for the projection $\mathfrak{Z}^n C \twoheadrightarrow C^{\circ k}$, and define a comultiplication Δ and counit ε on C as $\mathrm{proj}^{(2)}\gamma$ and $\mathrm{proj}^{(0)}\gamma$ respectively. ☐

Definition A.3.13 (Coderivation)**.** Let C be a $\mathfrak{Z}^n$-coalgebra with structure map $\Delta : C \longrightarrow \mathfrak{Z}^n C$. A morphism $d : C \longrightarrow C$ is a *coderivation* if $d\Delta = \Delta d$, explicitly

$$\frac{d(c)_2}{\xi_1^{d(c)}} = \frac{d[c_2]}{\xi_1^c}, \tag{A.3.14}$$

where $c \in C$. We write $\mathrm{Coder}(C)$ for the space of coderivations of C. If C is cofree conilpotent, say $C = \mathfrak{Z}^n X$, then equation (A.3.14) becomes

$$\Delta d\left(\frac{x}{\xi}\right) = \frac{d\left[\frac{x}{\xi_2}\right]}{\xi_1}. \tag{A.3.15}$$

Proposition A.3.16 (Generalization of [LV12, proposition 6.3.15])**.** *Let $X \in \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$, and consider the cofree conilpotent $\mathfrak{Z}^n$-coalgebra $\mathfrak{Z}^n X$. Then the map*

$$\begin{aligned} \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}(\mathfrak{Z}^n X, \mathfrak{Z}^n X) &\longrightarrow \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}(\mathfrak{Z}^n X, X) \\ f &\longmapsto \varepsilon_X f \end{aligned}$$

restricts to an isomorphism $\mathrm{Coder}(\mathfrak{Z}^n X) \longrightarrow \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}(\mathfrak{Z}^n X, X)$*. Specifically, let $f : \mathfrak{Z}^n X \longrightarrow X$ be a morphism of $\mathbb{O}_n$-modules, and define d_f by*

$$d_f\left(\frac{x}{\xi}\right) := \frac{(\varepsilon_X; f)\left(\frac{x}{\xi_2}\right)}{\xi_1}. \tag{A.3.17}$$

Then d_f is a coderivation on $\mathfrak{Z}^n X$, and $d_{(-)} : \mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}(\mathfrak{Z}^n X, X) \longrightarrow \mathrm{Coder}(\mathfrak{Z}^n X)$ is inverse to the postcomposition by ε_X

Proof. We first check that d_f really is a coderivation:

$$\begin{aligned} \Delta d_f\left(\frac{x}{\xi}\right) &= \Delta\left(\frac{(\varepsilon_X; f)\left(\frac{x}{\xi_2}\right)}{\xi_1}\right) && \text{by definition} \\ &= \frac{\dfrac{(\varepsilon_X; f)\left(\frac{x}{\xi_3}\right)}{\xi_2}}{\xi_1} \\ &= \frac{d_f\left[\frac{x}{\xi_2}\right]}{\xi_1} && \text{by definition.} \end{aligned}$$

Next, we check that d_f coextends f, i.e. that $\varepsilon_X d_f = f$:

$$\begin{aligned} \varepsilon_X d_f\left(\frac{x}{\xi}\right) &= \varepsilon_X\left(\frac{(\varepsilon_X; f)\left(\frac{x}{\xi_2}\right)}{\xi_1}\right) && \text{by definition} \\ &= \frac{(\varepsilon_X; f)\left(\frac{x}{\xi_2}\right)}{\varepsilon(\xi_1)} \\ &= f\left(\frac{x}{\xi}\right) && \text{see equation (A.3.10).} \end{aligned}$$

Thus the map $d_{(-)}$ is injective. Finally we check that it is surjective, specifically that for $d \in \mathrm{Coder}(\mathfrak{Z}^n X)$ and $f := \varepsilon_X d$, we have $d = d_f$:

$$\begin{aligned} d_f\left(\frac{x}{\xi}\right) &= \frac{(\varepsilon_X; f)\left(\frac{x}{\xi_2}\right)}{\xi_1} && \text{by definition} \\ &= \frac{(\varepsilon_X; \varepsilon_X d)\left(\frac{x}{\xi_2}\right)}{\xi_1} && \text{by definition} \\ &= \frac{\varepsilon_X\left(d\left[\frac{x}{\xi_2}\right]\right)}{\xi_1} \end{aligned}$$

$$
\begin{aligned}
&= \varepsilon_{\mathfrak{Z}^n X}\left(\frac{d\left[\frac{x}{\xi_2}\right]}{\xi_1}\right) && \\
&= \varepsilon_{\mathfrak{Z}^n X}\Delta d\left(\frac{x}{\xi}\right) && \text{see equation (A.3.15)} \\
&= d\left(\frac{x}{\xi}\right) && \text{see equation (A.3.10).}
\end{aligned}
$$

□

Lemma A.3.18. *Let $X \in {}_{\mathrm{dg}}\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$, $f, g : X \longrightarrow \mathfrak{Z}^n X$, and $d_f, d_g \in \mathrm{Coder}(\mathfrak{Z}^n X)$ the unique coderivations coextending f and g, as defined in equation* (A.3.17)*.*

(1) If f has an odd degree, then $d_f d_f = d_{f d_f}$, explicitly,

$$
d_f d_f\left(\frac{x}{\xi}\right) = \frac{(\varepsilon_X; f d_f)\left(\frac{x}{\xi_2}\right)}{\xi_1} = d_{f d_f}\left(\frac{x}{\xi}\right).
$$

(2) If f and g both have (possibly distinct) odd degrees, then $d_f d_g + d_g d_f = d_{f d_g + g d_f}$.

(3) If f and g both have (possibly distinct) even degrees, then $d_f d_g - d_g d_f = d_{f d_g - g d_f}$.

Proof. We have

$$
\begin{aligned}
d_f d_f\left(\frac{x}{\xi}\right) &= d_f\left(\frac{(\varepsilon_X; f)\left(\frac{x}{\xi_2}\right)}{\xi_1}\right) \\
&= \frac{(\varepsilon_X; f)\left(d_f\left[\frac{x}{\xi_2}\right]\right)}{\xi_1} && \spadesuit \\
&= \sum_i \left(\prod_{j<i,l}(-1)^{x_{j,l}}\right)\frac{(\varepsilon_X; f)\left(\frac{x_1}{\xi_{2,1}}\right), \dots, \frac{x_{i-1}}{\xi_{2,i-1}}, d_f\left(\frac{x_i}{\xi_{2,i}}\right), \frac{x_{i+1}}{\xi_{2,i+1}}}{\xi_1} \\
&= \sum_i \sum_{j<i}\left(\prod_{j\le k<i,l}(-1)^{x_{k,l}}\right)\frac{\dots, f\left(\frac{x_j}{\xi_{2,j}}\right), \dots, d_f\left(\frac{x_i}{\xi_{2,i}}\right), \dots}{\xi_1} \\
&\quad + \sum_i \frac{\dots, f d_f\left(\frac{x_i}{\xi_{2,i}}\right), \dots}{\xi_1} \\
&\quad + \sum_i \sum_{j>i}(-1)^f\left(\prod_{i\le k<j,l}(-1)^{x_{k,l}}\right)\frac{\dots, d_f\left(\frac{x_i}{\xi_{2,i}}\right), \dots, f\left(\frac{x_j}{\xi_{2,j}}\right), \dots}{\xi_1} \\
&= \sum_i \frac{\dots, f d_f\left(\frac{x_i}{\xi_{2,i}}\right), \dots}{\xi_1} \\
&= \frac{(\varepsilon_X; f d_f)\left(\frac{x}{\xi_2}\right)}{\xi_1}.
\end{aligned}
$$

where ♠ follows from equation (A.3.15). The second and third points can be proved using the same technique. □

Definition A.4.1 (hom-algebra)**.** Let $A = (A, m, u) \in {}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R(\mathfrak{Z}^n)$ and $C = (C, \Delta, \varepsilon) \in {}^{\mathrm{co}}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R(\mathfrak{Z}^n)$. Define their hom dg-algebra as

$$\mathrm{hom}(C, A)_m := \prod_{\omega \in \mathbb{O}_n} {}_{\mathrm{dg}}\mathcal{M}\mathrm{od}(C_\omega, A_\omega)_m.$$

In other words, an element $f \in \mathrm{hom}(C, A)$ of degree m is a collection of degree m morphisms $f_\omega : C_\omega \longrightarrow A_\omega$ in ${}_{\mathrm{dg}}\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$. The differential on $\mathrm{hom}(C, A)$ is given by

$$\partial_{\mathrm{hom}(C,A)}(f) := \partial_A \cdot f - (-1)^f f \cdot \partial_C. \tag{A.4.2}$$

Definition A.4.3 (Convolution product)**.** Let $A = (A, m, u) \in {}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R(\mathfrak{Z}^n)$ and $C = (C, \Delta, \varepsilon) \in {}^{\mathrm{co}}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R(\mathfrak{Z}^n)$, and $f, g \in \mathrm{hom}(C, A)$. Their *convolution product* $f \star g$ [LV12, section 1.6 and 6.4.4] (noted $f \smile g$ and called *cup-product* in [Pro11]) is defined as the composite

$$C \xrightarrow{\Delta_{(1)}} C \circ_{(1)} C \xrightarrow{f \circ_{(1)} g} A \circ_{(1)} A \xrightarrow{m} A.$$

Explicitly, for $c \in C$,

$$(f \star g)(c) = (-1)^{gc_1} \frac{(u\varepsilon; g)(c_2)}{f(c_1)}. \tag{A.4.4}$$

Definition A.4.5 (Hom-algebra)**.** Let $A = (A, m, u) \in {}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R(\mathfrak{Z}^n)$ and $C = (C, \Delta, \varepsilon) \in {}^{\mathrm{co}}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R(\mathfrak{Z}^n)$. Following [LV12, section 1.6.1 and 6.4.1], define the Hom $\mathfrak{Z}^n$-algebra of C and A as

$$\mathrm{Hom}(C, A)_\omega := \bigoplus_{m \in \mathbb{N}} {}_{\mathrm{dg}}\mathcal{M}\mathrm{od}(C_\omega, A_\omega)_m.$$

We endow it with the same differential as $\mathrm{hom}(C, A)$, see equation (A.4.2). If course, we have to check that $\mathrm{Hom}(C, A)$ really is a dg $\mathfrak{Z}^n$-algebra. First, we define a multiplication m on it as follows: for a homogeneous element $\frac{g}{f} \in (\mathrm{Hom}(C, A) \circ \mathrm{Hom}(C, A))_\omega$ in the $\nu \,\square_i\, \nu_i = \omega$ component, let $\frac{g}{f} := m\left(\frac{g}{f}\right)$ be the composite

$$C_\omega \xrightarrow{\Delta} (C \circ C)_\omega \twoheadrightarrow C_\nu \bigotimes_i C_{\nu_i} \xrightarrow{f \otimes_i g_i} A_\nu \bigotimes_i A_{\nu_i} \xrightarrow{m} A_\omega,$$

given explicitly by

$$\frac{g}{f}(c) = (-1)^{gc_1} \frac{g(c_2)}{f(c_1)}. \tag{A.4.6}$$

By convention, $f(d) = 0$ if $d \notin C_\nu$, and likewise for g. Second, the unit $I \longrightarrow \mathrm{Hom}(C, A)$ maps $1\mathsf{Y}_\psi$ to the composite $C_{\mathsf{Y}_\psi} \xrightarrow{\varepsilon} I_{\mathsf{Y}_\psi} \xrightarrow{u} A_{\mathsf{Y}_\psi}$.

