Simultaneous Localization and Mapping in Unstructured Environments
A Set-Membership Approach

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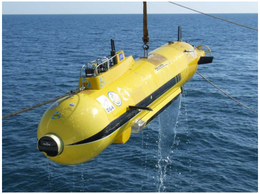
Outline

Context

Localization

- critical function
- needed to assert mission requirements
- huge impact on the mission plan
- cannot rely on GNSS systems

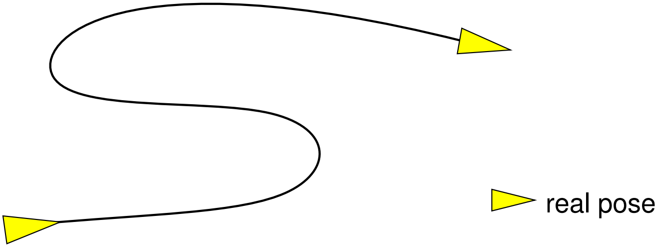


Daurade

Autonomous Underwater Navigation

- no external positioning system
- dead-reckoning like navigation
- weak prior knowledge environment
- long term mission > 24h

Localization



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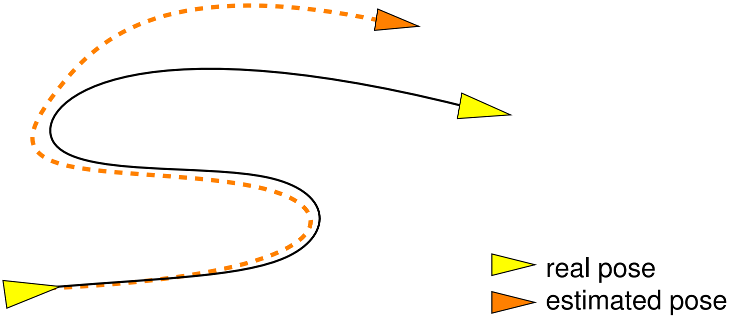

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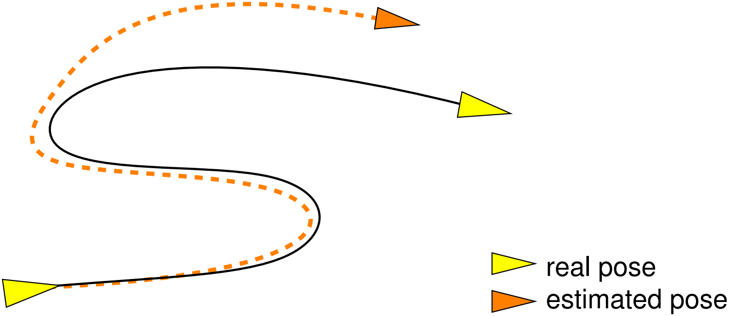


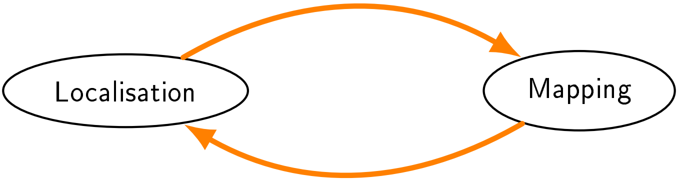
Challenges

- get the best estimation
- get reliable error bounds

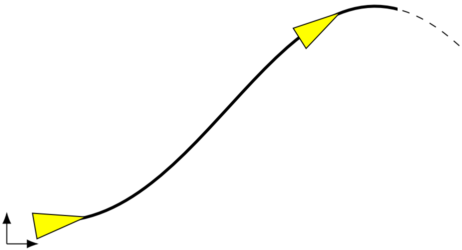
Simultaneous Localization And Mapping

Definition

The simultaneous localization and mapping (SLAM) problem, for an autonomous vehicle moving in an unknown environment, is to build a map of this environment while simultaneously using this map to compute its location.



Simultaneous Localization And Mapping



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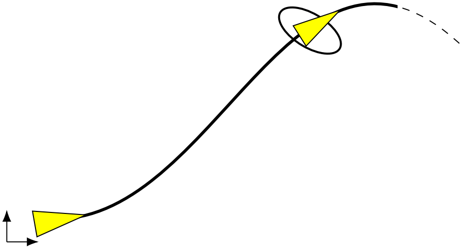

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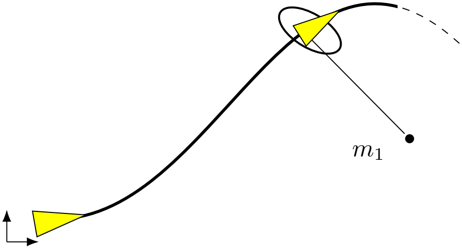

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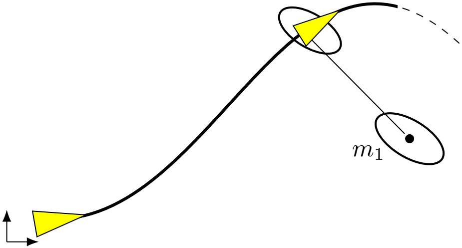

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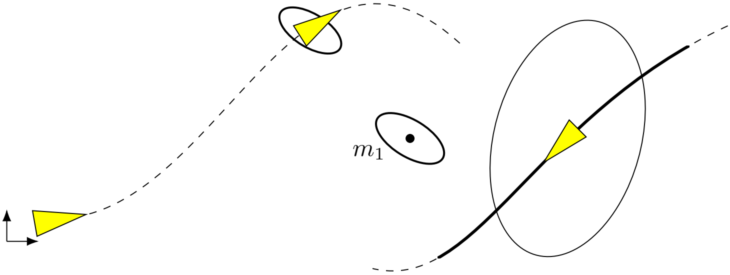

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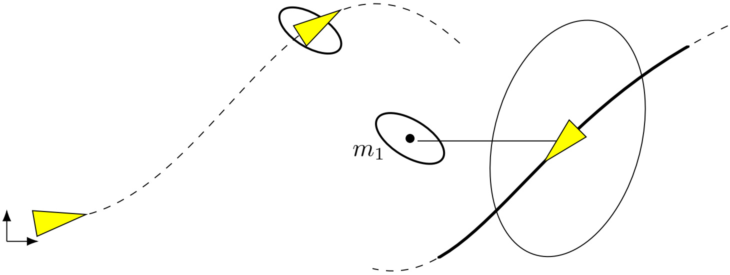

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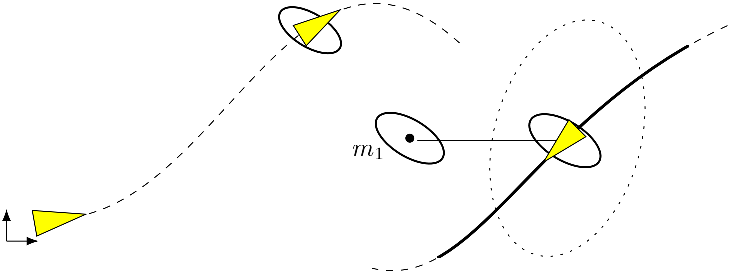
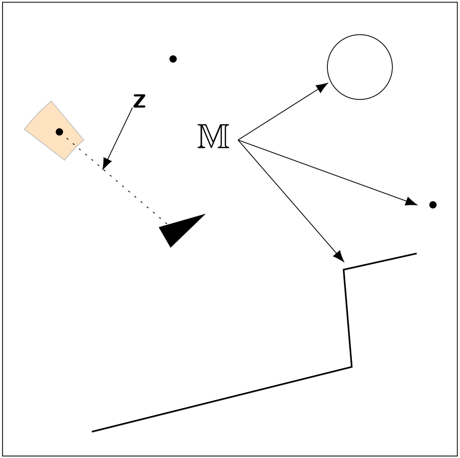


Feature-based localization

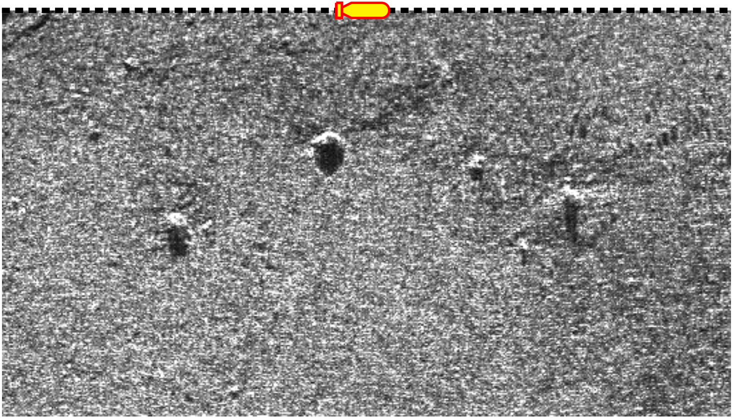
Observation equation

$$\begin{cases} \mathbf{z} = \mathbf{g}(\mathbf{x}, \mathbf{p}) \\ \mathbf{p} \in \mathbb{R}^n \end{cases}$$

- well studied in the literature
- data association problem
- features are hard to detect in underwater environment
- indistinguishable objects

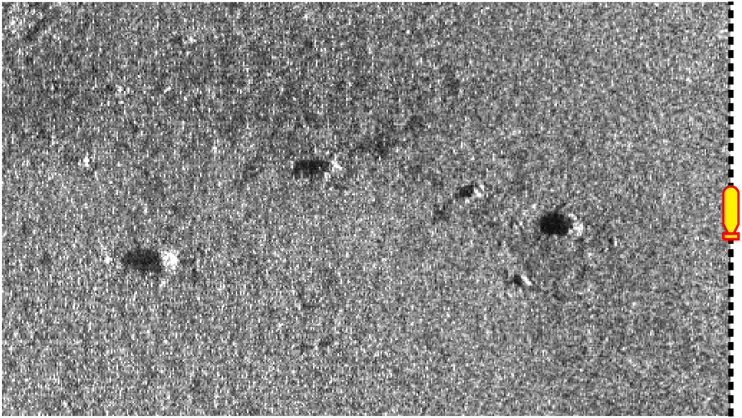


Side-scan sonar images



Images taken with *Daurade* (Klein 5000)

Side-scan sonar images



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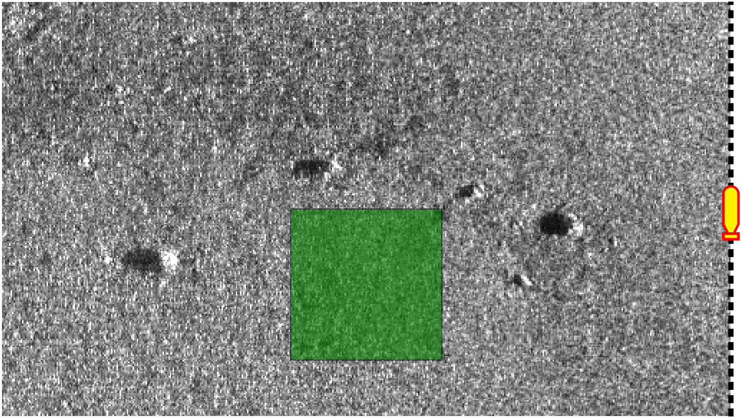

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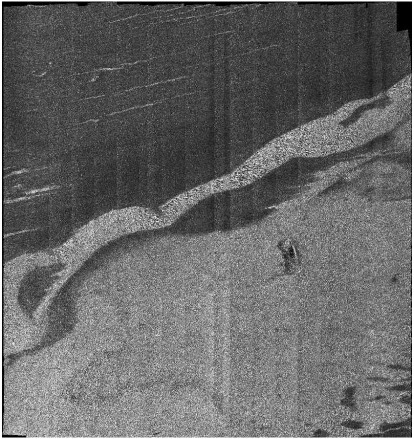
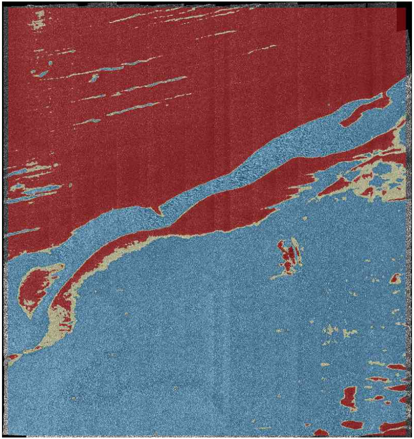
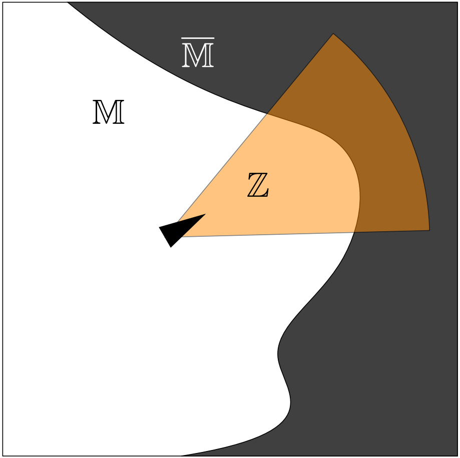
Side-scan sonar images
Douarnenez 2008 (700m x 700m)





