Analyse Harmonique Quaternionique et Fonctions Spéciales Classiques

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ANALYSE HARMONIQUE QUATERNIONIQUE ET FONCTIONS SPÉCIALES CLASSIQUES

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Résumé :

Ce travail s'inscrit dans l'étude des symétries d'espaces de dimension infinie. Il répond à des questions algébriques en suivant des méthodes analytiques. Plus précisément, nous étudions certaines représentations du groupe symplectique complexe dans des espaces fonctionnels. Elles sont caractérisées par leurs décompositions isotypiques relativement à un sous-groupe compact maximal. Ce travail décrit ces décompositions dans deux modèles : un modèle classique (dit compact) et un autre plus récent (dit non-standard). Nous montrons que cela établit un lien entre deux familles de fonctions spéciales (fonctions hypergéométriques et fonctions de Bessel) ; ces familles sont associées à des équations différentielles ordinaires d'ordre 2, fuchsiennes dans un cas et non fuchsiennes dans l'autre. Nous mettons aussi en évidence, dans le modèle non-standard, un lien avec certaines équations d'Emden-Fowler, ainsi qu'un opérateur différentiel simple qui agit sur les décompositions isotypiques.

Mots clés :

Groupe symplectique complexe, représentation unitaire, décomposition isotypique, modèle non-standard, fonctions hypergéométriques, fonctions de Bessel.

Abstract :

The general setting of this work is the study of symmetry groups of infinite-dimensional spaces. We answer algebraic questions, using analytical methods. To be more specific, we study certain representations of the complex symplectic group in functional spaces. These representations are characterised by their isotypic decompositions with respect to a maximal compact subgroup. In this work, we describe these decompositions in two different models : a classical model (compact picture) and a more recent one (non-standard picture). We show that this establishes a connection between two families of special functions (hypergeometric functions and Bessel functions) ; these families correspond to second order differential equations, which are Fuchsian in one case and non-Fuchsian in the other. We also establish a link with certain Emden-Fowler equations and exhibit a simple differential operator that acts on the isotypic decompositions.

Keywords :

Complex symplectic group, unitary representation, isotypic decomposition, non-standard model, hypergeometric functions, Bessel functions.

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Notation

The choices made here apply to all integers $n \in \mathbb{N} \setminus \{0\}$; let us consider an arbitrary value of n.

Given any two vectors $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ of $\mathbb{R}^n$, the standard Euclidean product is denoted by a simple dot and given by the usual sum :

$$x \cdot y = \sum_{i=1}^{n} x_i y_i$$

Given any two vectots $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ of $\mathbb{C}^n$, we set :

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i w_i$$

We point out that this does not correspond to the hermitian scalar product of $\mathbb{C}^n$.

We identify real, complex and quaternionic coordinates (in Chapter 1, we explain all the notions we need about quaternions and quaternionic linear algebra). The identifications work as follows :

- a vector $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \simeq \mathbb{R}^{2n}$ will correspond to the vector $z = x + iy \in \mathbb{C}^n$.
- a vector $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n \simeq \mathbb{C}^{2n}$ will correspond to the $h = z + jw \in \mathbb{H}^n$.

Consider any (non-zero) positive integers c and l. Denote by $\mathbb{K}$ the field $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Then :

- $\mathrm{M}_{l,c}(\mathbb{K})$ denotes the $\mathbb{K}$-vector space of $l \times c$ matrices (meaning matrices with l lines and c columns) whose coefficients all belong to $\mathbb{K}$.
- Given $g \in \mathrm{M}_{l,c}(\mathbb{K})$, ${}^t g \in \mathrm{M}_{c,l}(\mathbb{K})$ denotes the transpose of g.
- $\mathrm{M}(n, \mathbb{K})$ denotes the $\mathbb{K}$-vector space of all $n \times n$ matrices whose coefficients all belong to $\mathbb{K}$.
- $\mathrm{GL}(n, \mathbb{K})$ denotes the group of all $n \times n$ invertible matrices whose coefficients all belong to $\mathbb{K}$.

All zero matrices are simply denoted by 0. Also, we always identify matrices with the linear maps they correspond to with respect to the canonical basis of the underlying vector space.

The $n \times n$ identity matrix is denoted by I_n.

Whatever kind a matrix we're studying, notation $E_{r,s}$ always refers to the matrix whose terms are all 0, except for one term, the one that sits in line r and column s, which is equal to 1 ; such matrices are called *elementary matrices*.

It will always be assumed that the unit sphere S^{n-1} is equipped with the standard Euclidean measure, denoted by σ and induced by the Lebesgue measure of $\mathbb{R}^n$.

Given a measurable space X :

- $L^1(X)$ denotes the Banach space of equivalence classes of complex-valued integrable functions on X.
- $L^2(X)$ denotes the Hilbert space of equivalence classes of complex-valued square integrable functions on X.

Because $\mathbb{R}^{4n} \simeq \mathbb{C}^{2n} \simeq \mathbb{H}^n$, we will denote by S^{4n-1} the unit sphere seen as a subset of $\mathbb{R}^{4n}$, but also seen as a subset of $\mathbb{C}^{2n}$ or $\mathbb{H}^n$.

Given a topological space X, $C^0(X)$ will denote the complex vector space of all continuous functions $f : X \longrightarrow \mathbb{C}$.

We fix, once and for all, an integer n such that $n \geq 2$ and set :

$$m = n - 1$$

$$N = 2n$$

As for the bibliography, the references in no way mean to cover the huge litterature connected to this work ; they only appear if explicitely mentionned at some point.

Introduction

0.1 Introduction : French version

On peut dire que la théorie des représentations consiste à donner une forme tangible à des groupes abstraits, en identifiant leurs éléments à des opérateurs que l'on peut en quelque sorte visualiser. Une telle démarche permet de mieux comprendre la structure des groupes en question, mais, ce qui est plus intéressant peut-être, elle permet d'utiliser les connaissances dont on dispose au sujet de ces groupes pour résoudre des problèmes difficiles. Par exemple, étant donnée une équation différentielle, il se peut que les solutions forment un espace vectoriel sur lequel agit un certain groupe ; définissant alors une représentation, on peut essayer de décomposer l'espace en sous-espaces invariants irréductibles, de telle sorte que les combinaisons des éléments de ces sous-espaces reconstituent les solutions espérées. Cette idée est déjà présente de façon sous-jacente dans les travaux de J. Fourier. Dans son célèbre ouvrage [14], Fourier étudie l'équation de la chaleur et décompose les solutions en séries de fonctions trigonométriques. Evidemment, attribuer les mérites à une seule personne a toujours quelque chose d'injuste, dans la mesure où les idées nouvelles dépendent toujours de connaissances existantes (par exemple, Euler avait déjà travaillé sur les séries trigonométriques). Résumer deux siècles de mathématiques en quelques pages est naturellement impossible, si bien que nombre de personnes et contributions importantes seront sans doute injustement oubliées dans notre entrée en matière à caractère historique. Mais nous souhaitons uniquement ici esquisser les étapes importantes qui ont mené à la théorie des représentations telle qu'on la connaît aujourd'hui, afin de mieux situer le contexte de notre propre travail. Malgré les choix que nous avons dû faire au niveau des noms et travaux à citer, nous avons essayé de relater le plus fidèlement possible cette belle aventure mathématique. Les premières pages de cette introduction sont ainsi fortement basées sur la façon dont Mackey présente lui-même cette histoire dans son article [41] ; nous nous sommes aussi appuyés sur [24].

Cauchy, Poisson, Dirichlet, Riemann, Cayley et d'autres reprennent les idées de Fourier, arrivant notamment à d'autres applications en physique et aux harmoniques sphériques en dimension plus grande. A la fin du dix-neuvième siècle, la notion de groupe, introduite dans les travaux de Galois et Abel, devient prédominante en analyse et en géométrie, notamment sous l'impulsion de Klein, qui en présente l'importance dans une célèbre conférence à l'université d'Erlangen (voir [26]). Lie se met à étudier les actions de groupes topologiques. Frobenius se consacre aux représentations des groupes finis, montrant qu'elles sont unitarisables et totalement réductibles, étudiant les liens qu'il y a entre les caractères et les fonctions sur le groupe, s'intéressant à la représentations régulière et ainsi de suite. Weyl étudie les représentations des groupes compacts, obtenant des résultats similaires à ceux de Frobenius mais grâce à des outils plus sophistiqués, notamment la théorie de l'intégration développée par Lebesgue, Borel et Haar. En collaboration avec son étudiant Peter, il démontre le fameux théorème de Peter-Weyl, généralisation du théorème de Plancherel pour les séries de Fourier (voir [47]).

L'étape suivante vient de la notion d'algèbre de Lie. En effet, Lie avait étudié les actions de groupes d'un point de vue infinitésimal, en les interprétant au niveau des algèbres dites de Lie par la suite. Killing et E. Cartan avaient alors étudié les algèbres de Lie en tant que telles, aboutissant à la classification complète des algèbres de Lie simples de dimension finie. Puis, s'intéressant aux représentations de ces algèbres de Lie, E. Cartan avait caractérisé celles qui sont irréductibles et de dimension finie, ce en termes de plus hauts poids. Grâce à toutes ces avancées, Weyl approfondit la théorie des représentations des groupes compacts, obtenant par exemple sa formule des caractères et celle des dimensions.

De nombreux scientifiques contribuent au développement de la physique quantique au début du vingtième siècle (Planck, Schrödinger, Heisenberg, Von Neumann et ainsi de suite). Le formalisme qu'ils mettent en place est celui que l'on connaît aujourd'hui : un système correspond à un certain espace de Hilbert, dont les droites vectorielles sont les états ; les observables correspondent à des opérateurs auto-adjoints, qui définissent des mesures de probabilités. Weyl montre comment se servir de l'application exponentielle pour associer à un opérateur auto-adjoint toute une famille d'opérateurs unitaires, obtenant ainsi une représentation unitaire de la droite réelle vue comme groupe additif ; lorsque possible, diagonaliser les opérateurs auto-adjoints revient à décomposer la représentation unitaire.

L'étude des représentations unitaires de groupes quelconques (en autorisant les groupes non abéliens et les groupes non compacts) commence dans les années quarante, avec les travaux de Gelfand notamment. Se dégage l'idée générale qu'un groupe est en correspondance avec l'ensemble de ses représentations unitaires (voir par exemple [16]) et que l'on peut interpréter les actions de groupes dans des espaces topologiques X en termes de représentations des groupes en question dans des espaces de fonctions sur X. Chevalley rassemble et étend toute la théorie de Lie dans son ouvrage [7]. Dans les années cinquante, Harish-Chandra étudie les représentations des groupes de Lie semi-simples, tandis que Mackey developpe la théorie des représentations induites (voir [38], [39] et [40]). Au fil du temps et des contributions de nombre de mathématiciens et physiciens (mentionnés ou pas), deux grandes approches émergent :

- La première approche s'appuie sur la *méthode des orbites* de Kirillov (voir [25]) ; cette méthode consiste en une correspondance, valable pour une large classe de groupes de Lie, entre leurs représentations unitaires irréductibles et les orbites de leurs actions co-adjointes.

- La seconde approche s'appuie sur la *classification de Langlands*, qui dit, étant donné un sous-groupe parabolique d'un groupe réductif, que les représentations irréductibles admissibles du groupe sont en correspondance avec certains triplets de paramètres (ces paramètres sont liés à la notion de représentation induite - une référence en la matière est [27]).

Notre travail s'inscrit dans le cadre du second point de vue. Cela couvre un très grand nombre de situations. Donnons ci-dessous quelques exemples de propriétés connues dans ce cadre.

- Les représentations irréductibles des groupes de Lie semi-simples, connexes et compacts sont de dimension finie et paramétrées (comme E. Cartan l'a montré) par des listes finies d'entiers positifs, à savoir les plus hauts poids (voir [27], chapitres I et IV).

- Les représentations unitaires irréductibles non triviales des groupes linéaires connexes non compacts et semisimples sont de dimension infinie (voir [23], chapitre 11).

- La restriction d'une représentation unitaire irréductible d'un groupe linéaire connexe réductif à un sous-groupe compact maximal se décompose en une somme directe (hilbertienne) de sous-représentations, chacune d'elle étant elle-même une somme directe finie de représentations irréductibles qui sont équivalentes et de dimension finie (ce résultat provient du théorème de Peter-Weyl et des travaux d'Harish-Chandra - voir [27], chapitre VIII, paragraphe 2) ; en d'autres termes, irreductibilité et unitarité entraînent admissibilité.

- La notion de $(\mathfrak{g}, K)$-module est introduite par Harish-Chandra (on peut en trouver une présentation concise dans [52], chapitre 2). A l'aide de cette notion, il démontre un théorème remarquable ("subquotient theorem") ; on en donne ici une version un peu plus forte ("subrepresentation theorem"), obtenue par Casselman et Miličić dans [5] : étant donnée l'algèbre de Lie $\mathfrak{g}$ d'un groupe de Lie réductif connexe de centre fini, étant donnés un sous-groupe compact maximal et un sous-groupe parabolique minimal, tout $(\mathfrak{g}, K)$-module admissible et irréductible se plonge dans une représentation qui est induite à partir d'une représentation irréductible de dimension finie du sous-groupe parabolique en question.

En fait, tant de choses sont connues dans ce cadre que l'on pourrait penser que la théorie des représentations est à peu près achevée, ce qui bien sûr n'est pas le cas. En effet, comprendre un objet nécessite des hypothèses ; changer ces hypothèses modifie la théorie ; on peut vouloir par exemple travailler avec d'autres espaces que les espaces de Hilbert, avec des corps de caractéristique non nulle et ainsi de suite (nous n'irons pas dans ces directions). Egalement, les résultats ci-dessus traitent de représentations irréductibles ; il est naturel de se tourner vers celles qui ne le sont pas. Sans compter que de nombreux aspects des représentations sont intéressants à explorer. Listons ci-dessous quelques tâches auxquelles on peut vouloir s'atteler.

- Donner des exemples concrets de représentations apparaissant dans la classification de Langlands. Le théorème de Stone von Neumann (dans les années trente) classifie les représentations unitaires du groupe d'Heisenberg ; dans les années quarante, Gelfand, Naimark et Bargmann traitent de celles de $\mathrm{SL}(2, \mathbb{R})$ (voir [15] et [1]) ; dans les années soixante, au cours de l'étude des actions des groupes symplectiques, apparaît la représentation métaplectique (on pourra à ce sujet consulter [13], où l'on trouve au passage ce qui concerne le groupe d'Heisenberg) ; d'autres exemples sont développés plus loin. L'intérêt des exemples est, d'une part, de renforcer la légitimité d'une théorie, mais aussi, d'autre part, d'établir des ponts avec d'autres domaines scientifiques ; il se trouve que la théorie des représentations est liée à énormément de domaines.

- Etudier les représentations unitaires, sans supposer l'irréductibilité au départ, en observant comment elles se décomposent en irréductibles lorsque restreintes à tel ou tel sous-groupe. De nombreux travaux portent sur ce type de problème. On trouve par exemple le théorème général de Kostant sur les lois de branchement, l'étude de Howe des representations des groupes $\mathrm{O}(p,q)$, $\mathrm{U}(p,q)$ et $\mathrm{Sp}(p,q)$, les travaux de Kobayashi et Pevzner exprimant des opérateurs d'entrelacement en termes d'opérateurs différentiels (voir [36]) et ainsi de suite.

- Etudier de plus près les représentations de la classification de Langlands, déterminant par exemple celles qui sont unitaires ; de nombreux chercheurs travaillent ou ont travaillé sur ce thème (Zuckermann, Adams, Vogan...).

- Exploiter les liens entre telles représentations et tels domaines scientifiques. Notre travail s'intéresse à l'utilisation de fonctions spéciales. Le fait que des fonctions spéciales ont un rôle à jouer n'est pas nouveau. Effectivement, elles interviennent dans le calcul de coefficients matriciels : on rencontre par exemple des fonctions de Bessel pour une certaine action du groupe des déplacements du plan ISO(2) (voir [50], paragraphe 4.1). Elles apparaissent comme fonctions sphériques des espaces symétriques riemanniens. Egalement, sur le plan infinitésimal, les algèbres enveloppantes correspondent à des opérateurs différentiels, qui amènent à des équations différentielles, caractérisant parfois des fonctions spéciales connues. Ce sera justement notre approche : on utilisera les K-types et l'opérateur de Casimir pour atteindre des fonctions hypergéometriques et des fonctions de Bessel. On peut aussi mentionner la construction des opérateurs de brisure de symétrie (voir [36] et [29]) où les fonctions spéciales jouent de nouveau un rôle central.

En plus des exemples donnés ci-dessus, les fonctions spéciales ont ceci d'intéressant que ce sont des solutions d'équations particulières et qu'elles admettent différentes expressions en termes de séries, d'intégrales ou de récurrences. Sous certaines conditions, elles forment des bases orthonormales de certains espaces fonctionnels de Hilbert.

Gardant en mémoire notre intérêt pour les fonctions spéciales, venons-en aux choix que nous avons fait pour délimiter le cadre de notre travail.

Tout d'abord, le théorème des sous-représentations évoqué plus haut justifie à nos yeux que l'on se restreigne aux représentations induites.

Ensuite, à la recherche d'objets explicites, il semble approprié de choisir des groupes également concrets. Puisque la notion de groupe réductif est une généralisation d'une classe de groupes matriciels ayant comme point commun l'invariance sous une certaine involution, nous avons directement opté pour des groupes linéaires (réductifs, même simples en fait). Mais quel(s) groupe(s) linéaire(s) choisir exactement ?

Le dual unitaire de $\mathrm{GL}(n, \mathbb{R})$ a été largement étudié. Il est par exemple prouvé dans [49] que toute représentation unitaire irréductible de ce groupe peut être définie par un certain procédé d'induction à partir de produits tensoriels de représentations unitaires irréductibles des sous-groupes $\mathrm{GL}(m, \mathbb{R})$ de dimension plus petite. Parmi les sous-groupes de $\mathrm{GL}(n, \mathbb{R})$, ceux que l'on appelle sous-groupes classiques sont définis relativement à telle ou telle forme bilinéaire non dégénérée qu'ils laissent invariante.

Les représentations de $\mathrm{O}(n)$, $\mathrm{U}(n)$ et $\mathrm{Sp}(n)$ sont parfaitement identifiées (par leurs plus hauts poids, puisque ces groupes sont compacts - voir par exemple [53] et plus particulièrement [44] pour le groupe $\mathrm{Sp}(n)$) ; on y reviendra un peu plus loin. Les représentations de $\mathrm{O}(p, q)$ and $\mathrm{U}(p, q)$) ont aussi été largement étudiées. On trouve dans [22] l'étude des sous-modules de certaines représentations de ces groupes sur des espaces de fonctions homogènes. Dans une série d'articles, Kobayashi et Ørsted étudient la représentation unitaire minimale du groupe $\mathrm{O}(p, q)$, en décrivant les K-types minimaux en termes de fonctions spéciales ([32], [33] and [34]). On peut aussi regarder les sous-groupes qui préservent une forme antisymétrique/antihermitienne ; ce cas est aussi étudié dans [22], mais également dans [35] et [8], entre autres.

Ces travaux sont basés sur les structures des K-types, en raison de l'hypothèse d'admissibilité des représentations considérées. Ainsi, même si les représentations irréductibles de $\mathrm{O}(n)$, $\mathrm{U}(n)$ et $\mathrm{Sp}(n)$ sont bien connues, elles forment un ingrédient essentiel dans l'étude générale des représentations des groupes de Lie réductibles. Sans compter qu'il est notoire qu'elles sont liées aux fonctions spéciales, via les harmoniques sphériques.

Cela nous amène jusqu'ici à resserrer notre champs d'investigation aux K-types de repré-

sentations unitaires de sous-groupes linéaires classiques, obtenues par induction parabolique. Mais pourquoi avons-nous spécialement choisi $\mathrm{Sp}(n,\mathbb{C})$?
Dans les articles [35] et [8], les auteurs étudient des séries principales dégénérées de $\mathrm{Sp}(n,\mathbb{R})$ et $\mathrm{Sp}(n,\mathbb{C})$, realisées géométriquement sur des espaces de fonctions définies sur $\mathbb{R}^{2n}\setminus\{0\}$ et $\mathbb{C}^{2n}\setminus\{0\}$. En restreignant ces fonctions à la sphère, on établit un lien avec les harmoniques sphériques. Le cas de $\mathrm{Sp}(n,\mathbb{R})$ est différent du cas de $\mathrm{Sp}(n,\mathbb{C})$, mais les deux cas ont ceci de commun que la multiplication scalaire par les nombres de module 1 amorce la décomposition en irréductibles. Plus précisément, tout sous-espace invariant constitue un pas en avant dans la recherche d'une telle décomposition. Ce premier pas est illustré dans les sommes directes ci-dessous, paramétrées par les caractères de $\mathrm{O}(1)$ (resp. $\mathrm{U}(1)$), de sous-espaces invariants sous l'action à gauche de $\mathrm{O}(4n)$ (resp. $\mathrm{U}(2n)$) :

- $L^2(S^{4n-1}) = L^2_{\text{paires}}(S^{4n-1}) \oplus L^2_{\text{impaires}}(S^{4n-1})$;
- $L^2(S^{4n-1}) = \widehat{\bigoplus_{\delta\in\mathbb{Z}}} L^2_\delta(S^{4n-1})$, où chaque $L^2_\delta(S^{4n-1})$ est la somme hilbertienne de tous les espaces d'harmoniques sphériques de degré d'homogénéité (α,β) tel que $\delta = \beta-\alpha$.

Ainsi, on peut dire que deux actions dictent la décomposition recherchée de $L^2(S^{4n-1})$: l'action à gauche des matrices et la multiplication scalaire par les nombres de module 1. Ce schéma illustre cette double action :

$$\begin{array}{ccccc} \mathrm{O}(4n) & \curvearrowright & L^2(S^{4n-1}) & \curvearrowleft & \mathrm{O}(1) \\ \bigcup & & & & \bigcap \\ \mathrm{U}(2n) & \curvearrowright & L^2(S^{4n-1}) & \curvearrowleft & \mathrm{U}(1) \end{array}$$

Instinctivement, on aurait envie d'y ajouter une ligne qui schématiserait :

- l'action du groupe des isométries linéaires de $\mathbb{H}^n$ (isomorphe à $\mathrm{Sp}(n)$ et ainsi plongé dans le groupe $\mathrm{U}(2n)$, qui lui-même se plonge dans $\mathrm{O}(4n)$) ;
- une action de type multiplication scalaire du groupe des quaternions unitaires (isomorphe à $\mathrm{Sp}(1)$).

La ligne que l'on aurait envie de rajouter serait donc :

$$\mathrm{Sp}(n) \curvearrowright L^2(S^{4n-1}) \curvearrowleft \mathrm{Sp}(1)$$

Cela suggère donc une interaction intéressante entre deux actions : l'une de $\mathrm{Sp}(n)$ et l'autre de $\mathrm{Sp}(1)$, qui n'est pas abélien, contrairement à $\mathrm{O}(1)$ et $\mathrm{U}(1)$. Rassemblant toutes ces considérations, voici finalement la question centrale de notre travail :

Via l'action à gauche de $\mathrm{Sp}(n)$ *et une action de type multiplication scalaire de* $\mathrm{Sp}(1)$ *(à préciser), quels liens peut-on établir entre des fonctions spéciales et les K-types de la série principale dégénérée de* $\mathrm{Sp}(n,\mathbb{C})$ *?*

Remarque : tout cela revient à construire l'analogue de la décomposition en harmoniques sphériques, en prenant comme corps de base le corps des quaternions.

Il est à noter que l'action scalaire de $\mathrm{Sp}(1)$ n'est pas aussi simple que l'on pourrait croire : le fait que le corps des quaternions n'est pas commutatif implique de choisir entre la multiplication à gauche et la multiplication à droite (on choisira à droite) et cette multiplication des quaternions doit ensuite être ramenée dans le cadre du corps des complexes.
Remarquons que l'étude de certaines séries principales de groupes classiques sur le corps des quaternions est entreprise dans [18], [46] et [9].
De manière générale, notre façon d'atteindre des fonctions spéciales poursuit la philosophie initiée dans les travaux [31], [21], [30], [20] et [35], l'idée étant d'exploiter des opérateurs différentiels et des transformations de Fourier particulières.

Dans tout notre travail, on considère un entier quelconque $n \geq 2$ (la valeur minimale 2 vient du fait que l'on veut travailler avec un sous-groupe parabolique qui n'est pas minimal).
Dans le chapitre 3, nous explicitons la structure de $L^2(S^{4n-1})$ relativement aux actions de $\mathrm{Sp}(n)$ et $\mathrm{Sp}(1)$. Même si cette double structure des K-types apparaît déjà dans [22], le mérite de notre travail est de proposer un traitement personnel de la question, de façon totalement explicite et autosuffisante. De plus, nous mettons en avant et calculons des polynômes qui sont invariants sous l'effet des deux actions et importants pour la suite. Les résultats principaux du chapitre 3 (théorèmes 3.2, 3.6 et 3.13) peuvent être présentés ainsi :

Théorème A.

- *L'espace* $L^2(S^{4n-1})$ *se décompose, relativement à l'action à gauche de* $\mathrm{Sp}(n)$*, en une* *somme hilbertienne :*

$$L^2(S^{4n-1}) = \widehat{\bigoplus_{k\in\mathbb{N}}} \bigoplus_{\substack{(\alpha,\beta)\in\mathbb{N}\times\mathbb{N}\\ \alpha+\beta=k}} \bigoplus_{\gamma=0}^{min(\alpha,\beta)} V_\gamma^{\alpha,\beta}$$

Dans cette somme, $V_\gamma^{\alpha,\beta}$ est le sous-espace invariant irréducible engendré par les translatés du polynôme $P_\gamma^{\alpha,\beta}$ (en fait sa restriction à S^{4n-1}) défini par :

$$P_\gamma^{\alpha,\beta}(z,w) = w_1^{\alpha-\gamma}\overline{z_1}^{\beta-\gamma}(w_2\overline{z_1} - w_1\overline{z_2})^\gamma$$

Ici, (z,w) désignent les coodonnées sur $\mathbb{C}^n\times\mathbb{C}^n \simeq \mathbb{C}^{2n} \simeq \mathbb{R}^{4n}$, avec $z = (z_1,\ldots,z_n)$ et $w = (w_1,\ldots,w_n)$.

- *L'espace $L^2(S^{4n-1})$ se décompose, relativement à l'action à droite de $\mathrm{Sp}(1)$, en une somme hilbertienne :*

$$L^2(S^{4n-1}) = \widehat{\bigoplus_{k\in\mathbb{N}}} \bigoplus_{\gamma=0}^{E\left(\frac{k}{2}\right)} d_\gamma^k W_\gamma^k$$

où $E\left(\frac{k}{2}\right)$ désigne la partie entière de $\frac{k}{2}$. Dans cette somme, W_γ^k est un sous-espace invariant irréductible de dimension finie, ayant $P_\gamma^{k-\gamma,\gamma}$ pour vecteur de plus haut poids ; d_γ^k est un entier naturel et il y a d_γ^k sous-espaces invariants équivalents à W_γ^k.

L'énoncé ci-dessous correspond au théorème 3.18 :

Théorème B. *Considérons un entier $k\in\mathbb{N}$ et notons $\mathrm{H}^k(\mathbb{R}^{4n})$ l'espace des polynômes de $4n$ variables réelles, à coefficients complexes, harmoniques et de plus homogènes de degré k. Notons $1\times\mathrm{Sp}(n-1)$ le groupe des matrices de $\mathrm{Sp}(n)$ qui s'écrivent $\begin{pmatrix}1 & 0\\ 0 & A\end{pmatrix}$ avec $A\in\mathrm{Sp}(n-1)$.*
Alors si k est pair, il existe un unique élément de $\mathrm{H}^k(\mathbb{R}^{4n})$ (à un coefficient près) (on voit cet élément comme un polynôme en $2n$ variables complexes et en leurs conjuguées) qui est invariant par l'action à gauche de $1\times\mathrm{Sp}(n-1)$ et également par l'action à droite de $\mathrm{Sp}(1)$. Ce polynôme appartient à $V_\alpha^{\alpha,\alpha}$, avec $\alpha = \frac{k}{2}$; on dit qu'il est bi-invariant.

<u>Remarque</u> : on montre dans le paragraphe 3.2.3 comment calculer ce polynôme.
Ces théorèmes font usage de la théorie standard des groupes et algèbres de Lie, ainsi que de notions basiques, quoique moins standards peut-être, sur les quaternions. Nous fournissons en détails dans le chapitre 1 tous les pré-requis nécessaires, tout en faisant

un certain nombre de calculs préparatoires au niveau des groupes et algèbres de Lie symplectiques.

Le chapitre 2 présente la série principale dégénérée de $\mathrm{Sp}(n, \mathbb{C})$, qui fait l'objet de notre travail. Elle est obtenue par induction parabolique, que l'on explicite : choix du sous-groupe parabolique, choix du caractère et réalisations géométriques. On obtient une famille à deux paramètres de représentations $\pi_{i\lambda,\delta}$ de $G = \mathrm{Sp}(n, \mathbb{C})$, avec $(\lambda, \delta) \in \mathbb{R} \times \mathbb{Z}$. Elles sont définies de cette manière (en appelant $V_{i\lambda,\delta}$ l'espace de Hilbert sur lequel agit $\pi_{i\lambda,\delta}$) :

- Considérons l'espace vectoriel complexe $V^0_{i\lambda,\delta}$ de toutes les fonctions $f \in C^0(\mathbb{C}^N \setminus \{0\})$ telles que pour tout $c \in \mathbb{C}$ et tout $x \in \mathbb{C}^N \setminus \{0\}$:

$$f(cx) = \left(\frac{c}{|c|}\right)^{-\delta} |c|^{-i\lambda - N} f(x)$$

  Considérons sur cet espace l'action à gauche de G, définie pour tout $(g, f, x) \in G \times V^0_{i\lambda,\delta} \times (\mathbb{C}^N \setminus \{0\})$ par :

$$\pi_{i\lambda,\delta}(g) f(x) = f(g^{-1}x)$$

- $V_{i\lambda,\delta}$ et $\pi_{i\lambda,\delta}$ sont obtenus par complétion de $V^0_{i\lambda,\delta}$, par rapport à la norme $\| \cdot \|$ qui est définie par :

$$\|f\|^2 = \int_{S^{2N-1}} |f(x)|^2 dx$$

Les representations $\pi_{i\lambda,\delta}$ sont unitaires.

Comme nous l'avons dit plus haut, nous nous intéressons au lien qu'il existe entre fonctions spéciales et représentations. Le choix des groupes orthogonaux est judicieux car l'on sait comment associer des fonctions spéciales à des sous-espaces d'harmoniques sphériques, en imposant une contrainte supplémentaire (définissant la notion de fonction zonale) et en utilisant le Laplacien (voir [12], chapitre 9, paragraphes 3 and 4). Pour appliquer le même genre de méthode, il nous faut ajuster la contrainte supplémentaire parce que nous travaillons avec des quaternions ; c'est tout l'intérêt de l'action de $\mathrm{Sp}(1)$. Les fonctions spéciales que l'on obtient sont, comme dans le cas du groupe orthogonal, des fonctions hypergéométriques, mais cela ne fonctionne que pour des K-types particuliers. Tout cela fait l'objet du chapitre 4. Le résultat principal peut être formulé/résumé ainsi (pour plus de détails, voir théorème 4.8) :

Théorème C. *Etant donné $\alpha \in \mathbb{N}$, il existe dans $V_{\alpha}^{\alpha,\alpha}$ une fonction (unique à un coefficient près) que l'on sait déterminer explicitement grâce au théorème précédent et qui, après une certaine réduction du nombre de variables, se transforme en une solution de l'équation hypergéométrique suivante :*

$$u(1-u)\varphi'' + 2(1-nu)\varphi' + [\alpha^2 + (2n-1)\alpha]\varphi = 0$$

Une autre façon d'obtenir des fonctions spéciales est de passer par des transformations de Fourier adaptées, parce que les fonctions spéciales ont souvent des expressions intégrales. A l'aide d'une transformation de Fourier partielle, les auteurs de [35] définissent un *modèle non-standard* des représentations $\pi_{i\lambda,\delta}$, modèle grâce auquel ils expriment les fonctions constantes sur la sphère (formant le K-type le plus simple) en termes de fonctions de Bessel. Nous généralisons ceci, non seulement au cadre complexe, mais aussi au niveau d'un grand nombre de K-types, pour lesquels nous explicitons la forme non-standard de leurs vecteurs de plus haut poids. Cela constitue notre résultat principal. Avant d'en voir le contenu, précisons la transformation de Fourier partielle $\mathcal{F}$ (sur $L^2(\mathbb{C}^{2m+1})$) dont il est question dans le modèle non-standard. Elle est définie pour $f \in L^1(\mathbb{C}^{2m+1}) \cap L^2(\mathbb{C}^{2m+1})$ par :

$$\mathcal{F}(f)(s,u,v) = \int_{\mathbb{C}\times\mathbb{C}^m} f(\tau,u,\xi)\, e^{-2i\pi \mathrm{Re}(s\tau + \langle v,\xi\rangle)}\, d\tau d\xi$$

où l'on note (s,u,v) les coordonnées de $\mathbb{C}\times\mathbb{C}^m\times\mathbb{C}^m \simeq \mathbb{C}^{2m+1}$. On peut à présent énoncer notre théorème principal, c'est à dire le théorème 5.12, mais d'une manière un peu différente et allégée :

Théorème D (Modèle non-standard et fonctions de Bessel)**.** *Soit $n \in \mathbb{N}$ tel que $n \geq 2$; posons $N = 2n$ et $m = n-1$. Soient $k \in \mathbb{N}$ et un couple quelconque $(\alpha,\beta) \in \mathbb{N}^2$ tel que $\alpha+\beta = k$; posons alors $\delta = \beta - \alpha$. Pour $\lambda \in \mathbb{R}$, considérons la représentation $\pi_{i\lambda,\delta}$ du groupe* $\mathrm{Sp}(n,\mathbb{C})$ *et la fonction suivante :*

$$\begin{array}{cccc} f & : & \mathbb{C}^n \times \mathbb{C}^n \setminus \{(0,0)\} & \longrightarrow & \mathbb{C} \\ & & (z,w) & \longmapsto & (\|z\|^2 + \|w\|^2)^{-\frac{i\lambda+k}{2}-n}\, w_1^{\alpha}\, \overline{z_1}^{\beta} \end{array}$$

où w_1 représente la première coordonnée de w. Alors f est un vecteur de plus haut poids de la restriction de $\pi_{i\lambda,\delta}$ au sous-groupe $\mathrm{Sp}(n)$ *et sa forme non-standard, c'est à dire la fonction $\mathcal{F}\Big((\tau,u,\xi) \longmapsto f(1,u,2\tau,\xi)\Big)$, associe à tout (s,u,v) in $\mathbb{C}\times\mathbb{C}^m\times\mathbb{C}^m$ vérifiant*

$v \neq 0$ et $s \neq 0$ la valeur :

$$\frac{(-i\,\overline{s})^{\alpha}\,\pi^{i\lambda+\beta+n}}{2^{\frac{i\lambda+k}{2}+1}\,\Gamma(\frac{i\lambda+k}{2}+n)}\left(\frac{\sqrt{|s|^2+4\|v\|^2}}{\pi\sqrt{1+\|u\|^2}}\right)^{\frac{i\lambda+\delta}{2}} K_{\frac{i\lambda+\delta}{2}}\left(\pi\sqrt{1+\|u\|^2}\sqrt{|s|^2+4\|v\|^2}\right)$$

Remarque : on verra dans le chapitre 2 que la représentation $\pi_{i\lambda,\delta}$ est induite par un caractère d'un sous-groupe parabolique de G dont le radical nilpotent est isomorphe au groupe de Heisenberg complexe.

Nous terminons le chapitre 5 avec deux observations intéressantes :

- la première concerne un opérateur différentiel simple qui relie les formes non-standards de certains vecteurs de plus haut poids (théorème 5.13) ;
- la seconde concerne un lien entre la formule du théorème D et des équations différentielles dites d'Emden-Fowler (paragraphe 5.7.2).

Pour finir, nous exposons ce que notre travail apporte, à notre sens, en mettant en avant des questions sur lesquelles il s'ouvre.

0.2 Introduction: English version

In broad terms, the point of representation theory is to give tangible pictures of abstract groups, by identifying their elements with operators that one can visualise. This procedure helps understand the structure of the given groups, but, more importantly perhaps, uses the knowledge we have about these groups to solve important and difficult problems. For instance, one might look for solutions of a certain differential equation; these solutions might form a vector space on which some relevant group acts; this would define a representation and one would try to break the vector space into irreducible invariant subspaces; combining elements of these subspaces would hand out the desired solutions.

These ideas already appear in J. Fourier's works. In his famous book [14], J. Fourier studies the heat equation and decomposes its solutions into sums of trigonometric functions. Of course, to refer to a single person when discussing pioneering work is unfair, in the sense that ideas always depend on existing knowledge (for example, Euler had already worked on trigonometric series). To summarise two centuries of mathematical research in a few pages is obviously impossible, so many important names and contributions will certainly be left out in this introduction. But our intention here is merely to outline a few important stages that have led to representation theory as we know it today, in order to see how our work fits in a global picture of this theory. Though we have made choices in the facts and people we mention as parts of this exciting mathematical story, we have nonetheless tried to be as accurate as possible. The first pages of this introduction are based on Mackey's historical survey [41] and also on [24].

The ideas of Fourier were developed by Cauchy, Poisson, Dirichlet, Riemann, Cayley and others, leading for instance to spherical harmonics in arbitrary dimensions and more applications to physics. In the late 19th century, group theory, which had appeared in the works of Galois and Abel, was placed at the heart of analysis and geometry. Klein shared this point of view and explained it in his famous lecture at Erlangen (see [26]). Lie studied the actions of continuous groups. Frobenius decided to concentrate on representations of finite groups, showing that they are unitarisable and completely reducible, studying the connections between characters and functions on the group, looking at the regular representation and so forth. Weyl studied representations of compact groups, obtaining similar results to those of Frobenius, but using more sophisticated notions, in particular the theory of integration made available by Borel, Lebesgue and Haar. With his student Peter, he proved in [47] the famous Peter-Weyl theorem, which is a generalisation of the Plancherel formula for Fourier series.

The next step involved Lie algebras. Lie had studied actions of continuous groups from an infinitesimal point of view, interpreting them on the level of Lie algebras. Lie algebras were then studied in their own right by Killing and E. Cartan, whose works led to the complete classification of finite-dimensional simple Lie algebras. E. Cartan studied representations of Lie algebras and classified the irreducible finite-dimensional ones in terms of highest weights. This gave new tools to Weyl to study representations of compact groups in further detail, obtaining for instance the formulas known as the Weyl character formula and the Weyl dimension formula.

Many scientists contributed to quantum physics in the early 20th century (Schrödinger, Planck, Heisenberg, Von Neumann and so on). Their works built up the setting we know today: a physical system corresponds to a certain Hilbert space whose 1-dimensional subspaces are the states of the system; observables correspond to self-adjoint operators, which define probability distributions. The role of Hilbert spaces here justified the general interest in unitary representations: Weyl showed how to use the exponential map to assign to a self-adjoint operator a whole family of unitary operators, thereby defining a unitary representation of the additive group of reals; diagonalising self-adjoint operators meant decomposing the unitary representation.

The study of unitary representations of arbitrary groups (including non-abelian or non-compact groups) began in the 40's, with the works of Gelfand, amongst others. The general idea arose that there existed a correspondance between groups and the set of their unitary representations (see for instance [16]) and that one could study actions of groups on topological spaces X in terms of representations of these groups on spaces of functions on X. Chevalley published an account of Lie theory (see [7]), extending existing notions and results and introducing many new ideas. In the 50's, Harish-Chandra studied the representations of real semi-simple groups, while Mackey developed the theory of induced representations (see [38], [39] and [40]). From the contributions over the years of numerous mathematicians and physicists (who appear above or are unjustly unmentioned), two guidelines emerged:

- the *orbit method* of A. A. Kirillov (see [25]), which associates unitary irreducible representations of a large class of groups to orbits of their co-adjoint action.

- the *Langland's classification theorem*: given a parabolic subgroup of a reductive group G, the irreducible admissible representations of G are in one-to-one correspondence with certain triplets (called Langland parameters); these triplets involve induced representations (an acclaimed reference for this topic is [27]).