Proposition A.4.7 (Generalization of [LV12, proposition 6.4.2])**.** *Let* $A = (A, m, u) \in {}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R(\mathfrak{Z}^n)$ *and* $C = (C, \Delta, \varepsilon) \in {}^{\mathrm{co}}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R(\mathfrak{Z}^n)$. *With the structure of definition A.4.5,* $\mathrm{Hom}(C, A)$ *is a* $\circ$*-monoid.*

Proof. Let $\frac{g}{f} \in (\operatorname{Hom}(C, A) \circ \operatorname{Hom}(C, A))_\omega$ be a homogeneous element in the $\nu \,\square_i\, \nu_i = \omega$ component, and $c \in C_\omega$. We have

$$\frac{h}{\left(\frac{g}{f}\right)}(c) = (-1)^{hc_1} \frac{h(c_2)}{\frac{g}{f}(c_1)} = (-1)^{hc_1 + hc_2 + gc_1} \frac{\frac{h(c_3)}{g(c_2)}}{f(c_1)},$$

$$\frac{\left(\frac{h}{g}\right)}{f}(c) = (-1)^{gc_1 + hc_1} \frac{\frac{h}{g}(c_2)}{f(c_1)} = (-1)^{gc_1 + hc_1 + hc_2} \frac{\frac{h(c_3)}{g(c_2)}}{f(c_1)}.$$

For unitality, we have

$$\begin{aligned}
\frac{g}{u(1)}(c) &= (-1)^{gc_1} \frac{g(c_2)}{u\varepsilon(c_1)} && \text{by definition} \\
&= m \cdot (u \otimes g) \left(\frac{c_2}{\varepsilon(c_1)} \right) \\
&= m \cdot (u \otimes g) \left(\frac{c}{1} \right) && \text{see equation (A.3.2)} \\
&= \frac{g(c)}{1} \\
&= g(c) && \text{see equation (A.2.9),}
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
&\frac{u(1)}{f}(c) \\
&\quad = (-1)^{u\varepsilon c_1} \frac{u\varepsilon(c_2)}{f(c_1)} && \text{by definition} \\
&\quad = m \cdot \left(f \bigotimes_i u \right) \left(\frac{\varepsilon(c_2)}{c_1} \right) \\
&\quad = m \cdot \left(f \bigotimes_i u \right) \left(\frac{1}{c} \right) && \text{see equation (A.3.2)} \\
&\quad = (-1)^{uc} \frac{u(1)}{f(c)} \\
&\quad = \frac{u(1)}{f(c)} && \text{since } |u| = 0 \\
&\quad = f(c) && \text{see equation (A.2.9).}
\end{aligned}$$

$\square$

Proposition A.4.8 (Generalization of [LV12, proposition 6.4.3])**.** *Let $A = (A, m, u) \in {}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R(\mathfrak{Z}^n)$ and $C = (C, \Delta, \varepsilon) \in {}^{\mathrm{co}}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R(\mathfrak{Z}^n)$. With the structure of definition A.4.5, $\operatorname{Hom}(C, A)$ is a dg $\mathfrak{Z}^n$-algebra.*

Proof. We have to check that $\partial\left(\frac{g}{f}\right) = \frac{g}{\partial(f)} + (-1)^f \frac{\partial[g]}{f}$, or, unfolding the definitions, that

$$\partial \cdot \frac{g}{f} - (-1)^{f+g} \frac{g}{f} \cdot \partial \;=\; \frac{g}{\partial(f)} - (-1)^f \frac{\partial[g]}{f}.$$

On the one hand, we have

$$\begin{aligned}
\partial \cdot \left(\frac{g}{f}\right)(c) &= \partial\left((-1)^{gc_1} \frac{g(c_2)}{f(c_1)}\right) && \text{by definition} \\
&= (-1)^{gc_1}\left(\frac{g(c_2)}{\partial f(c_1)} + (-1)^{f+c_1} \frac{\partial[g(c_2)]}{f(c_1)}\right) && \text{see equation (A.1.28)} \\
&= (-1)^{gc_1} \frac{g(c_2)}{\partial f(c_1)} + (-1)^{f+gc_1+c_1} \frac{\partial[g(c_2)]}{f(c_1)},
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
(-1)^{f+g}\left(\frac{g}{f}\right) \cdot \partial(c) &= (-1)^{f+g} m \cdot \left(f \bigotimes_i g_i\right)\left(\frac{\partial(c)_2}{\partial(c)_1}\right) \\
&= (-1)^{f+g} m \cdot \left(f \bigotimes_i g_i\right)\left(\frac{c_2}{\partial(c_1)} + (-1)^{c_1} \frac{\partial[c_2]}{c_1}\right) && \spadesuit \\
&= (-1)^{f+g}\left((-1)^{g\partial(c_1)} \frac{g(c_2)}{f\partial(c_1)} + (-1)^{c_1+gc_1} \frac{g(\partial[c_2])}{f(c_1)}\right) \\
&= (-1)^{f+g}\left((-1)^{g+gc_1} \frac{g(c_2)}{f\partial(c_1)} + (-1)^{c_1+gc_1} \frac{g(\partial[c_2])}{f(c_1)}\right), \\
&= (-1)^{f+gc_1} \frac{g(c_2)}{f\partial(c_1)} + (-1)^{f+g+gc_1+c_1} \frac{g(\partial[c_2])}{f(c_1)},
\end{aligned}$$

where $\spadesuit$ follows from equation (A.3.3). Finally,

$$\begin{aligned}
\partial \cdot \left(\frac{g}{f}\right)(c) &= \partial \cdot \left(\frac{g}{f}\right)(c) - (-1)^{f+g}\left(\frac{g}{f}\right) \cdot \partial(c) \\
&= (-1)^{gc_1} \frac{g(c_2)}{\partial f(c_1)} + (-1)^{f+gc_1+c_1} \frac{\partial[g(c_2)]}{f(c_1)} \\
&\qquad - (-1)^{f+gc_1} \frac{g(c_2)}{f\partial(c_1)} - (-1)^{f+g+gc_1+c_1} \frac{g(\partial[c_2])}{f(c_1)} \\
&= (-1)^{gc_1} \frac{g(c_2)}{\partial(f)(c_1)} + (-1)^{f+gc_1+c_1} \frac{\partial[g](c_2)}{f(c_1)} \\
&= \left(\frac{g}{\partial(f)} + (-1)^f \frac{\partial[g]}{f}\right)(c).
\end{aligned}$$

$\square$

LEFT TWISTED COMPOSITE PRODUCT

Definition A.5.1 (Twisted differential)**.** A morphism $t : C \longrightarrow A$ induces a *twisting term* $\partial_t^l : A \circ C \longrightarrow A \circ C$ given by

$$\partial_t^l\left(\frac{c}{a}\right) := (-1)^{ta} \frac{\frac{c_2}{(u\varepsilon;t)(c_1)}}{a}. \qquad \text{(A.5.2)}$$

From this twisting term, we construct the *twisted differential* ∂_t on $A \circ C$ as $\partial_t := \partial_{A\circ C} + \partial_t^l$. Despite the name, ∂_t is not a differential in general, but we give a necessary and sufficient condition for it to be the case in theorem A.5.7.

Lemma A.5.3. *Let $t, t' : C \longrightarrow A$. We have $\partial_{t'}^l \cdot \partial_t^l = \partial_{t\star t'}^l$.*

Proof. Straightforward computations:

$$\partial_{t'}^l \partial_t^l\left(\frac{c}{a}\right) = (-1)^{ta}\partial_{t'}^l\left(\frac{\frac{c_2}{(u\varepsilon;t)(c_1)}}{a}\right) = (-1)^{ta+t'a+t'c_1}\frac{\frac{\frac{c_3}{(u\varepsilon;t')(c_2)}}{(u\varepsilon;t)(c_1)}}{a}$$

$$= (-1)^{(t\star t')a}\frac{\frac{c_2}{(u\varepsilon;t\star t')(c_1)}}{a} = \partial_{t\star t'}^l\left(\frac{c}{a}\right).$$

□

Lemma A.5.4. *If $t : C \longrightarrow A$ is of degree -1, then*

$$\partial_{A\circ C}\cdot\partial_t^l + \partial_t^l\cdot\partial_{A\circ C} = \partial_{\partial_{\hom(A,C)}(t)}^l$$

Proof. On the one hand

$$\partial\partial_t^l\left(\frac{c}{a}\right) = \partial\left((-1)^a\frac{\frac{c_2}{(u\varepsilon;t)(c_1)}}{a}\right)$$

$$= (-1)^a\frac{c_2}{\partial\left(\frac{(u\varepsilon;t)(c_1)}{a}\right)} + (-1)^a\frac{-(-1)^{c_1}\frac{\partial[c_2]}{(u\varepsilon;t)(c_1)}}{a}$$

$$= (-1)^a\frac{\frac{c_2}{(u\varepsilon;t)(c_1)}}{\partial(a)} + \frac{\frac{c_2}{\partial[(u\varepsilon;t)(c_1)]}}{a} - (-1)^{a+c_1}\frac{\frac{\partial[c_2]}{(u\varepsilon;t)(c_1)}}{a},$$

and on the other hand,

$$\partial_t^l\partial\left(\frac{c}{a}\right) = \partial_t^l\left(\frac{c}{\partial(a)}\right) + (-1)^a\frac{\partial[c]}{a}$$

$$
\begin{aligned}
&= -(-1)^{a} \begin{array}{c} c_2 \\ \hline (u\varepsilon;t)(c_1) \\ \hline\hline \partial(a) \end{array} + \begin{array}{c} \partial[c]_2 \\ \hline (u\varepsilon;t)(\partial[c]_1) \\ \hline\hline a \end{array} \\
&= -(-1)^{a} \begin{array}{c} c_2 \\ \hline (u\varepsilon;t)(c_1) \\ \hline\hline \partial(a) \end{array} + \begin{array}{c} c_2 \\ \hline (u\varepsilon;t)(\partial[c_1]) \\ \hline\hline a \end{array} + (-1)^{a+c_1} \begin{array}{c} \partial[c_2] \\ \hline (u\varepsilon;t)(c_1) \\ \hline\hline a \end{array}.
\end{aligned}
$$

Consequently,

$$
(\partial\partial^l_t + \partial^l_t\partial)\left(\frac{c}{a}\right) = \begin{array}{c} c_2 \\ \hline \partial[(u\varepsilon;t)(c_1)] \\ \hline\hline a \end{array} + \begin{array}{c} c_2 \\ \hline (u\varepsilon;t)(\partial[c_1]) \\ \hline\hline a \end{array} = \begin{array}{c} c_2 \\ \hline (u\varepsilon;\partial(t))(c_1) \\ \hline\hline a \end{array} = \partial^l_{\partial(t)}\left(\frac{c}{a}\right).
$$

□

Definition A.5.5 (Twisting morphism)**.** A morphism $t : C \longrightarrow A$ of degree -1 is called a *twisting morphism* if it satisfies the *Maurer–Cartan equation* [LV12, sections 2.1.3 and 6.4.8]

$$\partial(t) + t \star t = 0. \tag{A.5.6}$$

Moreover, if C is coaugmented with coaugmentation map $\eta : I \longrightarrow C$, we require that $t\eta = 0$, and dually, if A is augmented with augmentation map $\varepsilon : A \longrightarrow I$, we require $\varepsilon t = 0$. We write $\mathrm{Tw}(C, A)$ for the set of twisting morphisms from C to A.

Theorem A.5.7. *The twisted differential ∂_t on $A \circ C$ really is a differential (i.e. $\partial_t\partial_t = 0$) if and only if t is a twisting morphism.*

Proof. We have

$$\partial_t\partial_t = (\partial_{A\circ C} + \partial^l_t)(\partial_{A\circ C} + \partial^l_t) = \partial_{A\circ C}\partial^l_t + \partial^l_t\partial_{A\circ C} + \partial^l_t\partial^l_t = \partial^l_{\partial(t)+t\star t}$$

which is 0 if and only if t satisfies the Maurer–Cartan equation. □

Definition A.5.8 (Twisted composite product)**.** If $t \in \mathrm{Tw}(C, A)$, the *twisted composite product* $A \circ_t C$ is the dg $\mathbb{O}_n$-module $A \circ C$ endowed with the differential ∂_t. If C and A are differential graded, and $A \circ_t C$ is acyclic then we say that t is a *Koszul morphism* . We write $\mathrm{Kos}(C, A)$ for the set of Koszul morphisms from C to A.

