Space-time resonances and trapped waves.

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présentée par

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Résonances en espace-temps et ondes confinées

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dirigée par Isabelle Gallagher et Pierre Germain

Soutenue le 17 juin 2015 devant le jury composé de :

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Le principe est simple : une vibration du tonnerre avec une résonance maximum.
Franquin, Gaston Lagaffe

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Résumé
Cette thèse s'attache à l'existence et la dynamique en temps long de solutions d'équations des ondes bidimensionnelles dans lesquelles l'une des deux directions est confinée, soit par la géométrie du problème (équation des ondes sur $\mathbb{R} \times \mathbb{T}$), soit par un potentiel harmonique dans une direction. Après une présentation des méthodes utilisées pour l'analyse harmonique des équations dispersives (notamment la méthode dite des résonances en espace-temps), nous nous intéressons à l'équation des ondes avec masse non-nulle (ou de Klein-Gordon) sur $\mathbb{R} \times \mathbb{T}$ pour prouver un théorème d'existence et d'unicité en temps long et expliciter les méthodes employées pour traiter un problème anisotrope. Puis nous établissons un théorème d'existence en temps long pour l'équation des ondes sur $\mathbb{R}^2$ avec potentiel harmonique dans une direction. Nous approchons ensuite cette équation par une équation résonante : nous dérivons cette équation et nous prouvons un théorème d'existence en temps long, ainsi qu'un théorème d'approximation L^2.

Mots-clefs
Équations dispersives, équation des ondes, équation de Klein-Gordon, potentiel harmonique, résonances en espace-temps, système résonant, dynamique en temps long.

Abstract
This thesis focuses on long-time existence and long-time dynamics for the solutions of some bidimensional wave equations where one direction is trapped, either by the geometry of the problem (if we take a wave equation on $\mathbb{R} \times \mathbb{T}$) or by a harmonic potential in one direction. After presenting the methods used in the harmonic analysis of dispersive equations (with a special focus on the so-called space-time resonances method), we start by studying the wave equation with a mass on $\mathbb{R} \times \mathbb{T}$ (i.e. Klein-Gordon's equation) in order to prove a long-time existence and uniqueness theorem : it will allow us to explain in detail the methods used for an anisotropic problem. Then we establish a long-time existence and uniqueness theorem for the wave equation on $\mathbb{R}^2$ with a harmonic potential in one direction. Finally we approximate the long-time dynamics of this equation by a resonant equation : we derive this system, prove a long-time existence theorem and a L^2 approximation theorem.

Key words
Dispersive equations, wave equation, Klein-Gordon equation, harmonic potential, space-time resonances, resonant system, long-time dynamics.

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Chapitre 1

Introduction

Contents

Note for the non-French-speaking reader : Chapters 2 and after are self-contained and will not refer to this introduction.

Dans ce chapitre introductif, nous présentons le cadre d'étude dans lequel nous plaçons cette thèse, ainsi que les principaux résultats prouvés dans les chapitres suivants et leurs conséquences.

1.1 Étude à données petites d'équations dispersives

Nous nous intéressons dans cette thèse à deux équations basées sur l'équation des ondes $\partial_t^2 u - \Delta u = 0$.

1. Tout d'abord, nous considérons ladite équation sur $\mathbb{R} \times \mathbb{T}$, avec une masse μ (prise strictement positive), et avec une non-linéarité quadratique :

$$\begin{cases} \partial_t^2 u - \Delta u + \mu u &= Q(u), \\ u(0,x) &= u_0(x), \\ \partial_t u(0,x) &= u_1(x), \end{cases} \tag{O1}$$

   où

$$\begin{aligned} u &: (t, x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T} \mapsto u(t, x_1, x_2) \in \mathbb{R}, \\ u_0 &: (x_1, x_2) \in \mathbb{R} \times \mathbb{T} \mapsto u_0(x_1, x_2) \in \mathbb{R}, \\ u_1 &: (x_1, x_2) \in \mathbb{R} \times \mathbb{T} \mapsto u_1(x_1, x_2) \in \mathbb{R}, \\ Q &\text{ est une fonction quadratique de } u. \end{aligned}$$

2. Ensuite, nous nous intéressons à l'équation des ondes, sur $\mathbb{R}^2$, mais avec un potentiel harmonique dans la direction x_2, avec une non-linéarité quadratique :

$$\begin{cases} \partial_t^2 u - \Delta u + x_2^2 u + u &= Q(u), \\ u(0,x_1,x_2) &= u_0(x_1,x_2), \\ \partial_t u(0,x_1,x_2) &= u_1(x_1,x_2), \end{cases} \tag{O2}$$

   où

$$\begin{aligned} u &: (t, x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}^2 \mapsto u(t, x_1, x_2) \in \mathbb{R}, \\ u_0 &: (x_1, x_2) \in \mathbb{R}^2 \mapsto u_0(x_1, x_2) \in \mathbb{R}, \\ u_1 &: (x_1, x_2) \in \mathbb{R}^2 \mapsto u_1(x_1, x_2) \in \mathbb{R}, \\ Q &\text{ est une fonction quadratique de } u. \end{aligned}$$

Ces équations ont plusieurs similarités : ce sont des équations dites dispersives, basées sur l'équation des ondes, avec un confinement anisotrope.
Intéressons-nous dans un premier temps au caractère dispersif de ces équations. La notion de confinement sera développée ultérieurement.
Nous appellerons équation dispersive toute équation dont le terme linéaire présente des solutions sous la forme d'ondes planes, c'est-à-dire des solutions de la forme

$$(t, x) \mapsto e^{i(\omega t - k \cdot x)},$$

où ω et k sont deux réels, et telles que la *vitesse de groupe* $\frac{\partial \omega}{\partial k}$ dépende de k. Cette dépendance se traduit par $\mathrm{Hess}(\omega) \neq 0$: moins $\mathrm{Hess}(\omega)$ est singulière, plus l'équation est

dispersive.
Ainsi l'équation de la chaleur n'entre pas dans ce cadre (ω ou k devrait être complexe), mais l'équation de Schrödinger

$$\partial_t u - i\Delta u = 0, \tag{S}$$

admettant des ondes planes de la forme

$$e^{i(-k^2 t - k\cdot x)},$$

est une équation dispersive. De même, si l'on considère l'équation des ondes

$$\partial_t^2 u - \Delta u = 0, \tag{O}$$

on remarque qu'elle admet des solutions sous forme d'ondes planes de la forme

$$e^{i(|k|t \pm k\cdot x)}.$$

L'équation des ondes est donc dispersive en dimension supérieure ou égale à 2.
De manière plus générale, les équations s'écrivant sous la forme suivante sont des équations dispersives :

$$\partial_t u + L(i\partial)u = 0,$$

ou

$$\partial_t^2 - \left(L(i\partial)\right)^2 u = 0,$$

pour une fonction L vérifiant $L'' \neq 0$, si tant est que $L(i\partial)$ ait un sens. La simplicité de la géométrie dans laquelle nous travaillons ($\mathbb{R}^2$ ou $\mathbb{R}\times\mathbb{T}$) nous permet de définir aisément ce multiplicateur $L(i\partial)$, en utilisant la transformée de Fourier (notée $\mathcal{F}$ et définie précisément dans le chapitre 2 en (2.1.1)) :

$$L(i\partial)f = \mathcal{F}^{-1}\left(L(\xi)\mathcal{F}f(\xi)\right).$$

Par exemple, pour l'équation de Schrödinger, $L(\xi) = \xi^2$; pour l'équation des ondes il s'agit de $L(\xi) = |\xi|$. De manière générale, une onde plane solution d'une équation dispersive s'écrit sous la forme

$$e^{i(L(\xi)t - \xi x)}.$$

Pour ces équations, la vitesse de propagation d'un paquet d'ondes, i.e. une solution de l'équation centrée à la fréquence ξ, aussi appelée vitesse de groupe, est égale à $L'(\xi)$. Dans le cas de l'équation de Schrödinger, cette vitesse dépend de la fréquence (elle est égale à 2ξ) alors que pour l'équation des ondes en dimension 1 elle est constante sur $\mathbb{R}_-$ et sur $\mathbb{R}_+$ (en dimension supérieure elle est constante sur toute direction radiale).

Cette différence entre l'équation des ondes et l'équation de Schrödinger est fondamentale : lorsque la vitesse dépend de la fréquence, il y a dispersion de la solution, ce qui fait que cette dernière aura tendance à s'étaler.
Mathématiquement, cela se traduit par une décroissance de la norme L^∞ de la solution combinée avec la conservation de la norme L^2. Pour l'équation de Schrödinger dans $\mathbb{R}$, on a par exemple, pour une solution u de (S) de condition initiale u_0,

$$\|u(t)\|_{L^\infty} \leq \frac{C}{\sqrt{t}}\|u_0\|_{L^1}.$$

Une telle décroissance n'existe pas dans l'équation des ondes en dimension 1 : les ondes qu'elle engendre avancent toutes à la même vitesse, et la norme L^∞ d'une solution de

l'équation des ondes libre ne décroît pas (elle décroît en dimension supérieure).
En revanche – et c'est l'une des raisons pour lesquelles nous ne considérons pas tout-à-fait l'équation des ondes dans (O1) – si on s'intéresse à l'équation des ondes avec une masse (appelée aussi équation de Klein-Gordon)

$$\partial_t^2 u - \Delta u + \mu u = 0, \tag{KG}$$

on remarque que $L(\xi) = \sqrt{\xi^2 + \mu}$, de dérivée $L'(\xi) = \frac{\xi}{\sqrt{\xi^2+\mu}}$. On a ainsi un phénomène de dispersion, moins fort que celui lié à Schrödinger (étant donné que la vitesse d'une onde à une fréquence ξ tend vers 1 lorsque ξ tend vers l'infini), mais cela permet tout de même d'obtenir une décroissance L^∞ de la solution :

$$\|u(t)\|_{L^\infty} \leq \frac{1}{\sqrt{t}} \|u_0\|_{W^{3/2,1}} ,$$

où $W^{3/2,1}$ correspond à l'espace de Sobolev à régularité $3/2$ dans L^1 : ce coût en dérivée est en quelque sorte la traduction de la moindre efficacité de la dispersion dans Klein-Gordon.

Savoir que les solutions de l'équation libre décroissent en temps est d'une grande utilité lorsque vient l'étude du cas non-linéaire. Nous étudions dans cette thèse des non-linéarités quadratiques, à donnée initiale petite, c'est-à-dire des équations du type

$$\left\{\begin{array}{rcl} \partial_t u - i\Delta u &=& u^2, \\ u(0,x) &=& \varepsilon v_0(x), \end{array}\right. \tag{NLS}$$

avec ε un petit paramètre. Si l'on écrit $u = \varepsilon v$, l'équation que l'on obtient pour v est

$$\left\{\begin{array}{rcl} \partial_t v - i\Delta v &=& \varepsilon v^2, \\ v(0,x) &=& v_0(x), \end{array}\right.$$

En d'autres termes, pour ε assez petit le terme non-linéaire est négligeable et l'évolution de l'équation doit être gouvernée par la partie linéaire. C'est une des raisons pour laquelle nous considérons une non-linéarité quadratique : une non-linéarité d'ordre supérieure u^k deviendrait $\varepsilon^{k-1}v^k$, et serait en quelque sorte « plus négligeable » qu'une non-linéarité quadratique. La non-linéarité quadratique est, dans ce contexte de petites données, être le cas le plus difficile. De plus, il s'agit en quelque sorte d'un cas général : si l'on fait un développement limité de la nonlinéarité autour de 0, et que l'on suppose de plus qu'elle est invariante par translation, on obtient pour premier terme d'ordre > 1 un opérateur bilinéaire lui aussi invariant par translation. Or un opérateur bilinéaire invariant par translation est un opérateur de pseudo-produit, donc très proche d'une nonlinéarité « simple » en u^2.

La partie linéaire d'une équation dispersive étant bien comprise, les différentes techniques développées pour l'étude des équations dispersives s'attèlent à estimer au mieux la non-linéarité.

Une des méthodes les plus populaires consiste à utiliser des estimations dites de Strichartz. Introduites par Strichartz dans [56], elles consistent à estimer des normes $L^q_t L^p_x$ (L^q en temps/L^p en espace) de la non-linéarité. Cette méthode, que nous ne développerons pas ici, est très efficace dans un cadre hamiltonien puisqu'elle permet à moindre coût l'établissement d'un théorème d'existence locale, l'existence globale étant ensuite donnée par

l'existence d'une quantité conservée. Elle permet aussi d'obtenir des résultats à régularité plus faible que les méthodes développées dans cette thèse. Pour des exemples détaillés, on pourra se référer au livre de Tao [57] ainsi qu'à des articles comme [10].
Cependant, utiliser des estimations de Strichartz s'avère beaucoup moins efficace lorsque l'on étudie des systèmes non-hamiltoniens comme l'équation (NLS) ou les équations (O1) et (O2), et surtout lorsqu'on s'intéresse à une étude en temps long : en effet la décroissance obtenue par les inégalités de Strichartz est en général plus faible que celle donnée par les estimations dispersives. De par leur nature (elles sont basées sur des espaces L^q en temps), les estimations de Strichartz ne voient que des effets globaux des interactions dans le terme quadratique.

Plusieurs solutions ont été proposées pour analyser plus finement la non-linéarité. Nous nous focaliserons sur l'une des ces solutions : les résonances en espace-temps. Elle repose substantiellement sur l'un des théorèmes les plus simples et les plus élémentaires d'analyse harmonique : le lemme de la phase stationnaire.

Proposition 1.1.1. *Soient $u \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{C})$, $\varphi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$. On suppose que φ possède un unique point critique non dégénéré sur supp(u) noté x_c. Alors*

$$\int_{-\infty}^{+\infty} e^{i\lambda\varphi(x)} u(x) dx \sim_{\lambda\to+\infty} \sqrt{\frac{2\pi}{\lambda}} \frac{e^{i\frac{\pi}{4}\operatorname{sign}\varphi''(x_c)}}{|\varphi''(x_c)|^{1/2}} u(x_c) e^{i\lambda\varphi(x_c)}.$$

Sans rentrer dans les détails de la méthode – celle-ci sera développée dans le chapitre 2 –, tout vient du fait qu'une équation dispersive peut s'écrire sous la forme suivante (appelée formule de Duhamel) :

$$\hat{f}(t,\xi) := \hat{f}_0(\xi) + \int_0^t \int_{\mathbb{R}} e^{is\phi(\xi,\eta)} \hat{f}(\eta)\hat{f}(\xi-\eta) d\eta ds,$$

où $\phi(\xi,\eta) = L(\xi) \pm L(\eta) \pm L(\xi-\eta)$.
Afin d'estimer et prouver de la décroissance pour le terme intégral (correspondant à la non-linéarité de l'équation), le lemme de phase stationnaire est idoine : il faut donc savoir quand cette phase est stationnaire par rapport à la variable s (i.e. lorsque $\phi = 0$: ce sont les résonances en temps) ou par rapport à η (i.e. lorsque $\partial_\eta\phi = 0$: ce sont les résonances en espace). L'étude des zones d'annulation de la phase ϕ ou de sa dérivée $\partial_\eta\phi$ est donc une étape importante de cette méthode. Plus ces zones seront complexes, moins les estimations du terme intégral seront aisées : ainsi pour l'équation de Schrödinger en dimensions 2 et 3 ont été traités les cas où $\{\phi = 0\} \cap \{\partial_\eta\phi = 0\} = \{0\}$ (dans [24], [26]), mais le cas d'une non-linéarité égale à $|u|^2$, pour laquelle $\{\phi = 0\} \cap \{\partial_\eta\phi = 0\}$ est une droite, est toujours ouvert (on s'attend à une explosion en temps fini).

Cette méthode s'est révélée extrêmement fructueuse depuis son développement en 2009 par Germain, Masmoudi et Shatah qui dans [24] ont synthétisé deux techniques d'étude des équations dispersives : les formes normales de Shatah [52] et les champs de vecteurs de Klainerman [37]. Par exemple, pour des équations de Klein-Gordon quadratiques, outre l'article de Shatah [52], on peut se référer à [44] par exemple, ou, pour des systèmes, à [22] ou [36]. Dans le cadre de l'équation des ondes, on regardera notamment les articles de Pusateri [47] et Pusateri-Shatah [48].
Dans cette thèse, nous utiliserons la méthode des résonances en espace-temps pour l'étude des deux équations (O1) et (O2). Si l'utilisation de cette méthode est assez directe dans le

cas de (O1), elle se révèlera beaucoup plus délicate dans le cadre de l'étude de l'équation des ondes avec potentiel harmonique : en effet, pour l'équation (O2), l'ensemble résonant en espace-temps $\{\phi = 0\} \cap \{\partial_\eta \phi = 0\}$ est une droite dans certains cas, ce qui contraint à une étude très fine de la phase ϕ et à l'utilisation de nouvelles méthode d'étude de ces résonances.

1.2 Confinement d'ondes

Une particularité supplémentaire des équations (O1) et (O2) est que ces équations en deux dimensions présentent une situation de confinement dans une direction. Physiquement, pour confiner une onde, deux solutions sont possibles :

1. il est tout d'abord possible de les faire circuler dans un câble, c'est-à-dire dans $\mathbb{R}\times\mathbb{T}$: la première direction est libre mais la seconde, de par la géométrie du problème, est confinée. Il s'agit de la situation de l'équation (O1).
2. il est possible de les laisser évoluer dans une géométrie plate ($\mathbb{R}^2$ par exemple), mais avec un potentiel qui les confine dans une direction : c'est l'effet qu'a le potentiel harmonique de l'équation (O2). Ce phénomène s'illustre par exemple lorsque l'on soumet des condensats de Bose-Einstein à un potentiel harmonique : ils prennent alors une « forme de cigare » (cf [16]).

Ce confinement, et surtout cette anisotropie, demandent une étude de l'équation elle-même anisotrope. Prenons l'exemple de

$$\partial_t^2 u - \Delta_{\mathbb{R}^2} u + \mu u = u^2,$$

avec u fonction de $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{T}$. Deux transformations sont naturelles : la transformée de Fourier dans la première direction, le développement en série de Fourier dans la seconde. Un calcul rapide montre que si $u_m(t, x_1)$ est le m-ième coefficient de Fourier de u dans la direction x_2, l'équation se réécrit :

$$\partial_t^2 u_p - \Delta_{\mathbb{R}} u_p + p^2 u_p + \mu u_p = \sum_{m\in\mathbb{Z}} u_m u_{p-m}.$$

Ainsi on a transformé l'équation de départ en une infinité (dénombrable) d'équations de Klein-Gordon, avec $L(\xi) = \sqrt{\xi^2 + p^2 + \mu}$, c'est-à-dire avec une dispersion d'autant meilleure que p est grand. Ce phénomène propre aux espaces produits peut être analysé ainsi :

- une équation dispersive sur le tore n'a aucune chance de disperser, ce phénomène étant lié au fait que les hautes fréquences sont envoyées à l'infini plus rapidement que les autres.
- mais lorsque ce tore est couplé à un espace plat, dans $\mathbb{T}\times\mathbb{R}$ par exemple, les hauts modes de Fourier du tore vont contribuer à la vitesse de propagation dans la direction infinie (c'est ce qui est visible dans la formule $L(\xi) = \sqrt{\xi^2 + p^2 + \mu}$).

L'idée est substantiellement la même lorsqu'un potentiel harmonique vient faire office de potentiel confinant. En effet, de même que les séries de Fourier sont adaptés à l'étude du Laplacien sur $\mathbb{T}$, les séries d'Hermite sont adaptés à l'oscillateur harmonique $-\Delta + x^2$: nous appelons p-ième coefficient d'Hermite d'une fonction u la coordonnée $(u, \psi_p)_{L^2}$, où ψ_p est la p-ième fonction d'Hermite, c'est-à-dire le p-ième vecteur propre de $-\Delta + x^2$, associé à la valeur propre $2p + 1$ (p entier naturel). De même que précédemment, on ramène

(O2) à une infinité d'équations en dimension 1 avec $L_p(\xi) = \sqrt{\xi^2 + 2p + 2}$. Cependant, la transformation d'Hermite, contrairement à la transformée de Fourier, ne transforme pas la non-linéarité quadratique en une convolution. Cette difficulté supplémentaire doit être prise en charge et le sera notamment dans les appendices, par quelques compléments d'analyse harmonique.

Les cas d'équations dispersives en géométrie non plate ou avec des potentiels ont donné lieu à un nombre important de publications récentes. Pour des études de Klein-Gordon sur des tores ou des sphères, on pourra lire [12] ou [11] ; [31] quant à lui s'intéresse à Schrödinger sur un espace produit. Pour l'étude de potentiels harmoniques, on se réfèrera notamment à [7], [2], ou plus récemment à [32].

1.3 Principaux résultats

Nous présentons ici brièvement les trois principaux résultats qui sont prouvés dans la thèse. Dans le chapitre 2 nous montrons qu'une étude assez directe permet d'avoir des résultats d'existence et d'unicité à données petites, assurant – si la donnée initiale est de taille ε – un temps d'existence de l'ordre de ε (par une majoration directe) ou même $\varepsilon^{-4/3}$ en utilisant la dispersion.

Étude de l'équation des ondes avec masse non nulle sur $\mathbb{R} \times \mathbb{T}$. Nous établissons dans le théorème 3.2.8 un résultat

— d'existence et d'unicité à données petites.
— assurant un temps d'existence de l'ordre de ε^{-2} où ε est la taille de la donnée initiale.
— à régularité modérée, c'est-à-dire que l'on a impose une régularité $> 1/2$ dans la direction torique et $\geq 3/2$ dans la direction libre.

Ce résultat, bien qu'assez élémentaire, montre en substance la manière de traiter un problème anisotrope et la manière dont les résonances en espace et en temps interviennent (l'équation (O1) n'étant pas résonante en espace-temps, ceci explique le caractère élémentaire de la preuve du théorème).

Étude de l'équation des ondes avec potentiel harmonique. Contrairement au précédent système, l'équation des ondes quadratique avec potentiel harmonique est très résonante, dans le sens où elle présente, dans certaines configurations, une zone de résonances en espace-temps égale à une droite. Le théorème 4.1.10 prouvé dans la thèse est donc plus faible que le théorème 3.2.8. Il s'agit

— d'un théorème d'existence et d'unicité à données petites.
— d'un théorème assurant un temps d'existence de l'ordre de $\varepsilon^{-\frac{4}{3}+\delta}$ où ε est la taille de la donnée initiale et $\delta > 0$ est arbitrairement petit.
— d'un théorème à régularité modérée dans la direction piégée, mais haute dans la direction libre (on l'impose supérieure à $1/\delta$).

Ce résultat pourrait sembler faible en comparaison du résultat d'existence en temps $\varepsilon^{4/3}$ trouvé dans le Chapitre 2, théorème 2.4.4 : en réalité l'anisotropie de la situation empêche d'avoir aussi facilement ce type de résultat.
De plus les espaces fonctionnels utilisés dans le théorème 4.1.10 sont beaucoup plus pertinents pour l'étude de la dynamique en temps long de l'équation.

Enfin, l'étude fine des résonances développée dans le Chapitre 4 est fondamentale pour l'étude de ce que l'on appelle le système résonant associé à l'équation.

Dérivation et étude d'un système résonant. Afin de mieux comprendre la dynamique de l'équation (O2), nous dérivons une équation résonante associée à (O2).
L'idée de l'étude d'un système résonant a été initée pour des systèmes hyperboliques par Klainerman et Majda dans [39], ainsi que par Grenier dans [30] et Schochet dans [51] avec la méthode dite de « filtrage des fréquences ».
Cette idée a connu un regain de popularité il y a quelques années lorsque Ionescu et Pausader l'ont adaptée pour les équations dispersives dans [34] et [35], où ils étudient l'équation de Schrödinger non-linéaire sur $\mathbb{R} \times \mathbb{T}^3$.
Ici, nous nous inspirons plus précisément des travaux de Hani, Pausader, Tzvetkov, Visciglia dans [31] pour NLS sur $\mathbb{R} \times \mathbb{T}^d$ $(1 \leq d \leq 4)$, ou Hani et Thomann pour NLS avec confinement harmonique ([32]).
L'idée est de partir de la formulation fréquentielle d'une équation dispersive :

$$\hat{f}(t,\xi) = \hat{f}(0,\xi) + \int_0^t \int_\eta e^{is\phi(\xi,\eta)} \hat{f}(\eta)\hat{f}(\xi-\eta)d\eta ds,$$

et d'appliquer un lemme de phase stationnaire afin d'identifier les fréquences résonantes, i.e. les fréquences qui donnent naissance à des termes non-négligeables.
Dans cette thèse, nous dérivons l'équation résonante pour (O2), nous prouvons un théorème d'existence en temps long d'ordre ε^{-2}, à régularité modérée (Théorème 5.1.1). Enfin nous montrons que ce système résonant est une bonne approximation de l'équation initiale (Théorème 5.1.2).

1.4 Applications des résultats de la thèse – perspectives

Dans cette section, nous présentons plusieurs pistes d'étude se basant sur les résultats prouvés dans la thèse.

1.4.1 Étude précise de systèmes résonants

Le système résonant établi au chapitre 5, s'il a été étudié du point de vue de l'existence et de l'unicité en temps long, mériterait une étude plus approfondie de la dynamique. Par exemple, dans [31], les auteurs ont réussi à montrer l'existence de solutions à l'équation de Schrödinger cubique sur un espace produit avec norme de Sobolev croissant logarithmiquement en temps en trouvant d'abord de telles solutions au système résonant puis en utilisant leur théorème d'appoximation.
Nous pourrions, si nous parvenons à comprendre la dynamique du système résonant dérivé ici, obtenir des informations sur le comportement des solutions de (O2) du même acabit.

La difficulté vient de la complexité du système résonant : bien qu'a priori plus simple (dynamiquement parlant) que l'équation initiale, la présence des fonctions d'Hermite et la complexité de leur interaction rendent le système ardu à comprendre de prime abord.

Une solution pourrait être de s'intéresser d'abord à un système résonant pour les ondes sur $\mathbb{R} \times \mathbb{T}$ *sans masse*. L'équation (O1) n'est pas en soi pertinente : en effet, dès lors que

$\mu \neq 0$, il n'y a pas de résonances en espace-temps, donc pas de système résonant. C'est pourquoi on pourrait étudier

$$\begin{cases} \partial_t^2 u - \Delta u &= Q(u), \\ u(0,x) &= u_0(x), \\ \partial_t u(0,x) &= u_1(x), \end{cases} \tag{O3}$$

où

$$\begin{aligned} &u : (t, x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T} \mapsto u(t, x_1, x_2) \in \mathbb{R}, \\ &u_0 : (x_1, x_2) \in \mathbb{R} \times \mathbb{T} \mapsto u_0(x_1, x_2) \in \mathbb{R}, \\ &u_1 : (x_1, x_2) \in \mathbb{R} \times \mathbb{T} \mapsto u_1(x_1, x_2) \in \mathbb{R}, \\ &Q \text{ est une fonction quadratique de } u \end{aligned}$$

mais un autre problème apparaît : l'équation projetée sur le mode 0 n'est pas dispersive (car de symbole $L(\xi) = \sqrt{\xi^2 + 0^2} = |\xi|$). Il faudrait alors trouver un moyen de supprimer ce mode nul, sans ajouter de masse. Dans ce cas l'équation résonante pourrait être assez simple et aider à comprendre mieux l'équation résonante pour (O2).

1.4.2 Vers une application aux fluides géophysiques ?

L'étude de l'équation des ondes avec potentiel harmonique, bien qu'intéressante en elle-même, a d'abord été entamée par l'auteur comme une première étape de l'étude des ondes dites de Poincaré dans le système de Shallow-Water-Coriolis. Il s'agit d'un modèle adimensionné pour l'étude des ondes équatoriales en eaux peu profondes :

$$\begin{cases} \partial_t \eta + \frac{1}{\varepsilon} \nabla \cdot u + \bar{u} \cdot \nabla \eta + \varepsilon^2 \nabla \cdot (\eta u) &= 0, \\ \partial_t u + \frac{1}{\varepsilon^2} x_2 u^\perp + \frac{1}{\varepsilon} \nabla \eta + \bar{u} \cdot \nabla u + u \cdot \nabla \bar{u} + \varepsilon^2 u \cdot \nabla u &= 0, \end{cases}$$

où η est la hauteur relative de l'eau par rapport à une hauteur de référence, u la vitesse relative du fluide par rapport à une vitesse stationnaire $\bar{u}$. Le terme $x_2 u^\perp$ représente la force de Coriolis : près de l'équateur il satisfait l'approximation dite du *betaplan*, en se comportant comme une fonction linéaire de la latitude. Le petit paramètre ε représente à la fois les nombres de Rossby (rapport entre la vitesse de rotation de la Terre et la vitesse du fluide) et de Froude (mesurant l'influence de la gravité).

Plusieurs travaux récents traitent de cette équation : Gallagher et Saint-Raymond l'ont étudiée dans le cadre classique (sans le paramètre ε) dans [18] et Cheverry, Paul, Gallagher et Saint-Raymond ont étudié le cadre semi-classique (i.e. $\varepsilon \to 0$) dans [8] et [17].

Si l'on prend $\bar{u} = 0$, on obtient :

$$\varepsilon^2 \partial_t U + A(x_2, \varepsilon D) U = \varepsilon^3 Q(U),$$

où

$$U := \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} \text{ with } \left\{ \begin{array}{rcl} \eta & = & \left((1+\frac{\varepsilon^3 u_0}{2})^2 - 1\right)\varepsilon^{-3}, \\ u & = & (u_1, u_2), \end{array} \right.$$

$$A(x_2, \varepsilon D) := \begin{pmatrix} 0 & \varepsilon\partial_1 & \varepsilon\partial_2 \\ \varepsilon\partial_1 & 0 & -x_2 \\ \varepsilon\partial_2 & x_2 & 0 \end{pmatrix},$$

$$Q(U) := -\begin{pmatrix} u_1 & \frac{1}{2}u_0 & 0 \\ \frac{1}{2}u_0 & u_1 & 0 \\ 0 & 0 & u_1 \end{pmatrix} \varepsilon\partial_1 U - \begin{pmatrix} u_2 & 0 & \frac{1}{2}u_0 \\ 0 & u_2 & 0 \\ \frac{1}{2}u_0 & 0 & u_2 \end{pmatrix} \varepsilon\partial_2 U.$$

Dans [18] et [8] les auteurs se sont ramenés à des propagateurs scalaires, en pseudodiagonalisant l'opérateur A. Ils ont déterminé les valeurs propres comme solutions de l'équation polynomiale suivante de paramètres (ξ, n) et d'inconnue τ :

$$\tau^3 - (\xi_1^2 + \varepsilon(2n+1))\tau + \varepsilon\xi_1 = 0. \tag{1.4.1}$$

Les valeurs propres d'ordre ε correspondent aux ondes dites de Rossby et leur piégeage est prouvé dans [8].
Les valeurs propres d'ordre 1 sont appelées ondes de Poincaré. Les auteurs de [8] prouvent un résultat de dispersion semi-classique pour ces ondes, mais de nombreuses questions restent en suspens : peut-on avoir un théorème d'existence en temps long pour les ondes de Poincaré ? Quelle est la dynamique de ces ondes en temps long ?

Par les formules de Cardan, les solutions de (1.4.1) sont

$$\lambda_k = 2\sqrt{\frac{\xi_1^2 + \varepsilon(2n+1)}{3}} \cos\left(\frac{1}{3}\arccos\left(\frac{-\varepsilon\xi_1}{2}\sqrt{\frac{27}{(\xi_1^2+\varepsilon(2n+1))^3}}\right) + \frac{2k\pi}{3}\right),$$

avec $k = 0, 1, 2$, λ_1 le mode de Rossby et λ_0, λ_2 les modes de Poincaré.
Lorsque ε tend vers 0, on obtient l'asymptotique suivante :

$$\lambda_0(\xi_1, n) = \sqrt{\xi_1^2 + \varepsilon(2n+2)} + o(\varepsilon^2)$$

$$\lambda_2(\xi_1, n) = -\sqrt{\xi_1^2 + \varepsilon(2n+2)} + o(\varepsilon^2)$$

Les vecteurs propres associés à ces valeurs propres sont calculés dans [18] :

$$\Psi_{\xi_1,n,\pm}(x_1,x_2) := C_{\xi_1,n,\pm} e^{i\xi x_1} \begin{pmatrix} \frac{-i}{\tau_P(\xi_1,n,\pm)+\xi}\sqrt{\frac{\varepsilon n}{2}}\psi_{n-1}(x_2) + \frac{i}{\tau_P(\xi_1,n,\pm)-\xi}\sqrt{\frac{\varepsilon(n+1)}{2}}\psi_{n+1}(x_1) \\ \psi_n(x_2) \\ \frac{i}{\tau_P(\xi_1,n,\pm)+\xi}\sqrt{\frac{\varepsilon n}{2}}\psi_{n-1}(x_2) + \frac{i}{\tau_P(\xi_1,n,\pm)-\xi}\sqrt{\frac{\varepsilon(n+1)}{2}}\psi_{n+1}(x_2) \end{pmatrix},$$

où ψ_n est la n-ième fonction d'Hermite.

Voilà pourquoi nous considérons comme première approximation des ondes de Poincaré l'équation

$$\partial_t^2 u - \Delta u + x_2^2 u + u = u^2.$$

En effet,

1. les valeurs propres associées à l'équation sont $\pm\sqrt{\xi_1^2+2n+2}$.
2. le vecteur propre correspondant à la valeur propre $\pm\sqrt{\xi_1^2+2n+2}$ est la fonction $(x_1,x_2)\mapsto e^{i\xi x_1}\psi_n(x_2)$.

Ainsi, il paraît pertinent de s'intéresser à l'introduction du paramètre semi-classique ε dans l'équation (O2), et de passer à un cadre vectoriel (prenant ainsi en compte les vecteurs propres $\Psi_{\xi_1,n,\pm}$).

1.5 Organisation de la thèse

La thèse s'organise autour de trois parties largement indépendantes, mais d'un niveau technique croissant, et d'appendices.

1. dans le Chapitre 2, nous exposons une démarche générale d'étude d'une équation dispersive dans le cadre de données petites, ainsi que la méthode des résonances en espace-temps. Nous prenons comme exemple l'équation de Schrödinger quadratique.
2. dans le Chapitre 3, nous nous intéressons à l'équation des ondes sur $\mathbb{R}\times\mathbb{T}$ avec masse non nulle (O1) et établissons l'existence et l'unicité en temps long.
3. le Chapitre 4, quant à lui, s'intéresse à l'équation des ondes avec potentiel harmonique (O2) : nous y prouvons l'existence et l'unicité en temps long. Le Chapitre 5 poursuit l'étude de cette équation en dérivant un système résonant. On y prouve l'existence et l'unicité en temps long ainsi que la validité de l'approximation d'une solution de (O2) par une solution du système résonant.
4. enfin, trois appendices exposent les résultats techniques nécessaires à la thèse. D'abord, l'appendice A s'attache à prouver plusieurs résultats d'analyse harmonique (multiplicateurs de Fourier, phase stationnaire, fonctions d'Hermite). L'appendice B étudie en détail les résonances pour l'équation des ondes avec potentiel harmonique. Enfin, l'appendice C montre un résultat de paraproduit pour les fonctions d'Hermite.

Chapter 2

Existence theorems for dispersive equations

Contents

The goal of this section is to introduce the main tools and methods used in the study of dispersive equations. In order to illustrate the concepts handled in this part, we are going to use a very simple example of dispersive equation: the quadratic 1-D Schrödinger equation

$$\left\{\begin{array}{rcl}\partial_t u - i\Delta u & = & Q(u,u),\\ u(0,x) & = & u_0(x),\end{array}\right. \tag{2.0.1}$$

with $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $Q(u)$ a quadratic term in u, $\bar{u}$, and $u_0 : \mathbb{R} \to \mathbb{R}$ a small initial data. After discussing the general methods for establishing existence theorems (use of the profile of a solution, Duhamel formula, contraction estimates), we are going to prove three existence theorems for our toy model (2.0.1):

1. a first one (Theorem 2.3.1) giving an existence time of order ε^{-1} (where ε is the size of the initial data) in H^N. This theorem only uses the hyperbolic structure of the problem (i.e. energy estimates are possible).
2. a second one (Theorem 2.4.4) giving an existence time of order $\varepsilon^{-\frac{4}{3}}$ in $H^N \cap L^1$. This theorem uses the dispersive properties of the equation.
3. a third one (Theorem 2.5.5) giving an existence time of order ε^{-2} in $H^N \cap L^2(\langle x \rangle)$ (where $L^2(\langle x \rangle)$ is a weighted L^2 space defined in 2.2.2). This theorem uses the space-time resonances method.

We are proving Theorems giving longer and longer existence times, but for more and more localized data (functions in $L^2(\langle x \rangle)$ are more localized around the origin than L^1 functions, which are more localized than L^2 functions). The methods used to obtain longer existence time use the frequency structure of the equation: the dispersive properties needed for Theorem 2.4.4 use the harmonic properties of the linear part, the space-time resonances use the harmonic properties of the nonlinear one.

2.1 The notion of profile

Since we are working with small data and a quadratic nonlinearity, our problem has to be seen as *weakly nonlinear*: indeed, if $u = \varepsilon v$ with v of size 1, (2.0.1) becomes

$$\partial_t v - i\Delta v = \varepsilon Q(v).$$

The solution of the linear problem ($\varepsilon = 0$)

$$\left\{\begin{array}{rcl}\partial_t u - i\Delta u & = & 0,\\ u(t=0) & = & u_0,\end{array}\right.$$

is given by

$$u(t,x) = e^{-it\Delta} u_0(x),$$

where $e^{it\Delta}$ is the linear propagator associated to (2.0.1), defined thanks to the Fourier transform:

$$e^{it\Delta} u_0(x) := (2\pi)^d \int_{\mathbb{R}} e^{ix.\xi} e^{-it\xi^2} \hat{u}_0(\xi) d\xi,$$

with

$$\hat{f}(t,\xi) = \mathcal{F}(f)(t,\xi) := \int_{\mathbb{R}} f(t,x) e^{-ix\xi} dx. \tag{2.1.1}$$

In the weakly nonlinear regime, we can expect the linear dynamics to be dominant, in particular in the long-time scale. This incites us to consider the backwards linear evolution of a solution of the nonlinear equation: this is what we call the *profile* of a solution of (2.0.1).

Definition 2.1.1. *The profile f of a solution u of (2.0.1) is defined by*

$$f(t,x) := e^{-it\Delta}u(t,x).$$

In the weakly nonlinear setting, f is slowly varying, its temporal variation being governed by the nonlinearity:

$$\begin{aligned}\partial_t f &= \partial_t\left(e^{-it\Delta}u(t,x)\right)\\ &= -i\Delta e^{-it\Delta}u(t,x) + e^{-it\Delta}\partial_t u(t,x).\end{aligned}$$

Since u satisfies (2.0.1), $\partial_t u(t,x) = i\Delta u + Q(u,u)$. Then

$$\partial_t f = -i\Delta e^{-it\Delta}u(t,x) + e^{-it\Delta}\left(i\Delta u + Q(u,u)\right),$$

which gives

$$\partial_t f = e^{-it\Delta}Q(u,u),$$

or, if we write everything in terms of f,

$$\partial_t f = e^{-it\Delta}Q\left(e^{it\Delta}f, e^{it\Delta}f\right). \tag{2.1.2}$$

2.2 Use of a contraction mapping principle

Similarly to what is done for the study of differential equations, we begin the study of a dispersive equation by an *integral formula* for our equation: this formula is called a *Duhamel formula.*

If we integrate (2.1.2) between 0 and t, we get

$$f(t,x) = f(0,x) + \int_0^t e^{-is\Delta}Q\left(e^{is\Delta}f(s,x), e^{is\Delta}f(s,x)\right)ds, \tag{2.2.1}$$

or, in terms of u,

$$u(t,x) = e^{it\Delta}f(0,x) + \int_0^t e^{i(t-s)\Delta}Q\left(u(s,x),u(s,x)\right)ds, \tag{2.2.2}$$

Equation (2.2.1) is a Duhamel formula for (2.0.1): it allows us to rewrite (2.0.1) as a fixed point equation

$$f = A(f), \tag{2.2.3}$$

where $A(f) = f_0 + \mathcal{A}(f)$, and

$$\mathcal{A}(f)(t,x) := \int_0^t e^{-is\Delta}Q\left(e^{is\Delta}f(s,x), e^{is\Delta}f(s,x)\right)ds. \tag{2.2.4}$$

We can similarly write it as $u = \widetilde{A}(u)$, with $\widetilde{A}(u) = e^{it\Delta}u_0 + \widetilde{\mathcal{A}}(u)$. Solving the fixed point problem (2.2.3) can be done by proving that A (or $\widetilde{A}$) is a contraction in a ball $B_S(0,\varepsilon)$ of a well-chosen space S i.e.

$$\forall (f,g)\in B_S(0,\varepsilon),\ \ \|A(f)-A(g)\|_S \le c\,\|f-g\|_S\,,\ \text{ with } c<1.$$

Actually it suffices to prove an inequality of the form

$$\|A(f)\|_S \leq C\,\|f\|_S^2\,, \text{ with } C \in \mathbb{R}. \tag{2.2.5}$$

Then for $\varepsilon < \frac{1}{2C}$, for $\|f_0\|_S \leq \frac{\varepsilon}{2}$, A is a contraction of the ball of radius ε and then has a unique fixed point.

Remark 2.2.1. *The time of existence depends strongly on the space chosen and of the dependence in time of the constant C in (2.2.5). If C does not depend on time and S is of the form $L^\infty(\mathbb{R}_+, S')$ with S' a functional space on the spatial variable, then we have a global existence. The following sections will give three examples of local existence in time.*

Throughout this chapter and this thesis, we are going to consider Sobolev spaces. Here we recall their definition.

Definition 2.2.2. *If m is an integer and $1 \leq p \leq \infty$, the Sobolev space $W^{m,p}(\mathbb{R})$ is defined as the space of L^p functions with $f', f'', \ldots, f^{(m)}$ in L^p. If $p = 2$, $W^{m,p}$ is denoted H^m. For s non integer, $W^{s,p}$ is defined by interpolating $W^{[s],p}$ and $W^{[s]+1,p}$, where $[s]$ is the integer part of s. In the case $p = 2$, the $H^s = W^{s,2}$ norm is defined equivalently by*

$$\|f\|_{H^s} := \left(\int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$

We define in a similar way the homogeneous Sobolev space $\dot{H}^s(\mathbb{R})$:

$$\|f\|_{H^s} := \left(\int_{\mathbb{R}} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$

If $\omega : \mathbb{R} \to \mathbb{C}$, we define the weighted Sobolev space $H^s(\omega)$ by

$$f \in H^s(\omega) \Leftrightarrow \omega f \in H^s.$$

2.3 A simple theorem: direct energy estimates

Here we give a first example of an existence theorem established with a contraction estimate, without any fine study of the structure of the equation.

Theorem 2.3.1. *Let $N > 1/2$. There exists $C > 0$ such that for all $\varepsilon > 0$, for all u_0 with $\|u_0\|_{H^N} \leq \varepsilon/2$, there exists a solution u to (2.0.1) in the space $L^\infty([0,T), H^N)$ with*

$$T := \frac{1}{2C\varepsilon},$$

and

$$\|u\|_{L^\infty([0,T),H^N)} \leq \varepsilon.$$

Moreover, u is unique in the ball of radius ε of H^N.