Shape-based localization



Observation equation

$$\mathbb{Z} = \mathbf{g}\left(\mathbf{x}, \mathbb{M}\right)$$

- each point of the measurement can be labeled
- occupancy grids in a probabilistic context
- no data association problem
- use reliable information

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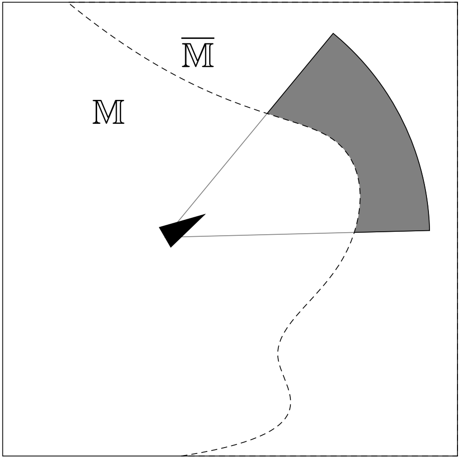

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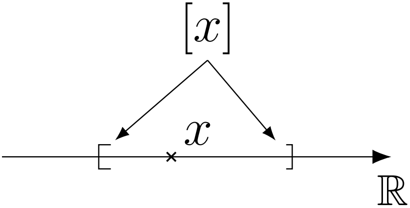
Outline

Notation

- x an element of $\mathbb{R}$ *e.g.* 1, 10.2,
- $[x]$ an interval of $\mathbb{R}$ *e.g.* $[1, 4]$



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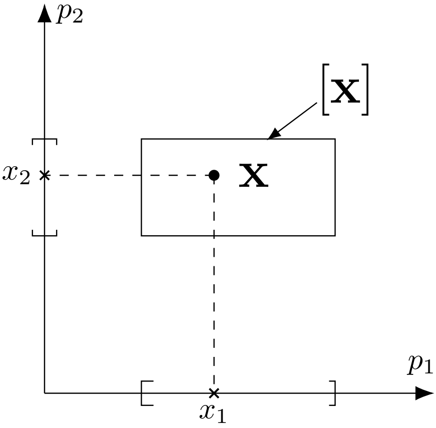
- $\mathbf{x}$ a vector of $\mathbb{R}^n$ *e.g.* $(1,2,3)^\intercal$
- $[\mathbf{x}]$ a box of $\mathbb{R}^n$ *e.g.* $[1,4]\times[2,5]$



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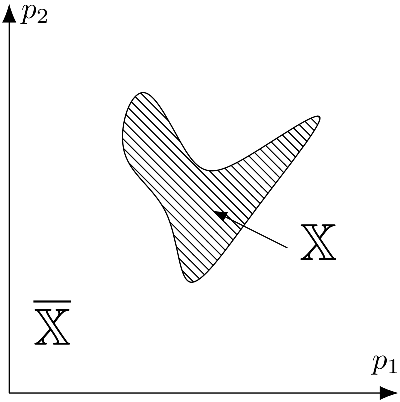
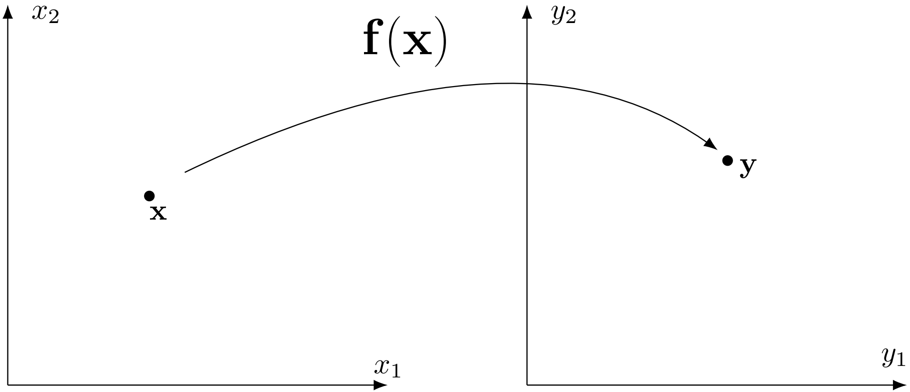
- $[\mathbf{x}]$ a box of $\mathbb{R}^n$ *e.g.* $[1, 4] \times [2, 5]$
- $\mathbb{X}$ a subset of $\mathbb{R}^n$
- $\overline{\mathbb{X}}$ the complement set of $\mathbb{X}$



Notation

- f : a function from $\mathbb{R}^n$ to $\mathbb{R}$
- $\mathbf{f}$: a function from $\mathbb{R}^n$ to $\mathbb{R}^m$
- $\mathbf{f}(\mathbb{X}) = \{\mathbf{f}(\mathbf{x}) \mid \mathbf{x} \in \mathbb{X}\}$



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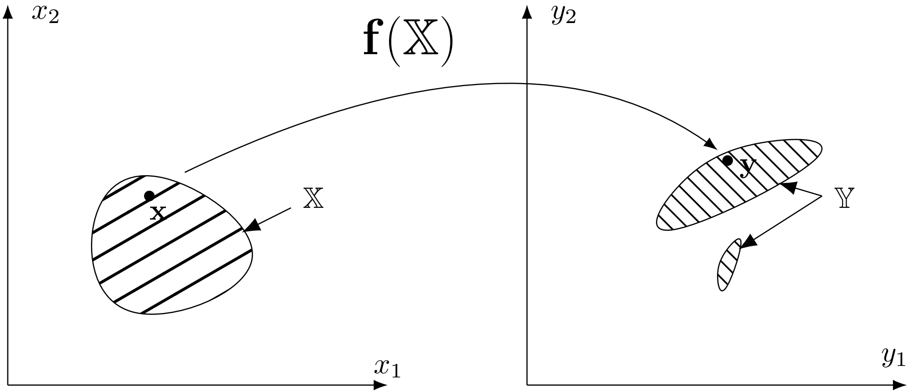
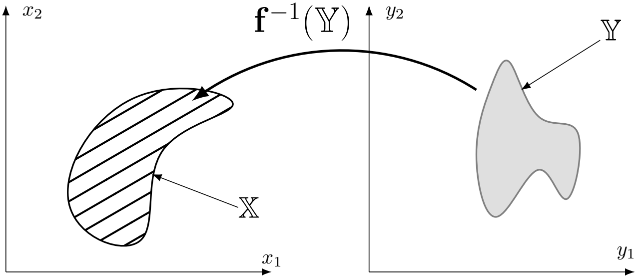


Set inversion

Set inversion

With $\mathbb{Y} \in \mathbb{R}^p$ and $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$, the set inversion problem aims at characterizing the set:

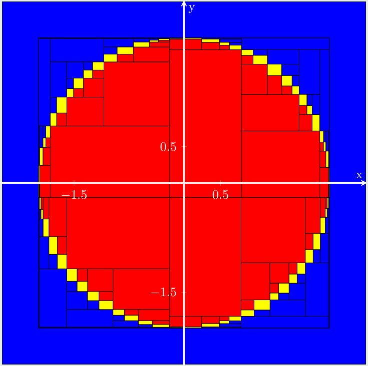
$$\begin{aligned} \mathbb{X} &= \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{f}(\mathbf{x}) \in \mathbb{Y}\} \\ &= \mathbf{f}^{-1}(\mathbb{Y}) \end{aligned}$$



Example

Example with a disk

With $f(\mathbf{x}) = \sqrt{x_1^2 + x_2^2}$:



$$\mathbb{X} = f^{-1}([0, 2])$$

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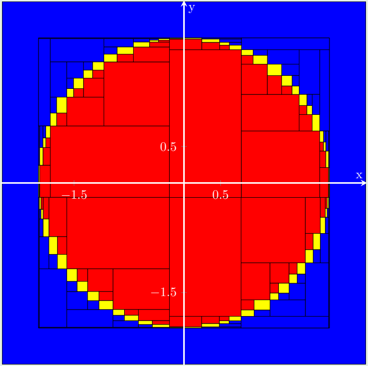

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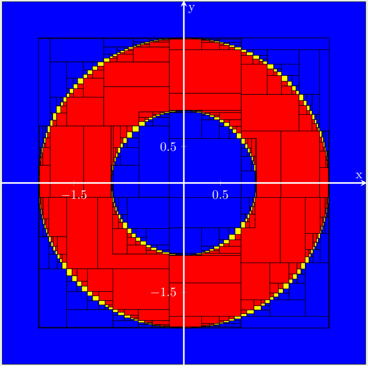

$\mathbb{X} = f^{-1}([1,2])$

Constraint network

$$
\begin{cases}
\textbf{Variables:} \\
\quad \mathbf{x}(t_1), \mathbf{x}_0, d_1, d_2 \\
\textbf{Constraints:} \\
\quad \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) \\
\quad \mathbf{x}(0) = \mathbf{x}_0 \\
\quad d_1 = g_{\mathbf{m}_1}(\mathbf{x}(t_1)) \\
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\textbf{Domains:} \\
\quad \mathbb{X}_{t_1}, [\mathbf{x}_0], [d_1], [d_2]
\end{cases}
$$

with $g_{\mathbf{m}}(\mathbf{x}) = \sqrt{(m_1 - x_1)^2 + (m_2 - x_2)^2}$

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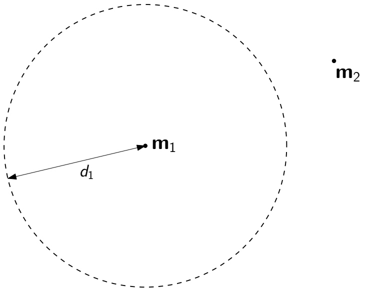

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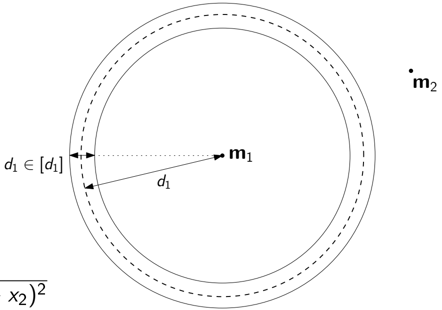

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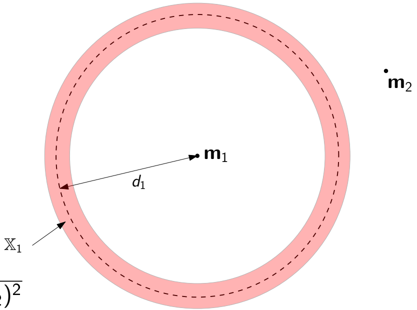

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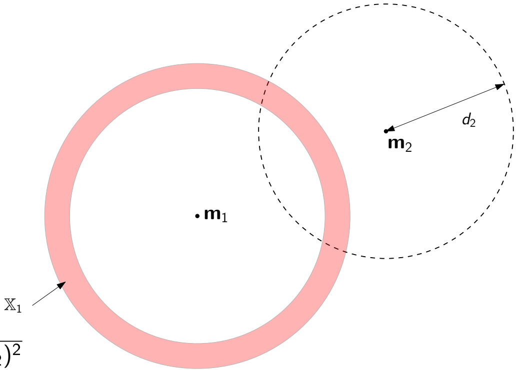

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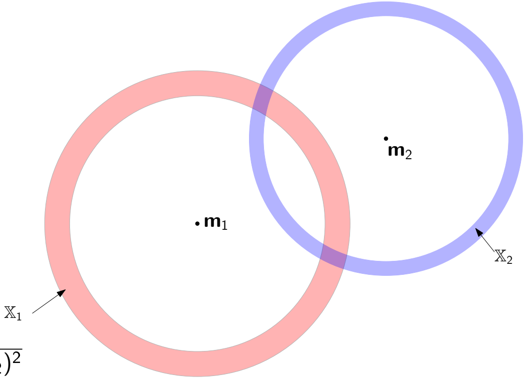

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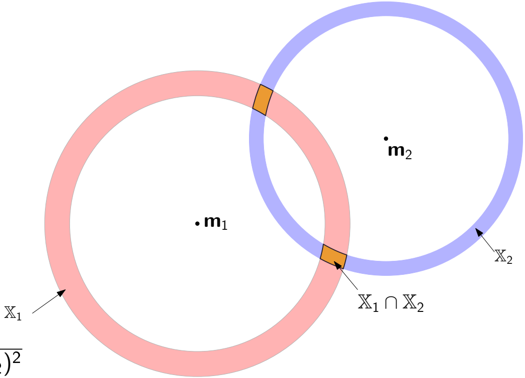

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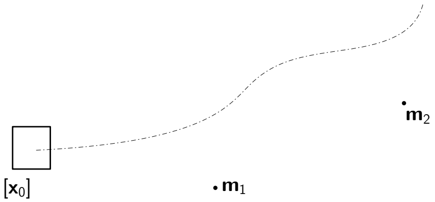