Our work comes under the second approach. This covers a huge number of situations and many general results are well known, such as:

- The irreducible representations of compact connected semisimple Lie groups are finite-dimensional and parametrised, as Cartan showed, by finite lists of non-negative integers (the highest weights) (see [27], chapter I and IV).

- Non-trivial irreducible unitary representations of non-compact linear connected groups are infinite-dimensional (see [23], chapter 11).

- An irreducible unitary representation of a linear connected reductive group, when restricted to a certain maximal compact subgroup, decomposes into a Hilbert sum of subrepresentations, each of which consists of a finite direct sum of equivalent irreducible representations which are finite-dimensional (this result is due to the Peter-Weyl theorem and to the works of Harish-Chandra - see [27], Section 2 of Chapter VIII); in other words, irreducibility plus unitarity imply admissibility.

- To discuss suitable representations, Harish-Chandra introduced $(\mathfrak{g}, K)$-modules (a short and efficient presentation of these modules can be found in chapter 2 of [52]). He obtained a remarkable theorem, of which we give a stronger version (the *subrepresentation theorem*) proved by Casselman and Miličić in [5]: consider the Lie algebra $\mathfrak{g}$ of a connected reductive Lie group whose center is finite (the class of group this theorem applies to is in fact larger); consider a maximal compact subgroup and a minimal parabolic subgroup; then an irreducible admissible $(\mathfrak{g}, K)$-module always embeds in a representation which is induced from some irreducible finite-dimensional one of the parabolic subgroup.

In fact, so much is known that one could almost feel that representation theory is near to complete. Which of course is not true. For one thing, understanding an object requires assumptions; changing these assumptions changes the theory; one could decide to work with other spaces than Hilbert spaces, fields of positive characteristic and so forth (we will not go in these directions). Then, above results deal with irreducible representations, which comes with the question: what can be said about reducible ones ? Besides, there are a great many things one can wish to investigate, in particular:

- Give tangible examples of the representations of Langland's classification: the Stone von Neumann theorem (in the 30's) classified unitary representations of the Heisenberg group, those of $\mathrm{SL}(2,\mathbb{R})$ were studied by Gelfand, Naimark and Bargamnn in the 40's (see [15] and [1]), actions of symplectic groups were studied in the 60's and lead to the metaplectic representation (one can find an account of this in [13], along with representations of the Heisenberg group); more examples will be discussed further on. The point of examples is of course to justify the interest one might have in a theory and also to give some insight on various areas it is connected to; it so happens that representation theory is connected to many.

- Study unitary representations, without assuming irreducibility, looking at the way they break into irreducibles and what they become when restricted to various subgroups. Many people work in this field: amongst many other results, one has Kostant's general branching theorem, Howe's study of representations of the groups $\mathrm{O}(p,q)$, $\mathrm{U}(p,q)$ and $\mathrm{Sp}(p,q)$, works of T. Kobayashi and Pevzer in terms of differential operators (see [36])and so forth.

- Compute various features of the representations given by Langland's classification and determine the ones that are unitary; many people have worked in this direction (Zuckermann, Adams, Vogan...).

- Make use of connections there might exist between representations and other objects. The present work investigates the appearance of special functions in representation theory. The fact that such functions do appear is hardly a discovery. They can emerge from the computation of matrix coefficients: for example, the role of Bessel functions is shown in chapter 4 of [50] (section 4.1) for some action of

ISO(2) on smooth functions on the circle. They can appear as spherical functions of Riemannian symmetric spaces. Also, on an infinitesimal level, enveloping algebras correspond to differential operators that lead to differential equations that sometime characterise well known special functions. This will be our approach: we will use K-types and the Casimir operator to reach hypergeometric and Bessel equations. Special functions also play a key role in the construction of symmetry breaking operators (see [36] and [29]).

The interesting thing about special functions is that they are solutions of particular differential equations and that they can be expressed in various ways: as series, integrals and through recursive methods. Under certain conditions, they form orthonormal bases of certain functional Hilbert spaces.

Bearing in mind the role of special functions in representation theory, here are a number of considerations that guided us towards the setting we have chosen for our work.

First, the subrepresentation theorem justified, to us, the choice of induced representations. Then, to study the appearance of explicit objects, it seems appropriate to select concrete groups: the notion of reductive group is a generalisation of a number of matrix groups that are preserved by some involution; why not choose straight away the explicit matrix groups? Which brings us to the actual choice of a matrix group.

The unitary dual of $\mathrm{GL}(n,\mathbb{R})$ has been studied in great detail. For instance, it is proved in [49] that every irreducible unitary representation of these groups can be defined by some induction process involving tensor products of irreducible unitary representations of subgroups $\mathrm{GL}(m,\mathbb{R})$ of lower dimensions. Looking at matrix subgroups of $\mathrm{GL}(n,\mathbb{R})$, the classical ones are defined with respect to non-degenerate bilinear forms they leave invariant. Representations of $\mathrm{O}(n)$ and $\mathrm{U}(n)$ are thoroughly understood (in terms of highest weights, because these groups are compact – see for instance [53] and more specifically [44] for $\mathrm{Sp}(n)$); we shall come back to them in a moment. Representations of the subgroups $\mathrm{O}(p,q)$ and $\mathrm{U}(p,q)$ have been studied extensively. Submodules of some representations of these groups on spaces of homogeneous functions are described in [22]. In a series of papers ([32], [33] and [34]), Kobayashi and Ørsted studied the minimal unitary representation of $\mathrm{O}(p,q)$, describing the minimal K-types in terms of special functions. One can look at subgroups that preserve anti-symmetric/anti-Hermitian forms; this is studied in

[22], but also in [35] and [8], amongst other works.

These works are based on the structure of K-types, which comes from the admissibility assumption of the given representations. So, although the irreducible representations of $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ are well known, they are an essential ingredient in the general study of representations of reductive groups. Added to which it is well known that special functions are related to them via spherical harmonics.

So we are drawn to the study of K-types of unitary parabolically induced representations of classical linear subgroups. But why choose $\mathrm{Sp}(n,\mathbb{C})$?

In [35] and [8], the authors studied degenerate principle series of $\mathrm{Sp}(n,\mathbb{R})$ and $\mathrm{Sp}(n,\mathbb{C})$ that are geometrically realised on spaces of functions defined on $\mathbb{R}^{2n}\setminus\{0\}$ and $\mathbb{C}^{2n}\setminus\{0\}$. Restricting these functions to the unit sphere connects them to spherical harmonics. The cases of $\mathrm{Sp}(n,\mathbb{R})$ and $\mathrm{Sp}(n,\mathbb{C})$, are different, but share a same feature: scalar multiplication by numbers of modulus 1 guides us to irreducibles. To be more specific, any invariant subspace takes one a step forward towards a K-type decomposition. The direct sums below illustrate this. In these sums, the parameters correspond to the characters of $\mathrm{O}(1)$ (resp. $\mathrm{U}(1)$) and the summands are invariant subspaces under the left action of $\mathrm{O}(4n)$ (resp. $\mathrm{U}(2n)$):

- $L^2(S^{4n-1}) = L^2_{\text{even}}(S^{4n-1}) \oplus L^2_{\text{odd}}(S^{4n-1})$;
- $L^2(S^{4n-1}) = \widehat{\bigoplus_{\delta\in\mathbb{Z}}} L^2_\delta(S^{4n-1})$, where each space $L^2_\delta(S^{4n-1})$ is the Hilbert sum of all subspaces of spherical harmonics of homogeneous degrees (α,β) such that $\delta=\beta-\alpha$.

So two actions seem to rule the decomposition of $L^2(S^{4n-1})$ we are looking for: left action of matrices and scalar multiplication by numbers of modulus 1. This diagram captures the situation:

$$\begin{array}{ccccc} \mathrm{O}(4n) & \curvearrowright & L^2(S^{4n-1}) & \curvearrowleft & \mathrm{O}(1) \\ \cup & & & & \cap \\ \mathrm{U}(2n) & \curvearrowright & L^2(S^{4n-1}) & \curvearrowleft & \mathrm{U}(1) \end{array}$$

Of course, one instinctively wants to add a line with:

- the group of linear isometries of $\mathbb{H}^n$, which is isomorphic to $\mathrm{Sp}(n)$ and thus imbeds in $\mathrm{U}(2n)$, which itself imbeds in $\mathrm{O}(4n)$;

- some scalar type of action of the group of unit quaternions (which is isomorphic to $\mathrm{Sp}(1)$).

In other words, the line would be:

$$\mathrm{Sp}(n) \curvearrowright L^2(S^{4n-1}) \curvearrowleft \mathrm{Sp}(1)$$

This suggests an interesting interaction between the left action of $\mathrm{Sp}(n)$ and a scalar-type action of the group $\mathrm{Sp}(1)$, which, unlike $\mathrm{O}(1)$ and $\mathrm{U}(1)$, is not abelian.
Putting all these considerations together, the central question of this work is:

Via the left action of $\mathrm{Sp}(n)$ *and the right action of* $\mathrm{Sp}(1)$*, how do special functions connect to* K*-types of the degenerate principal series of* $\mathrm{Sp}(n,\mathbb{C})$*?*

Remark: what we are trying to do is generalise spherical harmonics by changing fields, choosing $\mathbb{H}$ instead of $\mathbb{R}$ or $\mathbb{C}$.

It must be pointed out that the scalar type action of $\mathrm{Sp}(1)$ is not as simple as one might expect: non-commutativity of the field of quaternions requires a choice of sides for scalar multiplication (we choose right multiplication) and this multiplication in the quaternion setting must then be translated back into the complex setting.
Let us point out that principal series of classical groups over the field of quaternions are discussed in [18], [46] and [9].
Generally speaking, works such as [31], [21], [30], [20] and [35] serve as guidelines to obtain special functions: we make use of differential operators and Fourier transforms.

Throughout this work, we consider an integer $n \geq 2$ (the minimal value 2 comes from the fact that we wish to work with a non-minimal parabolic subgroup).
In chapter 3, we make the structure of $L^2(S^{4n-1})$ explicit with respect to both the left action of $\mathrm{Sp}(n)$ and the right action of $\mathrm{Sp}(1)$. Although this double K-type structure appears in [22], the credit one can give to the present work is to offer a personal treatment of the subject, totally explicit and self-contained. Moreover, we bring forward polynomials which are invariant under both actions and we show how to compute them. They are the ingredient that takes us to hypergeometric functions. The main results we obtain in chapter 3 are:

Theorem A.

- *With respect to the left action of* $\mathrm{Sp}(n)$*, the space* $L^2(S^{4n-1})$ *decomposes into the following Hilbert sum:*

$$L^2(S^{4n-1}) = \widehat{\bigoplus_{k\in\mathbb{N}}} \bigoplus_{\substack{(\alpha,\beta)\in\mathbb{N}\times\mathbb{N} \\ \alpha+\beta=k}} \bigoplus_{\gamma=0}^{min(\alpha,\beta)} V_\gamma^{\alpha,\beta}$$

In this sum, $V_\gamma^{\alpha,\beta}$ *is the irreducible invariant subspace generated by the left translates of the polynomial* $P_\gamma^{\alpha,\beta}$ *(in fact its restriction to* S^{4n-1}*) which is defined by:*

$$P_\gamma^{\alpha,\beta}(z,w) = w_1^{\alpha-\gamma}\overline{z_1}^{\beta-\gamma}(w_2\overline{z_1} - w_1\overline{z_2})^\gamma$$

Here, (z,w) *denote the coodinates on* $\mathbb{C}^n \times \mathbb{C}^n \simeq \mathbb{C}^{2n} \simeq \mathbb{R}^{4n}$*, taking* $z = (z_1,\dots,z_n)$ *and* $w = (w_1,\dots,w_n)$*.*

- *With respect to the right action of* $\mathrm{Sp}(1)$*, the space* $L^2(S^{4n-1})$ *decomposes into the following Hilbert sum:*

$$L^2(S^{4n-1}) = \widehat{\bigoplus_{k\in\mathbb{N}}} \bigoplus_{\gamma=0}^{E\left(\frac{k}{2}\right)} d_\gamma^k W_\gamma^k$$

where $E\left(\frac{k}{2}\right)$ *denotes the integer part of* $\frac{k}{2}$*. In this sum,* W_γ^k *is a finite-dimensional irreducible invariant subspace, which contains* $P_\gamma^{k-\gamma,\gamma}$ *as a highest weight vector;* d_γ^k *is a positive integer and there are* d_γ^k *invariant subspaces that are equivalent to* W_γ^k*.*

The following statement corresponds to theorem 3.18:

Theorem B. *Consider an integer* $k \in \mathbb{N}$ *and denote by* $\mathrm{H}^k(\mathbb{R}^{4n})$ *the space of harmonic polynomials of* $4n$ *real variables with complex coefficients and which are homogeneous of degree* k*. Denote by* $1\times\mathrm{Sp}(n-1)$ *the group of matrices of* $\mathrm{Sp}(n)$ *that can be written* $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$ *with* $A \in \mathrm{Sp}(n-1)$*.*
Then if k *is even, there exists a unique element of* $H^k(\mathbb{R}^{4n})$ *(up to a constant) (we see this element as a polynomial of* $2n$ *complex variables together with their conjugates) which is invariant under the left action of* $1\times\mathrm{Sp}(n-1)$ *and also under the right action of* $\mathrm{Sp}(1)$*. Taking* $\alpha = \frac{k}{2}$*, this polynomial belongs to* $V_\alpha^{\alpha,\alpha}$*. We call it a bi-invariant polynomial.*

Remark: we show in Section 3.2.3 how to compute this polynomial.

These theorems involve standard theory of Lie groups and Lie algebras and basic, but perhaps less standard, knowledge of quaternions. Chapter 1 details all the necessary background and in the process determines and computes many things that are needed further on.

Chapter 2 introduces the actual degenerate principle series we are interested in. It is obtained by parabolic induction which we make explicit: choice of parabolic subgroup, choice of character and geometric realisations. We obtain a two parameter family of representations of $G = \mathrm{Sp}(n, \mathbb{C})$, denoted by $\pi_{i\lambda,\delta}$ with $(\lambda, \delta) \in \mathbb{R} \times \mathbb{Z}$ and defined as follows (denoting by $V_{i\lambda,\delta}$ the carrying space of $\pi_{i\lambda,\delta}$):

- Consider the space $V^0_{i\lambda,\delta}$ of all functions $f \in C^0(\mathbb{C}^N \setminus \{0\})$ such that for all $c \in \mathbb{C}$ and $x \in \mathbb{C}^N \setminus \{0\}$:
$$f(cx) = \left(\frac{c}{|c|}\right)^{-\delta} |c|^{-i\lambda - N} f(x)$$
Consider the left action of G on this space, defined by
$$\pi_{i\lambda,\delta}(g)f(x) = f(g^{-1}x)$$
for all $(g, f, x) \in G \times V^0_{i\lambda,\delta} \times (\mathbb{C}^N \setminus \{0\})$.

- $V_{i\lambda,\delta}$ and $\pi_{i\lambda,\delta}$ are then obtained by completion of $V^0_{i\lambda,\delta}$ with respect to the norm $\|\cdot\|$ defined by:
$$\|f\|^2 = \int_{S^{2N-1}} |f(x)|^2 dx$$

Representations $\pi_{i\lambda,\delta}$ are unitary.

As mentioned earlier on, our interest lies in the link one can establish between representations and special functions. What makes the choice of unitary groups all the more appropriate for us is that it is well known how to associate special functions to irreducible spaces of spherical harmonics, by adding an additional invariance constraint (defining zonal functions) and using the Laplace operator (see [12], sections 3 and 4). To use similar methods with symplectic groups requires to slightly adjust this additional constraint; this is precisely the point of the right action of Sp(1). The special functions we end up with, as for the orthogonal groups, are hypergeometric functions, but this only works in specific K-types. These methods are developed in chapter 4. Here is a partial statement of the main result (see theorem 4.8 for more details):

Theorem C. *Given $\alpha \in \mathbb{N}$, the previous theorem shows us how to explicitely compute in $V_\alpha^{\alpha,\alpha}$ a function (unique up to a constant) which can, after a suitable reduction of variables, be written as a solution of the following hypergeometric equation:*

$$u(1-u)\varphi'' + 2(1-nu)\varphi' + [\alpha^2 + (2n-1)\alpha]\varphi = 0$$

Another way to obtain special functions is to apply Fourier transforms, simply because special functions can be expressed in many ways, ways that often involve integrals. Using a partial Fourier transform, the authors of [35] define a non-standard model of $\pi_{i\lambda,\delta}$ which enables them to describe constant functions on the sphere (elements of the lowest K-type) in terms of Bessel functions. We generalise this, not only by adapting it to a complex setting, but also by dealing with a much wider set of K-types, finding explicit formulas for the non-standard forms of their highest weight vectors. This is our main result. Before we actually get to it, let us define the partial Fourier transform $\mathcal{F}$ (on $L^2(\mathbb{C}^{2m+1})$) that is required in the non-standard model (or *non-standard picture*). It is defined for $f \in L^1(\mathbb{C}^{2m+1}) \cap L^2(\mathbb{C}^{2m+1})$ by:

$$\mathcal{F}(f)(s,u,v) = \int_{\mathbb{C}\times\mathbb{C}^m} f(\tau,u,\xi)\, e^{-2i\pi \mathrm{Re}(s\tau + \langle v,\xi\rangle)}\, d\tau d\xi$$

where (s,u,v) denote the coordinates of $\mathbb{C}\times\mathbb{C}^m\times\mathbb{C}^m \simeq \mathbb{C}^{2m+1}$. We can now state our main theorem, namely Theorem 5.12, but in a slightly different and lighter way:

Theorem D (Non-standard picture and Bessel functions)**.** *Let $n \in \mathbb{N}$ be such that $n \geq 2$; set $N = 2n$ and $m = n-1$. Let $k \in \mathbb{N}$ and any pair $(\alpha,\beta) \in \mathbb{N}^2$ be such that $\alpha+\beta = k$; set $\delta = \beta - \alpha$. For $\lambda \in \mathbb{R}$, consider the representation $\pi_{i\lambda,\delta}$ of the group* $\mathrm{Sp}(n,\mathbb{C})$ *and the following function:*

$$\begin{array}{rccc} f & : & \mathbb{C}^n\times\mathbb{C}^n \setminus \{(0,0)\} & \longrightarrow & \mathbb{C} \\ & & (z,w) & \longmapsto & (\|z\|^2+\|w\|^2)^{-\frac{i\lambda+k}{2}-n}\, w_1^\alpha\, \overline{z_1}^\beta \end{array}$$

where w_1 denotes the first coordinate of w. Then f generates a finite-dimensional subspace under the action of $\pi_{i\lambda,\delta}\big|_{\mathrm{Sp}(n)}$; f is a highest weight vector of this subspace and the non-standard form of f, meaning the function $\mathcal{F}\Big((\tau,u,\xi) \longmapsto f(1,u,2\tau,\xi)\Big)$, assigns to all

(s,u,v) in $\mathbb{C}\times\mathbb{C}^m\times\mathbb{C}^m$ such that $v\neq 0$ and $s\neq 0$ the following value:

$$\frac{(-i\,\overline{s})^{\alpha}\,\pi^{i\lambda+\beta+n}}{2^{\frac{i\lambda+k}{2}+1}\,\Gamma(\frac{i\lambda+k}{2}+n)}\left(\frac{\sqrt{|s|^2+4\|v\|^2}}{\pi\sqrt{1+\|u\|^2}}\right)^{\frac{i\lambda+\delta}{2}} K_{\frac{i\lambda+\delta}{2}}\left(\pi\sqrt{1+\|u\|^2}\sqrt{|s|^2+4\|v\|^2}\right)$$

Remark: we will see in Chapter 2 that $\pi_{i\lambda,\delta}$ is induced by a character of a parabolic subgroup of G whose nilradical is isomorphic to the complex Heisenberg group.

Chapter 5 ends with two interesting observations:

- there exists a simple differential operator that connects the non-standard forms of certain highest weight vectors (Theorem 5.13);
- the formula given in Theorem D is linked to differential equations known as Emden-Fowler equations (Section 5.7.2).

Finally, we discuss what we feel has been gained with our work and what perspectives it leaves us with.

Chapter 1

Preliminaries: Lie theory and quaternions

Dans ce chapitre, nous rappelons toutes les notions dont nous avons besoin dans notre travail.

Dans le paragraphe 1.1, nous rappelons un certain nombre de généralités : actions de groupes, représentations, réductibilité, action naturelle d'un groupe linéaire sur l'espace ambiant, action à gauche sur les fonctions, groupes et algèbres de Lie, crochet et exponentielle de matrices, groupes linéaires connexes réductifs/semi-simples, représentations unitaires et continues, exemple de la représentation régulière gauche, dual unitaire d'un groupe, vecteurs C^∞ et différentielle d'une représentation en l'élément neutre. On explique enfin que le théorème de Peter-Weyl est à la base de la notion de K-type et d'un théorème d'Harish-Chandra qui peut se formuler ainsi :

Soient G un groupe linéaire connexe réductif, K un sous-groupe maximal compact K et π une représentation unitaire irréductible de G. Alors la restriction $\pi\Big|_K$ de π au sous-groupe K se décompose en une somme hilbertienne de sous-représentations irréductibles de dimensions finies :

$$\pi|_K \cong \sum_{\tau\in\widehat{K}}^{\oplus} n_\tau \tau$$

où, pour tout élément τ de $\widehat{K}$, n_τ désigne un entier tel que $0 \leq n_\tau \leq \dim\tau < \infty$; on appelle cet entier la multiplicité de τ.

La somme ci-dessus est une façon symbolique de dire que si l'on décompose la représentation $\pi|_K$ en sous-représentations que l'on regroupe par classes d'équivalence, alors n_τ d'entre elles appartiennent à la classe τ. Les classes d'équivalence τ telles que $n_\tau \neq 0$ sont appelées K-types de π. Cette décomposition est une sorte de signature de la représentation π ; la notion qui formalise cela est celle d'équivalence infinitésimale, mais nous n'en dirons pas plus (on peut consulter [27] à ce sujet, paragraphe 2 du chapitre VIII et paragraphe 1 du chapitre IX).

Dans le paragraphe 1.2, nous définissons le groupe symplectique $G = \mathrm{Sp}(n, \mathbb{C})$, son algèbre de Lie $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$, le sous-groupe compact maximal $K = \mathrm{Sp}(n)$ de G et son algèbre de Lie $\mathfrak{k} = \mathfrak{sp}(n)$.

Dans le paragraphe 1.3, nous présentons des éléments de la théorie de Lie sur laquelle s'appuie notre travail, mais que nous appliquons directement à $\mathfrak{g}$ et à G. Listons ci-dessous les points évoqués.

- En termes de structure : sous-algèbre de Cartan, racines et espaces propres correspondants. Les racines sont des combinaisons linéaires des formes linéaires L_r qui associent à une matrice diagonale son $r^{\text{ième}}$ coefficient diagonal. On peut voir une telle combinaison comme un n-uplet d'entiers et définir un produit scalaire sur les combinaisons linéaires en question en appliquant simplement le produit scalaire euclidien aux n-uplets. On définit enfin les racines positives, en explicitant la demi-somme des racines positives, qui est la forme linéaire ρ_K donnée par la formule (1.3).

- Ensuite nous montrons comment complexifier une représentation de $\mathfrak{k}$ pour obtenir une représentation de $\mathfrak{g}$ et déterminer les poids (qu'on peut aussi voir comme des n-uplets d'entiers), les sous-espaces propres correspondants, les vecteurs de plus hauts poids et les espaces invariants irréductibles qu'ils engendrent (sous l'effet de la représentation considérée) ; la dimension de ces espaces se calcule avec la formule de Weyl (1.4).

- Nous introduisons ensuite l'opérateur de Casimir de l'algèbre de Lie enveloppante de $\mathfrak{k}$, avec la définition d'une base de matrices, des calculs d'exponentielles avec ces matrices, la définition d'un produit scalaire et d'une base orthonormée et enfin le calcul de la valeur propre de cet opérateur correspondant à un sous-espace irréductible invariant par une représentation donnée de K (voir proposition 1.6).

Dans le chapitre 1.4, nous donnons une définition des quaternions, en les envisageant comme des couples de nombres complexes et en précisant les notions de conjugué et de module. Nous mettons ensuite en place l'algèbre linéaire quaternionique, insistant sur le choix important de la multiplication scalaire à droite et en montrant comment identifier des matrices de $\mathrm{M}(n, \mathbb{H})$ avec des matrices de $\mathrm{M}(2n, \mathbb{C})$.

Enfin, dans le chapitre 1.4.4, grâce aux identifications évoquées ci-dessus, on ré-interprète les matrices de $\mathrm{Sp}(n)$ rencontrées précédemment en termes de matrices quaternioniques, en refaisant des calculs d'exponentielles et en démontrant une propriété utile sur les matrices triangulaires par blocs (proposition 1.14).

1.1 Basic definitions and properties

Given a set X, denote by $\mathrm{Bij}(X)$ the set of bijections from X onto X; it is a group with respect to composition. An *action* of a group G on X is a group homomorphism from G into $\mathrm{Bij}(X)$.

Given a vector space V, a *representation* of a group G on V is an action π of G on V such that all maps $\pi(g)$ are linear isomorphisms of V. The space V is the *carrying space* of π; the *dimension* of π is the dimension of V (possibly infinite). A subspace W of V is *invariant* (or *stable*) under the action of π if for all $g \in G$: $\pi(g)W \subseteq W$; in this case, the restriction $\pi|_W$ is a *subrepresentation* of π. The representation π is *algebraically irreducible* if it has no invariant subspaces other than $\{0\}$ and V itself; in this case we also say that V is irreducible. An invariant subspace W of V which is irreducible is called an *irreducible of* V (we will often say *component* instead); the corresponding subrepresentation is called an irreducible of the initial one.

Take $\mathbb{K}$ to be the field $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and denote by V the vector space $\mathbb{K}^n$. Consider a subgroup G of $\mathrm{GL}(n, \mathbb{K})$. The *natural action* of G on V is simply defined by standard matrix multiplication: a matrix $g \in G$ assigns to an element $x \in V$ the element $gx \in V$. Now consider a subset S of V which is stable under the natural action of G and the complex vector space $\mathcal{F}(S, \mathbb{C}) = \{f : S \longrightarrow \mathbb{C}\}$. Because S is stable under the natural action of G, given $f \in \mathcal{F}(S, \mathbb{C})$ and $g \in G$, one can define a new element of $\mathcal{F}(S, \mathbb{C})$:

$$\begin{array}{rccc} L(g)f: & S & \longrightarrow & \mathbb{C} \\ & x & \longmapsto & f\left(g^{-1}x\right) \end{array}$$

By varying f, we define a bijection:

$$\begin{array}{rccc} L(g): & \mathcal{F}(S,\mathbb{C}) & \longrightarrow & \mathcal{F}(S,\mathbb{C}) \\ & f & \longmapsto & L(g)f \end{array}$$

This induces another map:

$$\begin{array}{rccc} L: & G & \longrightarrow & \mathrm{Bij}\,(\mathcal{F}(S,\mathbb{C})) \\ & g & \longmapsto & L(g) \end{array}$$

This map L is an action of G on $\mathrm{Bij}\,(\mathcal{F}(S,\mathbb{C}))$. Because each $L(g)$ is in fact an operator of $\mathcal{F}(S,\mathbb{C})$, L is a representation of G on $\mathcal{F}(S,\mathbb{C})$; we will simply call it the *left action of* G *on* $\mathcal{F}(S,\mathbb{C})$. We say that an element f of $\mathcal{F}(S,\mathbb{C})$ is *left-invariant* if f is invariant under the left action of G, meaning that $L(g)f = f$ for all $g \in G$.

Given a subspace $\mathcal{F}'$ of $\mathcal{F}(S,\mathbb{C})$ which is stable under the left action of G, the map

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Bij}\,(\mathcal{F}') \\ g & \longmapsto & L(g)\big|_{\mathcal{F}'} \end{array}$$

is of course also a representation; it will be referred to as the *left action of* G *on* $\mathcal{F}'$ and also denoted by L.

According to the kind of spaces one is interested in, some additional structure is necessary to be technically able to study group representations on those spaces. Our setting will be that of Lie groups acting on complex Hilbert spaces.

A *Lie group* G is a group that has the structure of a smooth manifold such that multiplication and inversion are smooth. For instance, the group $\mathrm{GL}(n,\mathbb{K})$ and all its closed subgroups are (linear) Lie groups (see Theorem 0.15 in [28]); when connected and stable under conjugate transpose, taking $\mathbb{K}$ to be $\mathbb{R}$ or $\mathbb{C}$, they are called *linear connected reductive groups* and if their center is also finite they are *linear connected semisimple groups.*

Being a manifold, G has tangent spaces at all of its elements. In particular, denote by $\mathfrak{g}$, or $\mathrm{Lie}(G)$, the tangent space at the identity element. It can be shown that $\mathfrak{g}$ is a *Lie algebra* with respect to a bilinear skew-symmetric product called the *commutator*, that involves a general definition of the exponential map. We do not enter the details here, because all we need to know in this work is that when G is a linear Lie group:

- The commutator of its Lie algebra $\mathfrak{g}$ is just the usual commutator $[\cdot,\cdot]$ of matrices, defined by $[X,Y] = XY - YX$; this in fact defines the *adjoint representation* ad of $\mathfrak{g}$: $\mathrm{ad}(X)Y = [X,Y]$.

- The exponential map $\exp : \mathfrak{g} \longrightarrow G$ just assigns to each $X \in \mathfrak{g}$ the usual matrix exponential $\exp X = I + X + \frac{X^2}{2} + \frac{X^3}{3!} + \frac{X^4}{4!} + \dots$.

The choice of Hilbert spaces is not a random choice: the scalar product enables one to use orthogonality to study invariant subspaces, from the corresponding norm arise continuity and differentiability, unitarity of operators have led to extensive results, quantum theory is based on Hilbert spaces, finite-dimensional spaces can always be turned into Hilbert spaces and so on.
So consider a Lie group G and a Hilbert space H.
From now on, when considering a representation π of G on a Hilbert space H, we automatically assume that:

- the linear isomorphisms $\pi(g)$ are bounded;
- the map
$$\begin{array}{ccc} G \times H & \longrightarrow & H \\ (g, x) & \longmapsto & \pi(g)x \end{array}$$
is continuous.

The notion of reducibility is also slightly adjusted: π is said to be irreducible if there are no closed invariant subspaces of H other than $\{0\}$ and H. Also, π is said to be *unitary* if all operators $\pi(g)$ are unitary operators of H. If one considers another representation π' of G, on a Hilbert space H', then π and π' are *equivalent* if there exists a bounded linear isomorphism $f : H \longrightarrow H'$ such that f^{-1} is also bounded and such that for all $g \in G$ the following diagram commutes:
$$\begin{array}{ccc} H & \xrightarrow{\pi(g)} & H \\ \Big\downarrow f & & \Big\downarrow f \\ H' & \xrightarrow[\pi'(g)]{} & H' \end{array}$$
We often say in this case, though it is a slight abuse of language, that H and H' are equivalent. The set of equivalence classes of all irreducible unitary representations of G is called the *unitary dual* of G and denoted by $\widehat{G}$. We identify $\widehat{G}$ with any set of representatives of its equivalence classes: we might write $\tau \in \widehat{G}$, though we are actually thinking of the equivalence class of τ; we might consider a sum over all $\tau \in \widehat{G}$, implicitely meaning that we have chosen a set of representatives and that each τ is one of them.

A vector $v \in H$ is a C^∞*-vector* of a representation π if the map $g \longrightarrow \pi(g)v$ is C^∞ on G. We denote by $C^\infty(\pi)$ the set of such vectors. It can be shown that it is a dense subspace of H. In particular, one can define on this subspace $C^\infty(\pi)$ the differential at the identity element $d\pi$ of π by

$$d\pi(X)v = \frac{d}{dt}\Big|_{t=0} [\pi(\exp tX)v]$$

for all $(X, v) \in \mathfrak{g} \times C^\infty(\pi)$. Elements X are thereby seen as first order differential operators. In our work, we will in fact not need to worry about C^∞-vectors, because calculations will always be done for restrictions of representations to finite-dimensional invariant subspaces of H and it can be shown that each of them coincides with the space of C^∞-vectors it contains (see [27], Chapter III, Section 4, Corollary 3.16).

We finish this section with perhaps the most important example, then a historical and founding theorem and finally a major consequence; they all revolve around topological compactness.

This proposition follows [27] (Chapter I, Section 3, fourth example):

1.1 Proposition (Important example). *Consider a compact Lie group K, a left-invariant measure μ of K, the Hilbert space $L^2(K, \mu)$ and the left action of K on $L^2(K, \mu)$, defined (as we saw earlier on) by*

$$L(k)f(x) = f(k^{-1}x)$$

for all $(k, f, x) \in K \times L^2(K, \mu) \times K$ Then L is a continuous unitary representation of K on $L^2(K, \mu)$, called the left-regular representation of K.

Then comes a theorem that is also proved in [27] (Section 5 of Chapter I, Theorem 1.12):

1.2 Theorem (Peter-Weyl Theorem). *Let K be a compact Lie group. Let π be a unitary representation of K on a Hilbert space H. For each $\tau \in \widehat{K}$, let H_τ denote the sum of all irreducible invariant subspaces of H that define subrepresentations that are equivalent to τ (if there are none, just set $H_\tau = \{0\}$). Denote by n_τ the lowest number of such subspaces one can choose to ensure that they add up to H_τ (if $H_\tau = \{0\}$, just set $n_\tau = 0$). This number n_τ is called the multiplicity of τ. We point out that $n_\tau \in \mathbb{N} \cup \{\infty\}$. An element τ of $\widehat{K}$ is called a K-type if $n_\tau \neq 0$. Let E_τ denote the orthogonal projection on H_τ. Then:*

1. *All representations $\tau \in \widehat{K}$ are finite-dimensional.*

2. *If two representations τ and τ' of $\widehat{K}$ are inequivalent, then:*

$$E_\tau E_{\tau'} = E_{\tau'} E_\tau = 0$$

3. *Every $h \in H$ satisfies: $h = \sum_{\tau \in \widehat{K}} E_\tau h$.*

Another way to state Theorem 1.2 is to say that π decomposes into a sum of subrepresentations that are parametrised by $\widehat{K}$; the one that corresponds to a given parameter $\tau \in \widehat{K}$ decomposes into n_τ finite-dimensional irreducible unitary subrepresentations that are equivalent to τ; this kind of decomposition is written:

$$\pi \cong \sum_{\tau \in \widehat{K}}^{\oplus} n_\tau \tau$$

Theorem 1.2 can also be used to study representations of groups that are not compact, simply by restricting to compact subgroups. Under certain assumptions, multiplicities are finite:

1.3 Theorem. *Suppose G is a linear connected reductive group and let K be a maximal compact subgroup of G. Consider an element π of $\widehat{G}$. Apply Theorem 1.2 to the restriction $\pi|_K$ of π to the subgroup K:*

$$\pi|_K \cong \sum_{\tau \in \widehat{K}}^{\oplus} n_\tau \tau$$

Then the multiplicities n_τ are finite. More accurately: $n_\tau \leq \dim\tau < \infty$.

Remarks:

- This theorem is part of Harish-Chandra's work. For a proof, one can read Section 2 of Chapter VIII in [27] (Theorem 8.1).

- In the way we have stated this theorem, we implicitely use the fact that all maximal compact subgroups of G are conjugate (see for instance [4], Chapter VII, Theorem 1.2).

1.2 Symplectic groups

First of all, it will be convenient to split matrices into blocks. A $2n \times 2n$ complex matrix g, that is, an element of $\mathrm{M}(2n, \mathbb{C})$, will be split into four $n \times n$ complex matrices A, B, C, D:

$$g = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

In particular, we shall be interested in the matrix:

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

We denote by ω the *standard symplectic form* on $\mathbb{C}^{2n}$, defined by:

$$\omega(X, X') = {}^tXJX' = \langle y, x' \rangle - \langle x, y' \rangle$$

where $X = (x, y)$ and $X' = (x', y')$ belong to $\mathbb{C}^n \times \mathbb{C}^n$.
A $2n \times 2n$ complex matrix g is said to be *symplectic* if

$${}^tgJg = J$$

This is equivalent to saying that g preserves ω, meaning:

$$\forall (X, X') \in \mathbb{C}^{2n} \times \mathbb{C}^{2n} : \ \omega(gX, gX') = \omega(X, X')$$

The group of such matrices is the *complex symplectic group* (or just *symplectic group*), denoted by:

$$G = \mathrm{Sp}(n, \mathbb{C})$$

It is straightforward to check that a $2n \times 2n$ complex matrix $g = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ is symplectic if and only if:

$${}^tAB = {}^tBA \ , \ {}^tCD = {}^tDC \text{ and } {}^tAD - {}^tBC = I \qquad (1.1)$$

The group G is a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$ and is therefore a Lie group. It is in fact a linear connected semisimple group. Its Lie algebra is:

$$\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C}) = \left\{ X \in \mathrm{M}(2n, \mathbb{C}) \ / \ {}^tXJ + JX = 0 \right\}$$

or, expressed otherwise:

$$\mathfrak{g} = \left\{ X = \begin{pmatrix} A & C \\ B & -{}^tA \end{pmatrix} \in \mathrm{M}(2n, \mathbb{C}) \ / \ B \text{ and } C \text{ are symmetric} \right\}$$

Define $K = \mathrm{Sp}(n) = \mathrm{Sp}(n, \mathbb{C}) \bigcap \mathrm{U}(2n)$ (as usual $\mathrm{U}(2n)$ denotes the group of unitary complex $2n \times 2n$ matrices). Unitarity and preservation of the standard symplectic form imply immediately that the group K consists of all matrices of $\mathrm{U}(2n)$ which can be written

$$\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}$$

where of course $\overline{M}$ denotes the (complex) conjugate of a given matrix M. It is well known (see for instance Theorem 6.5.2 in [42]) that K is compact and simply connected (therefore connected). It can be shown that K is a *maximal compact subgroup*, via what is called the Cartan decomposition of $\mathfrak{g}$ (which we say nothing of, just referring the reader to [27], Section 1, Proposition 1.2); we will often refer to K as the maximal compact subgroup of G, although such a subgroup is not unique. Being a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$, K is itself a Lie group; denote by $\mathfrak{k} = \mathfrak{sp}(n)$ its Lie algebra (it is a Lie subalgebra of $\mathfrak{g}$). This Lie algebra consists of all skew-hermitian (skew for short) elements of $\mathfrak{g}$:

$$\mathfrak{k} = \left\{ X = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \in \mathrm{M}(2n, \mathbb{C}) \ / \ A \text{ is skew and } B \text{ is symmetric} \right\}$$

1.3 Lie theory applied to $\mathrm{Sp}(n)$ and $\mathfrak{sp}(n)$

1.3.1 Roots

Let us point out that the complexification of $\mathfrak{k}$ is $\mathfrak{g}$ and that $\mathfrak{g}$ is a complex semisimple Lie algebra (for a definition of semisimplicity, we refer the reader to [48], Chapter 2).

We summarise some well known facts below, applying the general theory of complex semisimple Lie algebras to the particular case of $\mathfrak{g}$. We follow [27] (Chapter IV) and [48] (Chapter 2), using the explicit setting of $\mathfrak{k} = \mathfrak{sp}(n)$ and $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$.

Let $\mathfrak{h}$ be the usual Cartan subalgebra of $\mathfrak{g}$ consisting of diagonal elements of $\mathfrak{g}$. If r belongs to $\{1, ..., n\}$, denote by L_r the linear form that assigns to an element of $\mathfrak{h}$ its r^{th} diagonal term. Denote by Δ_K the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. By definition, if ν is a root then the corresponding *root space* is:

$$\mathfrak{g}_\nu = \{X \in \mathfrak{g} / \forall H \in \mathfrak{h} : \ \mathrm{ad}(H)(X) = \nu(H)X\}$$

The set Δ_K consists of the following linear forms, where r and s denote integers that belong to $\{1, ..., n\}$:

- $L_r - L_s$ with $r < s$;
- $-L_r + L_s$ with $r < s$;
- $L_r + L_s$ with $r < s$;
- $-L_r - L_s$ with $r < s$;
- $2L_r$;
- $-2L_r$.