Remark 2.3.2. *This result does not give a very long existence time, but it only needs the solution to be in H^N.*

Proof :
The proof of this theorem does not rely on the use of the profile of the solution: it only uses the very basic properties of the linear propagator and the simplicity of the nonlinearity (in particular the fact that it does not involve derivatives of u). Given the form of the Duhamel formula (2.2.1), we want to estimate the H^N norm of

$$\widetilde{\mathcal{A}}(u)(t) := \int_0^t e^{i(t-s)\Delta} Q\left(u(s,x), u(s,x)\right) ds.$$

First of all,

$$\left\| \widetilde{\mathcal{A}}(u)(t) \right\|_{H^N} \leq \int_0^t \left\| e^{i(t-s)\Delta} Q\left(u(s), u(s)\right) \right\|_{H^N} ds.$$

Since the operator $e^{i(t-s)\Delta}$ is an L^2 isometry and commutes with $(-\Delta)^{N/2}$, we have

$$\left\| \widetilde{\mathcal{A}}(u)(t) \right\|_{H^N} \leq \int_0^t \left\| Q\left(u(s), u(s)\right) \right\|_{H^N} ds. \tag{2.3.1}$$

Because Q is quadratic and because of Leibniz' rule, there exists a constant C such that

$$\left\| Q\left(u(s), u(s)\right) \right\|_{H^N} \leq C \left\| u(s) \right\|_{L^\infty} \left\| u(s) \right\|_{H^N}. \tag{2.3.2}$$

Then, by the Sobolev embedding $H^N(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ when $N > 1/2$, we have

$$\left\| u(s) \right\|_{L^\infty} \leq \left\| u(s) \right\|_{H^N},$$

i.e.

$$\left\| u(s) \right\|_{L^\infty} \leq \left\| u(s) \right\|_{H^N}. \tag{2.3.3}$$

Estimate (2.3.2) then becomes

$$\left\| Q\left(u(s), u(s)\right) \right\|_{H^N} \leq C \left\| u(s) \right\|_{H^N}^2, \tag{2.3.4}$$

which leads to

$$\left\| \widetilde{\mathcal{A}}(u)(t) \right\|_{H^N} \leq C \int_0^t \left\| u(s) \right\|_{H^N}^2 ds \tag{2.3.5}$$

Then let $T > 0$. If we take the $L^\infty([0,T))$ norm of (2.3.5), we obtain

$$\left\| \widetilde{\mathcal{A}}(u) \right\|_{L^\infty([0,T),H^N)} \leq CT \left\| u \right\|_{L^\infty([0,T),H^N)}^2. \tag{2.3.6}$$

Hence, if $\|u\|_{L^\infty([0,T),H^N)} \leq \varepsilon$ and if $T < \frac{1}{2C\varepsilon}$, then $\left\| \widetilde{\mathcal{A}}(u) \right\|_{L^\infty([0,T),H^N)} < \varepsilon/2$, and $\widetilde{A}$ is a contraction on the ball of radius ε of $L^\infty([0,T), H^N$. This ends the proof of Theorem 2.3.1. ■

2.4 Use of the dispersive effect

Theorem 2.3.1 uses only the fact that the propagator for Schrödinger's equation is a L^2 isometry and that we are able to make energy estimates.
In order to improve this result, one can study more finely the properties of the linear propagator, in particular its decay properties, ie. the *dispersion.*

2.4.1 Heuristics of dispersion

Physical observations show that in a large variety of frameworks (Schrödinger's equation, Korteweg-de Vries' equation, wave equation, etc.), a free wave will tend to spread out, leading to a L^∞ decay in time. This phenomenon called dispersion occurs in so-called dispersive equations, in particular in equations of the form

$$\partial_t u + iL(D)u = 0, \tag{2.4.1}$$

with $D = i\partial$, $L : \mathbb{R} \mapsto \mathbb{R}$ and $L(D)$ is the Fourier multiplier associated to L, i.e. for all function f, $\mathcal{F}(L(D)f)(\xi) = L(\xi)\hat{f}(\xi)$. To call this equation dispersive we moreover ask $\mathrm{Hess}(L) \neq 0$.
Now we consider an initial data localized at a frequency ξ_0, and we determine the velocity of the resulting solution. This solution is a wave packet, i.e. of the form,

$$u(t,x) = \int_{\mathbb{R}} A(\xi - \xi_0) e^{i(x\cdot\xi - \omega(\xi)t)} d\xi, \tag{2.4.2}$$

with A localizing around 0 and exponentially decreasing.

Its group velocity is defined by

$$v_g := \frac{d\omega}{dk}.$$

Since u is a solution of (2.4.1), we have $\omega = -L(k)$, hence the group velocity is $-L'(k)$. For a dispersive equation, it is not constant: then different frequencies will travel at different speeds and a spreading of the solution is expected.

For Schrödinger's equation, $L(\xi) = \xi^2$: it is considered as dispersive. In particular, the group velocity at frequency ξ is -2ξ.

2.4.2 Dispersion for Schrödinger's equation

For Schrödinger's equation, we can write

$$e^{-itL(D)} u_0 = u_0 * K_t(x),$$

with

$$K_t(x) = \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{i|x|^2/4t}. \tag{2.4.3}$$

Hence, by Young's inequality

$$\left\| e^{-it\Delta} u_0 \right\|_{L^\infty} \leq C \frac{1}{\sqrt{t}} \left\| u_0 \right\|_{L^1}. \tag{2.4.4}$$

This is the dispersion inequality for Schrödinger's equation : the decrease of the L^∞ norm of a solution corresponds to the expected spreading mentionned in the heuristics.

Remark 2.4.1. *Having an explicit formula with a convolution kernel is specific to Schrödinger's equation. For other considerations on dispersion inequalities, see Section 2.6.*

2.4.3 An improved theorem

We are going to use the L^∞ decay of the Schrödinger propagator in order to obtain a better existence time for (2.0.1), for more localized data : dispersion needs functions in L^1.
First of all, we define a functional space adapted to this L^∞ decay.

Definition 2.4.2. *For $N > 1/2$, $T > 0$, we define the space S_T^N by the following norm*

$$\|u\|_{S_T^N} = \sup_{t\in[0,T)} \left(\|u(t)\|_{H^N(\mathbb{R})} + t^{1/4} \|u(t)\|_{L^\infty(\mathbb{R})} \right). \tag{2.4.5}$$

Remark 2.4.3. *If u is in S_T^N, then for all $t < T$, we have*

$$\|u(s)\|_{H^N} \leq \|u\|_{S_T^N},$$
$$\|u(s)\|_{L^\infty} \leq s^{-1/4} \|u\|_{S_T^N}.$$

In particular the L^∞ norm of u is controlled by a decreasing-in-time function.

The space S_T^N being defined, we can state the following result:

Theorem 2.4.4. *Let $N > 1/2$. There exists $C > 0$ such that for all $\varepsilon > 0$, for all u_0 with*

$$\|u_0\|_{H^N} + \|u_0\|_{L^1} \leq \varepsilon/2,$$

there exists a solution u to (2.0.1) in the space S_T^N with

$$T := (2C\varepsilon)^{-\frac{4}{3}},$$

and

$$\|u\|_{S_T^N} \leq \varepsilon.$$

Moreover, u is unique in the ball of radius ε of S_T^N.

Remark 2.4.5. *Even if the definition of the norm S_T^N do not explicitely show it, the hypotheses of Theorem 2.4.4 have to be thought as "f is in H^N" and "f is in L^1" (with a prescribed decay), where f is the profile of the solution. Actually, when proving Theorem 2.4.4, we are going to controll the L^1 norm of the profile. This corresponds to a stronger localization than in Theorem 2.3.1 (a function in L^1 is more localized than a one in L^2).*

Proof :
Let $T > 0$. Since the space S_T^N is defined with multiple norms, we are goint to estimate the H^N norm and the L^∞ norm of $\widetilde{\mathcal{A}}(u)$ (defined in 2.2.4) separately.
During the whole proof, C will stand for a universal constant but might be different from line to line.

High-regularity estimate. By (2.3.1) and (2.3.2), we have

$$\left\|\widetilde{\mathcal{A}}(u)(t)\right\|_{H^N} \leq C \int_0^t \|u(s)\|_{L^\infty} \|u(s)\|_{H^N} \, ds. \tag{2.4.6}$$

Then, given Remark 2.4.3, we know that

$$\|u(s)\|_{H^N} \leq \|u\|_{S_T^N},$$
$$\|u(s)\|_{L^\infty} \leq s^{-1/4} \|u\|_{S_T^N},$$

hence

$$\begin{aligned}\left\|\widetilde{\mathcal{A}}(u)(t)\right\|_{H^N} &\leq C \int_0^t s^{-1/4} \|u\|_{S_T^N}^2 \, ds \\ &\leq C t^{3/4} \|u\|_{S_T^N}^2 .\end{aligned}$$

Taking the supremum in time leads to

$$\sup_{t\in[0,T)} \left\|\widetilde{\mathcal{A}}(u)(t)\right\|_{H^N} \leq C T^{3/4} \|u\|_{S_T^N}^2 . \tag{2.4.7}$$

L^∞ estimate. Now, we have a look at $\left\|\widetilde{\mathcal{A}}(u)(t)\right\|_{L^\infty}$:

$$\left\|\widetilde{\mathcal{A}}(u)(t)\right\|_{L^\infty} \leq \int_0^t \left\|e^{i(t-s)\Delta} Q\left(u(s), u(s)\right)\right\|_{L^\infty} ds. \tag{2.4.8}$$

By the dispersion inequality (2.4.4),

$$\left\|e^{i(t-s)\Delta} Q\left(u(s), u(s)\right)\right\|_{L^\infty} \leq C \frac{1}{\sqrt{t-s}} \left\|Q\left(u(s), u(s)\right)\right\|_{L^1} . \tag{2.4.9}$$

Then, Hölder's inequality gives

$$\left\|Q\left(u(s), u(s)\right)\right\|_{L^1} \leq C \|u(s)\|_{L^2}^2 . \tag{2.4.10}$$

Using (2.4.9) and (2.4.10), (2.4.8) becomes

$$\left\|\widetilde{\mathcal{A}}(u)(t)\right\|_{L^\infty} \leq C \int_0^t \frac{1}{\sqrt{t-s}} \|u(s)\|_{L^2}^2 \, ds.$$

Since $\|u(s)\|_{L^2} \leq \|u\|_{S_T^N}$, we obtain

$$\left\|\widetilde{\mathcal{A}}(u)(t)\right\|_{L^\infty} \leq C\sqrt{t}\, \|u\|_{S_T^N}^2 .$$

Taking the supremum in time gives

$$\sup_{t\in[0,T)} \left(t^{1/4} \left\|\widetilde{\mathcal{A}}(u)(t)\right\|_{L^\infty}\right) \leq C T^{3/4} \|u\|_{S_T^N}^2 . \tag{2.4.11}$$

Conclusion. Gathering (2.4.7) and (2.4.11) gives

$$\left\|\widetilde{\mathcal{A}}(u)\right\|_{S_T^N} \leq C T^{3/4} \|u\|_{S_T^N}^2 .$$

This concludes the proof of Theorem 2.4.4 by the contraction argument of Section 2.2. ■

2.5 The notion of space-time resonances

2.5.1 Duhamel Formula in the phase space

Even if Theorem 2.4.4 is better than Theorem 2.3.1 in the sense that it uses the structure of the linear propagator of (2.0.1), it may be improved by a finer study of the nonlinear term. Indeed, the nature of the interactions brought by the quadratic nonlinearity are not visible when looking at the Duhamel formula (2.2.1).
If instead of (2.2.1) we look at its Fourier transform, we obtain the following:

$$\hat{f}(t,\xi) = \hat{f}(0,\xi) + \int_0^t e^{is\xi^2}\mathcal{F}\left(Q\left(e^{is\Delta}f(s), e^{is\Delta}f(s)\right)\right)(\xi)ds. \tag{2.5.1}$$

The question is now to determine $\mathcal{F}\left(Q\left(e^{is\Delta}f(s), e^{is\Delta}f(s)\right)\right)(\xi)$. We are going to state three particular examples:

1. if $Q(u,u) = (\bar{u})^2$, then
$$\mathcal{F}\left(Q\left(e^{is\Delta}f(s), e^{is\Delta}f(s)\right)\right)(\xi) = \int_{\mathbb{R}} e^{is\eta^2}\overline{\hat{f}(s,\eta)}e^{is(\xi-\eta)^2}\overline{\hat{f}(s,\xi-\eta)}d\eta,$$
and the Duhamel formula becomes
$$\hat{f}(t,\xi) = \hat{f}(0,\xi) + \int_0^t e^{is\phi_{--}(\xi,\eta)}\int_{\mathbb{R}}\overline{\hat{f}(s,\eta)\hat{f}(s,\xi-\eta)}d\eta ds, \tag{2.5.1-a}$$
with $\phi_{--}(\xi,\eta) := \xi^2 + \eta^2 + (\xi-\eta)^2$.
2. if $Q(u,u) = u^2$, then the Duhamel formula becomes
$$\hat{f}(t,\xi) = \hat{f}(0,\xi) + \int_0^t e^{is\phi_{++}(\xi,\eta)}\int_{\mathbb{R}}\hat{f}(s,\eta)\hat{f}(s,\xi-\eta)d\eta ds, \tag{2.5.1-b}$$
with $\phi_{++}(\xi,\eta) := \xi^2 - \eta^2 - (\xi-\eta)^2$.
3. if $Q(u,u) = |u|^2$, then we get the Duhamel formula becomes
$$\hat{f}(t,\xi) = \hat{f}(0,\xi) + \int_0^t e^{is\phi_{-+}(\xi,\eta)}\int_{\mathbb{R}}\hat{f}(s,\eta)\overline{\hat{f}(s,\xi-\eta)}d\eta ds, \tag{2.5.1-c}$$
with $\phi_{-+}(\xi,\eta) := \xi^2 - \eta^2 + (\xi-\eta)^2$.

In all these situations, the pattern is similar: the quadratic interaction corresponds to two input frequencies, η and $\xi-\eta$, giving birth to the output frequency ξ, with a oscillating factor $e^{is\phi(\xi,\eta)}$. The study of the phase $s\phi(\xi,\eta)$ may help us undestand the interaction between η and $\xi-\eta$, and in particular highlight the so-called *resonant* interactions.

Remark 2.5.1. *In the other examples of dispersive equations, the Duhamel formula is of the same kind:*

$$\hat{f}(t,\xi) = \hat{f}_0(\xi) + \int_0^t\int_{\mathbb{R}} e^{-is\phi(\xi,\eta)}\hat{f}(\eta)\hat{f}(\xi-\eta)d\eta ds, \tag{2.5.2}$$

with

$$\phi := L(\xi) \pm L(\eta) \pm L(\xi-\eta).$$

Resonant interactions have been studied from two different points of view in the 80s by Shatah in [52] and Klainerman in [37]: the global point of view used in this thesis has been initiated by Germain, Masmoudi and Shatah in [24]. We are first going to give a physical flavor of what resonances are, and then mathematically translate these physical heuristics, in order to prove a long-time existence theorem (Theorem 2.5.5).

2.5.2 Resonant interactions

2.5.2.a Time resonance phenomenon

In physics, the phenomenon of resonances corresponds to a forcing of a system at one of its eigenfrequencies. The resonance occurs when a plane wave (solution of the free evolution problem) coincides with the forcing term.
For example, in the following example

$$\begin{cases} u'(t) - i\omega u(t) & = & (v(t))^p\,, \\ v(t) & = & e^{it\alpha}, \\ u_0 & \text{given,} & \end{cases} \tag{2.5.3}$$

the solution of the system is bounded, except when $\alpha = \frac{\omega}{p}$: here we have a linear growth of the solution, because we are in a resonant configuration.
However, the free evolution in (2.5.3) is almost too basic to be interesting: indeed it only gives birth to one kind of oscillating solutions, at a frequency ω. Let us consider a slightly more sophisticated example where the free evolution generates an infinite number of plane waves:

$$\begin{cases} \partial_t u(t,x) + i\Delta u(t,x) & = & v(t,x)w(t,x), \\ v(t,x) & = & e^{i(\alpha x - \alpha^2 t)}, \\ w(t,x) & = & e^{i(\beta x - \beta^2 t)}. \end{cases} \tag{2.5.4}$$

Here we chose v and w to be plane waves, i.e. solutions of the free equation. A plane wave coincides with the forcing term if there exists ξ such that

$$e^{i(\xi x - \xi^2 t)} = e^{i(\alpha x - \alpha^2 t)} e^{i(\beta x - \beta^2 t)} = e^{i\left((\alpha+\beta)x - (\alpha^2+\beta^2)t\right)},$$

i.e. $\xi = \alpha + \beta$ and $\xi^2 = \alpha^2 + \beta^2$, or

$$(\alpha+\beta)^2 - \alpha^2 - \beta^2 = 0.$$

This condition can be transposed in the framework of Duhamel formula (2.5.1-b): the two input frequencies are η and $\xi - \eta$ instead of α and β and the resonant condition writes

$$\phi(\xi,\eta) = 0.$$

Mathematically speaking, it corresponds to considering the Duhamel formula (2.5.1) – in the case u^2 to simplify notations – as an oscillating integral in time

$$\int_0^t e^{-is\phi(\xi,\eta)} \hat{f}(s,\eta)\hat{f}(s,\xi-\eta)ds,$$

with $\phi = \phi_{++}$, and doing a stationary phase lemma on this integral tells us that the frequencies leading to higher order terms are the one cancelling $\partial_s\left(-is\phi(\xi,\eta)\right)$, i.e. the frequencies (ξ,η) such that

$$\phi(\xi,\eta) = 0.$$

In this case, we say that the interaction is time-resonant (because the oscillating integral is in time).

2.5.2.b Space resonances

In physics, some interaction may be time resonant, but will not see each other after a long time, because they travel at different speeds. So as to see it, we are using the framework of wave packets defined in (2.4.2). We consider the system

$$\begin{cases} \partial_t u(t,x) + i\Delta u(t,x) &=& v(t,x)w(t,x), \\ v(t,x) &=& \text{wave packet centered in } \alpha, \\ w(t,x) &=& \text{wave packet centered in } \beta. \end{cases} \tag{2.5.5}$$

The form of the solution u is then

$$u(t,x) = e^{-it\Delta}u_0(x) + \int_0^t e^{is\Delta}v(s,x)w(s,x)ds. \tag{2.5.6}$$

Since v and w are wave packets, we can think of them as,

$$\begin{cases} v(t,x) &=& e^{-(x-\alpha t)^2 + i\alpha(x-\alpha t)}, \\ w(t,x) &=& e^{-(x-\beta t)^2 + i\beta(x-\beta t)}. \end{cases} \tag{2.5.7}$$

(it is the case, up to a scaling, if $v(0,x)$ and $w(0,x)$ are gaussians).
Now if we look at the L^2 norm of $e^{is\Delta}v(s,x)w(s,x)$, we easily remark that if $\alpha \neq \beta$, it will go to 0 when t goes to infinity. On the contrary, if $\alpha = \beta$ and the system is time-resonant, this norm will be constant: hence the solution u given by 2.5.6 will go to infinity as t goes to infinity.
This consideration is adaptable to Equation (2.5.1-b), if we replace the input frequency α by η and β by $\xi - \eta$. The condition $\alpha = \beta$ then writes $\eta - (\xi - \eta) = 0$, i.e.

$$\partial_\eta \phi(\xi, \eta) = 0.$$

From a mathematical point of view, it corresponds to considering the spatial integral as an oscillating integral:

$$\int_{\mathbb{R}} e^{-is\phi(\xi,\eta)} \hat{f}(\eta)\hat{f}(\xi - \eta)d\eta.$$

A stationary phase lemma on this integral gives that the frequencies creating leading order contributions are the ones cancelling $\partial_\eta \left(-is\phi(\xi,\eta)\right)$, i.e. the frequencies (ξ,η) such that

$$\partial_\eta \phi(\xi, \eta) = 0.$$

In this case, we say that the interaction is space resonant (because the oscillating integral is in η).

2.5.2.c The space-time resonances method

The toy models studied in Sections 2.5.2.a and 2.5.2.b highlight the importance of the vanishing (or the non-vanishing) of the phase ϕ or its derivative $\partial_\eta \phi$. Let us develop mathematically what happens in those two different cases. We are going to use the following version of Duhamel's formula (2.5.1):

$$\hat{f}(t,\xi) = \hat{f}_0(\xi) + \int_0^t \int_{\mathbb{R}} e^{-is\phi(\xi,\eta)} \hat{f}(\eta)\hat{f}(\xi-\eta)d\eta ds.$$

Our aim is to be able to estimate this integral in a better way than in proofs of Theorems 2.3.1 and 2.4.4, by performing some transforms, depending on the behavior of ϕ and $\partial_\eta \phi$.

If the phase ϕ does not vanish, we can write

$$e^{-is\phi} = \frac{i}{\phi}\partial_s\left(e^{-is\phi}\right) \tag{2.5.8}$$

and perform an integration by parts in s. This *normal form transformation* (according to Shatah's terminology in [52], inspired by the dynamical systems classical terminology) will lead to an expression of the form

$$\hat{f}(t,\xi) = \hat{f}_0(\xi) + \int_{\mathbb{R}} e^{-it\phi(\xi,\eta)}\hat{f}(t,\eta)\hat{f}(t,\xi-\eta)d\eta + \int_0^t\int_{\mathbb{R}}\frac{1}{\phi}e^{-is\phi(\xi,\eta)}\partial_s\hat{f}(\eta)\hat{f}(\xi-\eta)d\eta ds$$
$$+ \text{ symmetric or easier terms.}$$

A short calculation shows that

$$\partial_s\hat{f} = e^{is\xi^2}\mathcal{F}(u^2).$$

Thus, the nonlinearity $f\partial_s f$ is now a cubic one, easier to deal with since we are working with small data.

If the derivative of the phase $\partial_\eta\phi$ does not vanish, we write

$$e^{-is\phi} = \frac{1}{-is\partial_\eta\phi}\partial_\eta\left(e^{-is\phi}\right), \tag{2.5.9}$$

and perform an integration by parts in η, to obtain

$$\begin{aligned}\hat{f}(t,\xi) = \hat{f}_0(\xi) &+ \int_0^t\int_{\mathbb{R}}\partial_\eta\left(\frac{1}{-is\partial_\eta\phi}\right)e^{-is\phi(\xi,\eta)}\hat{f}(\eta)\hat{f}(\xi-\eta)d\eta ds\\ &+ \int_0^t\int_{\mathbb{R}}\frac{1}{-is\partial_\eta\phi}e^{-is\phi(\xi,\eta)}\partial_\eta\hat{f}(\eta)\hat{f}(\xi-\eta)d\eta ds\\ &+ \text{ symmetric or easier terms.}\end{aligned}$$

This allows to gain a power of s and improve the long-time existence. This corresponds to Klainerman's vector fields method, developed in [37].

Remark 2.5.2. *Performing an integration by parts in η will lead to terms of the form $\partial_\eta\hat{f}(s,\eta)$, which is, in the physical space, $xf(s,x)$: this is one of the reasons for the strong localization hypothesis appearing in Theorem 2.5.5.*
This point is quite easily understandable if we focus on the interaction between wave packets:

- *If the solution is localized and we take two wave packets with different speeds, they are both localized around 0 at time 0. So we can be sure that after a large enough time, the packets will not interact, and we will be able to quantify their "non-interaction" as a function of time (this is the $\frac{1}{s}$ term we find by integrating by parts).*
- *But if we don't assume the data to be localized, it is possible for two wave packets to come from very far away from zero and interact after an arbitrary long time.*

In the general case, we have zones where ϕ vanishes (call this zone $\mathcal{T}$, the time resonant set), where $\partial_\eta \phi$ vanishes ($\mathcal{S}$, the space resonant set) and a zone where both vanish: call it $\mathcal{R}$, the space-time resonant set. This zone is problematic because none of the integration by parts presented before are feasible.

If the space-time resonant set is of measure 0, it is reasonable to think that this difficulty will be handled by an *adapted localization*: we introduce a function χ^t equal to 1 around $\mathcal{R}$, the localization being narrower as time grows. The integral

$$\int_0^t \int_{\mathbb{R}} \chi^s(\xi,\eta) e^{-is\phi(\xi,\eta)} \hat{f}(\eta)\hat{f}(\xi-\eta) d\eta ds$$

should be small.
Then in order to deal with

$$\int_0^t \int_{\mathbb{R}} (1-\chi) e^{-is\phi(\xi,\eta)} \hat{f}(\eta)\hat{f}(\xi-\eta) d\eta ds,$$

we cut again the space with a function θ^s localizing around the space resonant set (for example): hence we are reduced to estimate

$$\int_0^t \int_{\mathbb{R}} (1-\chi^s)\theta^s e^{-is\phi(\xi,\eta)} \hat{f}(\eta)\hat{f}(\xi-\eta) d\eta ds, \tag{2.5.10}$$

and

$$\int_0^t \int_{\mathbb{R}} (1-\chi^s)(1-\theta^s) e^{-is\phi(\xi,\eta)} \hat{f}(\eta)\hat{f}(\xi-\eta) d\eta ds, \tag{2.5.11}$$

In (2.5.10), since we are around the space resonant set but outside the space-time resonant set, the phase ϕ does not vanish: we perform an integration by parts in time. In (2.5.11), $\partial_\eta\phi$ does not vanish, and we perform an integration by parts in η.

One problem is then to be able to estimate terms of the form

$$\int_0^t \int_{\mathbb{R}} m(\xi,\eta) e^{-is\phi(\xi,\eta)} \hat{f}(\eta)\hat{f}(\xi-\eta) d\eta ds, \tag{2.5.12}$$

so-called bilinear Fourier multipliers. Ideally they would satisfy Hölder-like estimates (i.e. they would be continuous $L^p \times L^q \to L^r$ with $1/p + 1/q = 1/r$). The results needed in this thesis are gathered in Appendix A.

All the difficulty will be to deal with these three different zones: we have to identify them (i.e. to study the phase), cut off carefully the frequency space and deal with each zone. We give the following definitions:

Definition 2.5.3. *Let $\phi(\xi,\eta)$ be a real phase.*

(i) The time resonant set *is the set $\mathcal{T} := \{(\xi,\eta) \in \mathbb{R}^2 | \phi(\xi,\eta) = 0\}$.*

(ii) The space resonant set *is the set $\mathcal{S} := \{(\xi,\eta) \in \mathbb{R}^2 | \partial_\eta\phi(\xi,\eta) = 0\}$.*

(iii) The space coresonant set *is the set $\widetilde{\mathcal{S}} := \{(\xi,\eta) \in \mathbb{R}^2 | \partial_\xi\phi(\xi,\eta) = 0\}$.*

(iv) The space-time resonant set *is the set $\mathcal{R} := \{(\xi,\eta) \in \mathbb{R}^2 | \phi(\xi,\eta) = \partial_\eta(\xi,\eta) = 0\}$.*

Remark 2.5.4. *The space coresonant set is not directly involved in the study of resonances. However, since the fixed-point spaces are often built on weighted norms, we may have to differentiate with respect to ξ the integral term in the Duhamel formula* (4.2.3)*: this will make terms of the form $\partial_\xi\phi$ appear. Hence it can be interesting to compare $\partial_\xi\phi$ to $\partial_\eta\phi$: this explains that we also want to define and study the space coresonant set $\widetilde{\mathcal{S}}$.*

2.5.3 An example treated with space-time resonances

In order to illustrate the method described in Section 2.5.2.c, we are going to establish an existence and uniqueness theorem in the same spirit as Theorems 2.3.1 and 2.4.4, but with an improved existence time. So as to avoid technical details and to have a quite short proof, we are going to consider the following Schrödinger equation:

$$\begin{cases} \partial_t u - i(\Delta-1)u &= \bar{u}^2, \\ u(0,x) &= u_0(x), \end{cases} \tag{2.5.13}$$

with $u:\mathbb{R}_+\times\mathbb{R}\to\mathbb{C}$ and $u_0:\mathbb{R}\to\mathbb{C}$ a small initial data.

Theorem 2.5.5. *Let $N>1/2$. There exists $C>0$ such that for all $\varepsilon>0$, for all u_0 with $\|u_0\|_{H^N}+\|u_0\|_{L^2(\langle x\rangle)}\leq\varepsilon/2$, there exists a solution u to (2.0.1) in the space $L^\infty([0,T),H^N)\cap L^\infty([0,T),L^2(\langle x\rangle))$ with*

$$T:=(2C\varepsilon)^{-2},$$

and

$$\sup_{t\in[0,T)}\|u(t)\|_{H^N}+\|u(t)\|_{L^2(\langle x\rangle)}\leq\varepsilon.$$

Moreover, u is unique in the ball of radius ε of $L^\infty([0,T),H^N)\cap L^\infty([0,T),L^2(\langle x\rangle))$.

Proof :
First of all, it is not hard to see that $-\Delta+1$ has the same L^2-continuity and dispersion properties as $-\Delta$ (Proposition 2.4.4), i.e. there exists $C>0$ such that

$$\left\|e^{-it(\Delta-1)}f\right\|_{L^2}\leq C\,\|f\|_{L^2}\,, \tag{2.5.14}$$

$$\left\|e^{-is(\Delta-1)}f\right\|_{L^\infty}\leq\frac{C}{\sqrt{t}}\,\|f\|_{L^1}\,. \tag{2.5.15}$$

We take those results for granted.
Then we are going to write

$$S_T^N=L^\infty([0,T),H^N)\cap L^\infty([0,T),L^2(\langle x\rangle)).$$

In order to prove Theorem 2.5.5, we are going to follow the following steps:

1. establishing a Duhamel formula in the frequency space
2. identifying the phase and its behavior
3. establishing the high-regularity estimates
4. using the space-time resonances method to deal with the weighted norm

Let us follow these steps. Note that the same constant C may have different values from line to line.

1. In order to give the Duhamel formula for (2.5.13), we first define the profile f of a solution of (2.5.13):
$$f(t,x):=e^{-it(\Delta-1)}u(t,x). \tag{2.5.16}$$

Then, adapting (2.2.4) gives the following Duhamel formula for (2.5.13):

$$f(t,x) = f_0(x) + \mathcal{A}(f)(t,x), \tag{2.5.17}$$

$$\text{with } \mathcal{A}(f)(t,x) = \int_0^t e^{-is(\Delta-1)}\left(e^{-is(\Delta-1)}\bar{f}(s,x)\right)\left(e^{-is(\Delta-1)}\bar{f}(s,x)\right)ds. \tag{2.5.18}$$

The operator $\mathcal{A}$ can be rewritten in the frequency space as in (2.5.1-a):

$$\begin{aligned}\hat{f}(t,\xi) &= \hat{f}(0,\xi) + \int_0^t \int_{\mathbb{R}} e^{is\phi(\xi,\eta)}\overline{\hat{f}}(s,\eta)\overline{\hat{f}}(s,\xi-\eta)d\eta ds,\\ \text{with } \phi(\xi,\eta) &:= (\xi^2+1) + (\eta^2+1) + \left((\xi-\eta)^2+1\right)\\ &= \xi^2+\eta^2+(\xi-\eta)^2+3.\end{aligned} \tag{2.5.19}$$

2. The phase in Duhamel's formula is $\phi(\xi,\eta) := \xi^2+\eta^2+(\xi-\eta)^2+3$. It does not vanish: it is even bounded from below by 3. Hence we can say that Equation 2.5.13 is *non resonant in time*. We are going to take advantage of this property later by performing a normal form transform.
3. Now we start to make our contraction estimates. The easiest one is the high-regularity estimate: we are going to use the physical space Duhamel formula:

$$\|\mathcal{A}(f)(t)\|_{H^N} \leq \int_0^t \left\|e^{-is(\Delta-1)}\left(e^{-is(\Delta-1)}\bar{f}(s)\right)\left(e^{-is(\Delta-1)}\bar{f}(s)\right)\right\|_{H^N} ds.$$

Since $e^{-is(\Delta-1)}$ is H^N-continuous, we can write

$$\|\mathcal{A}(f)(t)\|_{H^N} \leq C\int_0^t \left\|\left(e^{-is(\Delta-1)}\bar{f}(s)\right)\left(e^{-is(\Delta-1)}\bar{f}(s)\right)\right\|_{H^N} ds.$$

Now, thanks to Leibniz' rule,

$$\|\mathcal{A}(f)(t)\|_{H^N} \leq C\int_0^t \left\|e^{-is(\Delta-1)}\bar{f}(s)\right\|_{H^N}\left\|e^{-is(\Delta-1)}\bar{f}(s)\right\|_{L^\infty} ds.$$

Because of the H^N-continuity of $e^{-is(\Delta-1)}$,

$$\left\|e^{-is(\Delta-1)}\bar{f}(s)\right\|_{H^N} \leq C\left\|\bar{f}(s)\right\|_{H^N} = C\left\|f(s)\right\|_{H^N}.$$

Dispersion for $e^{-is(\Delta-1)}$ in dimension 1 (2.5.15) then gives

$$\left\|e^{-is(\Delta-1)}\bar{f}(s)\right\|_{L^\infty} \leq C\frac{1}{\sqrt{s}}\left\|\bar{f}(s)\right\|_{L^1}.$$

But the L^1 norm can be bounded by a weighted norm: by Lemma A.0.1,

$$\begin{aligned}\|f(s)\|_{L^1} &\leq \sqrt{\|f\|_{L^2}\|f\|_{L^2(x)}}\\ &\leq \|f(s)\|_{L^2(\langle x\rangle)}.\end{aligned}$$

Hence,

$$\|\mathcal{A}(f)(t)\|_{H^N} \leq C\int_0^t \frac{1}{\sqrt{s}}\|f(s)\|_{H^N}\|f(s)\|_{L^2(\langle x\rangle)} ds.$$

Thus,

$$\begin{aligned}\|\mathcal{A}(f)(t)\|_{H^N} &\leq C\int_0^t \frac{1}{\sqrt{s}}\,\|f\|_{S_T^N}^2\,ds\\ &\leq C\sqrt{t}\,\|f\|_{S_T^N}^2\,.\end{aligned}$$

Taking a supremum in time then gives

$$\sup_{t\in[0,T)}\|\mathcal{A}(f)\|_{H^N} \leq C\sqrt{T}\,\|f\|_{S_T^N}^2\,. \tag{2.5.20}$$

4. Finally we have to find bounds for $\|\mathcal{A}(t)\|_{L^2(\langle x\rangle)}$. In order to do so, we first notice that

$$\|\mathcal{A}(t)\|_{L^2(\langle x\rangle)} \leq \|\mathcal{A}(t)\|_{L^2} + \|\mathcal{A}(t)\|_{L^2(x)}\,.$$

Since $\|\mathcal{A}(t)\|_{L^2}$ is bounded by $\|\mathcal{A}(t)\|_{H^N}$, we are allowed to focus on $\|\mathcal{A}(t)\|_{L^2(x)}$ only. Now,

$$\|\mathcal{A}(t)\|_{L^2(x)} = \left\|\partial_\xi\widehat{\mathcal{A}}(t)\right\|_{L^2}.$$

Here we have to use the expression of $\mathcal{A}$ in the frequency space given in 2.5.19 in order to compute $\partial_\xi\widehat{\mathcal{A}}(t)$:

$$\partial_\xi\widehat{\mathcal{A}}(t) = \widehat{\mathcal{A}}_1(t) + \widehat{\mathcal{A}}_2(t),$$

with

$$\begin{aligned}\widehat{\mathcal{A}}_1(t) &= \int_0^t\int_{\mathbb{R}} e^{is\phi(\xi,\eta)}\overline{\hat{f}}(s,\eta)\partial_\xi\overline{\hat{f}}(s,\xi-\eta)d\eta ds,\\ \widehat{\mathcal{A}}_2(t) &= \int_0^t\int_{\mathbb{R}} is\partial_\xi\phi(\xi,\eta)e^{is\phi(\xi,\eta)}\overline{\hat{f}}(s,\eta)\overline{\hat{f}}(s,\xi-\eta)d\eta ds.\end{aligned}$$

We are going to estimate each of these terms separately.

(i) First of all, $\mathcal{A}_1(t)$ can be expressed as

$$\mathcal{A}_1(t) = \int_0^t e^{-is(\Delta-1)}\left(e^{-is(\Delta-1)}\bar{f}(s,x)\right)\left(e^{-is(\Delta-1)}(x\bar{f})(s,x)\right)ds.$$

Since $e^{-is(\Delta-1)}$ is L^2 continuous, and by Hölder's inequality,

$$\|\mathcal{A}_1(t)\|_{L^2} \leq C\int_0^t\left\|e^{-is(\Delta-1)}\bar{f}(s)\right\|_{L^\infty}\left\|e^{-is(\Delta-1)}(x\bar{f})(s)\right\|_{L^2_x}ds.$$

Now, by (2.5.15) and Lemma A.0.1,

$$\left\|e^{-is(\Delta-1)}\bar{f}(s)\right\|_{L^\infty} \leq C\frac{1}{\sqrt{s}}\left\|\bar{f}(s)\right\|_{L^2(\langle x\rangle)}.$$

The L^2 continuity of $e^{-is(\Delta-1)}$ gives

$$\begin{aligned}\left\|e^{-is(\Delta-1)}(x\bar{f})(s)\right\|_{L^2} &\leq C\left\|(x\bar{f})(s)\right\|_{L^2_x}\\ &\leq C\,\|f(s)\|_{L^2(\langle x\rangle)}\,.\end{aligned}$$

Finally, we obtain

$$\|\mathcal{A}_1(f)(t)\|_{L^2} \leq C \int_0^t \frac{1}{\sqrt{s}} \|f(s)\|_{L^2(\langle x \rangle)} \|f(s)\|_{L^2(\langle x \rangle)} \, ds,$$

leading to the same conclusion as in (2.5.20),

$$\sup_{t \in [0,T)} \|\mathcal{A}_1(f)\|_{L^2} \leq C\sqrt{T} \, \|f\|^2_{S^N_T} . \tag{2.5.21}$$

(ii) Dealing with $\widehat{\mathcal{A}}_2(t)$ may seem harder, because the term under the integral is not obviously decreasing like $1/\sqrt{s}$. However, we know that $\phi(\xi,\eta)$ does not vanish and is even bounded from below uniformly on $\mathbb{R}$. We are going to take advantage of this fact by writing

$$e^{is\phi(\xi,\eta)} = \frac{1}{i\phi(\xi,\eta)} \partial_s \left(e^{is\phi(\xi,\eta)} \right),$$

and by performing an integration by parts: this gives

$$\widehat{\mathcal{A}}_2(t) = \widehat{\mathcal{A}}_2^1(t) + \widehat{\mathcal{A}}_2^2(t) + \widehat{\mathcal{A}}_2^3(t) + \widehat{\mathcal{A}}_2^4(t),$$

with

$$\begin{aligned}
\widehat{\mathcal{A}}_2^1(t) &= \int_{\mathbb{R}} it \frac{\partial_\xi \phi(\xi,\eta)}{i\phi(\xi,\eta)} e^{it\phi(\xi,\eta)} \overline{\hat{f}}(t,\eta) \overline{\hat{f}}(t,\xi-\eta) d\eta, \\
\widehat{\mathcal{A}}_2^2(t) &= \int_0^t \int_{\mathbb{R}} i \frac{\partial_\xi \phi(\xi,\eta)}{i\phi(\xi,\eta)} e^{is\phi(\xi,\eta)} \overline{\hat{f}}(s,\eta) \overline{\hat{f}}(s,\xi-\eta) d\eta ds, \\
\widehat{\mathcal{A}}_2^3(t) &= \int_0^t \int_{\mathbb{R}} is \frac{\partial_\xi \phi(\xi,\eta)}{i\phi(\xi,\eta)} e^{is\phi(\xi,\eta)} \partial_s \left(\overline{\hat{f}}(s,\eta) \right) \overline{\hat{f}}(s,\xi-\eta) d\eta ds, \\
\widehat{\mathcal{A}}_2^4(t) &= \int_0^t \int_{\mathbb{R}} is \frac{\partial_\xi \phi(\xi,\eta)}{i\phi(\xi,\eta)} e^{is\phi(\xi,\eta)} \overline{\hat{f}}(s,\eta) \partial_s \left(\overline{\hat{f}}(s,\xi-\eta) \right) d\eta ds.
\end{aligned}$$

• **A bilinear multiplier estimate.** The problem with the integrals $\widehat{\mathcal{A}}_2^i$ is that they are no longer convolutions, as $\widehat{\mathcal{A}}_1$ was for example: this is due to the presence of the symbol $\frac{\partial_\xi \phi(\xi,\eta)}{\phi(\xi,\eta)}$. Let us define

$$\begin{aligned}
m(\eta, \xi - \eta) &:= \frac{\partial_\xi \phi(\xi,\eta)}{\phi(\xi,\eta)} \\
&= \frac{4\xi - 2\eta}{\xi^2 + \eta^2 + (\xi-\eta)^2 + 3} \\
&= \frac{2\eta + 4(\xi-\eta)}{\xi^2 + \eta^2 + (\xi-\eta)^2 + 3},
\end{aligned}$$

and, for two functions a and b

$$T_m(a,b) := \mathcal{F}^{-1} \left(\int_{\mathbb{R}} m(\eta, \xi-\eta) \hat{a}(\eta) \hat{b}(\xi-\eta) d\eta \right).$$

The problem is now to have "Hölder-like" estimates for T_m : this is true and stated in the following proposition:

Proposition 2.5.6. *For all $1 < r < \infty$, $1 < p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, a in $L^p(\mathbb{R})$, b in $L^q(\mathbb{R})$, then $T_m(a,b)$ is in L^r and there exists a constant $C > 0$ such that*

$$\|T_m(a,b)\|_{L^r} \leq C \|a\|_{L^p} \|b\|_{L^q}.$$

Proof :
We are going to apply Coifman-Meyer's Theorem A.3.1. So as to do this, we write

$$m(\eta,\zeta) = \frac{4\zeta + 2\eta}{(\eta+\zeta)^2 + \eta^2 + \zeta^2 + 3},$$

and we have to bound $\partial_\eta^\alpha \partial_\zeta^\beta$ by the correct power of $1/(|\eta| + |\zeta|)$. If we look at the first partial derivatives, we obtain:

$$\begin{aligned}
\partial_\eta m(\eta,\zeta) &= \frac{2}{(\eta+\zeta)^2 + \eta^2 + \zeta^2 + 3} - \frac{(4\zeta+2\eta)(4\eta+\zeta)}{((\eta+\zeta)^2 + \eta^2 + \zeta^2 + 3)^2}, \\
\partial_\zeta m(\eta,\zeta) &= \frac{4}{(\eta+\zeta)^2 + \eta^2 + \zeta^2 + 3} - \frac{(4\zeta+2\eta)^2}{((\eta+\zeta)^2 + \eta^2 + \zeta^2 + 3)^2}.
\end{aligned}$$

Hence, we have

$$|\partial_\eta m(\eta,\zeta)| \leq \frac{C}{(|\eta| + |\zeta|)} \quad \text{and} \quad |\partial_\zeta m(\eta,\zeta)| \leq \frac{C}{(|\eta| + |\zeta|)}.$$

We can continue the process to get

$$\left|\partial_\eta^\alpha \partial_\zeta^\beta m(\eta,\zeta)\right| \leq \frac{C}{(|\eta| + |\zeta|)^{\alpha+\beta}}.$$

Thus m satisfies Coifman-Meyers' Theorem's hypoteses: this ends the proof of Proposition 2.5.6. ∎

Now, in order to apply Proposition 2.5.6, we write each of the $\mathcal{A}_2^i$ terms as bilinear multipliers:

$$\begin{aligned}
\mathcal{A}_2^1(t) &= te^{-it(\Delta-1)} T_m\left(e^{-it(\Delta-1)}\bar{f}(t), e^{-it(\Delta-1)}\bar{f}(t)\right), \\
\mathcal{A}_2^2(t) &= \int_0^t e^{-is(\Delta-1)} T_m\left(e^{-is(\Delta-1)}\bar{f}(s), e^{-it(\Delta-1)}\bar{f}(s)\right) ds, \\
\mathcal{A}_2^3(t) &= \int_0^t se^{-is(\Delta-1)} T_m\left(e^{-is(\Delta-1)}\partial_s\bar{f}(s), e^{-it(\Delta-1)}\bar{f}(s)\right) ds, \\
\mathcal{A}_2^4(t) &= \int_0^t se^{-is(\Delta-1)} T_m\left(e^{-is(\Delta-1)}\bar{f}(s), e^{-it(\Delta-1)}\partial_s\bar{f}(s)\right) ds.
\end{aligned}$$