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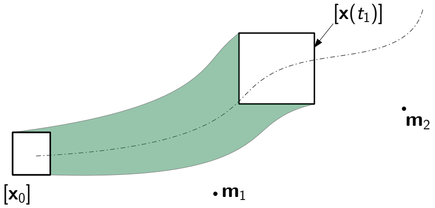

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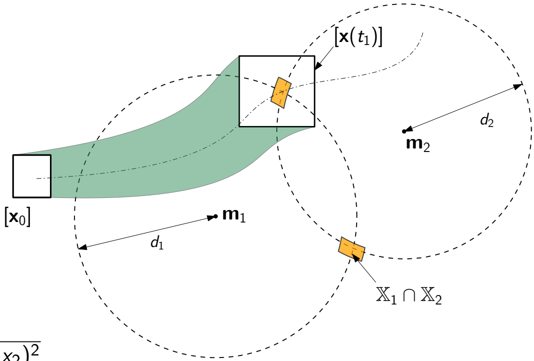

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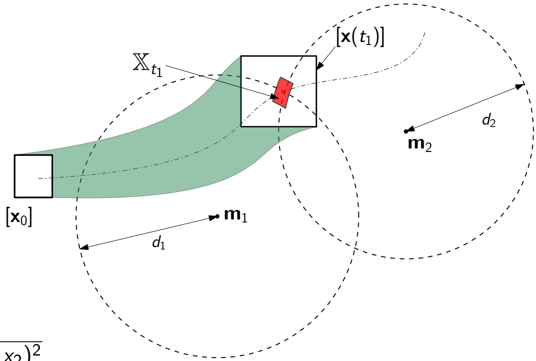
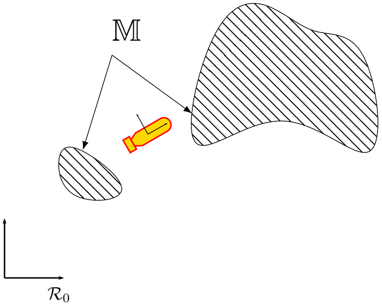


Constraint network

$$\begin{cases} \textbf{Variables:} \\ \quad \mathbf{x}(.), \mathbf{u}(.), \mathbb{Z}(t_i), \mathbb{M} \\ \textbf{Constraints:} \\ \qquad \dot{\mathbf{x}}(.) = \mathbf{f}(\mathbf{x}(.), \mathbf{u}(.)) \\ \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ \qquad \mathbb{Z}(t_i) = \mathbf{g}(\mathbf{x}(t_i), \mathbb{M}) \\ \textbf{Domains:} \\ \quad [\mathbf{x}(.)], [\mathbf{u}(.)], [\mathbb{Z}](t_i), [\mathbb{M}] \end{cases}$$

with $t_i \in \mathbb{T} \subset \mathbb{R}$



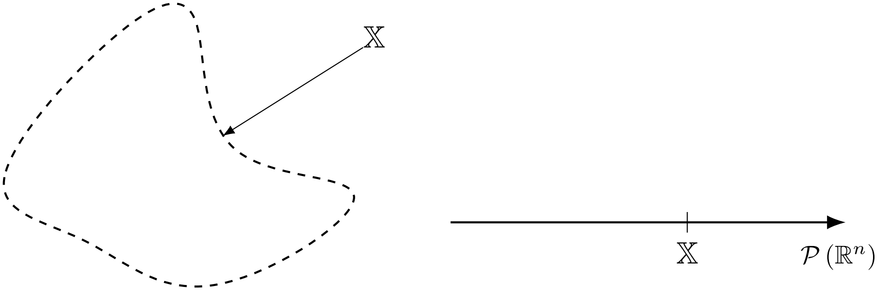
Outline

Thick set

Definition

A *thick set* $[\mathbb{X}]$ of $\mathbb{R}^n$ is an interval of $(\mathcal{P}(\mathbb{R}^n), \subset)$ such as :

$$\begin{aligned} [\mathbb{X}] &= [\mathbb{X}^-, \mathbb{X}^+] \\ &= \{\mathbb{X} \in \mathcal{P}(\mathbb{R}^n) \mid \mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+\} \end{aligned}$$



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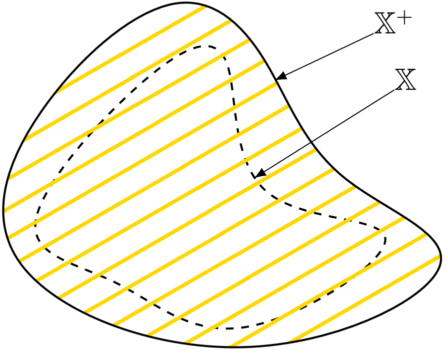

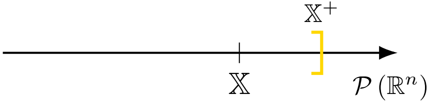

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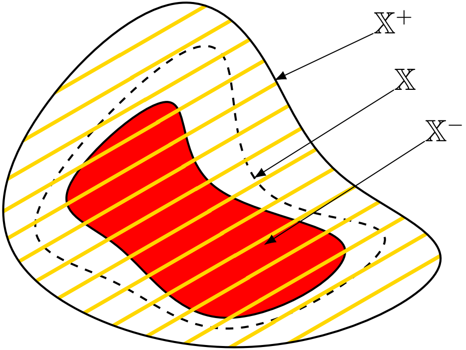

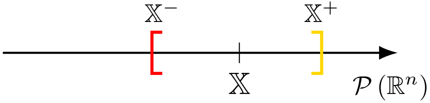

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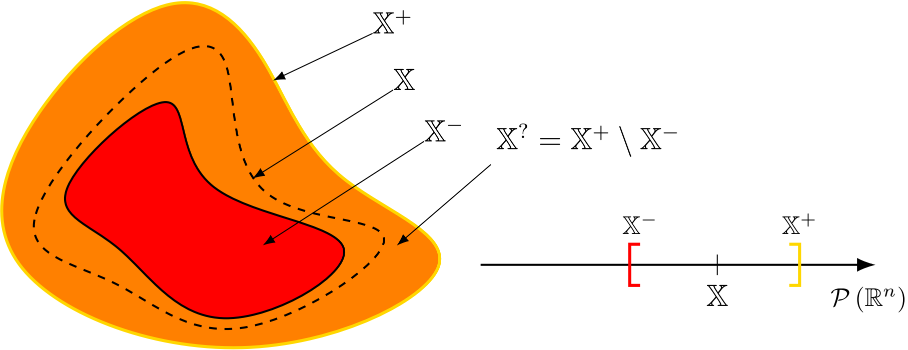

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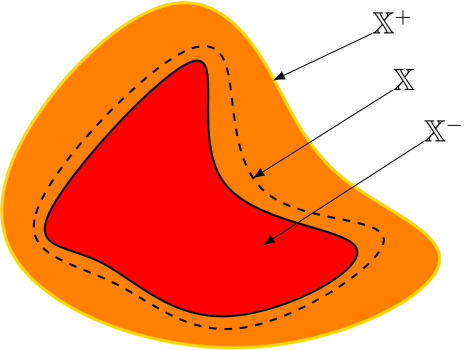

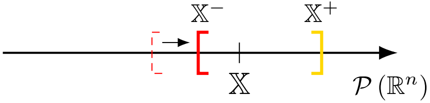

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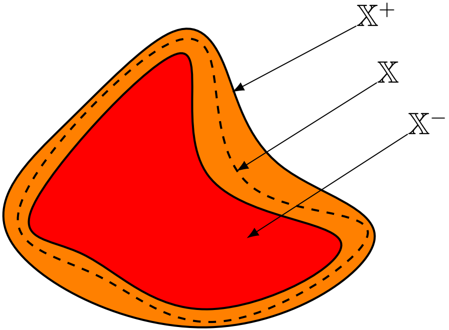
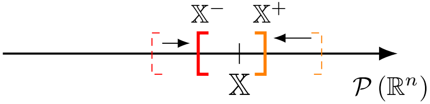
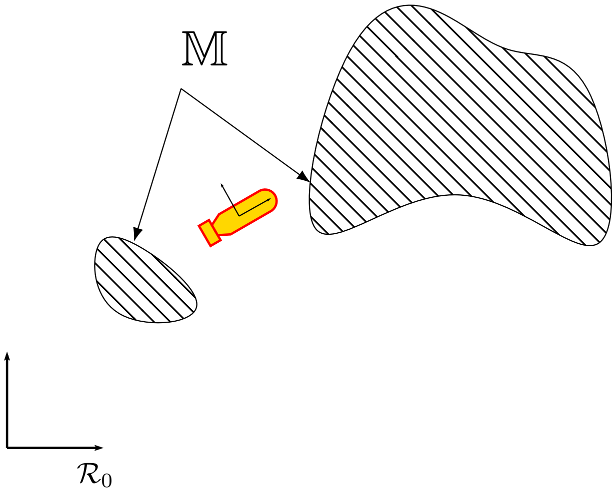




Examples of interval shapes



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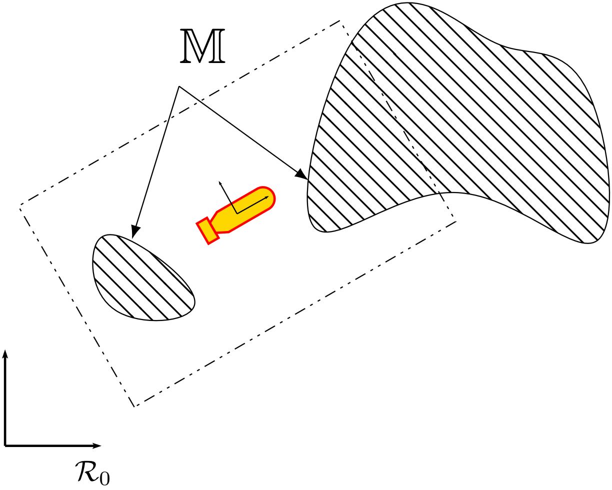

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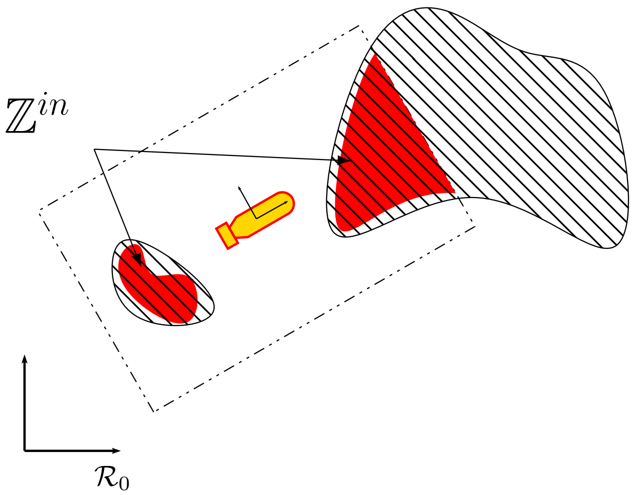

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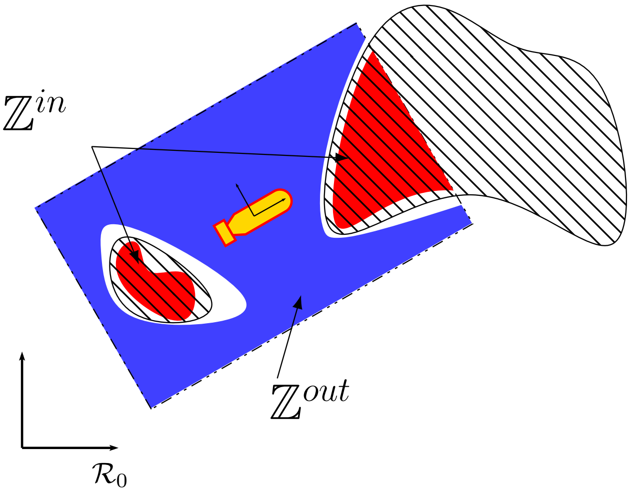

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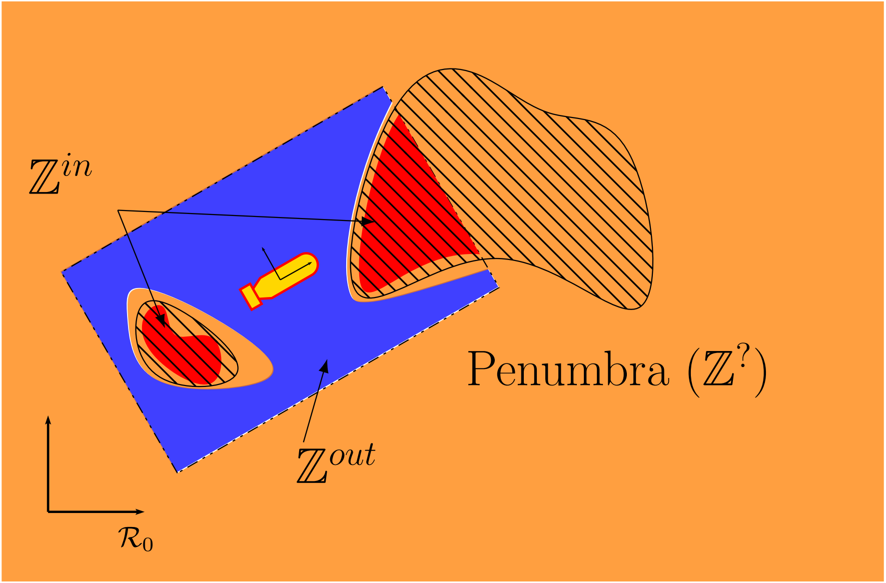