A.6 BAR AND COBAR

THE BAR CONSTRUCTION

Definition A.6.1 (Bar construction)**.** Let A be an augmented $\mathfrak{Z}^n$-algebra, with augmentation ideal $\bar{A}$. We define a $\mathfrak{Z}^n$-coalgebra $\mathrm{B}(A)$ (which shall be the underlying coalgebra of the *bar construction* of A defined in equation (A.6.6)) as follows:

$$\mathrm{B}(A) := \mathfrak{Z}^n(\blacktriangle\bar{A}). \tag{A.6.2}$$

We now follow [LV12, section 2.2.1 and 6.5.1] to construct a suitable differential on $\mathrm{B}(A)$. Firstly, if ∂_A is the differential of A, let $\partial_1 : \mathfrak{Z}^n(\blacktriangle\bar{A}) \longrightarrow \mathfrak{Z}^n(\blacktriangle\bar{A})$ be the unique coderivation coextending the composite on the left (as defined in proposition A.3.16):

$$\mathrm{B}(A) = \mathfrak{Z}^n(\blacktriangle\bar{A}) \xrightarrow{\varepsilon_{\blacktriangle\bar{A}}} \blacktriangle\bar{A} \xrightarrow{\partial_{\blacktriangle\bar{A}}} \blacktriangle\bar{A}, \qquad \partial_1\left(\frac{\blacktriangle a}{\xi}\right) := \frac{\partial_{\blacktriangle\bar{A}}[\blacktriangle a]}{\xi}.$$

It is explicitely given as on the right. Next, write $\gamma_A : \mathfrak{Z}^n A \longrightarrow A$ the structure map of A, and let γ'_A be the following composite:

$$\mathrm{B}(A) = \mathfrak{Z}^n(\blacktriangle\bar{A}) \twoheadrightarrow \blacktriangle\bar{A} \circ_{(1)} \blacktriangle\bar{A} \xrightarrow{\gamma_A} \blacktriangle\bar{A}.$$

Explicitly,

$$\gamma'_A\left(\frac{\blacktriangle a}{\xi}\right) = \begin{cases} (-1)^{a_1} \blacktriangle \frac{a_1, a_2}{\xi} & \text{if } \#\xi^\bullet = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{A.6.3}$$

where for $\#\xi_2^\bullet = 2$, we write the two elements sequence $\blacktriangle a$ as $\blacktriangle a_1, \blacktriangle a_2$, for $a_1, a_2 \in \bar{A}$, and $\blacktriangle a_1$ decorating the root node of ξ. Let $\partial_2 : \mathrm{B}(A) \longrightarrow \mathrm{B}(A)$ be the unique coderivation coextending γ'_A (as defined in proposition A.3.16).

Lemma A.6.4. *(1) The coderivation ∂_1 is a differential on $\mathrm{B}(A)$.*
(2) The coderivation ∂_2 is a differential on $\mathrm{B}(A)$.
(3) The differentials ∂_1 and ∂_2 anticommute, i.e. $\partial_1\partial_2 = -\partial_2\partial_1$.

Proof. (1) By lemma A.3.18, $\partial_1\partial_1$ is the unique coderivation coextending the following composite:

$$\mathfrak{Z}^n(\blacktriangle\bar{A}) \xrightarrow{\partial_1} \mathfrak{Z}^n(\blacktriangle\bar{A}) \xrightarrow{\varepsilon_{\blacktriangle\bar{A}}} \blacktriangle\bar{A} \xrightarrow{\partial_{\blacktriangle\bar{A}}} \blacktriangle\bar{A}$$

which is 0 since $\partial_{\blacktriangle\bar{A}}\partial_{\blacktriangle\bar{A}} = 0$.

(2) By lemma A.3.18 we have that $\partial_2\partial_2$ is the unique coderivation coextending $\gamma'_A\partial_2$, and we now show that $\gamma'_A\partial_2 = 0$. We have

$$\partial_2\partial_2\left(\frac{\blacktriangle a}{\xi}\right) = \partial_2\left(\frac{(\varepsilon_A;\gamma'_A)\left(\frac{\blacktriangle a}{\xi_2}\right)}{\xi_1}\right) = \frac{(\varepsilon_A;\gamma'_A)\left(\frac{(\varepsilon_A;\gamma'_A)\left(\frac{\blacktriangle a}{\xi_3}\right)}{\xi_2}\right)}{\xi_1} = \frac{(\varepsilon_A;\gamma'_A)\left(\partial_2\left[\frac{\blacktriangle a}{\xi_2}\right]\right)}{\xi_1}$$

since ∂_2 is a coderivation on a cofree conilpotent coalgebra, see equation (A.3.15). Let us consider the following term:

$$\gamma'_A\partial_2\left(\frac{\blacktriangle a}{\zeta}\right) = \gamma'_A\left(\frac{(\varepsilon_A;\gamma'_A)\left(\frac{\blacktriangle a}{\zeta_2}\right)}{\zeta_1}\right).$$

It is 0 whenever $\#\zeta^\bullet \neq 3$, thus let us assume that ζ has exactly 3 nodes. The sequence $\blacktriangle a$ can then be written $\blacktriangle b, \blacktriangle c, \blacktriangle d$ with $b, c, d \in \bar{A}$. Let us assume that ξ has height 3, i.e. that c "is below" d (the other case is that in which the height of ξ is 2, and thus where c and d are "disjoint", and can be treated similarly). Ignoring all 0 terms, we obtain

$$\gamma'_A\left(\frac{(\varepsilon_A;\gamma'_A)\left(\frac{\blacktriangle a}{\zeta_2}\right)}{\zeta_1}\right) = \gamma'_A\left(\frac{(-1)^b \blacktriangle \frac{b,c}{\zeta_{2,1}}, \blacktriangle d}{\zeta_1}\right) + \gamma'_A\left(\frac{\blacktriangle b, (-1)^c \blacktriangle \frac{c,d}{\zeta_{2,2}}}{\zeta_1}\right)$$

$$= (-1)^c \blacktriangle \frac{\frac{b,c}{\zeta_{2,1}},\ d}{\zeta_1} - (-1)^c \blacktriangle \frac{b,\ \frac{c,d}{\zeta_{2,2}}}{\zeta_1}$$

$$= 0$$

by associativity.

(3) Let $f := \partial_{\blacktriangle \bar{A}} \varepsilon_{\blacktriangle \bar{A}} \partial_2$ and $g := \gamma'_A \partial_1$. By lemma A.3.18, $\partial_1\partial_2 + \partial_2\partial_1$ is the unique coderivation coextending $f+g$, and we now show that it is 0. Take a homogeneous element $\frac{\blacktriangle a}{\xi} \in \mathfrak{Z}^n(\blacktriangle \bar{A})$. It is easy to see that $f\left(\frac{\blacktriangle a}{\xi}\right) = g\left(\frac{\blacktriangle a}{\xi}\right) = 0$ if $\#\xi^\bullet \neq 2$, so we assume $\#\xi^\bullet = 2$, and write the sequence $\blacktriangle a$ as $\blacktriangle a_1, \blacktriangle a_2$, with $a_1, a_2 \in \bar{A}$. We have

$$\begin{aligned}
f\left(\frac{\blacktriangle a}{\xi}\right) &= \partial_{\blacktriangle \bar{A}} \varepsilon_{\blacktriangle \bar{A}} \left(\frac{(\varepsilon_{\blacktriangle \bar{A}}; \gamma'_A)\left(\frac{\blacktriangle a}{\xi_2}\right)}{\xi_1} \right) \\
&= \partial_{\blacktriangle \bar{A}} \varepsilon_{\blacktriangle \bar{A}} \left(\frac{(-1)^{a_1} \blacktriangle \frac{a_1, a_2}{\xi}}{\mathsf{Y}_{\mathsf{t}\xi}} \right) \\
&= -(-1)^{a_1} \blacktriangle \partial_A \left(\frac{a_1, a_2}{\xi} \right) \\
&= -(-1)^{a_1} \blacktriangle \frac{\partial_A(a_1), a_2}{\xi} - \blacktriangle \frac{a_1, \partial_A(a_2)}{\xi} \\
g\left(\frac{\blacktriangle a}{\xi}\right) &= \gamma'_A \left(\frac{\partial_{\blacktriangle \bar{A}} [\blacktriangle a_1, \blacktriangle a_2]}{\xi} \right) \\
&= \gamma'_A \left(-\frac{\blacktriangle \partial_A(a_1), \blacktriangle a_2}{\xi} + (-1)^{a_1} \frac{\blacktriangle a_1, \blacktriangle \partial_A(a_2)}{\xi} \right) \\
&= (-1)^{a_1} \blacktriangle \frac{\partial_A(a_1), a_2}{\xi} + \blacktriangle \frac{a_1, \partial(a_2)}{\xi},
\end{aligned}$$

and therefore $f + g = 0$. □

Definition A.6.5 (Differential graded bar construction)**.** By lemma A.6.4, the morphism $\partial_1 + \partial_2$ is a differential on the coalgebra $\mathrm{B}(A)$ defined in equation (A.6.2). The *bar construction* of an augmented $A \in {}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R(\mathfrak{Z}^n)$ is the dg $\mathfrak{Z}^n$-coalgebra

$$\mathrm{B}(A) := \left(\mathfrak{Z}^n(\blacktriangle \bar{A}), \partial_1 + \partial_2\right). \tag{A.6.6}$$

Define now a degree (-1) morphism $\beta : \mathrm{B}(A) \longrightarrow A$ as the composite

$$\mathrm{B}(A) = \mathfrak{Z}^n(\blacktriangle \bar{A}) \xrightarrow{\varepsilon_{\blacktriangle \bar{A}}} \blacktriangle \bar{A} \xrightarrow{\blacktriangledown} \bar{A} \hookrightarrow A. \tag{A.6.7}$$

Explicitly, it maps a homogeneous element $\frac{\blacktriangle a}{\xi}$ to a if ξ is an endotope, and 0 otherwise.

Proposition A.6.8. *The morphism $\beta : \mathrm{B}(A) \longrightarrow A$ is a twisting morphism.*

Proof. We show that β satisfies the Maurer–Cartan equation (A.5.6), i.e. that $0 = \partial(\beta) + \beta \star \beta = \partial_A \cdot \beta + \beta \cdot \partial_1 + \beta \cdot \partial_2 + \beta \star \beta$. Let $\frac{\blacktriangle a}{\xi}$ be a homogeneous element of $\mathrm{B}(A)$. Applying definitions, we have

$$(\partial_A \cdot \beta)\left(\frac{\blacktriangle a}{\xi}\right) = \begin{cases} \partial_A(a) & \text{if } \#\xi^\bullet = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$
\begin{aligned}
(\beta\cdot\partial_1)\left(\frac{\blacktriangle a}{\xi}\right) &= \beta\left(\frac{\partial_{\blacktriangle\bar{A}}[\blacktriangle a]}{\xi}\right) = \begin{cases}-\partial_A(a) & \text{if } \#\xi^\bullet = 1,\\ 0 & \text{otherwise,}\end{cases}\\
(\beta\cdot\partial_2)\left(\frac{\blacktriangle a}{\xi}\right) &= (-1)^{a_1}\beta\left(\frac{\blacktriangle\frac{a_1,a_2}{\xi_2}}{\xi_1}\right) = \begin{cases}(-1)^{a_1}\frac{a_1,a_2}{\xi} & \text{if } \#\xi^\bullet = 2,\\ 0 & \text{otherwise,}\end{cases}\\
(\beta\star\beta)\left(\frac{\blacktriangle a}{\xi}\right) &= (-1)^{a_1+1}\frac{\beta\left(\frac{\blacktriangle a_1}{\xi_{2,1}}\right),\beta\left(\frac{\blacktriangle a_2}{\xi_{2,2}}\right)}{\xi_1} = \begin{cases}(-1)^{a_1+1}\frac{a_1,a_2}{\xi} & \text{if } \#\xi^\bullet = 2,\\ 0 & \text{otherwise,}\end{cases}
\end{aligned}
$$

where:

(1) in the cases in which $\#\xi^\bullet = 2$ we write a_1, a_2 for the two element sequence a,

(2) in the $\beta\cdot\partial_2$ case, ξ_1 and ξ_2 define the unique decompositions $\xi = \xi_1 \mathbin{\scriptstyle\square} \xi_2$ with $\#\xi_2 = 2$, which forces ξ_1 to be an endotope, and ξ_2 to be ξ,

(3) in the $\beta\star\beta$ case, ξ_1, $\xi_{2,1}$, and $\xi_{2,2}$ define the unique decompositions $\xi = \xi_1 \square_i \xi_{2,i}$, where $\#\xi_1 = 2$, which forces the $\xi_{2,i}$s to be endotopes, and ξ_1 to be ξ.