• Now we estimate each of the terms $\mathcal{A}_2^i$ for $i = 1,2,3$ ($\mathcal{A}_2^4$ being the same as $\mathcal{A}_2^3$).
First of all, since $e^{-it(\Delta-1)}$ is L^2 continuous,

$$\begin{aligned}
\left\|\mathcal{A}_2^1(t)\right\|_{L^2} &= \left\|te^{-it(\Delta-1)} T_m\left(e^{-it(\Delta-1)}\bar{f}(t), e^{-it(\Delta-1)}\bar{f}(t)\right)\right\|_{L^2} \\
&\leq Ct \left\|T_m\left(e^{-it(\Delta-1)}\bar{f}(t), e^{-it(\Delta-1)}\bar{f}(t)\right)\right\|_{L^2}.
\end{aligned}$$

By Proposition 2.5.6,

$$\left\| T_m \left(e^{-it(\Delta-1)} \bar{f}(t), e^{-it(\Delta-1)} \bar{f}(t) \right) \right\|_{L^2} \leq C \left\| e^{-it(\Delta-1)} \bar{f}(t) \right\|_{L^\infty} \left\| e^{-it(\Delta-1)} \bar{f}(t) \right\|_{L^2},$$

which leads, by the L^2-continuity of $e^{-it(\Delta-1)}$ and dispersion for $e^{-it(\Delta-1)}$ (2.5.15),

$$\begin{aligned} \left\| T_m \left(e^{-it(\Delta-1)} \bar{f}(t), e^{-it(\Delta-1)} \bar{f}(t) \right) \right\|_{L^2} &\leq C \frac{1}{\sqrt{t}} \left\| \bar{f}(t) \right\|_{L^\infty} \left\| \bar{f}(t) \right\|_{L^1} \\ &\leq \frac{C}{\sqrt{t}} \|f\|^2_{L^2(\langle x \rangle)}. \end{aligned}$$

This leads to

$$\left\| \mathcal{A}_2^1(t) \right\|_{L^2} \leq C\sqrt{t} \|f\|^2_{L^2(\langle x \rangle)}, \tag{2.5.22}$$

i.e.

$$\sup_{t\in[0,T)} \left\| \mathcal{A}_2^1(t) \right\|_{L^2} \leq C\sqrt{T} \|f\|^2_{S_T^N}. \tag{2.5.23}$$

By using the same arguments, we obtain

$$\sup_{t\in[0,T)} \left\| \mathcal{A}_2^2(t) \right\|_{L^2} \leq C\sqrt{T} \|f\|^2_{S_T^N}. \tag{2.5.24}$$

Dealing with $\mathcal{A}_2^3$ requires more work: by the same arguments as before, we obtain

$$\left\| \mathcal{A}_2^3(t) \right\|_{L^2} \leq C \int_0^t s \left\| \partial_s \bar{f}(s) \right\|_{L^2} \frac{1}{\sqrt{s}} \left\| \bar{f}(s) \right\|_{L^1} ds. \tag{2.5.25}$$

Now, the profile f satisfies

$$\partial_s(\bar{f}(s)) = e^{-is(\Delta-1)} u(s)^2.$$

This gives

$$\begin{aligned} \left\| \partial_s \bar{f}(s)(t) \right\|_{L^2} &= \left\| e^{-is(\Delta-1)} u(s)^2 \right\|_{L^2} \\ &\leq C \left\| u(s)^2 \right\|_{L^2}. \end{aligned}$$

Now by Hölder's inequality,

$$\left\| \partial_s \bar{f}(s) \right\|_{L^2} \leq C \|u(s)\|_{L^2} \|u(s)\|_{L^\infty}.$$

Since $u(s) = e^{is(\Delta-1)} f(s)$, we use the L^2 continuity of $e^{is(\Delta-1)}$ and the dispersion inequality 2.5.15 to obtain

$$\left\| \partial_s \bar{f}(s) \right\|_{L^2} \leq \frac{C}{\sqrt{s}} \|f(s)\|_{L^2} \|f(s)\|_{L^1}. \tag{2.5.26}$$

By (2.5.25) and (2.5.26), we have the following inequality:

$$\left\| \mathcal{A}_2^3(t) \right\|_{L^2} \leq C \int_0^t \|f(s)\|_{L^2} \left\| \bar{f}(s) \right\|^2_{L^1} ds.$$

Bounding $\|f(s)\|_{L^2}$ and $\left\|\bar{f}(s)\right\|_{L^1}$ by $\|f\|_{S_T^N}$ then gives, for all $t < T$,

$$\left\|\mathcal{A}_2^3(t)\right\|_{L^2} \leq Ct \|f\|_{S_T^N}^3 .$$

Taking the supremum in time finally gives

$$\sup_{t\in[0,T)} \left\|\mathcal{A}_2^3(t)\right\|_{L^2} \leq CT \|f\|_{S_T^N}^3 . \tag{2.5.27}$$

By Inequalities (2.5.23), (2.5.24), (2.5.27),

$$\sup_{t\in[0,T)} \|\mathcal{A}_2(t)\|_{L^2} \leq C\left(\sqrt{T}\|f\|_{S_T^N} + T\|f\|_{S_T^N}^2\right)\|f\|_{S_T^N} . \tag{2.5.28}$$

Then, by (2.5.21) and (2.5.28),

$$\sup_{t\in[0,T)} \|\mathcal{A}\|_{L^2(x)} \leq C\left(\sqrt{T}\|f\|_{S_T^N} + T\|f\|_{S_T^N}^2\right)\|f\|_{S_T^N} . \tag{2.5.29}$$

Finally, by (2.5.20) and (2.5.29), we have the following inequality:

$$\|\mathcal{A}(f)\|_{S_T^N} \leq C\left(\sqrt{T}\|f\|_{S_T^N} + T\|f\|_{S_T^N}^2\right)\|f\|_{S_T^N} . \tag{2.5.30}$$

Hence $\mathcal{A}$ is a contraction in the ball of size ε of S_T^N as long as

$$C\left(\sqrt{T}\varepsilon + T\varepsilon^2\right) < 1,$$

i.e. as long as

$$T < (C'\varepsilon)^{-2},$$

for a given constant C'. This ends the proof of Theorem 2.5.5. ■

2.6 Other models of dispersive equations

In this section, we are giving a taste of other frameworks where dispersive methods or space-time resonances method can apply, with a special emphasis on wave and Klein-Gordon's equation.

2.6.1 Fourier transforms

When we study a dispersive equation of the kind

$$\partial_t u - iLu = N(u),$$

with L a differential operator (in the spatial variable) and N a nonlinearity and we want to understand its harmonic properties, we need to find a decomposition adapted to this operator:

— if it is a flat differential polynomial, e.g. $L = \Delta$ on $\mathbb{R}$, then the *Fourier transform* is relevant.

— if it is a differential polynomial on the torus, then the *Fourier series* will be the good framework

$$c_p(t) := \int_0^{2\pi} f(t,x) e^{-ixp} dx.$$

— if $L = -\Delta + x^2$, i.e. L is the *harmonic oscillator*, the *Hermite coefficients* will be the tools to use:

$$f_p(t) := \int_{\mathbb{R}^d} f(x) \psi_p(x) dx,$$

where ψ_p is the p-th Hermite function ($p \in \mathbb{N}$), i.e. the eigenfunction of $-\Delta + x^2$ associated to the eigenvalue $2p+1$ (the reader may read more about these functions in Chapter 4, Definition 4.1.2).

However, in the two kinds of wave equations we are studying, the operators are strongly anisotropic : either it is the Laplacian on $\mathbb{R} \times \mathbb{T}$ (Chapter 3), or it is the flat Laplacian in one direction and the harmonic oscillator in the other one (Chapter 4). In those two cases we will have to define the appropriate transform in each direction. It does not seem relevant to define all those transforms in this section, the reader can refer to Definitions 3.2.1 and 3.2.3 page 50 or Definition 4.1.3 page 71 for more details.

2.6.2 Dispersion for wave and Klein-Gordon's equations

We are going to identify and discuss dispersion properties for two PDEs which are the basis of the models studied in this manuscript.

For the wave equation, i.e.

$$\left\{ \begin{array}{rcl} \partial_t^2 u - \Delta u & = & 0, \\ u(0,x) & = & u_0(x), \\ \partial_t u(0,x) & = & u_1(x), \end{array} \right. \tag{2.6.1}$$

we have $L(\xi) = |\xi|$.

Then the group velocity for a wave packet at frequency ξ is $\frac{\xi}{|\xi|}$. In dimension 1, it is constant on $\mathbb{R}_-$ and on $\mathbb{R}_+$: hence the wave equation in dimension 1 is not dispersive.

On the contrary, for a higher dimension d, the direction of this veclocity is not constant: this explains the dispersive effect we state in Proposition 2.6.1. More precisely, we remark that $\mathrm{Hess}(L) \neq 0$ but is of rank $d-1$: we have some dispersive effect, but not as strong as the one for Schrödinger's equation.

Thus we have the following inequality, well known for a long time (we can find an elementary and detailled proof in [1] for example):

Proposition 2.6.1. *Let $\mathcal{C} = \{\xi \in \mathbb{R}^d | r \leq |\xi| \leq R\}$ for some positive r and R such that $r < R$. A constant C then exists such that if u_0 and u_1 are supported in the annulus $\mathcal{C}$ then the associate solution u of the wave equation (2.6.1) in dimension d satisfies*

$$\forall t \neq 0, \ \|u(t)\|_{L^\infty} \leq C \frac{1}{t^{\frac{d-1}{2}}} \left(\|u_0\|_{L^1} + \|u_1\|_{L^1} \right). \tag{2.6.2}$$

This means in particular that the wave equation does not disperse in dimension 1. The dispersion power relates to the rank of $\mathrm{Hess}(L)$ which is equal to $d-1$.

For Klein-Gordon's equation, i.e.

$$\begin{cases} \partial_t^2 u - \Delta u + u &= 0, \\ u(0,x) &= u_0(x), \\ \partial_x u(0,x) &= u_1(x), \end{cases} \tag{2.6.3}$$

we have $L(\xi) = \sqrt{1+\xi^2}$.
Hence, the group velocity of a wave packet at frequency ξ is equal to $\frac{\xi}{\sqrt{1+\xi^2}}$, and Klein-Gordon's equation is dispersive. The following proposition is proven in [43].

Proposition 2.6.2. *Let $\mathcal{C}_\lambda = \{\xi \in \mathbb{R} | \lambda \leq |\xi| \leq 2\lambda\}$ for some positive λ. A constant C then exists such that if $\hat{u}_0$ and $\hat{u}_1$ are supported in the annulus $\mathcal{C}_\lambda$ then the associate solution u of the wave equation (2.6.3) in dimension d satisfies*

$$\forall t \neq 0,\ \|u(t)\|_{L^\infty} \leq C\lambda^{\frac{d}{2}+1} \frac{1}{t^{\frac{d}{2}}} \left(\|u_0\|_{L^1} + \|u_1\|_{L^1} \right). \tag{2.6.4}$$

In particular, we have

$$\left\| e^{-it\sqrt{-\Delta+1}} u_0 \right\|_{L^\infty} \leq C \|u_0\|_{W^{\frac{3}{2},1}}. \tag{2.6.5}$$

Remark 2.6.3. *Passing from (2.6.4) to (2.6.5) is not completely obvious: the Klein-Gordon equation is of order 2 in time and linear propagators have been described only for equations of order 1 in time. We explain the way to deal with order 2 in time equations in 3, Section 3.3 page 52.*

Remark 2.6.4. *The dispersion inequality for Klein-Gordon's equation is similar to the one for Schrödinger's equation: the decay in time is the same. However, it requires more regularity ($W^{3/2,1}$ instead of L^1). This can be explained the higher capacity of Schrödinger's equation to send high frequencies at infinity: indeed, the group velocity for Schrödinger is -2ξ whereas the one for Klein-Gordon is $\frac{\xi}{\sqrt{1+\xi^2}}$: this last one is bounded, contrary to the first one. This is why we need to weaken the weight of high frequencies in Klein-Gordon's equation, by taking a higher regularity norm.*

2.6.3 Some results obtained with space-time resonances

First of all, for three expository articles on space-time resonances with different points of view, the reader can refer to [53], [21] or [41].

A wide variety of dispersive equations has been studied from a space-time resonances point of view. Germain, Masmoudi and Shatah started with Schrödinger's equation, in dimension 3 in [24] and in dimension 2 in [26]. The phase for Schrödinger's equation is quite easy to compute, and so are the time, space, and space-time resonant sets:

— if the nonlinearity is equal to $\bar{u}^2$ as in (2.5.1-a), then the phase is $\phi = \xi^2 + \eta^2 + (\xi-\eta)^2$. The time resonant set is $\{\xi = \eta = 0\}$, so is the space-time resonant one.

— if the nonlinearity is equal to u^2 as in (2.5.1-b), then the then the phase is $\phi = \xi^2 - \eta^2 - (\xi-\eta)^2$. Hence the space resonant set is $\{\xi = 2\eta\}$ and the time resonant set is $\{\eta.(\xi-\eta) = 0\}$: this leads to a space-time resonant set equal to $\{\xi = \eta = 0\}$.

— if the nonlinearity is equal to $u\bar{u}$ as in (2.5.1-c), then the phase is $\phi = \xi^2 - \eta^2 + (\xi - \eta)^2$. Hence the space resonant set is $\{\xi = 0\}$, the time resonant set is $\{\eta.\xi = 0\}$ and the space-time resonant set equal to $\{\xi = 0\}$.

The first two cases are way easier than the third one: in the first case, the time resonant set is a single point. In the second one none of the space or time resonant set is reduced to a point, but the space-time resonant set is. That is why in those two cases a global existence theorem has been proven. In [40], the author rewrote the proof of [24] in the case of a nonlinearity equal to $\bar{u}^2$, and established an existence theorem in a simple space, i.e. in $H^s(\langle x \rangle)$, for all $s > -1/2$ (critical exponent).
Germain, Masmoudi and Shatah then proved existence theorems for capillarity waves in [25] or gravity waves in [27]. The same ideas helped to study a Klein-Gordon system ([21]) or for the Euler-Maxwell system in [23].
A special focus on wave equations was made in the last few years: see the works of Pusateri [47], Pusateri and Shatah [48] or Pocovnicu [46].
Space-time resonances also occur in more theoretical harmonic analysis problems: for example, Bernicot and Germain studied bilinear dispersive estimates in [4] and [5].

2.6.4 Comparison of some results on wave equation and Klein-Gordon

Our main existence results are of the following form:

1. For the wave equation on $\mathbb{R} \times \mathbb{T}$, we have a long time existence and uniqueness result (Theorem 3.2.8), with a time ε^{-2}, (ε being the size of the initial data), in a space of regularity $M > 1/2$ in the periodic direction, $N \geq 3/2$ in the free direction and with a weight $\langle x \rangle$ in the free direction.
2. For the wave equation on $\mathbb{R}^2$ with a harmonic potential we also have a long time existence and uniqueness result (Theorem 4.1.10), with a time $\varepsilon^{-\frac{4}{3+\delta}}$ ($\delta > 0$), in a space of regularity $M > 3$ in the trapped direction, $N > 1/\delta + 3/2 + 2M$ in the free direction with a weight $\langle x \rangle$.
   Even if this result may be compared to the "improved theorem" 2.4.4, this theorem is stronger, for two reasons in particular: first the functional spaces are more robust and natural than the ones of 2.4.4 ; then Theorem 4.1.10 gives bilinear estimates which are essential for the study of a resonant system.

These results can be compared to some results already established in other frameworks:

1. For the wave equation in $\mathbb{R}^3$, with a cubic nonlinearity, Pusateri and Shatah established in [48] a global existence theorem in a $H^2(x) \cap H^1(x^2) \cap H^N$.
2. A similar result for a quadratic nonlinearity is impossible for a pure wave equation (i.e. without mass), since Glassey in [28] and Schaeffer in [49] proved that for a nonlinearity with a degree below the so-called Strauss critical exponent p_c ($p_c = 1 + \sqrt{2}$ in dimension 3, $p_c = \frac{1}{2}(3 + \sqrt{17})$ in dimension 2), blow up occurs (for small initial data).
3. For Klein-Gordon's equation on the torus, Delort proved in [11] a long-time existence theorem in Sobolev spaces (unweighted because all the directions are compact). In the case of a quadratic nonlinearity and for the two dimensional torus, he managed to obtain a time of order $\varepsilon^{-2}|\log(\varepsilon)|^{-A}$ ($A > 0$ constant).

2.7 Notations used throughout the manuscript

Throughout this thesis we are going to use the following notations.

— for x a real number, we write

$$\langle x \rangle := \sqrt{1 + x^2}.$$

— For all $\eta \in \mathbb{R}$ and $m \in \mathbb{N}$,

$$\langle \eta \rangle_m := \sqrt{\eta^2 + 2m + 2}$$

— For all $\eta \in \mathbb{R}$, $m \in \mathbb{N}$, $\mu > 0$, we write

$$\langle \eta, m \rangle_\mu := \sqrt{\eta^2 + m^2 + \mu}$$

— if p is an integer, we define $\underline{p} = \max(1, |p|)$.
— if f and g are two functions, we write $f \lesssim g$ if there is a universal constant C such that $f \leq Cg$.
— if A and B are two functions, m and n two integers,

$$\left(A \lesssim_{m \leftrightarrow n} B(m,n)\right) \Leftrightarrow \left(A \lesssim B(m,n) + B(n,m)\right). \tag{2.7.1}$$

— we write $D = i\partial$.
— if $m : \mathbb{R}^2 \to \mathbb{R}$, we define the bilinear operator T_m associated to m by

$$T_m(f,g) := \mathcal{F}^{-1}\left(\int_{\mathbb{R}} m(\eta, \cdot - \eta)\hat{f}(\eta)\hat{g}(\cdot - \eta)d\eta\right). \tag{2.7.2}$$

The properties of this kind of multiplier are studied in Appendix A.

Chapter 3

The quadratic Klein-Gordon equation $\mathbb{R} \times \mathbb{T}$

Contents

3.1 Equation

In this chapter, we are going to consider a Klein-Gordon equation, on a product space $\mathbb{R} \times \mathbb{T}$. This is one of the simplest examples of a "wave equation with a mass" in an anisotropic setting, and it will help to get a clear understanding of the way of using space-time resonances, in particular the way of using them in an anisotropic framework.
We are considering the following equation

$$\left\{ \begin{array}{rcl} \partial_t^2 u - \Delta u + \mu u & = & u^2, \\ u(0,x) & = & u_0(x), \\ \partial_t u(0,x) & = & u_1(x), \end{array} \right. \tag{3.1.1}$$

with $u : (t, x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T} \mapsto u(t, x_1, x_2) \in \mathbb{R}$ and $\mu > 0$. As explained in Chapter 2, we are going to go through the following steps to study this equation :

— we have to build a functional framework adapted to the equation.
— a Duhamel formula has to be established.
— resonant sets are deduced from the phase given by the Duhamel formula.
— we establish contraction estimates.

In the case $\mu = 0$, this equation is a true wave equation. This case will not be studied here, we only focus on Klein-Gordon's equation.
Space-time resonances methods for quadratic Klein-Gordon equation have been used first by Shatah in [52]. Then several other results came for quadratic Klein-Gordon equations, see [44] for example, or for systems as in [22] or [36].
The case of non-flat geometry has also been studied, in particular Klein-Gordon equation on the torus or on a sphere: see [12] or [11] for more details.
However these results focus on isotropic models, i.e. on $\mathbb{T}^d$ or $\mathbb{S}^d$ and not on a product space as in our situation.

3.2 Functional framework and main result

3.2.1 Two transforms adapted to the equation

Given the product space structure, we need to define two different Fourier transforms in order to deal with Equation (3.1.1).

Definition 3.2.1. *Let $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T}$. For all $p \in \mathbb{Z}$, $t \in \mathbb{R}_+$, $x_1 \in \mathbb{R}$, we define the p-th Fourier coefficient of f, as follows:*

$$\hat{f}(t, x_1, p) := \int_0^{2\pi} f(t, x_1, x_2) e^{-ipx_2} dx_2. \tag{3.2.1}$$

Remark 3.2.2. *When it is clear from the context, we drop x_1 in $\hat{f}(s, x_1, m)$ and simply write $\hat{f}(s, m)$.*

Definition 3.2.3. *Let $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T}$. For all $s \in \mathbb{R}_+$, $\xi \in \mathbb{R}$, $p \in \mathbb{Z}$, we define the global Fourier transform of f as follows:*

$$\widetilde{f}(s, \xi, p) := \int_{\mathbb{R}} \hat{f}(s, x_1, p) e^{-ix_1\xi} dx_1. \tag{3.2.2}$$

3.2.1.a Functional spaces

Given the anisotropic framework in which we are working, we are going to define our Sobolev spaces in an anisotropic way. We do not recall the Sobolev spaces and the weighted Sobolev spaces as defined in Definition 2.2.2.

Remark 3.2.4. *In particular, we are going to consider norms of this kind:*

$$\left\| \hat{f}(s,m) \right\|_{H^N_{\mathbb{R}}(\langle x_1 \rangle)},$$

which can be controlled as follows

$$\left\| \hat{f}(s,m) \right\|_{H^N_{\mathbb{R}}(\langle x_1 \rangle)} \leq \left(\int_{\mathbb{R}} \langle \xi \rangle^{2N} \left| \widetilde{f}(s,\xi,m) \right|^2 d\xi \right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}} \langle \xi \rangle^{2N} \left| \partial_\xi \widetilde{f}(s,\xi,m) \right|^2 d\xi \right)^{\frac{1}{2}}.$$

Definition 3.2.5. *If M is a real number, we define for a 2π periodic function g its H^M norm by*

$$\|g\|_{H^M} = \left\| \left(p^M \hat{g}(p) \right)_{p \in \mathbb{Z}} \right\|_{\ell^2(\mathbb{Z})}.$$

In the case of Equation (3.1.1) we are going to focus on one space in particular:

Definition 3.2.6. *For M and N positive real numbers, for f defined on $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{T}$, we define the $H^M_{\mathbb{T}} H^N_{\mathbb{R}}(\langle x_1 \rangle)$ as the ℓ^2 norm of $\left(p^M \left\| \hat{f}(s,p) \right\|_{H^N(\langle x_1 \rangle)} \right)_{p \in \mathbb{Z}}$.*

Remark 3.2.7. *The space $H^M_{\mathbb{T}} H^N_{\mathbb{R}}(\langle x_1 \rangle)$ can also be defined by the form*

$$\left\| \langle x_1 \rangle \langle \partial_2 \rangle^M \langle \partial_1 \rangle^N f \right\|_{L^2}.$$

3.2.2 Main result

The existence theorem for this equation is the following:

Theorem 3.2.8. *Let $\varepsilon > 0$. Let*

$$T = C\varepsilon^{-2},$$

with C a universal constant.
Then, given M and N integers satisfying

$$M > \frac{1}{2},\ N \geq \frac{3}{2}, \tag{3.2.3}$$

if (u_0, u_1) satisfies

$$\|u_0\|_{H^{M+1}_{\mathbb{T}} H^{N+1}_{\mathbb{R}}(\langle x_1 \rangle)} + \|u_1\|_{H^M_{\mathbb{T}} H^N_{\mathbb{R}}(\langle x_1 \rangle)} \leq \frac{\varepsilon}{2},$$

then there exists a unique solution u to (3.1.1) defined on $[0,T]$ with

$$\sup_t \|u(t)\|_{H^{M+1}_{\mathbb{T}} H^{N+1}_{\mathbb{R}}(\langle x_1 \rangle)} \leq \varepsilon. \tag{3.2.4}$$

We notice that the existence time is of order ε^{-2}, which is better than the two simple theorems 2.3.1 and 2.4.4 established in Chapter 2.

3.3 Duhamel formula

Establishing the Duhamel formula for this equation is trickier than what we saw for NLS in Chapter 2. Indeed, Equation (3.1.1) is of order 2 in time, contrary to NLS which is of order 1 in time.
In order to go back to an equation of order one, let us define the following auxiliary functions:

Definition 3.3.1. *The* left-traveling *part of u (resp. the* right-traveling *part) denoted u_+ (resp. u_-) is defined by*

$$u_\pm := \partial_t u \pm i\sqrt{-\Delta+\mu}\ u.$$

Remark 3.3.2. *We make two important remarks on the left and right traveling parts of u:*

1. *The left and the right traveling parts $u_\pm$ satisfy*

$$\partial_t u_\pm \mp i\sqrt{-\Delta+\mu}\ u_\pm = -u^2. \tag{3.3.1}$$

2. *For all t,*

$$\left[(u(t),\partial_t u(t)) \in H^{M+1}_{\mathbb{T}} H^{N+1}_{\mathbb{R}}(\langle x_1\rangle) \times H^M_{\mathbb{T}} H^N_{\mathbb{R}}(\langle x_1\rangle)\right] \Leftrightarrow \left[u_\pm(t) \in H^M_{\mathbb{T}} H^N_{\mathbb{R}}(\langle x_1\rangle)\right]. \tag{3.3.2}$$

Projecting (3.3.1) on $e^{i(\xi x_1 + p x_2)}$ gives

$$\partial_t \widetilde{u}_\pm(t,\xi,p) \mp i\sqrt{\xi^2+p^2+\mu}\ \widetilde{u}_\pm(t,\xi,p) = \widetilde{(u^2)}(t,\xi,p). \tag{3.3.3}$$

Now we consider the profile

$$\widetilde{f}_\pm(t,\xi,p) := e^{\mp it\sqrt{\xi^2+p^2+\mu}}\widetilde{u}_\pm(t,\xi,p). \tag{3.3.4}$$

Then $\widetilde{f}_\pm(t,\xi,p)$ satisfies

$$\partial_t \widetilde{f}_\pm(t,\xi,p) = e^{\mp it\sqrt{\xi^2+p^2+\mu}}\widetilde{(u^2)}(t,\xi,p). \tag{3.3.5}$$

Equation (3.3.5) can be rewritten using the integral form:

$$\widetilde{f}_\pm(t,\xi,p) = \widetilde{f}_\pm(0,\xi,p) + \int_0^t e^{\mp is\sqrt{\xi^2+p^2+\mu}}\widetilde{(u^2)}(s,\xi,p)ds.$$

The Fourier transform of a product being a convolution, this leads to

$$\widetilde{(u^2)}(s,\xi,p) = 2\pi \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} \widetilde{u}(s,\eta,m)\widetilde{u}(s,\xi-\eta,p-m)d\eta.$$

Then, since

$$\widetilde{u}_+(s,\eta,m) - \widetilde{u}_-(s,\eta,m) = 2i\left(\eta^2+m^2+M\right)^{\frac{1}{2}}\widetilde{u}(s,\eta,m),$$

the formula becomes

$$\widetilde{(u^2)}(s,\xi,p) = 2\pi \sum_m \sum_{\alpha,\beta=\pm1} \alpha\beta \int_{\mathbb{R}} \frac{\widetilde{u}_\alpha(s,\eta,m)}{\langle \eta,m\rangle_\mu}\frac{\widetilde{u}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta,p-m\rangle_\mu}d\eta, \tag{3.3.6}$$

with $\langle \eta,m\rangle_\mu := \sqrt{\eta^2+m^2+\mu}$. Then from (3.3.6) we get the Duhamel formula for $\widetilde{f}_\pm$:

Proposition 3.3.3. *Equation (3.1.1) is equivalent to the fixed point formula for* $f = (f_-, f_+)$*:*

$$f = A(f),$$

with $A(f) = f_0 + \mathcal{A}(f)$*, and*

$$\mathcal{A}_\pm(f)(t,\xi,p) = 2\pi \sum_m \sum_{\alpha,\beta=\pm 1} \alpha\beta \int_0^t \int_{\mathbb{R}} e^{\mp is\phi^{\alpha,\beta}_{m,p}} \frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle \eta, m\rangle_\mu} \frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta, p-m\rangle_\mu} d\eta ds, \tag{3.3.7}$$

where the phase $\phi^{\alpha,\beta}_{m,n,p}$ *is defined by*

$$\phi^{\alpha,\beta}_{m,p} := \langle \xi, p\rangle_\mu + \alpha\langle \eta, m\rangle_\mu + \beta\langle \xi-\eta, p-m\rangle_\mu. \tag{3.3.8}$$

3.4 Study of the phase

In the case $\mu > 0$, we have the following result (a similar result has been proven by Shatah in [52]).

Proposition 3.4.1. *In the case* $\mu > 0$*, the phase* ϕ *never vanishes. Moreover, we have, for all real numbers* ξ, η *and integers* m, p*,*

$$\left|\phi^{\alpha,\beta}_{m,p}(\xi,\eta)\right| \gtrsim \frac{\mu}{\sqrt{\xi^2 + p^2 + \mu}}.$$

Proof :
For all $\alpha, \beta = \pm 1$, the cancellation set of $\phi^{\alpha,\beta}$ is included in the conic

$$(m^2+\mu)\xi^2 - (2pm+\mu)\xi\eta + (p^2+\mu)\eta^2 = \mu(pm - p^2 - m^2 - \frac{3}{4}\mu).$$

The left-hand side is always greater than 0 whereas the right-hand side is of the sign of $-\mu$. Hence this equation has no solution whenever $\mu > 0$.

• If $\alpha = \beta = 1$, then we immediately have $\phi^{\alpha,\beta} > \sqrt{\mu}$.
• If $\alpha = 1$ and $\beta = -1$, evaluating $\phi^{\alpha,\beta}$ in $\xi = \eta = p = m = 0$ gives that $\phi^{\alpha,\beta}$ is always positive. In particular, if $X = (\xi, p)$ and $Y = (\eta, m)$,

$$\begin{aligned}\phi^{+-}_{m,p}(\xi,\eta) &= \sqrt{|X|^2+\mu} + \sqrt{|Y|^2+\mu} - \sqrt{|X-Y|^2+\mu} \\ &\geq \sqrt{|X|^2+\mu} + \sqrt{|Y|^2+\mu} - \sqrt{\left||X|+|Y|\right|^2+\mu}.\end{aligned}$$

Writing $x = |X|$ and $y = |Y|$ leads to studying the two (real and nonnegative) variables function

$$\psi(x,y) := \sqrt{x^2+\mu} + \sqrt{y^2+\mu} - \sqrt{(x+y)^2+\mu}.$$

For x fixed, $\partial_y\psi(x,y) = \frac{y}{\sqrt{y^2+\mu}} - \frac{x+y}{\sqrt{(x+y)^2+\mu}}$. This quantity vanishes if and only if

$$\left\{\begin{array}{ccl} y^2 & = & (x+y)^2, \\ y & \text{and} & x+y \text{ have the same sign,}\end{array}\right.$$

which is impossible unless $x = 0$.
If $x = 0$, $\psi(x, y) \geq \sqrt{\mu}$. If not, $\partial_y \psi$ has a constant negative sign (by looking at its sign at $y = 0$). Then $y \mapsto \phi(x, y)$ is decreasing and it suffices to study the asymptotical regime $y \to \infty$ to bound $\phi^{\alpha,\beta}$ from below. A Taylor-Lagrange formula gives

$$\sqrt{(x+y)^2 + \mu} - \sqrt{y^2 + \mu} = x + r(x, y), \text{ with } r(x, y) \leq 0.$$

Hence

$$\psi(x, y) \geq \sqrt{x^2 + \mu} - x - r(x, y) \geq \sqrt{x^2 + \mu} - x.$$

Another Taylor-Lagrange formula gives

$$\psi(x, y) \geq \frac{\mu}{x},$$

in the asymptotic regime $x \to \infty$.
• If $\alpha = -1$ and $\beta = 1$, the conclusion is identical by exchanging the roles of (η, m) and $(\xi - \eta, p - m)$.
• If $\alpha = \beta = -1$, then ϕ is always negative. By using the same notations X, Y, x, y, we have, if $\psi_2(x, y) := \sqrt{x^2 + \mu} - \sqrt{y^2 + \mu} - \sqrt{(x - y)^2 + \mu}$,

$$\partial_y \psi(x, y) = -\frac{y}{\sqrt{y^2 + \mu}} - \frac{y - x}{\sqrt{(x - y)^2 + \mu}}.$$

It is vanishes for $y = \frac{x}{2}$. Given that $\psi_2(x, y) \to -\infty$ when $y \to \infty$, we only have to bound from below $\psi_2(x, y)$ when $y = 0$ and when $y = x/2$:

— when $y = 0$, $\psi_2(x, y) = -\sqrt{\mu}$.

— when $y = \frac{x}{2}$,

$$\begin{aligned}\psi_2(x, y) &= \sqrt{x^2 + \mu} - 2\sqrt{\left(\frac{x}{2}\right)^2 + \mu} \\ &= |x| \left(\sqrt{1 + \frac{\mu}{x^2}} - \sqrt{1 + \frac{4\mu}{x^2}} \right) \\ &= -\frac{3}{2}\frac{\mu}{x} + r(x),\end{aligned}$$

with $r(x) \geq 0$ and $r(x) \leq \frac{3}{2}\frac{\mu}{x + \sqrt{\mu}}$. Hence

$$|\psi_2(x, y)| \gtrsim \frac{3}{2}\frac{\mu}{x}.$$

This proves the proposition. ■

3.5 Proof of the existence theorem

3.5.1 Main result and strategy

3.5.1.a Main result

Following the strategy explained in Chapter 2, Section 2.2, we are going to prove the following inequality for the Duhamel operator $\mathcal{A}$ defined in (3.3.7):

Proposition 3.5.1. *For all M, N real numbers satisfying (3.2.3),*

$$\|\mathcal{A}(f)(t)\|_{H^M_{\mathbb{T}}H^N_{\mathbb{R}}(\langle x_1\rangle)} \lesssim \sqrt{t}\sup_{0\le s\le t}\|f(s)\|^2_{H^M_{\mathbb{T}}H^N_{\mathbb{R}}(\langle x_1\rangle)} + t\sup_{0\le s\le t}\|f(s)\|^3_{H^M_{\mathbb{T}}H^N_{\mathbb{R}}(\langle x_1\rangle)}. \quad (3.5.1)$$

Going from (3.5.1) to Theorem 3.2.8 is a straight application of the method of Section 2.2. For now on we only focus on proving Proposition 3.5.1.

3.5.1.b An intermediate proposition

In order to prove Proposition 3.5.1, we are going to prove an intermediate result. First, we give the following definition:

Definition 3.5.2. *For all $m,p\in\mathbb{Z}$, $\alpha,\beta=\pm1$, $\xi\in\mathbb{R}$, for $t\in\mathbb{R}_+$, we define*

$$\mathcal{I}^{\alpha,\beta}_{m,p}(t,\xi) := \int_0^t\int_{\mathbb{R}} e^{\mp is\phi^{\alpha,\beta}_{m,p}}\frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle\eta,m\rangle_\mu}\frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle\xi-\eta,p-m\rangle_\mu}d\eta ds. \quad (3.5.2)$$

Remark 3.5.3. *Note that given this definition,*

$$\widetilde{A(f)}(t) = 2\pi\sum_m\sum_{\alpha,\beta=\pm1}\alpha\beta\mathcal{I}^{\alpha,\beta}_{m,p}(t,\xi). \quad (3.5.3)$$

Proposition 3.5.4. *For all $N\ge 3/2$, for all $p\in\mathbb{Z}$, $m\in\mathbb{Z}$, $\alpha=\pm1$, $\beta=\pm1$,*

$$\left\|\mathcal{I}^{\alpha,\beta}_{m,p}(t)\right\|_{H^N_{\mathbb{R}}(\langle x_1\rangle)} \lesssim \mathcal{Q}^{\alpha,\beta}_{m,p}(t)+\mathcal{C}^{\alpha,\beta}_{m,p}(t)+\mathcal{R}^{\alpha,\beta}_{m,p}(t), \quad (3.5.4)$$

with

— $\mathcal{Q}^{\alpha,\beta}_{m,p}(t)$ *is the quadratic integral term*

$$\mathcal{Q}^{\alpha,\beta}_{m,p}(t) := \int_0^t\frac{1}{\sqrt{s}}\left\|\hat{f}(s,m)\right\|_{H^N_{\mathbb{R}}(\langle x_1\rangle)}\left\|\hat{f}(s,p-m)\right\|_{H^N_{\mathbb{R}}(\langle x_1\rangle)}ds, \quad (3.5.5)$$

— $\mathcal{C}^{\alpha,\beta}_{m,p}(t)$ *is the cubic integral term*

$$\mathcal{C}^{\alpha,\beta}_{m,p}(t) := \int_0^t\left\|\hat{f}(s,m)\right\|^2_{H^N_{\mathbb{R}}(\langle x_1\rangle)}\left\|\hat{f}(s,p-m)\right\|_{H^N_{\mathbb{R}}(\langle x_1\rangle)}ds, \quad (3.5.6)$$

— $\mathcal{R}^{\alpha,\beta}_{m,p}(t)$ *is the boundary term*

$$\mathcal{R}^{\alpha,\beta}_{m,p}(t) := \sqrt{t}\left\|\hat{f}_\alpha(t,m)\right\|_{H^N_{\mathbb{R}}(\langle x_1\rangle)}\left\|\hat{f}_\beta(t,p-m)\right\|_{H^N_{\mathbb{R}}(\langle x_1\rangle)}. \quad (3.5.7)$$

Proof of Proposition 3.5.1 assuming Proposition 3.5.4 :
We are going to focus on the term $\mathcal{Q}^{\alpha,\beta}_{m,p}(t)$ defined in (3.5.5), the method being exactly the same for $\mathcal{C}^{\alpha,\beta}_{m,p}(t)$ and $\mathcal{R}^{\alpha,\beta}_{m,p}(t)$.
The proof essentielly relies on the fact that $H^M(\mathcal{T})$ is an algebra for $M>1/2$. However, we are going to detail it for two reasons :

1. the ideas are of the same kind as the ones developed in the "summation theorem" C.1.1.
2. we have to be really careful about the time dependence and the uniformity of the bounds we are establishing.

Because of formula (3.5.3), we simply want to sum over m and take the $H^M_{\mathbb{T}}$ norm, i.e. we are reduced to proving the following inequality :

$$\left\| \left(p^M \sum_{m\in\mathbb{Z}} \left(\left\| \hat{f}(s,m) \right\|_{H^N_{\mathbb{R}}(\langle x_1 \rangle)} \left\| \hat{f}(s,p-m) \right\|_{H^N_{\mathbb{R}}(\langle x_1 \rangle)} \right) \right)_{p\in\mathbb{Z}} \right\|_{\ell^2} \lesssim \|f(s)\|^2_{H^M_{\mathcal{T}} H^N_{\mathbb{R}}(\langle x_1 \rangle)} . \tag{3.5.8}$$

In order to prove this inequality, let us write

$$\left\| \hat{f}(s,m) \right\|_{H^N_{\mathbb{R}}(\langle x_1 \rangle)} = m^{-M} \|f(s)\|_{H^M_{\mathbb{T}} H^N_{\mathbb{R}}(\langle x_1 \rangle)} a_m(s),$$

with $(a_m(s))_{m\in\mathbb{Z}}$ a sequence in the unit ball of ℓ^2. Then

$$p^M \sum_{m\in\mathbb{Z}} \left(\left\| \hat{f}(s,m) \right\|_{H^N_{\mathbb{R}}(\langle x_1 \rangle)} \left\| \hat{f}(s,p-m) \right\|_{H^N_{\mathbb{R}}(\langle x_1 \rangle)} \right) = \|f(s)\|^2_{H^M_{\mathbb{T}} H^N_{\mathbb{R}}(\langle x_1 \rangle)} b_p(s),$$

with

$$b_p(s) := p^M \sum_{m\in\mathbb{Z}} m^{-M} a_m(s)(p-m)^{-M} a_{p-m}(s). \tag{3.5.9}$$

Then it remains to prove that the sequence $\left(p^M \sum_{m\in\mathbb{Z}} m^{-M} a_m(s)(p-m)^{-M} a_{p-m}(s) \right)_{p\in\mathbb{Z}}$ is in the ball of ℓ^2 of diameter independent of $a_m(s)$ (given that $(a_m(s))_{m\in\mathbb{Z}}$ is in the unit ball of ℓ^2). Let us rewrite

$$b_p(s) = b^1_p(s) + b^2_p(s),$$

with

$$\begin{aligned} b^1_p(s) &:= p^M \sum_{\substack{m\in\mathbb{Z} \\ m\le p-m}} m^{-M} a_m(s)(p-m)^{-M} a_{p-m}(s), \\ b^2_p(s) &:= p^M \sum_{\substack{m\in\mathbb{Z} \\ m\ge p-m}} m^{-M} a_m(s)(p-m)^{-M} a_{p-m}(s). \end{aligned}$$

Remark 3.5.5. *Here we are performing a kind of ℓ^2 paraproduct.*

When $m \le p-m$, $|p| \le 2|p-m|$, and $|b^1_p(s)|$ can be bounded as follows:

$$\begin{aligned} \left| b^1_p(s) \right| &\le 2^M \sum_{\substack{m\in\mathbb{Z} \\ m\le p-m}} |p-m|^M |m|^{-M} |a_m(s)| |p-m|^{-M} |a_{p-m}(s)| \\ &\le \sum_{\substack{m\in\mathbb{Z} \\ m\le p-m}} \left(\frac{2}{|m|} \right)^M |a_m(s)||a_{p-m}(s)| \\ &\le \sum_{m\in\mathbb{Z}} \left(\frac{2}{|m|} \right)^M |a_m(s)||a_{p-m}(s)|. \end{aligned}$$

Then, by Young's inequality,

$$\left\| \left(p^M b_p \right)_{p\in\mathbb{Z}} \right\|_{\ell^2} \le \left\| \left(\left(\frac{2}{|m|} \right)^M |a_m(s)| \right)_{m\in\mathbb{Z}} \right\|_{\ell^1} \left\| \left(|a_m(s)| \right)_{m\in\mathbb{Z}} \right\|_{\ell^2},$$

with b_p defined in (3.5.9).
By hypothesis, $\|(|a_m(s)(s)|)_{m\in\mathbb{Z}}\|_{\ell^2} \leq 1$. Then by Hölder's inequality,

$$\left\| \left(\left(\frac{2}{|m|}\right)^M |a_m(s)| \right)_{m\in\mathbb{Z}} \right\|_{\ell^1} \leq \left\| \left(\left(\frac{2}{|m|}\right)^M \right)_{m\in\mathbb{Z}} \right\|_{\ell^2} \left\| \left(|a_m(s)| \right)_{m\in\mathbb{Z}} \right\|_{\ell^2},$$

which is bounded whenever $M > 1/2$ (because of (3.2.3)). Then $(b_p^1(s))_{p\in\mathbb{Z}}$ is bounded in ℓ^2 with a bound independent of $(a_m(s))_{m\in\mathbb{Z}}$.
The method being exactly the same for b_p^2, we get the inequality (3.5.8).
It remains to integrate in s and the proof of Proposition 3.5.1 is finished. ■
From now on we prove Proposition 3.5.4.

3.5.1.c Strategy for the proof of Proposition 3.5.4

We start by choosing two real numbers M and N satisfying condition (3.2.3), and m and p two integers.