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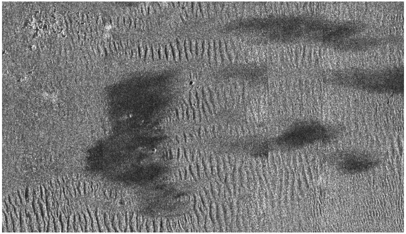

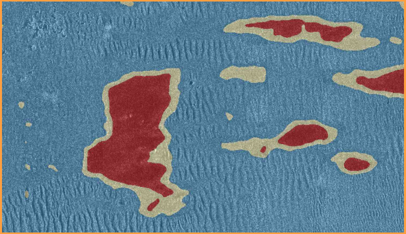

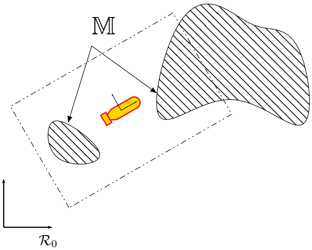

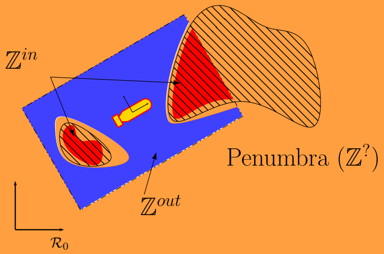

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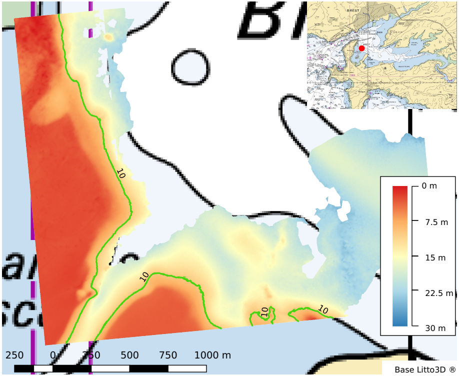
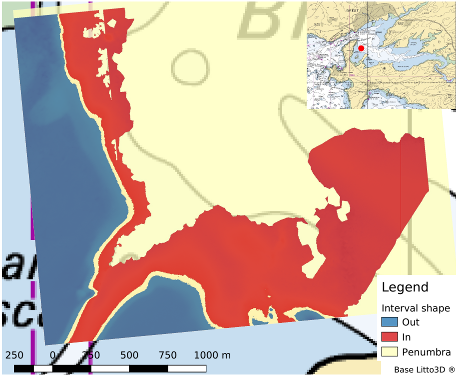


MNT from litto3D® database

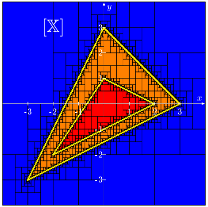


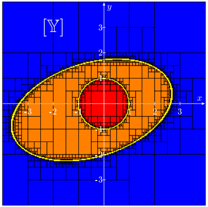
Interval shape representation

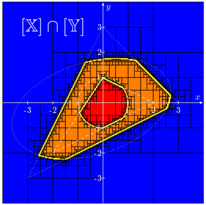
Operations on thick sets

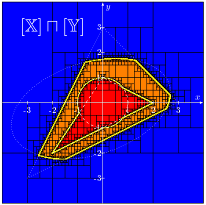
Two types of operations can be defined on thick sets. For instance:

- $$[\mathbb{X}] \cap [\mathbb{Y}] = \{\mathbb{Z} \in \mathcal{P}(\mathbb{R}^n) \,|\, \mathbb{Z} = \mathbb{X} \cap \mathbb{Y}, \mathbb{X} \in [\mathbb{X}], \mathbb{Y} \in [\mathbb{Y}]\}$$
  $$= \left[\mathbb{X}^- \cap \mathbb{Y}^-, \mathbb{X}^+ \cap \mathbb{Y}^+\right]$$
- $$[\mathbb{X}] \sqcap [\mathbb{Y}] = \{\mathbb{Z} \in \mathcal{P}(\mathbb{R}^n) \,|\, \mathbb{Z} \in [\mathbb{X}], \mathbb{Z} \in [\mathbb{Y}]\}$$
  $$= \left[\mathbb{X}^- \cup \mathbb{Y}^-, \mathbb{X}^+ \cap \mathbb{Y}^+\right]$$









Uncertain set inversion

Problem statement

Given $\mathbb{Y} \in \mathbb{R}^m$ and $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$, the set inversion problem aims at characterizing the set :

$$\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y})$$

In our case:

1. $\mathbb{Y} \in [\mathbb{Y}]$ is a thick set
2. $\mathbf{f} \in \mathbb{F} \subset \mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)$ is an uncertain function

The uncertain set inversion aims at finding the smallest thick set which encloses all feasible sets such as:

$$\exists \mathbf{f} \in \mathbb{F}, \exists \mathbb{Y} \in [\mathbb{Y}], \mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y})$$

Uncertain set inversion

Theorem

Given $f \in \mathbb{F} \subset \mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)$ and $[\mathbb{Y}] = [\mathbb{Y}^-, \mathbb{Y}^+]$, the smallest thick set solution of $[\mathbb{X}] = [\mathbf{f}]^{-1}([\mathbb{Y}])$ is:

$$\begin{aligned} [\mathbb{X}] &= [\mathbb{X}^-, \mathbb{X}^+] \\ &= \left[\bigcap_{\mathbf{f}\in\mathbb{F}} \mathbf{f}^{-1}(\mathbb{Y}^-), \bigcup_{\mathbf{f}\in\mathbb{F}} \mathbf{f}^{-1}(\mathbb{Y}^+)\right] \end{aligned}$$

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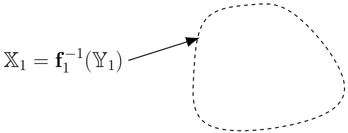

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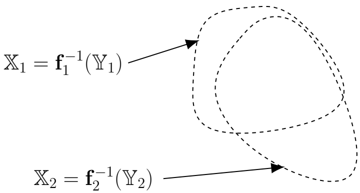

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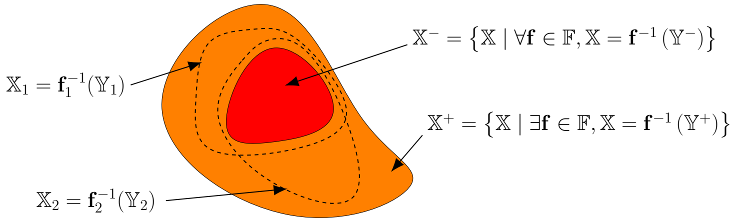
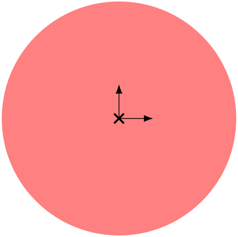


Example with an uncertain disk

Given $[y] = [0, 2]$, $[\mathbf{m}] \in [-0.5, 0.5]^2$ and

$$f_{[\mathbf{m}]}(\mathbf{x}) = \sqrt{(x_1 - [m_1])^2 + (x_2 - [m_2])^2}$$

We aim at finding the thick set $[\mathbb{X}] = f_{[\mathbf{m}]}^{-1}([y])$



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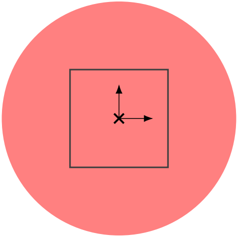

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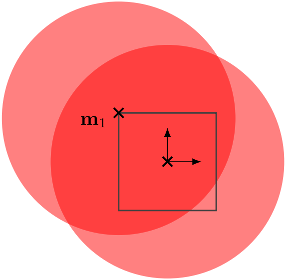

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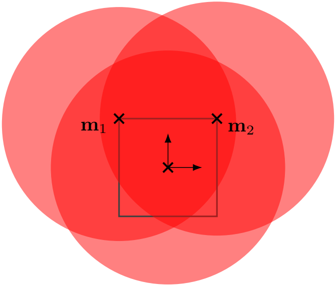

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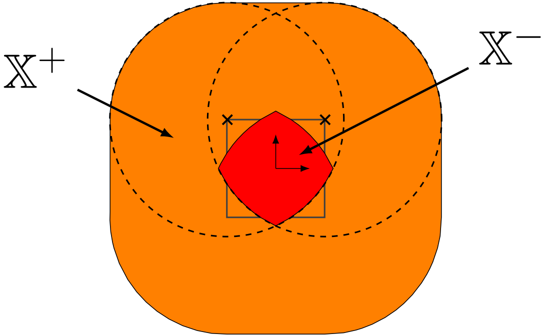

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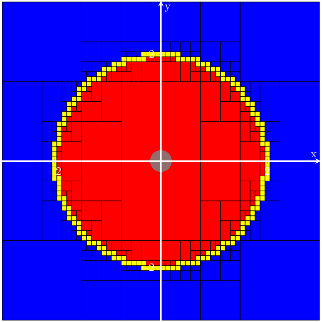

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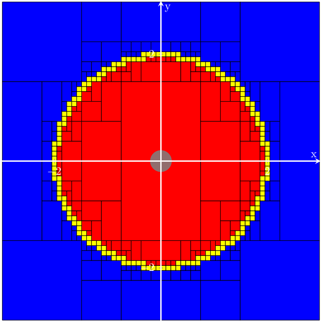

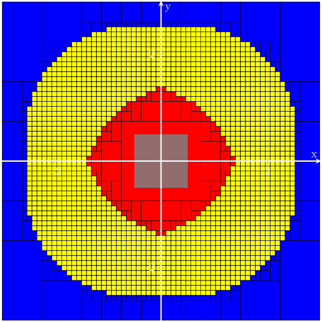

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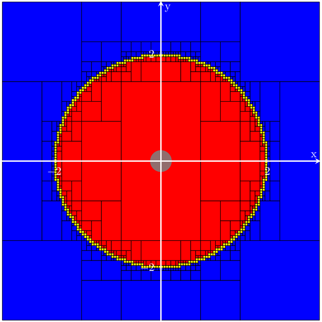

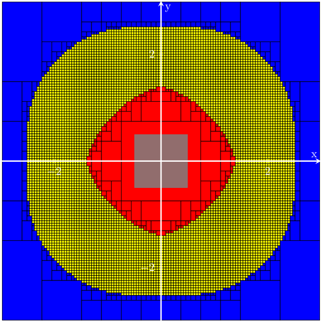

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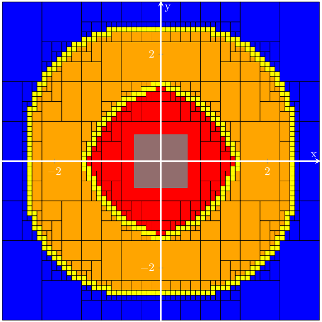
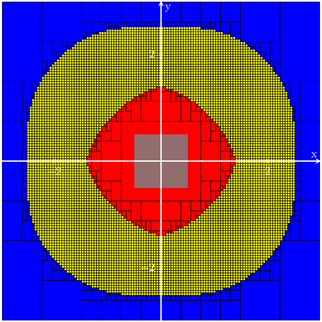




Outline

Shape registration and carving

Constraint network

$$\begin{cases} \mathbf{f}(\mathbb{A}, \mathbf{p}) = \mathbb{B}, \\ \mathbb{A} \in [\mathbb{A}], \\ \mathbb{B} \in [\mathbb{B}], \\ \mathbf{p} \in \mathbb{P} \end{cases}$$

where $\mathbf{f}_{\mathbf{p}}(\cdot) = \mathbf{f}(\mathbf{p}, \cdot)$ is a bijective function.