We see that $\partial_A\cdot\beta + \beta\cdot\partial_1 + \beta\cdot\partial_2 + \beta\star\beta = 0$ as desired. Let $\varepsilon : A \longrightarrow I$ be the augmentation map of A. By definition, β factors through $\bar{A} = \ker\varepsilon$ (see equation (A.6.7)), and thus $\varepsilon\beta = 0$. □

Theorem A.6.9. *The morphism $\beta : \mathrm{B}(A) \longrightarrow A$ is terminal among twisting morphisms over A. Explicitly, for every twisting morphism $t : C \longrightarrow A$, there exists a unique morphism of dg coalgebra $t^\flat : C \longrightarrow \mathrm{B}(A)$ such that $t = \beta t^\flat$:*

$$
\begin{tikzcd}
C \arrow[d, "\exists! t^\flat"'] \arrow[dr, dashed, "\forall t"] & \\
\mathrm{B}(A) \arrow[r, "\beta"'] & A.
\end{tikzcd}
$$

Consequently, we have an isomorphism ${}^{\mathrm{co}}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}^+_R(\mathfrak{Z}^n)(C, \mathrm{B}(A)) \cong \mathrm{Tw}(C, A)$ natural in both C and A.

Proof. By definition, t factors through the augmentation ideal $\bar{A}$ of A, and let u be the composite

$$C \xrightarrow{t} \bar{A} \xrightarrow{\blacktriangle} \blacktriangle\bar{A}.$$

Since $\mathrm{B}(A) = \mathfrak{Z}^n(\blacktriangle\bar{A})$ is cofree over $\blacktriangle\bar{A}$, there exists a unique morphism of coalgebra $t^\flat : C \longrightarrow \mathrm{B}(A)$ such that $u = \varepsilon_{\blacktriangle\bar{A}} t^\flat$. Postcomposing with

$$\blacktriangle\bar{A} \xrightarrow{\blacktriangledown} \bar{A} \hookrightarrow A,$$

we have $t = \beta t^\flat$. Conversely, if $g : C \longrightarrow \mathrm{B}(A)$ is a coalgebra morphism such that $t = \beta g$, then postcomposing with the suspension map $\bar{A} \longrightarrow \blacktriangle\bar{A}$ gives $u = \varepsilon_{\blacktriangle\bar{A}} g$, whence $g = t^\flat$.

We now show that $t^\flat$ is a morphism of dg coalgebras. Recall that as the unique factorization of u through the cofree coalgebra $\mathrm{B}(A)$, $t^\flat$ is the composite

$$C \xrightarrow{\Delta} \mathfrak{Z}^n C \xrightarrow{\mathfrak{Z}^n u} \mathfrak{Z}^n(\blacktriangle\bar{A}).$$

On the one hand, we have

$$(t^\flat\cdot\partial_C)(c) = (\mathfrak{Z}^n u\cdot\Delta\cdot\partial_C)(c) = (\mathfrak{Z}^n u\cdot\partial_{\mathfrak{Z}^n C}\cdot\Delta)(c) = \frac{\blacktriangle t\,(\partial_C[c_2])}{\xi_1^c},$$

and on the other hand,

$$(\partial_1 \cdot t^\flat)(c) = \frac{\partial_{\blacktriangle\bar{A}}\left[\blacktriangle t(c_2)\right]}{\xi_1^c} = -\frac{\blacktriangle\partial_A\left[t(c_2)\right]}{\xi_1^c},$$

and

$$\begin{aligned}(\partial_2 \cdot t^\flat)(c) &= \frac{(\varepsilon_{\blacktriangle\bar{A}}; \gamma'_A)\left(\frac{\blacktriangle t(c_2)}{\xi_{1,2}^c}\right)}{\xi_{1,1}^c} \\ &= \sum_{\substack{\xi_1^c = \zeta_1 \,\square_{[p_k]}\, \zeta_2 \\ \#\zeta_2 = 2}} (-1)^{c_{2,k}-1}\left(\prod_{k<i<k'}(-1)^{c_{2,i}}\right)\frac{\blacktriangle t(c_{2,1}),\ldots,\blacktriangle\frac{t(c_{2,k}),t(c_{2,k'})}{\zeta_2},\ldots}{\zeta_1} \qquad \spadesuit \\ &= -\frac{\blacktriangle(t;t\star t)(c_2)}{\xi_1^c},\end{aligned}$$

where in $\spadesuit$, $(\xi_1^c)^\bullet = \{[p_1] \prec [p_2] \prec \cdots\}$, and the sequence c_2 is as $c_{2,1},\ldots$ with $c_{2,i} \in C$ corresponding to $[p_i]$. Assembling the previous computations, we obtain

$$\begin{aligned}(t^\flat \cdot \partial_C - \partial_{\mathrm{B}(A)} \cdot f)(c) &= (t^\flat \cdot \partial_C - \partial_1 \cdot t^\flat - \partial_2 \cdot t^\flat)(c) \\ &= \frac{\blacktriangle t\left(\partial_C[c_2]\right)}{\xi_1^c} + \frac{\blacktriangle\partial_A\left[t(c_2)\right]}{\xi_1^c} + \frac{\blacktriangle(t;t\star t)(c_2)}{\xi_1^c} \\ &= \frac{\blacktriangle(t;t\cdot\partial_C + \partial_A\cdot t + t\star t)(c_2)}{\xi_1^c} \\ &= \frac{\blacktriangle(t;\partial(t) + t\star t)(c_2)}{\xi_1^c} \\ &= 0\end{aligned}$$

since t satisfies the Maurer–Cartan equation (A.5.6). □

Definition A.6.10 (Bar complex)**.** Since $\beta : \mathrm{B}(A) \longrightarrow A$ is a twisting morphism, the twisted composite $A \circ_\beta \mathrm{B}(A)$ is a dg $\mathbb{O}_n$-module, called the *bar complex* of A.

Remark A.6.11. In preparation for the next result, let us develop the twisting term $\partial_\beta^l : A \circ \mathrm{B}(A) \longrightarrow A \circ \mathrm{B}(A)$. Consider a homogeneous element of $A \circ \mathrm{B}(A)$:

$$\frac{\dfrac{\blacktriangle b}{\xi}}{a}.$$

Here, $\frac{\blacktriangle b}{\xi}$ is a sequence $\frac{\blacktriangle b_1}{\xi_1},\ldots$ of homogeneous elements of $\mathrm{B}(A)$. Assuming ξ_i is not degenerate, write

$$\xi_i = \curlyvee_{\mathsf{s}_{[]}\,\xi_i} \bigcirc_j \xi_{i,j}, \qquad \blacktriangle b_i = \blacktriangle b_{i,[]}, \blacktriangle b',$$

where $\blacktriangle b_{i,[]}$ is the first element of the sequence $\blacktriangle b_i$ (corresponding to the root node of ξ_i), and $\blacktriangle b'$ is the tail of the sequence. This just emphasizes the root node of ξ_i and the corresponding element of $\blacktriangle\bar{A}$. Now,

$$\partial_\beta^l\left(\frac{\dfrac{\blacktriangle b}{\xi}}{a}\right) = \sum_i\left(\prod_{j<i}(-1)^{b_{i,[]}\cdot\blacktriangle b_j}\right)\frac{\dfrac{\frac{\blacktriangle b_1}{\xi_1},\ldots,\frac{\blacktriangle b'_i}{\xi'_i},\ldots}{b_{i,[]}}}{a}$$

where the signs just result of the Koszul sign convention, and where if ξ_i is degenerate, the corresponding term in the sum is just 0. Intuitively, the i-th term removes the root element $b_{i,[]}$ from $\frac{\blacktriangle b_i}{\xi_i}$ and multiplies it with a.

Notation A.6.12. We now introduce a notation that will prove useful in the proof of the next result. Let $\frac{\blacktriangle a}{\xi} \in \mathrm{B}(A)$ be a homogeneous element, where $\blacktriangle a$ is a sequence $(\blacktriangle a_{[p]})_{[p]\in\xi^\bullet}$, and let $[e]$ be an inner edge address of ξ (note that this forces ξ to have at least two nodes). Equivalently, $[e]$ is a node address of ξ of the form $[p[q]]$, for $[p] \in \xi^\bullet$ and $[q] \in (\mathsf{s}_{[p]}\xi)^\bullet$. Then ξ can be decomposed as $\xi = \xi' \square_{[p]} \xi''$ where ξ'' has two nodes. In other words, the unique inner edge of ξ'' corresponds to the edge $[p[q]]$ of ξ. We introduce the following notation:

$$\frac{\blacktriangle a}{\xi}/[p[q]] := \left(\prod_{[p]<[r]<[p[q]]} (-1)^{(a_{[r]}+1)(a_{[p[q]]}+1)} \right) (-1)^{a_{[p]}} \\ \times \frac{\blacktriangle a_{[]}, \ldots, \blacktriangle \frac{a_{[p]}, a_{[p[q]]}}{\xi''}, \ldots}{\xi'}.$$

Intuitively, it is the tree of elements of $\blacktriangle\bar{A}$ where the edge at address $[p[q]]$ has been contracted, and the elements $a_{[p]}$ and $a_{[p[q]]}$ at the extremal nodes have been multiplied according to the algebra structure of A. The are given by the Koszul sign rule. For example, the differential ∂_2 can simply be expressed as

$$\partial_2 \left(\frac{\blacktriangle a}{\xi} \right) = \sum_{[p[q]]\in\xi^\bullet} \frac{\blacktriangle a}{\xi}/[p[q]],$$

where again, $[p[q]]$ ranges over the inner edge addresses of ξ.

Proposition A.6.13. *The universal twisting morphism $\beta : \mathrm{B}(A) \longrightarrow A$ is Koszul. In other words, the bar complex $A \circ_\beta \mathrm{B}(A)$ of A is acyclic.*

Proof. We construct a contracting homotopy $h : A \circ_\beta \mathrm{B}(A) \longrightarrow A \circ_\beta \mathrm{B}(A)$ (i.e. a morphism of $\mathbb{O}_n$-modules such that $\partial h + h\partial = \mathrm{id}$) as follows:

$$h \begin{pmatrix} \frac{\blacktriangle b}{\xi} \\ \hline a \end{pmatrix} := \frac{\frac{\blacktriangle a, \blacktriangle b}{\xi_+}}{1},$$

where if $a \in A_\omega$, then $\xi_+ := \mathsf{Y}_\omega \bigcirc_i \xi_i$. Recall that $\partial_{A\circ_\beta \mathrm{B}(A)} = \partial_{A\circ\mathrm{B}(A)} + \partial^l_\beta = (\partial_A \circ \mathrm{id}_{\mathrm{B}(A)}) + (\mathrm{id}_A \circ' \partial_1) + (\mathrm{id}_A \circ' \partial_2) + \partial^l_\beta$. We have:

$$(\partial_A \circ \mathrm{id}_{\mathrm{B}(A)}) \cdot h \begin{pmatrix} \frac{\blacktriangle b}{\xi} \\ \hline a \end{pmatrix} = (\partial_A \circ \mathrm{id}_{\mathrm{B}(A)}) \begin{pmatrix} \frac{\blacktriangle a, \blacktriangle b}{\xi_+} \\ \hline 1 \end{pmatrix} = 0,$$

$$(\mathrm{id}_A \circ' \partial_1) \cdot h \begin{pmatrix} \frac{\blacktriangle b}{\xi} \\ \hline a \end{pmatrix} = (\mathrm{id}_A \circ' \partial_1) \begin{pmatrix} \frac{\blacktriangle a, \blacktriangle b}{\xi_+} \\ \hline 1 \end{pmatrix} = \frac{\frac{\partial_{\blacktriangle\bar{A}}[\blacktriangle a, \blacktriangle b]}{\xi_+}}{1}$$

$$= -\frac{\frac{\blacktriangle \partial_A(a), \blacktriangle b}{\xi_+}}{1} + (-1)^{a+1} \frac{\frac{\blacktriangle a, \partial_{\blacktriangle\bar{A}}[\blacktriangle b]}{\xi_+}}{1},$$

$$(\mathrm{id}_A \circ' \partial_2) \cdot h \begin{pmatrix} \blacktriangle b \\ \hline \xi \\ \hline a \end{pmatrix} = (\mathrm{id}_A \circ' \partial_2) \begin{pmatrix} \blacktriangle a, \blacktriangle b \\ \hline \xi_+ \\ \hline 1 \end{pmatrix} = \sum_{[p[q]] \in \xi^\bullet} \frac{\frac{\blacktriangle a, \blacktriangle b}{\xi_+} / [p[q]]}{1},$$

$$\partial^l_\beta \cdot h \begin{pmatrix} \blacktriangle b \\ \hline \xi \\ \hline a \end{pmatrix} = \partial^l_\beta \begin{pmatrix} \blacktriangle a, \blacktriangle b \\ \hline \xi_+ \\ \hline 1 \end{pmatrix} = \begin{matrix} \blacktriangle b \\ \hline \xi \\ \hline a \end{matrix}.$$