In the Fourier analysis framework we are in, it is always more convenient to work with homogeneous norms (with both a homogeneous regularity and a homogeneous weight). Hence in order to establish inequalities on the $H^N_{\mathbb{R}}(\langle x_1 \rangle)$ norm, we are going to compare this norm to homogeneous norms:

$$\|f\|_{H^N_{\mathbb{R}}(\langle x_1\rangle)} \lesssim \|f\|_{L^2} + \|f\|_{L^2(x_1)} + \|f\|_{\dot{H}^N_{\mathbb{R}}} + \|f\|_{\dot{H}^N_{\mathbb{R}}(x_1)}.$$

In order to simplify the proof and to give the main ideas, we are only going to focus on the $\dot{H}^N_{\mathbb{R}}$ and the $\dot{H}^N_{\mathbb{R}}(x_1)$ norms, the two other ones being dealt with in the same fashion. Studying the $\dot{H}^N_{\mathbb{R}}$ norm should not be too difficult, it relies essentially on a fractional Leibniz' rule (or a paraproduct decomposition) and on the use of the dispersive decay of the solutions.
On the contrary, studying the $\dot{H}^N_{\mathbb{R}}(x_1)$ norm will make more problems appear: the weight x_1 corresponds to a differentiation in the Fourier space and will make direct estimates fail. In this case we will have to perform some integrations by parts, relying on a finer knowledge of the behavior of the phase. Studying weighted norms is nevertheless fundamental in order to establish the existence Theorem 3.2.8.
Proposition 3.5.4 will follow from Lemmas 3.5.6 and 3.5.8.

3.5.2 Unweighted norm: basic paraproduct decomposition

Lemma 3.5.6. *For all M, N satisfying (3.2.3),*

$$\left\|\mathcal{I}^{\alpha,\beta}_{m,p}(t)\right\|_{\dot{H}^N_{\mathbb{R}}} \lesssim \int_0^t \left(\frac{1}{\sqrt{s}} \frac{\left\|\hat{f}_\alpha(s,m)\right\|_{H^{3/2}(\langle x_1\rangle)}}{\sqrt{m^2+\mu}} \frac{\left\|\hat{f}_\beta(s,p-m)\right\|_{H^N}}{\sqrt{(p-m)^2+\mu}} \right) ds. \tag{3.5.10}$$

Remark 3.5.7. *In particular, we have $\left\|\mathcal{I}^{\alpha,\beta}_{m,p}(t)\right\|_{\dot{H}^N_{\mathbb{R}}} \lesssim \mathcal{Q}^{\alpha,\beta}_{m,p}(t)$.*

Proof :
We want to bound the following norm

$$\left\| \xi^N \int_{\mathbb{R}} e^{\mp is\phi^{\alpha,\beta}_{m,p}} \frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle \eta, m\rangle_\mu} \frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta, p-m\rangle_\mu} d\eta \right\|_{L^2}.$$

Here a paraproduct decomposition is the most appropriate: let χ be a smooth function satisfying Coifman-Meyer's theorem's hypotheses (Theorem A.3.1), localizing in the zone $|\eta| \lesssim |\xi-\eta|$, and let

$$\xi^N \int_{\mathbb{R}} e^{\mp is\phi^{\alpha,\beta}_{m,p}} \frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle \eta, m\rangle_\mu} \frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta, p-m\rangle_\mu} d\eta = I_\chi + I_{1-\chi},$$

with

$$\begin{aligned} I_\chi &:= \xi^N \int_{\mathbb{R}} \chi(\eta,\xi-\eta) e^{\mp is\phi^{\alpha,\beta}_{m,p}} \frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle \eta, m\rangle_\mu} \frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta, p-m\rangle_\mu} d\eta \\ I_{1-\chi} &:= \xi^N \int_{\mathbb{R}} (1-\chi(\eta,\xi-\eta)) e^{\mp is\phi^{\alpha,\beta}_{m,p}} \frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle \eta, m\rangle_\mu} \frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta, p-m\rangle_\mu} d\eta. \end{aligned}$$

These two terms being symmetric, we are only going to bound I_χ. We can write

$$\begin{aligned} I_\chi &= \int_{\mathbb{R}} \frac{\xi^N}{(\xi-\eta)^N} \chi(\eta,\xi-\eta) e^{\mp is\phi^{\alpha,\beta}_{m,p}} \frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle \eta, m\rangle_\mu} (\xi-\eta)^N \frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta, p-m\rangle_\mu} d\eta \\ &= T_m\left(\frac{e^{\pm is\alpha\sqrt{-\Delta_{x_1}+m^2+\mu}} \hat{f}_\alpha(s,m)}{\langle D, m\rangle_\mu}, |D|^N \frac{e^{\pm is\beta\sqrt{\Delta^2_{x_1}+(p-m)^2+\mu}} \hat{f}_\beta(s,p-m)}{\langle D, p-m\rangle_\mu} \right), \end{aligned}$$

with T_m the bilinear multiplier operator (defined in 2.7.2) of symbol

$$m(\eta,\xi-\eta) := \frac{\xi^N}{(\xi-\eta)^N} \chi(\eta,\xi-\eta) e^{\mp is\sqrt{\xi^2+p^2+\mu}}.$$

Since $e^{\mp is\sqrt{-\Delta+p^2+\mu}}$ is L^2-continuous and by Lemma A.3.3, the symbol m satisfies Hölder-like estimates: this leads to

$$\|I_\chi\|_{L^2} \lesssim \left\| \frac{e^{\pm is\alpha\sqrt{-\Delta_{x_1}+m^2+\mu}} \hat{f}_\alpha(m,s)}{\langle D, m\rangle_\mu} \right\|_{L^\infty} \left\| |D|^N \frac{e^{\pm is\beta\sqrt{\Delta^2_{x_1}+(p-m)^2+\mu}} \hat{f}_\beta(s,p-m)}{\langle D, p-m\rangle_\mu} \right\|_{L^2}.$$

Then, thanks to a combination of Lemma A.0.1 and the dispersion proposition A.1.1, we have the bound

$$\begin{aligned} \left\| \frac{e^{\pm is\alpha\sqrt{-\Delta_{x_1}+m^2+\mu}} \hat{f}_\alpha(s,m)}{\langle D, m\rangle_\mu} \right\|_{L^\infty} &\lesssim \frac{1}{\sqrt{s}} \left\| \frac{\hat{f}_\alpha(s,m)}{\langle D, m\rangle_\mu} \right\|_{W^{3/2,1}} \\ &\lesssim \frac{1}{\sqrt{s}} \left\| \frac{\hat{f}_\alpha(s,m)}{\langle D, m\rangle_\mu} \right\|^{\frac12}_{H^{3/2}} \left\| \frac{\hat{f}_\alpha(s,m)}{\langle D, m\rangle_\mu} \right\|^{\frac12}_{H^{3/2}(\langle x_1\rangle)} \\ &\lesssim \frac{1}{\sqrt{s}} \left\| \frac{\hat{f}_\alpha(s,m)}{\langle D, m\rangle_\mu} \right\|_{H^{3/2}(\langle x_1\rangle)}. \end{aligned}$$

Finally the Fourier multiplier Proposition A.1.4-a gives

$$\left\| \frac{e^{\pm is\alpha\sqrt{-\Delta_{x_1}+m^2+\mu}} \hat{f}_\alpha(s,m)}{\langle D, m\rangle_\mu} \right\|_{L^\infty} \lesssim \frac{1}{\sqrt{s}} \frac{1}{\sqrt{m^2+\mu}} \left\| \hat{f}_\alpha(s,m) \right\|_{H^{3/2}(\langle x_1\rangle)}. \tag{3.5.11}$$

The L^2 continuity of $e^{\pm is\beta\sqrt{\Delta_{x_1}^2+(p-m)^2+\mu}}$ and the Fourier multiplier Proposition A.1.4-a also give

$$\left\| |D|^N \frac{e^{\pm is\beta\sqrt{\Delta_{x_1}^2+(p-m)^2+\mu}}\hat{f}_\beta(s,p-m)}{\langle D,p-m\rangle_\mu}\right\|_{L^2} \lesssim \frac{1}{\sqrt{(p-m)^2+\mu}}\left\|\hat{f}_\beta(s,p-m)\right\|_{H^N}$$

This gives the following bound

$$\begin{aligned}&\left\|\xi^N\int_{\mathbb{R}} e^{\mp is\phi_{m,p}^{\alpha,\beta}}\frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle\eta,m\rangle_\mu}\frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle\xi-\eta,p-m\rangle_\mu}d\eta\right\|_{L^2}\\ &\lesssim \frac{1}{\sqrt{s}}\frac{1}{\sqrt{m^2+\mu}}\left\|\hat{f}_\alpha(s,m)\right\|_{H^{3/2}(\langle x_1\rangle)}\frac{1}{\sqrt{(p-m)^2+\mu}}\left\|\hat{f}_\beta(s,p-m)\right\|_{H^N}.\end{aligned}$$

Lemma 3.5.6 is now proven. ■

3.5.3 Weighted norm: towards space-time resonances

Lemma 3.5.8. *For all N satisfying (3.2.3),*

$$\left\|\mathcal{I}_{m,p}^{\alpha,\beta}(t)\right\|_{\dot{H}_{\mathbb{R}}^N(x_1)} \lesssim \mathcal{Q}_{m,p}^{\alpha,\beta}(t)+\mathcal{C}_{m,p}^{\alpha,\beta}(t)+\mathcal{R}_{m,p}^{\alpha,\beta}(t). \tag{3.5.12}$$

Proof :
A weight in the physical space corresponding to a derivative in the frequency space, we want to estimate the L^2 norm of

$$\int_0^t |\xi|^N\partial_\xi\left(\int_{\mathbb{R}} e^{\mp is\phi_{m,p}^{\alpha,\beta}}\frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle\eta,m\rangle_\mu}\frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle\xi-\eta,p-m\rangle_\mu}d\eta\right)ds = J_1+J_2+J_3,$$

where

$$\begin{aligned}J_1 &:= \int_0^t |\xi|^N\int_{\mathbb{R}} is\partial_\xi\phi_{m,p}^{\alpha,\beta}e^{\mp is\phi_{m,p}^{\alpha,\beta}}\frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle\eta,m\rangle_\mu}\frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle\xi-\eta,p-m\rangle_\mu}d\eta ds,\\ J_2 &:= \int_0^t |\xi|^N\int_{\mathbb{R}} e^{\mp is\phi_{m,p}^{\alpha,\beta}}\frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle\eta,m\rangle_\mu}\frac{\left(-\partial_\eta\widetilde{f}_\beta(s,\xi-\eta,p-m)\right)}{\langle\xi-\eta,p-m\rangle_\mu}d\eta ds,\\ J_3 &:= \int_0^t |\xi|^N\int_{\mathbb{R}} e^{\mp is\phi_{m,p}^{\alpha,\beta}}\frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle\eta,m\rangle_\mu}\widetilde{f}_\beta(s,\xi-\eta,p-m)\partial_\xi\left(\frac{1}{\langle\xi-\eta,p-m\rangle_\mu}\right)d\eta ds.\end{aligned}$$

Those three terms are quite different and three separate and independent methods will be needed to bound them:

- the integral J_3 is the easiest one and will be bounded in the same fashion as the H^N norm was bounded.
- the integral J_2 has a $\partial_\eta\widetilde{f}_\beta(s,\xi-\eta,p-m)$ term, which will "force" this term to be estimated in L^2. This situation can be problematic sometimes and we will need to perform an *integration by parts in η* to solve this problem.
- the integral J_1 is the most clearly problematic : the s factor makes the integral behave like $s^{3/2}$. This is typically the case where the space-time resonances method is powerful.

3.5.3.a Term J_3

We notice that

$$\partial_\xi\left(\frac{1}{\langle \xi-\eta, p-m\rangle_\mu}\right) = \frac{\eta-\xi}{((\xi-\eta)^2+(p-m)^2+\mu)^{\frac{3}{2}}},$$

so we are in the same situation than in section 3.5.2 and we have the following lemma:

Lemma 3.5.9. *For all N satisfying (3.2.3),*

$$\|J_3\|_{L^2_\xi} \lesssim \int_0^t \left(\frac{1}{\sqrt{s}} \frac{\left\|\hat{f}_\alpha(s,m)\right\|_{H^{3/2}(\langle x_1\rangle)}}{\sqrt{m^2+\mu}} \frac{\left\|\hat{f}_\beta(s,p-m)\right\|_{H^N}}{\sqrt{(p-m)^2+\mu}}\right) ds. \tag{3.5.13}$$

Remark 3.5.10. *We deduce from* (3.5.13) *that* $\|J_3\|_{L^2_\xi} \lesssim \mathcal{Q}^{\alpha,\beta}_{m,p}(t)$.

3.5.3.b Term J_2: use of an integration by parts in space

We want to prove the following Lemma:

Lemma 3.5.11. *For all N satisfying (3.2.3),*

$$\|J_2\|_{L^2_\xi} \lesssim \mathcal{Q}^{\alpha,\beta}_{m,p}(t) + \mathcal{C}^{\alpha,\beta}_{m,p}(t) + \mathcal{R}^{\alpha,\beta}_{m,p}(t). \tag{3.5.14}$$

As in Section 3.5.2, we are going to use the paraproduct cutoff χ. Let us write

$$J_2 := J_{2,\chi} + J_{2,1-\chi},$$

with

$$J_{2,\chi} := -|\xi|^N \int_{\mathbb{R}} \chi(\eta,\xi-\eta) e^{\mp is\phi^{\alpha,\beta}_{m,p}} \frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle\eta,m\rangle_\mu} \frac{\partial_\eta \widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle\xi-\eta,p-m\rangle_\mu} d\eta$$
$$J_{2,1-\chi} := -|\xi|^N \int_{\mathbb{R}} (1-\chi(\eta,\xi-\eta)) e^{\mp is\phi^{\alpha,\beta}_{m,p}} \frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle\eta,m\rangle_\mu} \frac{\partial_\eta \widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle\xi-\eta,p-m\rangle_\mu} d\eta$$

Here, there is no symmetry between $J_{2,\chi}$ and $J_{2,1-\chi}$: we will have to deal with those two terms separately, and prove the following result:

Lemma 3.5.12. *For all N satisfying (3.2.3),*

$$\|J_{2,\chi}\|_{L^2_\xi} \lesssim \int_0^t \left(\frac{1}{\sqrt{s}} \frac{\left\|\hat{f}_\alpha(s,m)\right\|_{H^{3/2}(\langle x_1\rangle)}}{\sqrt{m^2+\mu}} \frac{\left\|\hat{f}_\beta(s,p-m)\right\|_{H^N}}{\sqrt{(p-m)^2+\mu}}\right) ds, \tag{3.5.15}$$

$$\|J_{2,1-\chi}\|_{L^2_\xi} \lesssim \mathcal{Q}^{\alpha,\beta}_{m,p}(t) + \mathcal{C}^{\alpha,\beta}_{m,p}(t) + \mathcal{R}^{\alpha,\beta}_{m,p}(t). \tag{3.5.16}$$

Remark 3.5.13. *Similarly to what we concluded for Lemma 3.5.6 in Remark 3.5.7, we deduce from (3.5.15) that $\|J_{2,\chi}\|_{L^2_\xi} \lesssim \mathcal{Q}^{\alpha,\beta}_{m,p}(t)$. So Lemma 3.5.12 implies Lemma 3.5.11.*

Proof of Lemma 3.5.12 :

Term $J_{2,\chi}$. In the zone $\chi = 1$, we have $|\xi| \lesssim |\xi - \eta|$. This incites us to write

$$J_{2,\chi} = -\int_0^t \int_{\mathbb{R}} \frac{|\xi|^N}{|\xi-\eta|^N} \chi(\eta, \xi-\eta) e^{\mp is\phi^{\alpha,\beta}_{m,p}} \frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle \eta, m\rangle_\mu} \frac{|\xi-\eta|^N \partial_\eta \widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta, p-m\rangle_\mu} d\eta ds.$$

Then, since the symbol $\frac{|\xi|^N}{|\xi-\eta|^N}\chi(\eta,\xi-\eta)$ satisfies Hölder-like estimates (because of Proposition A.3.3), we can write

$$\|J_{2,\chi}\|_{L^2} \lesssim \int_0^t \left\| \frac{e^{\pm is\alpha\sqrt{-\Delta_{x_1}+m^2+\mu}} \hat{f}_\alpha(m,s)}{\langle D, m\rangle_\mu} \right\|_{L^\infty} \left\| |D|^N \frac{e^{\pm is\beta\sqrt{\Delta_{x_1}^2+(p-m)^2+\mu}} \left(x_1 \hat{f}_\beta(s,x_1,p-m)\right)}{\langle D, p-m\rangle_\mu} \right\|_{L^2_{x_1}} ds.$$

As in Section 3.5.2,

$$\left\| \frac{e^{\pm is\alpha\sqrt{-\Delta_{x_1}+m^2+\mu}} \hat{f}_\alpha(s,m)}{\langle D, m\rangle_\mu} \right\|_{L^\infty} \lesssim \frac{1}{\sqrt{s}} \frac{1}{\sqrt{m^2+\mu}} \left\| \hat{f}_\alpha(s,m) \right\|_{H^{3/2}(\langle x_1\rangle)},$$

and

$$\left\| |D|^N \frac{e^{\pm is\beta\sqrt{\Delta_{x_1}^2+(p-m)^2+\mu}} \left(x_1 \hat{f}_\beta(s,p-m)\right)}{\langle D, p-m\rangle_\mu} \right\|_{L^2_{x_1}} \lesssim \frac{1}{\sqrt{(p-m)^2+\mu}} \left\| \hat{f}_\beta(s,p-m) \right\|_{H^N}.$$

Hence the bound

$$\|J_{2,\chi}\|_{L^2} \lesssim \int_0^t \frac{1}{\sqrt{s}} \frac{1}{\sqrt{m^2+\mu}} \left\| \hat{f}_\alpha(s,m) \right\|_{H^{3/2}(\langle x_1\rangle)} \frac{1}{\sqrt{(p-m)^2+\mu}} \left\| \hat{f}_\beta(s,p-m) \right\|_{H^N} ds,$$

and Inequality (3.5.15).

Term $J_{2,1-\chi}$. The problem here is the following: if we use the same method as previously, we will get the following integral:

$$J_{2,\chi} = -\int_0^t \int_{\mathbb{R}} \frac{|\xi|^N}{|\eta|^N} \left(1 - \chi(\eta,\xi-\eta)\right) e^{\mp is\phi^{\alpha,\beta}_{m,p}} \frac{|\eta|^N \widetilde{f}_\alpha(s,\eta,m)}{\langle \eta, m\rangle_\mu} \frac{\partial_\eta \widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta, p-m\rangle_\mu} d\eta ds.$$

The only space where we are allowed to estimate

$$\mathcal{F}^{-1}\left(\frac{\partial_\eta \widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta, p-m\rangle_\mu} \right)$$

is L^2.

But if we try to bound

$$\mathcal{F}^{-1}\left(e^{\pm is\sqrt{\eta^2+m^2+\mu}} \frac{|\eta|^N \widetilde{f}_\alpha(s,\eta,m)}{\langle \eta, m\rangle_\mu} \right)$$

in L^∞, we will bound it by $\left\| |D|^N f_\alpha \right\|_{H^{3/2}(\langle x_1\rangle)}$ which is not controlled by the fixed point norm! This can be seen as a problem arising when xf is the high-frequency term.

In order to solve this problem, we are going to perform an integration by parts in η, by writing

$$-|\xi|^N \int_{\mathbb{R}} (1-\chi(\eta,\xi-\eta))e^{\mp is\phi^{\alpha,\beta}_{m,p}} \frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle \eta, m\rangle_\mu} \frac{\partial_\eta \widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta, p-m\rangle_\mu} d\eta = \sum_{i=1}^{5} \widetilde{J}_i,$$

where

$$\widetilde{J}_1 := -|\xi|^N \int_{\mathbb{R}} \left(\partial_1\chi(\eta,\xi-\eta) - \partial_2\chi(\eta,\xi-\eta)\right) e^{\mp is\phi^{\alpha,\beta}_{m,p}} \frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle \eta, m\rangle_\mu} \frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta, p-m\rangle_\mu} d\eta,$$

$$\widetilde{J}_2 := \pm|\xi|^N \int_{\mathbb{R}} (1-\chi(\eta,\xi-\eta)) is\partial_\eta\phi^{\alpha,\beta}_{m,p} e^{\mp is\phi^{\alpha,\beta}_{m,p}} \frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle \eta, m\rangle_\mu} \frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta, p-m\rangle_\mu} d\eta,$$

$$\widetilde{J}_3 := |\xi|^N \int_{\mathbb{R}} (1-\chi(\eta,\xi-\eta))e^{\mp is\phi^{\alpha,\beta}_{m,p}} \frac{\partial_\eta\widetilde{f}_\alpha(s,\eta,m)}{\langle \eta, m\rangle_\mu} \frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta, p-m\rangle_\mu} d\eta,$$

$$\widetilde{J}_4 := |\xi|^N \int_{\mathbb{R}} (1-\chi(\eta,\xi-\eta))e^{\mp is\phi^{\alpha,\beta}_{m,p}} \widetilde{f}_\alpha(s,\eta,m)\partial_\eta\left(\frac{1}{\langle \eta, m\rangle_\mu}\right) \frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta, p-m\rangle_\mu} d\eta,$$

$$\widetilde{J}_5 := |\xi|^N \int_{\mathbb{R}} (1-\chi(\eta,\xi-\eta))e^{\mp is\phi^{\alpha,\beta}_{m,p}} \frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle \eta, m\rangle_\mu} \widetilde{f}_\beta(s,\xi-\eta,p-m)\partial_\eta\left(\frac{1}{\langle \xi-\eta, p-m\rangle_\mu}\right) d\eta.$$

The terms $\widetilde{J}_1$, $\widetilde{J}_4$ and $\widetilde{J}_5$ are bounded, as in Section 3.5.2, by

$$\frac{1}{\sqrt{s}} \frac{1}{\sqrt{m^2+\mu}} \left\| \hat{f}_\alpha(s,m) \right\|_{H^{3/2}(\langle x_1\rangle)} \frac{1}{\sqrt{(p-m)^2+\mu}} \left\| \hat{f}_\beta(s,p-m) \right\|_{H^N}.$$

The term $\widetilde{J}_3$ can be dealt with as was the term $J_{1,\chi}$: hence it is bounded by

$$\frac{1}{\sqrt{s}} \frac{1}{\sqrt{m^2+\mu}} \left\| \hat{f}_\alpha(s,m) \right\|_{H^{3/2}(\langle x_1\rangle)} \frac{1}{\sqrt{(p-m)^2+\mu}} \left\| \hat{f}_\beta(s,p-m) \right\|_{H^N}.$$

Hence

$$\int_0^t \left(\widetilde{J}_1 + \widetilde{J}_3 + \widetilde{J}_4 + \widetilde{J}_5\right) ds \lesssim \mathcal{Q}^{\alpha,\beta}_{m,p}(t).$$

Finally, the term $\widetilde{J}_2$ is very similar to J_1, where recall that

$$J_1 := |\xi|^N \int_{\mathbb{R}} is\partial_\xi\phi^{\alpha,\beta}_{m,p} e^{\mp is\phi^{\alpha,\beta}_{m,p}} \frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle \eta, m\rangle_\mu} \frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta, p-m\rangle_\mu} d\eta.$$

It will be bounded in the next section. This ends the proof of Lemma 3.5.12. ■

3.5.3.c Term J_1: use of the space-time resonances method

The Lemma to prove is the same as in the previous sections:

Lemma 3.5.14. *For all N satisfying (3.2.3),*

$$\|J_1\|_{L^2_\xi} \lesssim \mathcal{Q}^{\alpha,\beta}_{m,p}(t) + \mathcal{C}^{\alpha,\beta}_{m,p}(t) + \mathcal{R}^{\alpha,\beta}_{m,p}(t). \quad (3.5.17)$$

In order to get the smallest possible growth of the integral J_1, we are going to use the result we saw in Section 3.4: the phase ϕ never vanishes and is easily bounded from below. If we write $e^{\mp is\phi^{\alpha,\beta}_{m,p}} = \frac{1}{i\phi^{\alpha,\beta}_{m,p}}\partial_s\left(e^{\mp is\phi^{\alpha,\beta}_{m,p}}\right)$, we have

$$J_1 = |\xi|^N \int_{\mathbb{R}} is\partial_\xi \phi^{\alpha,\beta}_{m,p}\frac{1}{i\phi^{\alpha,\beta}_{m,p}}\partial_s\left(e^{\mp is\phi^{\alpha,\beta}_{m,p}}\right)\frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle \eta,m\rangle_\mu}\frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta,p-m\rangle_\mu}d\eta.$$

Then an integration by parts gives

$$J_1 = J_{1,0} + J_{1,1} + J_{1,2} + J_{1,3},$$

with

$$\begin{aligned}
J_{1,0} &:= |\xi|^N \int_{\mathbb{R}} t\partial_\xi \phi^{\alpha,\beta}_{m,p}\frac{1}{\phi^{\alpha,\beta}_{m,p}}e^{\mp is\phi^{\alpha,\beta}_{m,p}}\frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle \eta,m\rangle_\mu}\frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta,p-m\rangle_\mu}d\eta,\\
J_{1,1} &:= -|\xi|^N \int_0^t\int_{\mathbb{R}} \partial_\xi \phi^{\alpha,\beta}_{m,p}\frac{1}{\phi^{\alpha,\beta}_{m,p}}e^{\mp is\phi^{\alpha,\beta}_{m,p}}\frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle \eta,m\rangle_\mu}\frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta,p-m\rangle_\mu}d\eta ds,\\
J_{1,2} &:= -|\xi|^N \int_0^t\int_{\mathbb{R}} s\partial_\xi \phi^{\alpha,\beta}_{m,p}\frac{1}{\phi^{\alpha,\beta}_{m,p}}e^{\mp is\phi^{\alpha,\beta}_{m,p}}\frac{\partial_s\widetilde{f}_\alpha(s,\eta,m)}{\langle \eta,m\rangle_\mu}\frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta,p-m\rangle_\mu}d\eta ds,\\
J_{1,3} &:= -|\xi|^N \int_0^t\int_{\mathbb{R}} s\partial_\xi \phi^{\alpha,\beta}_{m,p}\frac{1}{\phi^{\alpha,\beta}_{m,p}}e^{\mp is\phi^{\alpha,\beta}_{m,p}}\frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle \eta,m\rangle_\mu}\frac{\partial_s\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle \xi-\eta,p-m\rangle_\mu}d\eta ds.
\end{aligned}$$

Lemma 3.5.14 will be deduced from the following Lemma:

Lemma 3.5.15. *For all M, N satisfying (3.2.3),*

$$\|J_{1,0}\|_{L^2} \lesssim_{m\leftrightarrow(p-m)} \sqrt{t}\left\|\hat{f}_\alpha(s,m)\right\|_{H^N}\frac{1}{\sqrt{(p-m)^2+\mu}}\left\|\hat{f}_\beta(s,p-m)\right\|_{H^{3/2}(\langle x_1\rangle)}, \tag{3.5.18}$$

$$\|J_{1,1}\|_{L^2} \lesssim_{m\leftrightarrow(p-m)} \int_0^t \frac{1}{\sqrt{s}}\left\|\hat{f}_\alpha(s,m)\right\|_{H^{3/2}}\frac{1}{\sqrt{(p-m)^2+\mu}}\left\|\hat{f}_\beta(s,p-m)\right\|_{H^N(\langle x_1\rangle)}ds, \tag{3.5.19}$$

$$\|J_{1,2}\|_{L^2} \lesssim_{m\leftrightarrow(p-m)} \frac{1}{\sqrt{(p-m)^2+\mu}}\frac{1}{m^2+\mu}\sum_{\delta,\gamma=\pm1}\mathcal{J}^{\gamma,\delta,\beta}_{m,p}, \tag{3.5.20}$$

$$\|J_{1,3}\|_{L^2} \lesssim_{m\leftrightarrow(p-m)} \frac{1}{\sqrt{(p-m)^2+\mu}}\frac{1}{m^2+\mu}\sum_{\delta,\gamma=\pm1}\mathcal{J}^{\alpha,\gamma,\delta}_{m,p}, \tag{3.5.21}$$

with

$$\mathcal{J}^{\gamma,\delta,\beta}_{m,p} = \int_0^t \|f_\gamma(s,m)\|_{H^{3/2}}\|f_\delta(s,m)\|_{H^{3/2}(\langle x_1\rangle)}\|f_\beta(s,p-m)\|_{H^{3/2}(\langle x_1\rangle)}ds,$$

and $\lesssim_{m\leftrightarrow(p-m)}$ being defined in (2.7.1).

Remark 3.5.16. *From Lemma 3.5.15 we deduce that*

$$\begin{aligned}
\|J_{1,0}\|_{L^2} &\lesssim \mathcal{R}^{\alpha,\beta}_{m,p}(t),\\
\|J_{1,1}\|_{L^2} &\lesssim \mathcal{Q}^{\alpha,\beta}_{m,p}(t),\\
\|J_{1,2}\|_{L^2} &\lesssim \mathcal{C}^{\alpha,\beta}_{m,p}(t),\\
\|J_{1,3}\|_{L^2} &\lesssim \mathcal{C}^{\alpha,\beta}_{m,p}(t),
\end{aligned}$$

and we immediately get Lemma 3.5.14.

Proof of Lemma 3.5.15 :
We are only going to bound $J_{1,0}$, $J_{1,1}$ and $J_{1,2}$, the term $J_{1,3}$ being bounded in the same way as $J_{1,2}$ will be.

It is important to notice that the same multiplier appears in all those terms: by combining Coifman-Meyer estimates (Proposition A.3.1) and the paraproduct estimate (Proposition A.3.3), we know that the symbol

$$m(\eta,\xi-\eta)=\partial_\xi\phi^{\alpha,\beta}_{m,p}\frac{1}{\phi^{\alpha,\beta}_{m,p}}\frac{1}{\sqrt{\xi^2+p^2+\mu}}\frac{|\xi|^N}{|\xi-\eta|^N}\chi$$

satisfies Hölder-like estimates. However, this result is not enough: we want to be able to make paraproduct decompositions in one direction (with the function χ) and in two directions, with $\widetilde{\chi}$ localizing in the region $|(\eta,m)|\le|(\xi-\eta,p-m)|$. That is why we state the following straightforward result:

Proposition 3.5.17. *The following symbols satisfy Hölder-like estimates:*

$$\begin{aligned}
m_{\chi,\widetilde{\chi}}(\eta,\xi-\eta)&:=\partial_\xi\phi^{\alpha,\beta}_{m,p}\frac{1}{\phi^{\alpha,\beta}_{m,p}}\frac{1}{\sqrt{\xi^2+p^2+\mu}}\frac{|\xi|^N}{|\xi-\eta|^N}\chi\frac{\sqrt{\xi^2+p^2+\mu}}{\sqrt{(\xi-\eta)^2+(p-m)^2+\mu}}\widetilde{\chi}\\
m_{1-\chi,\widetilde{\chi}}(\eta,\xi-\eta)&:=\partial_\xi\phi^{\alpha,\beta}_{m,p}\frac{1}{\phi^{\alpha,\beta}_{m,p}}\frac{1}{\sqrt{\xi^2+p^2+\mu}}\frac{|\xi|^N}{|\eta|^N}(1-\chi)\frac{\sqrt{\xi^2+p^2+\mu}}{\sqrt{(\xi-\eta)^2+(p-m)^2+\mu}}\widetilde{\chi}\\
m_{\chi,1-\widetilde{\chi}}(\eta,\xi-\eta)&:=\partial_\xi\phi^{\alpha,\beta}_{m,p}\frac{1}{\phi^{\alpha,\beta}_{m,p}}\frac{1}{\sqrt{\xi^2+p^2+\mu}}\frac{|\xi|^N}{|\xi-\eta|^N}\chi\frac{\sqrt{\xi^2+p^2+\mu}}{\sqrt{\eta^2+m^2+\mu}}(1-\widetilde{\chi})\\
m_{1-\chi,1-\widetilde{\chi}}(\eta,\xi-\eta)&:=\partial_\xi\phi^{\alpha,\beta}_{m,p}\frac{1}{\phi^{\alpha,\beta}_{m,p}}\frac{1}{\sqrt{\xi^2+p^2+\mu}}\frac{|\xi|^N}{|\eta|^N}(1-\chi)\frac{\sqrt{\xi^2+p^2+\mu}}{\sqrt{\eta^2+m^2+\mu}}(1-\widetilde{\chi})
\end{aligned}$$

In order to simplify the proof, we are only going to bound the following integrals (the other ones being bounded in the same way):

$$\begin{aligned}
\widetilde{J}_{1,0}&:=\int_{\mathbb{R}}tm_{\chi,\widetilde{\chi}}(\eta,\xi-\eta)|\eta|^N\sqrt{\eta^2+m^2+\mu}e^{\mp is\phi^{\alpha,\beta}_{m,p}}\frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle\eta,m\rangle_\mu}\frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle\xi-\eta,p-m\rangle_\mu}d\eta,\\
\widetilde{J}_{1,1}&:=-\int_0^t\int_{\mathbb{R}}m_{\chi,\widetilde{\chi}}(\eta,\xi-\eta)|\eta|^N\sqrt{\eta^2+m^2+\mu}e^{\mp is\phi^{\alpha,\beta}_{m,p}}\frac{\widetilde{f}_\alpha(s,\eta,m)}{\langle\eta,m\rangle_\mu}\frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle\xi-\eta,p-m\rangle_\mu}d\eta ds,\\
\widetilde{J}_{1,2}&:=-\int_0^t\int_{\mathbb{R}}sm_{\chi,\widetilde{\chi}}(\eta,\xi-\eta)|\eta|^N\sqrt{\eta^2+m^2+\mu}e^{\mp is\phi^{\alpha,\beta}_{m,p}}\frac{\partial_s\widetilde{f}_\alpha(s,\eta,m)}{\langle\eta,m\rangle_\mu}\frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle\xi-\eta,p-m\rangle_\mu}d\eta ds.
\end{aligned}$$

More precisely, since $\sqrt{\eta^2+m^2+\mu}=:\langle\eta,m\rangle_\mu$, we are dealing with

$$\begin{aligned}
\widetilde{J}_{1,0}&:=\int_{\mathbb{R}}tm_{\chi,\widetilde{\chi}}(\eta,\xi-\eta)|\eta|^Ne^{\mp is\phi^{\alpha,\beta}_{m,p}}\widetilde{f}_\alpha(s,\eta,m)\frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle\xi-\eta,p-m\rangle_\mu}d\eta,\\
\widetilde{J}_{1,1}&:=-\int_0^t\int_{\mathbb{R}}m_{\chi,\widetilde{\chi}}(\eta,\xi-\eta)|\eta|^Ne^{\mp is\phi^{\alpha,\beta}_{m,p}}\widetilde{f}_\alpha(s,\eta,m)\frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle\xi-\eta,p-m\rangle_\mu}d\eta ds,\\
\widetilde{J}_{1,2}&:=-\int_0^t\int_{\mathbb{R}}sm_{\chi,\widetilde{\chi}}(\eta,\xi-\eta)|\eta|^Ne^{\mp is\phi^{\alpha,\beta}_{m,p}}\partial_s\widetilde{f}_\alpha(s,\eta,m)\frac{\widetilde{f}_\beta(s,\xi-\eta,p-m)}{\langle\xi-\eta,p-m\rangle_\mu}d\eta ds.
\end{aligned}$$

Bounds for $\widetilde{J}_{1,0}$. This term should be easy to bound since we do not have any integral in time. The bilinear operator associated to the symbol $m_{\chi,\widetilde{\chi}}(\eta, \xi - \eta)$ satisfying Hölder-like estimates, we have the following bound:

$$\left\| \widetilde{J}_{1,0} \right\|_{L^2} \lesssim t \left\| |D|^N e^{\pm is\sqrt{-\Delta+m^2+\mu}} \hat{f}_\alpha(t,m) \right\|_{L^2} \left\| e^{\pm is\sqrt{-\Delta+(p-m)^2+\mu}} \frac{\hat{f}_\beta(t,p-m)}{\langle D, p-m\rangle_\mu} \right\|_{L^\infty}.$$

Then, the same steps as in section 3.5.2, detailed page 58, gives

$$\left\| \widetilde{J}_{1,0} \right\|_{L^2} \lesssim \sqrt{t} \left\| \hat{f}_\alpha(t,m) \right\|_{H^N} \frac{1}{\sqrt{(p-m)^2+\mu}} \left\| \hat{f}_\beta(t,p-m) \right\|_{H^{3/2}(\langle x_1 \rangle)}.$$

Hence we get the inequality (3.5.18).

Bounds for $\widetilde{J}_{1,1}$. Similarly to what is done for $\widetilde{J}_{1,0}$, we get

$$\left\| \widetilde{J}_{1,1} \right\|_{L^2} \lesssim \int_0^t \frac{1}{\sqrt{s}} \left\| \hat{f}_\alpha(s,m) \right\|_{H^{3/2}} \frac{1}{\sqrt{(p-m)^2+\mu}} \left\| \hat{f}_\beta(s,p-m) \right\|_{H^N(\langle x_1 \rangle)} ds,$$

hence (3.5.19) is proven.

Bounds for $\widetilde{J}_{1,2}$. This term is quite different from the two previous ones and deserves a special treatment. Using the Proposition 3.5.17 for $m_{\chi,\widetilde{\chi}}$, we get

$$\left\| \widetilde{J}_{1,2} \right\|_{L^2} \lesssim \int_0^t s \left\| |D|^N e^{\pm is\sqrt{-\Delta+m^2+\mu}} \partial_s \hat{f}_\alpha(s,m) \right\|_{L^2} \left\| \frac{e^{\pm is\sqrt{-\Delta+(p-m)^2+\mu}} \hat{f}_\beta(s,p-m)}{\sqrt{-\Delta+(p-m)^2+\mu}} \right\|_{L^\infty} ds.$$

The second term is bounded as in 3.5.11:

$$\left\| \frac{e^{\pm is\sqrt{-\Delta+(p-m)^2+\mu}} \hat{f}_\beta(s,p-m)}{\sqrt{-\Delta+(p-m)^2+\mu}} \right\|_{L^\infty} \lesssim \frac{1}{\sqrt{s}} \frac{1}{\sqrt{(p-m)^2+\mu}} \left\| \hat{f}_\beta(s,p-m) \right\|_{H^{3/2}(\langle x_1 \rangle)}. \tag{3.5.22}$$

In order to bound the first term, we have to remember that

$$\begin{aligned} \partial_s \hat{f}_\alpha(s,m) &:= e^{-\alpha it\sqrt{-\Delta^2+m^2+\mu}} \left(\hat{u}(s,m)^2 \right) \\ &:= e^{-\alpha it\sqrt{-\Delta^2+2p+2}} \left(\left(\frac{\hat{u}_+(s,m) - \hat{u}_-(s,m)}{2\sqrt{-\Delta+m^2+\mu}} \right)^2 \right), \end{aligned}$$

with $\hat{u}_\pm(s,m) := e^{\pm i\sqrt{-\Delta+m^2_\mu}\hat{f}_\pm(s,m)}$. Then, since $e^{\pm it\sqrt{-\Delta^2+m^2+\mu}}$ is unitary,

$$\left\| |D|^N e^{\pm is\sqrt{-\Delta+m^2+\mu}} \partial_s \hat{f}_\alpha(s,m) \right\|_{L^2} = \left\| \left(\frac{\hat{u}_+(s,m) - \hat{u}_-(s,m)}{2\sqrt{-\Delta+m^2+\mu}} \right)^2 \right\|_{H^N}.$$

This means that

$$\left\| |D|^N e^{\pm is\sqrt{-\Delta+m^2+\mu}} \partial_s \hat{f}_\alpha(s,m) \right\|_{L^2} \lesssim \left\| \frac{\hat{u}_+(s,m) - \hat{u}_-(s,m)}{2\sqrt{-\Delta+m^2+\mu}} \right\|_{H^N} \left\| \frac{\hat{u}_+(s,m) - \hat{u}_-(s,m)}{2\sqrt{-\Delta+m^2+\mu}} \right\|_{L^\infty}. \tag{3.5.23}$$

By the Fourier multiplier proposition A.1.4-a, we have

$$\left\| \frac{\hat{u}_+(s,m) - \hat{u}_-(s,m)}{2\sqrt{-\Delta + m^2 + \mu}} \right\|_{H^N} \lesssim \frac{1}{\sqrt{m^2+\mu}} \left(\|\hat{u}_+(s,m)\|_{H^N} + \|\hat{u}_-(s,m)\|_{H^N} \right).$$

Since $e^{\pm is\sqrt{-\Delta+m^2+\mu}}$ is L^2-unitary, we get

$$\left\| \frac{\hat{u}_+(s,m) - \hat{u}_-(s,m)}{2\sqrt{-\Delta + m^2 + \mu}} \right\|_{H^N} \lesssim \frac{1}{\sqrt{m^2+\mu}} \left(\left\|\hat{f}_+(s,m)\right\|_{H^N} + \left\|\hat{f}_-(s,m)\right\|_{H^N} \right). \tag{3.5.24}$$

Similarly,

$$\left\| \frac{\hat{u}_+(s,m) - \hat{u}_-(s,m)}{2\sqrt{-\Delta + m^2 + \mu}} \right\|_{L^\infty} \lesssim \frac{1}{\sqrt{m^2+\mu}} \left(\|\hat{u}_+(s,m)\|_{L^\infty} + \|\hat{u}_-(s,m)\|_{L^\infty} \right). \tag{3.5.25}$$

Then the dispersion inequality $(A.1.1)$ in Proposition A.1.1 gives

$$\|\hat{u}_\pm(s,m)\|_{L^\infty} \lesssim \frac{1}{\sqrt{s}} \left\|\hat{f}_\pm(s,m)\right\|_{W^{3/2,1}}.$$

By Lemma A.0.1,

$$\|\hat{u}_\pm(s,m)\|_{L^\infty} \lesssim \frac{1}{\sqrt{s}} \left\|\hat{f}_\pm(s,m)\right\|_{H^{3/2}(\langle x_1 \rangle)}. \tag{3.5.26}$$

Then thanks to the inequalities (3.5.24), (3.5.25) and (3.5.26), the inequality (3.5.23) becomes

$$\left\| |D|^N e^{\pm is\sqrt{-\Delta+m^2+\mu}} \partial_s \hat{f}_\alpha(s,m) \right\|_{L^2} \lesssim \frac{1}{\sqrt{s}} \frac{1}{m^2+\mu} \sum_{\delta,\gamma=\pm1} \left\|\hat{f}_\gamma(s,m)\right\|_{H^N} \|f_\delta(s,m)\|_{H^{3/2}(\langle x_1 \rangle)}. \tag{3.5.27}$$

Then, (3.5.22) and (3.5.27) give the following inequality for $\widetilde{J}_{1,2}$:

$$\left\|\widetilde{J}_{1,2}\right\|_{L^2} \lesssim \frac{1}{\sqrt{(p-m)^2+\mu}} \frac{2}{m^2+\mu} \sum_{\delta,\gamma=\pm1} \mathcal{J}^{\gamma,\delta,\beta}_{m,p},$$

with

$$\mathcal{J}^{\gamma,\delta,\beta}_{m,p} = \int_0^t \left\|\hat{f}_\gamma(s,m)\right\|_{H^{3/2}} \left\|\hat{f}_\delta(s,m)\right\|_{H^{3/2}(\langle x_1 \rangle)} \left\|\hat{f}_\beta(s,p-m)\right\|_{H^{3/2}(\langle x_1 \rangle)} ds.$$

Hence the inequality (3.5.20): Lemma 3.5.15 is now proven. ■
Now we gather Lemmas 3.5.24, 3.5.25 and 3.5.26: this ends the proof of Lemma 3.5.8.■

Combined with Lemma 3.5.6, Lemma 3.5.8 gives the intermediate contraction estimate (Proposition 3.5.4). The contraction proposition 3.5.1 then follows, and Theorem 3.2.8 is proved.