Since

$$\mathbf{f}(\mathbb{A}) = \mathbb{B} \Leftrightarrow \begin{cases} \mathbf{f}(\mathbb{A}) \subset \mathbb{B} \\ \mathbb{A} \subset \mathbf{f}^{-1}(\mathbb{B}) \end{cases}$$

only the elementary constraint $\mathbf{f}(\mathbb{A}) \subset \mathbb{B}$ needs to be handled.
Two subproblems:

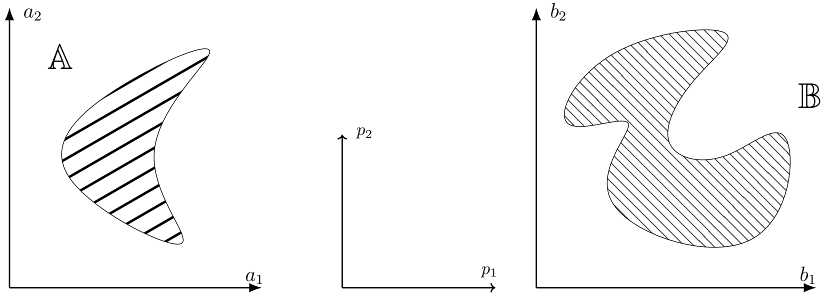
- the registration problem: find the smallest set $\mathbb{P}$ *w.r.t* $[\mathbb{A}]$ and $[\mathbb{B}]$
- the carving problem: contract $[\mathbb{A}]$ and $[\mathbb{B}]$ for a given $\mathbb{P}$

Registration

Problem statement

Given two shapes $\mathbb{A} \subset \mathbb{R}^n$, $\mathbb{B} \subset \mathbb{R}^m$ and a function $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$, we are looking for the set:

$$\mathbb{P} = \{\mathbf{p} \mid \mathbf{f}(\mathbb{A}, \mathbf{p}) \subset \mathbb{B}\}$$



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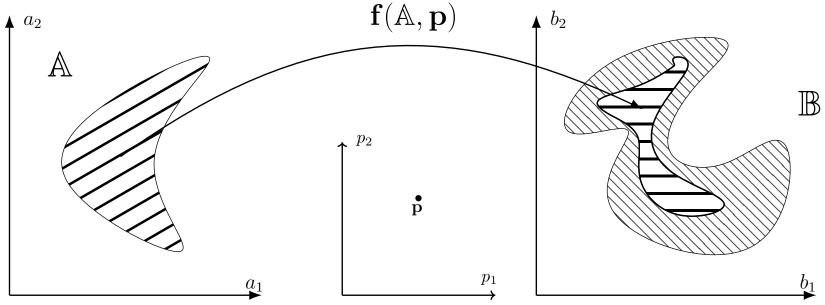

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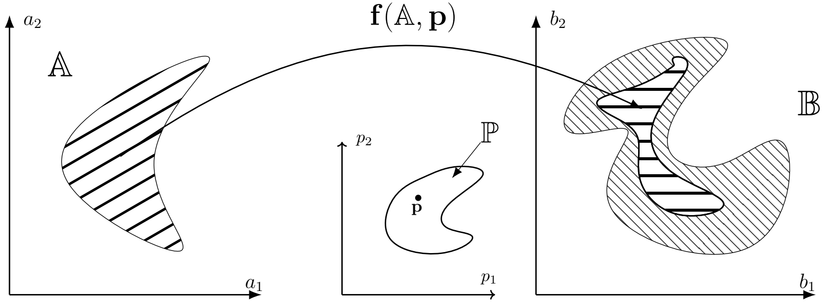

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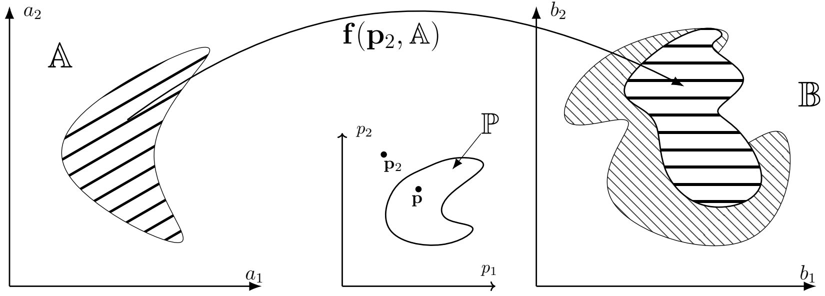


Registration

Proposition 1

Given $\mathbb{A} \in \mathcal{P}(\mathbb{R}^n)$, $\mathbb{B} \in \mathcal{P}(\mathbb{R}^n)$ and $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$, we have:

$$\begin{aligned} \mathbb{P} &= \{\mathbf{p} \in \mathbb{R}^p \mid \mathbf{f}(\mathbb{A}, \mathbf{p}) \subset \mathbb{B}\} \\ &= \left\{\mathbf{p} \in \mathbb{R}^p \mid \neg\left(\exists \mathbf{a} \in \mathbb{A}, (\mathbf{a}, \mathbf{p}) \in f^{-1}(\overline{\mathbb{B}})\right)\right\} \\ &= \overline{proj_{\mathbf{p}}\left\{(\mathbb{A} \times \mathbb{R}^p) \cap \mathbf{f}^{-1}(\overline{\mathbb{B}})\right\}} \end{aligned}$$

Proposition 2

Given $\mathbb{A} \in [\mathbb{A}^-, \mathbb{A}^+]$, $\mathbb{B} \in [\mathbb{B}^-, \mathbb{B}^+]$ and $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$, we have:

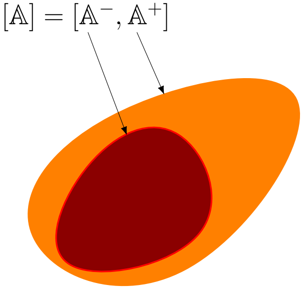
$$\{\mathbf{p} \mid \mathbf{f}(\mathbb{A}, \mathbf{p}) \subset \mathbb{B}\} \subset \left\{\mathbf{p} \mid \mathbf{f}(\mathbb{A}^-, \mathbf{p}) \subset \mathbb{B}^+\right\}$$

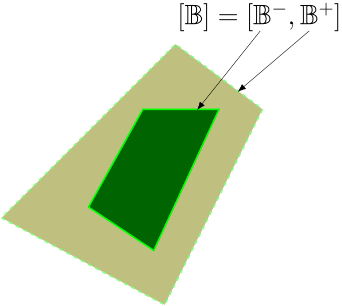
Shape carving

Proposition

Given two shapes $\mathbb{A} \in [\mathbb{A}]$, $\mathbb{B} \in [\mathbb{B}]$, $\mathbf{f} : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{p} \in [\mathbf{p}]$.
A contractor for $[\mathbb{A}]$ is:

$$\mathcal{C}([\mathbb{A}]) = [\mathbb{A}] \sqcap \mathbf{f}^{-1}_{[\mathbf{p}]}([\mathbb{B}])$$



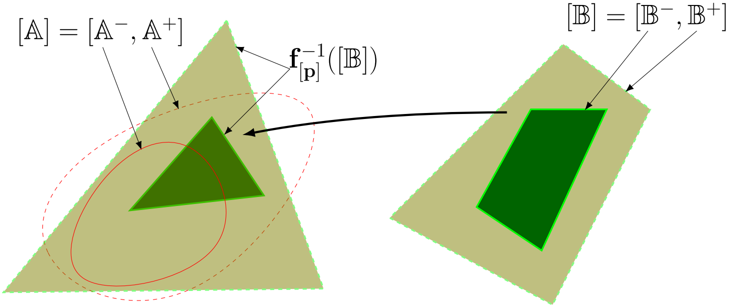


Shape carving

Proposition

Given two shapes $\mathbb{A} \in [\mathbb{A}]$, $\mathbb{B} \in [\mathbb{B}]$, $\mathbf{f} : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{p} \in [\mathbf{p}]$.
A contractor for $[\mathbb{A}]$ is:

$$\mathcal{C}([\mathbb{A}]) = [\mathbb{A}] \sqcap \mathbf{f}_{[\mathbf{p}]}^{-1}([\mathbb{B}])$$

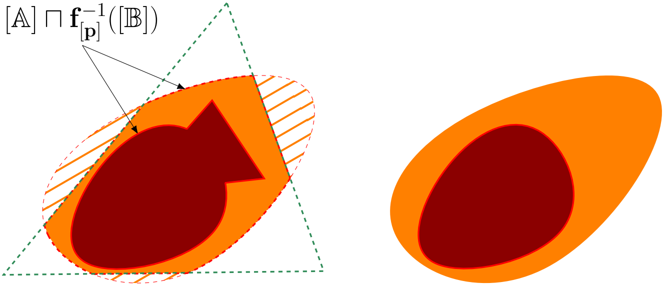


Shape carving

Proposition

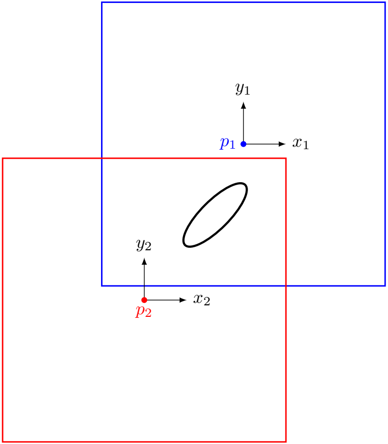
Given two shapes $\mathbb{A} \in [\mathbb{A}]$, $\mathbb{B} \in [\mathbb{B}]$, $\mathbf{f} : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{p} \in [\mathbf{p}]$.
A contractor for $[\mathbb{A}]$ is:

$$\mathcal{C}([\mathbb{A}]) = [\mathbb{A}] \sqcap \mathbf{f}^{-1}_{[\mathbf{p}]}([\mathbb{B}])$$

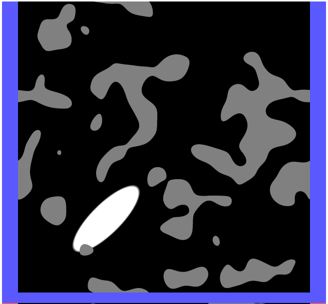


Example

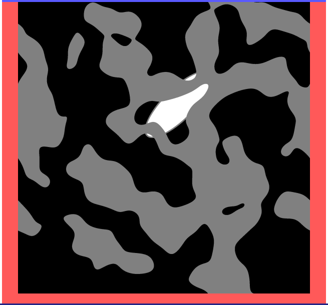
Find the feasible set of translations between $\mathbf{p}_1$ and $\mathbf{p}_2$ such as:



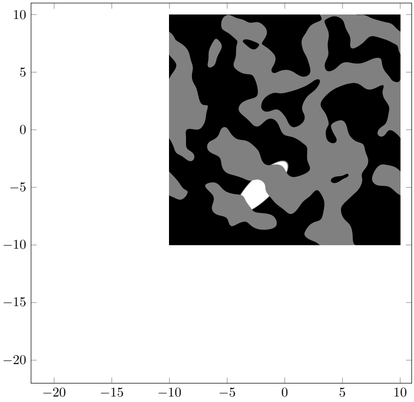
[$\mathbb{A}$]



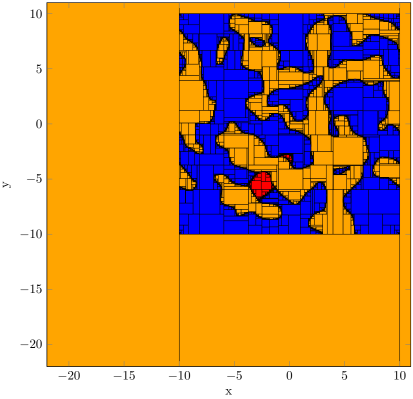
[$\mathbb{B}$]



Example



grayscale image



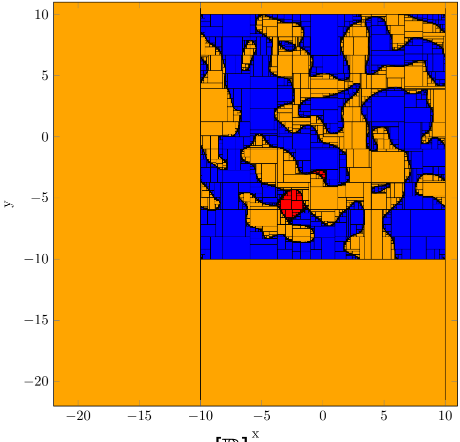
$[\mathbb{B}]$

Results

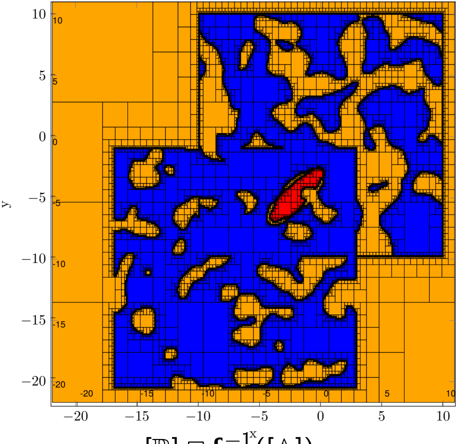
Registrations:

- $[\mathbf{p}_0] = [4, 11] \times [9, 16]$
- $[\mathbf{p}] = [6.85, 7.04] \times [10.829, 11.001]$

Carving:



$[\mathbb{B}]$



$[\mathbb{B}] \sqcap \mathbf{f}^{-1}_{[\mathbf{p}]}([\mathbb{A}])$

Outline

Shaped SLAM

Problem statement

Given $\mathbf{x} \in \mathbb{R}^n$ the state vector, $\mathbf{u} \in \mathbb{R}^m$ the input vector, $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ the evolution function, $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^r$ the observation function,

$$\begin{cases} \dot{\mathbf{x}}(t) & = & \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) & t \in \mathbb{R} \\ \mathbb{Z}(t_i) & = & \mathbf{g}_{\mathbf{x}(t_i)}(\mathbb{M}) & t_i \in \mathbb{T} \subset \mathbb{R} \\ \mathbf{x}(0) & = & \mathbf{x}_0 & \end{cases}$$

where $\mathbb{M} \in \mathcal{P}(\mathbb{R}^q)$ and $\mathbb{Z} \in \mathcal{P}(\mathbb{R}^r)$.
We assume that $\mathbf{g}_{\mathbf{x}(t_i)}$ is bijective.