On the other hand,

$$h \cdot (\partial_A \circ \mathrm{id}_{\mathrm{B}(A)}) \begin{pmatrix} \blacktriangle b \\ \hline \xi \\ \hline a \end{pmatrix} = h \begin{pmatrix} \blacktriangle b \\ \hline \xi \\ \hline \partial_A(a) \end{pmatrix} = \begin{matrix} \blacktriangle \partial(a), \blacktriangle b \\ \hline \xi_+ \\ \hline 1 \end{matrix},$$

$$h \cdot (\mathrm{id}_A \circ' \partial_1) \begin{pmatrix} \blacktriangle b \\ \hline \xi \\ \hline a \end{pmatrix} = (-1)^a h \begin{pmatrix} \partial_{\blacktriangle \bar{A}}[\blacktriangle b] \\ \hline \xi \\ \hline a \end{pmatrix} = (-1)^a \begin{matrix} \blacktriangle a, \partial_{\blacktriangle \bar{A}}[\blacktriangle b] \\ \hline \xi_+ \\ \hline 1 \end{matrix},$$

$$\begin{aligned} h \cdot (\mathrm{id}_A \circ' \partial_2) \begin{pmatrix} \blacktriangle b \\ \hline \xi \\ \hline a \end{pmatrix} &= (-1)^a h \begin{pmatrix} \partial_2 \left[\frac{\blacktriangle b}{\xi} \right] \\ \hline a \end{pmatrix} \\ &= (-1)^a \sum_{\substack{[p[q]] \in \xi_+^\bullet \\ [p] \neq []}} (-1)^{a+1} \frac{\frac{\blacktriangle a, \blacktriangle b}{\xi_+} / [p[q]]}{1}, \\ &= - \sum_{\substack{[p[q]] \in \xi_+^\bullet \\ [p] \neq []}} \frac{\frac{\blacktriangle a, \blacktriangle b}{\xi_+} / [p[q]]}{1}, \end{aligned}$$

$$\begin{aligned} h \cdot \partial^l_\beta \begin{pmatrix} \blacktriangle b \\ \hline \xi \\ \hline a \end{pmatrix} &= (-1)^a \sum_i \left(\prod_{j<i} (-1)^{b_{i,[]} \cdot \blacktriangle b_j} \right) h \begin{pmatrix} \frac{\blacktriangle b_1}{\xi_1}, \ldots, \frac{\blacktriangle b'_i}{\xi'_i}, \ldots \\ \hline b_{i,[]} \\ \hline\hline a \end{pmatrix} \\ &= (-1)^a \sum_{[q] \in (\mathsf{s}_{[]}\, \xi_+)^\bullet} (-1)^{a+1} \frac{\frac{\blacktriangle a, \blacktriangle b}{\xi_+} / [[q]]}{1} \\ &= - \sum_{[q] \in (\mathsf{s}_{[]}\, \xi_+)^\bullet} \frac{\frac{\blacktriangle a, \blacktriangle b}{\xi_+} / [[q]]}{1}, \end{aligned}$$

Note that $(\mathrm{id}_A \circ' \partial_2) \cdot h + h \cdot (\mathrm{id}_A \circ' \partial_2) + h \cdot \partial^l_\beta = 0$. The result follows. ∎

THE COBAR CONSTRUCTION

Definition A.6.14 (Cobar construction)**.** Let C be a coaugmented $\mathfrak{Z}^n$-coalgebra, with coaugmentation quotient $C \twoheadrightarrow \bar{C}$. If $c \in C$ we abuse notations and write c for its class in $\bar{C}$. We define a $\mathfrak{Z}^n$-algebra (which shall be the underlying algebra of the *cobar construction* of C defined in equation (A.6.18)) as follows:

$$\Omega(C) := \mathfrak{Z}^n(\blacktriangledown \bar{C}). \tag{A.6.15}$$

We now follow [LV12, section 2.2.5 and 6.5.5] to construct a suitable differential on $\Omega(C)$. Firstly, if ∂_C is the differential of C, let $\partial_1 : \mathfrak{Z}^n(\blacktriangledown\bar{C}) \longrightarrow \mathfrak{Z}^n(\blacktriangledown\bar{C})$ be the unique derivation extending the composite (as defined in proposition A.2.24)

$$\blacktriangledown\bar{C} \xrightarrow{\partial_{\blacktriangledown\bar{C}}} \blacktriangledown\bar{C} \xrightarrow{\eta_{\blacktriangledown\bar{C}}} \mathfrak{Z}^n(\blacktriangledown\bar{C}) = \Omega(C),$$

explicitly,

$$\partial_1(\blacktriangledown c) \coloneqq \frac{\partial_{\blacktriangledown\bar{C}}[\blacktriangledown c_2]}{\xi_1^c}.$$

Next, write $\Delta_C : C \longrightarrow \mathfrak{Z}^n C$ for the structure map of C, and let Δ'_C be the composite

$$\blacktriangledown\bar{C} \xrightarrow{\Delta_C} \blacktriangledown\bar{C} \circ_{(1)} \blacktriangledown\bar{C} \hookrightarrow \mathfrak{Z}^n(\blacktriangledown\bar{C}) = \Omega(C),$$

explicitly,

$$\Delta'_C(\blacktriangledown c) \coloneqq (-1)^{c_2}\frac{\blacktriangledown c_2, \blacktriangledown c_3}{\xi_1^c},$$

where $\frac{\blacktriangledown c_2, \blacktriangledown c_3}{\xi_1^c}$ ranges over the terms of $\Delta_C(c) = \frac{\blacktriangledown d}{\xi_1^c}$ where $\#(\xi_1^c)^\bullet = 2$, and where we write the sequence d as c_2, c_3, for $c_2, c_3 \in C$. Let $\partial_2 : \Omega(C) \longrightarrow \Omega(C)$ be the unique derivation extending Δ'_C (as defined in proposition A.2.24).

Lemma A.6.16. *(1) The derivation ∂_1 is a differential on $\Omega(C)$.*

(2) The derivation ∂_2 is a differential on $\Omega(C)$.

(3) The differentials ∂_1 and ∂_2 commute, i.e. $\partial_1\partial_2 = -\partial_2\partial_1$.

Proof. (1) By lemma A.2.26, $\partial_1\partial_1$ is the unique derivation extending the following composite

$$\blacktriangledown\bar{C} \xrightarrow{\partial_{\blacktriangledown\bar{C}}} \blacktriangledown\bar{C} \xrightarrow{\eta_{\blacktriangledown\bar{C}}} \mathfrak{Z}^n(\blacktriangledown\bar{C}) \xrightarrow{\partial_1} \mathfrak{Z}^n(\blacktriangledown\bar{C}),$$

which is 0 since $\partial_{\blacktriangledown\bar{C}}\partial_{\blacktriangledown\bar{C}} = 0$.

(2) By lemma A.2.26, $\partial_2\partial_2$ is the unique derivation extending $\partial_2\Delta'_C$, and we show that it is 0. We have

$$\begin{aligned}
\partial_2\Delta'_C(\blacktriangledown c) &= (-1)^{c_2}\partial_2\left(\frac{\blacktriangledown c_2, \blacktriangledown c_3}{\xi_1^c}\right) \\
&= (-1)^{c_2}\frac{(\eta_{\blacktriangledown\bar{C}}; \Delta'_C)(\blacktriangledown c_2, \blacktriangledown c_3)}{\xi_1^c} \\
&= (-1)^{c_2+c_{2,2}}\frac{\frac{\blacktriangledown c_{2,2}, \blacktriangledown c_{2,3}}{\xi_1^{c_2}}, \blacktriangledown c_3}{\xi_1^c} - (-1)^{c_{3,2}}\frac{\blacktriangledown c_2, \frac{\blacktriangledown c_{3,2}, \blacktriangledown c_{3,3}}{\xi_1^{c_3}},}{\xi_1^c},
\end{aligned}$$

and

$$\begin{aligned}
-(-1)^{c_{3,2}}\frac{\blacktriangledown c_2, \frac{\blacktriangledown c_{3,2}, \blacktriangledown c_{3,3}}{\xi_1^{c_3}},}{\xi_1^c} &= -\blacktriangledown 1 \otimes \blacktriangledown 1 \otimes \blacktriangledown 1 \otimes \frac{c_2, \Delta_{(1)}(c_3),}{\xi_1^c} \\
&= -\blacktriangledown 1 \otimes \blacktriangledown 1 \otimes \blacktriangledown 1 \otimes \frac{\Delta_{(1)}(c_2), c_3,}{\xi_1^c}
\end{aligned}$$

$$
\begin{aligned}
&= -(-1)^{c_2}\, \blacktriangledown 1 \otimes \blacktriangledown 1 \otimes \frac{\Delta_{(1)}(c_2), \blacktriangledown c_3,}{\xi_1^c} \\
&= -(-1)^{c_2}\, \blacktriangledown 1 \otimes \blacktriangledown 1 \otimes \frac{\frac{c_{2,2}, c_{2,3}}{\xi_1^{c_2}}, \blacktriangledown c_3}{\xi_1^c} \\
&= -(-1)^{c_2 + c_{2,2}} \frac{\frac{\blacktriangledown c_{2,2}, \blacktriangledown c_{2,3}}{\xi_1^{c_2}}, \blacktriangledown c_3}{\xi_1^c}.
\end{aligned}
$$

(3) Let $f := \partial_1 \Delta'_C$ and $g := \partial_2 \eta_{\blacktriangledown \bar{C}} \partial_{\blacktriangledown \bar{C}}$. By lemma A.2.26, $\partial_1\partial_2 + \partial_2\partial_1$ is the unique differential extending $f + g$, and we show that it is 0. We have

$$
\begin{aligned}
f(\blacktriangledown c) &= (-1)^{c_2} \partial_1 \left(\frac{\blacktriangledown c_2, \blacktriangledown c_3}{\xi_1^c} \right) \\
&= (-1)^{c_2} \frac{\partial_{\blacktriangledown \bar{C}} [\blacktriangledown c_2, \blacktriangledown c_3]}{\xi_1^c} \\
&= (-1)^{c_2} \frac{\partial_{\blacktriangledown \bar{C}}(\blacktriangledown c_2), \blacktriangledown c_3}{\xi_1^c} - \frac{\blacktriangledown c_2, \partial_{\blacktriangledown \bar{C}}(\blacktriangledown c_3)}{\xi_1^c} \\
&= -(-1)^{c_2} \frac{\blacktriangledown \partial_C(c_2), \blacktriangledown c_3}{\xi_1^c} + \frac{\blacktriangledown c_2, \blacktriangledown \partial_C(c_3)}{\xi_1^c} \\
&= \blacktriangledown 1 \otimes \blacktriangledown 1 \otimes \left(\frac{\partial_C(c_2), c_3}{\xi_1^c} + (-1)^{c_2} \frac{c_2, \partial_C(c_3)}{\xi_1^c} \right) \\
&= \blacktriangledown 1 \otimes \blacktriangledown 1 \otimes \partial_{\mathfrak{Z}^n C} \Delta_{(1)}(c) \\
&= \blacktriangledown 1 \otimes \blacktriangledown 1 \otimes \Delta_{(1)} \partial_C(c) \\
&= -\partial_2 \eta_{\blacktriangledown \bar{C}} \partial_{\blacktriangledown \bar{C}}(\blacktriangledown c) \\
&= -g(\blacktriangledown c).
\end{aligned}
$$

□

Definition A.6.17 (Differential graded cobar construction)**.** By lemma A.6.16, the morphism $\partial_1 + \partial_2$ is a differential on the algebra $\Omega(C)$ defined in equation (A.6.15). The *cobar construction* of $C \in {}^{\mathrm{co}}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}^+_R(\mathfrak{Z}^n)$ is the dg $\mathfrak{Z}^n$-algebra

$$
\Omega(A) := \left(\mathfrak{Z}^n(\blacktriangledown \bar{C}), \partial_1 + \partial_2 \right). \tag{A.6.18}
$$

Define now a degree (-1) morphism $\iota : C \longrightarrow \Omega(C)$ as the composite

$$
C \twoheadrightarrow \bar{C} \xrightarrow{-\blacktriangledown} \blacktriangledown \bar{C} \xrightarrow{\eta_{\blacktriangledown \bar{C}}} \mathfrak{Z}^n(\blacktriangledown \bar{C}) = \Omega(C). \tag{A.6.19}
$$

It maps $c \in C_\omega$ to $\frac{\blacktriangledown c}{\mathsf{Y}_\omega}$.