Because $\mathfrak{g}$ is semisimple, the root spaces are complex one-dimensional subspaces. One can check that the root spaces of the above roots are respectively generated by the following *root elements*:

- $U^+_{r,s} = E_{r,s} - E_{n+s,n+r}$ (with $r < s$)
- $U^-_{r,s} = E_{s,r} - E_{n+r,n+s}$ (with $r < s$)
- $V^+_{r,s} = E_{r,n+s} + E_{s,n+r}$ (with $r < s$)
- $V^-_{r,s} = E_{n+s,r} + E_{n+r,s}$ (with $r < s$)
- $D^+_r = E_{r,n+r}$
- $D^-_r = E_{n+r,r}$

where $E_{r,s}$ are elementary matrices. We will write

$$H_r = E_{r,r} - E_{n+r,n+r}$$

for each integer $r \in \{1, ..., n\}$. Let us point out that

$$\{H_r\}_{r\in\{1,...,n\}}$$

is a $\mathbb{C}$-basis of $\mathfrak{h}$ and that

$$\{H_r, D_r^+, D_r^-\}_{r\in\{1,\ldots,n\}} \cup \{U_{r,s}^+, U_{r,s}^-, V_{r,s}^+, V_{r,s}^-\}_{r,s\in\{1,\ldots,n\},r<s} \tag{1.2}$$

is a $\mathbb{C}$-basis of $\mathfrak{g}$.

Denote by $\mathfrak{h}_\mathbb{R}$ the real form of $\mathfrak{h}$ that consists of real matrices of $\mathfrak{h}$. If r belongs to $\{1, ..., n\}$, the restriction $L_r\Big|_{\mathfrak{h}_\mathbb{R}}$ is again denoted by L_r, for simplicity. Denote by $\mathfrak{h}'_\mathbb{R}$ the space of real-valued linear forms defined on $\mathfrak{h}_\mathbb{R}$. Any element σ of $\mathfrak{h}'_\mathbb{R}$ can be written $\sigma = \sum_{r=1}^n l_r L_r$, where each l_r is a real number; the linear form σ is said to be an *integral form* if each l_r is an integer. The space $\mathfrak{h}'_\mathbb{R}$ is endowed with a natural Euclidean product $< \cdot, \cdot >$ defined by:

$$< \sigma, \sigma' >= \sum_{r=1}^n l_r l'_r$$

We denote by $|\cdot|$ the corresponding norm.

Remark: in [27] (Chapter IV, Section 2), the inner product used on $\mathfrak{h}'_\mathbb{R}$ corresponds to $2 < \cdot, \cdot >$. The fact that we have omitted the coefficient 2 has no consequence in what we discuss further on: for instance, in the Weyl dimension formula (see next section) all the coefficients 2 just cancel out.

The restrictions of the various roots to $\mathfrak{h}_\mathbb{R}$ are real-valued and therefore belong to $\mathfrak{h}'_\mathbb{R}$ (in fact they are integral forms). They also completely define the corresponding roots; this is why we shall think of roots as elements of $\mathfrak{h}'_\mathbb{R}$ (without any loss of information).

An order can be chosen on Δ_K such that the positive roots are the linear forms:

- $L_r - L_s$ with $r < s$;
- $L_r + L_s$ with $r < s$;
- $2L_r$.

Denote by Δ_K^+ the set of positive roots and by ρ_K the half sum of the positive roots. It is straightforward to check:

$$\rho_K = nL_1 + (n-1)L_2 + ... + 2L_{n-1} + L_n$$

Identifying linear combinations of $L_1, \ldots, L_n$ with their coordinates in the basis $\{L_1, \ldots, L_n\}$, we will write:

$$\rho_K = (n, n-1, \ldots, 1) \tag{1.3}$$

1.3.2 Highest weights

When studying a finite-dimensional representation π of K on a complex vector space, one can study the infinitesimal action given by $d\pi$; this is an $\mathbb{R}$-linear representation of $\mathfrak{k}$. It can always be thought of as a $\mathbb{C}$-linear representation of $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$, by extending it to $\mathfrak{g}$ in the following natural way:

$$d\pi(X + iY) := d\pi(X) + id\pi(Y)$$

for $X, Y \in \mathfrak{k}$. Correspondance between the initial representation $d\pi$ and its *complexification* (meaning its extension to $\mathfrak{g}$ - we'll also say that we have *complexified* $d\pi$) is clearly one-to-one.

For a given $\mathbb{C}$-linear representation ϖ of $\mathfrak{g}$ on a complex finite-dimensional vector space V, one defines weights similarly to roots: weights are linear forms on $\mathfrak{h}$ that give the corresponding eigenvalues of joint eigenvectors of all the linear maps $\varpi(H)$, where H runs through $\mathfrak{h}$; if σ is a weight, the corresponding *weight space* (by definition it is not reduced to 0) is:

$$V_\sigma = \{v \in V / \forall H \in \mathfrak{h} : \ \varpi(H)(v) = \sigma(H)v\}$$

The structures of semisimple Lie algebras and of finite-dimensional representations of $\mathfrak{sl}(2, \mathbb{C})$ imply that the restrictions of the various weights to $\mathfrak{h}_\mathbb{R}$ are integral forms, thus belong to $\mathfrak{h}'_\mathbb{R}$. So we shall think of weights as elements of $\mathfrak{h}'_\mathbb{R}$. Then a *highest weight vector of (highest) weight* σ is a vector v that belongs to V_σ and that is cancelled by the action of all elements of $\mathfrak{n}^+$ under ϖ. Applying ϖ to v for all $X \in \mathfrak{g}$ then generates an irreducible invariant subspace $V_{irr}(v)$ of V. The linear form σ is called a highest weight because basic Lie theory shows that $\varpi|_{V_{irr}(v)}$ can have no higher weight than σ.

If v is a highest weight vector of the differential (at the unit element) of some finite-dimensional representation π of K, then basic Lie theory assures us that the same conclusion as above holds for π: applying $\pi(k)$ to v for all $k \in K$ generates the same subspace $V_{irr}(v)$. We shall say that σ is a highest weight of π (though it is in fact defined on $\mathfrak{g}$) and that v is a highest weight vector of π.

An element σ of $\mathfrak{h}'_\mathbb{R}$ is said to be *dominant* if for any positive root σ':

$$< \sigma, \sigma' > \geq 0$$

The highest weight theorem (see [27], Chapter IV, Theorem 4.28) applied to K sets a correspondence between irreducible representations of K and dominant integral forms: these are precisely the highest weights; they are the forms $\sigma = \sum_{r=1}^{n} l_r L_r$ such that $(l_1, ..., l_n) \in \mathbb{N}^n$ and $l_1 \geq l_2 \geq ... \geq l_n$ (we will identify σ and $(l_1, ..., l_n)$).
If σ is the highest weight of an irreducible representation of K, then the dimension d_σ of this representation is given by *Weyl's dimension formula* (see Theorem 4.48 of [27]):

$$d_\sigma = \frac{\prod_{\alpha \in \Delta_K^+} < \sigma + \rho_K, \alpha >}{\prod_{\alpha \in \Delta_K^+} < \rho_K, \alpha >} \tag{1.4}$$

1.3.3 Casimir operator

As we have seen, the Lie subalgebra $\mathfrak{k}$ consists of complex $N \times N$ matrices

$$\left(\begin{array}{c|c} A & -\overline{B} \\ \hline B & \overline{A} \end{array}\right)$$

such that A is a skew-Hermitian $n \times n$ matrix and B a symmetric $n \times n$ matrix.
Define (r and s denote integers that belong to $\{1, ..., n\}$):

- $A_r = iE_{r,r} - iE_{n+r,n+r}$
- $B_{r,s} = E_{r,s} - E_{s,r} + E_{n+r,n+s} - E_{n+s,n+r}$ (when $r \neq s$)
- $C_{r,s} = iE_{r,s} + iE_{s,r} - iE_{n+r,n+s} - iE_{n+s,n+r}$ (when $r \neq s$)
- $D_r = E_{n+r,r} - E_{r,n+r}$
- $E_r = iE_{n+r,r} + iE_{r,n+r}$
- $F_{r,s} = E_{n+r,s} + E_{n+s,r} - E_{r,n+s} - E_{s,n+r}$ (when $r \neq s$)
- $G_{r,s} = iE_{n+r,s} + iE_{n+s,r} + iE_{r,n+s} + iE_{s,n+r}$ (when $r \neq s$)

where, again, $E_{r,s}$ are elementary matrices.
One easily sees that:

- If $n \geq 2$ then $\{A_r, D_r, E_r\}_{r \in \{1,...,n\}} \cup \{B_{r,s}, C_{r,s}, F_{r,s}, G_{r,s}\}_{r,s \in \{1,...,n\}, r<s}$ is a basis of $\mathfrak{k}$ over $\mathbb{R}$.
- If $n = 1$ then $\{A_1, D_1, E_1\}$ is a basis of $\mathfrak{k}$ over $\mathbb{R}$.

Elements of $\mathfrak{k}$ correspond to elements of K via the exponential map. One easily obtains the following table for $n, r, s \in \{1, ..., n\}$:

Matrix M	M^2	M^3
A_r	$-I_r$	$-A_r$
D_r	$-I_r$	$-D_r$
E_r	$-I_r$	$-E_r$
$B_{r,s}$	$-K_{r,s}$	$-B_{r,s}$
$C_{r,s}$	$-K_{r,s}$	$-C_{r,s}$
$F_{r,s}$	$-K_{r,s}$	$-F_{r,s}$
$G_{r,s}$	$-K_{r,s}$	$-G_{r,s}$

where we denote by I_r the matrix $E_{r,r} + E_{n+r,n+r}$ and by $K_{r,s}$ the matrix $E_{r,r} + E_{s,s} + E_{n+r,n+r} + E_{n+s,n+s}$.

From this table we obtain the following exponentials for $t \in \mathbb{R}$:

1.4 Lemma.

- *For $r \in \{1, 2, ..., n\}$ and $M \in \{A_r, D_r, E_r\}$:*

$$\exp(-tM) = I - (\sin t)M + (\cos t - 1)I_r$$

- *For $r, s \in \{1, ..., n\}$ such as $r < s$ (here it is assumed that $n \geq 2$) and for $M \in \{B_{r,s}, C_{r,s}, F_{r,s}, G_{r,s}\}$:*

$$\exp(-tM) = I - (\sin t)M + (\cos t - 1)K_{r,s}$$

Define on $\mathfrak{k}$ the bilinear form β by:

$$\beta(X, Y) = -\frac{1}{2}\operatorname{Tr}(XY)$$

where Tr denotes the matrix trace form. One easily checks:

1.5 Lemma.

1. *The basis*

$$\mathcal{B}_{\mathfrak{k}} = \{A_r, D_r, E_r\}_{r\in\{1,\ldots,n\}} \cup \left\{\frac{B_{r,s}}{\sqrt{2}}, \frac{C_{r,s}}{\sqrt{2}}, \frac{F_{r,s}}{\sqrt{2}}, \frac{G_{r,s}}{\sqrt{2}}\right\}_{r,s\in\{1,\ldots,n\},r<s}$$

of $\mathfrak{k}$ is orthonormal with respect to β.

2. *The symmetric bilinear form β is real-valued and positive-definite; in other words it is an inner-product on $\mathfrak{k}$.*

3. *β is* ad*-invariant, meaning invariant under the adjoint action:*

$$\forall X, Y, Z \in \mathfrak{g}: \; \beta([X,Y],Z) = -\beta(Y,[X,Z])$$

Let us consider a finite-dimensional representation σ of K. Because β is a nondegenerate ad-invariant bilinear form on $\mathfrak{k}$ and because $\mathcal{B}_{\mathfrak{k}}$ is an orthonormal basis of $\mathfrak{k}$, we can define (as in [12], see section 7 of chapter VI) the Casimir operator of σ:

$$\Omega_\sigma = \sum_{r=1}^{dim(\mathfrak{k})} (d\sigma(X_r))^2 \tag{1.5}$$

where, for the time being, we denote by X_r the elements of $\mathcal{B}_{\mathfrak{k}}$.
We shall make (1.5) lighter by using enveloping algebra notations:

$$\Omega_\pi = \sum_{r=1}^{dim(\mathfrak{k})} X_r^2 \tag{1.6}$$

As we explained in Section 1.3.2, we can always extend $d\sigma$ to obtain, in a one-to-one fashion, a $\mathbb{C}$-linear representation of $\mathfrak{g}$. This enables us to write down elements of $\mathcal{B}_{\mathfrak{k}}$ as combinations of elements of the basis (1.2) of $\mathfrak{g}$ and consider the action on these combinations of the complexification of $d\sigma$, which in turn enables us to use information we have on weights, in particular highest weights. The formulas for the Casimir operator become quite simple when applied to a highest weight vector. If the representation σ is irreducible, then Schur's lemma combined with the well known fact (see for instance [12], proposition 6.7.1) that Ω_σ commutes with $d\sigma$ imply that Ω_σ is equal to a scalar multiple of the identity map. This scalar is easy to determine when considering a highest weight vector and the aim of this section is precisely to compute it.
One easily checks (using notations of Section 1.3.1 and the $\mathbb{C}$-basis (1.2) of $\mathfrak{g}$) that for all integers r and s in $\{1, ..., n\}$:

- $A_r = iH_r$
- $B_{r,s} = U^+_{r,s} - U^-_{r,s}$ $(r < s)$
- $C_{r,s} = iU^+_{r,s} + iU^-_{r,s}$ $(r < s)$

- $D_r = D_r^- - D_r^+$
- $E_r = iD_r^- + iD_r^+$
- $F_{r,s} = V_{r,s}^- - V_{r,s}^+ \ (r < s)$
- $G_{r,s} = iV_{r,s}^- + iV_{r,s}^+ \ (r < s)$

1.6 Proposition. *If σ is a finite-dimensional irreducible representation of K on a complex vector space and if λ is its highest weight, then:*

$$\Omega_\sigma = -\left(\sum_{r=1}^{n}\Big(\lambda^2(H_r) + 2\lambda(H_r)\Big) + \sum_{r=1}^{n-1} 2(n-r)\lambda(H_r)\right) \cdot Id$$

where Id denotes the identity map.

Proof:

By definition:

$$\Omega_\sigma = \sum_{r=1}^{n}\Big(A_r^2 + D_r^2 + E_r^2\Big) + \frac{1}{2}\sum_{1\le r<s\le n}\Big(B_{r,s}^2 + C_{r,s}^2 + F_{r,s}^2 + G_{r,s}^2\Big) \tag{1.7}$$

Using the formulas given just before this proposition, multiplying the brackets out (one must be careful: the operators do not commute) and cancelling out various terms, we get:

$$\Omega_\sigma = -\sum_{r=1}^{n}\Big(H_r^2 + 2D_r^-D_r^+ + 2D_r^+D_r^-\Big) - \sum_{1\le r<s\le n}\Big(U_{r,s}^+U_{r,s}^- + U_{r,s}^-U_{r,s}^+ + V_{r,s}^+V_{r,s}^- + V_{r,s}^-V_{r,s}^+\Big) \tag{1.8}$$

Apply Ω_σ to a highest weight vector v. Then, by definition of a highest weight vector, all terms such as $D_r^+(v)$, $U_{r,s}^+(v)$ and $V_{r,s}^+(v)$ are equal to 0. So (1.8) applied to v becomes:

$$\Omega_\sigma(v) = -\sum_{r=1}^{n}\Big(H_r^2(v) + 2D_r^+D_r^-(v)\Big) - \sum_{1\le r<s\le n}\Big(U_{r,s}^+U_{r,s}^-(v) + V_{r,s}^+V_{r,s}^-(v)\Big) \tag{1.9}$$

One can check the following commutation rules:

$$[D_r^+, D_r^-] = H_r \quad ; \quad [U_{r,s}^+, U_{r,s}^-] = H_r - H_s \quad ; \quad [V_{r,s}^+, V_{r,s}^-] = H_r + H_s \ .$$

Using these rules in (1.9) we get

$$\Omega_\sigma(v) = -\sum_{r=1}^{n}\Big(H_r^2(v) + 2H_r(v)\Big) - \sum_{1\le r<s\le n}\big(H_r(v) - H_s(v) + H_r(v) + H_s(v)\big)$$

which can be written:

$$\Omega_\sigma(v) = -\sum_{r=1}^{n}\left(H_r^2(v) + 2H_r(v)\right) - \sum_{r=1}^{n-1} 2(n-r)H_r(v) \tag{1.10}$$

Because λ is a weight and v a corresponding weight vector, we finally obtain:

$$\Omega_\sigma(v) = -\left(\sum_{r=1}^{n}\left(\lambda^2(H_r) + 2\lambda(H_r)\right) + \sum_{r=1}^{n-1} 2(n-r)\lambda(H_r)\right)(v)$$

One concludes with Schur's lemma.

End of proof.

1.4 Quaternions

1.4.1 One way to define quaternions

For the time being, we regard the set of *quaternions* $\mathbb{H}$ as the set $\mathbb{C}^2$. We write $(1,0) = 1_{\mathbb{H}}$ and $(0,1) = j_{\mathbb{H}}$. We equip $\mathbb{C}^2$ with the usual complex vector space structure, denoting scalar multiplication simply by a dot. A pair $(u,v) \in \mathbb{C}^2$ corresponds to the element $u \cdot 1_{\mathbb{H}} + v \cdot j_{\mathbb{H}}$. The following rules define a multiplication $\odot$ on $\mathbb{H}$ (we call it *quaternionic multiplication*):

- (i) $\forall (u,v) \in \mathbb{C}^2 :\ (u \cdot 1_{\mathbb{H}}) \odot (v \cdot 1_{\mathbb{H}}) = (uv) \cdot 1_{\mathbb{H}}$.
- (ii) $\forall (u,v) \in \mathbb{C}^2 :\ (u \cdot 1_{\mathbb{H}}) \odot (v \cdot j_{\mathbb{H}}) = (uv) \cdot j_{\mathbb{H}}$.
- (iii) $\forall (u,v) \in \mathbb{C}^2 :\ (u \cdot j_{\mathbb{H}}) \odot (v \cdot 1_{\mathbb{H}}) = (u\overline{v}) \cdot j_{\mathbb{H}}$. In particular, this implies the fundamental formula for all $v \in \mathbb{C}$:

$$j_{\mathbb{H}} \odot (v \cdot 1_{\mathbb{H}}) = \overline{v} \cdot j_{\mathbb{H}}$$

- (iv) $\forall (u,v) \in \mathbb{C}^2 :\ (u \cdot j_{\mathbb{H}}) \odot (v \cdot j_{\mathbb{H}}) = (-u\overline{v}) \cdot 1_{\mathbb{H}}$. In particular, this implies that $j_{\mathbb{H}} \odot j_{\mathbb{H}} = -1 \cdot 1_{\mathbb{H}}$, which we of course just write $j_{\mathbb{H}}^2 = -1_{\mathbb{H}}$.
- (v) $\forall (h,h',h'') \in \mathbb{H}^3$:
  - $h \odot (h' + h'') = (h \odot h') + (h \odot h'')$.
  - $(h + h') \odot h'' = (h \odot h') + (h' \odot h'')$.

These rules clearly imply that $1_{\mathbb{H}}$ is the left and right identity element for quaternionic multiplication. Let us also point out that quaternionic multiplication is not commutative, precisely because of rule (iii), but it is associative:

$$\forall (h, h', h'') \in \mathbb{H}^3 : (h \odot h') \odot h'' = h \odot (h' \odot h'')$$

Moreover, we obviously have:

$$\forall (u, h) \in \mathbb{C} \times \mathbb{H} : \; u \cdot h = (u \cdot 1_{\mathbb{H}}) \odot h$$

In this sense, if we identify complex numbers u with quaternions $u \cdot 1_{\mathbb{H}}$, we can say that quaternionic multiplication has absorbed complex scalar multiplication, which is therefore no longer needed: all the structure of $\mathbb{H}$ lies in addition and quaternionic multiplication. To make notations simpler, we will now:

- just write u instead of $u \cdot 1_{\mathbb{H}}$, given $u \in \mathbb{C}$;
- just write j instead of $j_{\mathbb{H}}$;
- omit all multiplication symbols.

Rule (iii) implies that any quaternion h can be written in two ways for suitable $(u, v) \in \mathbb{C}^2$: either $u + jv$ or $u + \overline{v} j$. So we could choose to write the second coefficients of quaternions either on the left of j or on the right of j. We choose the second possibility, because it is more convenient for quaternionic linear algebra. Finally, as we shall soon see, every non-zero quaternion has a unique inverse, which is both a left and right one. Now let us summarise the facts and definitions we have seen so far:

- The set of quaternions is $\mathbb{H} = \left\{ u + jv \;/\; (u, v) \in \mathbb{C}^2 \right\}$.
- $\mathbb{H}$ is a skew (non-commutative) field, with respect to addition and quaternionic multiplication.
- $\forall v \in \mathbb{C} : \; jv = \overline{v} j$.
- $j^2 = -1$.

Given a quaternion $h = u + jv$, one defines its *quaternionic conjugate* h^* (or *conjugate* for short):

$$h^* = \overline{u} - jv$$

1.7 Lemma. *Given any quaternions h_1 and h_2:*

1. $(h_1 + h_2)^* = h_1^* + h_2^*$
2. $(h_1 h_2)^* = h_2^* h_1^*$

The *modulus* of a quaternion h is then defined by :

$$|h| = \sqrt{hh^*} = \sqrt{|u|^2 + |v|^2}$$

This definition shows that every non-zero quaternion h, as announced previously, has a unique left inverse, a unique right inverse, and that both of these inverses coincide:

$$h^{-1} = \frac{h^*}{|h|^2}$$

Also, one obviously has:

1.8 Lemma. *Given any quaternions h_1 and h_2:*

1. $h_1 = 0 \Longleftrightarrow |h_1| = 0$
2. $|h_1 + h_2| \leq |h_1| + |h_2|$
3. $|h_1 h_2| = |h_1||h_2|$

A quaternion h will be called a *unit quaternion* if $|h| = 1$. We point out that the set of unit quaternions is a group with respect to quaternionic multiplication; we shall denote it by $\mathrm{U}_{\mathbb{H}}$; it is clearly diffeomorphic to the three-dimensionnal sphere.
Remark:
This complex version of quaternions corresponds exactly to the traditional and historical real version (used by W. R. Hamilton himself - see [19]). In the real version, quaternions are usually written $h = a + bi + cj + dk$, with $(a, b, c, d) \in \mathbb{R}^4$, their conjugates becoming $a - bi - cj - dk$ and their moduli $\sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}$. The correspondance between the real and complex versions works via the identifications $u = a + ib$ and $v = c - id$.

1.4.2 Quaternionic linear algebra

Consider any integer $p > 0$ and the set $\mathbb{H}^p$, equipped with the usual addition and the usual scalar multiplication (by quaternions). Multiplications can be applied on the left or on the right of coordinates; the only difference, compared to $\mathbb{R}^p$ or $\mathbb{C}^p$, is that left and right multiplication usually give different results ($\mathbb{H}$ is non-commutative). Considering left multiplication, we see that all the axioms of a vector space are satisfied, so we say that $\mathbb{H}^p$ is a *left quaternionic vector space*; similarly, one can focus on right multiplication and say that $\mathbb{H}^p$ is a *right quaternionic vector space.* In some way, which point of view we choose has no importance, because $\mathbb{H}^p$ is both types of vector spaces at the same time. However, linearity of functions must be specified: left linearity does not coincide with right linearity, again because $\mathbb{H}$ is not commutative. *We shall always work with right linearity*: for us, a *linear map* from some quaternionic vector space to another will always mean a right linear map. The reason for this choice lies in matrix multiplication, which we now look into.

Remember that $\mathrm{M}(p, \mathbb{H})$ denotes the vector space of $p \times p$ *quaternionic matrices* (that is, with quaternionic coefficients); but one needs to specify the vector space structure. Of course we equip $\mathrm{M}(p, \mathbb{H})$ with the usual matrix addition. Scalar multiplication (multiplication by quaternions) can be defined on the left or on the right. So clearly, $\mathrm{M}(p, \mathbb{H})$ is both a left and a right quaternionic vector space. The discussion is similar for vector spaces of $l \times c$ quaternionic matrices, given $(l, c) \in \mathbb{N}^2$.

To write down a matrix $A \in \mathrm{M}(p, \mathbb{H})$ whose coefficients are denoted by $a_{r,s}$, with r and s two integers between 1 (included) and n (included), we just write $A = [a_{r,s}]$. Here, we do not mention p explicitly because we know the size of the matrix from the context. If ever we need to consider other types of matrices, we must specify indices or simply write down all the coefficients.

Multiplying matrices by other matrices is defined as usual, but one must respect the order in which coefficients come.

As one expects, a matrix $A \in \mathrm{M}(p, \mathbb{H})$ is *invertible* if for some matrix $B \in \mathrm{M}(p, \mathbb{H})$ one has $AB = BA = I_p$, in which case B is the *inverse* of A and is denoted by A^{-1} instead. The set of invertible matrices is of course a group with respect to matrix multiplication and is denoted by $\mathrm{GL}(n, \mathbb{H})$.

1.4.3 Identifying complex and quaternionic coordinates

Again, consider any integer $p > 0$. As we have seen, we identify $\mathbb{C}^{2p}$ with $\mathbb{H}^p$ by identifying $(u, v) \in \mathbb{C}^p \times \mathbb{C}^p$ with $u + jv \in \mathbb{H}^p$. In the course of this identification, we shall use the following notations for quaternion vectors x:

- $x = (x_1, ..., x_p) \in \mathbb{H}^p$;
- $x_r = u_r + jv_r$, for all $r \in \{1, 2, \cdots, p\}$ and with $(u_r, v_r) \in \mathbb{C}^2$;
- $u = (u_1, \cdots, u_p) \in \mathbb{C}^p$ and $v = (v_1, \cdots, v_p) \in \mathbb{C}^p$;
- $x = u + jv$.

We define for all vectors x, y in $\mathbb{H}^p$:

- $\langle x, y\rangle_{\mathbb{H}} = \sum_{r=1}^{p} x_r y_r^* \in \mathbb{H}$
- $\|x\|^2 = \langle x, x\rangle_{\mathbb{H}} = \sum_{r=1}^{p} x_r x_r^* = \sum_{r=1}^{p} |x_r|^2$

1.9 Lemma. *Given any vectors $x, y \in \mathbb{H}^p$ and any quaternion h:*

- $x = 0 \Longleftrightarrow \|x\| = 0$
- $\|hx\| = \|xh\| = |h|\|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$

We refer to $\|\cdot\|$ as the *norm* of $\mathbb{H}^p$. It is the standard norm for $\mathbb{H}^p$ seen as a real (resp. left or right complex) vector space and it obviously turns $\mathbb{H}^p$ into a real (resp. left or right complex) Banach space.

Matrices $M \in \mathrm{M}(p, \mathbb{H})$ will be decomposed as $M = A + jB$, with A and B two complex $p \times p$ matrices. With these notations, such a matrix M can be identified with the following element of $\mathrm{M}(2p, \mathbb{C})$:

$$M' = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}$$

We say that M is the *pullback* of M' and we denote by $\mathrm{M}_{\mathbb{H}}(2p, \mathbb{C})$ the vector space that consists of all elements of $\mathrm{M}(2p, \mathbb{C})$ that can be written like M'. This procedure defines a map:

$$\begin{array}{rccc} E_{\mathbb{H}}^{\mathbb{C}} : & \mathrm{M}(p, \mathbb{H}) & \longrightarrow & \mathrm{M}_{\mathbb{H}}(2p, \mathbb{C}) \\ & M & \longmapsto & M' \end{array}$$

This map is bijective and respects matrix multiplication, meaning that, given any two elements M_1 and M_2 of $\mathrm{M}(p, \mathbb{H})$:

$$E_{\mathbb{H}}^{\mathbb{C}}(M_1 M_2) = E_{\mathbb{H}}^{\mathbb{C}}(M_1) E_{\mathbb{H}}^{\mathbb{C}}(M_2)$$

This implies that if M_2 is a right inverse of M_1, then $M_2' = E_{\mathbb{H}}^{\mathbb{C}}(M_2)$ is a right inverse of $M_1' = E_{\mathbb{H}}^{\mathbb{C}}(M_1)$. The same goes for left inverses and, because left and right inverses coincide in $\mathrm{GL}(2p, \mathbb{C})$:

1.10 Lemma. *If a matrix M of $\mathrm{M}(p, \mathbb{H})$ has a right inverse $M' \in \mathrm{M}(p, \mathbb{H})$, then M' is also a left inverse of M. This also works the other way round, swapping left and right.*

This is why we grouped both kinds of inverses in a single definition when we introduced invertible matrices earlier on. We denote by $\mathrm{GL}_{\mathbb{H}}(2p, \mathbb{C})$ the image of $\mathrm{GL}(p, \mathbb{H})$ under the map $E_{\mathbb{H}}^{\mathbb{C}}$; it is obviously a subgroup of $\mathrm{GL}(2p, \mathbb{C})$ and the restriction of $E_{\mathbb{H}}^{\mathbb{C}}$ to $\mathrm{GL}(p, \mathbb{H})$ is an isomorphism onto $\mathrm{GL}_{\mathbb{H}}(2p, \mathbb{C})$.
Using above notations, applying a matrix $M \in \mathrm{M}(p, \mathbb{H})$ to a vector $u + jv \in \mathbb{H}^p$ corresponds to applying $E_{\mathbb{H}}^{\mathbb{C}}(M) = M'$ to the vector $(u, v) \in \mathbb{C}^{2p}$.
If one looks back at the shape of elements of the compact group K given in section 1.2, one cannot miss the similarity with matrices such as the matrix M' above. This tells us that, via the map $E_{\mathbb{H}}^{\mathbb{C}}$, the group $\mathrm{Sp}(p)$ corresponds to the group of elements of $\mathrm{GL}(p, \mathbb{H})$ that perserve the norms of vectors of $\mathbb{H}^p$, that is, the group of linear isometries of $\mathbb{H}^p$. We again denote this group by $\mathrm{Sp}(p)$. In particular: $\mathrm{Sp}(1) \simeq \mathrm{U}_{\mathbb{H}}$.
Left and right multiplication by quaternions in $\mathbb{H}^p$ can also be read in $\mathbb{C}^{2p}$, but not as simple scalar multiplications: they combine the complex coordinates in a more subtle way. Indeed, multiplying on the right a quaternion $h = u + jv \in \mathbb{H}$ by another quaternion $q = a + jb \in \mathbb{H}$ gives:

$$hq = (ua - \overline{v}b) + j(va + \overline{u}b)$$

This formula also shows how left multiplication operates (h and q play symmetric roles). Let us summarise how the actions of quaternionic matrices and quaternionic scalars read in the complex setting:

1.11 Lemma. *Given $x = u + jv \in \mathbb{H}^p$, $q = a + jb \in \mathbb{H}$ and $M = A + jB \in \mathrm{M}(p, \mathbb{H})$:*

1. *$Mx \in \mathbb{H}^n$ corresponds to* $\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{C}^{2p}$

2. $xq \in \mathbb{H}^p$ *corresponds to* $(u_1 a - \overline{v}_1 b, \ldots, u_p a - \overline{v}_p b, v_1 a + \overline{u_r} b, \ldots, v_p a + \overline{u}_p b) \in \mathbb{C}^{2p}$.

Consider a quaternionic matrix $M \in \mathrm{M}(p, \mathbb{H})$ and denote by m_{rs} its coefficients. Similarly to previous notations, write $m_{rs} = u_{rs} + j v_{rs}$ and define:

$$\|M\|_{\mathrm{op}} = max\{|u_{rs}|, |v_{rs}|\}_{1 \le r,s \le n}$$

Obviously, $\|M\|_{\mathrm{op}}$ is equal to the standard norm of $E^{\mathbb{C}}_{\mathbb{H}}(M)$ in $\mathrm{M}(2p, \mathbb{C})$ which hands out the highest (complex) modulus amongst all (complex) coefficients of $E^{\mathbb{C}}_{\mathbb{H}}(M)$. This combined to lemma 1.9 implies:

1.12 Lemma. *Given any matrices* $M, M' \in \mathrm{M}(p, \mathbb{H})$ *and any quaternion* h*:*

- $M = 0 \Longleftrightarrow \|M\|_{\mathrm{op}} = 0$
- $\|hM\|_{\mathrm{op}} = \|Mh\|_{\mathrm{op}} = |h| \|M\|_{\mathrm{op}}$
- $\|M + M'\|_{\mathrm{op}} \le \|M\|_{\mathrm{op}} + \|M'\|_{\mathrm{op}}$

We refer to $\|\cdot\|_{\mathrm{op}}$ as the *norm* of $\mathrm{M}(p, \mathbb{H})$. It turns $\mathrm{M}(p, \mathbb{H})$ into a Banach space and the map $E^{\mathbb{C}}_{\mathbb{H}}$ into a homeomorphism, allowing one to define the exponential map exp on $\mathrm{M}(p, \mathbb{H})$ by the usual formula:

$$\exp(M) = \sum_{r=0}^{\infty} \frac{M^r}{r!}$$

One easily checks that the maps $E^{\mathbb{C}}_{\mathbb{H}}$ and exp commute; in other words, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{GL}(p, \mathbb{H}) \arrow[r, "E^{\mathbb{C}}_{\mathbb{H}}"] & \mathrm{GL}_{\mathbb{H}}(2p, \mathbb{C}) \\
\mathrm{M}(p, \mathbb{H}) \arrow[u, "\exp"] \arrow[r, "E^{\mathbb{C}}_{\mathbb{H}}"'] & \mathrm{M}_{\mathbb{H}}(2p, \mathbb{C}) \arrow[u, "\exp"']
\end{tikzcd}$$

1.4.4 Symplectic matrices seen as quaternionic matrices

Elements of $\mathrm{Sp}(n)$ (see section 1.4.3) can be seen, via the embedding map $E^{\mathbb{C}}_{\mathbb{H}}$, as quaternion matrices; in this process of identification, for simplicity, we will not change notations, writing in particular:

- $A_r = i E_{r,r}$

- $B_{r,s} = E_{r,s} - E_{s,r}$ (when $n \geq 2$ and $r \neq s$)
- $C_{r,s} = iE_{r,s} + iE_{s,r}$ (when $n \geq 2$ and $r \neq s$)
- $D_r = jE_{r,r}$
- $E_r = jiE_{r,r}$
- $F_{r,s} = jE_{r,s} + jE_{s,r}$ (when $n \geq 2$ and $r \neq s$)
- $G_{r,s} = jiE_{r,s} + jiE_{s,r}$ (when $n \geq 2$ and $r \neq s$)

Then $\{A_r, D_r, E_r\}_{r\in\{1,\dots,n\}} \cup \{B_{r,s}, C_{r,s}, F_{r,s}, G_{r,s}\}_{r,s\in\{1,\dots,n\},r<s}$ is a basis over $\mathbb{R}$ of $\mathfrak{k}$ (seen as a subset of $\mathrm{M}(n,\mathbb{H})$). Applying the exponential map and lemma 1.4, given $t \in \mathbb{R}$, we have:

1.13 Lemma.

- *For $r \in \{1, 2, \dots, n\}$ and $M \in \{A_r, D_r, E_r\}$:*

$$\exp(-tM) = I - (\sin t)M + (\cos t - 1)E_{r,r}$$

- *For $r, s \in \{1, \dots, n\}$ such as $r < s$ (here it is assumed that $n \geq 2$) and for any $M \in \{B_{r,s}, C_{r,s}, F_{r,s}, G_{r,s}\}$:*

$$\exp(-tM) = I - (\sin t)M + (\cos t - 1)(E_{r,r} + E_{s,s})$$

One can transport the inner product β defined in section 1.3.3 to matrices of $\mathfrak{k}$ seen as quaternionic matrices: for such matrices X and Y, one sets $\beta(X,Y) = \beta\left(E^{\mathbb{C}}_{\mathbb{H}}(X), E^{\mathbb{C}}_{\mathbb{H}}(Y)\right)$. Again:

- β is an inner product on $\mathfrak{k}$ (seen as a subset of $\mathrm{M}(n,\mathbb{H})$).
- The set

$$\mathcal{B}_{\mathfrak{k}} = \{A_r, D_r, E_r\}_{r\in\{1,\dots,n\}} \cup \left\{\frac{B_{r,s}}{\sqrt{2}}, \frac{C_{r,s}}{\sqrt{2}}, \frac{F_{r,s}}{\sqrt{2}}, \frac{G_{r,s}}{\sqrt{2}}\right\}_{r,s\in\{1,\dots,n\},r<s}$$

  is an orthonormal basis of $\mathfrak{k}$ (seen as a subset of $\mathrm{M}(n,\mathbb{H})$) with respect to β.

The following proposition will come out useful when studying stabilisers of actions of K:

1.14 Proposition.

1. Consider a quaternionic matrix $M = \begin{pmatrix} h & L \\ 0 & T \end{pmatrix}$*, with* $h \in \mathbb{H}$*,* $L \in \mathrm{M}_{1,n-1}(\mathbb{H})$ *and* $T \in \mathrm{M}(n-1, \mathbb{H})$*. Then* M *belongs to* K *if and only if* $L = 0$*,* $h \in \mathrm{Sp}(1)$ *and* $T \in \mathrm{Sp}(n-1)$*.*

2. Consider a quaternionic matrix $M = \begin{pmatrix} h & 0 \\ C & T \end{pmatrix}$*, with* $h \in \mathbb{H}$*,* $C \in \mathrm{M}_{n-1,1}(\mathbb{H})$ *and* $T \in \mathrm{M}(n-1, \mathbb{H})$*. Then* M *belongs to* K *if and only if* $C = 0$*,* $h \in \mathrm{Sp}(1)$ *and* $T \in \mathrm{Sp}(n-1)$*.*

Proof:

The proofs of both items work similarly, so we'll just deal with 1) (in fact, each item implies the other, via the transpose of matrices).
Suppose M belongs to K. One can decompose M into $M = U + jV$, with

$$U = \begin{pmatrix} h_1 & L_1 \\ 0 & T_1 \end{pmatrix} \in \mathrm{M}_n(\mathbb{C})$$

$$V = \begin{pmatrix} h_2 & L_2 \\ 0 & T_2 \end{pmatrix} \in \mathrm{M}_n(\mathbb{C})$$

Recalling the definition of the map $E_{\mathbb{H}}^{\mathbb{C}}$, we have:

$$E_{\mathbb{H}}^{\mathbb{C}}(M) = \begin{pmatrix} h_1 & L_1 & -\overline{h_2} & -\overline{L_2} \\ 0 & T_1 & 0 & -\overline{T_2} \\ h_2 & L_2 & \overline{h_1} & \overline{L_1} \\ 0 & T_2 & 0 & \overline{T_1} \end{pmatrix}$$

The matrix M belongs to K, so $E_{\mathbb{H}}^{\mathbb{C}}(M)$ is unitary:

$${}^t\overline{\Psi(M)} E_{\mathbb{H}}^{\mathbb{C}}(M) = E_{\mathbb{H}}^{\mathbb{C}}(M)\,{}^t\overline{\Psi(M)} = I$$

By looking at the top left coefficient of these products, one finds (the symbol $\|\cdot\|$ denotes the norm in $\mathbb{C}^{n-1}$ - we use it identifying line matrices and vectors):

$$|h_1|^2 + \|L_1\|^2 + |h_2|^2 + \|L_2\|^2 = 1$$

$$|h_1|^2 + |h_2|^2 = 1$$

This implies:

$$\|L_1\|^2 + \|L_2\|^2 = 0$$

Consequently, $L = 0$. From this it follows that h and T must belong respectively to $\mathrm{Sp}(1)$ and $\mathrm{Sp}(n-1)$.

This proves one implication; the converse is straightforward.

End of proof.

CHAPTER 2

Degenerate principal series of $\mathrm{Sp}(n, \mathbb{C})$

Dans ce chapitre, nous présentons la famille de représentations induites qui est l'objet central de ce travail.

Tout d'abord, nous choisissons un sous-groupe parabolique $Q = MAN$ que nous explicitons dans le paragraphe 1 ; M, A et N sont trois sous-groupes fermés de G, donc sont des groupes de Lie. On note de manière naturelle man les éléments de MAN.

Remarque : les notations N, n et m renvoient aussi à dimensions, mais les contextes empêchent toute ambiguïté.

On fait remarquer dans la proposition 2.1 que le sous-groupe Q est constitué de toutes les matrices de $\mathrm{Sp}(n, \mathbb{C})$ de la forme $q = \begin{pmatrix} \alpha & L \\ 0 & D \end{pmatrix}$ avec $\alpha \in \mathbb{C} \setminus \{0\}$, L une certaine matrice ligne de $\mathrm{M}_{1,2n-1}(\mathbb{C})$ et D une certaine matrice carrée de $\mathrm{M}(2n-1, \mathbb{C})$; pour les matrices de A, le coefficient α est un réel strictement positif et on le note $\alpha(a)$; pour les matrices m de M, le coefficient α est de norme 1 alors on le note plutôt $e^{i\theta(m)}$.