3.6 Conclusion and perspectives

3.6.1 Cubic resonances

Theorem 3.2.8 was quite straightforward to prove because there were no time resonances in the quadratic nonlinearity. After doing an integration by parts in time, we obtain a cubic term, which may be resonant: the study of these resonances is a good way to know if we can hope to improve the existence time in Theorem 3.2.8. The phase to study for cubic resonances is the following one

$$\begin{aligned}\psi_{m,n}^{\alpha,\beta,\gamma,\delta}(\xi,\eta,\zeta) :=& \sqrt{\xi^2+p^2+\mu} + \alpha\sqrt{\eta^2+m^2+\mu} + \beta\sqrt{(\xi-\eta)^2+(p-m)^2+\mu} \\ &+ \gamma\sqrt{\zeta^2+m^2+\mu} + \delta\sqrt{(\eta-\zeta)^2+m^2+\mu},\end{aligned}$$

with $\alpha, \beta, \gamma, \delta = \pm 1$.
This phase is not easy to study, contrary to what would happen in the flat geometry (Klein-Gordon in $\mathbb{R}^2$), where the phase is a real "convolution phase". The main reason is that when we write

$$\partial_s \hat{f}_\alpha(s,m) = e^{-\alpha i t\sqrt{-\Delta^2+2p+2}} \left(\frac{\hat{u}_+(s,m) - \hat{u}_-(s,m)}{2\sqrt{-\Delta+m^2+\mu}} \right)^2,$$

we are already considering states at the Fourier mode m (on the torus). Knowing where $\psi_{m,n}^{\alpha,\beta,\gamma,\delta}(\xi,\eta,\zeta)$ and its derivatives with respect to η and ζ vanish would allow us to apply (or not) the space-time resonances method and improve the existence time.

3.6.2 The non-massic case

Another natural question would be to study the case where $\mu = 0$, i.e.

$$\begin{cases} \partial_t^2 u - \Delta u &= Q(u), \\ u(0,x) &= u_0(x), \\ \partial_t u(0,x) &= u_1(x), \end{cases} \tag{3.6.1}$$

If $Q(u) = u^2$, another problem occurs: the 0-mode (in the periodic direction) is non-dispersive (contrary to the other ones which are dispersive, with dispersion relation $\sqrt{\xi^2+p^2}$). If we manage to get rid of the zero mode, then the study of the resonant interaction is rather simple.
The phase is the following

$$\phi_{m,p}^{\alpha,\beta} := \sqrt{\xi^2+p^2} + \alpha\sqrt{\eta^2+m^2} + \beta\sqrt{(\xi-\eta)^2+(p-m)^2}.$$

Then the space resonances set (i.e. the cancellation set of $\partial_\eta \phi_{m,n}^{\alpha,\beta}$) is given by the following chart

	$\alpha = \beta$	$\alpha = -\beta$
$\frac{p}{m} - 1 > 0$	$\xi = \frac{p}{m}\eta$	$\xi = \left(2 - \frac{p}{m}\right)\eta$
$\frac{p}{m} - 1 < 0$	$\xi = \left(2 - \frac{p}{m}\right)$	$\xi = \frac{p}{m}\eta$

(3.6.2)

(we assume that neither p, m or $p-m$ is 0). Then the space-time resonances condition writes

$$\begin{aligned} &\alpha = -1, \beta = -1, (0 < m < p \text{ or } p < m < 0) \\ \text{or } &\alpha = -1, \beta = 1, (0 < p < m \text{ or } m < p < 0). \end{aligned}$$

These resonances conditions will be useful to compute the equation's *resonant system*: we are giving more details in the adequate section, i.e. in Section 5.5.

Chapter 4

The quadratic wave equation in $\mathbb{R}^2$ with a trapping harmonic potential

Contents

4.1 Introduction

4.1.1 Presentation of the equation

The goal of this part is to study the equation

$$\begin{cases} \partial_t^2 u - \Delta u + x_2^2 u + u &= u^2, \\ u(0, x_1, x_2) &= u_0(x_1, x_2), \\ \partial_t u(0, x_1, x_2) &= u_1(x_1, x_2), \end{cases} \tag{4.1.1}$$

where $u : (t, x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}^2 \mapsto u(t, x_1, x_2) \in \mathbb{C}$. This equation is based on a wave equation with a harmonic potential in one direction. We chose the operator $-\Delta + x_2 + 1$ instead of $-\Delta + x_2 + \mu$ for two reasons. First of all it is the classical version of the semi-classical equation for Poincaré waves (see Chapter 1). Moreover, chosing $\mu = 1$ makes very interesting resonance phenomena occur (which is not the case if $\mu = 0$ for example).

Since in Chapter 3 we focused on the history of Klein-Gordon equation, we are going to state here some previous results about wave equation with a potential.

The study of wave equations with a potential has a pretty long history: for a review of the different dispersive effects, see the work of Schlag in [50], or [20] and [13] for specific and more recent examples. Some global existence theorems have been proven in the case of polynomially decreasing potentials: see the books of Strauss ([55]) and of Shatah and Struwe ([54]) for reviews or [19] for a specific example. These results mainly rely on the fact that a localized or decreasing-at-infinity potential should be invisible for solutions supported far from the origin: its effect should be either neglictible or well-understood from a global point of view. This is not the case for a harmonic potential, and other methods have to be considered. Harmonic potentials (i.e. non-decaying, non-localized) in a dispersive equation have been studied in the past years, in the particular case of Schrödinger's equation: see for example [7], [2], or more recently in [31] (which considers a toric geometry, quite close to the geometry created by the harmonic potential) or [32]. Considering a harmonic potential forces to consider the harmonic structure of the equation and to study the interaction of frequencies, i.e. the resonances, inside the nonlinearity.
This fine study of resonances was introduced by Klainerman in [37] and developped for example in [38]. To be more precise, we are going to use the new version of this study of resonances, developped by Germain, Masmoudi and Shatah in [24], and used for the wave equation by Pusateri and Shatah in [48].

Studying the equation (4.1.1) is therefore quite new. Equation (4.1.1) is semilinear –weakly nonlinear since we are studying it in the small data framework, but with a quadratic nonlinearity, which means that the resonant interactions will not be able to be compensated simply by using the weakness of the nonlinearity. Moreover, the geometry given by the harmonic potential, which is physically know as the "cigar-shaped" geometry in the case of Bose-Einstein condensates (for a cubic Schrödinger equation with a harmonic potential) gives birth to very specific resonant interactions and will force us to understand in detail the resonant zones in the frequency space.
This fine understanding allows us to understand better the dynamics of our equation: in particular we are able to find a resonant system for (4.1.1), in the spirit of what has been done in [31] or [32]. This system is simpler to study and we prove that it is a good approximation of the solutions of the original equation.

The end of Section 4.1 is devoted to presenting the main result of this chapter, i.e. the existence theorem 4.1.10. Section 4.2 gives a strategy for the proof of Theorem 4.1.10, which is detailed in Sections 4.3 to 4.7.

4.1.2 Mathematical framework

Since we want to prove long-time existence theorems in a high-regularity framework, we have to define the regularity spaces we are going to use: given the anisotropic structure of the operator $-\Delta + x_2^2$, we have to define anisotropic spaces and anisotropic transforms adapted to the differential operator, in the spirit of Chapter 3, Section 3.2.

Definition 4.1.1. *The n-th Hermite function ψ_n on $\mathbb{R}$ is defined as follows*

$$\psi_n(x) = (-1)^n e^{\frac{-x^2}{2}} \frac{d^n}{dx^n} \left(e^{-x^2}\right). \tag{4.1.2}$$

It is the n-th eigenfunction of the harmonic oscillator:

$$-\psi_n''(x) + x^2 \psi_n = (2n+1)\psi_n. \tag{4.1.3}$$

We also define the interaction term between three Hermite functions $\mathcal{M}(m,n,p)$:

$$\mathcal{M}(m,n,p) := \int_{\mathbb{R}} \psi_m(x)\psi_n(x)\psi_p(x)dx. \tag{4.1.4}$$

Remark 4.1.2. *We recall that the family $(\psi_n)_{n\in\mathbb{N}}$ is a hilbertian basis of $L^2(\mathbb{R})$. However, contrary to what happens for complex exponentials, the product of two Hermite functions is not a Hermite function. That is why we need to define the symbol $\mathcal{M}(m,n,p)$: its properties are studied in Appendix A, Section A.5.*

Definition 4.1.3. *The Fourier transform of a function g defined on $\mathbb{R}$ is given by*

$$\mathcal{F}(g)(\xi) = \hat{g}(\xi) := \int_{\mathbb{R}} e^{-ix\xi} g(x)dx.$$

The Fourier-Hermite *transform of a function f defined on $\mathbb{R}^2$ is defined by*

$$\widetilde{f_p}(t,\xi) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(t,x_1,x_2)e^{-ix_1\xi}\psi_p(x_2)dx_1dx_2,$$

where ψ_p is the p-th Hermite function defined in (4.1.2)*.*
Define also $f_p := \mathcal{F}_{x_1}^{-1}(\widetilde{f_p})$.

Given the form of the operator $(-\Delta + x_2^2 + 1)$, anisotropic regularity spaces will have to be defined. We are going to define two different kinds of "Hermite regularity spaces", depending on whether or not we give a global definition or a strongly anisotropic one: actually both will be related to each other.
The isotropic point of view consists in defining the regularity with the operator, in the same fashion that, for example, $\|f\|_{H^s} = \left\|(-\Delta)^{s/2}f\right\|_2$:

Definition 4.1.4. *For all integers N, for all f in $L^2(\mathbb{R}^2)$, we define the $\widetilde{\mathcal{H}}^N$ norm of f by*

$$\|f\|_{\widetilde{\mathcal{H}}^N} = \left\|(-\Delta + x_2^2 + 1)^N f\right\|_{L^2(\mathbb{R}^2)}.$$

However, given the way we are going to work (based on the study of the phase and on integration by parts), it will be useful to be able to study each direction separately. First of all the Hermite regularity in one direction is defined.

Definition 4.1.5. *Let M be an integer. Then the space at Hermite regularity M, written $\mathcal{H}^M(\mathbb{R})$, is defined by the following norm.*

$$\|f\|_{\mathcal{H}^M} := \left\|(-\partial_x^2 + x^2 + 1)^M f\right\|_{L^2(\mathbb{R})}. \tag{4.1.5}$$

This one-dimensional definition allows us to introduce the following space of functions defined on $\mathbb{R}^2$.

Definition 4.1.6. *Let N, M be two integers. The space $\mathcal{H}^M_{x_2} H^N_{x_1}$ (written $\mathcal{H}^M H^N$ in the rest of the paper) is defined on functions on $\mathbb{R}^2$ by the following norm*

$$\|f\|_{\mathcal{H}^M H^N} = \left\| \left(p^M \|f_p\|_{H^N_{x_1}}\right)_{p\in\mathbb{N}} \right\|_{\ell^2}.$$

Remark 4.1.7. *The order chosen to define $\mathcal{H}^M H^N$ is very important: in our proof we are going to start to work with a given Hermite mode, and then sum over all the Hermite modes.*

Finally, the following spaces will be defined so as to have lighter and simpler notations for time-dependent functions.

Definition 4.1.8. *Let M and N be two integers, t a non negative real number. Then the space B_t is defined for all g defined on $\mathbb{R}_+ \times \mathbb{R}$ by the norm:*

$$\|g(t)\|_{B_t} := \langle t\rangle^{-\frac{1}{2}} \|g(t)\|_{H^{\frac{3}{2}}(\langle x_1\rangle)}. \tag{4.1.6}$$

In order to simplify notations, we are going to write $\mathcal{B}(g)(s)$ for the following quantity:

$$\mathcal{B}(g)(s) := \sqrt{\|g(s)\|_{H^N} \|g(s)\|_{B_s}}. \tag{4.1.7}$$

The spaces $\mathcal{B}^M_t$ and $S^{M,N}_t$ are defined, for all function f defined on $\mathbb{R}_+ \times \mathbb{R}^2$, by the norms

$$\|f(t)\|_{\mathcal{B}^M_t} := \langle t\rangle^{-\frac{1}{2}} \|f(t)\|_{\mathcal{H}^M H^{\frac{3}{2}}(\langle x_1\rangle)}, \tag{4.1.8}$$

$$\|f(t)\|_{S^{M,N}_t} := \|f(t)\|_{\widetilde{\mathcal{H}}^N} + \|f(t)\|_{\mathcal{B}^M_t}. \tag{4.1.9}$$

Finally, the $S^{M,N}_t$ norm of a vector is defined as follows:

$$\left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\|_{S^{M,N}_t} = \|u\|_{S^{M,N}_t} + \|v\|_{S^{M,N}_t}. \tag{4.1.10}$$

The fixed-point space where we are going to work is then defined as follows: if $T > 0$, $\Sigma^{M,N}_T$ is defined by

$$\|f\|_{\Sigma^{M,N}_T} = \sup_{0\le t\le T} \|f(t)\|_{S^{M,N}_t}. \tag{4.1.11}$$

Remark 4.1.9. — *Since the eigenvalues of* $-\partial_x^2 + x^2 + 1$ *are* $(2n+2)_{n\in\mathbb{N}}$, *the following equivalence of norms holds*

$$\|f\|_{\mathcal{H}^M} \sim \left(\sum_{p\in\mathbb{N}} (2p+2)^{2M}|f_p|^2\right)^{\frac{1}{2}} \sim \left(\sum_{p\in\mathbb{N}} \underline{p}^{2M}|f_p|^2\right)^{\frac{1}{2}},$$

with $\underline{p} = \max(1,p)$.

— *We also have for all* f

$$\|f\|_{\mathcal{H}^{\frac{N}{2}}H^N} \leq \|f\|_{\widetilde{\mathcal{H}}^N} \leq \|f\|_{\mathcal{H}^N H^{2N}}.$$

— *if* $f \in S_t^{N,M}$, *then for all* $p \in \mathbb{N}$, *there exists* $(a_p(t))_{p\in\mathbb{N}}$, *such that*

$$\begin{aligned} \|f_p(t)\|_{H^N} &= \underline{p}^{-\frac{N}{2}} a_p(t)\, \|f(t)\|_{S_t^{N,M}}, \\ \|f_p(t)\|_{B_t} &= \underline{p}^{-M} a_p(t)\, \|f(t)\|_{S_t^{N,M}}, \end{aligned}$$

with $\|(a_p(t))_{p\in\mathbb{N}}\|_{\ell^2} \leq 1$.

4.1.3 Main result

The whole point of this chapter is to be able to have a long-time existence, that is to say an existence time of order ε^{-a} with $a > 1$, if ε is the size of the initial data. We are going to prove the following Theorem:

Theorem 4.1.10. *Let* $\delta > 0$, $\varepsilon > 0$. *Let*

$$T = C(\delta)\varepsilon^{-a}, \text{ with } a = \frac{4}{3(1+\delta)},$$

with $C(\delta)$ *depending on* δ *only.*
Then, given M *and* N *integers satisfying*

$$M > 3, \tag{4.1.12}$$
$$N > \frac{1}{\delta} + \frac{3}{2} + 2M, \tag{4.1.13}$$
$$\tag{4.1.14}$$

if (u_0, u_1) *satisfies*

$$\|u_0\|_{S_0^{N+1,M+1}} + \|u_1\|_{S_0^{N,M}} \leq \frac{\varepsilon}{2},$$

then there exists a unique solution u *in* $\Sigma_T^{M+1,N+1}$ *to* (4.1.1) *with*

$$\begin{aligned} \|u\|_{\Sigma_T^{M+1,N+1}} &\leq \varepsilon, \\ \|\partial_t u\|_{\Sigma_T^{M,N}} &\leq \varepsilon. \end{aligned} \tag{4.1.15}$$

This result comes from a fine study of resonance phenomena occurring in the equation and of the dispersive properties of the Klein-Gordon operator with a harmonic potential.

Remark 4.1.11. *One can remark that a similar result to Theorem 4.1.10 can be proven quite easily in the same fashion as Theorem 2.4.4 if instead of the space $S_t^{M,N}$ we take the space defined by*

$$\|f\|_{\widetilde{S}_t^{M,N}} := \|f\|_{\widetilde{\mathcal{H}}^N} + \frac{1}{\langle t\rangle^{\frac{1}{4}}} \|f\|_{\mathcal{H}^M W^{\frac{3}{2},1}}.$$

Proving this result relies on the dispersive estimate for the Klein-Gordon with a harmonic potential, i.e. a decay inequality for the L^∞ norm of the solution.
However it does not involve any study of resonances. It is really weaker than the result of Theorem 4.1.10, because it does not give any estimate on f in a weighted Sobolev norm. And having an estimate on weighted norms is fundamental when it comes to the study of the dynamics of the system, in particular when approximating it by a resonant system, as it is done in Chapter 5.

4.2 Strategy

Our goal is to rewrite Equation (4.1.1) as a fixed point problem

$$u = A(u),$$

where $A(u) = u_0 + \mathcal{A}(u)$. To determine $\mathcal{A}$, we establish a Duhamel formula for (4.1.1). Then we state the contraction estimates we want to prove in order to get Theorem 4.1.10. Finally we describe the space-time resonances of the system in order to give an idea of the strategy adopted in the following sections.

4.2.1 Duhamel formula

Studying the resonances relies on establishing a Duhamel formula for the equation, i.e. on writing (4.1.1) with an integral formulation. First of all we give a definition of the left and right-traveling parts adapted to (4.1.1):

Definition 4.2.1. *The* left-traveling *part of u (resp. the* right-traveling *part) denoted u_+ (resp. u_-) is defined by*

$$u_\pm := \partial_t u \pm i\left(-\Delta + x_2^2 + 1\right)^{1/2} u.$$

Remark 4.2.2. *It is important to remark that we have the following equivalence for all t:*

$$\Big((u(t), \partial_t u(t)) \in S_t^{N+1,M+1} \times S_t^{N,M}\Big) \Leftrightarrow \Big(u_\pm \in S_t^{N,M}\Big). \tag{4.2.1}$$

Thanks to Remark 4.2.2, we are going to reformulate the problem as follows:

Proposition 4.2.3. *A function u is a solution of* (4.1.1) *if and only if the profile $f = (f_+, f_-)$ satisfies*

$$f = A(f), \tag{4.2.2}$$

where $A(f) = f_0 + \mathcal{A}(f)$ and

$$\widetilde{\mathcal{A}(f)}_p(t,\xi) = \begin{pmatrix} \widetilde{\mathcal{A}(f)}_{+,p}(t,\xi) \\ \widetilde{\mathcal{A}(f)}_{-,p}(t,\xi) \end{pmatrix},$$

where

$$\widetilde{\mathcal{A}(f)}_{\pm,p}(t,\xi) = \sum_{m,n}\sum_{\alpha,\beta=\pm 1} \alpha\beta\mathcal{M}(n,m,p)\int_0^t\int_{\mathbb{R}} e^{\mp is\phi^{\alpha,\beta}_{m,n,p}}\frac{\widetilde{f}_{\alpha,m}(\eta)}{\langle\eta\rangle_m}\frac{\widetilde{f}_{\beta,n}(\xi-\eta)}{\langle\xi-\eta\rangle_n}d\eta ds, \quad (4.2.3)$$

with $\phi^{\alpha,\beta}_{m,n,p} = \langle\xi\rangle_p + \alpha\langle\eta\rangle_m + \beta\langle\xi-\eta\rangle_n$ *and* $\mathcal{M}$ *is the hermite functions interaction term* (4.1.4).

Remark 4.2.4. *Here, the frequency variables* ξ *or* η *have to be understood as* ξ_1 *or* η_1*, i.e. the frequency variable associated to the first space variable. We do not write this index in order to simplify notations.*
Moreover, for now on we are going to write $\widetilde{f}_{\alpha,m}(\eta)$ *instead of* $\widetilde{f}_{\alpha,m}(s,\eta)$ *when the dependence in* s *is obvious.*

Proof :
The left and the right traveling parts $u_\pm$ satisfy

$$\partial_t u_\pm \mp i\sqrt{-\Delta + x_2^2 + 1}\; u_\pm = -u^2. \quad (4.2.4)$$

Projected on the eigenfucntions $e^{i\xi x_1}\psi_n(x_2)$, the equation (4.2.4) becomes

$$\partial_t\widetilde{u}_{\pm,p}(t,\xi) \mp i\sqrt{\xi^2+2p+2}\;\widetilde{u}_{\pm,p}(t,\xi) = \widetilde{(u^2)}_p. \quad (4.2.5)$$

Now the profile $\widetilde{f}_{\pm,p}(t,\xi) = e^{\mp it\sqrt{\xi^2+2p+2}}\widetilde{u}_{\pm,p}(t,\xi)$ satisfies

$$\partial_t\widetilde{f}_{\pm,p}(t,\xi) = e^{\mp it\sqrt{\xi^2+2p+2}}\widetilde{(u^2)}_p(\xi). \quad (4.2.6)$$

Equation (4.2.6) can be rewritten using the integral form:

$$\widetilde{f}_{\pm,p}(t,\xi) = \widetilde{f}_{\pm,p}(0,\xi) + \int_0^t e^{\mp is\sqrt{\xi^2+2p+2}}\widetilde{(u^2)}_p(s,\xi)ds.$$

Now we study the $\widetilde{(u^2)}_p(s,\xi)$ so as to write it as a function of f.

$$\widetilde{(u^2)}_p(\xi) = \int_{\mathbb{R}}\int_{\mathbb{R}} u^2(t,x_1,x_2)e^{-ix_1\xi}\psi_p(x_2)dx_1dx_2.$$

We use the decomposition $u(x_1,x_2) = \sum_{m\in\mathbb{N}}\int_{\mathbb{R}}\widetilde{u}_m(\eta)e^{ix_1\eta}\psi_m(x_2)d\eta$, and Fubini's theorem to get:

$$\begin{aligned}
\widetilde{(u^2)}_p(\xi) &= \int_{\mathbb{R}}\int_{\mathbb{R}}\left(\sum_{m\in\mathbb{N}}\int_{\mathbb{R}}\widetilde{u}_m(\eta)e^{ix_1\eta}\psi_m(x_2)d\eta\right)^2 e^{-ix_1\xi}\psi_p(x_2)dx_1dx_2\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\sum_{m,n\in\mathbb{N}}\int_{\mathbb{R}}\widetilde{u}_m(\eta)e^{ix_1\eta}\psi_m(x_2)d\eta\int_{\mathbb{R}}\widetilde{u}_n(\zeta)e^{ix_1\zeta}\psi_n(x_2)d\zeta e^{-ix_1\xi}\psi_p(x_2)dx_1dx_2\\
&= \sum_{m,n}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\widetilde{u}_m(\eta)\widetilde{u}_p(\zeta)\int_{\mathbb{R}} e^{i(\eta+\zeta-\xi)x_1}dx_1\int_{\mathbb{R}}\psi_m(x_2)\psi_n(x_2)\psi_p(x_2)dx_2\right)d\eta d\zeta.
\end{aligned}$$

Then, since

$$\int_{\mathbb{R}} e^{i(\eta+\zeta-\xi)x_1}dx_1 = 2\pi\delta_{\xi-\eta-\zeta=0}, \quad (4.2.7)$$

we obtain

$$\widetilde{(u^2)}_p(\xi) = 2\pi \sum_{m,n} \int_{\mathbb{R}} \left(\widetilde{u}_m(\eta)\widetilde{u}_n(\xi-\eta) \int_{\mathbb{R}} \psi_m(x_2)\psi_n(x_2)\psi_p(x_2)dx_2 \right) d\eta$$
$$= 2\pi \sum_{m,n} \mathcal{M}(n,m,p) \int_{\mathbb{R}} \widetilde{u}_m(\eta)\widetilde{u}_n(\xi-\eta)d\eta.$$

Then,

$$\widetilde{u}_{+,m}(\eta) - \widetilde{u}_{-,m}(\eta) = 2i\sqrt{\eta^2+2m+2}\widetilde{u}_m.$$

Hence

$$\widetilde{(u^2)}_n(\xi) = 2\pi \sum_{m,p} \sum_{\alpha,\beta=\pm 1} \alpha\beta\mathcal{M}(n,m,p) \int_{\mathbb{R}} \frac{\widetilde{u}_{\alpha,m}(\eta)}{\langle\eta\rangle_m} \frac{\widetilde{u}_{\beta,p}(\xi-\eta)}{\langle\xi-\eta\rangle_p} d\eta. \tag{4.2.8}$$

Thanks to Equation (4.2.8), it is easy to get the formula for $\widetilde{f}_{\pm,p}$:

$$\widetilde{f}_{\pm,p}(t,\xi) = \widetilde{f}_{\pm,p}(0,\xi) + 2\pi \sum_{m,n} \sum_{\alpha,\beta=\pm 1} \alpha\beta\mathcal{M}(n,m,p) \int_0^t \int_{\mathbb{R}} e^{\mp is\phi^{\alpha,\beta}_{m,n,p}} \frac{\widetilde{f}_{\alpha,m}(\eta)}{\langle\eta\rangle_m} \frac{\widetilde{f}_{\beta,n}(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds,$$

where $\phi^{\alpha,\beta}_{m,n,p} = \langle\xi\rangle_p + \alpha\langle\eta\rangle_m + \beta\langle\xi-\eta\rangle_n$.
This ends the proof of Proposition 4.2.3. ■

4.2.2 Contraction estimates

As developed in Chapter 2, Section 2.2, it suffices to prove that the operator A is a contraction which maps the ball of radius ε for the $\Sigma_T^{M,N}$ norm into itself (for well-chosen M, N and T). More precisely, in order to get the existence time $T = \varepsilon^{-\frac{4}{3(1+\delta)}}$, we are going to prove the following theorem.

Theorem 4.2.5. *For all $\omega > 0$, for all M and N satisfying* (4.1.12) – (4.1.13)*, we have the following inequality:*

$$\|\mathcal{A}(f)\|_{S_t^{M,N}} \lesssim_\omega \int_0^t s^{-\frac{1}{4}+\omega} \|f\|^2_{S_s^{M,N}} + s^{\frac{1}{2}+\omega} \|f\|^3_{S_s^{M,N}} ds + C(t), \tag{4.2.9}$$

with $C(t) = \langle t\rangle^{\omega+\frac{1}{4}} \left(\|f(t)\|^2_{S_t^{M,N}} + \|f(1)\|^2_{S_1^{M,N}} \right)$.

Remark 4.2.6. *We can make two remarks on this property:*
— *the following inequality is a direct consequence of* (4.2.9)*:*

$$\|A(f)\|_{\Sigma_T^{M,N}} \lesssim_\omega \max\left(T^{\frac{3}{4}+\omega} \|f\|^2_{\Sigma_T^{M,N}}, T^{\frac{3}{2}+\omega} \|f\|^3_{\Sigma_T^{M,N}} \right). \tag{4.2.10}$$

with the same notations and hypothesis as in Theorem 4.2.5.
— *Going from Theorem 4.2.5 to Theorem 4.1.10 is then quite straightforward: the inequality* (4.2.10) *gives an existence time equal to (up to a constant):*

$$\min\left(\varepsilon^{\frac{1}{\frac{3}{4}+\omega}}, \varepsilon^{2.\frac{1}{\frac{3}{2}+\omega}} \right) = \min(\varepsilon^{-\frac{4}{3+4\omega}}, \varepsilon^{-\frac{4}{3+2\omega}}) = \varepsilon^{-\frac{4}{3+4\omega}}.$$

Then taking $\omega = \frac{3\delta}{4}$ gives the result.

For now on, our goal will be to prove Theorem 4.2.5.

4.2.3 Space-time resonances

In order to obtain a large existence time, we are going to use the method of *space-time resonances* as described in Chapter 2, Section 2.5. Here the phase to study is

$$\phi^{\alpha,\beta}_{m,n,p}(\xi,\eta) = \sqrt{\xi^2+2p+2} + \alpha\sqrt{\eta^2+2m+2} + \beta\sqrt{(\xi-\eta)^2+2n+2}. \tag{4.2.11}$$

The study of the resonances and of the resonant sets is done in Appendix B: here we only state the main theorem.

Theorem 4.2.7. *Let α and β be two elements of $\{-1,1\}$. Consider the phase $\phi(\xi,\eta) = \phi^{\alpha,\beta}_{m,n,p}(\xi,\eta)$ as defined in 4.2.11. Then the space resonant set is*

$$\mathcal{S} = \left\{ \left(\left(1 + \frac{\beta}{\alpha}\sqrt{\frac{2n+2}{2m+2}} \right)\eta, \eta \right), \eta \in \mathbb{R} \right\}.$$

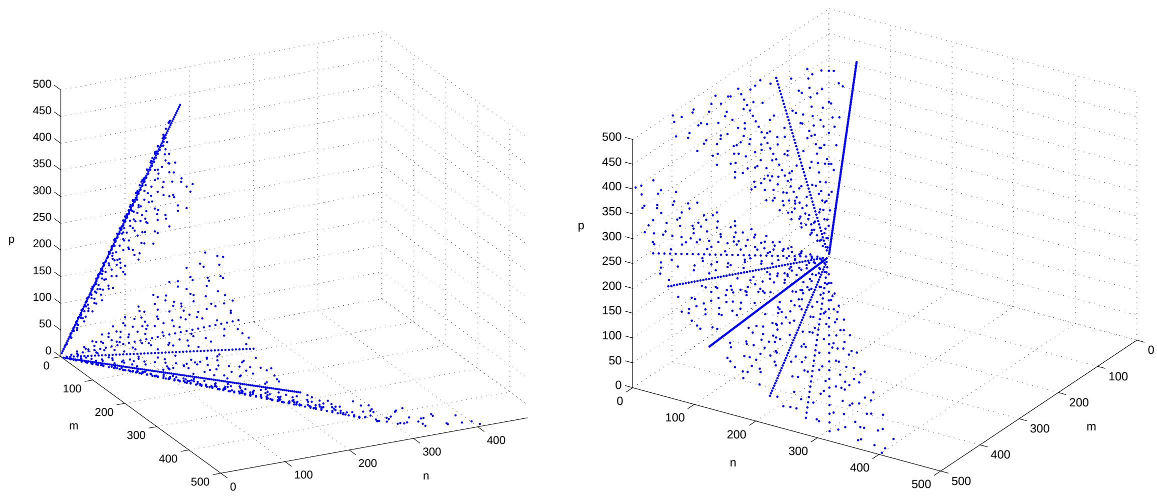
1. *In the case $(\alpha,\beta) = (1,1)$, there are no time resonances: $\mathcal{T} = \emptyset$.*
2. *Otherwise,*
   (a) *If $\alpha\beta p + \beta m < 0$ or $\alpha\beta p + \beta n < 0$, there are no time resonances.*
   (b) *If $\alpha\beta p + \beta m \geq 0$ and $\alpha\beta p + \beta n \geq 0$, there are space-time resonances if and only if the following condition is satisfied.*

$$\begin{cases} \alpha\beta p + \beta m + \alpha n \geq 0, \\ m^2+n^2+p^2-2mn-2pm-2pn-2m-2n-2p-3=0. \end{cases} \tag{C}$$

In that case, the space coresonant set is equal to the space resonant set: $\widetilde{\mathcal{S}} = \mathcal{S}$.

Hence we are in a rather new situation, compared to the situations described in Chapter 2, Section 2.6.3.
In our situation, like in the nonlinear Schrödinger equation with a $|u|^2$ nonlinearity, the space-time resonant set is a line; one more difficulty is that this line depends on the input and output Hermite modes. This kind of problem is a really new situation which will require a fine adaptation of the Germain-Masmoudi-Shatah method.

Remark 4.2.8. *Here is how the integers satisfying the condition $C_{\alpha,\beta}$ distribute (in the case $\alpha=\beta=1$: they all are on a surface of degree 2 but, more interesting, they seem to be uniformly distributed. Moreover, this surface looks like a cone: when m, n and p are large, the resonant condition reduces to its quadratic part.*



In order to study separately space and time resonant sets, a precise understanding of the phase and its derivatives is necessary: this work is done in Appendix B.
This study being done, we introduce the following functions, used in order to explicitly split the (ξ,η)-frequency space into several zones:

Definition 4.2.9. *We define χ to be a smooth function, homogeneous of degree 0 such that $\chi(\eta,\xi-\eta)=0$ for $|\xi-\eta|\leq 2|\eta|$.*
The function θ is defined as a smooth function supported on $[0,2]$, equal to 1 on $[0,1]$. For all $R>0$ we define $\theta_R(x)=\theta(\langle R\rangle x)$.
Moreover we impose χ and θ to satisfy Coifman-Meyer's theorem's hypotheses.

Remark 4.2.10. *The function χ will allow us to make a* paraproduct decomposition *adapted to the convolution. The function θ_R will be introduced in order deal separately with high and low frequencies, and the parameter R may be time-dependent.*

4.3 High regularity results

We start with the high-regularity contraction estimate. Our aim is to prove the following Proposition.

Proposition 4.3.1. *The operator $\mathcal{B}$ defined as*

$$\mathcal{B}(f):=\int_0^t e^{is\sqrt{-\Delta+x_2^2+1}}\sum_{\alpha,\beta=\pm1}\frac{e^{-\alpha is\sqrt{-\Delta+x_2^2+1}}f_\alpha}{\sqrt{-\Delta+x_2^2+1}}\frac{e^{-\beta is\sqrt{-\Delta+x_2^2+1}}f_\beta}{\sqrt{-\Delta+x_2^2+1}}ds. \tag{4.3.1}$$

satisfies, for all f, for all M satisfiying he condition (4.1.12) *and N satisfying* (4.1.13).

$$\|\mathcal{B}(f)\|_{\widetilde{H}^N}\lesssim\int_0^t s^{-\frac14}\,\|f\|^2_{S_s^{M,N}}\,ds.$$

Proving this inequality is a simple Hölder inequality $L^2\times L^\infty\to L^2$ combined with the global dispersion estimate (Proposition A.1.3) and the following product law (proven in [15]):

Proposition 4.3.2. *(product law for the space $\widetilde{\mathcal{H}}^N$) Let f and g be two elements of $\widetilde{\mathcal{H}}^N\cap L^\infty$, and $N>1/2$. Then*

$$\|fg\|_{\widetilde{\mathcal{H}}^N}\lesssim\|f\|_{\widetilde{\mathcal{H}}^N}\|g\|_{L^\infty}+\|f\|_{L^\infty}\|g\|_{\widetilde{\mathcal{H}}^N}.$$

Proof of Proposition 4.3.1 :
We have to bound the $\widetilde{\mathcal{H}}^N$ norm of the bilinear term $\mathcal{B}$ defined in (4.3.1). Since $e^{is\sqrt{-\Delta+x_2^2+1}}$ is $\widetilde{\mathcal{H}}^N$ continuous,

$$\|\mathcal{B}(f)\|_{\widetilde{\mathcal{H}}^N}\lesssim\int_0^t\sum_{\alpha,\beta=\pm1}\left\|\frac{e^{-\alpha is\sqrt{-\Delta+x_2^2+1}}f_\alpha}{\sqrt{-\Delta+x_2^2+1}}\frac{e^{-\beta is\sqrt{-\Delta+x_2^2+1}}f_\beta}{\sqrt{-\Delta+x_2^2+1}}\right\|_{\widetilde{\mathcal{H}}^N}ds.$$

Then, by the product law for $\widetilde{\mathcal{H}}^N$ (Proposition 4.3.2),

$$\begin{aligned}\|\mathcal{B}(f)\|_{\widetilde{\mathcal{H}}^N}\lesssim\int_0^t\sum_{\alpha,\beta=\pm1}&\left\|\frac{e^{-\alpha is\sqrt{-\Delta+x_2^2+1}}f_\alpha}{\sqrt{-\Delta+x_2^2+1}}\right\|_{\widetilde{\mathcal{H}}^N}\left\|\frac{e^{-\beta is\sqrt{-\Delta+x_2^2+1}}f_\beta}{\sqrt{-\Delta+x_2^2+1}}\right\|_{L^\infty(\mathbb{R}^2)}\\&+\left\|\frac{e^{-\alpha is\sqrt{-\Delta+x_2^2+1}}f_\alpha}{\sqrt{-\Delta+x_2^2+1}}\right\|_{L^\infty(\mathbb{R}^2)}\left\|\frac{e^{-\beta is\sqrt{-\Delta+x_2^2+1}}f_\beta}{\sqrt{-\Delta+x_2^2+1}}\right\|_{\widetilde{\mathcal{H}}^N}ds.\end{aligned}$$

Because of the $\widetilde{\mathcal{H}}^N$ continuity of $e^{is\sqrt{-\Delta+x_2^2+1}}$ and the $S_T^{N,M}$-continuity of $\frac{1}{\sqrt{-\Delta+x_2^2}}$ (Proposition A.1.4-b),

$$\left\| \frac{e^{\mp is\sqrt{-\Delta+x_2^2+1}} f_\pm}{\sqrt{-\Delta+x_2^2+1}} \right\|_{\widetilde{\mathcal{H}}^N} \lesssim \|f_\pm\|_{S_s^{M,N}}. \tag{4.3.2}$$

By the global dispersion inequality (Proposition A.1.3) and the $S_T^{N,M}$ continuity of $\frac{1}{\sqrt{-\Delta+x_2^2+1}}$ (Proposition A.1.4-b),

$$\left\| \frac{e^{-\beta is\sqrt{-\Delta+x_2^2+1}} f_\beta}{\sqrt{-\Delta+x_2^2+1}} \right\|_{L^\infty(\mathbb{R}^2)} \lesssim s^{-\frac{1}{4}} \|f\|_{S_s^{M,N}}. \tag{4.3.3}$$

Finally, gathering (4.3.2) and (4.3.3) leads to

$$\|\mathcal{B}(f)\|_{\widetilde{\mathcal{H}}^N} \lesssim \int_0^t s^{-\frac{1}{4}} \|f\|^2_{S_s^{M,N}}\, ds,$$

and Proposition 4.3.1 is proved. ■

4.4 General strategy for the $\mathcal{H}^M_{x_2} H^{\frac{3}{2}}_{x_1}(x_1)$ norm

We now focus on the weighted norm and we want to prove the following proposition:

Proposition 4.4.1. *If U is the bilinear operator defined as*

$$\mathcal{F}(U_{m,n}(a,b)) := \int_0^t \int \partial_\xi \left(e^{-is\phi} \frac{\hat{a}(\eta)}{\langle \eta \rangle_m} \frac{\hat{b}(\xi-\eta)}{\langle \xi-\eta \rangle_n} \right) d\eta ds$$

then for all $\delta > 0$, for all M, N satisfying the conditions $(4.1.12) - (4.1.13)$, there exists a constant $K(\delta)$ such that

$$\begin{aligned} &\frac{1}{\sqrt{t}} p^M \sum_{m,n} \mathcal{M}(m,n,p) \|U_{m,n}(f_n, f_m)\|_{H^{3/2}} \\ &\leq K(\delta) \Big(\int_0^t \left(s^{4\delta} s^{-\frac{1}{4}} \|f\|^2_{S_s^{M,N}} a_p(s) + s^{3\delta} s^{\frac{1}{2}} \|f\|^3_{S_s^{M,N}} b_p(s) \right) ds + C(t) c_p(t) \Big), \end{aligned} \tag{4.4.1}$$

with $C(t) = \langle t \rangle^{\frac{5\delta}{2}+\frac{1}{4}} \left(\|f(t)\|^2_{S_s^{M,N}} + \|f(1)\|^2_{S_1^{M,N}} \right)$ and where $(a_p(s))_{p\in\mathbb{N}}$, $(b_p(s))_{p\in\mathbb{N}}$ and $(c_p(t))_{p\in\mathbb{N}}$ are ℓ^2 sequences of norm bounded by 1.

Remark 4.4.2. *Taking the ℓ^2 norm (in p) of (4.4.1) allows us to write*

$$\left\| \sum_{m,n} \mathcal{M}(m,n,p) \frac{1}{\sqrt{t}} U_{m,n}(f_n, f_m) \right\|_{\mathcal{B}_t^M} \leq K(\delta) \left(\int_0^t \left(s^{4\delta} s^{-\frac{1}{4}} \|f\|^2_{S_s^{M,N}} + s^{3\delta} s^{\frac{1}{2}} \|f\|^3_{S_s^{M,N}} \right) ds + C(t) \right),$$

which corresponds to the inequality in Theorem 4.2.5.

Proof of Proposition 4.4.1 :
We know that

$$\frac{1}{\sqrt{t}}\left\|U(f_n,f_m)\right\|_{\dot{H}^{3/2}}=\frac{1}{\sqrt{t}}\left\||\xi|^{\frac{3}{2}}\int_0^t\int\partial_\xi\left(e^{-is\phi}\frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m}\frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n}\right)d\eta ds\right\|_{L^2_\xi}.$$

By the Leibniz rule the integral can be rewritten as follows:

$$|\xi|^{\frac{3}{2}}\int_0^t\int\partial_\xi\left(e^{-is\phi}\frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m}\frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n}\right)d\eta ds=I_{m,n}+J_{m,n}+K_{m,n},$$

where

$$\begin{aligned}I_{m,n}&:=|\xi|^{\frac{3}{2}}\int_0^t\int -is\partial_\xi\phi e^{-is\phi}\frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m}\frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n}d\eta ds,\\ J_{m,n}&:=|\xi|^{\frac{3}{2}}\int_0^t\int e^{-is\phi}\frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m}\frac{\partial_\xi\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n}d\eta ds,\\ K_{m,n}&:=|\xi|^{\frac{3}{2}}\int_0^t\int e^{-is\phi}\frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m}\frac{\hat{f}_n(\xi-\eta)(-\xi)}{((\xi-\eta)^2+2n+2)^{3/2}}d\eta ds.\end{aligned}$$

The integral term $J_{m,n}$ will be treated in Section 4.4.1 (Proposition 4.4.3) ; $K_{m,n}$ will be dealt with in Section 4.4.2 (Proposition 4.4.5). Estimating the integral term $I_{m,n}$ will be harder and explained in Section 4.4.3.
Although the estimates for $J_{m,n}$ and $K_{m,n}$ might not seem sharp in view of their proof, they fit in the ones for the $I_{m,n}$ term, which is the harder one to deal with.

4.4.1 Estimates for $J_{m,n}$

We are going to prove the following inequality:

Proposition 4.4.3. *There exists $(a_p(s))_{p\in\mathbb{N}}$ in the unit ball of ℓ^2 such that*

$$\frac{1}{\sqrt{t}}p^M\sum_{m,n\in\mathbb{N}}\mathcal{M}(m,n,p)\left\|J_{m,n}\right\|_{L^2}\lesssim\int_0^t\langle s\rangle^{-\frac{1}{4}+\frac{3\delta}{2}}\left\|f\right\|^2_{S^{M,N}_s}a_p ds,$$

for all N and M satisfying (4.1.12)-(4.1.13).

We are going to proceed in two steps: first, we establish n and m-dependent bounds on $\|J_{m,n}\|_{L^2}$; then it remains to sum these bounds.

4.4.1.a Bounds for $\|J_{m,n}\|_{L^2}$

Lemma 4.4.4. *The following bound holds.*

$$\frac{1}{\sqrt{t}}\left\|J_{m,n}\right\|_{L^2}\lesssim_{m\leftrightarrow n}\int_0^t\langle s\rangle^{-\frac{1}{4}+\frac{3\delta}{2}}\frac{\max(\underline{m},\underline{n})^{\frac{1}{4}}}{\sqrt{\underline{mn}}}\Big(\mathcal{B}(f_m)(s)+\left\|f_m(s)\right\|_{H^N}\Big)\left\|f_n(s)\right\|_{B_s}ds, \tag{4.4.2}$$

with $\lesssim_{m\leftrightarrow n}$ defined in (2.7.1) and $\mathcal{B}(f_m)(s)$ defined in (4.1.7).