Shaped SLAM

Problem statement

Given $\mathbf{x} \in \mathbb{R}^n$ the state vector, $\mathbf{u} \in \mathbb{R}^m$ the input vector, $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ the evolution function, $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^r$ the observation function,

$$\begin{cases} \dot{\mathbf{x}}(t) & = & \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) & t \in \mathbb{R} \\ \mathbb{Z}(t_i) & = & \mathbf{g}_{\mathbf{x}(t_i)}(\mathbb{M}) & t_i \in \mathbb{T} \subset \mathbb{R} \\ \mathbf{x}(0) & = & \mathbf{x}_0 & \end{cases}$$

where $\mathbb{M} \in \mathcal{P}(\mathbb{R}^q)$ and $\mathbb{Z} \in \mathcal{P}(\mathbb{R}^r)$.
We assume that $\mathbf{g}_{\mathbf{x}(t_i)}$ is bijective.

Example

With $\mathbf{x}(t) \in \mathbb{R}^2$ and $\mathbf{m} \in \mathbb{R}^2$:

$$\mathbf{g}_{\mathbf{x}(t)}(\mathbf{m}) = \mathbf{m} - \mathbf{x}(t)$$

$$\mathbf{g}^{-1}_{\mathbf{x}(t)}(\mathbf{z}) = \mathbf{z} + \mathbf{x}(t)$$

Inter-temporal formulation

Since for $(t_i, t_j) \in \mathbb{T}^2 \subset \mathbb{R}^2$, we have:

$$\begin{cases} \mathbb{Z}_i = \mathbf{g}_{\mathbf{x}(t_i)}(\mathbb{M}) \\ \mathbb{Z}_j = \mathbf{g}_{\mathbf{x}(t_j)}(\mathbb{M}) \end{cases} \Leftrightarrow \quad \mathbb{Z}_i = \mathbf{g}_{\mathbf{x}(t_i)} \circ \mathbf{g}^{-1}_{\mathbf{x}(t_j)}(\mathbb{Z}_j)$$

$$\Leftrightarrow \quad \mathbb{Z}_i = \mathbf{h}_{\mathbf{p}_{ij}}(\mathbb{Z}_j)$$

$\mathbb{M}$ can be removed from the unknowns of the problem.

Example

With $\mathbf{g}_{\mathbf{x}(t)}(\mathbf{m}) = \mathbf{m} - \mathbf{x}(t)$:

$$\mathbb{Z}_i + \mathbf{x}(t_i) = \mathbb{Z}_j + \mathbf{x}(t_j)$$
$$\Leftrightarrow \mathbb{Z}_i = \mathbb{Z}_j + \mathbf{x}(t_j) - \mathbf{x}(t_i)$$
$$\Leftrightarrow \mathbb{Z}_i = \mathbb{Z}_j + \mathbf{p}_{t_{ij}}$$

Constraint network

Variables:

$\mathbf{x}(\cdot)$, $\mathbf{u}(\cdot)$, $\mathbb{Z}_{t_i}$, $\mathbf{p}_{t_i t_j}$

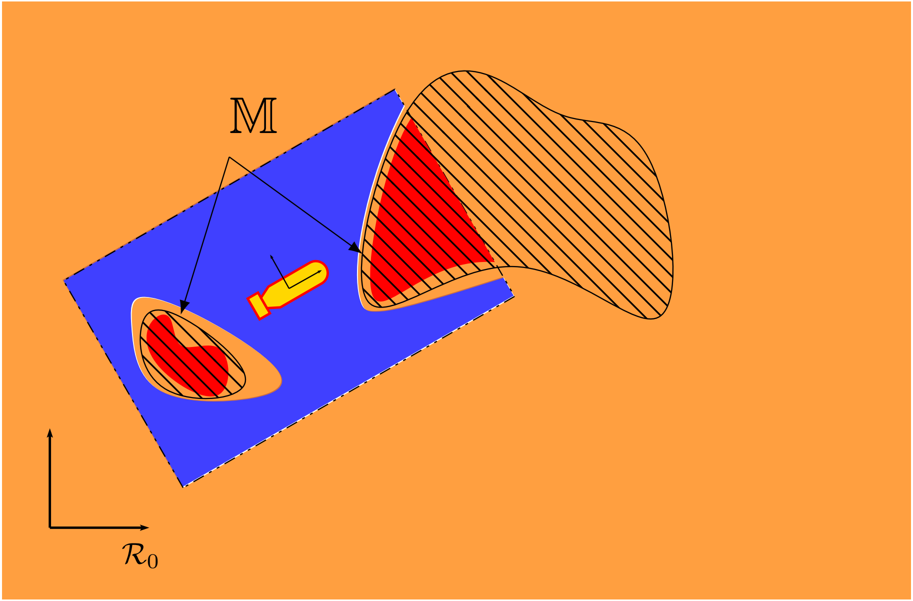
Constraints:

1. $\dot{\mathbf{x}}(\cdot) = \mathbf{f}(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$
2. $\mathbf{p}_{t_i t_j} = \mathbf{x}(t_j) - \mathbf{x}(t_i)$ with $(t_i, t_j) \in \mathbb{T}^2 \subset \mathbb{R}^2$
3. $\mathbb{Z}_{t_i} = \mathbb{Z}_{t_j} + \mathbf{p}_{t_i t_j}$

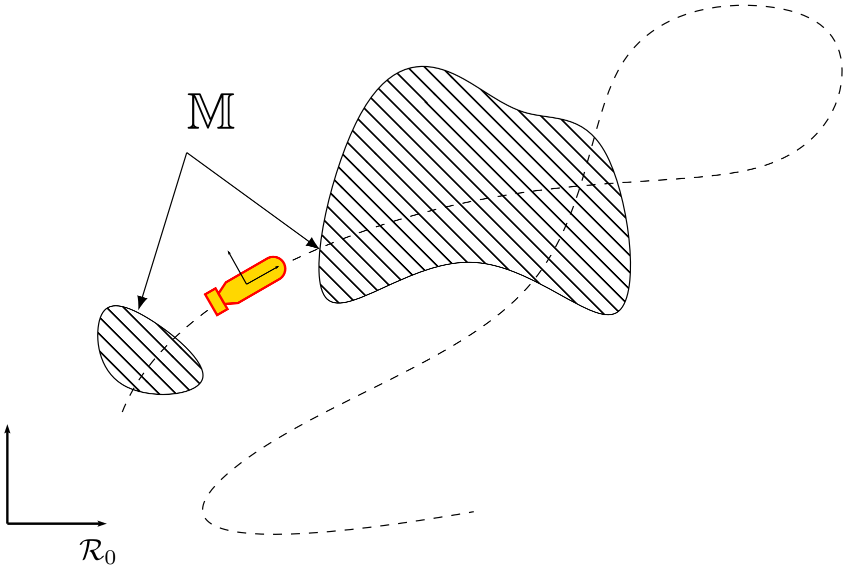
Domains:

$[\mathbf{x}](\cdot)$, $[\mathbf{u}](\cdot)$, $[\mathbb{Z}_i]$, $[\mathbf{p}_{t_i t_j}]$