Proposition A.6.20. *The morphism $\iota : C \longrightarrow \Omega(C)$ is a twisting morphism.*

Proof. We show that ι satisfies the Maurer–Cartan equation (A.5.6), i.e. that $0 = \partial(\iota) + \iota \star \iota = \partial_1 \iota + \partial_2 \iota + \iota \partial_C + \iota \star \iota$. For $c \in C_\omega$ we have

$$
\partial_1 \iota(c) = -\partial_1 \left(\frac{\blacktriangledown c}{\mathsf{Y}_\omega} \right) = \frac{\blacktriangledown \partial_C(c)}{\mathsf{Y}_\omega},
$$

$$\begin{aligned}
\partial_2\iota(c) &= -\partial_2\left(\frac{\blacktriangledown c}{\mathsf{Y}_\omega}\right) = -\Delta'_C(\blacktriangledown c), && \spadesuit\\
\iota\partial_C(c) &= -\frac{\blacktriangledown\partial_C(c)}{\mathsf{Y}_\omega},\\
(\iota\star\iota)(c) &= (-1)^{c_2}\frac{\iota(c_1),\iota(c_2)}{\xi_1^c} = (-1)^{c_2}\frac{\blacktriangledown c_1,\blacktriangledown c_2}{\xi_1^c} = \Delta'_C(C), && \diamondsuit
\end{aligned}$$

where in $\spadesuit$ we recall that ∂_2 extends Δ'_C, and in $\diamondsuit$, $\frac{c_1,c_2}{\xi_1^c}$ ranges over the terms of $\Delta_C(c) = \frac{d}{\xi_1^c}$ where $\#(\xi_1^c)^\bullet = 2$, and we write c_2, c_3 for the sequence d, where $c_2, c_3 \in \bar{C}$. We see that $\partial_1\iota + \partial_2\iota + \iota\partial_C + \iota\star\iota = 0$ as desired. Let $\eta : I \longrightarrow C$ be the coaugmentation map of C. By definition, ι factors through $\bar{C} = C/\operatorname{im}\eta$, whence $\iota\eta = 0$. $\square$

Theorem A.6.21. *The morphism $\iota : C \longrightarrow \Omega(C)$ is initial among twisting morphisms under C. Explicitly, for every twisting morphism $t : C \longrightarrow A$, there exists a unique morphism of dg algebra $t^\sharp : \Omega(C) \longrightarrow A$ such that $t = t^\sharp\iota$:*

$$\begin{tikzcd}
C \arrow[r, "\iota"] \arrow[dr, "\forall t"'] & \Omega(C) \arrow[d, dashed, "\exists! t^\sharp"] \\
 & A.
\end{tikzcd}$$

Consequently, we have an isomorphism ${}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}^+_R(\mathfrak{Z}^n)(\Omega(C), A) \cong \mathrm{Tw}(C, A)$ natural in both C and A.

Proof. By definition, t factors through the coaugmentation quotient $\bar{C}$ of C, and let u be the composite

$$\blacktriangledown\bar{C} \xrightarrow{\blacktriangle} \bar{C} \xrightarrow{-t} A.$$

Since $\Omega(C) = \mathfrak{Z}^n(\blacktriangledown\bar{C})$ is free over $\blacktriangledown\bar{C}$, there exust a unique morphism of algebras $t^\sharp : \Omega(C) \longrightarrow A$ such that $u = t^\sharp\eta_{\blacktriangledown\bar{C}}$. Precomposing with $-\blacktriangledown : C \longrightarrow \blacktriangledown\bar{C}$ gives $t = t^\sharp\iota$.

Conversely, if $g : \Omega(C) \longrightarrow A$ is an algebra morphism such that $t = g\iota$, then precomposing with $-\blacktriangle : \blacktriangledown\bar{C} \longrightarrow C$ gives $u = g\eta_{\blacktriangledown\bar{C}}$ whence $g = t^\sharp$.

We now show that $t^\sharp$ is a morphism of dg algebras. Recall that as the unique factorization of u through the free algebra $\Omega(C)$, $t^\sharp$ is the composite

$$\Omega(C) \xrightarrow{\mathfrak{Z}^n u} \mathfrak{Z}^n A \xrightarrow{\gamma} A,$$

where γ is the multiplication of A. Explicitly,

$$t^\sharp\left(\frac{\blacktriangledown c}{\xi}\right) := \frac{-t(c)}{\xi}.$$

We have

$$\begin{aligned}
\partial_A t^\sharp\left(\frac{\blacktriangledown c}{\xi}\right) &= \partial_A\left(\frac{-t(c)}{\xi}\right) = \frac{-(t;\partial_A t)(c)}{\xi},\\
t^\sharp\partial_1\left(\frac{\blacktriangledown c}{\xi}\right) &= t^\sharp\left(\frac{\partial_{\blacktriangledown\bar{C}}[\blacktriangledown c]}{\xi}\right) = -\frac{-(t;t\partial_C)(c)}{\xi}
\end{aligned}$$

$$t^\sharp \partial_2 \left(\frac{\blacktriangledown c}{\xi} \right) = t^\sharp \left(\frac{(\eta_{\blacktriangledown \bar{C}}; \Delta'_C)(\blacktriangledown c)}{\xi} \right) = \frac{(-t; (-t) \star (-t))(\blacktriangledown c)}{\xi}$$
$$= -\frac{-(t; t \star t)(\blacktriangledown c)}{\xi}$$

whence

$$\left(\partial_A t^\sharp - t^\sharp \partial_{\Omega(C)}\right) \left(\frac{\blacktriangledown c}{\xi} \right) = \frac{-(t; \partial_A t + t \partial_C + t \star t)(c)}{\xi} = 0$$

since t is a twisting morphism. □

A.7 FLATTENING

Definition A.7.1 (Linear flattening)**.** Recall from definition 11.3.5 the *flattening operator* $(-)^\vee : \mathbb{O}_{n,n+1} \longrightarrow \mathbb{O}_{2,3}$. It maps $\omega \in \mathbb{O}_n$ to the opetopic integer $\mathbf{k}$, where $k = \#\omega^\bullet$, and $\xi \in \mathbb{O}_{n+1}$ to the underlying tree of ξ, seen as a 3-opetope. This functor is faithful but not full in general.

In particular, we have a set map $(-)^\vee : \mathbb{O}_n \longrightarrow \mathbb{O}_2$, which induces a left adjoint $(-)^\vee : {}_{\mathrm{dg}}\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n} \longrightarrow {}_{\mathrm{dg}}\mathcal{M}\mathrm{od}_R^{\mathbb{O}_2} = {}_{\mathrm{dg}}\mathcal{M}\mathrm{od}_R^{\mathbb{N}}$ using definition A.2.1 and proposition A.2.2. Explicitly, for $X \in {}_{\mathrm{dg}}\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$, we have

$$X^\vee_{\mathbf{k}} = \bigoplus_{\#\omega^\bullet = k} X_\omega .$$

Proposition A.7.2. *The flattening operator* $(-)^\vee : {}_{\mathrm{dg}}\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n} \longrightarrow {}_{\mathrm{dg}}\mathcal{M}\mathrm{od}_R^{\mathbb{O}_2}$ *is additive and exact.*

Proof. Additivity is trivial. By proposition A.2.2, $(-)^\vee$ is left adjoint, thus it preserves all colimits, and in particular, direct sums and cokernels. To conclude, we show that it preserves kernels. For $f : X \longrightarrow Y$, note that f and $f^\vee$ have the same underlying set map. Therefore,

$$\begin{aligned}
\ker f^\vee_{\mathbf{k}} &= \bigoplus_{\#\omega^\bullet = k} (\ker f)_\omega \\
&= \bigoplus_{\#\omega^\bullet = k} \{x \in X_\omega \mid f(x) = 0\} \\
&= \bigoplus_{\#\omega^\bullet = k} \{x \in X_\omega \mid f^\vee(x) = 0\} \\
&= \{x \in X^\vee_{\mathbf{k}} \mid f^\vee(x) = 0\} \\
&= (\ker f^\vee)_{\mathbf{k}} .
\end{aligned}$$

□

Corollary A.7.3. *For* $X \in {}_{\mathrm{dg}}\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$*, we have* $H(X)^\vee \cong H(X^\vee)$ *naturally in* X*. Further,* $f : X \longrightarrow Y$ *is a quasi-isomorphism if only if* $f^\vee$ *is.*

Proof. The first claim follows directly from proposition A.7.2, while the second claim follows from the first, and the fact that $(-)^\vee$ preserves and reflects isomorphisms. □

Lemma A.7.4. *Let $X \in {}_{\mathrm{dg}}\mathcal{M}\mathrm{od}_R^{\mathbb{O}_n}$, and consider $\mathfrak{Z}^n X$. The following canonical map is injective:*

$$i : (\mathfrak{Z}^n X)^\vee \longrightarrow \mathfrak{Z}^2 X^\vee$$
$$\frac{x}{\xi} \longmapsto \frac{x}{\xi^\vee}.$$

Proof. Assume $i\left(\frac{x}{\xi}\right) = i\left(\frac{x}{\xi'}\right)$. First, since $\xi^\vee = \xi'^\vee$, ξ and ξ' have the same underlying tree. Write $\xi^\bullet = \{[p_1] < \cdots < [p_k]\}$ and $(\xi')^\bullet = \{[q_1] < \cdots < [q_k]\}$. In particular, x is a sequence $x_1, \ldots, x_k$ of elements of X. For $1 \leq j \leq k$, let $\omega_j \in \mathbb{O}_n$ be the opetope such that $x_j \in X_{\omega_j}$. Then by definition, $\mathsf{s}_{[p_j]} \xi = \omega_j = \mathsf{s}_{[q_j]} \xi'$. Therefore, in addition to having the same underlying tree, ξ and ξ' have the same source faces. Consequently, $\xi = \xi'$. □

Proposition A.7.5. *Let $A \in {}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R(\mathfrak{Z}^n)$ be an algebra with structure map $m : \mathfrak{Z}^n A \longrightarrow A$. Define a map $m' : \mathfrak{Z}^2 A^\vee \longrightarrow A^\vee$ as follows:*

$$m'\left(\frac{x}{\psi}\right) := \begin{cases} \frac{x}{\xi} & \textit{if there exists } \xi \in \mathbb{O}_{n+1} \textit{ st. } \frac{x}{\psi} = i\left(\frac{x}{\xi}\right), \\ 0 & \textit{otherwise.} \end{cases}$$

The tuple $(A^\vee, m')$ is a dg $\mathfrak{Z}^2$-algebra, a.k.a. a dg planar operad.