Le sous-groupe parabolique étant fixé, en suivant le procédé exposé dans [27] (chapitre VII, paragraphe 1) nous définissons dans le paragraphe 2.2.1.1 une famille, paramétrée par des couples $(\lambda, \delta) \in \mathbb{R} \times \mathbb{Z}$, de représentations induites $\pi_{i\lambda,\delta}$ sur des espaces de Hilbert $V_{i\lambda,\delta}$. Ces représentations unitaires $\pi_{i\lambda,\delta}$ sont obtenues en considérant d'abord l'action à gauche de G sur l'espace vectoriel

$$V^0_{i\lambda,\delta} \;=\; \left\{ f \in C^0(G) \Big/ \forall g \in G,\ \forall man \in MAN \ :\ f(gman) = e^{-i\delta\theta(m)} \Big(\alpha(a)\Big)^{-i\lambda - N} f(g) \right\}$$

puis en complétant cet espace par rapport à la norme $\| \cdot \|$ définie par

$$\|f\|^2 = \int_K |f(k)|^2 \, dk$$

où dk désigne la mesure de Haar (unique à un coefficient près) de K.

On peut réaliser $\pi_{i\lambda,\delta}$ dans d'autres espaces fonctionnels ; on dit alors que l'on change de modèle. Celui que nous venons de présenter est le modèle induit. Dans les paragraphes 2.2.1.2 et 2.2.1.3, nous donnons deux autres modèles, l'un dit compact et l'autre non-compact, toujours en suivant ce qui est fait dans le chapitre VII de [27].

Puis, nous introduisons d'autres modèles encore, qui sont ceux que l'on considère concrètement dans notre travail. La façon dont on passe des modèles précédents à ceux-là est expliquée dans le paragraphe 2.2.2.1, où l'on donne la nouvelle version du modèle induit. Il s'agit de réaliser géométriquement les fonctions de $V^0_{i\lambda,\delta}$, en identifiant les matrices de G, modulo un stabilisateur, à des vecteurs de $\mathbb{C}^N \setminus \{0\}$: cela se fait en considérant un vecteur de référence $e_1 = (1, 0, \dots, 0)$ et en y appliquant l'action naturelle de G. On obtient le *modèle induit* de notre travail avec un procédé similaire à celui vu plus haut : $V^0_{i\lambda,\delta}$ devient l'espace

$$V^0_{i\lambda,\delta} = \left\{ f \in C^0\left(\mathbb{C}^N \setminus \{0\}\right) \Big/ \forall c \in \mathbb{C} \setminus \{0\} : \ f(c\cdot) = \left(\frac{c}{|c|}\right)^{-\delta} |c|^{-i\lambda - N} f(\cdot) \right\}$$

sur lequel G agit à gauche, puis l'on obtient $\pi_{i\lambda,\delta}$ et $V_{i\lambda,\delta}$ en complétant cet espace par rapport à la norme $\|\cdot\|$ définie cette fois par

$$\|f\|^2 = \int_{S^{2N-1}} |f(x)|^2 \, d\sigma(x)$$

On dit que les fonctions de $V^0_{i\lambda,\delta}$ sont covariantes ; la propriété de covariance fait référence non pas à la continuité de ces fonctions mais à l'égalité qu'elles vérifient pour tout $c \in \mathbb{C} \setminus \{0\}$.

Dans les paragraphes 2.2.2.2 et 2.2.2.3, on décrit le *modèle compact* puis le *modèle non-compact*.

Pour le modèle compact, même si le paramètre λ est caché, on conserve la notation $V_{i\lambda,\delta}$ qui désigne cette fois l'espace de Hilbert

$$\{f \in L^2(S^{2N-1}) \ / \ \forall \theta \in \mathbb{R} \ : \ f(e^{i\theta}\cdot) = e^{-i\delta\theta} f(\cdot)\}$$

pour la norme $\|\cdot\|$ de nouveau définie par :

$$\|f\|^2 = \int_{S^{2N-1}} |f(x)|^2 \, d\sigma(x)$$

L'action de G n'est pas exactement l'action à gauche, le problème étant que l'action naturelle de G ne préserve pas tout le temps la sphère. En revanche, et cela nous suffira, si l'on restreint la représentation à K, alors on obtient vraiment une action à gauche.

Il est à noter que la restriction à la sphère des fonctions F de $V^0_{i\lambda,\delta}$, dans le modèle induit, donne toutes les fonctions continues f du modèle compact et que le procédé de restriction est réversible. Pour des fonctions F et f qui se correspondent, on a :

$$F(x) = \|x\|^{-i\lambda - N} f\left(\frac{x}{\|x\|}\right)$$

pour tout $x \in \mathbb{C}^N \setminus \{0\}$.

Pour le modèle non-compact, même si les deux paramètres λ et δ sont cachés, on conserve encore la notation $V_{i\lambda,\delta}$, qui désigne cette fois l'espace de Hilbert $L^2(\mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m)$, pour la mesure de Lebesgue $ds\,du\,dv$. De nouveau l'action de G n'est pas si simple, mais si l'on restreint la représentation au sous-groupe $\overline{N}$ constitué des transposées de matrices de N, alors on obtient vraiment une action à gauche.

Il est aussi à noter que le principe de covariance permet, étant donnée une fonction f de $V_{i\lambda,\delta}$, dans le modèle non-compact, de définir une fonction F sur $S^{2N-1} \cap (z_1 \neq 0)$ par :

$$F(z; w) = \left(\frac{z_1}{|z_1|}\right)^{-\delta} |z_1|^{-i\lambda - N} f\left(\frac{w_1}{2z_1}, \frac{z_2}{z_1}, \ldots, \frac{z_n}{z_1}, \frac{w_2}{z_1}, \ldots, \frac{w_n}{z_1}\right)$$

La fonction f est continue si et seulement si F l'est. Comme $S^{2N-1} \cap (z_1 = 0)$ est de mesure nulle sur la sphère, ce procédé permet de reconstituer tout l'espace $V_{i\lambda,\delta}$ dans le modèle compact et le modèle induit.

The heart of our work is the study of certain induced representations of G. Why it is interesting to concentrate on such representations was dicussed in the introduction. These representations are constructed by considering a representation of a parabolic subgroup and turning it into a representation of the whole group G. The theory that leads to induced representations would take too long to explain here, so we will just explicit the specific parabolic subgroup we will work with, along with the corresponding induced representations, referring the reader to [27] (sections 2 and 5 of Chapter V and Section 1 of Chapter VII) for a thorough lecture on the theory that underlies section 2.2.1.

2.1 A specific parabolic subgroup

We write $2n \times 2n$ matrices in the following way:

$$g = \begin{pmatrix} * & * & * & * \\ * & E & * & G \\ * & * & * & 0 \\ * & F & * & H \end{pmatrix}$$

where E, F, G, H are $m \times m$ matrices and where the stars denote suitable numbers, line matrices or column matrices.

Using those notations, we list below the subgroups of G we need to define the induced representations of G we intend to study.

- We denote by M the subgroup of G consisting of all matrices of the following type:

$$m = \left(\begin{array}{cc|cc} e^{i\theta} & 0 & 0 & 0 \\ 0 & A & 0 & C \\ \hline 0 & 0 & e^{-i\theta} & 0 \\ 0 & B & 0 & D \end{array}\right) \text{ with } \theta \in \mathbb{R} \text{ and } \left(\begin{array}{c|c} A & C \\ \hline B & D \end{array}\right) \in \mathrm{Sp}(m, \mathbb{C}).$$

  We then write $\theta = \theta(m)$.

- We denote by A the subgroup of G consisting of all matrices of the following type:

$$a = \left(\begin{array}{cc|cc} \alpha & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ \hline 0 & 0 & \alpha^{-1} & 0 \\ 0 & 0 & 0 & I_m \end{array}\right) \text{ with } \alpha \in]0, \infty[.$$

  We then write $\alpha = \alpha(a)$.

- We denote by N the subgroup of G consisting of all matrices of the following type:

$$n = \left(\begin{array}{cc|cc} 1 & {}^t u & 2s & {}^t v \\ 0 & I_m & v & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & -u & I_m \end{array}\right) \text{ with } (s, u, v) \in \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m .$$

The coefficient 2 in front of s is not really necessary in the definition, but we keep it, as customary, to signify the isomorphism between N and the so-called Heisenberg group H^{2m+1}: matrices n as above correspond to elements (s, u, v) of H^{2m+1}. Though we will say nothing more of it, this isomorphism underlies the non-compact picture to come.

- We define the *parabolic subgroup* Q as the group of elements $q \in G$ for which there exists $(m, a, n) \in M \times A \times N$ such that $q = man$; we write $Q = MAN$, calling this equality the *Langlands decomposition* of Q (it defines a diffeomorphism of Q onto $M \times A \times N$).

- We denote by $\overline{N}$ the subgroup of G that consists of all matrices ${}^t n$ for $n \in N$.

Remark: we are aware that symbols m, n and N are already used, but with context there can be no confusion.

We point out that notations N and $\overline{N}$ are swapped in [35]; we have not followed the choice of the authors, so as to stay close to notations used in [27].

The subgroups M and A are linear Lie groups (as N, $\overline{N}$ and Q) and:

- the Lie algebra of M is the subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ consisting of matrices of the following type:

$$X = \begin{pmatrix} i\theta & 0 & 0 & 0 \\ 0 & A & 0 & C \\ 0 & 0 & -i\theta & 0 \\ 0 & B & 0 & D \end{pmatrix} \text{ with } \theta \in \mathbb{R} \text{ and } \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in \mathfrak{sp}(m, \mathbb{C}) \ ;$$

- the Lie algebra of A is the subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ consisting of matrices of the following type:

$$Y = \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ with } \beta \in \mathbb{R} \ .$$

We have left out the Lie algebras of N and $\overline{N}$ because we will not explicitly use them. When defining induced representations, one can wish to change the carrying spaces. In order to do so, this proposition will prove useful:

2.1 Proposition. *Q is the subgroup of G consisting of all matrices of G in which the entries of the first column are all 0 except the one at the top.*

Proof:

If q belongs to MAN then one can write:

$$q = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & A & 0 & C \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & B & 0 & D \end{pmatrix} \begin{pmatrix} 1 & {}^t u & s & {}^t v \\ 0 & I & v & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -u & I \end{pmatrix}$$

where $\alpha \in \mathbb{C} \setminus \{0\}$, $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$ belongs to $\mathrm{Sp}(m, \mathbb{C})$ and where (s, u, v) belongs to $\mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$. This gives the following form of q (we'll say that a matrix of this form is a Q-form matrix):

$$q = \begin{pmatrix} \alpha & \alpha {}^t u & \alpha s & \alpha {}^t v \\ 0 & A & Av - Cu & C \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & B & Bv - Du & D \end{pmatrix}$$

and we see the first column of q satisfies the required condition.

Suppose now that a matrix q of G has all its entries in the first column equal to 0 except the top (diagonal) entry which is non zero; let us write:

$$q = \begin{pmatrix} \alpha & {}^t k & t & {}^t l \\ 0 & A & m & C \\ 0 & {}^t e & b & {}^t f \\ 0 & B & n & D \end{pmatrix}$$

where α is non zero, $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$ is some $m \times m$ matrix, (t, k, l) and (b, e, f) belong to $\mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$ and (m, n) to $\mathbb{C}^m \times \mathbb{C}^m$. Working with the properties of symplectic matrices given in formulas (1.1), one can prove that:

- $e = f = 0$;

- $b = \alpha^{-1}$;
- $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$ belongs to $\mathrm{Sp}(m, \mathbb{C})$.

Then, again using the conditions that a symplectic matrix must satisfy, one shows that:

- (i) $k\alpha^{-1} + {}^tAn = {}^tBm$;
- (ii) $l\alpha^{-1} + {}^tCn = {}^tDm$;

Comparing with the Q-form matrix obtained in the first part of the proof, we try to identify s, u, v. We see that we must have:

- $s = t\alpha^{-1}$;
- $u = k\alpha^{-1}$;
- $v = l\alpha^{-1}$.

So that (i) and (ii) become:

$$\begin{pmatrix} u \\ v \end{pmatrix} \begin{pmatrix} {}^tB & -{}^tA \\ {}^tD & -{}^tC \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix}$$

Inverting this gives:

$$\begin{pmatrix} m \\ n \end{pmatrix} \begin{pmatrix} -C & A \\ -D & B \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

So we see that $q = \begin{pmatrix} \alpha & \alpha{}^tu & \alpha s & \alpha{}^tv \\ 0 & A & m & C \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & B & n & D \end{pmatrix}$ is a Q-form matrix. Consequently we can write:

$$q = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & A & 0 & C \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & B & 0 & D \end{pmatrix} \begin{pmatrix} 1 & {}^tu & s & {}^tv \\ 0 & I & v & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -u & I \end{pmatrix}$$

and q finally does belong to $Q = MAN$.

End of proof.

2.2 Specific induced representations

Throughout this section we consider any $\lambda \in \mathbb{R}$ and $\delta \in \mathbb{Z}$.

2.2.1 Usual definitions

2.2.1.1 Induced picture

Consider $(\lambda, \delta) \in \mathbb{R} \times \mathbb{Z}$ and the character $\chi_{i\lambda,\delta}$ of the parabolic subgroup $Q = MAN$ (introduced in section 2.1) defined by:

$$\chi_{i\lambda,\delta}(man) = e^{i\delta\theta(m)} \left(\alpha(a)\right)^{i\lambda+N}$$

The corresponding induced representation (from Q to G), denoted by $\pi_{i\lambda,\delta} = \mathrm{Ind}_Q^G \chi_{i\lambda,\delta}$, is obtained by considering the left action of G on the complex vector space

$$V^0_{i\lambda,\delta} = \{f \in C^0(G) \,/\, \forall g \in G,\ \forall man \in MAN \ :\ f(gman) = \chi^{-1}_{i\lambda,\delta}(man)\, f(g)\}$$

and completing this space with respect to the norm $\|\cdot\|$ defined by:

$$\|f\|^2 = \int_K |f(k)|^2 \, dk$$

where dk denotes the Haar measure (unique up to a constant) of K. Functions of $V^0_{i\lambda,\delta}$ are said to be (λ, δ)-*covariant*, or just *covariant* for short, when the exact values of λ and δ are not needed or when the context makes them clear. The representations $\pi_{i\lambda,\delta}$ obtained by varying the parameters λ and δ are unitary and form a family called the *degenerate principal series* of G (the word degenerate refers to the fact that the parabolic subgroup we have chosen is not minimal). They were studied in [18], where it is proved that $\pi_{i\lambda,\delta}$ is irreducible if and only if $(\lambda, \delta) \neq (0, 0)$.

The representations $\pi_{i\lambda,\delta}$ obtained by varying the parameters λ and δ are unitary and form a family called the *degenerate principal series* of G (the word degenerate refers to the fact that the parabolic subgroup we have chosen is not minimal). They were studied in [18], where it is proved that $\pi_{i\lambda,\delta}$ is irreducible if and only if $(\lambda, \delta) \neq (0, 0)$.

One can change the carrying space and define representations that are equivalent to $\pi_{i\lambda,\delta}$. To specify which one of them is considered, one uses the word *picture* to mean description of action and carrying space. The above description is called the *induced picture*. The equivalences are due to some structure theory, which, as mentioned ealier on, we have decided not to discuss.

2.2.1.2 Compact picture

The compact picture is obtained by considering the complex vector space

$$V^0_{i\lambda,\delta} \;=\; \{f \in C^0(K) \;/\; \forall k \in K,\; \forall m \in M \;\cap K \;:\; f(km) = e^{-i\delta\theta(m)} f(k)\}$$

and completing it with respect to the norm $\|\cdot\|$ defined by:

$$\|f\|^2 = \int_K \;|f(k)|^2 \; dk$$

Identification of the carrying spaces of the induced and compact pictures is based on a specific decomposition of elements of G: structure theory shows that $G = KMAN$. The restriction to K then defines a one-to-one correspondance between functions of $V^0_{i\lambda,\delta}$ seen in the induced picture and functions of $V^0_{i\lambda,\delta}$ seen in the compact picture. The action of G here is not as simple as one might think: elements of G take elements of K possibly out of K. But an interesting fact to point out is that the restriction of the action to K is just the left action of K.

2.2.1.3 Non-compact picture

One can show that restricting elements of $V^0_{i\lambda,\delta}$ to $\overline{N}$ gives continuous functions in $L^2(\overline{N}, d\overline{n})$, where $d\overline{n}$ denotes the Haar measure (unique up to a constant) of $\overline{N}$.
So we choose as carrying space the whole of $L^2(\overline{N})$. Here as well, the action of G is not as simple as one might think: elements of G take elements of $\overline{N}$ possibly out of $\overline{N}$. But an interesting fact to point out is that the restriction of the action to $\overline{N}$ is just the left action of $\overline{N}$.
Identification of the carrying spaces of the induced and non-compact pictures is a little more subtle than for the compact picture, because based on a decomposition of G that only works for almost all $g \in G$: $\overline{N}MAN$ is dense in G.

2.2.2 Changing the carrying spaces

Another way to change the carrying spaces is to make use of the natural action of the group G on $\mathbb{C}^N$. Each of the three pictures described previously then leads to another picture; in the process, for simplicity, we will not rename the pictures and we will keep on writing $\pi_{i\lambda,\delta}$ for the induced representations.

2.2.2.1 Another induced picture

2.2 Definition (Induced picture of $\pi_{i\lambda,\delta}$)**.** It is obtained by considering the left action of G on the complex vector space

$$V^0_{i\lambda,\delta} \ = \ \left\{ f \in C^0\left(\mathbb{C}^N \backslash \{0\}\right) \Big/ \ \forall c \in \mathbb{C} \backslash \{0\} : \ f(c\cdot) = \left(\frac{c}{|c|}\right)^{-\delta} |c|^{-i\lambda - N} f(\cdot) \right\}$$

and completing this space with respect to the norm $\| \cdot \|$ defined by:

$$\|f\|^2 = \int_{S^{2N-1}} |f(x)|^2 \, d\sigma(x)$$

The completion of $V^0_{i\lambda,\delta}$ is denoted by $V_{i\lambda,\delta}$. Functions of $V^0_{i\lambda,\delta}$ are also said to be (λ, δ)-*covariant*, or just *covariant* for short, when the exact values of λ and δ are not needed or when the context makes them clear.

We have the same properties as in the initial induced picture:

- $\pi_{i\lambda,\delta}$ is unitary;
- $\pi_{i\lambda,\delta}$ is irreducible if and only if $(\lambda, \delta) \neq (0, 0)$.

It is also explained in [8], amongst other things, how $\pi_{0,0}$ decomposes into the direct sum of two irreducible invariant subspaces.

It is both interesting and important to understand how the correspondence works between both versions of the induced picture.

Though we said we would not change notation, it will be helpful to write $\tilde{V}_{i\lambda,\delta}$ (resp. $\tilde{\pi}_{i\lambda,\delta}$ and $\lceil \cdot \rceil$) instead of $V_{i\lambda,\delta}$ (resp. $\pi_{i\lambda,\delta}$ and $\| \cdot \|$) when considering the initial induced picture.

Consider the vector $e_1 = (1, 0, \ldots, 0)$ of $\mathbb{C}^N$. Consider the natural action of G on vectors of $\mathbb{C}^N \backslash \{0\}$ (matrices times column vectors). The stabiliser S of e_1 is then the subgroup of matrices of G such that the coefficients of the first column are all zero except the one at the top that has to be 1. Looking back at the proof of Proposition 2.1, we see that this stabiliser is a subgroup of the parabolic subgroup $Q = MAN$; more accurately, elements of S can be written $s = man$ with:

- $m = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & A & 0 & C \\ \hline 0 & 0 & 1 & 0 \\ 0 & B & 0 & D \end{array}\right)$ with $\left(\begin{array}{c|c} A & C \\ \hline B & D \end{array}\right) \in \mathrm{Sp}(m, \mathbb{C})$;

- $a = I_N$;
- $n = \left(\begin{array}{cc|cc} 1 & {}^t u & 2s & {}^t v \\ 0 & I_m & v & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & -u & I_m \end{array}\right)$ with $(s, u, v) \in \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$.

Consider a function $\tilde{f}$ that belongs to the space $\tilde{V}^0_{i\lambda,\delta}$.
Elements man of S satisfy $e^{i\theta(m)} = \alpha(a) = 1$, so covariance of $\tilde{f}$ implies that the function $\tilde{f}$ is constant on cosets of G/S and thus defines a function $\tilde{f}$ on G/S by:

$$f_S(gS) = \tilde{f}(g)$$

Because G/S is homeomorphic to $\mathbb{C}^N \setminus \{0\}$, one can then define a (complex-valued) function f on $\mathbb{C}^N \setminus \{0\}$ by:

$$f(x) \ = \ f_S(gS) \ = \ \tilde{f}(g) \ , \text{ where one chooses any } g \text{ that satisfies } g(e_1) = x \, .$$

Maps f_S and f are continuous. Also, given $c \in \mathbb{C} \setminus \{0\}$, one can write (again, for any g that satisfies $g(e_1) = x$)

$$f(cx) \ = \ f(cg(e_1)) \ = \ f\left(g(ce_1)\right) \ = \ f(gC(e_1)) \ = \ \tilde{f}(gC) \ = \ \left(\frac{c}{|c|}\right)^{-\delta} \ |c|^{-i\lambda - N} \, \tilde{f}(g)$$

and consequently

$$f(cx) = \left(\frac{c}{|c|}\right)^{-\delta} |c|^{-i\lambda - N} f(x) \tag{2.1}$$

where C is the matrix

$$C = \left(\begin{array}{cc|cc} c & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ \hline 0 & 0 & c^{-1} & 0 \\ 0 & 0 & 0 & I_m \end{array}\right)$$

that belongs to Q because it can be written $C = man$ with:

$$m = \left(\begin{array}{cc|cc} \frac{c}{|c|} & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ \hline 0 & 0 & \left(\frac{c}{|c|}\right)^{-1} & 0 \\ 0 & 0 & 0 & I_m \end{array}\right) \ , \ a = \left(\begin{array}{cc|cc} |c| & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ \hline 0 & 0 & |c|^{-1} & 0 \\ 0 & 0 & 0 & I_m \end{array}\right) \text{ and } n = I_N.$$

So far, we have associated, in a one-to-one fashion, covariant functions $\tilde{f}$ on G to continuous functions f on $\mathbb{C}^N \setminus \{0\}$ that satisfy (2.1) for all non-zero complex numbers c. This correspondence makes representations and identifications compatible, in the sense that we have a commutative diagram for each element g of G (to observe this commutativity, one just looks for instance at the image of some $x \in S^{2N-1}$ under some $\pi_{i\lambda,\delta}(g)f$ and uses the definition of the left action of G and the way functions are identified):

$$\begin{array}{ccc} \tilde{f} & \longrightarrow & \tilde{\pi}_{i\lambda,\delta}(g)\tilde{f} \\ \big\downarrow & & \big\downarrow \\ f & \longrightarrow & \pi_{i\lambda,\delta}(g)f \end{array}$$

Now, what is the relevant norm to choose for f?

Because K is a subgroup of $\mathrm{U}(N)$, the sphere S^{2N-1} and its Euclidean measure σ are invariant under the natural action of K. Restricting the natural action of K to S^{2N-1}, the stabiliser of e_1 is $S_K = S \cap K$. Thus the map (taking, as above, any g such that $ge_1 = x$)

$$\begin{array}{cccc} \Psi \ : & S^{2N-1} & \longrightarrow & G/S_K \\ & x & \longmapsto & gS_K \end{array}$$

is a homeomorphism between S^{2N-1} and the quotient K/S_K; let us denote equivalence classes by $[k]$ (k refering to any representative of the coset kS_K). The natural action of K on the sphere induces a natural action (simply denoted here by a dot) on K/S_K, defined on equivalence classes by (taking k_1 and k_2 in K):

$$k_1 \cdot [k_2] = [k_1 k_2]$$

The Euclidean measure σ can be transported to K/S_K so as to define a K-invariant measure μ (the image measure of σ):

$$\mu(B) = \sigma(\Psi^{-1}(B))$$

for all measurable subsets B of K/S_K.

Because S_K is a closed subgroup of K and because K is compact, the homogeneous space K/S_K has a unique left invariant Borel measure, up to a constant. This is due to a standard integration theorem, which one can find for instance in [28], Chapter VIII, Section 3, Theorem 8.36 (this theorem is in fact stated for real-valued functions, but it passes on to complex-valued ones). So measures μ and $\tilde{\mu}$ must be proportional. Moreover, the same theorem says that for all continuous functions ϕ on K one has

$$\int_K \phi(k)\, dk = \int_{K/S_K} \left[\int_{S_K} \phi(ks) ds \right] d\tilde{\mu}([k])$$

where ds denotes the Haar measure of S_K (induced by the normalised left Haar measure of K). From this we deduce (using the fact that $\tilde{f}$ is constant on each coset of K/S_K)

$$\int_K \left| \tilde{f}(k) \right|^2 \, dk \;=\; \text{some constant} \cdot \int_{S^{2N-1}} |f(x)|^2 \, d\sigma(x)$$

where the constant involves the proportionality constant mentioned above and the volume of the subgroup S_K. These considerations justify our choice of norm $\| \cdot \|$. Covariance makes it a proper norm, in particular: the norm of a covariant function is equal to 0 if and only the function vanishes almost everywhere.

Our discussion above has shown that $\lceil \cdot \rceil$ and $\| \cdot \|$ are proportional, so they define the same topology. One needs not worry about the proportionality coefficient, because it does not affect unitarity, in the sense that unitarity of operators in one picture corresponds to unitarity in the other picture.

2.2.2.2 Another compact picture

2.3 Definition (Compact picture of $\pi_{i\lambda,\delta}$)**.** The carrying space here is the Hilbert space

$$V_{i\lambda,\delta} \;=\; \{ f \in L^2(S^{2N-1}) \;/\; \forall \theta \in \mathbb{R} \;:\; f(e^{i\theta} \cdot) = e^{-i\delta\theta} f(\cdot) \}$$

with respect to the norm $\| \cdot \|$ defined by:

$$\|f\|^2 = \int_K |f(k)|^2 \, dk$$

We say that elements of $V_{i\lambda,\delta}$ are δ-*covariant* (or just *covariant* for short, when the exact values of δ is not needed or when the context makes it clear). Again, the action of G is not as simple as one might think, but its restriction to K is just the left action of K.

The parameter λ does not explicitly appear in the compact picture, but is hidden in the following observation. Restriction of functions F of $V^0_{i\lambda,\delta}$, in the induced picture, to S^{2N-1} establishes a one-to-one correspondence with continuous elements f of the space $V_{i\lambda,\delta}$ seen in the compact picture. This correspondence works as follows:

$$F(x) = \|x\|^{-i\lambda - N} f\left(\frac{x}{\|x\|} \right)$$

for $x \in \mathbb{C}^N \setminus \{0\}$.

2.2.2.3 Another non-compact picture

This picture is based on two observations:

- The image of $e_1 = (1, 0, \ldots, 0) \in C^N$ under the natural action of the subgroup $\overline{N}$ is the hyperplane $\mathcal{P} = \{1\} \times \mathbb{C}^m \times \mathbb{C} \times \mathbb{C}^m$: indeed, using notations of section 2.1 for elements $n \in N$, an element ${}^t n \in \overline{N}$ assigns to e_1 the point $(1, u, 2s, v) \in \mathcal{P}$.
- The Haar measure of $\overline{N}$ is the Lebesgue measure $ds\, du\, dv$.

So restricting functions of $V^0_{i\lambda,\delta}$, in the initial induced picture, to functions on $\overline{N}$, in the initial non-compact picture, corresponds to restricting functions of $V^0_{i\lambda,\delta}$, in another induced picture, to the hyperplane $\mathcal{P}$, thereby obtaining continuous function of $L^2(\mathcal{P})$ with respect to the Lebesgue measure $ds\, du\, dv$. These considerations naturally lead us to the following choice:

2.4 Definition (Non-compact picture of $\pi_{i\lambda,\delta}$)**.** The carrying space is $L^2(\mathcal{P})$, with respect to the Lebesgue measure $ds\, du\, dv$, where $(1, u, 2s, v)$ denote the coordinates on $\mathcal{P}$. Again, the action of G is not as simple as one might think, but the restriction of the action to $\overline{N}$ is just the left action of $\overline{N}$.

One has to be careful with the way the induced picture and the non-compact picture correspond to one another. As much as one can always restrict functions of $V^0_{i\lambda,\delta}$, in the induced picture, to obtain continuous square integrable functions on $\{1\} \times \mathbb{C}^m \times \mathbb{C} \times \mathbb{C}^m$, the reverse procedure via covariance only gives functions defined on $\mathbb{C}^N \cap (z_1 \neq 0)$. In more detail, an element f of $V_{i\lambda,\delta}$ in the non-compact picture defines a function F on $S^{2N-1} \cap (z_1 \neq 0)$ as goes:

$$F(z, w) = \left(\frac{z_1}{|z_1|}\right)^{-\delta} |z_1|^{-i\lambda - N} f\left(\frac{w_1}{2z_1}, \frac{z_2}{z_1}, \ldots, \frac{z_n}{z_1}, \frac{w_2}{z_1}, \ldots, \frac{w_n}{z_1}\right)$$

The function f is continuous if and only if F is. The measure of $S^{2N-1} \cap (z_1 = 0)$ is 0 in S^{2N-1}, so this procedure enables one to obtain the whole of $V_{i\lambda,\delta}$ in the compact and induced pictures.

Via the bijection

$$\begin{array}{ccc} \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m & \longrightarrow & \{1\} \times \mathbb{C}^m \times \mathbb{C} \times \mathbb{C}^m \\ (s, u, v) & \longmapsto & (1, u, 2s, v) \end{array}$$

we will identify the spaces $L^2(\mathcal{P})$ and $L^2(\mathbb{C}^{2m+1})$ (with respect to the Lebesgue measures $ds\,du\,dv$).

Remark: the parameters λ and δ do not actually appear in this picture; they are hidden in the restriction/extension process.

Chapter 3

Actions of $\mathrm{Sp}(n)$ and $\mathrm{Sp}(1)$

La notion de K-type est au cœur de tout notre travail, dont l'objectif est d'associer à certains d'entre eux des fonctions spéciales. Comme nous l'avons annoncé en introduction, il y aura deux actions à considérer : une action à gauche de $\mathrm{Sp}(n)$ (étudiée dans le paragraphe 3.1) et une à droite de $\mathrm{Sp}(1)$ (étudiée dans le paragraphe 3.2).

Pour l'action à gauche, on travaille dans le modèle compact, où l'espace de Hilbert est un sous-espace de $L^2(S^{4n-1})$. Il est bien connu que celui-ci se décompose en une somme directe hilbertienne, dont chaque terme est constitué de restrictions à la sphère de polynômes harmoniques homogènes de même degré ; ces restrictions sont les harmoniques sphériques ; on note H^k l'espace de tels polynômes de degré k et $\mathcal{Y}^k$ l'espace des harmoniques sphériques correspondantes. Les espaces $\mathcal{Y}^k$ sont des espaces invariants irréductibles pour l'action à gauche du groupe orthogonal $\mathrm{O}(4n)$.

Les identifications $\mathbb{R}^{4n} \simeq \mathbb{C}^{2n} \simeq \mathbb{H}^n$ permettent de considérer les polynômes de H^k comme des polynômes à variables complexes $z \in \mathbb{C}^{2n}$ (en tenant compte de leur conjuguées $\overline{z}$) et de considérer $\mathrm{U}(2n)$ comme un sous-groupe de $\mathrm{O}(4n)$. Ainsi, définissant $\mathrm{H}^{\alpha,\beta}$ comme l'espace des polynômes homogènes harmoniques de degré d'homogénéité (α, β) sur $\mathbb{C}^N$ (c'est à dire de degré α en z et β en $\overline{z}$) et en définissant $\mathcal{Y}^{\alpha,\beta}$ comme l'espace de leurs restrictions à la sphère, on peut considérer que $\mathrm{H}^{\alpha,\beta}$ est un sous-espace de $\mathrm{H}^{\alpha+\beta}$ et que $\mathcal{Y}^{\alpha,\beta}$ est un sous-espace de $\mathcal{Y}^{\alpha+\beta}$. Les espaces $\mathcal{Y}^{\alpha,\beta}$ sont invariants et irréductibles sous l'action à gauche de $\mathrm{U}(2n)$, donc la restriction de cette action à $\mathrm{Sp}(n)$ préserve encore les espaces $\mathcal{Y}^{\alpha+\beta}$, qui cette fois ne sont plus irréductibles. Comprendre la structure de $L^2(S^{4n-1})$ par rapport à l'action à gauche de $\mathrm{Sp}(n)$ nécessite donc simplement de se restreindre à chaque espace $\mathrm{H}^{\alpha,\beta}$. On fixe alors un couple quelconque (α, β).

La théorie des représentations montre que chaque composante irréductible est associée à un vecteur de plus haut poids (unique à un coefficient près). Le paragraphe 3.1.2 identifie, dans le théorème 3.2, ces vecteurs de plus haut poids $P_\gamma^{\alpha,\beta}$, ainsi que leurs poids $\sigma_\gamma^{\alpha,\beta}$; γ est un paramètre entier compris entre 0 et $\min(\alpha,\beta)$ (il compte les composantes, en quelque sorte). Dans le reste du chapitre 3.1 :

- nous calculons les dimensions des composantes irréductibles $V_\gamma^{\alpha,\beta}$ engendrées par les polynômes $P_\gamma^{\alpha,\beta}$, ce qui se fait à l'aide des plus haut poids et du théorème des dimensions de Weyl ;
- nous vérifions que ces dimensions épuisent la dimension de $\mathrm{H}^{\alpha,\beta}$, démontrant ainsi la somme directe donnée par le théorème 3.8, à savoir :

$$\pi_{i\lambda,\delta}\Big|_K \cong \sum_{\substack{(\alpha,\beta)\in\mathbb{N}^2\\ \delta=\beta-\alpha\\ \gamma\in I^{\alpha,\beta}}}^{\oplus} L\Big|_{V_\gamma^{\alpha,\beta}}$$

Pour l'action à droite de $\mathrm{Sp}(1)$, dans le paragraphe 3.2, on définit déjà cette action R précisément, en expliquant comment elle est liée à la multiplication scalaire à droite dans $\mathbb{H}^n$. L'action à droite d'une matrice $q = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \in \mathrm{Sp}(1)$ sur une fonction f à variables complexes $(z,w)\in\mathbb{C}^n\times\mathbb{C}^n \simeq \mathbb{C}^{2n}$ fonctionne ainsi :

$$R(q)f(z;w) = f\left(az - b\overline{w}, aw + b\overline{z}\right)$$

Pour en étudier les vecteurs de plus hauts poids, on complexifie R, obtenant une représentation de $\mathfrak{sl}(2,\mathbb{C})$, identifiant alors les éléments d'une certaine base de $\mathfrak{sl}(2,\mathbb{C})$ à des opérateurs différentiels : c'est l'objet de la proposition 3.11. Le théorème 3.12 montre que les plus hauts poids se trouvent parmi les $P_\gamma^{\alpha,\beta}$ et permet d'aboutir au théorème 3.13 qui explicite la décomposition de H^k (k valant $\alpha+\beta$) en irréductibles pour l'action à droite de $\mathrm{Sp}(1)$. Il est à noter que c'est bien H^k tout entier qu'il faut considérer, car les sous-espaces $\mathrm{H}^{\alpha,\beta}$ ne sont en général pas stables par l'action R.

La façon dont les actions à gauche et à droite interagissent est résumée dans le diagramme 3.1. Pour finir, nous montrons que dans ce diagramme, il existe une unique harmonique sphérique (à un coefficient près) qui soit invariante par les deux actions L et R à la fois (on dit qu'elle est bi-invariante). Une méthode par induction sur les coefficients permet de la calculer. Cela fait l'objet du paragraphe 3.2.3.

In this section, we work with the compact picture of the induced representations $\pi_{i\lambda,\delta}$. Because the restrictions $\pi_{i\lambda,\delta}\Big|_K$ coincide with the left action of K on $V_{i\lambda,\delta}$, we now simply write L instead of $\pi_{i\lambda,\delta}$, whatever the value of δ. If we change the values of δ, we can reconstruct the whole of $V = L^2\left(S^{2N-1}\right)$, as we will see. We intend to study how this Hilbert space decomposes into irreducibles under two actions: the left action L of $K = \mathrm{Sp}(n)$ and the right of $\mathrm{Sp}(1)$ we will define later on.

3.1 Left action of Sp(n)

3.1.1 Preliminaries

For the time being, denote the coordinates of $\mathbb{R}^{2N}$ by $(x_1, \ldots x_{2N})$. Consider the Laplace operator $\Delta_{\mathbb{R}}$ defined by:

$$\Delta_{\mathbb{R}} = \sum_{i=1}^{2N} \frac{\partial^2}{\partial x_i^2}$$

For $k \in \mathbb{N}$, denote by H^k the complex vector space of polynomial functions f defined on $\mathbb{R}^{2N}$, with complex coefficients and such that:

1. f is homogeneous of degree k;

2. f is harmonic, that is, $\Delta_{\mathbb{R}}(f) = 0$.

Denote by $\mathcal{Y}^k$ the complex vector space that consists in the restrictions to S^{2N-1} of elements of H^k; these restrictions are called *spherical harmonics*. It is well known that (see for example [12], chapter 9):

$$L^2\left(S^{2N-1}\right) = \widehat{\bigoplus_{k\in\mathbb{N}}} \mathcal{Y}^k \tag{3.1}$$

where the various spaces $\mathcal{Y}^k$ are orthogonal to one another in $L^2(S^{2N-1})$. It is also well known that the subspaces $\mathcal{Y}^d$ are stable under the left action of $\mathrm{SO}(2N)$ and define irreducible and pairwise inequivalent representations of $\mathrm{SO}(2N)$ (see chapter 9 of [51]). Because K can be seen as a subgroup of $\mathrm{SU}(N)$ which can itself be seen as a subgroup of $\mathrm{SO}(2N)$, the sphere S^{2N-1} is stable under the natural actions of K and $\mathrm{SU}(N)$ and one can therefore consider the left actions of K and $\mathrm{SU}(N)$ on $\mathcal{Y}^k$. It so happens that the right action of $\mathrm{Sp}(1)$ that we will define later also preserves $\mathcal{Y}^k$. So, to understand how

$L^2(S^{2N-1})$ decomposes under the left action of K and the right action of Sp(1), one just needs to concentrate on each $\mathcal{Y}^k$.

Let us switch to complex coordinates. Put $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $y = (x_{N+1}, \ldots, x_{2N}) \in \mathbb{R}^N$ and $z = x + iy = (z_1, \ldots, z_N) \in \mathbb{C}^N$. Because $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$, a function f of x and y can be written as a function F of the complex variable z and its conjugate $\bar{z}$:

$$f(x, y) = F(z, \bar{z})$$

An element u of SU(N) can be viewed as a matrix $\tilde{u}$ of SO($2N$), through the following embedding of the set of complex $N \times N$ matrices into the set of real $2N \times 2N$ matrices (the way we have identified $\mathbb{R}^{2N}$ and $\mathbb{C}^N$ is precisely designed to match this embedding): one decomposes u into two real matrices A and B, writing $u = A + iB$, then defines

$$\tilde{u} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

and checks that $\tilde{u}$ belongs to SO($2N$) and that the mapping $u \longrightarrow \tilde{u}$ is an injective group morphism. The left action of $\tilde{u}$ on f then transfers to F in the obvious way:

$$L(u)P(z, \bar{z}) = P(u^{-1}z, \overline{u^{-1}}\bar{z}) = P(u^{-1}z, {}^t u\bar{z}) \tag{3.2}$$

The last equality holds because u is unitary and therefore $\overline{u^{-1}} = {}^t u$.

In the coordinates $(z, \overline{z})$, the Laplace operator becomes:

$$\Delta_{\mathbb{C}} = 4 \sum_{i=1}^{N} \frac{\partial^2}{\partial z_i \partial \overline{z}_i}$$

For $\alpha, \beta \in \mathbb{N}$, consider the space $\mathrm{H}^{\alpha,\beta}$ of polynomials $P(z, \bar{z})$ ($z \in \mathbb{C}^N$) such that:

1. P is homogeneous of degree α in z and of degree β in $\bar{z}$;

2. $\Delta_{\mathbb{C}}(P) = 0$.