Proof :
Here we will perform two cutoffs:

1. the paraproduct decomposition, using the function χ, equal to 0 for $|\xi-\eta| \leq 2|\eta|$.
2. high-low frequency cut-off: $\theta_{s^\delta}(|\eta|)$ is the smooth function localizing in the zone $|\eta| \leq s^\delta$.

Write the L^2 norm of J as follows.

$$\begin{aligned}\|J_{m,n}\|_{L^2} \leq & \left\| |\xi|^{\frac{3}{2}} \int_0^t \int (1-\chi(\eta,\xi-\eta)) e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\partial_\xi \hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds \right\|_{L^2_\xi} \\ & + \left\| |\xi|^{\frac{3}{2}} \int_0^t \int \chi(\eta,\xi-\eta)\theta_{s^\delta}(|\eta^2|) e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\partial_\xi \hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds \right\|_{L^2_\xi} \\ & + \left\| |\xi|^{\frac{3}{2}} \int_0^t \int \chi(\eta,\xi-\eta)(1-\theta_{s^\delta}(|\eta^2|)) e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\partial_\xi \hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds \right\|_{L^2_\xi} \\ & = \|J_{1-\chi}\|_{L^2} + \|J_{\chi,l}\|_{L^2} + \|J_{\chi,h}\|_{L^2}.\end{aligned}$$

We are going to estimate separately each of those three terms.

— **Term** $J_{1-\chi}$

We can rewrite $J_{1-\chi}$ as

$$J_{1-\chi} = \int_0^t \int \left(\frac{|\xi|}{|\xi-\eta|}\right)^{\frac{3}{2}} (1-\chi(\eta,\xi-\eta)) e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{|\xi-\eta|^{\frac{3}{2}} \partial_\xi \hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds,$$

which allows us to write it as a bilinear Fourier operator:

$$\mathcal{F}^{-1}(J_{1-\chi}) = \int_0^t e^{is\langle D\rangle_p} T_{m^J_{1-\chi}} \left(e^{-is\langle D\rangle_m} \frac{f_m}{\langle D\rangle_m}, |D|^{\frac{3}{2}} e^{-is\langle D\rangle_n} \frac{x_1 f_n}{\langle D\rangle_n} \right) ds,$$

with $T_{m^J_{1-\chi}}$ the bilinear Fourier operator (defined in (2.7.2)) associated to the multiplier

$$m^J_{1-\chi}(\eta,\zeta) := \left|\frac{\eta+\zeta}{\zeta}\right|^{\frac{3}{2}} (1-\chi(\eta,\zeta)),$$

which satisfies Hölder-like inequalities thanks to Proposition A.3.3.
The differential operator $e^{is\langle D\rangle}$ is continuous from L^2 to L^2. Then

$$\|J_{1-\chi}\|_{L^2} \lesssim \int_0^t \left\| T_{m^J_{1-\chi}} \left(e^{-is\langle D\rangle_m} \frac{f_m}{\langle D\rangle_m}, |D|^{\frac{3}{2}} e^{-is\langle D\rangle_n} \frac{x_1 f_n}{\langle D\rangle_n} \right) \right\|_{L^2} ds.$$

Proposition A.3.3 implies

$$\|J_{1-\chi}\|_{L^2} \lesssim \int_0^t \left\| e^{-is\langle D\rangle_m} \frac{f_m}{\langle D\rangle_m} \right\|_{L^\infty} \left\| |D|^{\frac{3}{2}} e^{-is\langle D\rangle_n} \frac{x_1 f_n}{\langle D\rangle_n} \right\|_{L^2} ds. \tag{4.4.3}$$

Proposition A.1.6 implies that

$$\left\| e^{-is\langle D\rangle_m} \frac{f_m}{\langle D\rangle_m} \right\|_{L^\infty} \lesssim m^{-\frac{1}{4}} \langle s\rangle^{-\frac{1}{4}} \sqrt{\|f_m\|_{H^N} \|f_m\|_{B_s}}. \tag{4.4.4}$$

Thus Proposition A.1.4, inequality (A.1.4-a), gives

$$\left\| |D|^{\frac{3}{2}} e^{-is\langle D\rangle_n} \frac{x_1 f_n}{\langle D\rangle_n} \right\|_{L^2} \lesssim n^{-\frac{1}{2}} \left\| |D|^{\frac{3}{2}} e^{-is\langle D\rangle_n} x_1 f_n \right\|_{L^2} \lesssim n^{-\frac{1}{2}} \|f_n\|_{B_s}. \tag{4.4.5}$$

The two inequalities (4.4.4) and (4.4.5) allow us to rewrite (4.4.3) as

$$\|J_{1-\chi}\|_{L^2} \lesssim m^{-\frac{1}{4}} n^{-\frac{1}{2}} \int_0^t \langle s\rangle^{-\frac{1}{4}} \sqrt{\|f_m\|_{H^N} \|f_m\|_{B_s}} \|f_n\|_{B_s} ds. \tag{4.4.6}$$

— **Term** $J_{\chi,l}$
First of all since we are in the zone $\{\sqrt{|\xi|^2+|\eta|^2} \leq \langle s\rangle^\delta\} \cap \{|\xi-\eta| \leq |\eta|\}$, we can bound $|\xi|^{\frac{3}{2}}$ by $\langle s\rangle^{\frac{3\delta}{2}}$ (up to a constant). This gives

$$\begin{aligned}\|J_{\chi,l}\|_{L^2} &\lesssim \left\| \int_0^t \int \langle s\rangle^{\frac{3\delta}{2}} \chi(\eta,\xi-\eta)\theta_{s^\delta}(|\eta^2|) e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\partial_\xi \hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds \right\|_{L^2_\xi} \\ &\lesssim \int_0^t \langle s\rangle^{\frac{3\delta}{2}} \left\| T_{m^J_{\chi,l}} \left(e^{-is\langle D\rangle_m} \frac{f_m}{\langle D\rangle_m}, e^{-is\langle D\rangle_n} \frac{x_1 f_n}{\langle D\rangle_n} \right) \right\|_{L^2} ds,\end{aligned}$$

where $m^J_{\chi,l}(\xi,\eta) := \chi(\eta,\xi-\eta)\theta_{s^\delta}(|\eta^2|)$. Thanks to Coifman-Meyer estimates (Theorem A.3.1) we can write

$$\|J_{\chi,l}\|_{L^2} \lesssim \int_0^t \langle s\rangle^{\frac{3\delta}{2}} \left\| e^{-is\langle D\rangle_m} \frac{f_m}{\langle D\rangle_m} \right\|_{L^\infty} \left\| e^{-is\langle D\rangle_n} \frac{x_1 f_n}{\langle D\rangle_n} \right\|_{L^2} ds.$$

The conclusion is the same as previously: we find

$$\|J_{\chi,l}\|_{L^2} \lesssim \int_0^t \langle s\rangle^{\frac{3\delta}{2}-\frac{1}{4}} m^{-\frac{1}{4}} n^{-\frac{1}{2}} \|f_m\|_{B_s} \sqrt{\|f_n\|_{H^N} \|f_n\|_{B_s}} ds. \tag{4.4.7}$$

— **Term** $J_{\chi,h}$
Inequality (A.1.4-e) will be crucial to deal with high frequencies. Firstly, $J_{\chi,h}$ can be rewritten as

$$\begin{aligned}\|J_{\chi,h}\|_{L^2} &= \left\| \int_1^t \int \left|\frac{\xi}{\eta}\right|^{\frac{3}{2}} \chi(\eta,\xi-\eta)(1-\theta_{s^\delta}(|\eta^2|))|\eta|^{\frac{3}{2}} e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\partial_\xi \hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds \right\|_{L^2_\xi} \\ &\lesssim \int_0^t \left\| T_{m^J_{\chi,h}} \left((1-\theta_{s^\delta}(|D|))|D|^{\frac{3}{2}} e^{-is\langle D\rangle_m} \frac{f_m}{\langle D\rangle_m}, e^{-is\langle D\rangle_n} \frac{x_1 f_n}{\langle D\rangle_n} \right) \right\|_{L^2} ds,\end{aligned}$$

with $m^J_{\chi,h}(\eta,\zeta) := \left|\frac{\xi}{\eta}\right|^{\frac{3}{2}} \chi(\eta,\xi-\eta)(1-\theta_{s^\delta}(|\eta^2|))$. Then Theorem A.3.1 combined with Proposition A.3.3 give

$$\|J_{\chi,h}\|_{L^2} \lesssim \int_0^t \left\| (1-\theta_{s^\delta}(|D|))|D|^{\frac{3}{2}} e^{-is\langle D\rangle_m} \frac{f_m}{\langle D\rangle_m} \right\|_{L^\infty} \left\| e^{-is\langle D\rangle_n} \frac{x_1 f_n}{\langle D\rangle_n} \right\|_{L^2} ds. \tag{4.4.8}$$

First, by the multiplier estimate (A.1.4-a),

$$\left\| e^{-is\langle D\rangle_n} \frac{x_1 f_n}{\langle D\rangle_n} \right\|_{L^2} \lesssim n^{-\frac{1}{2}} \|f_n(s)\|_{B_s}. \tag{4.4.9}$$

Then, the Sobolev embedding $\|u\|_{L^\infty} \lesssim \|u\|_{H^N}$ for $N > 1/2$ – actually it is enough to use $\|u\|_{L^\infty} \lesssim \|u\|_{H^1}$ – allows us to deal with the other factor in (4.4.8):

$$\begin{aligned}\left\| (1-\theta_{s^\delta}(|D|))|D|^{\frac{3}{2}} e^{-is\langle D\rangle_m} \frac{f_m}{\langle D\rangle_m} \right\|_{L^\infty} &\lesssim \left\| (1-\theta_{s^\delta}(|D|))|D|^{\frac{5}{2}} e^{-is\langle D\rangle_m} \frac{f_m}{\langle D\rangle_m} \right\|_{L^2} \\ &\lesssim m^{-\frac{1}{2}} \left\| (1-\theta_R(D))|D|^{\frac{5}{2}} f_m \right\|_{L^2}.\end{aligned}$$

Thanks to Proposition A.1.4, inequality (A.1.4-e),

$$\left\|(1-\theta_{s^\delta}(|D|))|D|^{\frac{3}{2}}e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}\right\|_{L^\infty} \lesssim \frac{1}{\sqrt{m}}\langle s\rangle^{-\left(N-\frac{5}{2}\right)\delta}\left\|f_m\right\|_{H^N}. \tag{4.4.10}$$

Finally, using (4.4.9) and (4.4.10) in (4.4.8) gives

$$\begin{aligned}\frac{1}{\sqrt{t}}\left\|J_{\chi,h}\right\|_{L^2} &\lesssim \int_0^t \frac{1}{\sqrt{\underline{mn}}}\frac{1}{\langle s\rangle^{\delta\left(N-\frac{3}{2}\right)}}\left\|f_m\right\|_{H^N}\left\|f_n\right\|_{B_s} ds \\ &\lesssim \frac{1}{\sqrt{\underline{mn}}}\int_0^t \langle s\rangle^{-\frac{1}{4}}\left\|f_m\right\|_{H^N}\left\|f_n\right\|_{B_s} ds,\end{aligned}$$

as soon as $N \geq \frac{3}{2}+\frac{1}{4\delta}$ (which is true because of Hypothesis (4.1.13)).
Combining Inequalities (4.4.6), (4.4.7) and (4.4.8) concludes the proof of Lemma 4.4.4.■

4.4.1.b Summation

Now that Lemma 4.4.4 is proven, going back to Proposition 4.4.3 reduces to summing over the indices m and n, i.e. to proving the following result:

$$\underline{p}^M\sum_{m,n}\mathcal{M}(m,n,p)\frac{\max(\underline{m},\underline{n})^{\frac{1}{4}}}{\sqrt{\underline{mn}}}\Big(\mathcal{B}(f_m)(s)+\left\|f_m(s)\right\|_{H^N}\Big)\left\|f_n(s)\right\|_{B_s} \lesssim \left\|f(s)\right\|^2_{S_s^{M,N}} a_p(s), \tag{4.4.11}$$

with $\|(a_p(s))_{p\in\mathbb{N}}\|_{\ell^2} \leq 1$.

Proof :
First of all, for all integers p,

$$\left\|f_p(s)\right\|_{B_s} = \left\|f(s)\right\|_{S_s^{M,N}}\underline{p}^{-M}a_p(s),$$

with $(a_p(s))_{p\in\mathbb{N}}$ in the unit ball of ℓ^2. Similarly, there exists $(b_p(s))_{p\in\mathbb{N}}$ in the unit ball of ℓ^2 such that for all $p\in\mathbb{N}$,

$$\begin{aligned}\left\|f_p(s)\right\|_{H^N} &= \underline{p}^{-\frac{N}{2}}\left\|f(s)\right\|_{S_s^{M,N}} b_p(s) \\ &\leq \underline{p}^{-M}\left\|f(s)\right\|_{S_s^{M,N}} b_p(s),\end{aligned}$$

since $N \geq 2M$ thanks to condition (4.1.13). Finally, using that if $(a_p(s))_{p\in\mathbb{N}}, (b_p(s))_{p\in\mathbb{N}} \in \ell^2$, then $(\sqrt{a_pb_p(s)})_{p\in\mathbb{N}}$ is in ℓ^2, we can bound each of the factors

$$\begin{aligned}&\mathcal{B}(f_m)(s)\left\|f_n(s)\right\|_{B_s}, \ \left\|f_m(s)\right\|_{H^N}\left\|f_n(s)\right\|_{B_s}, \\ &\mathcal{B}(f_n)(s)\left\|f_m(s)\right\|_{B_s}, \ \left\|f_m(s)\right\|_{B_s}\left\|f_n(s)\right\|_{H^N},\end{aligned}$$

by

$$\underline{m}^{-M}\mu_m(s)\underline{n}^{-M}\eta_n(s)\left\|f\right\|^2_{S_s^{M,N}},$$

with $(\mu_m(s))_{m\in\mathbb{N}}$ and $(\eta_n(s))_{n\in\mathbb{N}}$ in the unit ball of ℓ^2. Then we can bound

$$\Big(\mathcal{B}(f_m)(s) + \|f_m(s)\|_{H^N}\Big)\|f_n(s)\|_{B_s} \lesssim \underline{m}^{-M}\alpha_m(s)\underline{n}^{-M}\beta_n(s)\|f(s)\|^2_{S^{M,N}_s},$$

with $(\alpha_m(s))_{m\in\mathbb{N}}$ and $(\beta_n(s))_{n\in\mathbb{N}}$ in the unit ball of ℓ^2. This inequality implies

$$\begin{aligned}&\underline{p}^M\sum_{m,n}\mathcal{M}(m,n,p)\frac{\max(\underline{m},\underline{n})^{\frac{1}{4}}}{\sqrt{\underline{mn}}}\Big(\mathcal{B}(f_m)(s) + \|f_m(s)\|_{H^N}\Big)\|f_n(s)\|_{B_s}\\ &\lesssim \|f(s)\|^2_{S^{N,M}_s}\underline{p}^M\sum_{m,n}\mathcal{M}(m,n,p)\frac{\max(\underline{m},\underline{n})^{\frac{1}{4}}}{\sqrt{\underline{mn}}}\underline{m}^{-M}\alpha_m(s)\underline{n}^{-N}\beta_n(s).\end{aligned}$$

Then we are in the framework of the resummation theorem C.1.1-1, with $a=\frac{1}{4}$, hence

$$\begin{aligned}&\underline{p}^M\sum_{m,n}\mathcal{M}(m,n,p)\frac{\max(\underline{m},\underline{n})^{\frac{1}{4}}}{\sqrt{\underline{mn}}}\Big(\mathcal{B}(f_m)(s) + \|f_m(s)\|_{H^N}\Big)\|f_n(s)\|_{B_s}\\ &\lesssim C_\gamma\|f\|^2_{S^{N,M}_s}\underline{p}^{-\frac{1}{2}+\gamma}a_p(s) \qquad (4.4.12)\\ &\lesssim \|f\|^2_{S^{M,N}_s}a_p(s), \qquad (4.4.13)\end{aligned}$$

with $\|(a_p(s))_{p\in\mathbb{N}}\|_{\ell^2}\leq 1$. Combining (4.4.12) and the integral inequality (4.4.2) ends the proof of Proposition 4.4.3. ■

4.4.2 Estimates for $K_{m,n}$

Here the property we need to prove is pretty similar than the one for J, but the methods are not exactly the same.

Proposition 4.4.5. *There exists $(a_p(s))_{p\in\mathbb{N}}$ in ℓ^2 such that*

$$\frac{1}{\sqrt{t}}\underline{p}^M\sum_{m,n\in\mathbb{N}}\mathcal{M}(m,n,p)\|K_{m,n}\|_{L^2}\lesssim\int_0^t s^{-\frac{1}{4}}\|f\|^2_{S^{M,N}_s}u_p(s)ds.$$

Proof :
We focus on proving the following lemma:

Lemma 4.4.6. *The following bound holds for all integers m and n, for all time $t\in\mathbb{R}_+$ and for all M, N satisfying (4.1.12)-(4.1.13):*

$$\frac{1}{\sqrt{t}}\|K_{m,n}\|_{L^2}\lesssim_{m\leftrightarrow n}\int_0^t s^{-\frac{1}{4}}\frac{\max(\underline{m},\underline{n})^{\frac{1}{4}}}{\sqrt{\underline{mn}}}\Big(\mathcal{B}(f_m)(s) + \|f_m(s)\|_{H^N}\Big)\|f_n(s)\|_{B_s}ds.$$

Proof of Lemma 4.4.6 :
We are going to proceed in a similar and simpler way. Write $K=K_\chi+K_{1-\chi}$, where

$$\begin{aligned}K_\chi &:= |\xi|^{\frac{3}{2}}\int_0^t\int\chi(\eta,\xi-\eta)e^{-is\phi}\frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m}\frac{\hat{f}_n(\xi-\eta)\times(-\xi)}{((\xi-\eta)^2+2n+2)^{3/2}}d\eta ds\\ K_{1-\chi} &:= |\xi|^{\frac{3}{2}}\int_0^t\int(1-\chi(\eta,\xi-\eta))e^{-is\phi}\frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m}\frac{\hat{f}_n(\xi-\eta)\times(-\xi)}{((\xi-\eta)^2+2n+2)^{3/2}}d\eta ds.\end{aligned}$$

We can write the following inequalities:

$$\|K_\chi\|_{L^2} \lesssim \int_0^t \left\| e^{is\langle D\rangle_p} T_{m_\chi^K}\left(|D|^{\frac{5}{2}} e^{-is\langle D\rangle_m} \frac{f_m}{\langle D\rangle_m}, e^{-is\langle D\rangle_n} \frac{f_n}{\langle D\rangle_n^3}\right)\right\|_{L^2} ds$$

$$\|K_{1-\chi}\|_{L^2} \lesssim \int_0^t \left\| e^{is\langle D\rangle_p} T_{m_{1-\chi}^K}\left(e^{-is\langle D\rangle_m} \frac{f_m}{\langle D\rangle_m}, |D|^{\frac{5}{2}} e^{-is\langle D\rangle_n} \frac{f_n}{\langle D\rangle_n^3}\right)\right\|_{L^2} ds,$$

where $m_\chi^K = \chi(\eta,\zeta)\frac{|\eta+\zeta|^{\frac{5}{2}}\chi}{|\eta|^{\frac{5}{2}}}$ and $m_{1-\chi}^K = (1-\chi(\eta,\zeta))\frac{|\eta+\zeta|^{\frac{5}{2}}\chi}{|\eta|^{\frac{5}{2}}}$. These symbols satisfy Hölder-like estimates (Proposition A.3.3); combined with the multiplier estimate (A.1.4-a) and the dispersion-multiplier lemma A.1.6 this proves Lemma 4.4.6. ■
Then summation is identical to what is done in 4.4.1.b: we skip it and end the proof of Proposition 4.4.5. ■

4.4.3 Estimates for $I_{m,n}$

The integral term $I_{m,n}$ concentrates the main difficulties of the proof: it corresponds to the case when the differentiation in ξ hits the complex exponential $e^{is\phi}$ and appears to make long-time estimates impossible given the additional power of s given by $\partial_\xi e^{is\phi}$. This is the reason why we are going to try to find a way to get additional decay in time. We write

$$\sum_{m,n} \mathcal{M}(m,n,p) I_{m,n} = I^{hf} + I^{hm} + I^{lf,lm},$$

where

1. I^{hf} corresponds to the *high frequency term*:

$$I^{hf} := \sum_{m,n} \mathcal{M}(m,n,p) I^{hf}_{m,n},$$

with

$$I^{hf}_{m,n} := -|\xi|^{\frac{3}{2}} \int_0^t \int (1-\theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|))\, is\partial_\xi\phi e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds,$$

the function $\theta_{s^\delta}(|\zeta|)$ being smooth and localizing in the zone $|\zeta| \leq s^\delta$.
High frequencies are quite easy to deal with: in fact, since we are in a high-regularity framework, this means that the high frequencies have a small amplitude. This can be understood with the high-frequency inequality in Proposition A.1.4, inequality (A.1.4-e).

2. I^{hm} corresponds to the *high Hermite modes*:

$$I^{hm} := \sum_{m,n \geq \langle t\rangle^\delta} \mathcal{M}(m,n,p) I^{hm}_{m,n},$$

with

$$I^{hm}_{m,n} := -|\xi|^{\frac{3}{2}} \int_0^t \int \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|) is\partial_\xi\phi e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds.$$

The idea of a high regularity leading to good estimates for high frequencies applies also in the framework of Hermite modes.

3. $I^{lf,lm}$ is the remaining term, corresponding to low frequencies and low Hermite modes:

$$I^{lf,lm} := \sum_{m \leq \langle t \rangle^{\delta}, n \in \mathbb{N}} \mathcal{M}(m,n,p) I^{lf}_{m,n} + \sum_{m \in \mathbb{N}, n \leq \langle t \rangle^{\delta}} \mathcal{M}(m,n,p) I^{lf}_{m,n}.$$

This last sum will be treated thanks to the space-time resonances method: in particular we are going to distinguish when there are space-times resonances (condition (C), page 77, satisfied) or when there are not space-time resonances (condition (C) not satisfied).

The situation is summed up in the following tree.

$$\sum_{m,n} \mathcal{M}(m,n,p) I_{m,n} := \sum_{m,n} \mathcal{M}(m,n,p) |\xi|^{\frac{3}{2}} \int_0^t \int -is\partial_\xi \phi e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle \eta \rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle \xi - \eta \rangle_n} d\eta ds$$

- High Fourier modes term I^{hf} section 4.5
- High Hermite modes term I^{hm} section 4.6
- Low Fourier and Hermite modes term $I^{lf,lm}$ section 4.7
  - Small times 4.7.2
  - Large times
    - Condition C satisfied 4.7.3
    - Condition C not satisfied 4.7.4

4.5 Estimates for high frequencies

Proposition 4.5.1. *There exists* $(a_p(s))_{p\in\mathbb{N}}$ *in* ℓ^2, $\|(a_p(s))_{p\in\mathbb{N}}\|_{\ell^2} \leq 1$ *such that for all* N, M *satisfying Conditions* (4.1.12) – (4.1.13),

$$\frac{1}{\sqrt{\langle t \rangle}} \underline{p}^M \sum_{m,n\in\mathbb{N}} \mathcal{M}(m,n,p) \left\| I^{hf}_{m,n} \right\|_{L^2} \lesssim \int_0^t s^{-\frac{1}{4}} \|f\|^2_{S^{M,N}_s} a_p(s) ds.$$

Proof :

First of all, we are going to prove the following result:

Lemma 4.5.2. *For all integers* m *and* n,

$$\frac{1}{\sqrt{\langle t \rangle}} \left\| I^{hf}_{m,n} \right\|_{L^2} \lesssim_{m \leftrightarrow n} \int_0^t \frac{\max(\underline{m}, \underline{n})^{\frac{1}{4}}}{\sqrt{\underline{mn}}} s^{-\frac{1}{4}} \|f_m(s)\|_{H^N} \mathcal{B}(f_n)(s) ds,$$

whenever N satisfies (4.1.13).

The sum over m and n will be skipped: we are in the framework of the summation theorem C.1.1, with the same parameters as in Section 4.4.1.b: Proposition 4.5.1 is deduced in the same way.

Proof of Lemma 4.5.2 :
Recall the expression of $I^{hf}_{m,n}$:

$$I^{hf}_{m,n} := -|\xi|^{\frac{3}{2}} \int_0^t \int (1-\theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|))\, is\partial_\xi \phi e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds.$$

Our idea is to say that if $|\xi-\eta|+|\eta|$ is large, it means that either $|\xi-\eta|$ is large or $|\eta|$ is large. In order to do so, we can remark that

$$1-\theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|) = (1-\theta_{s^\delta}(|\eta|)) + \theta_{s^\delta}(|\eta|)\,(1-\theta_{s^\delta}(|\xi-\eta|))\,.$$

Using the paraproduct decomposition (with the function χ, equal to 0 for $|\xi-\eta|\le 2|\eta|$) leads to the following splitting of $I^{hf}_{m,n}$:

$$I^{hf}_{m,n} = I^1+I^2+I^3+I^4,$$

where

$$\begin{aligned}
I^1 :=& |\xi|^{\frac{3}{2}} \int_0^t \int s\partial_\xi\phi\chi(\eta,\xi-\eta)\,(1-\theta_{s^\delta}(|\eta|))\, e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds,\\
I^2 :=& |\xi|^{\frac{3}{2}} \int_0^t \int s\partial_\xi\phi\chi(\eta,\xi-\eta)\theta_{s^\delta}(|\eta|)\,(1-\theta_{s^\delta}(|\xi-\eta|))\, e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds,\\
I^3 :=& |\xi|^{\frac{3}{2}} \int_0^t \int s\partial_\xi\phi(1-\chi(\eta,\xi-\eta))\,(1-\theta_{s^\delta}(|\eta|))\, e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds,\\
I^4 :=& |\xi|^{\frac{3}{2}} \int_0^t \int s\partial_\xi\phi(1-\chi(\eta,\xi-\eta))\theta_{s^\delta}(|\eta|)\,(1-\theta_{s^\delta}(|\xi-\eta|))\, e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds.
\end{aligned}$$

We shall only deal with the integral term I^1, the others can be dealt with in a similar way.

Using the expression of $\partial_\xi\phi = \frac{\xi}{\langle\xi\rangle_p} - \frac{\xi-\eta}{\langle\xi-\eta\rangle_n}$, we get

$$\begin{aligned}
I^1 =& \int_0^t \int \frac{|\xi|^{\frac{3}{2}}}{|\eta|^{\frac{3}{2}}} s\partial_\xi\phi\chi(\eta,\xi-\eta)(1-\theta_{s^\delta}(|\eta|))e^{-is\phi}|\eta|^{\frac{3}{2}} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds\\
=& \int_0^t \int \frac{|\xi|^{\frac{3}{2}}}{|\eta|^{\frac{3}{2}}} s\frac{\xi}{\langle\xi\rangle_p}(1-\theta_{s^\delta}(|\eta|))e^{-is\phi}|\eta|^{\frac{3}{2}} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds\\
&- \int_0^t \int \frac{|\xi|^{\frac{3}{2}}}{|\eta|^{\frac{3}{2}}} s\frac{\xi-\eta}{\langle\xi-\eta\rangle_n}(1-\theta_{s^\delta}(|\eta|))e^{-is\phi}|\eta|^{\frac{3}{2}} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds\\
=& I^{1,1}+I^{1,2}.
\end{aligned}$$

Let us focus only on $I^{1,1}$, $I^{1,2}$ being very similar.

The bilinear multiplier associated to the symbol $(\eta,\zeta)\mapsto \frac{|\eta+\zeta|^{\frac{3}{2}}}{|\eta|^{\frac{3}{2}}}\chi(\eta,\zeta)$ satisfies Hölder-type estimates thanks to Lemma A.3.3:

$$\left\|I^{1,1}\right\|_{L^2} \lesssim \frac{1}{\sqrt{mn}} \int_0^t s \left\| \frac{D}{\langle D\rangle_p} |D|^{3/2}(1-\theta_{s^\delta}(|D|))f_m \right\|_{L^2} \left\| e^{-is\langle iD\rangle_m} f_n \right\|_{L^\infty} ds.$$

Thanks to Proposition A.1.4, inequality (A.1.4-c), we know that the multiplier $\frac{|D|}{\langle D\rangle_p}$ is bounded in L^2. Hence we can write:

$$\left\| I^{1,1} \right\|_{L^2} \lesssim \frac{1}{\sqrt{mn}} \int_0^t s \left\| |D|^{3/2}(1-\theta_{s^\delta}(|D|)) f_m \right\|_{L^2} \left\| e^{-is\langle iD\rangle_n} f_n \right\|_{L^\infty} ds.$$

Then Proposition A.1.6 implies

$$\left\| e^{-is\langle iD\rangle_n} f_n \right\|_{L^\infty} \lesssim \underline{n}^{\frac{1}{4}} \langle s\rangle^{-\frac{1}{4}} \sqrt{\|f_n\|_{H^N} \|f_n\|_{B_s}}.$$

Thus

$$\left\| I^{1,1} \right\|_{L^2} \lesssim \frac{\underline{n}^{\frac{1}{4}}}{\sqrt{mn}} \int_0^t \langle s\rangle^{\frac{3}{4}} \left\| |D|^{3/2}(1-\theta_{s^\delta}(|D|)) f_m \right\|_{L^2} \sqrt{\|f_n\|_{H^N} \|f_n\|_{B_s}} ds.$$

Then use the high frequencies proposition A.1.4-e to write

$$\left\| I^{1,1} \right\|_{L^2} \lesssim \frac{\underline{n}^{\frac{1}{4}}}{\sqrt{mn}} \int_0^t \langle s\rangle^{\frac{3}{4}} \frac{1}{\langle s\rangle^{\delta(N-\frac{3}{2})}} \|f_m\|_{H^N} \sqrt{\|f_n\|_{H^N} \|f_n\|_{B_s}} ds$$

Finally, in order to estimate the norm in the space B_s, we have to divide by $\sqrt{\langle t\rangle}$. Then

$$\frac{1}{\sqrt{\langle t\rangle}} \left\| I^{1,1} \right\|_{L^2} \lesssim \int_0^t \frac{\max(\underline{m},\underline{n})^{\frac{1}{4}}}{\sqrt{mn}} \langle s\rangle^{-\frac{1}{4}} \|f_m\|_{H^N} \sqrt{\|f_n\|_{H^N} \|f_n\|_{B_s}} ds,$$

whenever $N > \frac{1}{\delta} + \frac{3}{2}$, which is true thanks to Condition (4.1.13). This ends the proof of Lemma 4.5.2. ■
This also ends the proof of Proposition 4.5.1. ■

4.6 Estimates for high Hermite modes

Proposition 4.6.1. *For all M, N satisfying (4.1.12)-(4.1.13), there exists $(a_p(s))_{p\in\mathbb{N}}$ in ℓ^2, $\|(a_p(s))_{p\in\mathbb{N}}\|_{\ell^2} \leq 1$ such that*

$$\frac{1}{\sqrt{\langle t\rangle}} \underline{p}^M \sum_{m\in\mathbb{N}, n\geq\langle t\rangle^\delta} \mathcal{M}(m,n,p) \left\| I^{lf}_{m,n} \right\|_{L^2} + \frac{1}{\sqrt{\langle t\rangle}} \underline{p}^M \sum_{m\geq\langle t\rangle^\delta, n\in\mathbb{N}} \mathcal{M}(m,n,p) \left\| I^{lf}_{m,n} \right\|_{L^2}$$
$$\lesssim \int_0^t \langle s\rangle^{-\frac{1}{4}} \|f\|^2_{S^{M,N}_s} a_p(s) ds.$$

Proof of Proposition 4.6.1 :
In all this section, we will be estimating the following term:

$$\frac{1}{\sqrt{\langle t\rangle}} \underline{p}^M \sum_{m\geq\langle t\rangle^\delta, n\in\mathbb{N}} \mathcal{M}(m,n,p) \left\| I^{lf}_{m,n} \right\|_{L^2},$$

the other one being bounded similarly.

4.6.1 Bounds for $\left\| I^{lf}_{m,n} \right\|_{L^2}$

Here the bounds for $\left\| I^{lf}_{m,n} \right\|_{L^2}$ will not be as sharp as possible: they only have to fit in the estimates we are going to get for the case "low frequencies, low Hermite modes" (Section 4.7).

Lemma 4.6.2. *For all m and n integers, N satisfying (4.1.13),*

$$\frac{1}{\langle \sqrt{t} \rangle} \left\| I^{lf}_{m,n} \right\|_{L^2} \lesssim \int_0^t s^{\frac{3\delta}{2}+\frac{1}{4}} \frac{\max(\underline{m},\underline{n})^{\frac{1}{4}}}{\sqrt{\underline{mn}}} \|f_m\|_{H^N} \mathcal{B}(f_n)(s) ds.$$

Proof :
The expression of $I^{lf}_{m,n}$ is

$$I^{lf}_{m,n} = -|\xi|^{\frac{3}{2}} \int_0^t \int \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|) is\partial_\xi \phi e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle \eta \rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle \xi-\eta \rangle_n} d\eta ds.$$

First of all, since $|\eta| \leq s^\delta$ and $|\xi-\eta| \leq s^\delta$,

$$\left\| I^{lf}_{m,n} \right\|_{L^2} \lesssim \int_0^t s^{\frac{3\delta}{2}} \left\| \int \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|) is\partial_\xi \phi e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle \eta \rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle \xi-\eta \rangle_n} d\eta \right\|_{L^2_\xi} ds.$$

Then the symbol $m(\eta,\zeta) = \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\zeta)\partial_\xi \phi(\eta,\zeta)$ is a Coifman-Meyer symbol, hence

$$\left\| I^{lf}_{m,n} \right\|_{L^2} \lesssim \int_0^t s^{\frac{3\delta}{2}} s \left\| e^{-is\langle D \rangle_m} \frac{f_m}{\langle D \rangle_m} \right\|_{L^2} \left\| e^{-is\langle D \rangle_n} \frac{f_n}{\langle D \rangle_m} \right\|_{L^\infty} ds.$$

Then we bound each of the factors:

1. the first term is bounded by Lemma A.1.4-a:

$$\begin{aligned} \left\| e^{-is\langle D \rangle_m} \frac{f_m}{\langle D \rangle_m} \right\|_{L^2} &\lesssim m^{-\frac{1}{4}} \|f_m\|_{L^2} \\ &\lesssim m^{-\frac{1}{4}} \|f_m\|_{H^N}. \end{aligned}$$

2. the second one is bounded by Lemma A.1.6:

$$\left\| e^{-is\langle D \rangle_n} \frac{f_n}{\langle D \rangle_m} \right\|_{L^\infty} \lesssim s^{-\frac{1}{4}} n^{-\frac{1}{4}} \sqrt{\|f_n\|_{H^N_T} \|f_n\|_{B_s}}.$$

Then,

$$\frac{1}{\langle \sqrt{t} \rangle} \left\| I^{lf}_{m,n} \right\|_{L^2} \lesssim \int_0^t s^{\frac{3\delta}{2}+\frac{1}{4}} \frac{\max(\underline{m},\underline{n})^{\frac{1}{4}}}{\sqrt{\underline{mn}}} \|f_m\|_{H^N} \sqrt{\|f_n\|_{H^N_T} \|f_n\|_{B_s}} ds,$$

and the lemma is proved. ■

4.6.2 Summing

Bounding

$$\|f_m(s)\|_{H^N} \mathcal{B}(f_n)(s)$$

by

$$\underline{m}^{-N} a_m(s) \underline{n}^{-M} b_n(s) \|f\|^2_{S_s^{M,N}},$$

with $(a_m(s))_{m\in\mathbb{N}}$ and $(b_n(s))_{n\in\mathbb{N}}$ in the unit ball of ℓ^2 less than 1 leads to

$$\begin{aligned}
&\underline{p}^M \sum_{m\geq\langle t\rangle^\delta, n\in\mathbb{N}} \mathcal{M}(m,n,p)\frac{1}{\langle\sqrt{t}\rangle}\left\|I^{lf}_{m,n}\right\|_{L^2}\\
&\lesssim \sum_{m\geq\langle t\rangle^\delta, n\in\mathbb{N}} \mathcal{M}(m,n,p)\frac{\max(\underline{m},\underline{n})^{\frac14}}{\sqrt{\underline{mn}}}\underline{m}^{-N}\underline{n}^{-M}\int_0^t s^{\frac{3\delta}{2}+\frac14} a_m(s)b_n(s)\|f\|^2_{S_s^{M,N}}\,ds\\
&\lesssim \underline{p}^M \sum_{m\geq\langle t\rangle^\delta, n\in\mathbb{N}} \mathcal{M}(m,n,p)\frac{\max(\underline{m},\underline{n})^{\frac14}}{\sqrt{\underline{mn}}}\underline{m}^{-M}\underline{n}^{-M}\int_0^t a_m(s)b_n(s)\underline{m}^{-(N-M)} s^{\frac{3\delta}{2}+\frac14}\|f\|^2_{S_s^{M,N}}\,ds\\
&\lesssim \underline{p}^M \sum_{m\geq\langle t\rangle^\delta, n\in\mathbb{N}} \mathcal{M}(m,n,p)\frac{\max(\underline{m},\underline{n})^{\frac14}}{\sqrt{\underline{mn}}}\underline{m}^{-M}\underline{n}^{-M}\int_0^t a_m(s)b_n(s)\langle t\rangle^{-(N-M)} s^{\frac{3\delta}{2}+\frac14}\|f\|^2_{S_s^{M,N}}\,ds.
\end{aligned}$$

By the summation theorem C.1.1-(1), we have

$$\underline{p}^M \sum_{m\geq\langle t\rangle^\delta, n\in\mathbb{N}} \mathcal{M}(m,n,p)\frac{1}{\langle\sqrt{t}\rangle}\left\|I^{lf}_{m,n}\right\|_{L^2} \lesssim \int_0^t \langle t\rangle^{-(N-M)} u_p(s) s^{\frac{3\delta}{2}+\frac14}\|f\|^2_{S_s^{M,N}}\,ds,$$

with $(\alpha_p(s))_{p\in\mathbb{N}}$ in ℓ^2.
If $N > 2M$, which is true by condition (4.1.13) then

$$\underline{p}^M \sum_{m\geq\langle t\rangle^\delta, n\in\mathbb{N}} \mathcal{M}(m,n,p)\frac{1}{\langle\sqrt{t}\rangle}\left\|I^{lf}_{m,n}\right\|_{L^2} \lesssim \int_0^t \alpha_p(s)\langle s\rangle^{-\frac14}\|f\|^2_{S_s^{M,N}}\,ds,$$

with $(\alpha_p(s))_{p\in\mathbb{N}}$ in ℓ^2. This ends the proof of Proposition 4.6.1.■

4.7 Estimates for low frequencies and low Hermite modes

Our aim is to prove the following proposition:

Proposition 4.7.1. *If M and N satisfy Conditions (4.1.12)-(4.1.13), there exists $(a_p(s))_{p\in\mathbb{N}}$, $(b_p(s))_{p\in\mathbb{N}}$ and $(c_p(s))$ in ℓ^2, with norm less than or equal to 1, such that*

$$\underline{p}^M \sum_{m,n\leq\langle t\rangle^\delta} \frac{1}{\sqrt{\langle t\rangle}}\left\|I^{lf}_{m,n}\right\|_{L^2} \lesssim \int_0^t \Big(Q(s)a_p(s) + C(s)b_p(s)\Big)ds + c_p(t)R(t),$$

with

$$\begin{aligned}
Q(s) &:= \langle s\rangle^{4\delta-\frac14}\|f\|^2_{S_s^{M,N}},\\
C(s) &:= \langle s\rangle^{3\delta+\frac12}\|f\|^3_{S_s^{M,N}},\\
R(t) &:= \langle t\rangle^{\frac{5\delta}{2}+\frac14}\left(\|f(t)\|^2_{S_s^{M,N}} + \|f(1)\|^2_{S_1^{M,N}}\right).
\end{aligned}$$

4.7.1 Intermediate result and summation

In this section we are going to prove the following proposition:

Proposition 4.7.2. *For all m and n integers, and $N \geq \frac{3}{2}$,*

$$\frac{1}{\sqrt{t}}\left\|I^{lf}_{m,n}\right\|_{L^2} \lesssim \int_0^t \Bigg(\frac{\max(\underline{m},\underline{n})^{3+\frac{3}{4}}}{\sqrt{mn}} \langle s\rangle^{3\delta} \langle s\rangle^{\frac{1}{4}} \|f_m(s)\|_{H^N} \mathcal{B}(f_n)(s) + \frac{\max(n,m)^{\frac{1}{2}}}{\sqrt{mn}} \langle s\rangle^{3\delta} \langle s\rangle^{\frac{3}{2}} \|f_m(s)\|_{H^N} \mathcal{B}(f_m)(s)\mathcal{B}(f_n)(s) \Bigg) ds + (\sqrt{n+1}+\sqrt{m+1})^2 \frac{\max(n,m)^{\frac{3}{4}}}{\sqrt{mn}} t^{\frac{5\delta}{2}+\frac{1}{4}} \left(A(t)+A(1)\right),$$

with $A(t) = \|f_m(t)\|_{H^N} \mathcal{B}(f_n)(s)$.

Going from Proposition 4.7.2 to Proposition 4.7.1 is quite easy: by Remark 4.1.9, we know that

1. the quadratic term can be bounded as follows:

$$\|f_m(s)\|_{H^N} \mathcal{B}(f_n)(s) \lesssim \underline{m}^{-2M} a_m(s)^2 \underline{n}^{-M} b_n(s) \|f\|^2_{S^{M,N}_s},$$

   with $(a_m(s))_{m\in\mathbb{N}}$ and $(b_n(s))_{n\in\mathbb{N}}$ in the unit ball of ℓ^2.

2. the cubic term is leads to a better bound:

$$\|f_m(s)\|_{H^N} \mathcal{B}(f_m)(s)\mathcal{B}(f_n)(s) \lesssim \underline{m}^{-M} a_m(s) \underline{n}^{-M} b_n(s) \|f\|^3_{S^{M,N}_s},$$

   with $(a_m(s))_{m\in\mathbb{N}}$ and $(b_n(s))_{n\in\mathbb{N}}$ in the unit ball of ℓ^2. Hence

$$\|f_m(s)\|_{H^N} \mathcal{B}(f_m)(s)\mathcal{B}(f_n)(s) \lesssim \underline{m}^{-2M} a_m(s) \underline{n}^{-M} b_n(s) \|f\|^3_{S^{M,N}_s},$$

   with $(a_m(s))_{m\in\mathbb{N}}$ and $(b_n(s))_{n\in\mathbb{N}}$ in the unit ball of ℓ^2.

3. a same bound can be found for the remaining term:

$$A(t) \lesssim \underline{m}^{-2M} a_m(s)^2 \underline{n}^{-M} b_n(s) \|f\|^2_{S^{M,N}_s},$$

   with $(a_m(s))_{m\in\mathbb{N}}$ and $(b_n(s))_{n\in\mathbb{N}}$ in the unit ball of ℓ^2.

We fit in the framework of the bounded resummation theorem C.1.1-(2), and Proposition 4.7.1 is proven.