Proof. We check the usual unitality and associativity axioms. For $a \in A_\omega$ and $k = \#\omega^\bullet$,

$$m'\eta(a) = m'\left(\frac{a}{\mathsf{Y}_{\mathbf{k}}}\right) = \frac{a}{\overline{\overline{\omega}}} = a.$$

Next, we show that the following square commutes:

$$\begin{tikzcd} \mathfrak{Z}^2\mathfrak{Z}^2 A^\vee \arrow[r, "\mathfrak{Z}^2 m'"] \arrow[d, "\mu"'] & \mathfrak{Z}^2 A^\vee \arrow[d, "m'"] \\ \mathfrak{Z}^2 A^\vee \arrow[r, "m'"'] & A^\vee. \end{tikzcd}$$

Let

$$x = \cfrac{a}{\cfrac{\psi'}{\psi}} \in \mathfrak{Z}^2\mathfrak{Z}^2 A^\vee.$$

Consider the following assumptions:

(A1) For all i, there exists $\frac{a_i}{\xi'_i} \in \mathfrak{Z}^n A$ such that $(\xi'_i)^\vee = \psi'_i$.

(A2) Writing $b_i := \frac{a_i}{\xi'_i}$, assume further that there exists $\frac{b}{\xi} \in \mathfrak{Z}^n A$ such that $\xi^\vee = \psi$.

We now distinguish cases.

(1) If assumption **(A1)** fails, say for index i, then by definition $m'\left(\frac{x_i}{\psi'_i}\right) = 0$, thus $m'(\mathfrak{Z}^2 m')(x) = 0$. On the other hand, there cannot exist $\frac{x}{\zeta} \in A^\vee$ such that $i\left(\frac{a}{\zeta}\right) = \frac{a}{\psi \square_i \psi'_i} = \mu(x)$, as it would allow us to construct an element $\frac{a_i}{\xi'_i} \in \mathfrak{Z}^n A$ such that $i\left(\frac{a_i}{\xi'_i}\right) = \frac{x_i}{\psi'_i}$ by taking ξ'_i to be the subtree of ζ corresponding to ψ'_i. Therefore, $m'\mu(x) = 0$ as well.

(2) If assumption **(A1)** holds but **(A2)** fails, then a similar argument shows that $m'(\mathfrak{Z}^2 m')(x) = 0 = m'\mu(x)$.
(3) Assume that **(A1)** and **(A2)** hold. Note that $(\xi \,\square_i\, \xi_i')^\vee = \psi \,\square_i\, \psi_i'$, and thus

$$m'(\mathfrak{Z}^2 m')\left(\frac{\dfrac{a}{\psi'}}{\psi}\right) = m'\left(\frac{\dfrac{a}{\xi'}}{\psi}\right) = \dfrac{\dfrac{a}{\xi'}}{\xi} = \frac{a}{\xi \,\square_i\, \xi_i'} = m'\left(\frac{a}{\psi \,\square\, \psi_i'}\right) = m'\mu\left(\frac{\dfrac{a}{\psi'}}{\psi}\right).$$

$\square$

Definition A.7.6 (Linear flattening, cont.)**.** In the light of proposition A.7.5, the flattening operator induces a functor $(-)^\vee : {}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R(\mathfrak{Z}^n) \longrightarrow {}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R(\mathfrak{Z}^2) = {}_{\mathrm{dg}}\mathcal{O}\mathrm{p}_R$. Likewise, if $C \in {}^{\mathrm{co}}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R(\mathfrak{Z}^n)$ is a dg coalgebra with structure map $\Delta : C \longrightarrow \mathfrak{Z}^n C$, then letting Δ' be the composite

$$C^\vee \xrightarrow{\Delta^\vee} (\mathfrak{Z}^n C)^\vee \xrightarrow{i} \mathfrak{Z}^2 C^\vee$$

gives rise to a dg cooperad $(C^\vee, \Delta')$. Thus we have a functor $(-)^\vee : {}^{\mathrm{co}}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R(\mathfrak{Z}^n) \longrightarrow {}^{\mathrm{co}}_{\mathrm{dg}}\mathcal{O}\mathrm{p}_R$.

Proposition A.7.7. *Let $A \in {}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R(\mathfrak{Z}^n)$, $C \in {}^{\mathrm{co}}_{\mathrm{dg}}\mathcal{A}\mathrm{lg}_R(\mathfrak{Z}^n)$, and $t : C \longrightarrow A$. Then t is a twisting morphism if and only if $t^\vee$ is.*

Proof. Note that $\partial(t)^\vee = \left(\partial_A t - (-1)^t t \partial_C\right)^\vee = \partial_{A^\vee} t^\vee - (-1)^{t^\vee} t^\vee \partial_{C^\vee}$. Likewise, $(t \star t)^\vee = t^\vee \star t^\vee$. Therefore, t satisfies the Maurer–Cartan equation (A.5.6) if and only if $t^\vee$ does. $\square$

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Bibliography

[AAG05] Michael Abbott, Thorsten Altenkirch, and Neil Ghani. Containers: constructing strictly positive types. *Theoret. Comput. Sci.*, 342(1):3–27, 2005.

[ACM19] Dimitri Ara, Denis-Charles Cisinski, and Ieke Moerdijk. The dendroidal category is a test category. *Mathematical Proceedings of the Cambridge Philosophical Society*, 167(1):107–121, 2019.

[Adá77] Jiří Adámek. Colimits of algebras revisited. *Bulletin of the Australian Mathematical Society*, 17(3):433–450, 1977.

[AGS12] Steve Awodey, Nicola Gambino, and Kristina Sojakova. Inductive types in homotopy type theory. In *Proceedings of the 2012 27th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 95–104. IEEE Computer Soc., Los Alamitos, CA, 2012.

[AK80] Jiří Adámek and Václav Koubek. Are colimits of algebras simple to construct? *Journal of Algebra*, 66(1):226–250, 1980.

[AMSW19] Jiří Adámek, Stefan Milius, Lurdes Sousa, and Thorsten Wissmann. On finitary functors. *Theory and Applications of Categories*, 34:Paper No. 35, 1134–1164, 2019.

[AR94] Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994.

[Ara12] Dimitri Ara. The groupoidal analogue $\widetilde{\Theta}$ to Joyal's category Θ is a test category. *Applied Categorical Structures*, 20(6):603–649, 2012.

[Awo10] Steve Awodey. *Category theory*, volume 52 of *Oxford Logic Guides*. Oxford University Press, Oxford, second edition, 2010.

[B67] Jean Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, pages 1–77. Springer, Berlin, 1967.

[BD86] Francis Borceux and Dominique Dejean. Cauchy completion in category theory. *Cahiers Topologie Géom. Différentielle Catég.*, 27(2):133–146, 1986.

[BD98] John C. Baez and James Dolan. Higher-dimensional algebra. III. n-categories and the algebra of opetopes. *Advances in Mathematics*, 135(2):145–206, 1998.

[Ber18] Julia Elizabeth Bergner. *The homotopy theory of $(\infty, 1)$-categories*, volume 90 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2018.

[BKV16] Krzysztof Bar, Aleks Kissinger, and Jamie Vicary. Globular: an online proof assistant for higher-dimensional rewriting. *Logical Methods in Computer Science*, 14:1–16, December 2016.

[BMW12] Clemens Berger, Paul-André Melliès, and Mark Weber. Monads with arities and their associated theories. *Journal of Pure and Applied Algebra*, 216(8-9):2029–2048, 2012.

[Bor94a] Francis Borceux. *Handbook of categorical algebra. 1*, volume 50 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Basic category theory.

[Bor94b] Francis Borceux. *Handbook of categorical algebra. 2*, volume 51 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Categories and structures.

[Bur71] Albert Burroni. T-catégories (catégories dans un triple). *Cahiers Topologie Géom. Différentielle*, 12:215–321, 1971.

[BV73] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*. Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin-New York, 1973.

[BW05] Michael Barr and Charles Wells. Toposes, triples and theories. *Reprints in Theory and Applications of Categories*, (12):x+288, 2005. Corrected reprint of the 1985 original [MR0771116].

[CG07] Eugenia Cheng and Michael Nicholas Gurski. The periodic table of n-categories for low dimensions. I. Degenerate categories and degenerate bicategories. In *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, pages 143–164. American Mathematical Society, Providence, RI, 2007.

[CG11] Eugenia Cheng and Michael Nicholas Gurski. The periodic table of n-categories II: Degenerate tricategories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 52(2):82–125, 2011.

[Che03a] Eugenia Cheng. The category of opetopes and the category of opetopic sets. *Theory and Applications of Categories*, 11:No. 16, 353–374, 2003.

[Che03b] Eugenia Cheng. The theory of opetopes via kelly-mac lane graphs. *arXiv e-prints*, April 2003.

[Che04a] Eugenia Cheng. Weak n-categories: comparing opetopic foundations. *Journal of Pure and Applied Algebra*, 186(3):219–231, 2004.

[Che04b] Eugenia Cheng. Weak n-categories: opetopic and multitopic foundations. *Journal of Pure and Applied Algebra*, 186(2):109–137, 2004.

[Che06] Eugenia Cheng. A relationship between trees and Kelly-Mac Lane graphs. *Mathematical Proceedings of the Cambridge Philosophical Society*, 141(1):33–56, 2006.

[Che13] Eugenia Cheng. A direct proof that the category of 3-computads is not Cartesian closed. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 54(1):3–12, 2013.

[CHM19a] Pierre-Louis Curien, Cédric Ho Thanh, and Samuel Mimram. A sequent calculus for opetopes. In *LICS '19: Proceedings of the 34th Annual ACM/IEEE Symposium on Logic in Computer Science*, 2019.

[CHM19b] Pierre-Louis Curien, Cédric Ho Thanh, and Samuel Mimram. Syntactic approaches to opetopes. *arXiv e-prints*, Mar 2019.

[CHR12] Antonio Martínez Cegarra, Benjamín A. Heredia, and Josué Remedios. Double groupoids and homotopy 2-types. *Applied Categorical Structures*, 20(4):323–378, 2012.

[Cis06] Denis-Charles Cisinski. *Les préfaisceaux comme modèles des types d'homotopie*. Number 308 in Astérisque. Société mathématique de France, October 2006.

[Cis19] Denis-Charles Cisinski. *Higher categories and homotopical algebra*, volume 180 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2019.

[CJ95] Aurelio Carboni and Peter Johnstone. Connected limits, familial representability and Artin glueing. *Mathematical Structures in Computer Science*, 5(4):441–459, 1995. Fifth Biennial Meeting on Category Theory and Computer Science (Amsterdam, 1993).

[CJ04] Aurelio Carboni and Peter Johnstone. Corrigenda for: Connected limits, familial representability and Artin glueing. *Mathematical Structures in Computer Science*, 14(1):185–187, 2004.

[CL04] Eugenia Cheng and Aaron Lauda. Higher-dimensional categories: an illustrated guide book. Available at `http://cheng.staff.shef.ac.uk/guidebook/index.html`, 2004.

[CM11a] Denis-Charles Cisinski and Georges Maltsiniotis. La catégorie Θ de Joyal est une catégorie test. *Journal of Pure and Applied Algebra*, 215(5):962–982, 2011.

[CM11b] Denis-Charles Cisinski and Ieke Moerdijk. Dendroidal sets as models for homotopy operads. *Journal of Topology*, 4(2):257–299, 2011.

[CM13] Denis-Charles Cisinski and Ieke Moerdijk. Dendroidal Segal spaces and ∞-operads. *Journal of Topology*, 6(3):675–704, 2013.

[Die17] Reinhard Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Berlin, fifth edition, 2017.

[Dug01] Daniel Dugger. Replacing model categories with simplicial ones. *Transactions of the American Mathematical Society*, 353(12):5003–5027, 2001.

[Fin16] Eric Finster. Opetopic.net. `http://opetopic.net`, May 2016.

[FM17] Eric Finster and Samuel Mimram. A Type-Theoretical Definition of Weak ω-Categories. In *32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–12, 2017.

[Fre09] Benoit Fresse. *Modules over operads and functors*, volume 1967 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.