Each $\mathrm{H}^{\alpha,\beta}$ is invariant under the left action of unitary matrices; the resulting representations of the unitary group SU(N) on $\mathrm{H}^{\alpha,\beta}$ are irreducible and pairwise inequivalent (again, see [51], chapter 11).

3.1 Proposition. *The dimension of* $\mathrm{H}^{\alpha,\beta}$ *is given by the following formula:*

$$\dim \mathrm{H}^{\alpha,\beta} = \frac{(\alpha+\beta+N-1)(\alpha+N-2)!(\beta+N-2)!}{(N-1)!(N-2)!\alpha!\beta!}$$

Via coordinate identifications, one can consider spaces $\mathrm{H}^{\alpha,\beta}$ as subspaces of $\mathrm{H}^{\alpha+\beta}$. Given $k \in \mathbb{N}$, this leads to the natural isomorphism (see [51], chapter 11):

$$\mathrm{H}^k \simeq \bigoplus_{\substack{(\alpha,\beta)\in\mathbb{N}^2 \\ \alpha+\beta=k}} \mathrm{H}^{\alpha,\beta} \tag{3.3}$$

We denote by $\mathcal{Y}^{\alpha,\beta}$ the space of restrictions of elements of $\mathrm{H}^{\alpha,\beta}$ to the unit sphere S^{4N-1}. From (3.1) and (3.3) we deduce:

$$L^2\left(S^{2N-1}\right) = \widehat{\bigoplus_{(\alpha,\beta)\in\mathbb{N}^2}} \mathcal{Y}^{\alpha,\beta}$$

Consequently, in the compact picture:

$$V_{i\lambda,\delta} = \widehat{\bigoplus_{\substack{(\alpha,\beta)\in\mathbb{N}^2 \\ \delta=\beta-\alpha}}} \mathcal{Y}^{\alpha,\beta} \tag{3.4}$$

We now discuss a basic yet important point: we can work with homogeneous harmonic polynomials rather than spherical harmonics. In order to explain why, let us fix any non-negative integers α and β and denote by $\mathcal{R}$ the map that restricts polynomials of $\mathrm{H}^{\alpha,\beta}$ to the unit sphere S^{2N-1}; the target space of $\mathcal{R}$ is precisely $\mathcal{Y}^{\alpha,\beta}$. Because of homogeneity, the map $\mathcal{R}$ is a bijection; it is in fact a linear isomorphism. Let us introduce two notations for the left action of K, according to which kind of functions we want it to apply to:

- π will denote the representation of K that corresponds to the left action on $\mathcal{Y}^{\alpha,\beta}$.
- π' will denote the representation of K that corresponds to the left action on $\mathrm{H}^{\alpha,\beta}$.

The restriction map $\mathcal{R}$ commutes with both left actions of K. Said otherwise, the following diagram commutes (with $k \in K$):

$$\begin{tikzcd} \mathrm{H}^{\alpha,\beta} \arrow[r, "\pi'(k)"] \arrow[d, "\mathcal{R}"] & \mathrm{H}^{\alpha,\beta} \arrow[d, "\mathcal{R}"] \\ \mathcal{Y}^{\alpha,\beta} \arrow[r, "\pi(k)"'] & \mathcal{Y}^{\alpha,\beta} \end{tikzcd}$$

Because the spaces $\mathrm{H}^{\alpha,\beta}$ and $\mathcal{Y}^{\alpha,\beta}$ are finite-dimensional, π and π' are differentiable. The differentials (at the identity) $d\pi$ and $d\pi'$ are also related by a commuting diagram (with $X \in \mathfrak{k}$):

$$\begin{tikzcd} \mathrm{H}^{\alpha,\beta} \arrow[r, "d\pi'(X)"] \arrow[d, "\mathcal{R}"] & \mathrm{H}^{\alpha,\beta} \arrow[d, "\mathcal{R}"] \\ \mathcal{Y}^{\alpha,\beta} \arrow[r, "d\pi(X)"'] & \mathcal{Y}^{\alpha,\beta} \end{tikzcd}$$

In other words, the linear isomorphism $\mathcal{R}$ intertwines $d\pi$ and $d\pi'$. This implies that weights of π coincide with weights of π', weight vectors of π correspond to weight vectors of π' and highest weight vectors of π correspond to highest weight vectors of π'.

This is why we now identify π and π', going back to our notation L and denoting by dL the differential of L (at the identity).

Remark: we will often identify polynomials of $\mathrm{H}^{\alpha,\beta}$ and the corresponding spherical harmonics of $\mathcal{Y}^{\alpha,\beta}$.

We now work with polynomials, to compute highest weight vectors and highest weights.

The space $\mathrm{H}^{\alpha,\beta}$ is stable under the action of K, but not necessarily irreducible: it can break up into (obviously) finite-dimensional irreducible components. Each component is generated by some highest weight vector under the action of K. Components being finite-dimensional, one can differentiate the restrictions of π to each of them. Remembering (3.2), if X belongs to the Lie algebra $\mathfrak{k}$ of K, if P belongs to the carrying space of L and if z belongs to $\mathbb{C}^N$, then by definition:

$$dL(X)(P)(z,\bar{z}) = \frac{d}{dt}\bigg|_{t=0} \left(P\left(\exp(-tX)z, {}^t\Big(\exp(tX)\Big)\bar{z}\right)\right)$$

The chain rule of differentiation applied to functions of real variables and then rewritten in terms of complex variables gives:

$$dL(X)(P)(z,\bar{z}) = \left(\begin{array}{c|c} \frac{\partial P}{\partial z}(z,\bar{z}) & \frac{\partial P}{\partial \bar{z}}(z,\bar{z}) \end{array} \right) \begin{pmatrix} \frac{d}{dt}\big|_{t=0} (\exp(-tX)z) \\ \frac{d}{dt}\big|_{t=0} ({}^t\left[\exp(tX)\right]\bar{z}) \end{pmatrix}$$

so that we have:

$$dL(X)(P)(z,\bar{z}) = \left(\begin{array}{c|c} \frac{\partial P}{\partial z}(z,\bar{z}) & \frac{\partial P}{\partial \bar{z}}(z,\bar{z}) \end{array} \right) \begin{pmatrix} -Xz \\ {}^tX\bar{z} \end{pmatrix} \tag{3.5}$$

The infinitesimal action dL can be complexified, in other words extended in the natural way to the complexification $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{C})$ of $\mathfrak{k} = \mathfrak{sp}(n)$. This complexification of dL is also defined by Formula (3.5), but this time with $X \in \mathfrak{g}$.

3.1.2 Highest weight vectors

We fix $k \in \mathbb{N}$ and $(\alpha,\beta) \in \mathbb{N}^2$ such that $\alpha + \beta = k$.

To make polynomial calculations clearer, we change our system of notation for complex variables: we write $(z,w) = (z_1,...,z_n,w_1,...,w_n)$ instead of our initial $z = (z_1,...,z_N)$, with of course $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and $w = (w_1, ..., w_n) \in \mathbb{C}^n$. We now state and prove a theorem which is the heart of this chapter:

3.2 Theorem. *Denote by* $I^{\alpha,\beta}$ *the set of integers* γ *such that* $0 \leq \gamma \leq \min(\alpha, \beta)$. *For* $\gamma \in I^{\alpha,\beta}$, *consider the polynomial* $P_\gamma^{\alpha,\beta}$ *defined by:*

$$P_\gamma^{\alpha,\beta}(z, w, \bar{z}, \bar{w}) = w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma} (w_2 \bar{z_1} - w_1 \bar{z_2})^\gamma$$

1. $P_\gamma^{\alpha,\beta}$ *belongs to* $\mathrm{H}^{\alpha,\beta}$.

2. *For any element* H *of the Cartan subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$*:*

$$dL(H) P_\gamma^{\alpha,\beta} = \left[(\alpha + \beta - \gamma) L_1 + \gamma L_2\right](H) P_\gamma^{\alpha,\beta}$$

   This means that $P_\gamma^{\alpha,\beta}$ *is a weight vector associated to the weight*

$$\sigma_\gamma^{\alpha,\beta} = (\alpha + \beta - \gamma, \gamma, 0, \ldots, 0) = (k - \gamma, \gamma, 0, \ldots, 0)$$

3. *If* i *and* j *denote positive integers, if* X *denotes* $E_{i,j} - E_{n+j,n+i}$ *or* $E_{i,n+j} + E_{j,n+i}$ *when* $1 \leq i < j \leq n$, *or* $E_{i,n+i}$ *when* $1 \leq i \leq n$, *then:*

$$dL(X) P_\gamma^{\alpha,\beta} = 0$$

   This means that the highest weight condition is satisfied.

In other words, $P_\gamma^{\alpha,\beta}$ *is a highest weight vector: under the action of* π, *it generates an irreducible invariant subspace* $V_\gamma^{\alpha,\beta}$ *of* $\mathrm{H}^{\alpha,\beta}$.

Proof:

For simplicity, let us just write P instead of $P_\gamma^{\alpha,\beta}$.

1. Multiplying out the brackets of P gives a list of monomials that are all homogeneous of degree γ in both w and $\bar{z}$. Combining this with the powers of the terms w_1 and $\bar{z_1}$ that sit before the brackets and the fact that no variable appears at the same time as its conjugate, we see that P belongs to $\mathrm{H}^{\alpha,\beta}(\mathbb{C}^N)$.

2. and 3. We use Formula (3.5) for the specific elements of $\mathfrak{g}$ we need to consider:

- if H belongs to $\mathfrak{h}$ and if $h_1, ..., h_n, -h_1, ..., -h_n$ denote the complex diagonal terms of H:

$$dL(H)P(z,w,\bar{z},\bar{w}) = \sum_{i=1}^{n} h_i \left(-z_i \frac{\partial P}{\partial z_i} + w_i \frac{\partial P}{\partial w_i} + \bar{z}_i \frac{\partial P}{\partial \bar{z}_i} - \bar{w}_i \frac{\partial P}{\partial \bar{w}_i} \right)$$

So:

$$dL(H)P(z,w,\bar{z},\bar{w}) = h_1 w_1 \frac{\partial P}{\partial w_1} + h_2 w_2 \frac{\partial P}{\partial w_2} + h_1 \bar{z}_1 \frac{\partial P}{\partial \bar{z}_1} + h_2 \bar{z}_2 \frac{\partial P}{\partial \bar{z}_2} \tag{3.6}$$

- if $X = E_{i,j} - E_{n+j,n+i}$ $(1 \le i < j \le n)$:

$$dL(X)P(z,w,\bar{z},\bar{w}) = -z_j \frac{\partial P}{\partial z_i} + w_i \frac{\partial P}{\partial w_j} + \bar{z}_i \frac{\partial P}{\partial \bar{z}_j} - \bar{w}_j \frac{\partial P}{\partial \bar{w}_i}$$

So:

  - for $j \ge 3$:

$$dL(X)P = 0$$

  - for $i = 1$ and $j = 2$ (only remaining case):

$$dL(X)P(z,w,\bar{z},\bar{w}) = w_1 \frac{\partial P}{\partial w_2} + \bar{z}_1 \frac{\partial P}{\partial \bar{z}_2} \tag{3.7}$$

- if $X = E_{i,n+j} + E_{j,n+i}$ $(1 \le i < j \le n)$:

$$dL(X)P(z,w,\bar{z},\bar{w}) = -w_j \frac{\partial P}{\partial z_i} - w_i \frac{\partial P}{\partial z_j} + \bar{z}_j \frac{\partial P}{\partial \bar{w}_i} + \bar{z}_i \frac{\partial P}{\partial \bar{w}_j}$$

So:

$$dL(X)P = 0$$

- If $X = E_{i,n+i}$ $(1 \le i \le n)$:

$$dL(X)P(z,w,\bar{z},\bar{w}) = -w_i \frac{\partial P}{\partial z_i} + \bar{z}_i \frac{\partial P}{\partial \bar{w}_i}$$

So:

$$dL(X)P = 0$$

All that remains to be done is compute the partial derivatives in formulas (3.6) and (3.7) to find:

- (i) $\forall\ H \in \mathfrak{h}$: $dL(H)P = [(\alpha + \beta - \gamma)h_1 + \gamma h_2]\, P$.

- (ii) $dL(E_{1,2} - E_{n+2,n+1})P = 0$.

To make computations easier we set $B = w_2\bar{z_1} - w_1\bar{z_2}$.

To establish (i), let us compute $dL(H)P(z, w, \bar{z}, \bar{w})$:

$$\begin{gathered} h_1 w_1 \left((\alpha-\gamma) w_1^{\alpha-\gamma-1} \bar{z_1}^{\beta-\gamma} B^\gamma - \gamma \bar{z_2} w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma} B^{\gamma-1}\right) \\ + \\ h_2 w_2 \left(\gamma \bar{z_1} w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma} B^{\gamma-1}\right) \\ + \\ h_1 \bar{z_1} \left((\beta-\gamma) w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma-1} B^\gamma + \gamma w_2 w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma} B^{\gamma-1}\right) \\ + \\ h_2 \bar{z_2} \left(-\gamma w_1 w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma} B^{\gamma-1}\right) \end{gathered} \tag{3.8}$$

Then we can organise the terms of (3.8) to get:

$$\begin{gathered} B^\gamma \left(h_1 (\alpha-\gamma) w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma} + h_1 (\beta-\gamma) w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma}\right) \\ + \\ B^{\gamma-1} h_1 \gamma w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma} (w_2 \bar{z_1} - w_1 \bar{z_2}) \\ + \\ B^{\gamma-1} h_2 \gamma w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma} (w_2 \bar{z_1} - w_1 \bar{z_2}) \end{gathered} \tag{3.9}$$

which can obviously be rewritten

$$[(\alpha+\beta-\gamma) h_1 + \gamma h_2]\, w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma} B^\gamma \tag{3.10}$$

We finally recognise the desired expression: $[(\alpha+\beta-\gamma)h_1 + \gamma h_2]\, P$.

To establish (ii), let us compute $dL(E_{1,2} - E_{n+2,n+1})P(z, w, \bar{z}, \bar{w})$:

$$\left(w_1 \frac{\partial P}{\partial w_2} + \bar{z_1} \frac{\partial P}{\partial \bar{z_2}}\right)(z, w, \bar{z}, \bar{w}) = w_1 \gamma \bar{z_1} w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma} B^{\gamma-1} - \bar{z_1} \gamma w_1 w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma} B^{\gamma-1}$$

which obviously equals 0.

End of proof.

Each component corresponds to a specific eigenvalue of the Casimir operator Ω_L of L. Proposition 1.6 and theorem 3.2 imply:

3.3 Corollary. *Consider the irreducible representation* $(L, V_\gamma^{\alpha,\beta})$*. We remind the reader that its highest weight is* $(\alpha+\beta-\gamma)L_1+\gamma L_2$*. Then (denoting by* Id *the identity map):*

$$\Omega_L = -\left[(\alpha+\beta-\gamma)^2+2n(\alpha+\beta)+\gamma^2-2\gamma\right]\cdot Id$$

3.1.3 Isotypic decomposition with respect to Sp(n)

Again we fix $k\in\mathbb{N}$ and $(\alpha,\beta)\in\mathbb{N}^2$ such that $\alpha+\beta=k$. We now want to establish that $\mathrm{H}^{\alpha,\beta}$ is the direct sum of all the subspaces $V_\gamma^{\alpha,\beta}$. We do this by computing dimensions and showing that the dimensions of the various $V_\gamma^{\alpha,\beta}$ add up to the dimension of $\mathrm{H}^{\alpha,\beta}$. We point out that in the formulas to come, we use the standard convention $0!=1$:

3.4 Proposition. *The dimension of* $V_\gamma^{\alpha,\beta}$ *is given by the following formula:*

$$\mathrm{dim}V_\gamma^{\alpha,\beta} = \frac{(k-\gamma+N-2)!\,(\gamma+N-3)!\,(k-2\gamma+1)\,(k+N-1)}{(k-\gamma+1)!\,\gamma!\,(N-1)!\,(N-3)!}$$

So we denote this dimension d_γ^k*, omitting the values of* α *and* β*.*

Proof:

We apply Weyl's dimension formula (1.4), choosing the highest weight $\sigma=\sigma_\gamma^{\alpha,\beta}$ (given by theorem 3.2), $\rho_K=(n,n-1,...,1)$ and using the three types of positive roots L_i-L_j, L_i+L_j and $2L_i$ introduced in section 1.3.1 of Chapter 1. With these choices, $\mathrm{dim}V_\gamma^{\alpha,\beta}$ is equal to:

$$\prod_{i<j}\frac{<\sigma+\rho_K,L_i-L_j>}{<\rho_K,L_i-L_j>}\prod_{i<j}\frac{<\sigma+\rho_K,L_i+L_j>}{<\rho_K,L_i+L_j>}\prod_{i=1}^n\frac{<\sigma+\rho_K,2L_i>}{<\rho_K,2L_i>} \tag{3.11}$$

Looking into each product, we see that the terms for $i\geq 3$ all cancel out so that (3.11) can be rewritten as:

$$\prod_{j=2}^n\frac{<\sigma+\rho_K,L_1-L_j>}{<\rho_K,L_1-L_j>}\prod_{j=3}^n\frac{<\sigma+\rho_K,L_2-L_j>}{<\rho_K,L_2-L_j>}$$

$$\prod_{j=2}^n\frac{<\sigma+\rho_K,L_1+L_j>}{<\rho_K,L_1+L_j>}\prod_{j=3}^n\frac{<\sigma+\rho_K,L_2+L_j>}{<\rho_K,L_2+L_j>} \tag{3.12}$$

$$\frac{<\sigma+\rho_K,2L_1>}{<\rho_K,2L_1>}\frac{<\sigma+\rho_K,2L_2>}{<\rho_K,2L_2>}$$

With $\sigma + \rho_K = (k-\gamma+n, \gamma+n-1, n-2, ..., 1)$, (3.12) becomes:

$$\frac{(k-\gamma+n)-(\gamma+n-1)}{n-(n-1)} \prod_{j=3}^{n} \frac{(k-\gamma+n)-(n-j+1)}{n-(n-j+1)} \prod_{j=3}^{n} \frac{((\gamma+n-1)-(n-j+1)}{(n-1)-(n-j+1)}$$

$$\frac{(k-\gamma+n)+(\gamma+n-1)}{n+(n-1)} \prod_{j=3}^{n} \frac{(k-\gamma+n)+(n-j+1)}{n+(n-j+1)} \prod_{j=3}^{n} \frac{((\gamma+n-1)+(n-j+1)}{(n-1)+(n-j+1)} \tag{3.13}$$

$$\frac{2(k-\gamma+n)}{2n} \frac{2(\gamma+n-1)}{2(n-1)}$$

Formula (3.13) can be rewritten as:

$$\frac{k-2\gamma+1}{1} \prod_{j=3}^{n} \frac{k-\gamma+j-1}{j-1} \prod_{j=3}^{n} \frac{\gamma+j-2}{j-2}$$

$$\frac{k+N-1}{N-1} \prod_{j=3}^{n} \frac{k-\gamma+N-j+1}{N-j+1} \prod_{j=3}^{n} \frac{\gamma+N-j}{N-j} \tag{3.14}$$

$$\frac{k-\gamma+n}{n} \frac{\gamma+n-1}{n-1}$$

Formula (3.14) can be reorganised as:

$$\prod_{j=2}^{n} \frac{k-\gamma+j}{j} \prod_{j=1}^{n-1} \frac{\gamma+j}{j}$$

$$\prod_{j=3}^{n} \frac{k-\gamma+N+1-j}{N+1-j} \prod_{j=3}^{n} \frac{\gamma+N-j}{N-j} \tag{3.15}$$

$$\frac{(k-2\gamma+1)(k+N-1)}{(N-1)}$$

Writing products in terms of factorials we get the following expression:

$$\frac{(k-\gamma+n)!}{(k-\gamma+1)!n!} \frac{(\gamma+n-1)!}{\gamma!(n-1)!}$$

$$\frac{(k-\gamma+N+1-3)!(N+1-n-1)!}{(k-\gamma+N+1-n-1)!(N+1-3)!} \frac{(\gamma+N-3)!(N-n-1)!}{(\gamma+N-n-1)!(N-3)!} \tag{3.16}$$

$$\frac{(k-2\gamma+1)(k+N-1)}{(N-1)}$$

Formula (3.16) becomes:

$$\frac{(k-\gamma+n)!}{(k-\gamma+1)!n!} \frac{(\gamma+n-1)!}{\gamma!(n-1)!}$$

$$\frac{(k-\gamma+N-2)!n!}{(k-\gamma+n)!(N-2)!} \frac{(\gamma+N-3)!(n-1)!}{(\gamma+n-1)!(N-3)!}$$

$$\frac{(k-2\gamma+1)(k+N-1)}{(N-1)}$$

One finishes the proof by noticing various cancellations.

End of proof.

3.5 Proposition. *The dimension of* $\mathrm{H}^{\alpha,\beta}$ *can be written as the following sum:*

$$\dim \mathrm{H}^{\alpha,\beta} = \sum_{\gamma \in I_\gamma^{\alpha,\beta}} \dim V_\gamma^{\alpha,\beta}$$

Proof:

We start by supposing $\alpha \leq \beta$, so that $\gamma \leq \alpha$. Using propositions 3.1 and 3.4 and after several cancelations, we just need to prove this equality (as previously mentioned, whenever 0! occurs, one takes $0! = 1$):

$$\textstyle\sum_{\gamma=0}^{\alpha} \frac{(k-\gamma+N-2)!(\gamma+N-3)!(k-2\gamma+1)}{(k-\gamma+1)!\gamma!} = \frac{(\alpha+N-2)!(\beta+N-2)!}{(N-2)\alpha!\beta!} \tag{3.17}$$

If $\beta \leq \alpha$, so that $\gamma \leq \beta$, we need to prove:

$$\textstyle\sum_{\gamma=0}^{\beta} \frac{(k-\gamma+N-2)!(\gamma+N-3)!(k-2\gamma+1)}{(k-\gamma+1)!\gamma!} = \frac{(\alpha+N-2)!(\beta+N-2)!}{(N-2)\alpha!\beta!}$$

These two formulas are in fact equivalent, because of the symmetrical roles of α and β. So one just has to prove Formula (3.17); in this formula, let us set $\mu = k+1$ and $A = N-3$, which changes Formula (3.17) into

$$\textstyle\sum_{\gamma=0}^{\alpha} \frac{(A+\mu-\gamma)!(A+\gamma)!(\mu-2\gamma)}{(\mu-\gamma)!\gamma!} = \frac{(\alpha+A+1)!(\mu-\alpha+A)!}{(A+1)\alpha!(\mu-\alpha-1)!} \tag{3.18}$$

Forgetting about the exact expression of μ, we shall prove Formula (3.18) by induction on α, proving the following statement:

$$\forall \alpha \in \mathbb{N},\ \forall \mu \geq 2\alpha+1: \text{ formula (3.18) is true} \tag{3.19}$$

One easily sees that Statement (3.19) is true for $\alpha = 0$.

Let us now suppose that it is true for given $\alpha \in \mathbb{N}$ and consider $\alpha+1$ and some $\mu \geq 2(\alpha+1)+1$. Then:

$$\textstyle\sum_{\gamma=0}^{\alpha+1} \frac{(A+\mu-\gamma)!(A+\gamma)!(\mu-2\gamma)}{(\mu-\gamma)!\gamma!}$$

separates into

$$\left[\textstyle\sum_{\gamma=0}^{\alpha} \frac{(A+\mu-\gamma)!(A+\gamma)!(\mu-2\gamma)}{(\mu-\gamma)!\gamma!}\right] + \frac{(A+\mu-\alpha-1)!(A+\alpha+1)!(\mu-2\alpha-2)}{(\mu-\alpha-1)!(\alpha+1)!}$$

Because $\mu \geq 2(\alpha+1)+1 \geq 2\alpha+1$, we can apply the induction hypothesis to the sum; we obtain:

$$\tfrac{(\alpha+A+1)!(\mu-\alpha+A)!}{(A+1)\alpha!(\mu-\alpha-1)!} + \tfrac{(A+\mu-\alpha-1)!(A+\alpha+1)!(\mu-2\alpha-2)}{(\mu-\alpha-1)!(\alpha+1)!} =$$

$$\tfrac{(\alpha+A+1)!(\mu-\alpha+A-1)!}{(\mu-\alpha-1)!\alpha!}\left[\tfrac{\mu-\alpha+A}{A+1} + \tfrac{\mu-2\alpha-2}{\alpha+1}\right] \tag{3.20}$$

One can check that the brackets can be written as

$$\tfrac{(\mu-\alpha-1)(\alpha+A+2)}{(A+1)(\alpha+1)}$$

so that (3.20) becomes:

$$\tfrac{(\alpha+A+1)!(\mu-\alpha+A-1)!}{(\mu-\alpha-1)!\alpha!}\tfrac{(\mu-\alpha-1)(\alpha+A+2)}{(A+1)(\alpha+1)} =$$

$$\tfrac{(\alpha+A+2)!(A+\mu-\alpha-1)!(\mu-\alpha-1)}{(A+1)(\mu-\alpha-1)!(\alpha+1)!} =$$

$$\tfrac{[(\alpha+1)+A+1]![\mu-(\alpha+1)+A]!}{(A+1)(\alpha+1)![\mu-(\alpha+1)-1]!}$$

This finishes the induction step and thus the proof.

End of proof.

We can now give the isotypic decomposition of the restrcition of L to $\mathrm{H}^{\alpha,\beta}$:

3.6 Theorem. $\mathrm{H}^{\alpha,\beta} = \bigoplus_{\gamma\in I^{\alpha,\beta}} V_\gamma^{\alpha,\beta}$

Proof:

The Peter-Weyl theorem 1.2 implies that two inequivalent subrepresentations of a unitary representation of a compact Lie group must act on orthogonal subspaces. Therefore the spaces $V_\gamma^{\alpha,\beta}$ are pairwise orthogonal (the corresponding subrepresentations each have a different weight, so they are pairwise inequivalent). As Proposition 3.5 says that the dimensions of all the $V_\gamma^{\alpha,\beta}$ add up to the dimension of $\mathrm{H}^{\alpha,\beta}$, Theorem 3.6 follows.

End of proof.

3.7 Remark. We see at the same time that each K-type has multiplicity 1 because the highest weights of the various $V_\gamma^{\alpha,\beta}$ are all different.

To summarise the whole of Section 3.1, putting it back into context with regards to our induced representations $\pi_{i\lambda,\delta}$, we can say that we have proved:

3.8 Theorem (Isotypic decomposition of $\pi_{i\lambda,\delta}$)**.** *Consider any* $\lambda \in \mathbb{R}$ *and* $\delta \in \mathbb{Z}$*. Then:*

$$\pi_{i\lambda,\delta}\Big|_K \cong \sum_{\substack{(\alpha,\beta)\in\mathbb{N}^2 \\ \delta=\beta-\alpha \\ \gamma\in I^{\alpha,\beta}}}^{\oplus} L\Big|_{V_\gamma^{\alpha,\beta}}$$

In this isotypic decomposition, the multiplicity of each K*-type is* 1 *(we say that the* K*-types are multiplicity free).*

3.2 Right action of Sp(1)

3.2.1 Isotypic decomposition with respect to Sp(1)

Let us fix $k \in \mathbb{N}$ and consider the space H^k. Again, denote by

$$(z, w) = (z_1, \ldots, z_n, w_1, \ldots, w_n)$$

the coordinates of $\mathbb{C}^N$, by $h = (h_1, \ldots, h_n)$ the coordinates of $\mathbb{H}^n$ and use the identification $h = (z + jw) \in \mathbb{H}^n \longleftrightarrow (z, w) \in \mathbb{C}^N$.

Right multiplication of a quaternionic vector h by a quaternion q consists simply in multiplying all quaternionic coordinates by q, obtaining the quaternionic vector hq. Consider any subset S of $\mathbb{H}^n$ and assume that it is stable under right multiplication by unit quaternions. Let $\mathcal{F}$ be any subset of $\{f : S \longrightarrow \mathbb{C}\}$. Then the *right action* R of $\mathrm{U}_{\mathbb{H}}$ on $\mathcal{F}$ is defined by

$$R(q)f(x) = f(xq)$$

for all $(q, f, x) \in \mathrm{U}_{\mathbb{H}} \times \mathcal{F} \times S$. As before, we write $R(q)f$ instead of $[R(q)](f)$.

Consider a unit quaternion $q = a + jb \in \mathrm{U}_{\mathbb{H}}$. The rules of quaternionic multiplication imply (as explained in Section 1.4.3), given $h = z + jw \in \mathbb{H}^n$:

$$hq = (az - b\overline{w}) + j(aw + b\overline{z}) \tag{3.21}$$

The subset S of $\mathbb{H}^n$ identifies with a subset of $\mathbb{C}^N$, that we also denote by S. As explained in Chapter 1 (section 1.4.3) the quaternion q (seen as a 1×1 matrix) identifies with the matrix

$$\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \in \mathrm{Sp}(1)$$

that also denote this matrix by q. These considerations explain our choice of action of $\mathrm{Sp}(1)$ on functions of the complex variables (z, w):

3.9 Definition. The *right action* of $\mathrm{Sp}(1)$ on $\mathcal{F}$, again denoted by R, is defined for all $q \in \mathrm{Sp}(1)$ and $(z, w) \in S$ by:

$$R(q)f(z, w) = f\left(az - b\overline{w}, aw + b\overline{z}\right)$$

An element f of $\mathcal{F}$ is *right-invariant* if it is invariant under the right action of $\mathrm{Sp}(1)$, meaning that for all $q \in \mathrm{Sp}(1)$: $R(q)f = f$.
Let us come back to spherical harmonics, fixing any $k \in \mathbb{N}$. One can show:

3.10 Proposition. *The right actions of* $\mathrm{Sp}(1)$ *on* H^k *and* $\mathcal{Y}^k$ *define continuous unitary representations of* $\mathrm{Sp}(1)$*; we identify these representations, denoting both by* R*. We point out that* H^k *is stable under* R*.*

As we have already seen, to study irreducible invariant subspaces under the action of a compact group, one complexifies its Lie algebra. The complexification of $\mathfrak{sp}(1) = \mathfrak{su}(2)$ is:

$$\mathfrak{su}(2) \oplus i\,\mathfrak{su}(2) = \mathfrak{sl}(2, \mathbb{C})$$

A basis over the field $\mathbb{R}$ of $\mathfrak{su}(2)$ is given by the three matrices :

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad ; \quad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad ; \quad C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

A basis over the field $\mathbb{C}$ of $\mathfrak{sl}(2, \mathbb{C})$ is given by the three matrices :

- $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0 + i(-A) \in \mathfrak{su}(2) \oplus i\mathfrak{su}(2)$
- $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{-C}{2} + i\left(\frac{-B}{2}\right) \in \mathfrak{su}(2) \oplus i\mathfrak{su}(2)$
- $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{C}{2} + i\left(\frac{-B}{2}\right) \in \mathfrak{su}(2) \oplus i\mathfrak{su}(2)$

We have for $t \in \mathbb{R}$ the following exponentials:

- $exp(-tA) = \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix}$

- $exp\left(\frac{-tB}{2}\right) = \begin{pmatrix} \cos\left(\frac{t}{2}\right) & -i\,\sin\left(\frac{t}{2}\right) \\ -i\,\sin\left(\frac{t}{2}\right) & \cos\left(\frac{t}{2}\right) \end{pmatrix}$

- $exp\left(\frac{-tC}{2}\right) = \begin{pmatrix} \cos\left(\frac{t}{2}\right) & \sin\left(\frac{t}{2}\right) \\ -\sin\left(\frac{t}{2}\right) & \cos\left(\frac{t}{2}\right) \end{pmatrix}$

These exponentials belong to Sp(1) and therefore correspond to specific unit quaternions $q_t = a_t + jb_t$, with $(a_t, b_t) \in \mathbb{C} \times \mathbb{C}$ (see Lemma 1.11 to recall identification between quaternion and complex matrices):

- $\exp(-tA)$ corresponds to $q_t = e^{-it} + 0j$;
- $\exp\left(\frac{-tB}{2}\right)$ corresponds to $q_t = \cos\left(\frac{t}{2}\right) - j\,i\,\sin\left(\frac{t}{2}\right)$;
- $\exp\left(\frac{-tC}{2}\right)$ corresponds to $q_t = \cos\left(\frac{t}{2}\right) - j\sin\left(\frac{t}{2}\right)$.

Now extend representation dR of $\mathfrak{sp}(1)$ to a complex representation $d_{\mathbb{C}}R$ of $\mathfrak{sl}(2,\mathbb{C})$ by defining for all $Z = X + iY \in \mathfrak{sl}(2,\mathbb{C})$ (taking $(X,Y) \in \mathfrak{sp}(1) \times \mathfrak{sp}(1)$):

$$d_{\mathbb{C}}R(Z) = dR(X) + idR(Y)$$

Remember that by definition, given any $P \in H^k$, $X \in \mathfrak{sp}(1)$ and $(z,w) \in \mathbb{C}^n \times \mathbb{C}^n$:

$$dR(X)P(z,w) = \left.\frac{d}{dt}\right|_{t=0} \left[R\left(\exp(tX)\right)P(z,w)\right]$$

Therefore, given any $Z \in \mathfrak{sl}(2,\mathbb{C})$, $d_{\mathbb{C}}R(Z)$ can be seen as a differential operator; given $P \in H^k$, we shall denote by $Z \cdot P$ the polynomial obtained by applying the differential operator Z to P. Using above formulas that link e, f, h to A, B, C, above exponential formulas for $-tA$, $\frac{-tB}{2}$, $\frac{-tC}{2}$ and definitions of dR and $d_{\mathbb{C}}R$, one can prove:

3.11 Proposition. *Simply denote by e, f, g the respective differential operators:*

$$d_{\mathbb{C}}R(e)\,,\; d_{\mathbb{C}}R(f)\,,\; d_{\mathbb{C}}R(h)$$

Then:

- $e = \sum_{r=1}^{n} \left(w_r \frac{\partial}{\partial z_r} - z_r \frac{\partial}{\partial w_r}\right)$

- $f = \sum_{r=1}^{n} \left(\overline{z_r} \frac{\partial}{\partial w_r} - \overline{w_r} \frac{\partial}{\partial z_r} \right)$

- $h = \sum_{r=1}^{n} \left(z_r \frac{\partial}{\partial z_r} - \overline{z_r} \frac{\partial}{\partial \overline{z_r}} + w_r \frac{\partial}{\partial w_r} - \overline{w_r} \frac{\partial}{\partial \overline{w_r}} \right)$

Let us study the direct sum $\mathrm{H}^k = \bigoplus_{s=0}^{k} \mathrm{H}^{k-s,s}$. We know that each $H^{k-s,s}$ breaks into $\bigoplus_{\gamma=0}^{\min(k-s,s)} V_\gamma^{k-s,s}$. Using formulas of Lemma 3.11, one shows:

3.12 Theorem. *Consider integers k, α, β, γ such that $0 \leq \gamma \leq \min(\alpha, \beta)$ and $\alpha + \beta = k$. Then*

- $h \cdot P_\gamma^{\alpha,\beta} = (\alpha - \beta) P_\gamma^{\alpha,\beta}$

- *When* $\gamma \leq \beta$: $e \cdot P_\gamma^{\alpha,\beta} = (\beta - \gamma) P_\gamma^{\alpha+1,\beta-1}$

- *When* $\gamma \leq \alpha$: $f \cdot P_\gamma^{\alpha,\beta} = (\alpha - \gamma) P_\gamma^{\alpha-1,\beta+1}$

3.2.2 K-type diagram

Figure 3.1 captures the contents of theorems 3.2 and 3.12. In this diagram, the thick black dots represent the highest weight vectors $P_\gamma^{\alpha,\beta}$ $(\alpha + \beta = k)$.
Let us take a closer look at Figure 3.1. For any fixed integer γ such that $0 \leq \gamma \leq E\left(\frac{k}{2}\right)$ (the function E assigns to a real number its integer part):

- The arrow beneath γ points to a **vertical set of components** whose direct sum defines an invariant subspace V_γ^k (obviously not irreducible) under the left action of Sp(n):
$$V_\gamma^k = \bigoplus_{\alpha=\gamma}^{k-\gamma} V_\gamma^{\alpha,k-\alpha}$$

- The **components in** V_γ^k all correspond to the **highest weight** $(k-\gamma, \gamma, 0, \cdots, 0)$ (given by Theorem 3.2) and are therefore equivalent; in particular they all have the **same dimension** d_γ^k.

- A **vertical set of thick black dots** defines a basis of an irreducible invariant subspace W_γ^k under the right action of Sp(1) (of dimension $k - 2\gamma + 1$):
$$W_\gamma^k = \mathrm{Vect}_{\mathbb{C}} \{ P_\gamma^{\alpha,k-\alpha} \}_{\alpha=\gamma,\ldots,k-\gamma} \subset V_\gamma$$

The polynomial $P_\gamma^{k-\gamma,\gamma}$ is a highest weight vector of W_γ^k and corresponds to the highest weight $k-2\gamma$.

- Because the left action of Sp(n) and the right action of Sp(1) commute, applying L to W_γ^k gives irreducible invariant subspaces of R contained in V_γ^k; these subspaces define subrepresentations of R which are equivalent to the restriction of R to W_γ^k; therefore, they all have the same dimension $k-2\gamma+1$.

Now, using the fact that the left action L of Sp(n) is transitive in each box, we deduce:

3.13 Theorem. *The isotypic decomposition of* H^k *with respect to the right action* R *of* Sp(1) *is:*

$$\mathrm{H}^k = \bigoplus_{\gamma=0}^{E\left(\frac{k}{2}\right)} d_\gamma^k \, W_\gamma^k$$

In particular, the multiplicity of W_γ^k *is equal to* d_γ^k.

3.14 Remark. The structure of H^k, which we have studied with respect to the left action of Sp(n) and the right action of Sp(1), appears in [22] (Proposition 5.1), where it is expressed in terms of tensor products (using our system of notation):

$$\mathrm{H}^k \Big|_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)} = \sum_{\gamma=0}^{E\left(\frac{k}{2}\right)} V_\gamma^{k-\gamma,\gamma} \otimes W_\gamma^k$$

Nonetheless, the credit one can give to our work is to explain this structure in a personal and self-contained way.

3.2.3 Bi-invariant polynomials

This result follows from Theorem 3.13:

3.15 Corollary. *Consider* $k \in \mathbb{N}$ *and suppose* k *is even, writing* $\alpha = \frac{k}{2}$.

- *The component* $V_\alpha^{\alpha,\alpha}$ *is the only one that contains one-dimensional invariant subspaces with respect to* R.

- *The* 1*-dimensional complex vector space generated by* $P_\alpha^{\alpha,\alpha}$ *is stable under the right action* R *of* Sp(1)*; in fact, one easily checks that* $P_\alpha^{\alpha,\alpha}$ *is right-invariant.*

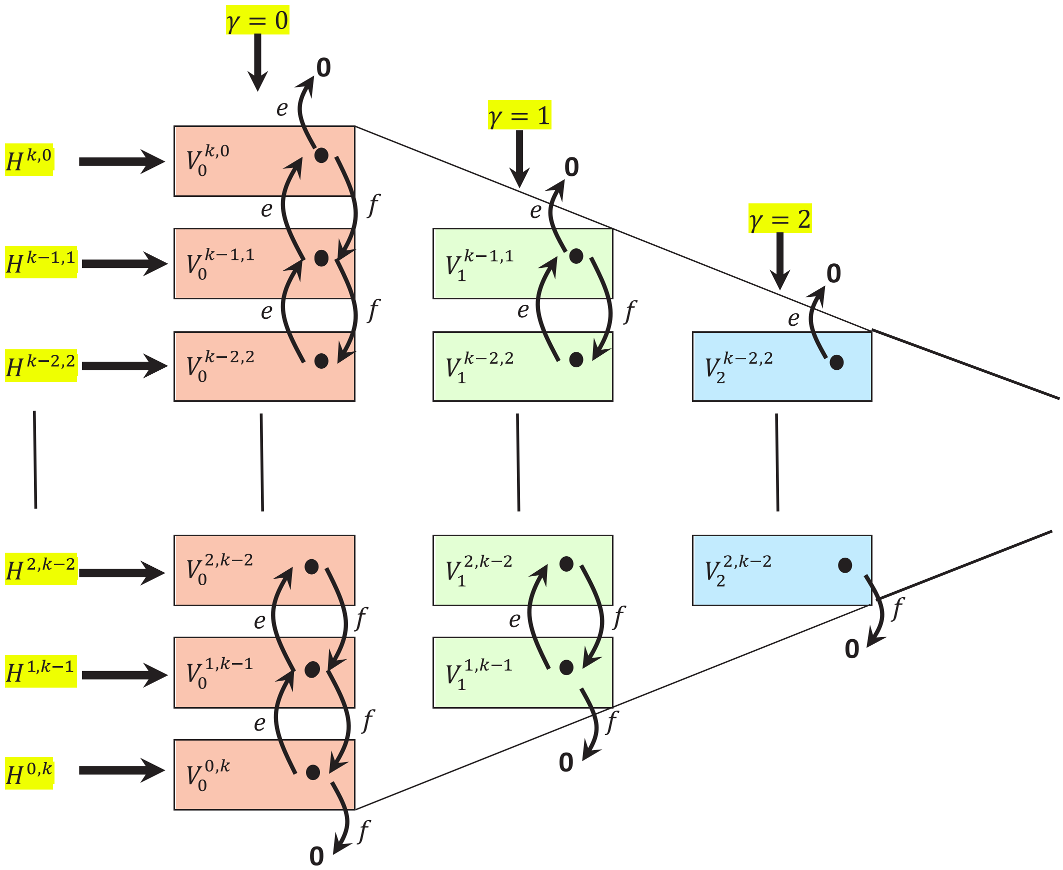

Figure 3.1: combining left action of Sp(n) and right action of Sp(1) within H^k, assuming here $k \geq 5$.