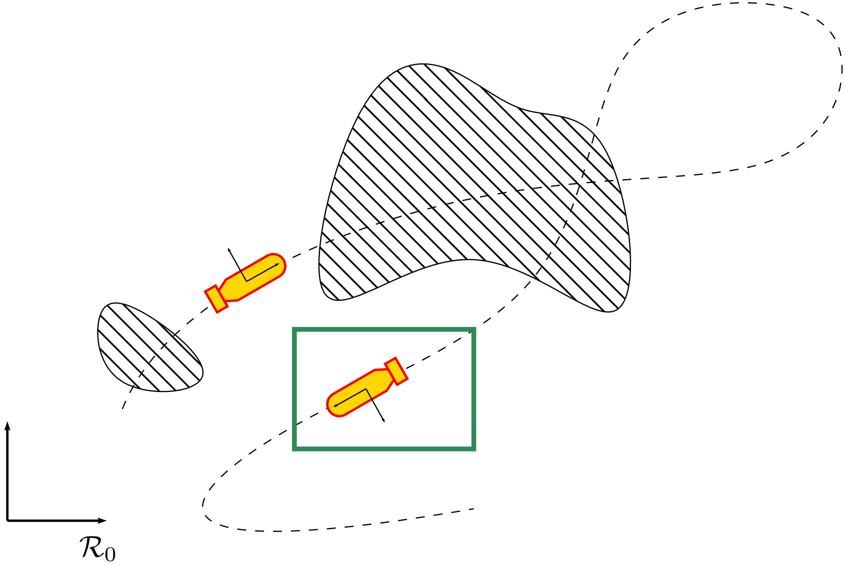
Inter-temporal formulation



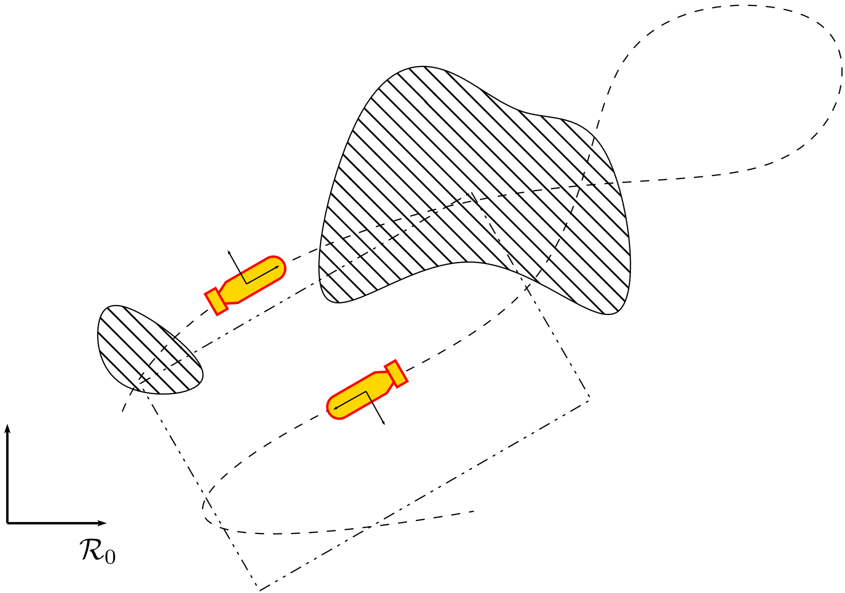
Inter-temporal formulation



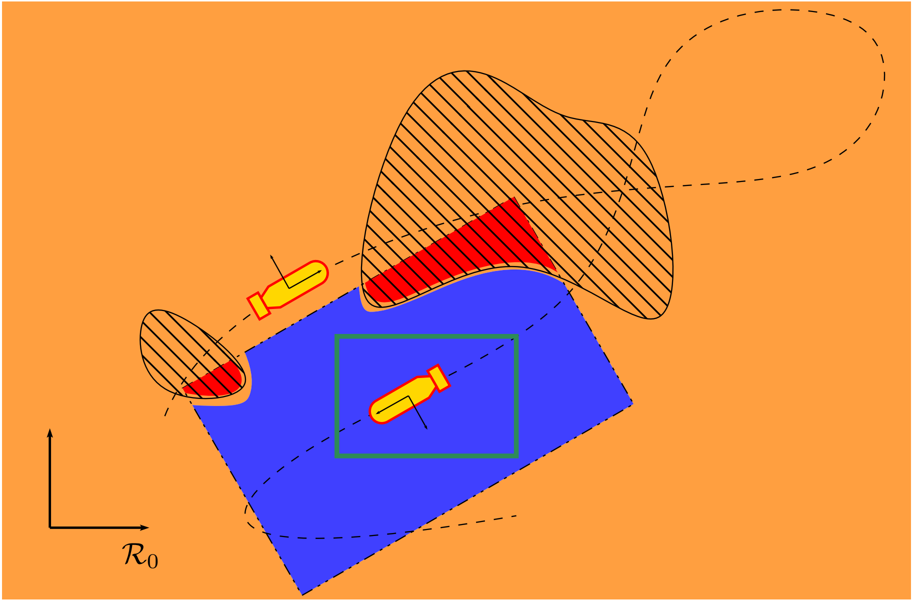
Inter-temporal formulation



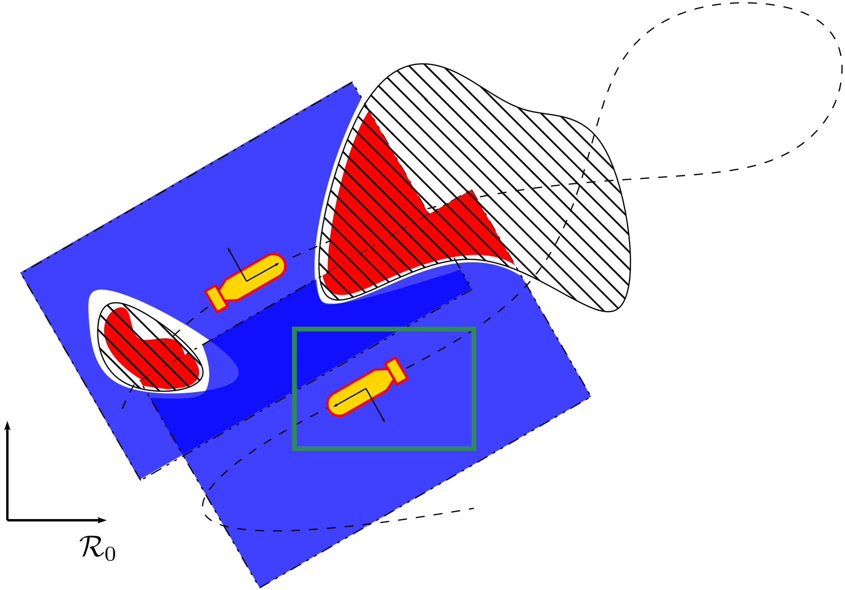
Inter-temporal formulation



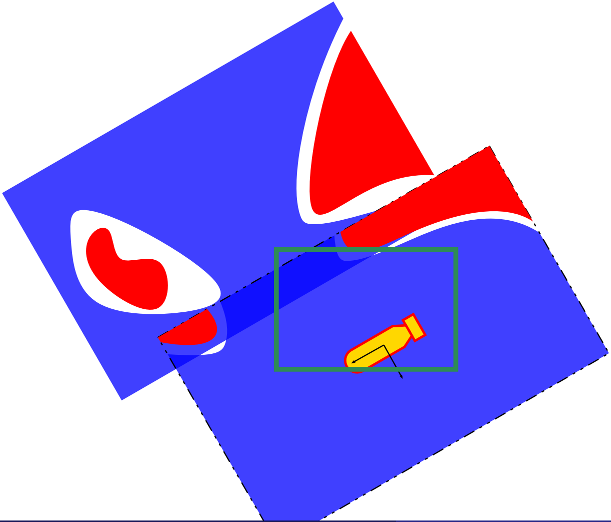
Inter-temporal formulation



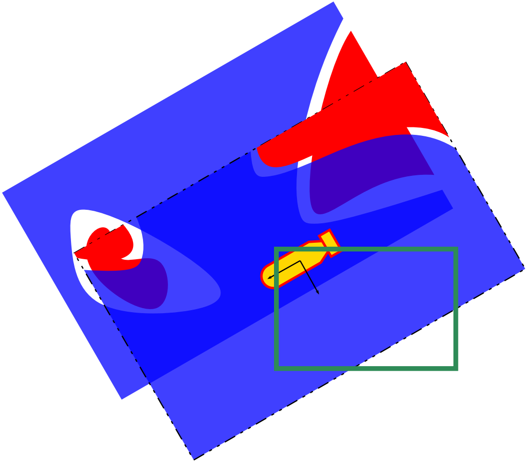
Inter-temporal formulation



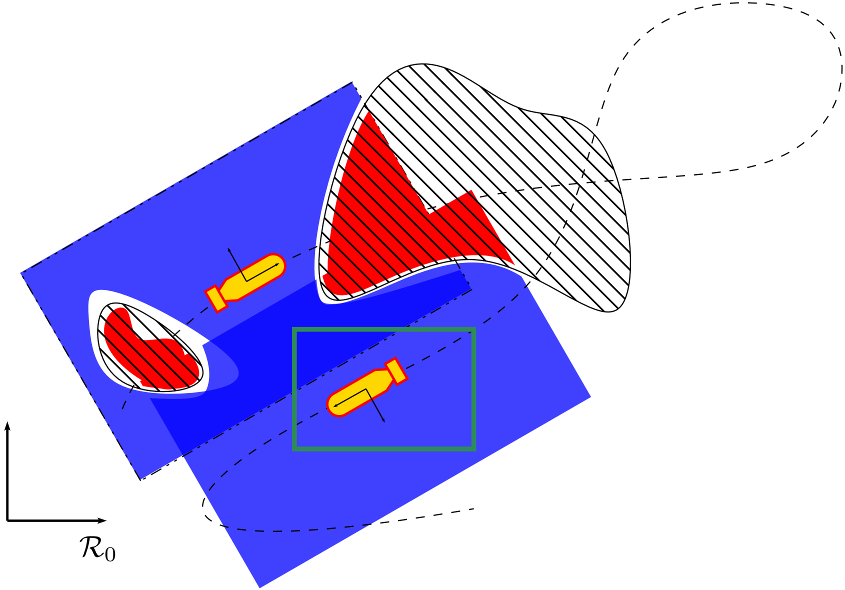
Inter-temporal formulation



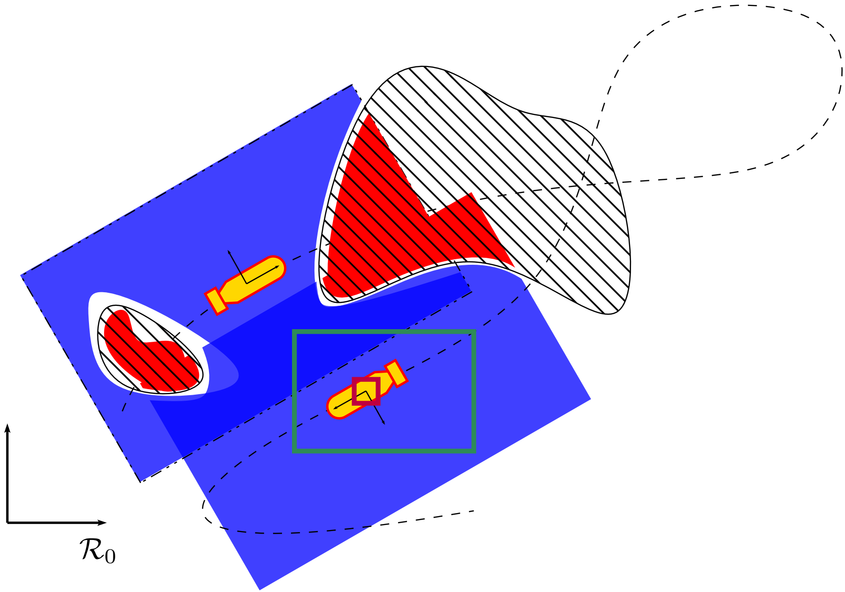
Inter-temporal formulation



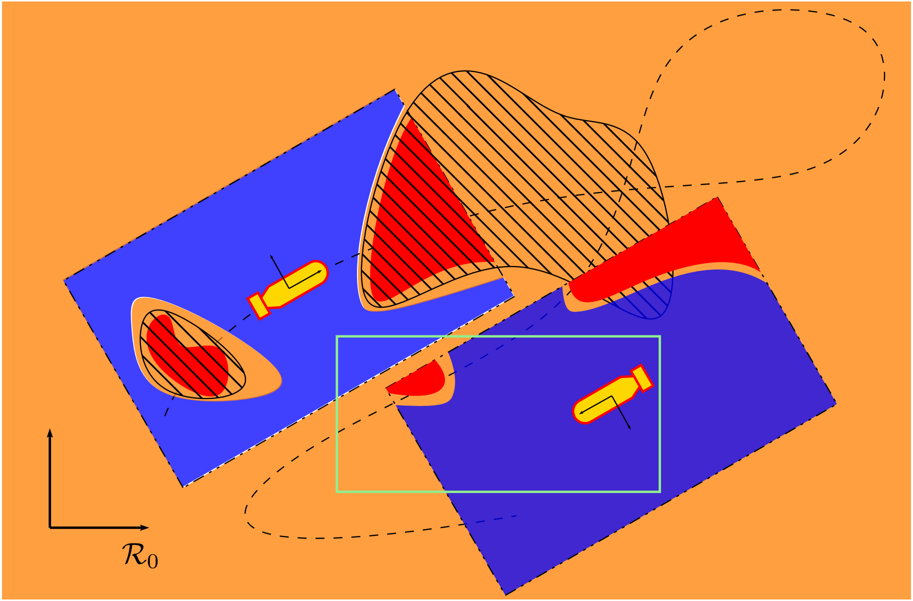
Inter-temporal formulation



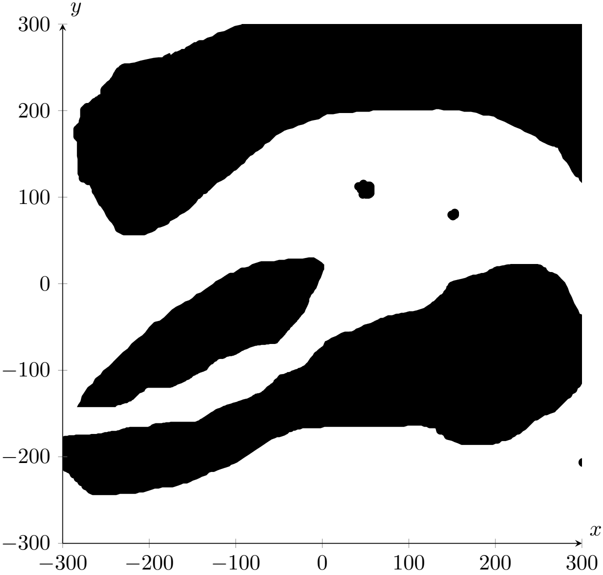
Inter-temporal formulation



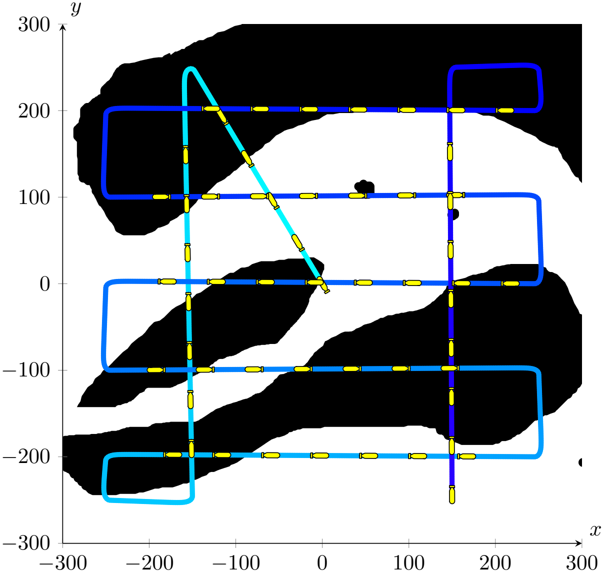
Inter-temporal formulation



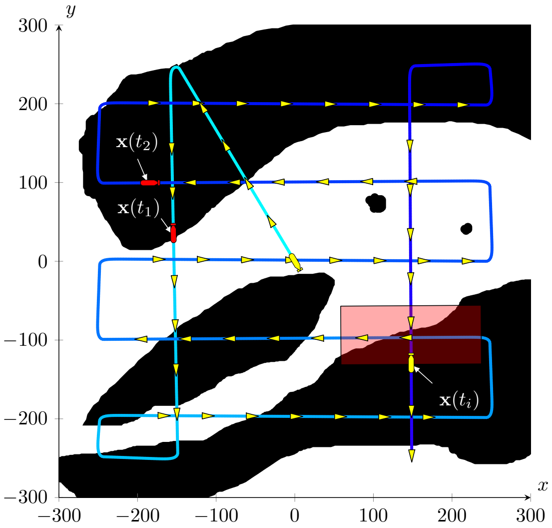
Simulated test case



Simulated test case



Test case


y
300
200
100
0
−100
−200
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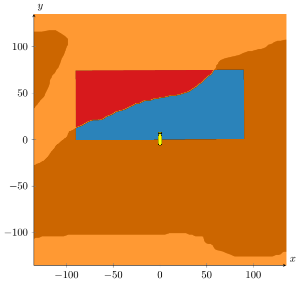
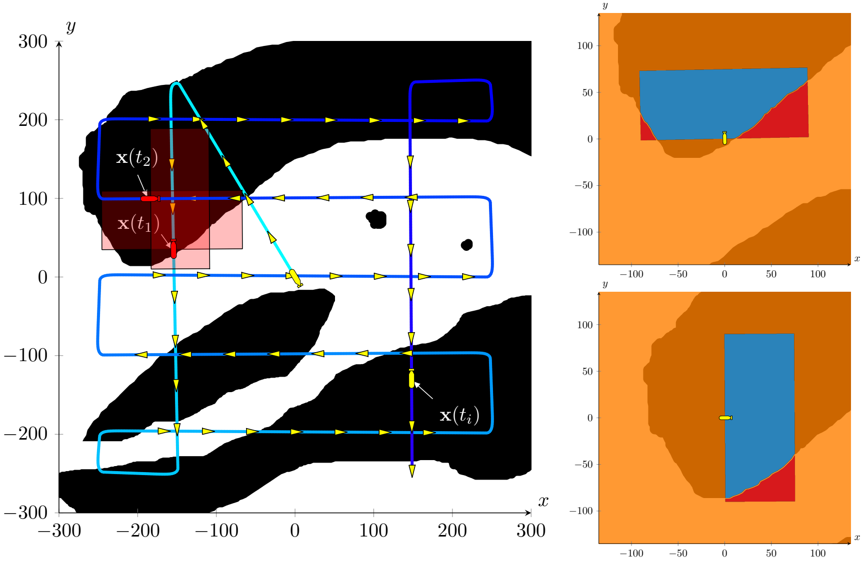
−300
−200
−100
0
100
200
300
x
x(t2)
x(t1)
x(ti)