[GH04] Nicola Gambino and Martin Hyland. Wellfounded trees and dependent polynomial functors. In *Types for proofs and programs*, volume 3085 of *Lecture Notes in Comput. Sci.*, pages 210–225. Springer, Berlin, 2004.

[GHK17] David Gepner, Rune Haugseng, and Joachim Kock. ∞-Operads as Analytic Monads. *arXiv e-prints*, page arXiv:1712.06469, December 2017.

[GJ09] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition [MR1711612].

[GK13] Nicola Gambino and Joachim Kock. Polynomial functors and polynomial monads. *Mathematical Proceedings of the Cambridge Philosophical Society*, 154(1):153–192, 2013.

[GM09] Yves Guiraud and Philippe Malbos. Higher-dimensional categories with finite derivation type. *Theory and Applications of Categories*, 22:No. 18, 420–478, 2009.

[GPS95] Robert Gordon, John Power, and Ross Street. Coherence for tricategories. *Memoirs of the American Mathematical Society*, 117(558):vi+81, 1995.

[Gro83] Alexander Grothendieck. Pursuing stacks. Available at `https://thescrivener.github.io/PursuingStacks/`, 1983.

[GU71] Peter Gabriel and Friedrich Ulmer. *Lokal präsentierbare Kategorien*. Lecture Notes in Mathematics, Vol. 221. Springer-Verlag, Berlin-New York, 1971.

[Gur06] Michael Nicholas Gurski. *An algebraic theory of tricategories*. ProQuest LLC, Ann Arbor, MI, 2006. Thesis (Ph.D.)–The University of Chicago.

[GZ67] Peter Gabriel and Michel Zisman. *Calculus of fractions and homotopy theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.

[Hen19] Simon Henry. Non-unital polygraphs form a presheaf category. *Higher Structures*, 3(1):248–291, 2019.

[Her01] Claudio Hermida. From coherent structures to universal properties. *Journal of Pure and Applied Algebra*, 165(1):7–61, 2001.

[Her19] Ada Hermelink. Cocompleteness of categories of algebras – an overview. Available at `https://math.uchicago.edu/~may/REU2019/REUPapers/Hermelink.pdf`, November 2019.

[Hir09] Philip S Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Soc., 2009.

[HKK01] K. A. Hardie, K. H. Kamps, and R. W. Kieboom. A homotopy bigroupoid of a topological space. *Applied Categorical Structures*, 9(3):311–327, 2001.

[HL19] Cédric Ho Thanh and Chaitanya Leena Subramaniam. Opetopic algebras I: Algebraic structures on opetopic sets. *arXiv e-prints*, Nov 2019.

[HL20a] Cédric Ho Thanh and Chaitanya Leena Subramaniam. Opetopic algebras ii: Homotopy-coherent opetopic algebras. In preparation, 2020.

[HL20b] Cédric Ho Thanh and Chaitanya Leena Subramaniam. Opetopic algebras III: Presheaf models of homotopy-coherent opetopic algebras. *arXiv e-prints*, Jan 2020.

[HMP00] Claudio Hermida, Michael Makkai, and John Power. On weak higher dimensional categories. I. 1. *Journal of Pure and Applied Algebra*, 154(1-3):221–246, 2000. Category theory and its applications (Montreal, QC, 1997).

[HMP02] Claudio Hermida, Michael Makkai, and John Power. On weak higher-dimensional categories. I. 3. *Journal of Pure and Applied Algebra*, 166(1-2):83–104, 2002.

[HMZ02] Victor Harnik, Michael Makkai, and Marek Zawadowski. Multitopic sets are the same as many-to-one computads. Available at `http://www.math.mcgill.ca/makkai/m_1_comp/Manytoonecomputads.pdf`, 2002.

[HMZ08] Victor Harnik, Michael Makkai, and Marek Zawadowski. Computads and multitopic sets. *arXiv e-prints*, 2008.

[Ho 18a] Cédric Ho Thanh. opetopy. `https://github.com/altaris/opetopy`, April 2018.

[Ho 18b] Cédric Ho Thanh. The equivalence between many-to-one polygraphs and opetopic sets. *arXiv e-prints*, Jun 2018.

[Hof97] Martin Hofmann. Syntax and semantics of dependent types. In *Semantics and logics of computation (Cambridge, 1995)*, volume 14 of *Publ. Newton Inst.*, pages 79–130. Cambridge University Press, Cambridge, 1997.

[Hor15] Geoffroy Horel. A model structure on internal categories in simplicial sets. *Theory and Applications of Categories*, 30:Paper No. 20, 704–750, 2015.

[Hov99] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1999.

[HP07] Martin Hyland and John Power. The category theoretic understanding of universal algebra: Lawvere theories and monads. In *Computation, meaning, and logic: articles dedicated to Gordon Plotkin*, volume 172 of *Electron. Notes Theor. Comput. Sci.*, pages 437–458. Elsevier Sci. B. V., Amsterdam, 2007.

[Jar06] John F. Jardine. Categorical homotopy theory. *Homology, Homotopy and Applications*, 8(1):71–144, 2006.

[Joy08] André Joyal. Notes on quasi-categories. Available at `https://www.math.uchicago.edu/~may/IMA/Joyal.pdf`, June 2008.

[JT07] André Joyal and Myles Tierney. Quasi-categories vs Segal spaces. In *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, pages 277–326. American Mathematical Society, Providence, RI, 2007.

[KJBM10] Joachim Kock, André Joyal, Michael Batanin, and Jean-François Mascari. Polynomial functors and opetopes. *Advances in Mathematics*, 224(6):2690–2737, 2010.

[Koc11] Joachim Kock. Polynomial functors and trees. *International Mathematics Research Notices*, 2011(3):609–673, January 2011.

[Koc12] Joachim Kock. Data types with symmetries and polynomial functors over groupoids. *arXiv e-prints*, page arXiv:1210.0828, October 2012.

[Lac11] Stephen Lack. An embedding theorem for adhesive categories. *Theory and Applications of Categories*, 25:No. 7, 180–188, 2011.

[Lan18] Victor Lanvin. Irif cake website. `https://cake.irif.beer`, 2018.

[Law04] F. William Lawvere. Functorial semantics of algebraic theories and some algebraic problems in the context of functorial semantics of algebraic theoriesc. *Reprints in Theory and Applications of Categories*, (5):1–121, 2004. Reprinted from Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 869–872 [MR0158921] and ıt Reports of the Midwest Category Seminar. II, 41–61, Springer, Berlin, 1968 [MR0231882].

[Lei04] Tom Leinster. *Higher Operads, Higher Categories.* Cambridge University Press, 2004.

[Lem10] Alexis Lemaire. *Application de l'hypercalculie et de l'informatique quantique gravifique à l'intelligence artificielle générale.* Phd thesis, Université de Reims, 2010.

[Lin69] F. E. J. Linton. Coequalizers in categories of algebras. In *Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67)*, pages 75–90. Springer, Berlin, 1969.

[Lod12] Jean-Louis Loday. Algebras, operads, combinads. Available at `https://irma-web1.math.unistra.fr/~loday/`, March 2012.

[Lor19] Fosco Loregian. Coend calculus. *arXiv e-prints*, December 2019.

[LP09] Stephen Lack and John Power. Gabriel-Ulmer duality and Lawvere theories enriched over a general base. *Journal of Functional Programming*, 19(3-4):265–286, 2009.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*, volume 346. Springer Science & Business Media, 2012.

[M03] François Métayer. Resolutions by polygraphs. *Theory and Applications of Categories*, 11:No. 7, 148–184, 2003.

[Mal09] Georges Maltsiniotis. La catégorie cubique avec connexions est une catégorie test stricte. *Homology, Homotopy and Applications*, 11(2):309–326, 2009.

[May72] J. Peter May. *The geometry of iterated loop spaces*. Springer-Verlag, Berlin-New York, 1972. Lectures Notes in Mathematics, Vol. 271.

[May99] J. Peter May. *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999.

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[MP00] Ieke Moerdijk and Erik Palmgren. Wellfounded trees in categories. In *Proceedings of the Workshop on Proof Theory and Complexity, PTAC'98 (Aarhus)*, volume 104, pages 189–218, 2000.

[MW07] Ieke Moerdijk and Ittay Weiss. Dendroidal sets. *Algebraic & Geometric Topology*, 7:1441–1470, 2007.

[MW09] Ieke Moerdijk and Ittay Weiss. On inner Kan complexes in the category of dendroidal sets. *Advances in Mathematics*, 221(2):343–389, 2009.

[MZ08] Michael Makkai and Marek Zawadowski. The category of 3-computads is not Cartesian closed. *Journal of Pure and Applied Algebra*, 212(11):2543–2546, 2008.

[Pal04] Thorsten Palm. Dendrotopic sets. In *Galois theory, Hopf algebras, and semiabelian categories*, volume 43 of *Fields Inst. Commun.*, pages 411–461. American Mathematical Society, Providence, RI, 2004.

[Pow89] John Power. A general coherence result. *Journal of Pure and Applied Algebra*, 57(2):165–173, 1989.

[Pro11] Alain Prouté. A_∞-structures. Modèles minimaux de Baues-Lemaire et Kadeishvili et homologie des fibrations. *Reprints in Theory and Applications of Categories*, (21):1–99, 2011. Reprint of the 1986 original, With a preface to the reprint by Jean-Louis Loday.

[Qui67] Daniel G. Quillen. *Homotopical algebra.* Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967.

[Rez01] Charles Rezk. A model for the homotopy theory of homotopy theory. *Transactions of the American Mathematical Society*, 353(3):973–1007, 2001.

[Rie17] Emily Riehl. *Category theory in context.* Courier Dover Publications, 2017.

[Rob16] David Michael Roberts. A bigroupoid's topology (or, Topologising the homotopy bigroupoid of a space). *Journal of Homotopy and Related Structures*, 11(4):923–942, 2016.

[Rot09] Joseph J. Rotman. *An introduction to homological algebra.* Universitext. Springer, New York, second edition, 2009.

[RSS01] Charles Rezk, Stefan Schwede, and Brooke Shipley. Simplicial structures on model categories and functors. *American Journal of Mathematics*, 123(3):551–575, 2001.

[RV14] Emily Riehl and Dominic Verity. The theory and practice of Reedy categories. *Theory and Applications of Categories*, 29:256–301, 2014.

[Seg68] Graeme Segal. Classifying spaces and spectral sequences. *Institut des Hautes Études Scientifiques. Publications Mathématiques*, (34):105–112, 1968.

[SGA72] SGA4-1. *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos.* Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[Ste12] Richard Steiner. Opetopes and chain complexes. *Theory and Applications of Categories*, 26:No. 19, 501–519, 2012.

[Str76] Ross Street. Limits indexed by category-valued 2-functors. *Journal of Pure and Applied Algebra*, 8(2):149–181, 1976.

[Str00] Ross Street. The petit topos of globular sets. *Journal of Pure and Applied Algebra*, 154(1-3):299–315, 2000. Category theory and its applications (Montreal, QC, 1997).

[Tri06] Todd Trimble. Notes on tetracategories. `http://math.ucr.edu/home/baez/trimble/tetracategories.html`, October 2006.

[Uni13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics.* `https://homotopytypetheory.org/book`, 2013.

[VdB13] Benno Van den Berg. A note on semisimplicial sets. Available at `https://ncatlab.org/ufias2012/files/semisimplicialsets.pdf`, April 2013.

[Web07] Mark Weber. Familial 2-functors and parametric right adjoints. *Theory and Applications of Categories*, 18:No. 22, 665–732, 2007.

[Web14] Mark Weber. Operads as polynomial 2-monads. *arXiv e-prints*, page arXiv:1412.7599, December 2014.

[Wei07] Ittay Weiss. *Dendroidal Sets*. PhD thesis, Utrecht University, September 2007.

[Wei11] Ittay Weiss. From operads to dendroidal sets. In *Mathematical foundations of quantum field theory and perturbative string theory*, volume 83 of *Proc. Sympos. Pure Math.*, pages 31–70. American Mathematical Society, Providence, RI, 2011.

[Wu16] Gang Wu. Koszul algebras and koszul duality. Master's thesis, University of Ottawa, August 2016.