- *The component* $V_\alpha^{\alpha,\alpha}$ *decomposes into* d_α^k *one-dimensional irreducible subspaces, along which* R *is just the identity representation. This implies that all elements of* $V_\alpha^{\alpha,\alpha}$ *are right invariant.*

Another consequence of previous section is:

3.16 Corollary. *Consider an element* f *of* $L^2(S^{2N-1})$*. Then* f *is right invariant if and only if* f *belongs to the following Hilbert sum:*

$$\widehat{\bigoplus_{\alpha\in\mathbb{N}} V_\alpha^{\alpha,\alpha}}$$

Let us now consider the subgroup $1 \times \mathrm{Sp}(n-1) \subset \mathrm{Sp}(n)$: it consists of block diagonal matrices $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$, where $A \in \mathrm{Sp}(n-1)$.

3.17 Definition. Consider any $(\alpha,\beta) \in \mathbb{N}^2$ such that $\alpha+\beta = k$. A polynomial of $\mathrm{H}^{\alpha,\beta}$ and its corresponding spherical harmonic in $\mathcal{Y}^{\alpha,\beta}$ are said to be *bi-invariant* if:

- they are invariant under the left action of $1 \times \mathrm{Sp}(n-1)$;
- they are also invariant under the right action of $\mathrm{Sp}(1)$.

3.18 Theorem.

1. *Consider any* $(\alpha,\beta) \in \mathbb{N}^2$ *and any integer* γ *such that* $0 \le \gamma \le \min(\alpha,\beta)$*.*

   Denote by $\mathrm{Inv}_\gamma^{\alpha,\beta}$ *the complex vector space that consists of all polynomials of* $V_\gamma^{\alpha,\beta}$ *which are invariant under the left action of* $1 \times \mathrm{Sp}(n-1)$*. Then:*

   $$\dim_{\mathbb{C}}(\mathrm{Inv}_\gamma^{\alpha,\beta}) = \alpha+\beta-2\gamma+1$$

   We point out that this dimension is higher than or equal to 1 *and that it equals* 1 *if and only if* $\alpha=\beta=\gamma$*.*

2. *Existence in* H^k *of a bi-invariant polynomial implies that* k *is even.*

3. *Assume that* k *is even and write* $\alpha = \frac{k}{2}$*. Then* $V_\alpha^{\alpha,\alpha}$ *contains a unique, up to a constant, bi-invariant polynomial; the other components do not contain any such polynomial.*

Proof:

Item 1: our proof relies on a Zhelobenko's branching theorem with respect to the pair $(\mathrm{Sp}(n), \mathrm{Sp}(n-1))$, namely Theorem 4 of Chapter XVIII in [53]. This theorem implies that the number of times an irreducible representation of $1 \times \mathrm{Sp}(n-1) \simeq \mathrm{Sp}(n-1)$ with highest weight $c = (c_1, \cdots, c_{n-1})$ occurs in an irreducible representation of $\mathrm{Sp}(n)$ with highest weight $a = (a_1, \cdots, a_n)$ is equal to the number of non-negative integral n-tuples $b = (b_1, \cdots, b_n)$ such that the following two rows of inequalities are satisfied:

$$\begin{array}{ccccc} a_1 \geq b_1 & \geq a_2 \geq b_2 & \geq a_3 \geq b_3 & \geq \cdots \geq a_n & \geq b_n \geq 0 \\ b_1 & \geq c_1 \geq b_2 & \geq c_2 \geq b_3 & \geq \cdots \geq c_{n-1} & \geq b_n \end{array}$$

Remember that the highest weight of the left action of $\mathrm{Sp}(n)$ on $V_\gamma^{\alpha,\beta}$ is the n-tuple $a = (\alpha+\beta-\gamma, \gamma, 0, \cdots, 0)$.
Also, the highest weight theorem (see for instance [27], Section 7 of Chapter IV) implies that a polynomial $P \in V_\gamma^{\alpha,\beta}$ is invariant under the left action of $1 \times \mathrm{Sp}(n-1)$ if and only if the restriction of the left action of $1 \times \mathrm{Sp}(n-1)$ to the 1-dimensionnal space $\mathrm{Vect}_{\mathbb{C}}(P)$ defines an irreducible representation of $\mathrm{Sp}(n-1)$ with highest weight $c = (0, \cdots, 0)$.
Taking $a = (\alpha+\beta-\gamma, \gamma, 0, \cdots, 0)$ and $c = (0, \cdots, 0)$, the two rows of inequalities become:

$$\begin{array}{ccccc} \alpha+\beta-\gamma \geq b_1 & \geq \gamma \geq b_2 & \geq 0 = b_3 & = \cdots = 0 & = b_n = 0 \\ b_1 & \geq 0 = b_2 & = 0 = b_3 & = \cdots = 0 & = b_n \end{array}$$

Thus the only non-zero coordinate of the n-tuple b is b_1 and we are left with counting the number of integers b_1 between $\alpha+\beta-\gamma$ and γ. This number is $\alpha+\beta-2\gamma+1$. Because $\gamma \in [0, \min(\alpha, \beta)]$, this number is higher than or equal to 1 and it equals 1 if and only if $\alpha = \beta = \gamma$.

Items 2 and 3: they just require to combine Item 1 and Corollary 3.15.

End of proof.

Let us fix any $\alpha \in \mathbb{N}$ and write $k = 2\alpha$. We now set to determine the unique bi-invariant polynomial of H^k (up to a constant) explicitly.
Define the following set:

$$\mathcal{A}_\alpha = \{(a, b) \in \mathbb{N}^2 \ / \ a + b = \alpha\}$$

Let us consider the polynomials U, V, P defined by:

- $U(z,w) = z_1\overline{z_1} + w_1\overline{w_1}$ (we will often write U instead of $U(z,w)$);
- $V(z,w) = \sum_{r=2}^{n} (z_r\overline{z_r} + w_r\overline{w_r})$ (we will often write V instead of $V(z,w)$);
- $P(z,w) = \sum_{A\in\mathcal{A}_\alpha} \nu_A\, U^a\, V^b$, where the various ν_A denote complex scalars (undetermined at this stage).

The polynomial P is obviously:

- homogeneous, of homogeneous degree (α,α);
- bi-invariant.

If P is harmonic, then, following Corollary 3.15, P necessarily belongs to the component $V_\alpha^{\alpha,\alpha}$ of H^k. To ensure that P is indeed harmonic, we need to apply the complex Laplace operator and make sure P is in its kernel. Remember that this operator is:

$$\Delta_{\mathbb{C}} = 4\sum_{r=1}^{n}\left(\frac{\partial^2}{\partial z_r \partial\overline{z_r}} + \frac{\partial^2}{\partial w_r \partial\overline{w_r}}\right)$$

One easily checks that:

$$\Delta_{\mathbb{C}} P(z,w) \;=\; \nu_A(a^2+a)\, U^{a-1}\, V^b \;+\; \nu_A(b^2+(2n-3)b)\, U^a\, V^{b-1}$$

Given $A\in\mathcal{A}$, let us introduce the following notations:

- $C_1(A) = \nu_A(a^2+a)$ and $T_1(A) \;=\; U^{a-1}\, V^b$;
- $C_2(A) = \nu_A(b^2+(2n-3)b)$ and $T_2(A) \;=\; U^a\, V^{b-1}$.

With these notations we have:

$$\Delta_{\mathbb{C}} P(z;w) = 4\sum_{A\in\mathcal{A}} [C_1(A)\cdot T_1(A) + C_2(A)\cdot T_2(A)] \tag{3.22}$$

By multiplying the brackets out in powers of U and V, we can write down the list of monomials that come from $T_i(A)$, given $A=(a,b)\in\mathcal{A}$ and $i\in\{1,2\}$; let us denote by $\mathcal{M}_i(A)$ the set such monomials.

3.19 Proposition. *Consider two pairs $A=(a,b)$ and $A'=(a',b')$ such that $A\neq A'$. Then:*

1. $\mathcal{M}_1(A)\cap\mathcal{M}_1(A') = \emptyset$

2. $\mathcal{M}_2(A) \cap \mathcal{M}_2(A') = \emptyset$

3. $[\mathcal{M}_1(A) \cap \mathcal{M}_2(A') \neq \emptyset] \Leftrightarrow [A' = (a-1, b+1)]$

4. $[\mathcal{M}_2(A) \cap \mathcal{M}_1(A') \neq \emptyset] \Leftrightarrow [A' = (a+1, b-1)]$

Proof:

The monomials of $\mathcal{M}_i(A)$ (resp. $\mathcal{M}_i(A')$), again with $i \in \{1, 2\}$, split into a first part which is a homegeneous polynomial of the variables $z_1, w_1, \overline{z}_1, \overline{w}_1$ and of homogeneous degree (a, a) (resp. (a', a')) and a second part which is a homegeneous polynomial of the variables $z_2, \ldots, z_n, w_2, \ldots, w_n, \overline{z}_2, \ldots \overline{z}_n, \overline{w}_2, \ldots \overline{w}_n$ and of homogeneous degree (b, b) (resp. (b', b')). For monomials to coincide, their homogeneous degrees must of course coincide, which is enough to establish (1), (2), (3) and (4).

End of proof.

Consider two pairs $A = (a, b) \in \mathcal{A}$ and $A' = (a', b') \in \mathcal{A}$ such that $A \neq A'$.

- If a monomial appears in both $\mathcal{M}_1(A)$ and $\mathcal{M}_2(A')$ (implying $A' = (a-1, b+1)$), then the coefficients it appears with are $C_1(A)$ times some combinatorial coefficient and $C_2(A')$ times the same combinatorial coefficient. Thus we must have:

$$C_1(A) + C_2(A') = 0$$

- If a monomial appears in both $\mathcal{M}_2(A)$ and $\mathcal{M}_1(A')$ (implying $A' = (a+1, b-1)$), then the coefficients it appears with are $C_2(A)$ times a combinatorial coefficient and $C_1(A')$ times the same combinatorial coefficient. Thus we must have:

$$C_2(A) + C_1(A') = 0 \tag{3.23}$$

Above considerations finally show us how to choose coefficients ν_A in the definition of P so as to ensure that P is harmonic. Figure 3.2 helps understand the following steps (assuming that $\alpha \geq 1$):

- List all pairs of $\mathcal{A}_\alpha$ from left to right: $(\alpha, 0)$, $(\alpha-1, 1)$, $\ldots$, $(1, \alpha-1)$, $(0, \alpha)$.

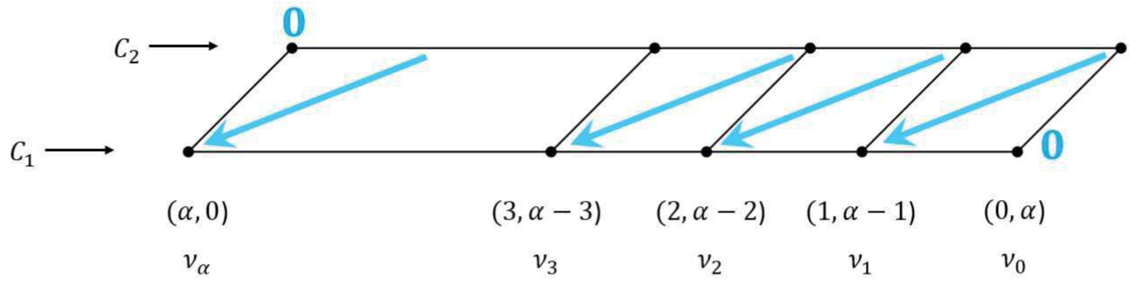

Figure 3.2: Induction method to find bi-invariant polynomials

- Start for instance by the pair $A_0 = (0, \alpha)$ on the far right and assign to it any coefficient $\nu_{A_0} = \nu_0$; it determines $C_1(A_0)$ (which is in fact 0) and $C_2(A_0)$.
- Move on to the pair $A_1 = (1, \alpha - 1)$ on the left and compute the only suitable coefficient $\nu_{A_1} = \nu_1$ by using Equation (3.23).
- And so on, until the pair $A_\alpha = (\alpha, 0)$ has been reached and its coefficient $\nu_{A_\alpha} = \nu_\alpha$ computed.

Let us refer to this as the *induction method*.

3.20 Theorem. *Given any even integer $k \in \mathbb{N}$ and writing $\alpha = \frac{k}{2}$, the induction method computes the unique (up to a constant) bi-invariant polynomial of H^k; this polynomial belongs to $V_\alpha^{\alpha,\alpha}$.*

3.2.1 Examples.

- For $\alpha = 1$, one finds $P = (1 - n)\, U \,+\, V$.
- For $\alpha = 2$, one finds $P = \frac{(2n-1)(n-1)}{3}\, U^2 \,+\, (1 - 2n)\, UV \,+\, V^2$.

Chapter 4

Compact picture and hypergeometric functions

On travaille de nouveau avec le modèle compact, en considérant l'action à gauche de $\mathrm{Sp}(n)$ sur les espaces $\mathrm{H}^{\alpha,\beta}$ tels que les sommes $\alpha+\beta$ soient toutes égales à un entier $k \in \mathbb{N}$ donné. On suppose en fait que k est pair et on considère la valeur particulière $\alpha = \frac{k}{2}$.

Le but de ce chapitre est d'exploiter l'opérateur de Casimir et des propriétés d'invariance d'une fonction bien choisie pour décrire celle-ci comme une solution de l'équation hypergéométrique. La fonction en question, notée f_P dans le paragraphe 4.2, est l'unique harmonique sphérique bi-invariante de H^k, donnée par le théorème 3.18 et calculable à l'aide du théorème 3.20 ; elle appartient à $V_\alpha^{\alpha,\alpha}$.

L'invariance à droite de f_P entraîne que l'on peut associer à f_P une fonction f définie sur l'espace projectif $\mathrm{P}^{n-1}(\mathbb{H})$. Cela nécessite bien sûr de définir ce dernier. Puisque l'on travaille avec la multiplication à droite, la droite quaternionique (vectorielle) qui passe par un point $x \in \mathbb{H}^n$ est l'ensemble $x\,\mathbb{H} = \{xh \ / \ h \in \mathbb{H}\}$ et $\mathrm{P}^{n-1}(\mathbb{H})$ est l'ensemble de toutes les droites quaternioniques.

La proposition 4.1 montre comment associer à une droite quaternionique $x\,\mathbb{H}$ un représentant de son orbite sous l'action à gauche de $1 \times \mathrm{Sp}(n-1)$, paramétré par un réel $\theta \in \left[0, \frac{\pi}{2}\right]$; la droite qui représente l'orbite est $x(\theta)\,\mathbb{H}$, où $x(\theta)$ désigne le point de coordonnées $(\cos\theta, 0, \ldots, 0, \sin\theta)$ dans $\mathbb{H}^n$. La proposition 4.1 montre aussi comment calculer θ (on notera bien sûr $x = (x_1, \ldots, x_n) \in \mathbb{H}^n$) :

- si $x_1 \neq 0$, alors $\theta = \arctan\left(\|x'\|\right)$, où $x' = \left(\frac{x_2}{x_1}, ..., \frac{x_n}{x_1}\right)$;
- si $x_1 = 0$, alors $\theta = \frac{\pi}{2}$.

Le diagramme 4.1 illustre ce procédé, qui permet ensuite d'associer à la fonction f une nouvelle fonction F de la variable θ.

La fonction f_P appartient à une composante irréductible et par conséquent est une fonction propre de l'opérateur de Casimir Ω_L associé à l'action à gauche L de $\mathrm{Sp}(n)$; notons Λ la valeur propre correspondante. Expliciter cela revient à écrire une équation différentielle du second ordre :

$$\sum_{X \in \mathcal{B}_{\mathfrak{k}}} \frac{\partial^2}{\partial t^2}\Big|_{t=0} \Big(L\Big(\exp(-tX)\Big) f_P \Big) = \Lambda f_P$$

où $\mathcal{B}_{\mathfrak{k}}$ est une base orthonormée de $\mathfrak{k}$. En appliquant cette équation à $x(\theta)$ on obtient (4.3). La réduction du nombre de variables dont on vient de parler permet d'écrire une équation équivalente, plus simple, à une seule variable. Pour comprendre comment, il faut regarder l'équation (4.3) de plus près. Les termes de la somme de gauche sont du type

$$\frac{\partial^2}{\partial t^2}\Big|_{t=0} \Big(f_P\Big(\exp(-tX)x(\theta)\Big)\Big)$$

où X appartient à la base orthonormée $\mathcal{B}_{\mathfrak{k}}$. L'exponentielle $\exp(-tX)$ appliquée à $x(\theta)$ donne un point image de $\mathbb{H}^n$ qui ne correspond plus forcément au même paramètre θ via la proposition 4.1. La valeur du nouveau paramètre est un nombre qui dépend de θ, de X et de t ; on note $\xi_{X,\theta}(t)$ cette valeur. L'équation différentielle équivalente à (4.3) que l'on obtient est alors :

$$\sum_{X \in \mathcal{B}_{\mathfrak{k}}} \frac{\partial^2}{\partial t^2}\Big|_{t=0} \Big(F\Big(\xi_{X,\theta}(t)\Big)\Big) = \Lambda \cdot F(\theta)$$

Tout cela fait l'objet des paragraphes 4.1 et 4.2.

Le paragraphe 3 explicite l'équation différentielle ci-dessus : on a vu plus haut que l'on sait calculer chaque $\xi_{X,\theta}(t)$ et que l'on peut donc calculer chaque dérivée seconde. Les calculs sont longs (et fastidieux), mais débouchent sur une équation différentielle intéressante, à savoir (4.5). Un changement de variable révèle l'équation hypergéométrique donnée dans le théorème (4.8), théorème qui constitue le résultat essentiel de ce chapitre :

$$u(1-u)\varphi''(u) + (2 - Nu)\varphi'(u) - \frac{\Lambda}{2}\varphi(u) = 0$$

Comme f_P appartient à $V_{\alpha}^{\alpha,\alpha}$, le corollaire 3.3 permet de calculer la valeur de Λ : elle est égale à $-(2\alpha^2 + (4n-2)\alpha)$, ce qui rend l'équation hypergéométrique totalement explicite.

4.1 Quaternionic projective space

In this work, straight lines are defined with respect to right-multiplication: given $x \in \mathbb{H}^n$, the *quaternionic line* through x is the vector space

$$x\,\mathbb{H} = \{xh \;/\; h \in \mathbb{H}\}$$

The *quaternionic projectice space* $\mathrm{P}^{n-1}(\mathbb{H})$ is naturally the set of quaternionic lines of $\mathbb{H}^n$. Quaternionic matrices act naturally on quaternionic lines: given $x \in \mathbb{H}^n$, an invertible matrix $M \in \mathrm{GL}(n, \mathbb{H})$ assigns to $x\,\mathbb{H}$ the quaternionic line through Mx. We call this action the *natural action* of $\mathrm{GL}(n, \mathbb{H})$ on $\mathrm{P}^{n-1}(\mathbb{H})$. We now study the orbits under the restriction of this action to the subgroup $1 \times \mathrm{Sp}(n-1)$. Given $x\,\mathbb{H} \in \mathrm{P}^{n-1}(\mathbb{H})$, we denote by $\mathcal{O}(x\,\mathbb{H})$ the orbit of $x\,\mathbb{H}$. We show in the next proposition that the orbits can be parametrised by a single real variable.

4.1 Proposition. *Consider any $x = (x_1, ..., x_n) \in \mathbb{H}^n$. There exists a unique $\theta \in \left[0, \frac{\pi}{2}\right]$ such that, denoting $x(\theta) = (\cos\theta, 0, ..., 0, \sin\theta) \in \mathbb{H}^n$, the quaternionic line $x(\theta)\,\mathbb{H}$ belongs to $\mathcal{O}(x\,\mathbb{H})$. Moreover, θ is explicitly given by the following formulas:*

- *if $x_1 \neq 0$, then $\theta = \arctan\left(\|x'\|\right)$, taking $x' = \left(\frac{x_2}{x_1}, ..., \frac{x_n}{x_1}\right)$;*
- *if $x_1 = 0$, then $\theta = \frac{\pi}{2}$.*

Proof:

Existence:

Suppose first that $x_1 \neq 0$. Then, writing $x' = \left(x_2\frac{1}{x_1}, ..., x_n\frac{1}{x_1}\right)$, we have:

$$x\,\mathbb{H} = (x_1, ..., x_n)\,\mathbb{H} = \left(x_1\frac{1}{x_1}, x_2\frac{1}{x_1}, ..., x_n\frac{1}{x_1}\right)\mathbb{H} = (1, x')\mathbb{H}$$

Because $1 \times \mathrm{Sp}(n-1)$ acts transitively on spheres of $\mathbb{H}^{n-1}$, there exists a matrix $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, with $k \in \mathrm{Sp}(n-1)$, that takes $(1, x')\mathbb{H}$ to the quaternionic line $(1, 0, ..., 0, \|x'\|)\,\mathbb{H}$ (remember that $\|\cdot\|$ denotes the norm of $\mathbb{H}^{n-1}$). This line can be rewritten

$$(1, 0, ..., 0, \|x'\|)\,\mathbb{H} = \left(\frac{1}{\sqrt{1+\|x'\|^2}}, 0, ..., 0, \frac{\|x'\|}{\sqrt{1+\|x'\|^2}}\right)\mathbb{H} = (\cos\theta, 0, ..., 0, \sin\theta)\,\mathbb{H}$$

for some $\theta \in \left[0, \frac{\pi}{2}\right[$. Figure 4.1 illustrates this.

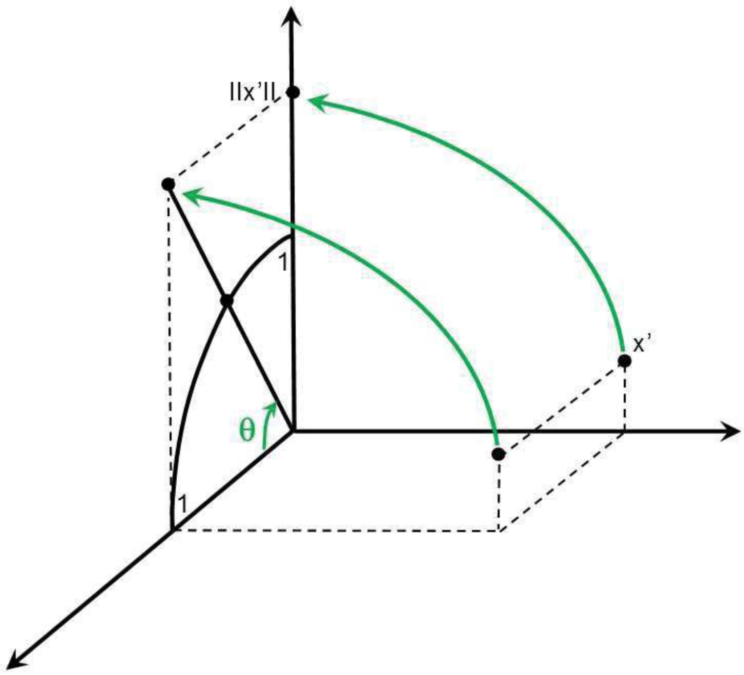

Figure 4.1: A single variable θ to parametrise orbits

If now $x_1 = 0$, thus $\|x'\| = 1$, then one can choose k so that $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ carries $x\,\mathbb{H}$ onto $(0, ...0, 1)\,\mathbb{H}$, which gives $\theta = \frac{\pi}{2}$.

This establishes existence and formulas of θ.

Uniqueness:

Suppose θ and θ' belong to $\left[0, \frac{\pi}{2}\right]$ and that the quaternionic lines $x(\theta)\,\mathbb{H}$ and $x(\theta')\,\mathbb{H}$ belong to $\mathcal{O}(x\,\mathbb{H})$. Then there exists $k \in 1 \times \mathrm{Sp}(n-1)$ such that $kx(\theta')\,\mathbb{H} = x(\theta)\,\mathbb{H}$, which implies that there exists $q \in \mathrm{U}_{\mathbb{H}}$ such that:

$$kx(\theta') = x(\theta)q \tag{4.1}$$

Suppose that $n \geq 3$, and write

$$k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & B \\ 0 & L & b \end{pmatrix}$$

with $b \in \mathbb{H}$, $A \in \mathrm{M}_{n-2,n-2}(\mathbb{H})$, $L \in \mathrm{M}_{1,n-2}(\mathbb{H})$ and $B \in \mathrm{M}_{n-2,1}(\mathbb{H})$.

Then (4.1) implies: We have:

$$\begin{pmatrix} \cos\theta' \\ B\sin\theta' \\ b\sin\theta' \end{pmatrix} = \begin{pmatrix} q\cos\theta \\ 0 \\ q\sin\theta \end{pmatrix}$$

We now split the rest of the proof into three cases.

- Case 1: $\theta, \theta' \in \left[0, \frac{\pi}{2}\right[$. Then $q = \frac{\cos\theta'}{\cos\theta}$, $B\sin\theta' = 0$ and:

$$b\sin\theta' = \frac{\sin\theta\cos\theta'}{\cos\theta} \tag{4.2}$$

  Subcase 1: $\theta' \neq 0$. Then B must be 0 and Proposition 1.14 (see Chapter 1) implies $L = 0$ but more importantly for us $|b| = 1$. Thus from (4.2) it follows that $|\tan\theta| = |\tan\theta'|$, implying $\tan\theta = \tan\theta'$ and $\theta = \theta'$.

  Subcase 2: $\theta' = 0$. Then $\sin\theta' = 0$ and $\cos\theta' = 1$, so (4.2) becomes $\tan\theta = 0$ and finally $\theta = 0 = \theta'$.

- Case 2: $\theta = \frac{\pi}{2}$. Then $\cos\theta = 0$ implies $\cos\theta' = 0$ and finally $\theta' = \frac{\pi}{2} = \theta$.

- Case 3: $\theta' = \frac{\pi}{2}$. Then $\cos\theta' = 0$ implies $q\cos\theta = 0$, thus $\cos\theta = 0$ and finally $\theta = \frac{\pi}{2} = \theta'$ (because $q \neq 0$).

There remains the case $n = 2$, for which we choose k as $\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$ with $b \in \mathrm{U}_{\mathbb{H}}$. Similarly to what is detailed above, we also obtain $\theta = \theta'$.

End of proof.

4.2 Reducing the number of variables

From now on, throughout the rest of this chapter, the setting is the following:

- We fix an even integer $k \in \mathbb{N}$ and write $\alpha = \frac{k}{2}$.

- We consider the space H^k and its unique, up to a constant, bi-invariant polynomial P, which in fact belongs to $V_\alpha^{\alpha,\alpha} \subset \mathrm{H}^{\alpha,\alpha}$ (see Theorem 3.20).

- We denote by f_P the spherical harmonic that corresponds to P, that is:

$$f_P = P|_{S^{2N-1}} \in \mathcal{Y}^{\alpha,\alpha}$$

We remind the reader that, by definition, the right action of $\mathrm{Sp}(1)$ corresponds, in the quaternionic setting, to right scalar multiplication by unit quaternions.
Invariance under the right action of $\mathrm{Sp}(1)$ enables f_P to descend to a function f on $\mathrm{P}^{n-1}(\mathbb{H})$:

$$\begin{array}{rcccl} f & : & \mathrm{P}^{n-1}(\mathbb{H}) & \longrightarrow & \mathbb{C} \\ & & x\,\mathbb{H} & \mapsto & f_P(x), \text{where } x \text{ is assumed to belong to } S^{2N-1} \end{array}$$

Transferring, in the way one expects, the left action of K on spherical harmonics to a left action, also denoted by L, of K on complex-valued functions defined on $\mathrm{P}^{n-1}(\mathbb{H})$, we write:

$$L(k)f(x\mathbb{H}) = f\Big((k^{-1}x)\mathbb{H}\Big)$$

for all $(k, x) \in K \times S^{2N-1}$.
Invariance of f_P under the left action of $1 \times \mathrm{Sp}(n-1)$ implies that f is invariant along the orbits of the left action of $1 \times \mathrm{Sp}(n-1)$ on $\mathrm{P}^{n-1}(\mathbb{H})$. Therefore, f descends to a function on the classifying set $\mathcal{O}$ of orbits, leading in turn to the following new function (via Proposition 4.1):

$$\begin{array}{rcccl} F & : & \left[0, \frac{\pi}{2}\right] & \longrightarrow & \mathbb{C} \\ & & \theta & \mapsto & F(\theta) = f\left(x(\theta)\mathbb{H}\right) = f_P\left(x(\theta)\right) \end{array}$$

Now, apply Formula (1.6) to write down the Casimir operator Ω_L of L. This operator acts on f_P like this, given an element $x \in S^{4n-1}$:

$$\Omega_L(f_P)(x) = \sum_{X \in \mathcal{B}_{\mathfrak{k}}} \frac{\partial^2}{\partial t^2}\bigg|_{t=0} \Big(\pi(\exp(-tX))(f_P)(x)\Big) = \sum_{X \in \mathcal{B}_{\mathfrak{k}}} \frac{\partial^2}{\partial t^2}\bigg|_{t=0} \Big(f_P\Big(\exp(-tX)x\Big)\Big)$$

as long as each expression $\frac{\partial^2}{\partial t^2}\Big|_{t=0} \left[f_P\left(\exp(-tX)x\right)\right]$ above is well defined, which is the case because f_P is smooth.
Corollary 3.3 says that the component $V_\alpha^{\alpha,\alpha}$ is associated to the eigenvalue $\Lambda = -(2\alpha^2 + (4n-2)\alpha)$ of Ω_L:

$$\Omega_L(f_P)(x) = \Lambda \cdot f_P(x)$$

Given $\theta \in \left[0, \frac{\pi}{2}\right]$, one can apply this equation to $x = x(\theta)$ (see Section 4.1 for notation):

$$\sum_{X \in \mathcal{B}_{\mathfrak{k}}} \frac{\partial^2}{\partial t^2}\Big|_{t=0} \Big(f_P\left(\exp(-tX)x(\theta)\right) \Big) = \Lambda \cdot f_P\left(x(\theta)\right) \tag{4.3}$$

One can convert this into a differential equation satisfied by the function F of the real variable θ. But this requires to know for each $X \in \mathcal{B}_{\mathfrak{k}}$ and each $t \in \mathbb{R}$, which parameter $\xi_{X,\theta}(t) \in \left[0, \frac{\pi}{2}\right]$ labels the orbit of $\exp(-tX)x(\theta)\mathbb{H}$ under the left action of $1 \times \mathrm{Sp}(n-1)$. Proposition 4.1 gives us the value of $\xi_{X,\theta}(t)$. Indeed, denoting by $(y_1, ..., y_n)$ the coordinates of $\exp(-tX)x(\theta)$:

- when $y_1 \neq 0$, we have $\xi_{X,\theta}(t) = \arctan \|y'\|$, where $y' =$ stands for $\left(\frac{y_2}{y_1}, ..., \frac{y_n}{y_1}\right)$;
- when $y_1 = 0$, we have $\xi_{X,\theta}(t) = \frac{\pi}{2}$.

Then (4.3) can be written:

$$\sum_{X \in \mathcal{B}_{\mathfrak{k}}} \frac{\partial^2}{\partial t^2}\Big|_{t=0} \Big(F\big(\xi_{X,\theta}(t)\big) \Big) = \Lambda \cdot F(\theta) \tag{4.4}$$

We point out that in this equation, the expressions that appear in the sum are automatically well defined. Finally, all we have to do now is determine the coordinates of the various $\exp(-tX)x(\theta)$ and use above formulas for $\xi_{X,\theta}(t)$.

4.3 Hypergeometric equation

We continue to work with the setting and notations of Section 4.2.

In Equation (4.3), all terms corresponding to elements of the Lie algebra of $1 \times \mathrm{Sp}(1)$ vanish, precisely because f_P is invariant under the left action of $1 \times \mathrm{Sp}(1)$. So the relevant elements of $\mathcal{B}_{\mathfrak{k}}$ are :

$$Z_i = A_1\ ,\ Z_j = D_1\ ,\ Z_{ji} = E_1\ ,\ X_r = \frac{B_{1,r}}{\sqrt{2}}\ ,\ Y_{r,i} = \frac{C_{1,r}}{\sqrt{2}}\ ,\ Y_{r,j} = \frac{F_{1,r}}{\sqrt{2}}\ ,\ Y_{r,ji} = \frac{G_{1,r}}{\sqrt{2}}$$

where r denotes an integer such that $2 \leq r \leq n$; let us denote by $\mathcal{I}$ the set of these integers.

Exponential formulas of Lemma 1.13 give, taking $r \in \mathcal{I}$, $t \in \mathbb{R}$ and $\eta \in \{i, j, ji\}$:

$$\exp(-tZ_\eta) = \begin{pmatrix} \cos t - \eta \sin t & 0 \\ 0 & I_m \end{pmatrix}$$

$$\exp(-tX_r) = \begin{pmatrix} \cos\tau & \cdots & -\sin\tau & \cdots \\ \vdots & \ddots & \vdots & \\ \sin\tau & \cdots & \cos\tau & \cdots \\ \vdots & & \vdots & \ddots \end{pmatrix}$$

$$\exp(-tY_{r,\eta}) = \begin{pmatrix} \cos\tau & \cdots & -\eta\sin\tau & \cdots \\ \vdots & \ddots & \vdots & \\ -\eta\sin\tau & \cdots & \cos\tau & \cdots \\ \vdots & & \vdots & \ddots \end{pmatrix}$$

where

- the nonexplicited entries are 1 on the diagonal and 0 elsewhere;
- the dots show lines r and rows r;
- the variable τ denotes $\frac{t}{\sqrt{2}}$.

Dealing first with matrices Z_η:

The following proposition is straightforward:

4.2 Proposition.

1. $\exp(-tZ_\eta)x(\theta) = ((\cos t - \eta \sin t)\cos\theta, 0, ..., 0, \sin\theta)$
2. *Suppose* $\theta \in \left[0, \frac{\pi}{2}\right[$. *Then:*

$$\xi_{Z_\eta,\theta}(t) = \theta \quad ; \quad \xi'_{Z_\eta,\theta}(0) = 0 \quad ; \quad \xi''_{Z_\eta,\theta}(0) = 0 \; .$$

Dealing now with matrices X_r and $Y_{r,\eta}$:

Case 1: $r = n$

This proposition is also straightforward:

4.3 Proposition.

1. $\exp(-tX_n)x(\theta) = (\cos\tau\cos\theta - \sin\tau\sin\theta, 0, ..., 0, \sin\tau\cos\theta + \cos\tau\sin\theta)$

2. *Suppose* $\theta \in \left]0, \frac{\pi}{2}\right[$. *Then for* t *small enough we have* $cos\tau \cos\theta - \sin\tau \sin\theta \neq 0$ *and*

$$\xi_{X_n,\theta}(t) = \tau + \theta \in \left]0, \frac{\pi}{2}\right[\ .$$

Also:

$$\xi'_{X_n,\theta}(0) = \frac{1}{\sqrt{2}} \quad ; \quad \xi''_{X_n,\theta}(0) = 0 \ .$$

4.4 Proposition. *Consider* $\eta \in \{i, j, ji\}$.

1. $\exp(-tY_{n,\eta})x(\theta) = (\cos\tau\cos\theta - \eta\sin\tau\sin\theta, 0, ..., 0, -\eta\sin\tau\cos\theta + \cos\tau\sin\theta)$

2. *Suppose* $\theta \in \left]0, \frac{\pi}{2}\right[$. *Then* $\cos\tau\cos\theta - \eta\sin\tau\sin\theta \neq 0$ *and*

$$\xi_{Y_{n,\eta},\theta}(t) = \arctan\sqrt{\frac{\cos^2\tau\sin^2\theta + \sin^2\tau\cos^2\theta}{\cos^2\tau\cos^2\theta + \sin^2\tau\sin^2\theta}} \in \left]0, \frac{\pi}{2}\right[$$

Also:

$$\xi'_{Y_{n,\eta},\theta}(0) = 0 \quad ; \quad \xi''_{Y_{n,\eta},\theta}(0) = \frac{1}{\tan(2\theta)} \ .$$

Proof:

A simple matrix multiplication proves Item 1. Let us now look at Item 2. Because $\cos\theta$ and $\sin\theta$ are both non-zero and $\cos\tau$ and $\sin\tau$ cannot be simultaneously equal to 0, one necessarily has

$$cos\tau\cos\theta - \eta\sin\tau\sin\theta \neq 0$$

Then the square root formula comes from Proposition 4.1 and the fact that, given any $(a, b, \eta) \in \mathbb{R} \times \mathbb{R} \times \{i, j, ji\}$:

$$|a + \eta b| = \sqrt{a^2 + b^2}$$

The reason why $\xi_{Y_\eta,\theta}(t)$ belongs to the open interval $\left]0, \frac{\pi}{2}\right[$ is due to the following facts:

- arctan maps $[0, +\infty[$ onto $\left[0, \frac{\pi}{2}\right[$;
- the square root expression cannot take the value 0 because, again, neither $\cos\theta$ nor $\sin\theta$ are 0 and $\cos\tau$ and $\sin\tau$ cannot simultaneously be equal to 0.

Now let us now compute $\xi'_{Y_\eta,\theta}(0)$ and $\xi'_{Y_\eta,\theta}(0)$. For this purpose, define functions T, U, V, W, R, S, ϕ by (t, x, y are real variables):

- $T(t) = \frac{t}{\sqrt{2}}$

- $U(x) = \cos^2 x \sin^2\theta + \sin^2 x \cos^2\theta$
- $V(x) = \cos^2 x \cos^2\theta + \sin^2 x \sin^2\theta$
- $W = \frac{U}{V}$
- $R(y) = \sqrt{y}$
- $S = R \circ W$
- $\phi = \arctan \circ S$

One obviously has

$$U(0) = \sin^2\theta \ ; \ V(0) = \cos^2\theta \ ; \ \theta W(0) = \tan^2(\theta) \ ; \ S(0) = \tan\theta \ ; \ \phi(0) = \theta$$

and easily checks

$$U'(0) = 0 \ ; \ V'(0) = 0 \ ; \ U''(0) = 2\cos(2\theta) \ ; \ V''(0) = -2\cos(2\theta).$$

By writing $W' = \frac{U'V - UV'}{V^2}$ and $S' = \frac{W'}{2\sqrt{W}}$ and by differentiating W' and S' one obtains:

$$W'(0) = 0 \ ; \ W''(0) = \frac{2\cos(2\theta)}{\cos^4\theta} \ ; \ S'(0) = 0 \ ; \ S''(0) = \frac{\cos(2\theta)}{\cos^4\theta\tan\theta} \ .$$

By writing $\phi' = \frac{S'}{1+S^2}$ and by differentiating ϕ' one gets:

$$\phi'(0) = 0 \ ; \ \phi''(0) = \frac{2}{\tan(2\theta)} \ .$$

Finally, writing $\xi = \phi \circ T$, we have:

- $\xi'_{Y_{\eta,\theta}}(t) = (\phi \circ T)'(t) = \phi'(T(t)) \cdot T'(t) = \frac{1}{\sqrt{2}} \cdot \phi'(T(t))$
- $\xi''_{Y_{\eta,\theta}}(t) = \frac{1}{\sqrt{2}} \cdot (\phi' \circ T)'(t) = \frac{1}{\sqrt{2}} \cdot \phi''(T(t)) \cdot T'(t) = \frac{1}{2} \cdot \phi''(T(t))$
- $\xi'_{Y_{\eta,\theta}}(0) = 0$
- $\xi''_{Y_{\eta,\theta}}(0) = \frac{1}{\tan(2\theta)}$.