y
100
50
0
−50
−100
−100
−50
0
50
100
x

Test case


y
300
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x
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−100
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300
$\mathbf{x}(t_2)$
$\mathbf{x}(t_1)$
$\mathbf{x}(t_i)$
y
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x
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y
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50
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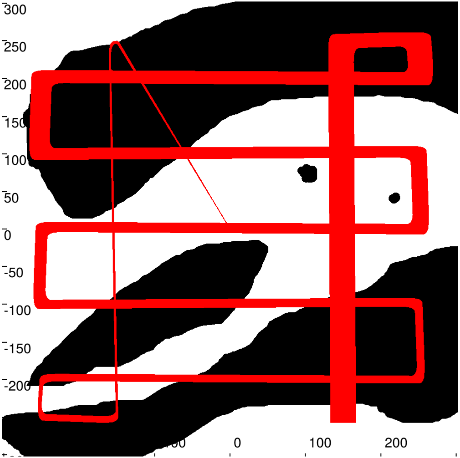
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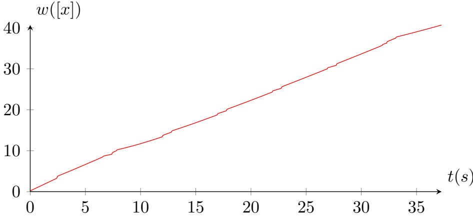
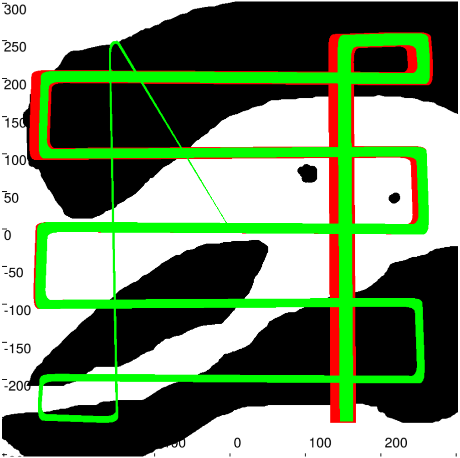

$w([x])$
40
30
20
10
0
$t(s)$
0
5
10
15
20
25
30
35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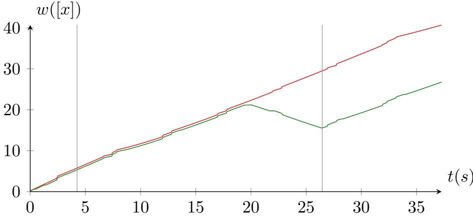
$w([x])$
40
30
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10
0
$t(s)$
0
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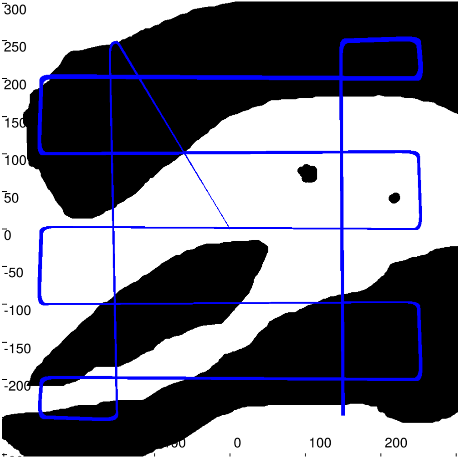
25
30
35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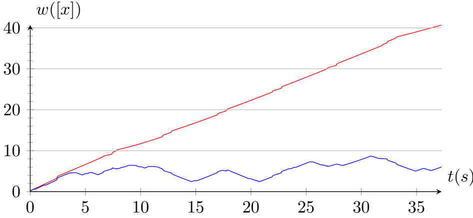
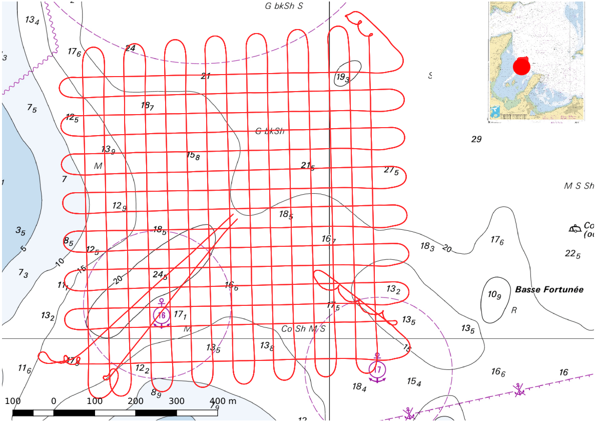
100
200

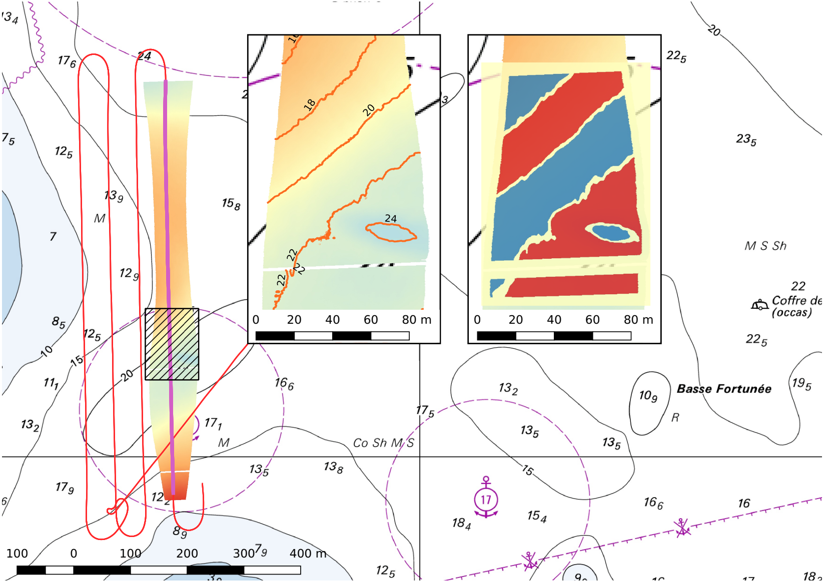

$w([x])$
40
30
20
10
0
0
5
10
15
20
25
30
35
$t(s)$

Experimental mission with *Daurade*

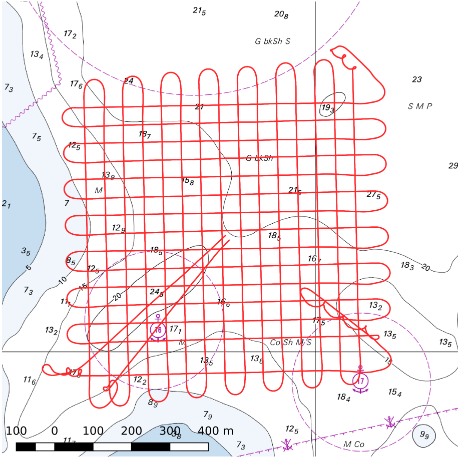
- 4h30 experimental mission in the bay of Roscanvel
- 30 km long trajectory
- data collected using a SeaBat 7125 Multibeam Echo Sounder

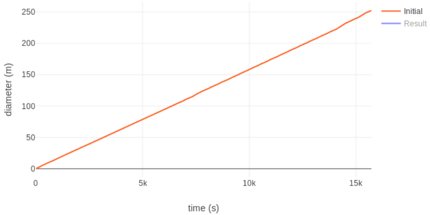


Experimental mission with *Daurade*

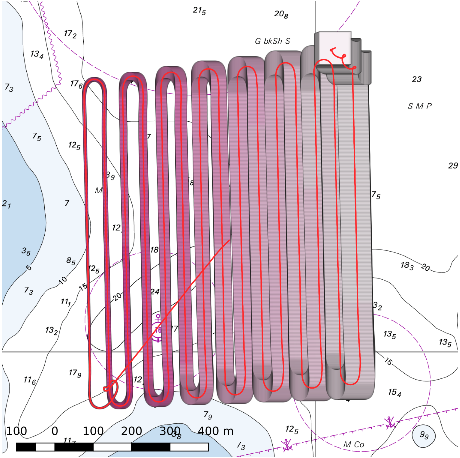


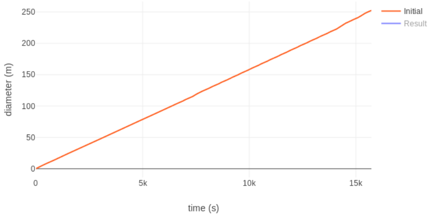
Experimental mission with *Daurade*



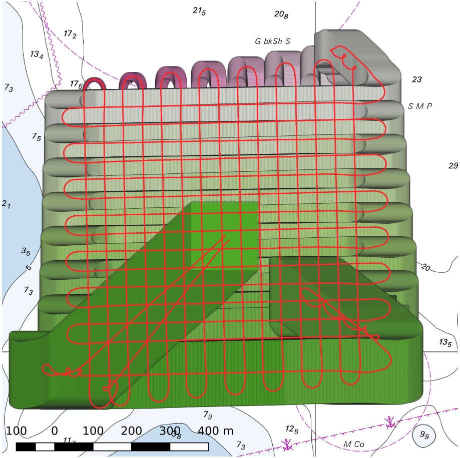


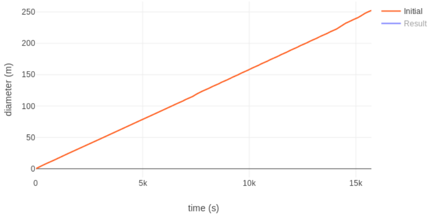
Experimental mission with *Daurade*



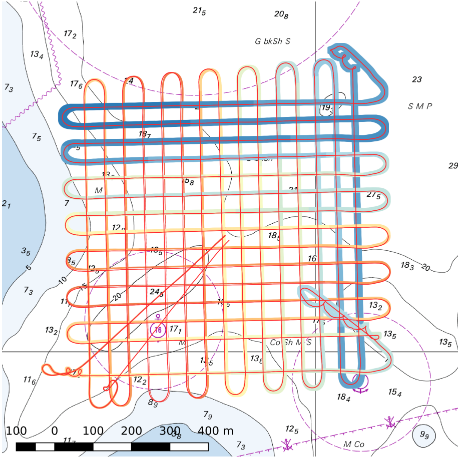


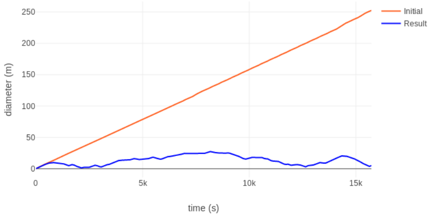
Experimental mission with *Daurade*





Experimental mission with *Daurade*





Conclusion and Prospects

Conclusion:

- shape related constraints
- development of new tools and algorithms
- application on robotics examples
- implementation on *pyibex*

Prospects:

- multi sensors, multi views, multi resolution
- need for reliable classifiers
- real-time implementation
- 3D shapes

Publications

Journal papers:

- **A Minimal Contractor for the Polar Equation; Application to Robot Localization.**
  B. Desrochers and L. Jaulin. *Engineering Applications of Artificial Intelligence (2016)*
- **Computing a guaranteed approximation the zone explored by a robot.**
  B. Desrochers and L. Jaulin. *IEEE Transaction on Automatic Control (2017)*
- **Thick set inversion.**
  B. Desrochers and L. Jaulin. *Artificial Intelligence (2017)*

Conferences papers:

- **Relaxed intersection of thick sets.**
  B. Desrochers and L. Jaulin. *SCAN'16, Uppsala*
- **Thick separators.**
  L. Jaulin and B. Desrochers. *COPROD'16, Uppsala*
- **Minkowski operations of sets with application to robot localization.**
  B. Desrochers and L. Jaulin. *SNR'2017, Uppsala, 2017* .
- **Chain of set inversion problems; Application to reachability analysis.**
  B. Desrochers and L. Jaulin. *IFAC'2017, Toulouse*

Simultaneous Localization and Mapping in Unstructured Environments
A Set-Membership Approach

— Thank you for your attention! —