The proof of Proposition 4.7.2 can be summed up as follows:

1. first of all we say a word about small times (section 4.7.2).
2. then, for large times, we have to split the space in two zones: around the space resonant set and far from it. But the way of dealing with those zones will depend on whether Condition (C) (page 77) is satisfied or not.
   (a) if this condition is satisfied,
      i. near the space resonant set, we are going to take advantage of the narrowness of the zone (Section 4.7.3.a).

ii. outside the space resonant set, we are going to perform an integration by parts in η and gain some powers of time (Section 4.7.3.b).

(b) if the condition is not satisfied,

i. near the space resonant set, we are going to take advantage of the non cancellation of the phase and perform an integration by parts in time (Section 4.7.4.a).

ii. outside the space resonant set, we are going to perform an integration by parts in η and gain some powers of time (Section 4.7.4.b).

4.7.2 Small times

Establishing contraction estimates for small times is not a big matter when we study weakly nonlinear dispersive equations. In our situation, we have the following theorem:

Proposition 4.7.3. *For all* $0 < t < 1$*,*

$$\frac{1}{\sqrt{\langle t \rangle}} \left\| I^{lf}_{m,n}(t) \right\|_{L^2} \lesssim_{m \leftrightarrow n} \frac{\max(\underline{m}, \underline{n})^{\frac{1}{4}}}{\sqrt{\underline{mn}}} \int_0^t s^{\frac{3}{4}} \left\| f_m \right\|_{H^N} \mathcal{B}(f_n)(s) ds. \tag{4.7.1}$$

Let us remark that since we are considering times smaller than 1, we have $s^{\frac{3}{4}} \leq \langle s \rangle^{-\frac{1}{4}}$: this is the reason why Proposition 4.7.3 will imply Proposition 4.7.2.

Proof :
First of all, so as to deal with the frequency $|\xi|^{\frac{3}{2}}$, we use the paraproduct decomposition. Wite $I^{lf}_{m,n} = I_\chi + I_{1-\chi}$, where

$$\begin{aligned} I_\chi &= |\xi|^{\frac{3}{2}} \int_0^t \int -is\chi(\eta, \xi-\eta) \partial_\xi \phi e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle \eta \rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle \xi-\eta \rangle_n} d\eta ds, \\ I_{1-\chi} &= |\xi|^{\frac{3}{2}} \int_0^t \int -is \left(1 - \chi(\eta, \xi-\eta)\right) \partial_\xi \phi e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle \eta \rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle \xi-\eta \rangle_n} d\eta ds. \end{aligned}$$

Then, since $\partial_\xi \phi = \frac{\xi}{\langle \xi \rangle_p} - \frac{\xi-\eta}{\langle \xi-\eta \rangle_n}$, write $I = I^1_\chi + I^1_{1-\chi} + I^2_\chi + I^2_{1-\chi}$ where

$$\begin{aligned} I^1_\chi &:= |\xi|^{\frac{3}{2}} \int_0^t \frac{\xi}{\langle \xi \rangle_p} \int -is\chi(\eta, \xi-\eta) \phi e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle \eta \rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle \xi-\eta \rangle_n} d\eta ds, \\ I^1_{1-\chi} &:= |\xi|^{\frac{3}{2}} \int_0^t \frac{\xi}{\langle \xi \rangle_p} \int -is \left(1 - \chi(\eta, \xi-\eta)\right) \phi e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle \eta \rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle \xi-\eta \rangle_n} d\eta ds, \\ I^2_\chi &:= |\xi|^{\frac{3}{2}} \int_0^t \int -is\chi(\eta, \xi-\eta) \phi e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle \eta \rangle_m} \frac{\xi-\eta}{\langle \xi-\eta \rangle_n} \frac{\hat{f}_n(\xi-\eta)}{\langle \xi-\eta \rangle_n} d\eta ds, \\ I^2_{1-\chi} &:= |\xi|^{\frac{3}{2}} \int_0^t \int -is \left(1 - \chi(\eta, \xi-\eta)\right) \phi e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle \eta \rangle_m} \frac{\xi-\eta}{\langle \xi-\eta \rangle_n} \frac{\hat{f}_n(\xi-\eta)}{\langle \xi-\eta \rangle_n} d\eta ds. \end{aligned}$$

Those integrals can be rewritten as bilinear multipliers

$$I_\chi^1 := \int_0^t s e^{is\langle D\rangle_p} \frac{D}{\langle D\rangle_p} T_{m_\chi}\left(|D|^{\frac{3}{2}} e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}, e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right) ds,$$
$$I_{1-\chi}^1 := \int_0^t s e^{is\langle D\rangle_p} \frac{D}{\langle D\rangle_p} T_{m_{1-\chi}}\left(e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}, |D|^{\frac{3}{2}} e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right) ds,$$
$$I_\chi^2 := \int_0^t s e^{is\langle D\rangle_p} T_{m_\chi}\left(|D|^{\frac{3}{2}} e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}, \frac{D}{\langle D\rangle_n} e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right) ds,$$
$$I_{1-\chi}^2 := \int_0^t s e^{is\langle D\rangle_p} T_{m_{1-\chi}}\left(e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}, |D|^{\frac{3}{2}} \frac{D}{\langle D\rangle_n} e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right) ds,$$

with $m_\chi(\eta,\zeta) = \chi(\eta,\zeta)\frac{|\eta+\zeta|^{\frac{3}{2}}}{|\eta|^{\frac{3}{2}}}$ and $m_{1-\chi}(\xi,\eta) = (1-\chi(\eta,\xi-\eta))\,\frac{|\eta+\zeta|^{\frac{3}{2}}}{|\zeta|^{\frac{3}{2}}}$.
Firstly, the multiplier lemma A.1.4-a gives the following:

$$\left\|I_\chi^1\right\|_{L^2} \lesssim \int_0^t s \left\|T_{m_\chi}\left(|D|^{\frac{3}{2}} e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}, e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right)\right\|_{L^2} ds.$$

Then, because of Proposition A.3.3, m_χ and $m_{1-\chi}$ satisfy Hölder-like estimates:

$$\left\|I_\chi^1\right\|_{L^2} \lesssim \int_0^t s \left\||D|^{\frac{3}{2}} e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}\right\|_{L^2} \left\|e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right\|_{L^\infty} ds. \tag{4.7.2}$$

Thanks to the Fourier multiplier lemma A.1.4-a,

$$\left\||D|^{\frac{3}{2}} e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}\right\|_{L^2} \leq \underline{m}^{-\frac{1}{2}} \left\||D|^{\frac{3}{2}} f_m\right\|_{L^2} \leq \underline{m}^{-\frac{1}{2}} \left\|f_m\right\|_{H^N}. \tag{4.7.3}$$

Thanks to the dispersion-multiplier lemma A.1.6,

$$\left\|e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right\|_{L^\infty} \leq s^{-\frac{1}{4}}\underline{n}^{-\frac{1}{4}}\sqrt{\|f_n\|_{H^N}\|f_n\|_{B_s}}. \tag{4.7.4}$$

Bounds (4.7.2), (4.7.3) and (4.7.4) thus give:

$$\left\|I_\chi^1\right\|_{L^2} \lesssim \int_0^t s^{\frac{3}{4}}\underline{m}^{-\frac{1}{2}}\underline{n}^{-\frac{1}{4}}\|f_m\|_{H^N}\sqrt{\|f_n\|_{H^N}\|f_n\|_{B_s}} ds,$$

which implies (4.7.1).
The three other integrals can be estimated in the same way: this ends the proof of Lemma 4.7.3. ■

4.7.3 Large times. Estimates for $I^{lf}_{m,n}$ with condition (C) satisfied

We are going to prove the following result:

Lemma 4.7.4. *For all m and n integers such that condition (C) is satisfied, for all M, N satisfying (4.1.12)-(4.1.13) and for all $t \geq 1$,*

$$\frac{1}{\sqrt{t}}\left\|I^{lf}_{m,n}\right\|_{L^2} \lesssim \frac{\max(n,m)^2}{\sqrt{\underline{mn}}}\int_1^t \langle s\rangle^{3\delta}\langle s\rangle^{-\frac{1}{4}}\|f_m\|_{H^N}\sqrt{\|f_n\|_{H^N}\|f_n\|_{B_s}} ds. \tag{4.7.5}$$

Recall that the study of resonances in Appendix B (Lemma B.1.3) implies that if condition (C) is satisfied, then the space-resonant set is the space-time-resonant set, and is the straight line $\{\xi = \Lambda_{m,n}\eta\}$, where

$$\Lambda_{m,n} = 1 + \sqrt{\frac{n+1}{m+1}}. \tag{4.7.6}$$

Hence it seems natural to distinguish two zones: close to this set and far away from it. Let $\varphi^s(\xi,\eta) := \theta(\sqrt{\langle s\rangle}\partial_\eta\phi(\xi,\eta))$ where θ is equal to 1 around 0: φ^s localizes in the zone

$$-\frac{1}{\sqrt{\langle s\rangle}} \leq \partial_\eta\phi(\xi,\eta) \leq \frac{1}{\sqrt{\langle s\rangle}}.$$

Let us now write $I^{lf}_{m,n} = I^{lf,r}_{m,n} + I^{lf,nr}_{m,n}$, where

— $I^{lf,r}_{m,n}$ is the low-frequency, resonant term.

$$I^{lf,r}_{m,n} := |\xi|^{\frac{3}{2}} \int_1^t \int s\partial_\xi\phi\theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)\varphi^s(\xi,\eta)e^{-is\phi}\frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m}\frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n}d\eta ds \tag{4.7.7}$$

— $I^{lf,nr}_{m,n}$ is the low-frequency, non resonant term.

$$I^{lf,nr}_{m,n} := |\xi|^{\frac{3}{2}} \int_1^t \int s\partial_\xi\phi\theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)(1-\varphi^s(\xi,\eta))e^{-is\phi}\frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m}\frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n}d\eta ds \tag{4.7.8}$$

4.7.3.a Around the space resonant set,

We are going to use the narrowness of the zone where we are localizing in order to prove the following result (which implies Lemma 4.7.4).

Lemma 4.7.5. *For all m and n integers, for all M satisfying (4.1.12) and N satisfying (4.1.13),*

$$\frac{1}{\sqrt{\langle t\rangle}}\left\|I^{lf,r}_{m,n}\right\|_{L^2} \lesssim \int_1^t \langle s\rangle^{3\delta}\frac{\max(n,m)^{\frac{3}{4}}}{\sqrt{mn}}\langle s\rangle^{-\frac{1}{4}}\|f_m\|_{H^N}\sqrt{\|f_n\|_{H^N}\|f_n\|_{B_s}}ds. \tag{4.7.9}$$

Proof :
First of all, we use the fact that in the zone where $\theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|) \neq 0$, we have $|\eta| \lesssim t^\delta$ and $|\xi-\eta| \lesssim t^\delta$, so $|\xi| \lesssim t^\delta$, hence:

$$\left\|I^{lf,r}_{m,n}\right\|_{L^2} \lesssim t^{\frac{3\delta}{2}}\left\|\int_1^t \int s\partial_\xi\phi\theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)\varphi^s(\xi,\eta)e^{-is\phi}\frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m}\frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n}d\eta ds\right\|_{L^2}.$$

Let us write

$$S(\xi,\eta) := \sqrt{s}\partial_\xi\phi\theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)\varphi^s(\xi,\eta). \tag{4.7.10}$$

Our aim is to get nice Hölder-like estimates for the symbol S, which is a bilinear multiplier localizing in a narrow zone, around a curve: this is the point of the paper of Bernicot and Germain [3], and in particular Theorem A.3.5 and its refined version A.3.8.

1. Study of the symbol S
Here we are interested in size and width of the support of S and in derivative estimates. We have the following lemma:

Lemma 4.7.6. *The symbol S satisfies the following properties:*

— *S is supported in a ball of radius $\rho = s^\delta$,*

— *S is supported in a band of width $\omega = \frac{s^{\frac{3\delta}{2}}}{(2m+2)\sqrt{s}}$,*

— *the derivatives of S satisfy the following inequalities:*

$$\left|\partial_\xi^a \partial_\eta^b S\right| \lesssim \sqrt{s}^{a+b}.$$

Hence the symbol S satisfies the hypotheses of Theorem A.3.8, with $\rho = s^\delta$, $\omega = \frac{s^{3\delta}\sqrt{\max(\underline{m},\underline{n})}}{\sqrt{s}}$, $\mu = \frac{1}{\sqrt{s}}$.

Proof of Lemma 4.7.6 :

— First we can remark that S is supported in a ball of radius s^δ.

— Then, we have to determine the width of the support of S, that is to say the width of the zone $|\partial_\eta \phi| \le \frac{1}{\sqrt{s}}$. So as to do this, use the asymptotics for $\partial_\eta \phi$ computed in Appendix B.

1. In the zone $|\eta| \ll \sqrt{\underline{m}}$, (B.1.4) applies, and the witdh of this zone is bounded by

$$\frac{\sqrt{\min(m,n)}}{\sqrt{s}}.$$

2. In the zone $|\eta| \ge c\sqrt{\underline{m}}$ ($c \in \mathbb{R}$), since $\eta^2 \lesssim s^\delta \ll \sqrt{s}$, we are in the asymptotics of (B.1.6). Hence the width of the band $|\partial_\eta \phi| \le \frac{1}{\sqrt{s}}$ is less than $\frac{s^{\frac{3\delta}{2}}}{(2m+2)\sqrt{s}}$.

This completes the proof.

— Finally, we have to estimate the derivatives of S. Thanks to Lemma B.1.3, we know that on the band $|\partial_\eta \phi| \le \frac{1}{\sqrt{s}}$, $|\partial_\xi \phi| \le \frac{1}{\sqrt{t}}$. Hence the inequality is satisfied for $a = b = 0$.

Then we have to study $\partial_\xi^a \partial_\eta^b \left(\sqrt{s}\partial_\xi \phi \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)\varphi^s\right)$:

— any derivative of ξ and η of $\partial_\eta \phi$ or $\partial_\xi \phi$ is a sum of fractions of negative order in ξ, in η, and in m, n and p. As an example, we have $\partial_\eta(\partial_\eta \phi) = \frac{2m+2}{\langle \eta \rangle_m^3} + \frac{2n+2}{\langle \xi-\eta \rangle_n^3}$, $\partial_\xi(\partial_\eta \phi) = -\frac{2n+2}{\langle \xi-\eta \rangle_n^3}$. Then

$$\left|\partial_\xi^a \partial_\eta^b \left(\partial_\eta \phi\right)\right| \lesssim 1,$$
$$\left|\partial_\eta^a \partial_\eta^b \left(\partial_\eta \phi\right)\right| \lesssim 1.$$

— then $|\partial_\xi^a \partial_\eta^b \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)| \lesssim \sqrt{s}^{a+b}$.

— finally,

$$\partial_\eta \varphi^s = \sqrt{t}\partial_\eta(\partial_\eta \phi)\theta'\left(\sqrt{s}\partial_\eta \phi\right)$$
$$\partial_\xi \varphi^s = \sqrt{s}\partial_\xi(\partial_\xi \phi)\theta'\left(\sqrt{t}\partial_\xi \phi\right).$$

We just proved that the different derivatives of $\partial_\eta \phi$ were bounded by a universal constant. This leads to $|\partial_\xi^a \partial_\eta^b \varphi^s| \lesssim \sqrt{s}^{a+b}$.

Leibniz' rule concludes the proof.

Lemma 4.7.6 is now proved. ■

2. Estimates.
The term to estimate is

$$\left\| \int_1^t \int s\partial_\xi \phi \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)\varphi^s(\xi,\eta)e^{-is\phi}\frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m}\frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n}d\eta ds\right\|_{L^2}$$
$$\lesssim \int_1^t \left\| \int e^{is\langle\xi\rangle_p} s\partial_\xi \phi \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)\varphi^s(\xi,\eta)e^{is\langle\eta\rangle_m}\frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m}e^{is\langle\xi-\eta\rangle_n}\frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n}d\eta\right\|_{L^2} ds$$
$$\lesssim \int_1^t \sqrt{s}\left\| T_S\left(e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}, e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right)\right\|_{L^2} ds,$$

with T_S the bilinear Fourier multiplier associated to S as defined in (2.7.2). Lemma 4.7.6 and Theorem A.3.8 lead to the following estimate.

$$\left\| T_S\left(e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}, e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right)\right\|_{L^2} \lesssim \sqrt{\max(\underline{m},\underline{n})}s^{\frac{3\delta}{2}}\left\|e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}\right\|_{L^2}\left\|e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right\|_{L^\infty}.$$

Now by the modified dispersion proposition A.1.6, we get the following inequality.

$$\left\| T_S\left(e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}, e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right)\right\|_{L^2} \lesssim \sqrt{\max(\underline{m},\underline{n})}s^{\frac{3\delta}{2}}\left\|\frac{f_m}{\langle D\rangle_m}\right\|_{L^2}\underline{n}^{-\frac{1}{4}}s^{-\frac{1}{4}}\sqrt{\|f_n\|_{H^N}\|f_n\|_{B_s}}.$$

Then, thanks to the linear multiplier inequality (A.1.4-a), we obtain the final inequality.

$$\left\| T_S\left(e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}, e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right)\right\|_{L^2} \lesssim \frac{\sqrt{\max(\underline{m},\underline{n})}}{\sqrt{\underline{m}}\underline{n}^{1/4}}s^{\frac{3\delta}{2}-\frac{1}{4}}\|f_m\|_{H^N}\sqrt{\|f_n\|_{H^N}\|f_n\|_{B_s}}.$$

Now it remains to integrate over s, and divide by $\sqrt{t}$ to get the B norm: we obtain

$$\frac{1}{\sqrt{t}}\left\|I^{lf,r}_{m,n}\right\|_{L^2} \lesssim \frac{1}{\sqrt{t}}\int_1^t \sqrt{s}\sqrt{\max(\underline{m},\underline{n})}\frac{1}{\sqrt{\underline{m}}\underline{n}^{1/4}}s^{\frac{3\delta}{2}-\frac{1}{4}}\|f_m\|_{H^N}\sqrt{\|f_n\|_{H^N}\|f_n\|_{B_s}}ds,$$

which proves Lemma 4.7.5.■

4.7.3.b Outside the space resonant set,

We have to take advantage fo the non-cancellation of $\partial_\eta\phi$: we are going to prove the following result.

Lemma 4.7.7. *For all m and n integers, $N \geq 3/2$, $t \geq 1$,*

$$\frac{1}{\sqrt{\langle t\rangle}}\left\|I^{lf,nr}_{m,n}\right\|_{L^2} \lesssim \frac{\max(n,m)^2}{\sqrt{\underline{mn}}}\int_1^t \langle s\rangle^{3\delta}\langle s\rangle^{-\frac{1}{4}}\|f_m\|_{H^N}\sqrt{\|f_n\|_{H^N}\|f_n\|_{B_s}}ds. \quad (4.7.11)$$

Proof :

1. Space resonances method. Write

$$e^{-is\phi} = \frac{i}{s}\frac{1}{\partial_\eta \phi}\partial_\eta\left(e^{-is\phi}\right).$$

The term $I^{lf,nr}_{m,n}$ can be rewritten as follows, for $t > 1$.

$$I^{lf,nr}_{m,n} = |\xi|^{\frac{3}{2}} \int_1^t \int s\partial_\xi\phi\theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)(1-\varphi^s)\frac{i}{s}\frac{1}{\partial_\eta\phi}\partial_\eta\left(e^{-is\phi}\right)\frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m}\frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n}d\eta ds.$$

An integration by parts in η leads to

$$I^{lf,nr}_{m,n} = \mathcal{I}^1 + \mathcal{I}^2 + \mathcal{I}^3 + \mathcal{I}^4,$$

where

$$\mathcal{I}^1 := |\xi|^{\frac{3}{2}} \int_1^t \int \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)(1-\varphi^s)i\frac{\partial_\xi\phi}{\partial_\eta\phi}\frac{\partial_\eta^2\phi}{\partial_\eta\phi}e^{-is\phi}\frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m}\frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n}d\eta ds,$$

$$\mathcal{I}^2 := -|\xi|^{\frac{3}{2}} \int_1^t \int \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)(1-\varphi^s)i\frac{\partial_\xi\phi}{\partial_\eta\phi}e^{-is\phi}\partial_\eta\left(\frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m}\right)\frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n}d\eta ds,$$

$$\mathcal{I}^3 := -|\xi|^{\frac{3}{2}} \int_1^t \int \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)(1-\varphi^s)i\frac{\partial_\xi\phi}{\partial_\eta\phi}e^{-is\phi}\frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m}\partial_\eta\left(\frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n}\right)d\eta ds,$$

$$\mathcal{I}^4 := -|\xi|^{\frac{3}{2}} \int_1^t \int \partial_\eta\Big(\theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)(1-\varphi^s)\partial_\xi\phi\Big)i\frac{1}{\partial_\eta\phi}e^{-is\phi}\frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m}\frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n}d\eta ds.$$

We will only prove

$$\frac{1}{\sqrt{\langle t\rangle}}\left\|\mathcal{I}^1\right\|_{L^2} \lesssim \frac{\max(n,m)^2}{\sqrt{mn}}\int_1^t \langle s\rangle^{3\delta}\langle s\rangle^{-\frac{1}{4}}\|f_m\|_{H^N}\sqrt{\|f_n\|_{H^N}\|f_n\|_{B_s}}ds,$$

the inequalities for $\mathcal{I}^j$, $j = 2,3,4$ being treated similarly. The term $\mathcal{I}^1$ is actually the harder to deal with since the fraction $\frac{\partial_\eta^2\phi}{\partial_\eta\phi}$ gets big close to the space-time resonant zone. The terms of the form $\partial_\eta f$ appearing in the terms $\mathcal{I}^j$, $j = 2,3,4$ are not really problematic: the difficulty coming from them is compensated by the absence of $\frac{\partial_\eta^2\phi}{\partial_\eta\phi}$.

2. Estimates for $\mathcal{I}^1$. The main problem arising here is to be able to find a bilinear estimate for the symbol

$$S := \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)(1-\varphi^s)\frac{\partial_\xi\phi}{\partial_\eta\phi}\frac{\partial_\eta^2\phi}{\partial_\eta\phi}. \tag{4.7.12}$$

This symbol does not enter directly in the framework of the Bernicot-Germain theorem A.3.8. In order to understand better the behaviour of this multiplier, we split the frequency space along the level lines of $\partial_\eta\phi$. So as to do this cutoff in a smooth way, let us define the following functions.

Definition 4.7.8. *Let ω be a real function supported in $\left[\frac{1}{2},1\right]$ such that*

$$\forall x \neq 0,\ \sum_{j\in\mathbb{Z}}(\omega(2^j x) + \omega(-2^j x)) = 1.$$

Define the following functions

$$\mathbb{I}_{a\sim b} = \omega(\frac{a}{b}),$$
$$\mathbb{I}_{a\leq m} = \sum_{2^k\leq m} \omega(2^k a).$$

Now we can write

$$(1-\varphi^s)\frac{\partial_\xi\phi}{\partial_\eta\phi}\frac{\partial_\eta^2\phi}{\partial_\eta\phi} = \sum_{1/2\leq 2^j\leq\sqrt{s}} S_j^+ + S_j^-, \tag{4.7.13}$$

where

$$S_j^\pm = \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)\mathbb{I}_{\partial_\eta\phi\sim\pm 2^{-j}}(1-\varphi^s)\frac{\partial_\xi\phi}{\partial_\eta\phi}\frac{\partial_\eta^2\phi}{\partial_\eta\phi}. \tag{4.7.14}$$

Then, we split dyadically the frequency space: the asymptotics of ϕ and its derivatives strongly depend on the comparison between the size of the frequencies and the size of $|\partial_\eta\phi|$: in Lemma B.1.5 we obtain three different asymptotical regimes, depending on a parameter $\varrho(m,j,\eta)$ defined by

$$\varrho(m,j,\eta) := \frac{\eta^2}{2^j m}. \tag{4.7.15}$$

To deal with this, we are going to use the smooth functions $\mathbb{I}_{\sqrt{|\xi|^2+|\eta|^2}\sim 2^k}$ and $\mathbb{I}_{|\xi|^2+|\eta|^2\leq m}$. This is why we need to define the following symbol:

$$S_{j,k}^\pm = \mathbb{I}_{\partial_\eta\phi\sim\pm 2^{-j}}\mathbb{I}_{\sqrt{|\xi|^2+|\eta|^2}\sim 2^k}(1-\varphi^s)\frac{\partial_\xi\phi}{\partial_\eta\phi}\frac{\partial_\eta^2\phi}{\partial_\eta\phi}.$$

Let us finally rewrite the symbol S defined in (4.7.12) in order to take into account the different asymptotics for $\partial_\eta\phi$. Write

$$S = M^1 + M^2 + M^3,$$

where M^1, M^2 and M^3 are defined as follows.

1. The symbol M^1 corresponds to small values of $|\xi|, |\eta|$.

$$M^1 = \mathbb{I}_{|\xi|^2+|\eta|^2\leq\frac{m}{2}}\left(\sum_{1/2\leq 2^j\leq\sqrt{s}} S_j^+ + S_j^-\right).$$

2. The symbol M^2 corresponds to small values of the parameter $\varrho(m,j,2^k)$ defined in (4.7.15).

$$M^2 = (1-\mathbb{I}_{|\xi|^2+|\eta|^2\leq\frac{m}{2}})\sum_{1/2\leq 2^j\leq\sqrt{s}}\ \sum_{k|\varrho(m,j,2^k)\ll 1} S_{j,k}^\pm.$$

3. Finally the symbol M^3 corresponds to the remaining terms, i.e. large values of $\varrho(m,j,\eta)$.

$$M^3 = (1-\mathbb{I}_{|\xi|^2+|\eta|^2\leq\frac{m}{2}})\sum_{1/2\leq 2^j\leq\sqrt{s}}\ \sum_{k|\varrho(m,j,2^k)\gtrsim 1} S_{j,k}^\pm.$$

If $q \in \{1,2,3\}$, we write J^q for

$$J^q := \int_1^t \int M^q(\xi,\eta) e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds.$$

With these notations, we remark that

$$\mathcal{I}^1 = |\xi|^{\frac{3}{2}}(J^1 + J^2 + J^3).$$

1. Estimates for M^1.

Lemma 4.7.9. *(Bilinear estimate for low frequencies) We have the following estimate.*

$$\frac{1}{\sqrt{\langle t\rangle}} \left\| |\xi|^{\frac{3}{2}} J^1 \right\|_{L^2} \lesssim \frac{\underline{n}^{\frac{1}{4}}}{\sqrt{\underline{mn}}} \min(\sqrt{\underline{m}}, \sqrt{\underline{n}}) \int_1^t s^{\frac{3}{2}\delta} s^{-\frac{1}{4}} \|f_m\|_{H^N} \sqrt{\|f_n\|_{H^N} \|f_n\|_{B_s}} ds. \quad (4.7.16)$$

Remark 4.7.10. *Remark that the inequality* (4.7.16) *is stronger than the inequality* (4.7.11) *in Lemma 4.7.7.*

Proof :
Let J_j^1 be the following quantity.

$$J_j^{1,\pm} := \int \mathbb{I}_{|\xi|^2+|\eta|^2 \le \frac{\underline{m}}{2}} S_j^{\pm} e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta,$$

where $S_j^{\pm}$ is defined in (4.7.14).

First we will establish a bilinear estimate for the symbol $\mathbb{I}_{|\xi|^2+|\eta|^2 \le \frac{\underline{m}}{2}} S_j^-$ which will be denoted $\widetilde{S}_j$ for the sake of simplicity. Given the central symmetry for the level lines of $\partial_\eta \phi$, estimates for $\widetilde{S}_j^-$ will also be valid for $\widetilde{S}_j^+$.

Let us adopt the following strategy: first we study completely the symbol $\widetilde{S}_j$, then we rescale it to fit in the Bernicot-Germain theorem's hypotheses.

Lemma 4.7.11. *The symbol $\widetilde{S}_j$ satisfies the following properties:*

— *$\widetilde{S}_j$ is supported in a ball of radius $\rho = \sqrt{\frac{\underline{m}}{2}}$,*
— *$\widetilde{S}_j$ is supported in a band of width $\omega = 2^{-j}\min(\sqrt{\underline{m}}, \sqrt{\underline{n}})$,*
— *the derivatives of $\widetilde{S}_j$ satisfy the following inequalities: for all a, b integers,*

$$\left|\partial_\xi^a \partial_\eta^b \widetilde{S}_j\right| \lesssim 2^j (2^j)^{a+b}.$$

Hence the symbol $2^{-j}\widetilde{S}_j$ satisfies the hypotheses of Theorem A.3.8, with

$$\rho = \sqrt{\frac{\underline{m}}{2}},$$
$$\omega = 2^{-j}\min(\sqrt{\underline{m}}, \sqrt{\underline{n}}),$$
$$\mu = 2^{-j}.$$

Proof of Lemma 4.7.11 :

— It is clear that the support of $\widetilde{S}_j$ is included in a ball of radius less than $\sqrt{\frac{m}{2}}$.

— Thanks to Lemma B.1.4, we know that if $|\eta| \leq \sqrt{\frac{m}{2}}$ then the width of the band $\partial_\eta \phi \sim -2^{-j}$ is of bounded by $2^{-j}\min(\sqrt{m}, \sqrt{n})$.

— To prove that the symbol satisfies $\left|\partial_\xi^a \partial_\eta^b S_j\right| \lesssim 2^j (2^j)^{a+b}$, we first recall the formula for S_j.

$$S_j(\xi,\eta) = \omega(2^j \partial_\eta \phi)\theta\left(\frac{\sqrt{|\xi|^2+|\eta|^2}}{\sqrt{m+1}}\right)\theta(s^{-\delta}|\xi|)\theta(s^{-\delta}|\eta|)(1-\theta(\sqrt{s}\partial_\eta\phi))\frac{\partial_\xi\phi}{\partial_\eta\phi}\frac{\partial_\eta^2\phi}{\partial_\eta\phi}.$$

We focus on the effect of the operator ∂_η on S_j.

— First, $\partial_\eta\left(\omega(2^j\partial_\eta\phi)\right) = 2^j\partial_\eta^2\phi\omega'(2^j\partial_\eta\phi)$. Since $\partial_\eta^a\phi$ is bounded by 2 for all $a > 1$, this leads to

$$\left|\partial_\eta\left(\omega(2^j\partial_\eta\phi)\right)\right| \lesssim 2^j.$$

— Then, if $\pi > 0$,

$$\left|\partial_\eta\left(\theta(\pi\sqrt{|\xi|^2+|\eta|^2})\right)\right| = \left|\frac{\eta}{\sqrt{|\xi|^2+|\eta|^2}}\pi\theta'(\pi\sqrt{|\xi|^2+|\eta|^2})\right| \lesssim \pi,$$

i.e we have the following estimate (for times $s > 1$).

$$\left|\partial_\eta\left(\theta\left(\frac{\sqrt{|\xi|^2+|\eta|^2}}{\sqrt{m+1}}\right)\theta(s^{-\delta}\sqrt{|\xi|^2+|\eta|^2})\right)\right| \lesssim \frac{1}{m+1} + s^{-\delta} \lesssim 1.$$

— Now we have to bound $\partial_\eta\left(\theta(\sqrt{s}\partial_\eta\phi)\right)$. Given the definition of θ, there exists c such that

$$\theta(x) = 1 \text{ for } -1-c \leq x \leq -1 \text{ or } 1 \leq x \leq 1+c.$$

This means that

$$\partial_\eta\left(\theta(\sqrt{s}\partial_\eta\phi)\right) \neq 0 \text{ for } \frac{1}{\sqrt{s}} \leq |\partial_\eta\phi| \leq \frac{1+c}{\sqrt{s}}.$$

Hence if we are in the zone $\partial_\eta\phi \sim -2^{-j}$, the quantity

$$\left|\partial_\eta\left(\theta(\sqrt{s}\partial_\eta\phi)\right)\right|$$

is different from 0 if and only if $2^j \sim \sqrt{s}$: so it is bounded (up to a constant) by $\sqrt{s}$. Finally we can write the following inequality.

$$\left|\partial_\eta\left(\theta(\sqrt{s}\partial_\eta\phi)\right)\right| \lesssim 2^j.$$

— Finally, $\left|\frac{\partial_\xi\phi}{\partial_\eta\phi}\right| \lesssim 1$ (Lemma B.1.3) and

$$\begin{aligned}\left|\partial_\eta\left(\frac{\partial_\xi\phi}{\partial_\eta\phi}\right)\right| =& \left|\frac{\partial_\eta\partial_\xi\phi}{\partial_\eta\phi} - \frac{\partial_\xi\phi\partial_\eta\phi}{(\partial_\eta\phi)^2}\right| \\ \lesssim& \frac{1}{|\partial_\eta\phi|} \\ \lesssim& 2^j.\end{aligned}$$

Similarly, we have $\left|\frac{\partial_\eta^2\phi}{\partial_\eta\phi}\right| \lesssim 2^j$ and

$$\left|\partial_\eta\left(\frac{\partial_\eta^2\phi}{\partial_\eta\phi}\right)\right| \lesssim 2^{2j}.$$

This finally gives the desired estimate (for $a = 1, b = 0$ but it generalizes easily). Lemma 4.7.11 is now proved. ■

Rewriting $J_j^{1,\pm}$ as a bilinear operator gives

$$\left\|J_j^1\right\|_{L^2} = 2^j\left\|T_{2^{-j}\widetilde{S}_j}\left(e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}, e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right)\right\|_{L^2_\xi}. \tag{4.7.17}$$

Using Theorem A.3.8, we obtain

$$\begin{aligned}
&\left\|T_{2^{-j}\widetilde{S}_j}\left(e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}, e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right)\right\|_{L^2_\xi} \\
&\lesssim \max\left(1,\frac{\omega}{\mu}\right)(\rho\omega)^{\frac{1}{2}+\frac{1}{2}-1}\left\|e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}\right\|_{L^2}\left\|e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right\|_{L^\infty} \\
&\lesssim \min(\sqrt{\underline{m}},\sqrt{\underline{n}})\left\|e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}\right\|_{L^2}\left\|e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right\|_{L^\infty}.
\end{aligned}$$

Then use the Fourier multiplier Proposition A.1.4-a, to get:

$$\left\|e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}\right\|_{L^2} \lesssim \frac{1}{\sqrt{\underline{m}}}\left\|f_m\right\|_{L^2} \tag{4.7.18}$$

$$\lesssim \frac{1}{\sqrt{\underline{m}}}\left\|f_m\right\|_{H^N}. \tag{4.7.19}$$

Similarly, Proposition A.1.6 implies

$$\left\|e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right\|_{L^\infty} \lesssim \underline{n}^{-\frac{1}{4}}s^{-\frac{1}{4}}\sqrt{\left\|f_n\right\|_{H^N}\left\|f_n\right\|_{B_s}}. \tag{4.7.20}$$

By (4.7.18) and (4.7.20),

$$\left\|T_{2^{-j}\widetilde{S}_j}\left(e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}, e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right)\right\|_{L^2_\xi} \lesssim \frac{\min(\underline{m},\underline{n})}{\underline{n}^{\frac{1}{4}}\sqrt{\underline{m}}}\left\|f_m\right\|_{H^N}\sqrt{\left\|f_n\right\|_{H^N}\left\|f_n\right\|_{B_s}}. \tag{4.7.21}$$

Now by (4.7.17), (4.7.21) and since $|\xi| \le s^\delta$, we get

$$\left\||\xi|^{\frac{3}{2}}J_j^1\right\|_{L^2} \lesssim 2^j\frac{\underline{n}^{\frac{1}{4}}}{\sqrt{\underline{mn}}}\min(\sqrt{\underline{m}},\sqrt{\underline{n}})s^{\frac{3}{2}\delta}s^{\frac{1}{4}}\frac{1}{\sqrt{s}}\left\|f_m\right\|_{H^N}\sqrt{\left\|f_n\right\|_{H^N}\left\|f_n\right\|_{B_s}}. \tag{4.7.22}$$

Then, since

$$\frac{1}{\sqrt{t}}\left\||\xi|^{\frac{3}{2}}\int_1^t\sum_{\frac{1}{2}\le 2^j\le\sqrt{s}}J_j^1(s)ds\right\|_{L^2} \lesssim \frac{1}{\sqrt{t}}\int_1^t\sum_{\frac{1}{2}\le 2^j\le\sqrt{s}}\left\|J_j^1(s)\right\|_{L^2}ds,$$

(4.7.22) gives

$$\begin{aligned}
&\frac{1}{\sqrt{t}}\left\| |\xi|^{\frac{3}{2}} \int_1^t \sum_{\frac{1}{2}\leq 2^j\leq\sqrt{s}} J_j^1(s)ds \right\|_{L^2} \\
&\lesssim \frac{\underline{n}^{\frac{1}{4}}}{\sqrt{\underline{mn}}} \min(\sqrt{\underline{m}},\sqrt{\underline{n}}) \int_1^t \|f_m\|_{H^N} \sqrt{\|f_n\|_{H^N}\|f_n\|_{B_s}} \frac{1}{\sqrt{s}} s^{\frac{3}{2}\delta} s^{\frac{1}{4}} \frac{1}{\sqrt{s}} \sum_{\frac{1}{2}\leq 2^j\leq\sqrt{s}} 2^j ds \\
&\lesssim \frac{\underline{n}^{\frac{1}{4}}}{\sqrt{\underline{mn}}} \min(\sqrt{\underline{m}},\sqrt{\underline{n}}) \int_1^t s^{\frac{3}{2}\delta} s^{-\frac{1}{4}} \|f_m\|_{H^N} \sqrt{\|f_n\|_{H^N}\|f_n\|_{B_s}} ds,
\end{aligned}$$

which concludes the proof of Lemma 4.7.9. ■

2. Estimates for M^2.

Lemma 4.7.12. *(Bilinear estimate for high frequencies and small values of $\varrho(m,j,\eta)$) We have the following inequality.*

$$\frac{1}{\sqrt{\langle t\rangle}} \left\| |\xi|^{\frac{3}{2}} J^2 \right\|_{L^2} \lesssim \frac{\max(\underline{m},\underline{n})^{\frac{1}{4}}}{\underline{m}\sqrt{\underline{mn}}} \int_1^t s^{\frac{3}{2}\delta} s^{-\frac{1}{4}} \|f_m\|_{H^N} \sqrt{\|f_n\|_{H^N}\|f_n\|_{B_s}} ds. \tag{4.7.23}$$

Remark 4.7.13. *The inequality* (4.7.23) *is stronger than the inequality* (4.7.11) *in Lemma 4.7.7.*

Proof :

Recall that:

$$J^2 = \int_1^t \sum_{\frac{2^k}{\sqrt{\underline{m}}2^j}\ll 1,\ 2^k\gtrsim\sqrt{\underline{m}}} J_{j,k}^{\pm}(s)ds = \int_1^t \sum_{\frac{2^k}{\sqrt{\underline{m}}2^j}\ll 1,\ 2^k\gtrsim\sqrt{\underline{m}}} \int S_{j,k}^{\pm} e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds.$$

We start by stating multilinear estimates for the symbol $S_{j,k} = S_{j,k}^{-}$ (the case $S_{j,k}^{+}$ is similar). We skip the proof of the following result, very similar to Lemma 4.7.11.

Lemma 4.7.14. *The symbol $S_{j,k}$ satisfies the following properties:*

- *$S_{j,k}$ is supported in a ball of radius $\rho = 2^k$,*
- *$S_{j,k}$ is supported in a band of width $\omega = \frac{2^{3k}2^{-j}}{2m+2}$,*
- *the derivatives of $S_{j,k}$ satisfy the following inequalities: for all a,b integers,*

$$\left|\partial_\xi^a \partial_\eta^b S_{j,k}\right| \lesssim 2^j (2^j)^{a+b}.$$

Hence the symbol $2^{-j}S_{j,k}$ satisfies the hypotheses of Theorem A.3.8, with

$$\begin{aligned}
\rho &= 2^k, \\
\omega &= 2^{-j}\frac{2^{3k}}{2m+2}, \\
\mu &= 2^{-j}.
\end{aligned}$$

If we rewrite $J_{j,k}$ as a bilinear operator,

$$\|J_{j,k}\|_{L^2} = 2^j \left\| T_{2^{-j}S_{j,k}}\left(e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}, e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right)\right\|_{L^2_\xi},$$

we can apply Theorem A.3.8 to obtain

$$\begin{aligned}&\left\| T_{2^{-j}S_j}\left(e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}, e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right)\right\|_{L^2_\xi}\\ &\lesssim \frac{2^{3k}}{2m+2}\left\| e^{-is\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}\right\|_{L^2}\left\| e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right\|_{L^\infty},\end{aligned}$$

by the dilation lemma A.2.2.
Then, by the dispersive estimate and Proposition A.1.4, inequality (A.1.4-a),

$$\left\||\xi|^{\frac{3}{2}} J_{j,k}\right\|_{L^2} \lesssim 2^j \frac{\underline{n}^{\frac{1}{4}}}{\sqrt{\underline{mn}}}\frac{2^{3k}}{2m+2} s^{\frac{3}{2}\delta} s^{\frac{1}{4}}\frac{1}{\sqrt{s}}\|f_m\|_{H^N}\sqrt{\|f_n\|_{H^N}\|f_n\|_{B_s}}.$$

Now we sum over k:

$$\sum_{2^k\leq s^\delta}\left\||\xi|^{\frac{3}{2}} J_{j,k}\right\|_{L^2} \lesssim 2^j \frac{\underline{n}^{\frac{1}{4}}}{m\sqrt{\underline{mn}}} s^{\frac{3}{2}\delta} s^{\frac{1}{4}}\frac{1}{\sqrt{s}}\|f_m\|_{H^N}\sqrt{\|f_n\|_{H^N}\|f_n\|_{B_s}}.$$

Finally by sum over j and integrating in time,

$$\frac{1}{\sqrt{t}}\left\|\int_1^t \sum_{2^k\leq s^\delta}\sum_{\frac{1}{2}\leq 2^j\leq\sqrt{s}} J_{j,k}(s)ds\right\|_{L^2} \lesssim \frac{\max(\underline{m},\underline{n})^{\frac{1}{4}}}{\underline{m}\sqrt{\underline{mn}}}\int_1^t s^{\frac{3}{2}\delta} s^{-\frac{1}{4}}\|f_m\|_{H^N}\sqrt{\|f_n\|_{H^N}\|f_n\|_{B_s}}ds.$$

This ends the proof of Lemma 4.7.12. ■

4. Estimates for M^3.

Lemma 4.7.15. *(Bilinear estimate for high frequencies, and small values of j) We have the following inequality.*

$$\frac{1}{\sqrt{\langle t\rangle}}\left\||\xi|^{\frac{3}{2}} J^3\right\|_{L^2} \lesssim \frac{\max(\underline{m},\underline{n})^{\frac{1}{4}}}{\sqrt{\underline{mn}}}\max\left(\sqrt{\frac{n}{m}},1\right)\int_1^t s^{3\delta} s^{-\frac{1}{4}}\|f_m\|_{H^N}\sqrt{\|f_n\|_{H^N}\|f_n\|_{B_s}}ds. \tag{4.7.24}$$

Remark 4.7.16. *Inequality* (4.7.24) *is stronger than the inequality* (4.7.11) *in Lemma 4.7.7.*

Proof :
Recall that:

$$J^3 = \int_1^t \sum_{\frac{2^k}{\sqrt{\underline{m}}2^j}\gtrsim 1,\ 2^k\gtrsim\sqrt{\underline{m}}} J^{\pm}_{j,k}(s)ds = \int_1^t \sum_{\frac{2^k}{\sqrt{\underline{m}}2^j}\gtrsim 1,\ 2^k\gtrsim\sqrt{\underline{m}}}\int S^{\pm}_{j,k}e^{-is\phi}\frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m}\frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n}d\eta ds.$$

First we establish multilinear estimates for the symbol $S_{j,k} = S^-_{j,k}$ (as previously, the case $S^+_{j,k}$ is similar): we are not going to give the proof of the following result but it depends on the asymptotics found in (B.1.7) and (B.1.8).