Throughout the calculations, one can check that all expressions are well defined.

End of proof.

Case 2: $2 \leq r < n$ **(when** $n \geq 3$ **only)**

4.5 Proposition.

1. $\exp(-tX_r)x(\theta) = (\cos\tau\cos\theta, 0, ..., 0, \sin\tau\cos\theta, 0, ..., 0, \sin\theta)$, *where the term* $\sin\tau\cos\theta$ *is the* r^{th} *coordinate.*

2. *Take* $\theta \in \left]0, \frac{\pi}{2}\right[$. *If* $\tau \in \mathbb{R}$ *is such that* $|\tau|$ *is small enough, then:*

$$\xi_{X_r,\theta}(t) = \arctan\left(\frac{1}{\cos\tau}\sqrt{\sin^2\tau + \tan^2\theta}\right) \quad ; \quad \xi'_{X_r,\theta}(0) = 0 \quad ; \quad \xi''_{X_r,\theta}(0) = \frac{1}{2\tan\theta} \ .$$

Proof:

The proof is straightforward; here are the steps:

- Define $U(\tau) = \sin^2\tau + \tan^2\theta$ and check: $U(0) = \tan^2(\theta)$, $U'(0) = 0$ and $U''(0) = 2$.
- Define $V = \sqrt{U}$ and check $V(0) = \tan\theta$, $V'(0) = 0$ and $V''(0) = \frac{1}{\tan\theta}$.
- Define $W(\tau) = \cos\tau$ and $Z = \frac{V}{W}$; check $Z(0) = \tan\theta$, $Z'(0) = 0$ and $Z''(0) = \frac{2}{\sin(2\theta)}$.
- Define $A = \arctan \circ Z$ and check $A(0) = \theta$, $A'(0) = 0$ and $A''(0) = \frac{1}{\tan\theta}$.

These are the calculations with respect to the variable τ; remembering that $\tau = \frac{t}{\sqrt{2}}$ establishes Item 2.

End of proof.

Above calculations also establish:

4.6 Proposition. *Consider* $\eta \in \{i, j, ji\}$.

1. $\exp(-tY_{r,\eta})x(\theta) = (\cos\tau\cos\theta, 0, ..., 0, -\eta\sin\tau\cos\theta, 0, ..., 0, \sin\theta)$, *where the term* $-\eta\sin\tau\cos\theta$ *is the* r^{th} *coordinate.*

2. *Take* $\theta \in \left]0, \frac{\pi}{2}\right[$. *If* $\tau \in \mathbb{R}$ *is such that* $|\tau|$ *is small enough, then:*

$$\xi_{X_r,\theta}(t) = \arctan\left(\frac{1}{\cos\tau}\sqrt{\sin^2\tau + \tan^2\theta}\right) \quad ; \quad \xi'_{X_r,\theta}(0) = 0 \quad ; \quad \xi''_{X_r,\theta}(0) = \frac{1}{2\tan\theta} \ .$$

Assembling all terms in (4.4)

Equation (4.4) can be written:

$$\sum_{\substack{\eta \in \{i,j,ji\} \\ 2 \leq r \leq n \\ M \in \{X_r, Y_{r,\eta}, Z_\eta\}}} F''\left(\xi_{M,\theta}(0)\right) \cdot \left[\xi'_{M,\theta}(0)\right]^2 + F'\left(\xi_{M,\theta}(0)\right) \cdot \xi''_{M,\theta}(0) = \Lambda \cdot F(\theta)$$

Combining this to propositions 4.2, 4.3, 4.4, 4.5 and 4.6 finally gives for $\theta \in \left]0, \frac{\pi}{2}\right[$:

$$F''(\theta) + \left(\frac{6}{\tan(2\theta)} + \frac{4n-8}{\tan\theta}\right) \cdot F'(\theta) - 2\Lambda F(\theta) = 0 \tag{4.5}$$

If one considers the smooth diffeomorphism (onto)

$$\begin{array}{cccc} \psi & : & \left]0, \frac{\pi}{2}\right[& \longrightarrow &]0,1[\\ & & \theta & \longmapsto & \cos^2\theta \end{array}$$

and applies the change of variables $u = \psi(\theta)$, then one obtains the standard hypergeometric equation given in the theorem below, which summarises this chapter and is one of our main results. Before we state it:

4.7 Definition. The *reduced version* of f_P is the function $\varphi_P = F \circ \psi^{-1}$.

4.8 Theorem. *Consider $\alpha \in \mathbb{N}$ and $k = 2\alpha$. Consider the unique (up to a constant) bi-invariant spherical harmonic of $\mathcal{Y}^k$. Then the restriction of its reduced version to the open interval $]0,1[$ satisfies the hypergeometric equation:*

$$u(1-u)\varphi''(u) + (2 - Nu)\varphi'(u) - \frac{\Lambda}{2}\varphi(u) = 0$$

where $\Lambda = -(2\alpha^2 + (4n-2)\alpha)$.

Remark: at the end of this work, as we draw our conlusions and discuss the occurence of special functions, we recall a few basic facts about hypergeometric equations.

CHAPTER 5

Non-standard picture and Bessel functions

Le but de ce chapitre est de mettre en évidence des K-types de la série principale dégénérée qui admettent une description en termes d'autres fonctions spéciales, à savoir ici des fonctions de Bessel. Comme elles servent à calculer de nombreuses intégrales, ayant elles-mêmes des expressions intégrales, l'idée principale est d'exploiter une transformation dont la construction est basée sur une transformation de Fourier complexe ; ce faisant, on définit un autre modèle des représentations de notre série principale dégénérée de $\mathrm{Sp}(n,\mathbb{C})$: le modèle non-standard.

Dans [35], les auteurs utilisent une transformation de Fourier partielle, qui a des vertus en termes d'opérateurs d'entrelacement. Au cours de leur travail, ils montrent qu'un certain K-type s'exprime à l'aide de la fonction de Bessel K_0. D'où l'idée de se servir d'un type similaire de transformation de Fourier, mais adaptée au cadre complexe. Cette adaptation est faite dans [8], où il est encore question d'opérateurs d'entrelacement, mais pas de fonctions spéciales (ce qui nous intéresse justement ici). Notons $\mathcal{F}$ la transformation de Fourier partielle en question, définie précisément dans le paragraphe 5.1.

Cette transformation de Fourier $\mathcal{F}$ est définie sur $L^2(\mathbb{C}^{2m+1})$; le cadre qu'il nous faut n'est donc plus le modèle compact, mais le modèle non-compact. Cela peut sembler problématique, car nous avons identifié les K-types dans le modèle compact, avec des harmoniques sphériques. Mais n'oublions pas que l'on peut passer d'un modèle à l'autre grâce à la propriété de covariance (voir le chapitre 2 pour se remémorer cette propriété). Ainsi, si l'on s'intéresse à une certaine composante $V_\gamma^{\alpha,\beta}$, dont le vecteur de plus haut poids est la restriction à la sphère $g_\gamma^{\alpha,\beta}$ de $P_\gamma^{\alpha,\beta}$, alors l'homologue de $g_\gamma^{\alpha,\beta}$ dans le modèle non-compact se détermine en prolongeant $g_\gamma^{\alpha,\beta}$ en une fonction g définie sur $\mathbb{C}^N \setminus \{0\}$ par

$$g(x) = |x|^{-i\lambda-N} g_\gamma^{\alpha,\beta}\left(\frac{x}{|x|}\right)$$

puis en restreignant g à l'hyperplan $\{1\} \times \mathbb{C}^m \times \mathbb{C} \times \mathbb{C}^m$, que l'on identifie à $\mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$ en suivant la règle $(1, u, 2s, v) \longleftrightarrow (s, u, v)$.

Fixons maintenant $k \in \mathbb{N}$. La famille de K-types que nous étudions (présentée dans le paragraphe 5.2) est constituée de tous les $V_0^{\alpha,\beta}$ tels que $\alpha + \beta = k$, autrement dit tous les K-types correspondant au paramètre $\gamma = 0$. Comme ce paramètre est désormais fixé à 0, on omet de le préciser dans les notations, que l'on simplifie un peu : on écrit $g_{\alpha,\beta}$ au lieu de $g_0^{\alpha,\beta}$ et l'on note $G_{\alpha,\beta}$ la fonction homologue dans le modèle non-compact.

A ce stade, on a donc des vecteurs de plus haut poids exprimés dans le modèle non-compact. Le modèle non-standard est le conjugué du modèle non-compact par la transformation de Fourier $\mathcal{F}$, c'est à dire que $\mathcal{F}$ entrelace la représentation $\pi_{i\lambda,\delta}$ vue dans le modèle non-compact et la représentation $\pi_{i\lambda,\delta}$ vue dans le modèle non-standard.

La forme non-standard de la fonction $g_{\alpha,\beta}$ est donc tout simplement $\mathcal{F}(G_{\alpha,\beta})$. Le but de ce chapitre est de déterminer explicitement cette fonction.

Dans le paragraphe 5.3, nous détaillons toutes les définitions et propriétés techniques nécessaires pour la suite du travail (fonctions de Bessel, calculs d'intégrales classiques et ainsi de suite).

Dans les paragraphes 5.4 et 5.5 on applique à $G_{\alpha,\beta}$ la transformation $\mathcal{F}_\xi$ puis la transformation $\mathcal{F}_\tau$, aboutissant au calcul explicite d'une intégrale élaborée (théorème 5.11) et, dans le paragraphe 5.6, au récapitulatif qui constitue notre résultat principal (théorème 5.12) et que l'on peut résumer ainsi :

La forme non-standard $\mathcal{F}(G_{\alpha,\beta})$ de $g_{\alpha,\beta}$ est définie pour tous les triplets (s, u, v) de $\mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$ tels que $v \neq 0$ et $s \neq 0$ par :

$$\mathcal{F}(G_{\alpha,\beta})(s,u,v) =$$

$$\frac{(-i\,\overline{s})^\alpha\,\pi^{i\lambda+\beta+n}}{2^{\frac{i\lambda+k}{2}+1}\,\Gamma(\frac{i\lambda+k}{2}+n)}\left(\frac{\sqrt{|s|^2+4\|v\|^2}}{\pi\sqrt{1+\|u\|^2}}\right)^{\frac{i\lambda+\delta}{2}} K_{\frac{i\lambda+\delta}{2}}\left(\pi\sqrt{1+\|u\|^2}\sqrt{|s|^2+4\|v\|^2}\right)$$

Ce chapitre se termine par deux résultats qui suggèrent de nouvelles pistes de recherche :

- le premier (théorème 5.13) met en évidence un opérateur différentiel simple qui permet de relier certains vecteurs de plus haut poids ;
- le second (proposition 5.17) établit un lien entre la formule du théorème 5.11 et des équations différentielles dites d'Emden-Fowler.

We use the definition of the non-standard picture given in [8] (Section 5.1, where it is actually called the non-standard model). It was initiated by the authors of [35], who studied a degenerate principle series of the real symplectic group $\mathrm{Sp}(n,\mathbb{R})$ and the minimal representations of $\mathrm{O}(p,q)$ in [34].
Let us consider any fixed $(\lambda,\delta)\in\mathbb{R}\times\mathbb{Z}$. Remember that these parameters seemingly disappear in the compact and non-compact pictures, but are hidden in the way one extends functions to switch from these pictures to the induced picture. In this section, the parameter δ is often implicit.

5.1 Non-standard picture

Let us denote by (s,u,v) the coordinates on $\mathbb{C}\times\mathbb{C}^m\times\mathbb{C}^m\simeq\mathbb{C}^{2m+1}$ and define some partial Fourier transforms along specific variables of $\mathbb{C}^{2m+1}$:

- Denote by L^1_τ the set of equivalence classes of functions $g:\mathbb{C}^{2m+1}\longrightarrow\mathbb{C}$ such that for almost all $(u,v)\in\mathbb{C}^m\times\mathbb{C}^m$: $\int_{\mathbb{C}}|g(\tau,u,v)|\,d\tau<+\infty$. For such functions g and such pairs (u,v) we set:

$$\mathcal{F}_\tau(g)(s,u,v)=\int_{\mathbb{C}}g(\tau,u,v)\,e^{-2i\pi\mathrm{Re}(s\tau)}\,d\tau$$

- Denote by L^1_ξ the set of equivalence classes of functions $g:\mathbb{C}^{2m+1}\longrightarrow\mathbb{C}$ such that for almost all $(s,u)\in\mathbb{C}\times\mathbb{C}^m$: $\int_{\mathbb{C}^m}|g(s,u,\xi)|\,d\xi<+\infty$. For such functions g and such pairs (s,u) we set:

$$\mathcal{F}_\xi(g)(s,u,v)=\int_{\mathbb{C}^m}g(s,u,\xi)\,e^{-2i\pi\mathrm{Re}<v,\xi>}\,d\xi$$

Density of $L^1_\tau \cap L^2(\mathbb{C}^{2m+1})$ and $L^1_\xi \cap L^2(\mathbb{C}^{2m+1})$ in $L^2(\mathbb{C}^{2m+1})$ implies that above formulas completely define two partial Fourier transforms, written respectively:

$$\mathcal{F}_\tau : L^2(\mathbb{C}^{2m+1}) \longrightarrow L^2(\mathbb{C}^{2m+1})$$
$$\mathcal{F}_\xi : L^2(\mathbb{C}^{2m+1}) \longrightarrow L^2(\mathbb{C}^{2m+1})$$

We now define the partial Fourier transform on which is based the non-standard picture:

$$\mathcal{F} = \mathcal{F}_\tau \circ \mathcal{F}_\xi \tag{5.1}$$

For integrable functions $f : \mathbb{C}^{2m+1} \longrightarrow \mathbb{C}$ one can write:

$$\mathcal{F}(f)(s,u,v) = \int_{\mathbb{C}\times\mathbb{C}^m} f(\tau,u,\xi)\, e^{-2i\pi \mathrm{Re}(s\tau+\langle v,\xi\rangle)}\, d\tau d\xi \tag{5.2}$$

Finally:

5.1 Definition. The *non-standard picture* of $\pi_{i\lambda,\delta}$ has $L^2(\mathbb{C}^{2m+1})$ as carrying space. The action of G is then the conjugate under $\mathcal{F}$ of the action of G in the non-compact picture; in other words, $\mathcal{F}$ intertwines the action of G in the non-compact picture and the action of G in the non-standard picture.

5.2 Aim of this chapter: specific highest weight vectors

Decomposition (3.4) and Theorem 3.6 tell us what the K-types of $\pi_{i\lambda,\delta}$ are: they are the isotypic components of the left action of K which sit in the spaces $\mathrm{H}^{\alpha,\beta}$ such that $\delta = \beta - \alpha$. This point of view comes from the compact picture. Though this picture has helped us describe all K-types, it is interesting to see what they look like in other pictures, in particular the non-standard one. This is the point of the rest of this chapter. Recalling notations of Theorem 3.2, for technical issues we restrict to K-types labelled by $\gamma = 0$, namely the components $V_0^{\alpha,\beta}$ whose highest weight vectors are

$$P_0^{\alpha,\beta}(z,w,\bar{z},\bar{w}) = w_1^\alpha \bar{z_1}^\beta$$

Because we intend to use the right action of $\mathrm{Sp}(1)$, we consider an entire space H^k of harmonic polynomials.

So we now fix (for the rest of this chapter) any $k \in \mathbb{N}$ and consider values of α and β such that:

$$\alpha + \beta = k$$

We denote by $g_{\alpha,\beta}$ the restriction of $P_0^{\alpha,\beta}$ to the unit sphere S^{2N-1}.
Let us call g the function in the induced picture that corresponds to $g_{\alpha,\beta}$. It extends $g_{\alpha,\beta}$, meaning: $g_{|_{S^{2N-1}}} = g_{\alpha,\beta}$. Remember that g must satisfy the covariance relation for all non-zero complex numbers c:

$$g(c\,\cdot) = \left(\frac{c}{|c|}\right)^{-\delta} |c|^{-i\lambda-N}\, g$$

We define

$$a(s,u) = \sqrt{1+4|s|^2+\|u\|^2}$$
$$r(s,u,v) = \sqrt{a^2(s,u)+\|v\|^2}$$

By restricting g to the complex hyperplane $\{1\} \times \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$, we associate to g the function $G_{\alpha,\beta}$ defined on $\mathbb{C}^{2m+1}$ by:

$$\begin{aligned}
G_{\alpha,\beta}(s,u,v) &= g(1,u,2s,v)\\
&= (r(s,u,v))^{-i\lambda-N}\, g\left(\frac{1}{r(s,u,v)}, \frac{u}{r(s,u,v)}, \frac{2s}{r(s,u,v)}, \frac{v}{r(s,u,v)}\right)\\
&= \left(\frac{1}{r(s,u,v)}\right)^{i\lambda+N} g_{\alpha,\beta}\left(\frac{1}{r(s,u,v)}, \frac{u}{r(s,u,v)}, \frac{2s}{r(s,u,v)}, \frac{v}{r(s,u,v)}\right)
\end{aligned}$$

Finally, due to the total homogeneity degree k of $P_0^{\alpha,\beta}$:

$$G_{\alpha,\beta}(s,u,v) = \frac{(2s)^\alpha}{(a^2(s,u)+\|v\|^2)^{\frac{i\lambda+k}{2}+n}} \tag{5.3}$$

The function $G_{\alpha,\beta}$ is the non-compact form of $g_{\alpha,\beta}$. We point out that we automatically know that $G_{\alpha,\beta}$ belongs to the Hilbert space $L^2(\mathbb{C}^{2m+1})$: this follows from the identifications between the carrying spaces in the various pictures.
The aim of the rest of this chapter is to determine the non-standard form $\mathcal{F}(G_{\alpha,\beta})$ of $G_{\alpha,\beta}$ (we shall use the composition formula we have seen: $\mathcal{F} = \mathcal{F}_\tau \circ \mathcal{F}_\xi$). The explicit non-standard form we end up with is given in Theorem 5.11 and also in Theorem 5.12, which is a recap that puts all these calculations back into context. Calculations will involve Bessel functions and various technical results which, for convenience, we put together in the next section.

5.3 Bessel functions and useful formulas

We shall need to compute various integrals. Some will be expressed in terms of Bessel functions. Let us recall definitions of these functions. In these definitions, following for instance [37] (sections 5.3 and 5.7), we take ν to be any complex number (called the *order*) and the variable z to belong to $\mathbb{C} \setminus L$, where L is the half line of complex numbers $\zeta \neq 0$ such that $-\pi < \operatorname{Arg}(\zeta) < \pi$:

1. The *Bessel function of the first kind* is the function J_ν defined by:

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{\nu+2k}$$

   We point out that for $\nu \in \mathbb{Z}$, $J_{-\nu}(z) = (-1)^\nu J_\nu(z)$.

2. The *Bessel function of the second kind* is the function Y_ν defined by:

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}$$

   when $\nu \notin \mathbb{Z}$ and, when $\nu \in \mathbb{Z}$, by

$$Y_\nu(z) = \lim_{\substack{\epsilon \to \nu \\ 0 < |\epsilon - \nu| < 1}} Y_\epsilon(z)$$

   We point out that for $\nu \in \mathbb{Z}$, $Y_{-\nu}(z) = (-1)^\nu Y_\nu(z)$.

3. The *modified Bessel function of the first kind* is the function I_ν defined by:

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{\nu+2k}$$

   We point out that for $\nu \in \mathbb{Z}$, $I_{-\nu}(z) = I_\nu(z)$.

4. The *modified Bessel function of the third kind* is the function K_ν defined by

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}$$

   when $\nu \notin \mathbb{Z}$ and, when $\nu \in \mathbb{Z}$, by

$$K_\nu(z) = \lim_{\substack{\epsilon \to \nu \\ 0 < |\epsilon - \nu| < 1}} K_\epsilon(z)$$

   We point out that for $\nu \in \mathbb{Z}$, $K_{-\nu}(z) = K_\nu(z)$.

Remarks:

- The Gamma function is a meromorphic function that has no zeros and whose poles are $0, -1, -2 \cdots$ (they are all simple poles). In the series above, some coefficients may involve terms $\frac{1}{\Gamma(x)}$ for certain poles x (in which case there are only finitely many coefficients of this sort); but this is not a problem, since such coefficients are then simply 0 (because $|\Gamma(x)| = +\infty$).
- At the end of this work, we recall the differential equations that are satisfied by the various Bessel functions.

When the order is a nonnegative integer, one can define Bessel functions as integrals (see [17], Chapter 8, Section 8.411, Formula 1.[11], or [10], Chapter VII, Section 7.3.1, Formula (2)):

5.2 Proposition (Bessel's integral representation)**.** *When $\nu \in \mathbb{N}$, one can define J_ν as follows:*

$$J_\nu(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz\sin\theta}\, e^{-i\nu\theta} d\theta$$

As a consequence, by writing

$$\cos a\, \cos\theta \,+\, \sin a\, \sin\theta \;=\; \cos(a-\theta) \;=\; \sin\left(\frac{\pi}{2} - a + \theta\right)$$

and using a change of variables one can prove:

5.3 Corollary. *Given any $\nu \in \mathbb{N}$, $\rho > 0$ and $a > 0$:*

$$\int_0^{2\pi} e^{i\nu\theta}\, e^{-i\rho(\cos a\, \cos\theta + \sin a\, \sin\theta)}\, d\theta \;=\; 2\pi e^{i\nu\left(a - \frac{\pi}{2}\right)}\, J_\nu(\rho)$$

The following equalities are stated in [10] (Chapter VII, Section 7.11, items (25) and (26)); one can also find them in [37] (Chapter 5, Section 5.7):

5.4 Proposition.

- $K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z} K_\nu(z)$;
- $K_{\nu-1}(z) + K_{\nu+1}(z) = -2K'_\nu(z)$.

As a consequence: $\frac{\partial K_\nu}{\partial z}(z) = \frac{\nu}{z} K_\nu(z) - K_{\nu+1}(z)$.

Formulas in this proposition are stated in [11] (Chapter VIII: Formula (20) of Section 8.5 and Formula (35) of Section 8.14); one can also find them in [17] (Section 6.565, Formula 4., page 678 and Section 6.596, Formula 7.[8], page 693):

5.5 Proposition (Two integral formulas involving Bessel functions)**.**

- *For any real number* $y > 0$ *and any complex numbers* a, ν, μ *such that* $\mathrm{Re}(a) > 0$ *and* $-1 < \mathrm{Re}(\nu) < 2\mathrm{Re}(\mu) + \frac{3}{2}$*, one has:*

$$\int_0^{\infty} x^{\nu+\frac{1}{2}} \left(x^2 + a^2\right)^{-\mu-1} J_\nu(xy)\sqrt{xy}\; dx =$$

$$\frac{a^{\nu-\mu} y^{\mu+\frac{1}{2}} K_{\nu-\mu}(ay)}{2^\mu \Gamma(\mu+1)}$$

- *For any real number* $y > 0$ *and any complex numbers* a, β, ν, μ *such that* $\mathrm{Re}(a) > 0$, $\mathrm{Re}(\beta) > 0$ *and* $\mathrm{Re}(\nu) > -1$*, one has:*

$$\int_0^{\infty} x^{\nu+\frac{1}{2}} \left(x^2 + \beta^2\right)^{-\frac{\mu}{2}} K_\mu\left(a(x^2+\beta^2)^{\frac{1}{2}}\right) J_\nu(xy)\sqrt{xy}\; dx =$$

$$a^{-\mu}\beta^{\nu+1-\mu} y^{\nu+\frac{1}{2}} (a^2+y^2)^{\frac{\mu}{2}-\frac{\nu}{2}-\frac{1}{2}} K_{\mu-\nu-1}\left(\beta(a^2+y^2)^{\frac{1}{2}}\right)$$

This next formula is stated in [10] (Section 7.4.1, Item (4), page 24):

5.6 Proposition (Asymptotic expansion for Modified Bessel functions)**.** *For any fixed* $P \in \mathbb{N}\backslash\{0\}$ *and* $\nu \in \mathbb{C}$*:*

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \left(\left[\sum_{p=0}^{P-1} \frac{\Gamma\left(\frac{1}{2}+\nu+p\right)}{p!\; \Gamma\left(\frac{1}{2}+\nu-p\right)} (2z)^{-p} \right] + \mathrm{O}\left(|z|^{-P}\right) \right)$$

The following proposition can be derived from Section 9.6 of [12]:

5.7 Proposition (The Bochner formula)**.** *Consider any integer* $p \geq 2$*. For* $\xi' \in S^{p-1}$ *and* $s > 0$*:*

$$\int_{S^{p-1}} e^{-2i\pi s \xi\cdot\xi'} d\sigma(\xi) = 2\pi s^{1-\frac{p}{2}} J_{\frac{p}{2}-1}(2\pi s)$$

where $d\sigma$ *denotes the Euclidean measure of* S^{p-1} *and* $\xi \cdot \xi'$ *denotes the euclidean scalar product of* $\mathbb{R}^p$ *applied to the elements of the sphere* ξ *and* ξ' *seen as elements of* $\mathbb{R}^p$*.*

We end this section by recalling two standard results: the first gives sufficient conditions to differentiate integrals with respect to parameters and the second relates the standard polar change of coordinates.

5.8 Proposition (Parameters and integrals). *Consider an integer $p \geq 1$, a pair $(a,b) \in \mathbb{R}^2$ such that $a < b$ and a function*

$$f : \mathbb{R}^p \times]a,b[\longrightarrow \mathbb{C}$$

Denote by (x,λ) the coordinates on $\mathbb{R}^p \times]a,b[$ (λ is the parameter). Assume that:

- *the function $x \longmapsto f(x,\lambda)$ is integrable on $\mathbb{R}^p$ for all fixed $\lambda \in]a,b|$;*
- *the function $\lambda \longmapsto f(x,\lambda)$ is differentiable for all fixed $x \in \mathbb{R}^p$;*
- *there exists an integrable function $g : \mathbb{R}^p \longrightarrow [0,\infty[$ such that for all $(x,\lambda) \in \mathbb{R}^p \times]a,b[$:*
$$\left| \frac{\partial f}{\partial \lambda}(x,\lambda) \right| \leq g(x)$$

Then, for all $\lambda \in]a,b[$, the function $x \longmapsto \frac{\partial f}{\partial \lambda}(x,\lambda)$ is integrable on $\mathbb{R}^p$, the function

$$\begin{array}{rcl} \phi \ : \ ]a,b[& \longrightarrow & \mathbb{C} \\ \lambda & \longmapsto & \int_{\mathbb{R}^p} f(x,\lambda)\, dx \end{array}$$

is differentiable and

$$\phi'(\lambda) = \int_{\mathbb{R}^p} \frac{\partial f}{\partial \lambda}(x,\lambda)\, dx$$

5.9 Proposition (Integrals and polar coordinates). *Consider any integer $p \geq 2$ and a measurable function $f : \mathbb{R}^p \longrightarrow \mathbb{C}$ (with respect to the Lebesgue measure of $\mathbb{R}^p$). Write elements $x \in \mathbb{R}^p \setminus \{0\}$ as $r\xi$ with $r > 0$ and $\xi \in S^{p-1}$.*
Then f is integrable on $\mathbb{R}^p$ if and only if the function $(r,\xi) \in]0,+\infty[\times S^{p-1} \longmapsto f(r\xi)$ is integrable on $]0,+\infty[\times S^{p-1}$ with respect to the measure $r^{p-1} dr\, d\sigma(\xi)$, where dr refers to the Lebesque measure of $\mathbb{R}$ and $d\sigma(\xi)$ to the euclidean measure of S^{p-1}.
When integrability is satisfied, we have:

$$\int_{\mathbb{R}^p} f(x) dx = \int_0^\infty \left(\int_{S^{p-1}} f(r\xi) d\sigma(\xi) \right) r^{p-1} dr$$

We now set to prove the main result of this chapter. The calculations are long and technical. We spread them over two sections: Section 5.4 (which applies $\mathcal{F}_\xi$) and Section 5.5 (which applies $\mathcal{F}_\tau$). In Section 5.6, we summarise in a single and self contained statement what we have achieved; this will be our main theorem, namely Theorem 5.12.

5.4 First transform

By definition:

$$\mathcal{F}_\xi(G_{\alpha,\beta})(s,u,v) = \int_{\mathbb{C}^m} \frac{(2s)^\alpha}{(a^2(s,u)+\|\xi\|^2)^{\frac{i\lambda+k}{2}+n}}\, e^{-2i\pi \mathrm{Re}<v,\xi>}\, d\xi \tag{5.4}$$

In real coordinates, writing $\xi = x+iy$ and $v = a+ib$ (elements x,y,a,b each belong to $\mathbb{R}^m$) and identifying ξ and v with the elements (x,y) and (a,b) of $\mathbb{R}^m\times\mathbb{R}^m$, formula (5.4) reads:

$$\mathcal{F}_\xi(G_{\alpha,\beta})(s,u,v) = \int_{\mathbb{R}^m\times\mathbb{R}^m} \frac{(2s)^\alpha}{(a^2(s,u)+\|x\|^2+\|y\|^2)^{\frac{i\lambda+k}{2}+n}}\, e^{-2i\pi(a\cdot x-b\cdot y)}\, dxdy \tag{5.5}$$

We remind the reader that the dots in the exponential term refer to the usual Euclidian scalar product of $\mathbb{R}^m$.

Proposition 5.9 allows us to switch to polar coordinates, identifying (x,y) (resp. $(a,-b)$) with the unique point rM (resp. $r'M'$) of $\mathbb{R}^{2m}$ such that M (resp. M') belongs to $S^{2m-1}\subset\mathbb{R}^{2m}$ and $r = \sqrt{\|x\|^2+\|y\|^2} = \|\xi\|$ (resp. $r' = \sqrt{a^2+(-b)^2} = \|v\|$). This is possible because the function

$$(r,M)\in]0,+\infty[\times S^{2m-1} \longmapsto G(s,u,rM)\, e^{-2i\pi M\cdot M'}$$

is integrable on $]0,+\infty[\times S^{2m-1}$ with respect to the measure $r^{2m-1}dr\, d\sigma(\xi)$. This follows from the inequality

$$\left|\frac{r^{2m-1}}{(a^2(s,u)+r^2)^{\frac{i\lambda+k}{2}+n}}\right| \leq \frac{1}{r^{k+3}}$$

and Riemann's usual criterion for integrability.

Here, we denote by $M\cdot M'$ the Euclidean scalar product of $\mathbb{R}^{2m}$ applied to the points M and M' of the sphere S^{2m-1} seen as vectors of $\mathbb{R}^{2m}$. The polar coordinates change the integral of Formula (5.5) into:

$$\int_0^\infty \left(\int_{S^{2m-1}} \frac{(2s)^\alpha}{(a^2(s,u)+r^2)^{\frac{i\lambda+k}{2}+n}} e^{-2i\pi rr'M\cdot M'} d\sigma(M)\right) r^{2m-1}dr \tag{5.6}$$

Integral (5.6) can be written:

$$\int_0^\infty \frac{(2s)^\alpha}{(a^2(s,u)+r^2)^{\frac{i\lambda+k}{2}+n}} \left(\int_{S^{2m-1}} e^{-2i\pi rr'M\cdot M'} d\sigma(M)\right) r^{2m-1}dr \tag{5.7}$$

Proposition 5.7 then changes (5.7) into:

$$\int_0^\infty \frac{(2s)^\alpha}{(a^2(s,u)+r^2)^{\frac{i\lambda+k}{2}+n}} 2\pi(rr')^{1-m} J_{m-1}(2\pi rr') r^{2m-1} dr \tag{5.8}$$

Because $r' = \|v\|$, (5.8) becomes:

$$2^{\alpha+1}\pi s^\alpha \|v\|^{1-m} \int_0^\infty \frac{r^m}{(a^2(s,u)+r^2)^{\frac{i\lambda+k}{2}+n}} J_{m-1}(2\pi\|v\|r) dr \tag{5.9}$$

We now want to apply Proposition 5.5. But it uses another notation system than ours. To understand how to switch from one to the other, let us define new variables x, y, μ by:

$$x = r \ ; \ y = 2\pi\|v\| \ ; \ \mu = \frac{i\lambda+k}{2} + n - 1 \ ; \ \nu = m-1$$

Then (5.9) becomes:

$$2^{\alpha+1}\pi s^\alpha \|v\|^{1-m} y^{-\frac{1}{2}} \int_0^\infty \frac{x^{\nu+\frac{1}{2}}}{(a^2(s,u)+r^2)^{\mu+1}} J_{m-1}(xy)\sqrt{xy} dx \tag{5.10}$$

Proposition 5.5 (first formula) now gives (as long as $\|v\| > 0$)

$$2^{\alpha+1}\pi s^\alpha \|v\|^{1-m} y^{-\frac{1}{2}} \frac{a^{\nu-\mu} y^{\mu+\frac{1}{2}} K_{\nu-\mu}(ay)}{2^\mu \Gamma(\mu+1)}$$

which, back to our own notation choices, is equal to

$$\frac{2^{\alpha+1} s^\alpha \pi^{\frac{i\lambda+k}{2}+n}}{\Gamma\left(\frac{i\lambda+k}{2}+n\right)} \left(\frac{\|v\|}{a(s,u)}\right)^{\frac{i\lambda+k}{2}+1} K_{-\left(\frac{i\lambda+k}{2}+1\right)}(2\pi a(s,u)\|v\|)$$

Because $K_\nu = K_{-\nu}$ whatever the value of ν, we have proved so far:

5.10 Proposition. *Given any $\lambda \in \mathbb{R}$ and any $(\alpha,\beta) \in \mathbb{N}^2$, consider the non-compact version $G_{\alpha,\beta}$ of the highest weight vector $g_{\alpha,\beta}$. Then, for all (s,u,v) in $\mathbb{C}\times\mathbb{C}^m\times\mathbb{C}^m$ such that $v \neq 0$:*

$$\mathcal{F}_\xi(G_{\alpha,\beta})(s,u,v) = 2^{\alpha+1} s^\alpha \frac{\pi^{\Lambda+n}}{\Gamma(\Lambda+n)} \left(\frac{\|v\|}{a(s,u)}\right)^{\Lambda+1} K_{\Lambda+1}(2\pi a(s,u)\|v\|)$$

where $k = \alpha+\beta$ and $\Lambda = \frac{i\lambda+k}{2}$.

5.5 Second transform

Given any $(s,u,v) \in \mathbb{C}\times\mathbb{C}^m\times\mathbb{C}^m$ (we assume $v \neq 0$), we want to apply $\mathcal{F}_\tau$ to $\mathcal{F}_\xi(G_{\alpha,\beta})$ by writing

$$\mathcal{F}_\tau\left[\mathcal{F}_\xi(G_{\alpha,\beta})\right](s,u,v) = \int_{\mathbb{C}} \mathcal{F}_\xi(G_{\alpha,\beta})(\tau,u,v)\, e^{-2i\pi \mathrm{Re}(s\tau)}\, d\tau \tag{5.11}$$

Let us first check that the function $\tau \longmapsto \mathcal{F}_\xi(G_{\alpha,\beta})(\tau,u,v)$ is indeed integrable. In other words, we wish to know whether the integral

$$\int_{\mathbb{C}} |\mathcal{F}_\xi(G_{\alpha,\beta})(\tau,u,v)|\, d\tau$$

or, more explicitely,

$$\int_{\mathbb{C}} |2^{\alpha+1}\tau^\alpha \frac{\pi^{\Lambda+n}}{\Gamma(\Lambda+n)} \left(\frac{\|v\|}{a(\tau,u)}\right)^{\Lambda+1} K_{\Lambda+1}(2\pi a(\tau,u)\|v\|)|\, d\tau \tag{5.12}$$

is finite or not. Let us split (5.12) into two integrals:

- one integral, denoted by I_1, over the unit ball B_1 of $\mathbb{C}$;
- one integral, denoted by I_2, over $\mathbb{C} \setminus B_1$.

Because $\|v\| \neq 0$ and $a(\tau,u) \geq 1$, the function that sits under integral (5.12) is continuous on B_1 and therefore I_1 is finite. Proposition 5.6 (using the integer $P = 1$) and Proposition 5.10 allow us to write $|\mathcal{F}_\xi(G_{\alpha,\beta})(\tau,u,v)|$ in the following way (A, B, C denote positive constants and $B \geq 1$):

$$\frac{A\,|\tau|^\alpha}{(B+4|\tau|^2)^{\frac{k+3}{4}}}\; e^{-C(B+4|\tau|^2)^{\frac{1}{2}}} \left[1 + \mathrm{O}\left(\frac{1}{C(B+4|\tau|^2)^{\frac{1}{2}}}\right)\right] \tag{5.13}$$

Then for $|\tau|$ large enough and a suitable constant $D > 0$,

$$\frac{A}{|\tau|^{\frac{k+3}{2}-\alpha}}\; e^{-C|\tau|} \left(1 + \frac{D}{|\tau|}\right)$$

is an upper-bound of (5.13). The exponential term forces convergence of I_2 and this finally establishes the desired integrability. We point out that this integrability will let us switch to polar coordinates.

Let us use notations a, b, x, y again, this time taking them to simply be real numbers such that $s = a + ib$ and $\tau = x + iy$. Then $\mathrm{Re}(s\tau) = ax - by$ and (5.11) becomes

$$\int_{\mathbb{R}^2} \frac{2^{\alpha+1}\tau^\alpha \pi^{\Lambda+n}}{\Gamma(\Lambda+n)} \left(\frac{\|v\|}{a(\tau,u)}\right)^{\Lambda+1} K_{\Lambda+1}(2\pi a(\tau,u)\|v\|)\, e^{-2i\pi(ax-by)}\, dxdy$$

which can be re-organised as

$$\frac{2^{\alpha+1}\pi^{\Lambda+n}\|v\|^{\Lambda+1}}{\Gamma(\Lambda+n)} \int_{\mathbb{R}^2} \frac{\tau^\alpha K_{\Lambda+1}(2\pi a(\tau,u)\|v\|)}{[a(\tau,u)]^{\Lambda+1}}\, e^{-2i\pi(ax-by)}\, dxdy \tag{5.14}$$

Let us again use polar coordinates (outside the origin):

- $(x, y) = rv_\theta$ with $r > 0$, $\theta \in \mathbb{R}$ and $v_\theta = (\cos\theta, \sin\theta)$; accordingly, $\tau = re^{i\theta}$.
- $(a, -b) = r'v_{\theta'}$ with $r' > 0$, $\theta' \in \mathbb{R}$ and $v_{\theta'} = (\cos\theta', \sin\theta')$.

Let us point out that we obviously have $r = |\tau|$ and $r' = |s|$. Let us also adapt notation $a(s, u)$ to these polar coordinates, writing $a(r, u)$ instead of $a(\tau, u)$:

$$a(r, u) = \sqrt{1 + 4r^2 + \|u\|^2}$$

Integral 5.14 can now be written:

$$\frac{2^{\alpha+1}\pi^{\Lambda+n}\|v\|^{\Lambda+1}}{\Gamma(\Lambda+n)} \int_0^\infty \frac{r^\alpha K_{\Lambda+1}(2\pi a(r,u)\|v\|)}{[a(r,u)]^{\Lambda+1}} \left(\int_0^{2\pi} e^{i\alpha\theta}\, e^{-2i\pi rr'(\cos\theta\cos\theta'+\sin\theta\sin\theta')} d\theta\right) rdr \tag{5.15}$$

Following Corollary 5.3, the inner integral

$$\int_0^{2\pi} e^{i\alpha\theta}\, e^{-2i\pi rr'(\cos\theta\,\cos\theta'+\sin\theta\,\sin\theta')}\, d\theta$$

is equal to:

$$2\pi e^{i\alpha\left(\theta' - \frac{\pi}{2}\right)}\, J_\alpha(2\pi rr') \tag{5.16}$$

Remembering that $r' = |s|$ and $\theta' = \operatorname{Arg}(\overline{s})$, (5.16) is equal to:

$$2\pi e^{i\alpha\left(\operatorname{Arg}(\overline{s}) - \frac{\pi}{2}\right)}\, J_\alpha(2\pi r|s|)$$

This turns (5.15) into:

$$\frac{2^{\alpha+2}\pi^{\Lambda+n+1}\|v\|^{\Lambda+1}e^{i\alpha\left(\operatorname{Arg}(\overline{s}) - \frac{\pi}{2}\right)}}{\Gamma(\Lambda+n)} \int_0^\infty \frac{r^{\alpha+1}\, K_{\Lambda+1}(2\pi a(r,u)\|v\|)}{[a(r,u)]^{\Lambda+1}}\, J_\alpha(2\pi r|s|)\, dr \tag{5.17}$$

We can now apply the second formula of proposition 5.5. To help follow notation choices made in this proposition, we set:

- $x = 2r$ and $dx = 2dr$;
- $\beta = \sqrt{1 + \|u\|^2} > 0$;
- $a = 2\pi\|v\| > 0$ (careful: this variable a is not what we have denoted $a(r, u)$);
- $y = \pi|s| > 0$;
- $\nu = \alpha$;

- $\mu = \Lambda + 1$.

Plugging these expressions in (5.17) and using Proposition 5.5, we finally achieve what we announced at the end of Section 5.2: compute the non-standard form $\mathcal{F}_\tau\left[\mathcal{F}_\xi(G_{\alpha,\beta})\right]$ of $G_{\alpha,\beta}$ (therefore of $g_{\alpha,\beta}$). The formula we obtain is:

5.11 Theorem (An integral formula)**.**

$$\mathcal{F}(G_{\alpha,\beta})(s,u,v) \;=$$
$$\int_{\mathbb{C}\times\mathbb{C}^m} \frac{(2\tau)^\alpha}{(1+4|\tau|^2+\|u\|^2+\|\xi\|^2)^{\frac{i\lambda+k}{2}+n}}\, e^{-2i\pi \mathrm{Re}(s\tau+\langle v,\xi\rangle)}\, d\tau d\xi \;=$$
$$\frac{(-i\,\overline{s})^\alpha\, \pi^{i\lambda+\beta+n}}{2^{\frac{i\lambda+k}{2}+1}\,\Gamma(\frac{i\lambda+k}{2}+n)} \left(\frac{\sqrt{|s|^2+4\|v\|^2}}{\pi\sqrt{1+\|u\|^2}}\right)^{\frac{i\lambda+\delta}{2}} K_{\frac{i\lambda+\delta}{2}}\left(\pi\sqrt{1+\|u\|^2}\sqrt{|s|^2+4\|v\|^2}\right)$$

where:

- $(k,\alpha,n)\in\mathbb{N}^3$, $0\le\alpha\le k$ *and* $n\ge 2$ *;*
- $(s,u,v)\in\mathbb{C}\times\mathbb{C}^m\times\mathbb{C}^m$ *and* $(\tau,\xi)\in\mathbb{C}\times\mathbb{C}^m$ *;*
- $s\neq 0$ *and* $v\neq 0$*.*

5.6 Statement of main result

The following result, which serves as a recap, binds together the formula of Theorem 5.11 and all the ingredients that are needed to understand the true meaning of this formula. In this theorem, we consider three groups M, A and N, respectively isomorphic to the groups $\mathrm{U}(1)\times\mathrm{Sp}(m,\mathbb{C})$, $\mathbb{R}_\times^{+,*}$ (multiplicative group of positive reals) and $\mathrm{H}_\mathbb{C}^{2m+1}$ (complex Heisenberg group), embedded in $\mathrm{Sp}(n,\mathbb{C})$ as follows (denoting elements of M, A, N respectively by m, a, n):

$$m = \left(\begin{array}{cc|cc} e^{i\theta(m)} & 0 & 0 & 0 \\ 0 & A & 0 & C \\ \hline 0 & 0 & e^{-i\theta(m)} & 0 \\ 0 & B & 0 & D \end{array}\right) \quad a = \left(\begin{array}{cc|cc} \alpha(a) & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ \hline 0 & 0 & [\alpha(a)]^{-1} & 0 \\ 0 & 0 & 0 & I_m \end{array}\right) \quad n = \left(\begin{array}{cc|cc} 1 & {}^t u & 2s & {}^t v \\ 0 & I_m & v & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & -u & I_m \end{array}\right)$$

with $\theta(m)\in\mathbb{R}$, $\alpha(a)\in]0,\infty[$, $(s,u,v)\in\mathbb{C}\times\mathbb{C}^m\times\mathbb{C}^m$ and $\left(\begin{array}{c|c} A & C \\ \hline B & D \end{array}\right)\in\mathrm{Sp}(m,\mathbb{C})$. We consider the group $Q = MAN$.

5.12 Theorem (Non-standard picture and Bessel functions)**.**

Consider $n \in \mathbb{N}$ *such that* $n \geq 2$ *and set* $N = 2n$*,* $m = n-1$*.*

Let G *be the group* $\mathrm{Sp}(n, \mathbb{C})$ *and* $Q = MAN$ *its parabolic subgroup introduced above; let* K *be the subgroup* $\mathrm{Sp}(n)$*.*

Consider $(\lambda, \delta) \in \mathbb{R} \times \mathbb{Z}$ *and the character* $\chi_{i\lambda,\delta}$ *defined on* Q *by:*

$$\chi_{i\lambda,\delta}(man) = e^{i\delta\theta(m)} \Big(\alpha(a)\Big)^{i\lambda+N}$$

Consider the degenerate principal series representations $\pi_{i\lambda,\delta} = \mathrm{Ind}_Q^G \chi_{i\lambda,\delta}$ *of* G *(see Section 2.2.1.1 of Chapter 2 for details).*

Consider $k \in \mathbb{N}$ *and any* $(\alpha, \beta) \in \mathbb{N}^2$ *such that* $\alpha + \beta = k$*; set* $\delta = \beta - \alpha$*.*

Consider the irreducible (finite-dimensional) subrepresentation of $\pi_{i\lambda,\delta}\Big|_K$ *whose highest weight is* $(k, 0, \cdots, 0)$*. Then the corresponding highest weight vector (up to a constant) is given:*

- *in the compact picture by the function*

$$\begin{array}{cccccc} g_{\alpha,\beta} & : & S^{2N-1} \subset \mathbb{C}^n \times \mathbb{C}^n & \longrightarrow & \mathbb{C} \\ & & (z, w) & \longmapsto & w_1^\alpha \bar{z_1}^\beta \end{array}$$

- *in the non-compact picture by the function*

$$\begin{array}{cccccc} G_{\alpha,\beta} & : & \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m & \longrightarrow & \mathbb{C} \\ & & (s, u, v) & \longmapsto & \frac{(2s)^\alpha}{(1+4|s|^2+\|u\|^2+\|v\|^2)^{\frac{i\lambda+k}{2}+n}} \end{array}$$

- *in the non-standard picture by the function* $\mathcal{F}(G_{\alpha,\beta})$*, which is defined for all* (s, u, v) *in* $\mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$ *such that* $v \neq 0$ *and* $s \neq 0$ *by:*

$$\mathcal{F}(G_{\alpha,\beta})(s, u, v) \ =$$

$$\frac{(-i\,\overline{s})^\alpha \, \pi^{i\lambda+\beta+n}}{2^{\frac{i\lambda+k}{2}+1}\,\Gamma(\frac{i\lambda+k}{2}+n)} \left(\frac{\sqrt{|s|^2+4\|v\|^2}}{\pi\sqrt{1+\|u\|^2}}\right)^{\frac{i\lambda+\delta}{2}} K_{\frac{i\lambda+\delta}{2}}\left(\pi\sqrt{1+\|u\|^2}\sqrt{|s|^2+4\|v\|^2}\right)$$

Remark: we point out that the symbol N refers to a group, symbols n, m to elements of groups, while the three symbols also refer to dimensions; context makes intended meanings of these symbols clear.

5.7 Two interesting properties

In this section, we make two observations that seem relevant to us: we feel they may prove useful in further investigation of the K-types of the various representations $\pi_{i\lambda,\delta}$ in the non-standard picture.

5.7.1 A differential operator that connects certain K-types

If we look back at Figure 3.1 of Chapter 3, we see that the highest weight vectors we have studied in this present chapter are those that correspond to the components of the column on the far left. We saw in Theorem 3.12 that the operator e applies a highest weight vector onto the one immediately above (up to a constant). When restricting to the highest weight vectors of the left column, the action of e can be interpreted, with respect to their non-standard versions given by Theorem 5.12, as a differential operator that acts on Fourier transforms; this present section makes this clear.
To study properties of $\mathcal{F}$ with respect to differentiablity and multiplication by coordinates, we will use the following notations:

- (τ, u, ξ) will denote the coordinates on $\mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$ with respect to initial functions f;
- (s, u, v) will then denote the coordinates on $\mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$ with respect to $\mathcal{F}(f)$.

5.13 Theorem (Differential induction formula)**.** *Suppose $k \geq 2$ and $\beta \geq 2$. Then:*

$$\mathcal{F}(G_{\alpha+1,\beta-1})(s,u,v) = \frac{2}{-i\pi}\frac{\partial}{\partial s}\Big(\mathcal{F}(G_{\alpha,\beta})(s,u,v)\Big)$$

Proof:

For $0 \leq \alpha \leq k-1$ and $(\tau, u, \xi) \in \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$:

$$\begin{aligned}
G_{\alpha+1,\beta-1}(\tau,u,\xi) &= \Big(r(\tau,u,\xi)\Big)^{-i\lambda-N-k} g_{\alpha+1,\beta-1}(1,u;2\tau,\xi) \\
&= \Big(r(\tau,u,\xi)\Big)^{-i\lambda-N-k}(2\tau)^{\alpha+1} \\
&= 2\tau\,\Big(r(\tau,u,\xi)\Big)^{-i\lambda-N-k}(2\tau)^{\alpha} \\
&= 2\tau\; G_{\alpha,\beta}(\tau,u,\xi)
\end{aligned}$$

Theorem 5.13 then follows lemmas 5.14 and 5.15, both of which are given below.

End of proof.

5.14 Lemma. *If $\beta \geq 2$, then, given any $u \in \mathbb{C}^m$, the maps*

$$(\tau, \xi) \in \mathbb{C} \times \mathbb{C}^m \longmapsto G_{\alpha,\beta}(\tau, u, \xi)$$

$$(\tau, \xi) \in \mathbb{C} \times \mathbb{C}^m \longmapsto \tau\, G_{\alpha,\beta}(\tau, u, \xi)$$

are both integrable.

Proof:

Let us call ψ the map $(\tau, \xi) \longmapsto \tau\, G_{\alpha,\beta}(\tau, u, \xi)$. We point out that integrability of this map forces integrability for the other.

There is no problem as to the integrability of ψ on the unit ball $B_1(0)$ whose center is the origin. So one just needs to check integrability on the set $\mathbb{R}^{2n} \setminus B_1(0)$.

Given a non-zero element (τ, ξ) of $\mathbb{C} \times \mathbb{C}^m \simeq \mathbb{C}^n$, there exists a unique $r \in]0, +\infty[$ and a unique $M \in S^{2n-1}$ such that if $(\tau_M, \xi_M) \in \mathbb{C} \times \mathbb{C}^m$ denote the coordinates of M seen as an element of $\mathbb{C} \times \mathbb{C}^m$, then $(\tau, \xi) = r(\tau_M, \xi_M)$. Identifying complex coordinates on $\mathbb{C} \times \mathbb{C}^m$ with real coordinates on $\mathbb{R}^2 \times \mathbb{R}^{2m} \simeq \mathbb{R}^{2n}$ and applying proposition 5.9, we see that ψ is integrable if the function

$$(r, M) \in]0, +\infty[\times S^{2n-1} \longmapsto r\, G_{\alpha,\beta}(r\tau_M, u, r\xi_M)$$

is itself integrable on $]0, +\infty[\times S^{2n-1}$, with respect to the measure $r^{2n-1} dr\, d\sigma(M)$ and for all $u \in \mathbb{C}^m$. We have:

$$\begin{aligned} |r\, G_{\alpha,\beta}(r\tau_M, u, r\xi_M)|\ r^{2n-1} &= \left| \frac{2^\alpha (r\tau_M)^\alpha\, r^{2n}}{(1 + 4|r\tau_M|^2 + \|u\|^2 + \|r\xi_M\|^2)^{\frac{i\lambda+k}{2}+n}} \right| \\ &\leq \frac{2^\alpha r^{\alpha+2n}}{(r^2)^{\frac{k}{2}+n}} \\ &= \frac{2^\alpha}{r^\beta} \end{aligned}$$

This finishes the proof, because $\beta \geq 2$ and $r \geq 1$.

End of proof.

5.15 Lemma. *Consider a function $f : \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m \longrightarrow \mathbb{C}$. Assume that the maps*

$$(\tau, \xi) \longmapsto f(\tau, u, \xi)$$

$$(\tau, \xi) \longmapsto \tau\, f(\tau, u, \xi)$$

are both integrable on $\mathbb{C} \times \mathbb{C}^m$ for all $u \in \mathbb{C}^m$. Consider the map $\psi : (\tau, u, \xi) \longmapsto \tau\, f(\tau, u, \xi)$. Then:

$$\mathcal{F}(\psi)(s, u, v) \;=\; \frac{1}{-i\pi} \frac{\partial}{\partial s}\Big(\mathcal{F}(f)(s, u, v)\Big)$$

Proof:

Switch to real variables:

- $s = x_s + iy_s$, with $(x_s, y_s) \in \mathbb{R}^2$;
- $u = x_u + iy_u$, with $(x_u, y_u) \in \mathbb{R}^m \times \mathbb{R}^m$;
- $v = x_v + iy_v$, with $(x_v, y_v) \in \mathbb{R}^m \times \mathbb{R}^m$.

Similarly for the integration variables τ and ξ that enter the definition of the non-standard Fourier transform:

- $\tau = x_\tau + iy_\tau$, with $(x_\tau, y_\tau) \in \mathbb{R}^2$;
- $\xi = x_\xi + ix_\xi$ with $(x_\xi, x_\xi) \in \mathbb{R}^m \times \mathbb{R}^m$.

To make expressions in the integral shorter, define:

- $w = (x_\tau, x_u, x_\xi, y_\tau, y_u, y_\xi)$;
- $dw = dx_\tau\, dx_u\, dx_\xi\, dy_\tau\, dy_u\, dy_\xi$.

We remind the reader that the dot simply refers to the standard euclean product (of $\mathbb{R}^m$ here). One last notation before we actually begin the proof. With all these notations, definition 5.2 reads:

$$\mathcal{F}(s, u, v) = \int_{\mathbb{R}^{2n}} f_{\mathbb{R}}(w) \exp^{-2i\pi(x_s x_\tau - y_s y_\tau + x_v \cdot x_\xi - y_v \cdot y_\xi)}\, dw$$

Coordinates x_s and y_s are just parameters that appear in the integral. Lemma 5.14 and proposition 5.8 enable us to differentiate this integral with respect to both parameters and obtain:

$$\frac{\partial}{\partial x_s}\Big(\mathcal{F}(s,u,v)\Big) = \int_{\mathbb{R}^{2n}} -2i\pi x_\tau f_{\mathbb{R}}(w)\exp^{-2i\pi(x_s x_\tau - y_s y_\tau + x_v\cdot x_\xi - y_v\cdot y_\xi)}\,dw$$

$$\frac{\partial}{\partial y_s}\Big(\mathcal{F}(s,u,v)\Big) = \int_{\mathbb{R}^{2n}} 2i\pi y_\tau f_{\mathbb{R}}(w)\exp^{-2i\pi(x_s x_\tau - y_s y_\tau + x_v\cdot x_\xi - y_v\cdot y_\xi)}\,dw$$

This finishes the proof because, by definition, the operator $\frac{\partial}{\partial s}$ is equal to $\frac{1}{2}\left(\frac{\partial}{\partial x_s} - i\frac{\partial}{\partial y_s}\right)$.

End of proof.

5.16 Corollary. *Consider integers k,α,β such that $k\geq 2$, $\alpha+\beta=k$ and $\beta\geq 1$. Then:*

$$\mathcal{F}(G_{\alpha,\beta})(s,u,v) = \left(\frac{2}{-i\pi}\right)^{\alpha}\frac{\partial^\alpha}{\partial s^\alpha}\Big(\mathcal{F}(G_{0,k})(s,u,v)\Big)$$

Remark: as we have seen in the proof, multiplying by 2τ, one switches from $G_{\alpha,\beta}$ to $G_{\alpha+1,\beta-1}$. This corresponds to the action of the operator e studied in section 3.2.1.

The following diagram shows how the operators of this section relate to one another, connecting functions and their transforms (assuming that $\beta\geq 2$):

$$\begin{array}{ccccc}
g_{\alpha+1,\beta-1} & \longrightarrow & G_{\alpha+1,\beta-1} & \longrightarrow & \mathcal{F}(G_{\alpha+1,\beta-1}) \\
\uparrow{\scriptstyle \frac{1}{\beta}e} & & \uparrow{\scriptstyle 2\tau Id} & & \uparrow{\scriptstyle \frac{2}{-i\pi}\frac{\partial}{\partial s}} \\
g_{\alpha,\beta} & \longrightarrow & G_{\alpha,\beta} & \longrightarrow & \mathcal{F}(G_{\alpha,\beta})
\end{array}$$

5.7.2 Underlying Emden-Fowler equations

Theorem 5.12 gives the following expression for the highest weight vectors $g_{\alpha,\beta}$ in the non standard picture:

$$\frac{(-i\,\overline{s})^\alpha\,\pi^{i\lambda+\beta+n}}{2^{\frac{i\lambda+k}{2}+1}\,\Gamma(\frac{i\lambda+k}{2}+n)}\left(\frac{\sqrt{|s|^2+4\|v\|^2}}{\pi\sqrt{1+\|u\|^2}}\right)^{\frac{i\lambda+\delta}{2}} K_{\frac{i\lambda+\delta}{2}}\left(\pi\sqrt{1+\|u\|^2}\sqrt{|s|^2+4\|v\|^2}\right)$$

One inevitably notices a sort of two variables structure in this expression. Indeed, the particular value $\alpha=0$ and the square root terms lead us to study the functions:

$$\begin{array}{rccc}
\psi_\nu: &]0,+\infty[\times]0,+\infty[& \longrightarrow & \mathbb{C} \\
 & (x,y) & \longmapsto & \left(\frac{x}{y}\right)^\nu K_\nu(xy)
\end{array}$$

where the parameter ν is taken to be any complex number.

For any given $\nu \in \mathbb{C}$, $x_0 > 0$ and $y_0 > 0$, define the functions:

$$\begin{array}{rccc} \varphi_{x_0,\nu} : &]0,+\infty[& \longrightarrow & \mathbb{C} \\ & y & \longmapsto & \psi_\nu(x_0, y) \end{array}$$

$$\begin{array}{rccc} \phi_{y_0,\nu} : &]0,+\infty[& \longrightarrow & \mathbb{C} \\ & x & \longmapsto & \psi_\nu(x, y_0) \end{array}$$

Because the function K_ν is a modified Bessel function of the third kind, we have:

$$K_\nu''(\xi) \,+\, \frac{1}{\xi} K_\nu'(\xi) \,-\, \left(1 + \frac{\nu^2}{\xi^2}\right) K_\nu(\xi) \,=\, 0$$

This equation (which is in fact recalled in the conclusions part of this work) is true whether ξ is considered as a real or a complex variable; here, we choose $\xi \in \mathbb{R}$. Combining this equation with the first and second derivatives of the functions $\varphi_{x_0,\nu}$ and $\phi_{y_0,\nu}$, one can check the following proposition.

5.17 Proposition. *Given any $\nu \in \mathbb{C}$, $x_0 > 0$, $y_0 > 0$ and taking $y > 0$:*

$$\varphi_{x_0,\nu}''(y) \,+\, \frac{(1+2\nu)}{y}\, \varphi_{x_0,\nu}'(y) \,-\, x_0^2\, \varphi_{x_0,\nu}(y) \,=\, 0$$

$$\phi_{y_0,\nu}''(x) \,+\, \frac{(1-2\nu)}{x}\, \phi_{y_0,\nu}'(x) \,-\, y_0^2\, \phi_{y_0,\nu}(x) \,=\, 0$$

Both equations can be written as (taking $a \in \mathbb{C}$, $b > 0$ and $t > 0$):

$$u''(t) \,+\, \frac{a}{t}\, u'(t) \,-\, b\, u(t) \,=\, 0 \tag{5.18}$$

Such equations belong to the family of *Emden-Fowler equations* (or *Lane-Emden equations*), which appear in various forms and have been studied in many works (see for instance [6], [2], [3], [43] and [45]).

The general solutions of (5.18) can be written as the following combinations (with t-dependent coefficients) of Bessel functions of the first and second kind:

$$u(t) \,=\, C_1\, t^{\frac{1-a}{2}}\, J_{\frac{a-1}{2}}\left(-it\sqrt{b}\right) \,+\, C_2\, t^{\frac{1-a}{2}}\, Y_{\frac{a-1}{2}}\left(-it\sqrt{b}\right)$$

This implies that $\varphi_{x_0,\nu}$ and $\varphi_{x_0,\nu}$ are such combinations, which gives new formulas for the function ψ_ν, thus other formulas for the non-standard version of the highest weight vector of Theorem 5.12.

Conclusions

Dans cette étude des K-types des séries principales dégénérées de $\mathrm{Sp}(n, \mathbb{C})$ et du lien qu'elles entretiennent avec les fonctions spéciales, nous avons :

- établi dans le chapitre 3 une sorte de cartographie des K-types, montrant comment chaque espace $\mathcal{Y}^k$ se décompose en sous espaces invariants irréductibles pour l'action à gauche de $\mathrm{Sp}(n)$ mais aussi pour l'action à droite de $\mathrm{Sp}(1)$, en mettant en avant la façon dont ces deux structures se croisent et, en corollaire, en obtenant une caractérisation de l'invariance d'une fonction de $L^2(S^{2N-1})$ par l'action à droite de $\mathrm{Sp}(1)$.

- introduit la notion de polynôme harmonique bi-invariant (allant de pair avec la notion d'harmonique sphérique bi-invariante) en précisant les conditions d'existence et d'unicité de ces polynômes et en donnant un algorithme sur les coefficients qui permet de les calculer explicitement.

- montré dans le chapitre 4 que lorsque k est pair, l'unique (à un coefficient près) fonction bi-invariante de $\mathcal{Y}^k$ est associée à une équation hypergéométrique (dont nous rappelons la définition plus loin) ;

- montré dans le chapitre 5 qu'il était possible pour beaucoup de K-types de leur associer des fonctions de Bessel modifiées (accompagnées malgré tout d'un facteur fonctionnel), ce par l'intermédiaire d'un certain type de transformation de Fourier et d'un opérateur différentiel très simple (fournissant au passage le calcul d'une intégrale élaborée).

- établi à la fin du chapitre 5 un lien avec des équations différentielles dites d'Emden-Fowler.

Nous ne rappelons pas ici les énoncés des résultats, puisque les résultats essentiels sont donnés de façon simplifiée dans l'introduction (théorèmes A, B, C et D) et de façon détaillée tout au long des chapitres, les principaux résultats étant :

- les théorèmes 3.2, 3.6, 3.13 et 3.20 (chapitre 3), pour la structure des K-types relatives aux deux actions en question ;

- le théorème 4.8 (chapitre 4), pour l'équation hypergéométrique ;
- le théorème 5.12, pour les fonctions de Bessel ;
- le théorème 5.13 pour un opérateur différentiel qui relie certains K-types.
- la proposition 5.17 pour les équations d'Emden-Fowler.

Une question légitime se pose : que pouvons nous tirer de ce travail ? L'intérêt des résultats obtenus se présente sous plusieurs facettes.

Tout d'abord, les séries principales dégénérées de $\mathrm{Sp}(n,\mathbb{C})$ sont des représentations unitaires significatives et nous en avons une meilleure vision à présent : leur K-types, ingrédients essentiels, sont bien identifiés ; beaucoup d'entre eux ont des éléments qui peuvent être vus comme des manifestations de fonctions spéciales importantes que nous avons su expliciter. En ce sens les fonctions spéciales précisent nos connaissances sur les séries principales dégénérées.

Inversement, la théorie des représentations a permis de regrouper dans un même tableau des fonctions spéciales de natures différentes : fonctions hypergéométriques et fonctions de Bessel. Chacune de ces fonctions est solution d'une certaine équation différentielle ordinaire linéaire d'ordre 2 à coefficients polynomiaux. Rappelons quelques généralités sur les fonctions spéciales (on pourra se reporter à [37] - chapitres 5 et 9).

- Etant donnés des nombres complexes a, b et c (on supposera $c \notin 0, -1, -2, \cdots$) et la variable complexe z (on supposera $0 < |z| < 1$), une solution (au voisinage d'un point régulier) de *l'équation hypergéométrique*

$$z(1-z)\varphi'' + [c-(a+b+1)z]\varphi' - ab\varphi = 0 \tag{5.19}$$

est la *fonction hypergéométrique* $F(a,b\,;\,c\,;\,\cdot)$ définie par :

$$F(a,b\,;\,c\,;\,z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!} \tag{5.20}$$

Dans cette série, les symboles de Pochhammer $(x)_k$ font référence à des produits finis : x désignant un réel et k un entier positif, $(x)_k$ vaut $x(x+1)...(x+k-1)$ lorsque $k > 0$ et 1 lorsque $k = 0$. On reconnaît dans le théorème 4.8 une équation hypergéométrique de paramètres a, b, c avec $ab = \frac{\Lambda}{2}$, $a+b+1 = N$ et $c = 2$. On remarque que Λ étant négatif, la série hypergéométrique associée est un polynôme.

- Etant donné un nombre complexe ν et la variable complexe z (on suppose que z n'est pas un réel négatif ou nul) :
  - Les fonctions J_ν et Y_ν (définies dans le paragraphe 5.3) sont des solutions linéairement indépendantes de l'équation différentielle :

$$\varphi'' + \frac{1}{z}\varphi' + \left(1 - \frac{\nu^2}{z^2}\right)\varphi = 0 \tag{5.21}$$

  - Les fonctions I_ν et K_ν (définies dans le paragraphe 5.3) sont des solutions linéairement indépendantes de l'équation différentielle :

$$\varphi'' + \frac{1}{z}\,\varphi' - \left(1 + \frac{\nu^2}{z^2}\right)\varphi = 0 \tag{5.22}$$

Dans notre travail, nous avons rencontré :

- dans le chapitre 4, une équation hypergéométrique, donc du type (5.19) ;
- dans le théorème 5.12, des fonctions de Bessel modifiées de troisième espèce (même si accompagnées d'un coefficient fonctionnel), correspondant donc à une équation de type (5.22).

Ces équations sont de natures différentes : l'équation (5.19) possède trois singularités régulières (0, 1 et ∞) tandis que (5.22) a deux singularités (0 et ∞), dont l'une n'est pas régulière (à savoir ∞). Etablir des relations entre leurs solutions a donc un intérêt en soi. Qu'il y ait un rapport entre fonctions hypergéométriques et fonctions de Bessel n'est pas nouveau ; en effet, pour $\nu \in \mathbb{C}$, on a l'égalité (voir [50], paragraphe 3.5.6) :

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)}\, F\left(\nu + 1\,;\, -\frac{z^2}{4}\right)$$

où, étant donné un nombre complexe c, $F(c;\cdot)$ désigne une autre fonction hypergéométrique, définie comme dans la formule 5.20 mais en remplaçant $(a)_k(b)_k$ par 1. Ce que notre travail montre d'intéressant, c'est qu'il existe des relations entre de nombreuses fonctions hypergéométriques et fonctions de Bessel (modifiées), relations imposées par la théorie des représentations.

Notre travail s'ouvre ainsi sur des questions qui pourraient se révéler fructueuses :

- ayant rencontré les équations hypergéométriques dans notre travail, comment pouvons-nous exploiter les formes explicites de leurs solutions ?
- ayant rencontré les fonctions de Bessel modifiées, comment pouvons-nous exploiter l'équation différentielle qu'elles vérifient et ont-elles ici une autre signification d'un point de vue théorie des représentations ?
- comment exploiter la théorie des représentations pour affiner le lien mis en évidence entre les fonctions spéciales rencontrées ?
- comment exploiter le lien avec les équations d'Emden-Fowler ?

Bibliographie

[1] V. Bargmann. Irreducible unitary representations of the Lorentz group. *Ann. of Math. (2)*, 48 :568–640, 1947. (Cited on pages 10 and 22.)

[2] R. Bellman. *Stability theory of differential equations.* Courier Corporation, 2008. (Cited on page 128.)

[3] O.P. Bhutani and K. Vijayakumar. On certain new and exact solutions of the Emden-Fowler equation and Emden equation via invariant variational principles and group invariance. *The ANZIAM Journal*, 32(4) :457–468, 1991. (Cited on page 128.)

[4] A. Borel. *Semisimple groups and Riemannian symmetric spaces*, volume 16. Springer, 1998. (Cited on page 37.)

[5] W. Casselman and D. Miličić. Asymptotic behavior of matrix coefficients of admissible representations. *Duke Math. J*, 49(4) :869–930, 1982. (Cited on pages 10 and 21.)

[6] S. Chandrasekhar. An introduction to the study of stellar structure. 1967. (Cited on page 128.)

[7] C. Chevalley. *Theory of Lie groups*, volume 1. Princeton University Press, 1946. (Cited on pages 9 and 20.)

[8] P. Clare. On the degenerate principal series of complex symplectic groups. *J. Funct. Anal.*, 262(9) :4160–4180, 2012. (Cited on pages 12, 13, 24, 66, 109 and 111.)

[9] G. van Dijk and A. Pasquale. Canonical representations of Sp(1,n) associated with representations of Sp(1). *Comm. in Mathematical Physics*, 202(3) :651–667, 1999. (Cited on pages 14 and 25.)

[10] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi. *Higher transcendental functions*, volume II. 1953. (Cited on pages 115 and 116.)

[11] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi. *Tables of integral transforms*, volume II. 1954. (Cited on page 116.)

[12] J. Faraut. *Analysis on Lie groups*, volume 110 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2008. An introduction. (Cited on pages 16, 27, 45, 75 and 116.)

[13] G.B. Folland. *Harmonic analysis in phase space*, volume 122 of *Ann. Math. Studies*. Princeton University Press, Princeton, NJ, 1989. (Cited on pages 10 and 22.)

[14] J. Fourier. *Theorie analytique de la chaleur.* Paris : Chez Firmin Didot, Père et Fils, 1822. (Cited on pages 7 and 19.)

[15] I. Gelfand and M. Neumark. Unitary representations of the Lorentz group. *Acad. Sci. USSR. J. Phys*, 10 :93–94, 1946. (Cited on pages 10 and 22.)

[16] I. Gelfand and D. Raikov. Irreducible unitary representations of locally bicompact groups. *Matematicheskii Sbornik*, 55(2) :301–316, 1943. (Cited on pages 9 and 20.)

[17] I.S. Gradshteyn and I.M. Ryzhik. *Table of integrals, series, and products.* Elsevier/Academic Press, Amsterdam, seventh edition, 2007. (Cited on pages 115 and 116.)

[18] K.I. Gross. The dual of a parabolic subgroup and a degenerate principal series of Sp(n,C). *Amer. J. Math.*, 93 :398–428, 1971. (Cited on pages 14, 25 and 64.)

[19] W.R. Hamilton. On quaternions ; or on a new system of imaginaries in algebra. *Philosophical Magazine Series 3*, 25(169) :489–495, 1844. (Cited on page 49.)

[20] J. Hilgert, T. Kobayashi, G. Mano, and J. Möllers. Orthogonal polynomials associated to a certain fourth order differential equation. *The Ramanujan Journal*, 26(3) :295–310, 2011. (Cited on pages 14 and 25.)

[21] J. Hilgert, T. Kobayashi, G. Mano, and J. Möllers. Special functions associated with a certain fourth-order differential equation. *The Ramanujan Journal*, 26(1) :1–34, 2011. (Cited on pages 14 and 25.)

[22] R.E. Howe and E.C. Tan. Homogeneous functions on light cones : the infinitesimal structure of some degenerate principal series representations. *Bull. Amer. Math. Soc. (N.S.)*, 28(1) :1–74, 1993. (Cited on pages 12, 14, 23, 24, 25 and 90.)

[23] J.-S. Huang. *Lectures on representation theory.* World scientific, 1999. (Cited on pages 9 and 21.)

[24] H. Kalf. On the expansion of a function in terms of spherical harmonics in arbitrary dimensions. *Bull. Belgian Math. Soc. Simon Stevin*, 2(4) :361–380, 1995. (Cited on pages 7 and 19.)

[25] A.A. Kirillov. *Lectures on the orbit method*, volume 64. American Mathematical Society Providence, RI, 2004. (Cited on pages 9 and 21.)

[26] F. Klein. Le programme d'Erlangen. *Jacques Gabay*, 1991. (Cited on pages 8 and 19.)

[27] A.W. Knapp. *Representation theory of semisimple groups.* Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2001. An overview based on examples, Reprint of the 1986 original. (Cited on pages 9, 10, 21, 32, 36, 37, 39, 41, 43, 57, 58, 59, 61 and 93.)

[28] A.W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics.* Birkhäuser Boston Inc., Boston, MA, second edition, 2002. (Cited on pages 34 and 68.)

[29] T. Kobayashi, T. Kubo, and M. Pevzner. *Conformal symmetry breaking operators for differential forms on spheres*, volume 2170 of *Lecture Notes in Math.* Springer, 2016. (Cited on pages 11 and 23.)

[30] T. Kobayashi and G. Mano. *The Schrödinger model for the minimal representation of the indefinite orthogonal group O(p,q)*, volume 213. Memoirs of the AMS, 2011. (Cited on pages 14 and 25.)

[31] T. Kobayashi and J. Möllers. An integral formula for L^2-eigenfunctions of a fourth-order Bessel-type differential operator. *Integral Transforms and Special Functions*, 22(7) :521–531, 2011. (Cited on pages 14 and 25.)

[32] T. Kobayashi and B. Ørsted. Analysis on the minimal representation of O(p,q) I. realization via conformal geometry. *Adv. in Math.*, 180(2) :486–512, 2003. (Cited on pages 12 and 23.)

[33] T. Kobayashi and B. Ørsted. Analysis on the minimal representation of O(p,q) II. branching laws. *Adv. in Math.*, 180(2) :513–550, 2003. (Cited on pages 12 and 23.)

[34] T. Kobayashi and B. Ørsted. Analysis on the minimal representation of O(p,q) : III. Ultrahyperbolic equations on $R^{p,q}$. *Adv. in Math.*, 180(2) :551–595, 2003. (Cited on pages 12, 23 and 111.)

[35] T. Kobayashi, B. Ørsted, and M. Pevzner. Geometric analysis on small unitary representations of GL(n,R). *J. Funct. Anal.*, 260(6) :1682–1720, 2011. (Cited on pages 12, 13, 14, 17, 24, 25, 28, 61, 109 and 111.)

[36] T. Kobayashi and M. Pevzner. Differential symmetry breaking operators : I. General theory and F-method. *Selecta Math.*, 22(2) :801–845, 2016. (Cited on pages 11, 22 and 23.)

[37] N.N. Lebedev. Special functions & their applications, Dover publications. (Cited on pages 114, 115 and 130.)

[38] G.W. Mackey. On induced representations of groups. *Amer. J. Math.*, 73 :576–592, 1951. (Cited on pages 9 and 20.)

[39] G.W. Mackey. Induced representations of locally compact groups I. *Ann. Math.*, Vol. 55, No. 1 :101–139, 1952. (Cited on pages 9 and 20.)

[40] G.W. Mackey. Induced representations of locally compact groups II. The frobenius reciprocity theorem. *Ann. Math.*, pages 193–221, 1953. (Cited on pages 9 and 20.)

[41] G.W. Mackey. Harmonic analysis as the exploitation of symmetry-a historical survey. *Bull. of the AMS*, 3(1) :543–698, 1980. (Cited on pages 7 and 19.)

[42] R. Mneimné and F. Testard. *Introduction à la théorie des groupes de Lie classiques*. Collection Méthodes. [Methods Collection]. Hermann, Paris, 1986. (Cited on page 39.)

[43] B. Muatjetjeja and C.M. Khalique. Exact solutions of the generalized Lane-Emden equations of the first and second kind. *Pramana*, 77(3) :545–554, 2011. (Cited on page 128.)

[44] P. Pajas and R. Rączka. Degenerate representations of the symplectic groups. I. The compact group Sp(n). *J. of Math. Physics*, 9(8) :1188–1201, 1968. (Cited on pages 12 and 23.)

[45] K. Parand, M. Nikarya, and J.A. Rad. Solving non-linear Lane-Emden type equations using Bessel orthogonal functions collocation method. *Celestial Mechanics and Dynamical Astronomy*, 116(1) :97–107, 2013. (Cited on page 128.)

[46] A. Pasquale. Maximal degenerate representations of SL(n+1,H). *J. Lie Theory*, 9(2) :369–382, 1999. (Cited on pages 14 and 25.)

[47] F. Peter and H. Weyl. Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe. *Math. Annalen*, 97(1) :737–755, 1927. (Cited on pages 8 and 20.)

[48] H. Samelson. *Notes on Lie algebras*. Universitext. Springer-Verlag, New York, second edition, 1990. (Cited on page 39.)

[49] M. Tadić. GL(n,C) and GL(n,R). *Contemporary Mathematics*, 14 :285, 2009. (Cited on pages 12 and 23.)

[50] N.Ja. Vilenkin and A.U. Klimyk. *Representations of Lie groups and special functions - Volume 1 : simplest Lie groups, special functions and integral transforms*, volume 72 of *Mathematics and Its Applications (Soviet Series)*. Springer-Science+Business Media, B.V., 1993. (Cited on pages 11, 22 and 131.)

[51] N.Ja. Vilenkin and A.U. Klimyk. *Representations of Lie groups and special functions - Volume 2 : class I representations, special functions and integral transforms*, volume 74 of *Mathematics and Its Applications (Soviet Series)*. Springer-Science+Business Media, B.V., 1993. (Cited on pages 75, 76 and 77.)

[52] D.A. Vogan Jr. *Unitary representations of reductive Lie groups*, volume 118 of *Ann. Math. Studies*. Princeton University Press, Princeton, NJ, 1987. (Cited on pages 10 and 21.)

[53] D.P. Želobenko. *Compact Lie groups and their representations*, volume 40. American Mathematical Soc., 1973. (Cited on pages 12, 23 and 93.)

Analyse Harmonique Quaternionique et Fonctions Spéciales Classiques

Ce travail s'inscrit dans l'étude des symétries d'espaces de dimension infinie. Il répond à des questions algébriques en suivant des méthodes analytiques. Plus précisément, nous étudions certaines représentations du groupe symplectique complexe dans des espaces fonctionnels. Elles sont caractérisées par leurs décompositions isotypiques relativement à un sous-groupe compact maximal. Ce travail décrit ces décompositions dans deux modèles : un modèle classique (dit compact) et un autre plus récent (dit non-standard). Nous montrons que cela établit un lien entre deux familles de fonctions spéciales (fonctions hypergéométriques et fonctions de Bessel) ; ces familles sont associées à des équations différentielles ordinaires d'ordre 2, fuchsiennes dans un cas et non fuchsiennes dans l'autre. Nous mettons aussi en évidence, dans le modèle non-standard, un lien avec certaines équations d'Emden-Fowler, ainsi qu'un opérateur différentiel simple qui agit sur les décompositions isotypiques.

Mots-clés : groupe symplectique complexe, représentation unitaire, décomposition isotypique, modèle non-standard, fonctions hypergéométriques, fonctions de Bessel.

Quaternionic Harmonic Analysis and Classical Special Functions

The general setting of this work is the study of symmetry groups of infinite-dimensional spaces. We answer algebraic questions, using analytical methods. To be more specific, we study certain representations of the complex symplectic group in functional spaces. These representations are characterised by their isotypic decompositions with respect to a maximal compact subgroup. In this work, we describe these decompositions in two different models: a classical model (compact picture) and a more recent one (non-standard picture). We show that this establishes a connection between two families of special functions (hypergeometric functions and Bessel functions); these families correspond to second order differential equations, which are Fuchsian in one case and non-Fuchsian in the other. We also establish a link with certain Emden-Fowler equations and exhibit a simple differential operator that acts on the isotypic decompositions.

Keywords : complex symplectic group, unitary representation, isotypic decomposition, non-standard model, hypergeometric functions, Bessel functions.

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