Lemma 4.7.17. *The symbol $S_{j,k}$ satisfies the following properties:*
- *$S_{j,k}$ is supported in a ball of radius $\rho = 2^k$,*
- *$S_{j,k}$ is supported in a band of width $\omega = 2^{\frac{j}{2}}\sqrt{2n+2} \lesssim 2^{\frac{j}{2}} \max(\underline{m}, \underline{n})^{\frac{1}{2}}$,*
- *the derivatives of $S_{j,k}$ satisfy the following inequalities: for all a, b integers,*

$$\left|\partial_\xi^a \partial_\eta^b S_{j,k}\right| \lesssim 2^j (2^j)^{a+b}.$$

Hence the symbol $2^{-j} S_{j,k}$ satisfies the hypotheses of Theorem A.3.8, with

$$\begin{aligned} \rho &= 2^k, \\ \omega &= 2^{\frac{j}{2}} \sqrt{\max(\underline{m}, \underline{n})}, \\ \mu &= 2^{-j}. \end{aligned}$$

This leads to

$$\left\| |\xi|^{\frac{3}{2}} J^2_{j,k} \right\|_{L^2} \lesssim 2^j \frac{\max(\underline{m},\underline{n})^{\frac{1}{4}}}{\sqrt{\underline{mn}}} 2^{\frac{3j}{2}} \max(\sqrt{\underline{n}}, \sqrt{\underline{m}}) \int_1^t s^{\frac{3}{2}\delta} s^{\frac{1}{4}} \frac{1}{\sqrt{s}} \|f_m\|_{H^N} \sqrt{\|f_n\|_{H^N} \|f_n\|_{B_s}} ds.$$

Here we are in the regime where $2^j \lesssim \frac{2^k}{\sqrt{\underline{m}}} \lesssim \frac{s^\delta}{\sqrt{\underline{m}}}$. Hence the inequality rewrites as follows.

$$\left\| |\xi|^{\frac{3}{2}} J^2_{j,k} \right\|_{L^2} \lesssim 2^j \frac{\max(\underline{m},\underline{n})^{\frac{1}{4}}}{\sqrt{\underline{mn}}} \max\left(\sqrt{\frac{n}{m}}, 1\right) \int_1^t s^{3\delta} s^{\frac{1}{4}} \frac{1}{\sqrt{s}} \|f_m\|_{H^N} \sqrt{\|f_n\|_{H^N} \|f_n\|_{B_s}} ds.$$

Then summing over j and k and integrating leads to

$$\begin{aligned} &\frac{1}{\sqrt{t}} \left\| \int_1^t \sum_{2^k \le s^\delta} \sum_{\frac{1}{2} \le 2^j \le \sqrt{s}} J_j^3(s) ds \right\|_{L^2} \\ &\lesssim \frac{\max(\underline{m},\underline{n})^{\frac{1}{4}}}{\sqrt{\underline{mn}}} \max\left(\sqrt{\frac{n}{m}}, 1\right) \int_1^t s^{3\delta} s^{-\frac{1}{4}} \|f_m\|_{H^N} \sqrt{\|f_n\|_{H^N} \|f_n\|_{B_s}} ds. \end{aligned}$$

Then Lemma 4.7.15 is proved. ■

Then gathering Lemma 4.7.9, Lemma 4.7.12 and Lemma 4.7.15 leads to Lemma 4.7.7.■

4.7.4 Case where $p > m$, $p > n$ but condition (C) is not satisfied

We are going to prove the following lemma:

Lemma 4.7.18. *For all m and n integers satisfying (4.1.12)-(4.1.13) and such that condition (C) is not satisfied,*

$$\begin{aligned} \frac{1}{\sqrt{t}} \left\| I^{lf}_{m,n} \right\|_{L^2} \lesssim & \frac{\max(\underline{m},\underline{n})^{3+\frac{3}{4}}}{\sqrt{\underline{mn}}} \int_1^t s^{3\delta} s^{-\frac{1}{4}} \|f_m\|_{H^N} \mathcal{B}(f_n)(s) ds \\ &+ \frac{\max(n,m)^{\frac{1}{2}}}{\sqrt{\underline{mn}}} \int_1^t s^{3\delta} s^{\frac{3}{2}} \|f_m\|_{H^N} \mathcal{B}(f_m)(s) \mathcal{B}(f_n)(s) ds \\ &+ (\sqrt{n+1} + \sqrt{m+1})^2 \frac{\max(n,m)^{\frac{3}{4}}}{\sqrt{\underline{mn}}} t^{\frac{5\delta}{2}+\frac{1}{4}} (A(t) + A(1)), \end{aligned}$$

with $A(t) = \|f_m(t)\|_{H^N} \mathcal{B}(f_n)(t)$.

Recall the situation: in the case where $p > m$, $p > n$ but condition (C) is not satisfied, there are no space-time resonances. When we are close to the space-resonant straight line, a normal form transformation should help.

However, one of the main problems is that we do not have $|\partial_\xi \phi| \leq |\partial_\eta \phi|$. So as to deal with this new configuration, we are going to loosen the constraint on the narrowness of the zone close to $\mathcal{S}$: this will make the estimates outside this zone easier. Inside it, we will be able to use the time-resonances method since ϕ does not vanish.
We are performing two different cutoffs:

— θ is a compactly supported $\mathcal{C}^\infty$ function equal to 1 on $[-1, 1]$

— $\theta_{s^\delta}(|\eta|) = \theta\left(\frac{|\eta|}{s^\delta}\right)$

— We have to choose a new function ψ^s localizing around the space resonant set. Our idea is to take the widest zone which does not meet the space-resonant set. Proposition B.1.4 will be very useful: if ψ localizes in a neigborhood of size $\frac{c}{(\sqrt{n+1}+\sqrt{m+1})^2}\frac{1}{R}$ of $\mathcal{S}$, we can be sure that we will not meet the time-resonant set.
If we adapt the proof of Lemma 4.7.6, we know that the zone $|\partial_\eta \phi| \leq d$ is of width $\sqrt{\max(\underline{m}, \underline{n})} s^{3\delta} d$. Consequently the function ψ^s can be chosen equal to

$$\psi^s(\xi, \eta) = \theta\left(c'\sqrt{\max(\underline{m}, \underline{n})} s^{3\delta} (\sqrt{n+1} + \sqrt{m+1})^2 s^\delta |\partial_\eta \phi|\right). \tag{4.7.25}$$

Then write

$$I^{lf}_{m,n} = I^{lf,r}_{m,n} + I^{lf,nr}_{m,n},$$

where

— $I^{lf,r}_{m,n}$ is the space-resonant term.

$$I^{lf,r}_{m,n} := |\xi|^{\frac{3}{2}} \int_1^t \int s \partial_\xi \phi \theta_{s^\delta}(|\eta|) \theta_{s^\delta}(|\xi - \eta|) \psi^s e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle \eta \rangle_m} \frac{\hat{f}_n(\xi - \eta)}{\langle \xi - \eta \rangle_n} d\eta ds,$$

— $I^{lf,nr}_{m,n}$ is the non space-resonant term:

$$I^{lf,nr}_{m,n} := |\xi|^{\frac{3}{2}} \int_1^t \int s \partial_\xi \phi \theta_{s^\delta}(|\eta|) \theta_{s^\delta}(|\xi - \eta|)(1 - \psi^s) e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle \eta \rangle_m} \frac{\hat{f}_n(\xi - \eta)}{\langle \xi - \eta \rangle_n} d\eta ds,$$

4.7.4.a Around the space resonant set ($I^{lf,r}_{m,n}$).

Lemma 4.7.19. *For all m and n integers, M and N integers satisfying (4.1.12)-(4.1.13), $t \geq 1$,*

$$\frac{1}{\sqrt{t}} \left\| I^{lf,r}_{m,n} \right\|_{L^2} \lesssim \mathcal{B} + \mathcal{Q} + \mathcal{C},$$

where $\mathcal{B}$ is the boundary term:

$$\mathcal{B} := (\sqrt{n+1} + \sqrt{m+1})^2 \frac{\max(n, m)^{\frac{3}{4}}}{\sqrt{\underline{mn}}} t^{\frac{5\delta}{2} + \frac{1}{4}} \left(A(t) + A(1)\right),$$

with

$$A(t) := \|f_m(t)\|_{H^N} \mathcal{B}(f_n)(t),$$

$\mathcal{Q}$ is the quadratic term:

$$\mathcal{Q} := \frac{\max(\underline{m},\underline{n})^{3+\frac{3}{4}}}{\sqrt{\underline{mn}}} \int_1^t s^{3\delta} s^{-\frac{1}{4}} \|f_m\|_{H^N} \mathcal{B}(f_n)(s) ds,$$

and $\mathcal{C}$ is the cubic one:

$$\mathcal{C} := \frac{\max(n,m)^{\frac{1}{2}}}{\sqrt{\underline{mn}}} \int_1^t s^{3\delta} s^{\frac{3}{2}} \|f_m\|_{H^N} \mathcal{B}(f_m)(s)\mathcal{B}(f_n)(s) ds.$$

Proof :
First of all, use the boundedness in the frequency space.

$$\left\| I^{lf,r}_{m,n} \right\|_{L^2} \lesssim t^{\frac{3\delta}{2}} \left\| \int_1^t \int s\partial_\xi \phi \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)\psi^s(\xi,\eta) e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds \right\|_{L^2}.$$

A. Integration by parts in t. Now we are going to use that there are no time resonances on the support of ψ^s, i.e. that ϕ does not vanish. This will allow us to write the following equality:

$$e^{-is\phi} = \frac{1}{-i\phi} \partial_s \left(e^{-is\phi} \right).$$

Then write that

$$\begin{aligned}
&t^{\frac{3\delta}{2}} \int_1^t \int s\partial_\xi \phi \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)\psi^s(\xi,\eta) e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds \\
&= t^{\frac{3\delta}{2}} \left\| \int_1^t \int s\partial_\xi \phi \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)\psi^s(\xi,\eta) \frac{1}{-i\phi} \partial_s \left(e^{-is\phi} \right) \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds \right\|_{L^2},
\end{aligned}$$

and perform an integration by parts. This operation leads to

$$t^{\frac{3}{2}\delta} \int_1^t \int s\partial_\xi \phi \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)\psi^s(\xi,\eta) \frac{1}{-i\phi} \partial_s \left(e^{-is\phi} \right) \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds = \sum_{i=0}^4 I^i_{l,r},$$

where

$$\begin{aligned}
I^0_{l,r} &:= t^{\frac{3}{2}\delta} \left[\int s\partial_\xi \phi \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)\psi^s(\xi,\eta) \frac{1}{-i\phi} e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta \right]_1^t, \\
I^1_{l,r} &:= t^{\frac{3}{2}\delta} \int_1^t \int \partial_\xi \phi \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)\psi^s(\xi,\eta) \frac{1}{-i\phi} e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds, \\
I^2_{l,r} &:= t^{\frac{3}{2}\delta} \int_1^t \int s\partial_\xi \phi \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)\psi^s(\xi,\eta) \frac{1}{-i\phi} e^{-is\phi} \frac{\partial_s\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds, \\
I^3_{l,r} &:= t^{\frac{3}{2}\delta} \int_1^t \int s\partial_\xi \phi \theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)\psi^s(\xi,\eta) \frac{1}{-i\phi} e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\partial_s\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds, \\
I^4_{l,r} &:= t^{\frac{3}{2}\delta} \int_1^t \int s\partial_\xi \phi \partial_s \left[\theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)\psi^s(\xi,\eta) \right] \frac{1}{-i\phi} e^{-is\phi} \frac{\hat{f}_m(\eta)}{\langle\eta\rangle_m} \frac{\hat{f}_n(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds.
\end{aligned}$$

B. A preliminary result. The terms $I^j_{l,r}$ can be written as bilinear multipliers associated with the same symbol. We are going to take advantage of it and prove a general result about the multiplier associated to the following symbol:

$$S(\xi,\eta) := \frac{1}{(\sqrt{n+1}+\sqrt{m+1})^2 t^\delta}\partial_\xi\phi(\xi,\eta)\theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)\psi^s(\xi,\eta)\frac{1}{\phi(\xi,\eta)}.$$

Lemma 4.7.20. *For all g and h we have the following inequality.*

$$\left\|T_S\left(e^{-is\langle D\rangle_m}\frac{g}{\langle D\rangle_m}, e^{-is\langle D\rangle_n}\frac{h}{\langle D\rangle_n}\right)\right\|_{L^2} \lesssim \frac{\max(n,m)^{\frac{3}{4}}}{\sqrt{mn}} s^{-\frac{1}{4}}\left\|g\right\|_{L^2}\sqrt{\left\|h\right\|_{H^N}\left\|h\right\|_{B_s}},$$

with T_S the bilinear operator associated to S as defined in (2.7.2).

Proof :
We want to apply Theorem A.3.8: to do this, we have to estimate the size of the support of S and its behavior with derivation operators:

Lemma 4.7.21. *The symbol S satisfies the hypotheses of Theorem A.3.8, with*

$$\rho = s^\delta,\ \omega = \frac{1}{(\sqrt{n+1}+\sqrt{m+1})^2 s^\delta},\ \mu = \left(\sqrt{\max(\underline{m},\underline{n})}s^{3\delta}(\sqrt{n+1}+\sqrt{m+1})^2 s^\delta\right)^{-1}.$$

Proof of Lemma 4.7.21 :
We are only going to detemrine a value for μ, i.e. we are going to prove that

$$\left|\partial_\xi^a\partial_\eta^b S(\xi,\eta)\right| \lesssim \left(\sqrt{\max(\underline{m},\underline{n})}s^{3\delta}(\sqrt{n+1}+\sqrt{m+1})^2 s^\delta\right)^{a+b}.$$

So as to prove this inequality, we need to understand the effect of differentiation on each factor in S.

- $|\partial_\xi\phi| \leq 2$ and $\left|\partial_\xi^a\partial_\eta^b\phi\right| \leq 2$ for all a, b such that $a+b\geq 1$.
- $\left|\partial_\xi^a\partial_\eta^b\theta\right| \lesssim \frac{1}{s^{(a+b)\delta}} \lesssim 1$ for $s>1$.
- $\phi_{nr} = \theta\left(\sqrt{\max(\underline{m},\underline{n})}s^{3\delta}(\sqrt{n+1}+\sqrt{m+1})^2 s^\delta|\partial_\eta\phi|\right)$ hence, given the boundedness of derivatives of $\partial_\eta\phi$,

$$\left|\partial_\xi^a\partial_\eta^b\psi^s\right| \lesssim \left(\sqrt{\max(\underline{m},\underline{n})}s^{3\delta}(\sqrt{n+1}+\sqrt{m+1})^2 s^\delta\right)^{a+b}.$$

- Finally, by Proposition B.1.4,

$$\frac{1}{|\phi|} \lesssim (\sqrt{n+1}+\sqrt{m+1})^2 s^\delta$$

and

$$\begin{aligned}\left|\partial_\eta\left(\frac{1}{\phi}\right)\right| &= \left|\frac{\partial_\eta\phi}{\phi^2}\right| \\ &\lesssim \frac{1}{|\phi|^2} \\ &\lesssim \left((\sqrt{n+1}+\sqrt{m+1})^2 s^\delta\right)^2.\end{aligned}$$

This ends the proof of Lemma 4.7.21. ■

Then applying Theorem A.3.8 leads to

$$\left\| T_S\left(e^{-is\langle D\rangle_m}\frac{g}{\langle D\rangle_m}, e^{-is\langle D\rangle_n}\frac{h}{\langle D\rangle_n}\right)\right\|_{L^2} \lesssim \sqrt{\max(\underline{m},\underline{n})}s^{3\delta}\left\|\frac{g}{\langle D\rangle_m}\right\|_{L^2}\frac{\underline{n}^{\frac14}}{\sqrt{s}}\left\|\frac{h}{\langle D\rangle_n}\right\|_{W^{\frac32,1}}.$$

Thanks to the linear multiplier estimate (A.1.4-a), we finally obtain

$$\begin{aligned}\left\| T_S\left(e^{-is\langle D\rangle_m}\frac{g}{\langle D\rangle_m}, e^{-is\langle D\rangle_n}\frac{h}{\langle D\rangle_n}\right)\right\|_{L^2} &\lesssim \sqrt{\max(\underline{m},\underline{n})}s^{3\delta}\frac{\max(n,m)^{\frac14}}{\sqrt{mn}}\frac{1}{\sqrt{s}}\left\|g\right\|_{L^2}\left\|h\right\|_{W^{\frac32,1}}\\ &\lesssim \sqrt{\max(\underline{m},\underline{n})}s^{3\delta}\frac{\max(n,m)^{\frac14}}{\sqrt{mn}}s^{-\frac14}\left\|g(s)\right\|_{L^2}\mathcal{B}(h)(s).\end{aligned}$$

The result is the same when exchanging the roles of g and h. This ends the proof of Lemma 4.7.20. ■

C. Application to $I^0_{l,r}$. The integral $I^0_{l,r}$ can be rewritten as follows.

$$\begin{aligned} I^0_{l,r} =& t^{\frac32\delta}\int t\partial_\xi\phi\theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)\psi^t(\xi,\eta)\frac{1}{-i\phi}e^{-it\phi}\frac{\hat f_m(t,\eta)}{\langle\eta\rangle_m}\frac{\hat f_n(t,\xi-\eta)}{\langle\xi-\eta\rangle_n}d\eta\\ &-\int \partial_\xi\phi\theta_{s^\delta}(|\eta|)\theta_{s^\delta}(|\xi-\eta|)\psi^1(\xi,\eta)\frac{1}{-i\phi}e^{-i\phi}\frac{\hat f_m(1,\eta)}{\langle\eta\rangle_m}\frac{\hat f_n(1,\xi-\eta)}{\langle\xi-\eta\rangle_n}d\eta\\ =& t^{\frac52\delta+1}(\sqrt{n+1}+\sqrt{m+1})^2T_S\left(e^{-it\langle D\rangle_m}\frac{f_m(t)}{\langle D\rangle_m}, e^{-it\langle D\rangle_n}\frac{f_n(t)}{\langle D\rangle_n}\right)\\ &-(\sqrt{n+1}+\sqrt{m+1})^2T_S\left(e^{-i\langle D\rangle_m}\frac{f_m(1)}{\langle D\rangle_m}, e^{-i\langle D\rangle_n}\frac{f_n(1)}{\langle D\rangle_n}\right).\end{aligned}$$

Then thanks to Lemma 4.7.20 we have the following inequality:

$$\frac{1}{\sqrt{t}}\left\|I^0_{l,r}\right\|_{L^2} \lesssim (\sqrt{n+1}+\sqrt{m+1})^2 t^{\frac{5\delta}{2}+\frac14}\frac{\max(n,m)^{\frac34}}{\sqrt{\underline{mn}}}\left(A(t)+A(1)\right), \tag{4.7.26}$$

with $A(t) = \left\|f_m(t)\right\|_{H^N}\sqrt{\left\|f_n(t)\right\|_{H^N}\left\|f_n(t)\right\|_{B_t}}$.

D. Estimates for $I^1_{l,r}$. First we give the following formula for $I^1_{l,r}$:

$$I^1_{l,r} = t^{\frac52\delta}(\sqrt{n+1}+\sqrt{m+1})^2\int_1^t T_S\left(e^{-it\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}, e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right)ds.$$

Then dividing by $\sqrt{t}$ and using Lemma 4.7.20 gives

$$\begin{aligned}\frac{1}{\sqrt{t}}\left\|I^1_{l,r}\right\|_{L^2} &\lesssim \frac{1}{\sqrt{t}}t^{\frac52\delta}(\sqrt{n+1}+\sqrt{m+1})^2\int_1^t\left\|T_S\left(e^{-it\langle D\rangle_m}\frac{f_m}{\langle D\rangle_m}, e^{-is\langle D\rangle_n}\frac{f_n}{\langle D\rangle_n}\right)\right\|_{L^2}ds\\ &\lesssim t^{\frac52\delta}(\sqrt{n+1}+\sqrt{m+1})^2\int_1^t s^{-\frac{3\delta}{2}}\frac{\max(n,m)^{\frac34}}{\sqrt{\underline{mn}}}\left\|f_m\right\|_{L^2}s^{-\frac14}\sqrt{\left\|f_n\right\|_{H^N}\left\|f_n\right\|_{B_s}}ds.\end{aligned}$$

Then we have the following inequality:

$$\frac{1}{\sqrt{t}}\left\|I^1_{l,r}\right\|_{L^2} \lesssim (\sqrt{n+1}+\sqrt{m+1})^2\frac{\max(n,m)^{\frac{3}{4}}}{\sqrt{mn}}\int_1^t s^\delta s^{-\frac{1}{4}}\left\|f_m\right\|_{H^N}\sqrt{\left\|f_n\right\|_{H^N}\left\|f_n\right\|_{B_s}}ds. \tag{4.7.27}$$

E. Estimates for $I^2_{l,r}$. We proceed as for $I^1_{l,r}$ and get the following.

$$\frac{1}{\sqrt{t}}\left\|I^1_{l,r}\right\|_{L^2} \lesssim t^{\frac{5}{2}\delta}(\sqrt{n+1}+\sqrt{m+1})^2\int_1^t s\frac{\max(n,m)^{\frac{3}{4}}}{\sqrt{mn}}s^{-\frac{1}{4}}\left\|\partial_s f_m\right\|_{L^2}\sqrt{\left\|f_n\right\|_{H^N}\left\|f_n\right\|_{B_s}}ds.$$

Now remark that

$$\begin{aligned}\partial_s f_{+,m} &= e^{-is\langle D\rangle_m}u_m^2\\ &= e^{-is\langle D\rangle_m}\left(\frac{u_{+,m}-u_{-,m}}{\langle D\rangle_m}\right)^2,\end{aligned}$$

where $u_{\pm,m} = e^{\mp is\langle D\rangle_m}f_{\pm,m}$. Now we use the method explained in details in Chapter 3, Section 3.5.3.c. For the sake of simplicity we write the following inequality:

$$\begin{aligned}\left\|\partial_s f_m\right\|_{L^2} &= \left\|e^{-is\langle D\rangle_m}u_m^2\right\|_{L^2}\\ &= \left\|u_m^2\right\|_{L^2}\\ &\lesssim \left\|u_m\right\|_{L^2}\left\|u_m\right\|_{L^\infty}.\end{aligned}$$

Then given the expression of u_m and the dispersion inequality $(A.1.1)$, this inequality holds.

$$\begin{aligned}\left\|\partial_s f_m\right\|_{L^2} &\lesssim \frac{m^{\frac{1}{4}}}{\sqrt{s}}\left\|f_m\right\|_{L^2}\left\|f_m\right\|_{W^{\frac{3}{2},1}}\\ &\lesssim \frac{m^{\frac{1}{4}}}{s^{\frac{1}{4}}}\sqrt{\left\|f_m\right\|^3_{H^N}\left\|f_m\right\|_{B_s}}.\end{aligned}$$

This leads to

$$\begin{aligned}\frac{1}{\sqrt{t}}\left\|I^1_{l,r}\right\|_{L^2} \lesssim\; & t^{\frac{5}{2}\delta}(\sqrt{n+1}+\sqrt{m+1})^2\\ &\times\int_1^t s\frac{\max(n,m)^{\frac{3}{4}}}{\sqrt{mn}}s^{-\frac{1}{4}}\frac{m^{\frac{1}{4}}}{s^{\frac{1}{4}}}\sqrt{\left\|f_m\right\|^3_{H^N}\left\|f_m\right\|_{B_s}}\sqrt{\left\|f_n\right\|_{H^N}\left\|f_n\right\|_{B_s}}ds.\end{aligned} \tag{4.7.28}$$

F. Estimates for $I^3_{l,r}$ and $I^4_{l,r}$ will be skipped since they can be treated as $I^2_{l,r}$, even if we differentiate the function θ_s.

We finally gather Inequalities (4.7.26), (4.7.27) and (4.7.28): Lemma 4.7.19 is now proved.■

4.7.4.b Outside the resonant set.

The lemma to prove is the following one:

Lemma 4.7.22. *For all m and n integers, $N \geq 3/2$, $t \geq 0$,*

$$\frac{1}{\sqrt{t}} \left\| I^{lf,nr}_{m,n} \right\|_{L^2} \lesssim \frac{\max(\underline{m}, \underline{n})^{3+\frac{3}{4}}}{\sqrt{mn}} \int_1^t s^{3\delta} s^{\frac{1}{4}} \left\| f_m \right\|_{H^N} \sqrt{\left\| f_n \right\|_{H^N} \left\| f_n \right\|_{B_s}} ds,$$

where $I^{lf,nr}_{m,n}$ is defined in (4.7.8), page 94.

Here the estimates are almost a copy-paste of the method developped page 96, with these two changes.

1. The term $|\partial_\xi \phi|$ is no longer smaller than $|\partial_\eta \phi|$. Then the quantity $\left|\frac{\partial_\xi \phi}{\partial_\eta \phi}\right|$ is bounded by 2^j in the zone $\partial_\eta \phi \sim -2^{-j}$.
2. We also define the symbols $S^{\pm}_j$, $S^{\pm}_{j,k}$, M^1, M^2, M^3, etc. However, given the change of localization around the space resonant set (cf. the definition of ψ^s (4.7.25)), the equality (4.7.13) becomes

$$(1 - \psi^s) \frac{\partial_\xi \phi}{\partial_\eta \phi} \frac{\partial^2_\eta \phi}{\partial_\eta \phi} = \sum_{1/2 \leq 2^j \leq \sqrt{\max(\underline{m},\underline{n}) t^{4\delta} (\sqrt{n+1} + \sqrt{m+1})^2}} S^+_j + S^-_j, \tag{4.7.29}$$

Since

$$\frac{1}{\sqrt{s}} \sum_{2^j \leq \sqrt{\max(\underline{m},\underline{n}) t^{4\delta} (\sqrt{n+1} + \sqrt{m+1})^2}} 2^{2j} \lesssim \max(\underline{m}, \underline{n})^3 \frac{t^{8\delta}}{\sqrt{s}},$$

we have the following estimate:

$$\frac{1}{\sqrt{t}} \left\| I^{lf,nr}_{m,n} \right\|_{L^2} \lesssim \frac{\max(\underline{m}, \underline{n})^{3+\frac{3}{4}}}{\sqrt{mn}} \int_1^t s^{\frac{3\delta}{2}} s^{-\frac{1}{4}} \left\| f_m \right\|_{H^N} \sqrt{\left\| f_n \right\|_{H^N} \left\| f_n \right\|_{B_s}} ds.$$

Lemmas 4.7.19 and 4.7.22 give Lemma 4.7.18.

Then, Proposition 4.7.2 is proven by combining Propositions 4.7.3, 4.7.4 and 4.7.18.

Remark 4.7.23. *The case where $p \leq m$ or $p \leq n$ should also be treated separately, but estimating this term is very similar to what we did in the case "$p > m$, $p > n$ but condition C not satisfied" (Section 4.7.4): actually it is even easier since we do not have time resonances.*

Finally, Proposition 4.5.1, Proposition 4.6.1 and Proposition 4.7.1 lead to Proposition 4.4.1.
Propositions 4.4.1 and 4.3.1 give Theorem 4.2.5, and consequently Theorem 4.1.10 is proven. ■

Chapter 5

Resonant system

Contents

5.1 Introduction

In the previous chapter we proved an existence theorem with the method of space-time resonances, which depends on a pointwise-in-time study of the Duhamel formula (contrary to Strichartz-based methods). This fine study can help to understand the dynamics of the equation: for example Germain Masmoudi and Shatah establish scattering results for Schrödinger's equation in [24].

Here we do not have any scattering result since we do not have a global existence result. In order to understand the dynamics of 4.1.1, it may be useful to derive a simpler system, generating the dynamics of the original equation. This study of a resonant system has been initiated in hyperbolic equations by Klainerman and Majda in [39] for incompressible fluids, then by Grenier [30] and Schochet [51] with the so-called "filtering method" for highly rotating fluids.
Using this notion of resonant system in the framework of dispersive equations is more recent: Ionescu and Pausader in [34] and [35] studied the nonlinear Schrödinger equation on $\mathbb{R} \times \mathbb{T}^3$; other studies have been made by Hani, Pausader, Tzvetkov, Visciglia in [31] for NLS in $\mathbb{R} \times \mathbb{T}^d$ $(1 \leq d \leq 4)$, by Pausader, Tzvetkov and Wang for NLS on $\mathbb{S}^3$ ([45]) and more recently by Hani and Thomann in for the NLS with a harmonic trapping ([32]). In all of these articles, the main idea is that the dynamics of a system is governed by the resonant frequencies. So as to understand this idea, assume that a quadratic dispersive PDE has the following Duhamel formula:

$$\hat{f}(t,\xi) = \hat{f}(0,\xi) + \int_0^t \int e^{is\phi(\xi,\eta)} \hat{f}(\eta)\hat{f}(\xi-\eta)d\eta ds. \tag{5.1.1}$$

Assume that for all ξ, there is one $\eta_0(\xi)$ such that $\partial_\eta \phi(\xi, \eta_0(x)) = 0$. Then, a Stationary Phase Lemma will give

$$\begin{aligned}\int e^{is\phi(\xi,\eta)} \hat{f}(\eta)\hat{f}(\xi-\eta)d\eta =& \frac{C}{\sqrt{s}} e^{is\phi(\xi,\eta_0(\xi))} \hat{f}(\eta_0(\xi))\hat{f}(\xi-\eta_0(\xi)) \\ &+ \text{remainder decreasing with time.}\end{aligned}$$

Moreover, if $\phi(\xi, \eta_0(\xi)) = 0$, the integral

$$\int_0^t e^{is\phi(\xi,\eta_0(\xi))} \hat{f}(\eta_0(\xi))\hat{f}(\xi-\eta_0(\xi))ds$$

is an oscillating integral, and it is bounded as t goes to infinity thanks to Riemann-Lebesgue's Lemma. Hence, if Ξ is the set of ξ such that $\phi(\xi, \eta_0(\xi)) = 0$ and $\mathbb{I}_{\xi\in\Xi}$ the indicator function of Ξ, the leading term in the Duhamel formula is

$$\mathbb{I}_{\xi\in\Xi} \int_0^t \hat{f}(\eta_0(\xi))\hat{f}(\xi-\eta_0(\xi))ds.$$

We will call the equation

$$\hat{f}(t,\xi) = \hat{f}(0,\xi) + \mathbb{I}_{\xi\in\Xi} \int_0^t \hat{f}(\eta_0(\xi))\hat{f}(\xi-\eta_0(\xi))ds \tag{5.1.2}$$

the *resonant equation*. This equation is simpler since we restricted the original one to some *resonant modes*. In the case of anisotropic models as ours (with one free direction

and one direction trapped by a harmonic potential), the resonant equation keeps the same form but with trickier resonant conditions.
A good resonant system has to satisfy the two following properties:

1. it has to be a good approximation of the initial equation, i.e. if f is a solution of (5.1.1) and g is a solution of (5.1.2) with the same initial data, then $f-g$ goes to zero as t goes to infinity (if we are in the lucky case of a global existence).
2. we should be able to understand its dynamics. For example, in [31], Hani, Pausader, Tzvetkov, Visciglia were able to build solutions of the resonant system with growing Sobolev norms, and consequently prove that the initial equation had solutions with growing Sobolev norms.

In this manuscript we focus on the derivation of the resonant system, the existence of long-time solutions for this system and the validity of the approximation of the initial equation by this system.
The resonant equation associated to (4.1.1) is

$$\widetilde{f}_{\pm,p}(t,\xi)=\widetilde{f}_{\pm,p}(0,\xi)+\int_0^t \sum_{\substack{m,n\in\mathbb{Z}\\ \alpha,\beta\in\{\pm1\}\\ m\neq n \text{ or } \alpha\neq-\beta\\ (C)\text{ satisfied}}} \frac{\mathcal{M}(m,n,p)}{\sqrt{s|\partial_\eta^2\phi(\xi,\lambda^{\alpha,\beta}_{m,n}\xi)|}}\frac{\widetilde{f}_{\alpha,m}(\lambda^{\alpha,\beta}_{m,n}\xi)}{\langle\lambda^{\alpha,\beta}_{m,n}\xi\rangle_m}\frac{\widetilde{f}_{\beta,n}((1-\lambda^{\alpha,\beta}_{m,n})\xi)}{\langle(1-\lambda^{\alpha,\beta}_{m,n})\xi\rangle_n}ds, \tag{5.1.3}$$

with $\lambda^{\alpha,\beta}_{m,n}:=\frac{1}{1+\alpha\beta\sqrt{\frac{n+1}{m+1}}}$ and (C) is the resonant condition appearing in Theorem 4.2.7.
Since the system is simpler given it involves only selected interacting modes, we are able to prove a better existence and uniqueness result than for the original one.

Theorem 5.1.1. *Let $\varepsilon>0$, $T=C/\varepsilon^2$ with C a universal constant. Then, given M, N and κ satisfying*

$$M>6,\ N\geq\frac{3}{2},\ \kappa=1 \text{ or } 2, \tag{5.1.4}$$

if $f_0=(f_{0,+},f_{0,-})$ is an initial data with $\|f_0\|_{\mathcal{H}^MH^N(\langle x_1\rangle^\kappa)}\leq\varepsilon/2$, then there exists one and only one solution $f=(f_+,f_-)$ to the resonant system on the interval $[0,T)$, belonging to the $L^\infty_t\Big([0,T),\mathcal{H}^MH^N(\langle x_1\rangle^\kappa)\Big)$. Moreover, for all $t\in[0,T)$, $\|f_\pm\|_{\mathcal{H}^MH^N(\langle x_1\rangle^\kappa)}\leq\varepsilon$.

Moreover we are able to prove that this resonant system is a good approximation of the initial equation:

Theorem 5.1.2. *Let $0<\alpha<\frac{5}{3}$, $0<\omega<1-\frac{3}{5}\alpha$, $\varepsilon>0$. Let N and M satisfying (4.1.12), (4.1.13) with in addition $N\geq 9-\frac{1}{4}$. Let $0\leq M_0<M-\frac{1}{8}$.*
There exists a $C(\alpha,\omega)$ such that, for ε small enough, if $T=C(\alpha,\omega)\varepsilon^{-\frac{4}{3+\omega}}$, if
— f is a solution to the initial system in $\Sigma^{M,N}_T$ with initial data f_0 in the ball of center 0 and radius $\varepsilon/2$ of $S^{M,N}_0$,
— g is a solution to the resonant system in $\Sigma^{M,N}_T$ with the same initial data,
then we have, for all $t\leq T$,

$$\|(f-g)(t)\|_{\mathcal{H}^{M_0}L^2}\leq\varepsilon^\alpha.$$

Remark 5.1.3. *We have to compare the size of $f-g$ to the variation of f and g during a time $\varepsilon^{-\frac{4}{3+\omega}}$: we prove in Theorem 4.2.5 that the one for f is of order*

$$\int_0^{\varepsilon^{-\frac{4}{3+\omega}}} s^{-\frac{1}{4}}\varepsilon^2 ds\sim\varepsilon^{2-\frac{3}{3+\omega}}.$$

Similarly, the increase of g is of order

$$\int_0^{\varepsilon^{-\frac{4}{3+\omega}}} s^{-\frac{1}{2}}\varepsilon^2 ds \sim \varepsilon^{2-\frac{2}{3+3\omega}} << \varepsilon^{2-\frac{3}{3+\omega}}.$$

But if ω is small enough, we have $\varepsilon^\alpha << \varepsilon^{2-\frac{2}{3+3\omega}}$, which means that the size of $f-g$ is small compared to the variation of f and g.

5.2 Derivation of the resonant system

The Duhamel formula for (4.1.1) is the following

$$\widetilde{f}_{\pm,p}(t,\xi) = \widetilde{f}_{\pm,p}(0,\xi) + D_1(f,f),$$

where

$$D_1(f,f) = \sum_{m,n}\sum_{\alpha,\beta=\pm 1} \alpha\beta\mathcal{M}(n,m,p)\int_0^t\int_{\mathbb{R}} e^{\mp is\phi^{\alpha,\beta}_{m,n,p}}\frac{\widetilde{f}_{\alpha,m}(\eta)}{\langle\eta\rangle_m}\frac{\widetilde{f}_{\beta,n}(\xi-\eta)}{\langle\xi-\eta\rangle_n}d\eta ds.$$

It describes the interaction between Fourier modes (input frequency η and $\xi-\eta$, output frequency ξ) or between Hermite modes (input modes m and n, output mode p). We saw in the study of existence and uniqueness for the original equation that some modes are *resonant*: more particularly, they can be *space resonant* when $\partial_\eta\phi^{\alpha,\beta}_{m,n,p}(\xi,\eta) = 0$ or *time resonant* when $\phi^{\alpha,\beta}_{m,n,p}(\xi,\eta) = 0$. These resonant interactions must be the ones governing the dynamics of the whole equation.
In this section we are going to determine these resonant interactions and the corresponding resonant equation.

Space resonant interactions. The first step is to remove the interactions which are non resonant in space, i.e. the interactions such that $\partial_\eta\phi \neq 0$. In Appendix B, the cancellation of $\partial_\eta\phi$ is studied in detail :

1. if $\alpha = -\beta$ and $m = n$, then $\partial_\eta\phi$ is identically zero for $\xi = 0$, and does not vanishes for $\xi \neq 0$. Moreover ϕ never vanishes.
2. otherwise, for all ξ there exists one and only one $\eta_0(\xi)$ such that $\partial_\eta\phi(\xi,\eta_0(\xi)) = 0$:

$$\eta_0(\xi) := \lambda^{\alpha,\beta}_{m,n}\xi, \text{ where } \lambda^{\alpha,\beta}_{m,n} = \frac{1}{1+\alpha\beta\sqrt{\frac{n+1}{m+1}}}.$$

Hence it is natural to approximate the Duhamel formula as follows

1. first, if $m = n$ and $\alpha = -\beta$, the non-cancellation of $\partial_\eta\phi$ allows us to perform an integration by parts in η, and consequently gain a decay in s. That is why we are allowed to remove those Hermite modes

$$D_1(f,f) \sim \sum_{\substack{m,n\in\mathbb{N}\\ \alpha,\beta\in\{\pm 1\}\\ m\neq n \text{ or } \alpha\neq-\beta}} \alpha\beta\mathcal{M}(n,m,p)\int_0^t\int_{\mathbb{R}} e^{\mp is\phi^{\alpha,\beta}_{m,n,p}}\frac{\widetilde{f}_{\alpha,m}(\eta)}{\langle\eta\rangle_m}\frac{\widetilde{f}_{\beta,n}(\xi-\eta)}{\langle\xi-\eta\rangle_n}d\eta ds.$$

2. then if $m \neq n$ or $\alpha \neq \beta$, we are in the framework of stationary phase Lemma: the behavior of the oscillating integral in η

$$\int_{\mathbb{R}} e^{\mp is\phi^{\alpha,\beta}_{m,n,p}} \frac{\widetilde{f}_{\alpha,m}(\eta)}{\langle\eta\rangle_m} \frac{\widetilde{f}_{\beta,n}(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta.$$

is governed by the frequencies on which $\partial_\eta \phi$ vanish, i.e.

$$\begin{aligned}&\int_{\mathbb{R}} e^{\mp is\phi^{\alpha,\beta}_{m,n,p}} \frac{\widetilde{f}_{\alpha,m}(\eta)}{\langle\eta\rangle_m} \frac{\widetilde{f}_{\beta,n}(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta \\ &\sim \frac{C}{\sqrt{s|\partial^2_\eta \phi(\xi,\eta_0)|}} e^{\mp is\phi^{\alpha,\beta}_{m,n,p}(\xi,\eta_0(\xi))} \frac{\widetilde{f}_{\alpha,m}(\eta_0(\xi))}{\langle\eta_0(\xi)\rangle_m} \frac{\widetilde{f}_{\beta,n}(\xi-\eta_0(\xi))}{\langle\xi-\eta_0(\xi)\rangle_n},\end{aligned}$$

for some constant C. Hence the following approximation for $D_1(f,f)$:

$$D_1(f,f) \sim \sum_{\substack{m,n\in\mathbb{N} \\ \alpha,\beta\in\{\pm 1\} \\ m\neq n \text{ or } \alpha\neq-\beta}} \alpha\beta\mathcal{M}(n,m,p) \int_0^t \frac{Ce^{\mp is\phi^{\alpha,\beta}_{m,n,p}(\xi,\eta_0(\xi))}}{\sqrt{s|\partial^2_\eta \phi(\xi,\eta_0)|}} \frac{\widetilde{f}_{\alpha,m}(\eta_0(\xi))}{\langle\eta_0(\xi)\rangle_m} \frac{\widetilde{f}_{\beta,n}(\xi-\eta_0(\xi))}{\langle\xi-\eta_0(\xi)\rangle_n} ds.$$

Time resonances. Now the space nonresonant interactions have been removed, the approximate formula we obtained is a sum of oscillating integrals of the form

$$\int_0^t e^{\mp is\phi^{\alpha,\beta}_{m,n,p}(\xi,\eta_0(\xi))} F^{\alpha,\beta}_{m,n,p}(s,\xi) ds.$$

The integrals for which $\phi^{\alpha,\beta}_{m,n,p}$ is different from 0 are more likely to be neglectable compared to the ones where it is. But $\phi^{\alpha,\beta}_{m,n,p}$ vanishes on the zero set of $\partial_\eta \phi^{\alpha,\beta}_{m,n,p}$ if and only if the condition (C) is satisfied :

$$m^2 + n^2 + p^2 - 2mn - 2pm - 2pn - 2m - 2n - 2p - 3 = 0. \tag{C}$$

Hence if (C) is not satisfied, the integral

$$\int_0^t \frac{Ce^{\mp is\phi^{\alpha,\beta}_{m,n,p}(\xi,\eta_0(\xi))}}{\sqrt{s|\partial^2_\eta \phi(\xi,\eta_0)|}} \frac{\widetilde{f}_{\alpha,m}(\eta_0(\xi))}{\langle\eta_0(\xi)\rangle_m} \frac{\widetilde{f}_{\beta,n}(\xi-\eta_0(\xi))}{\langle\xi-\eta_0(\xi)\rangle_n} ds$$

is an oscillating integral. Hence the approximation

$$D_1(f,f) \sim \int_0^t \sum_{\substack{m,n\in\mathbb{Z} \\ \alpha,\beta\in\{\pm 1\} \\ m\neq n \text{ or } \alpha\neq-\beta \\ (C) \text{ satisfied}}} \frac{C}{\sqrt{s|\partial^2_\eta \phi(\xi,\eta_0)|}} \frac{\widetilde{f}_{\alpha,m}(\lambda^{\alpha,\beta}_{m,n}\xi)}{\langle\lambda^{\alpha,\beta}_{m,n}\xi\rangle_m} \frac{\widetilde{f}_{\beta,n}((1-\lambda^{\alpha,\beta}_{m,n})\xi)}{\langle(1-\lambda^{\alpha,\beta}_{m,n})\xi\rangle_n} ds.$$

The *resonant equation* is the following one

$$\widetilde{f}_{\pm,p}(t,\xi) = \widetilde{f}_{\pm,p}(0,\xi) + \int_0^t \sum_{\substack{m,n\in\mathbb{Z} \\ \alpha,\beta\in\{\pm 1\} \\ m\neq n \text{ or } \alpha\neq-\beta \\ (C) \text{ satisfied}}} \frac{C}{\sqrt{t|\partial^2_\eta \phi(\xi,\eta_0)|}} \frac{\widetilde{f}_{\alpha,m}(\lambda^{\alpha,\beta}_{m,n}\xi)}{\langle\lambda^{\alpha,\beta}_{m,n}\xi\rangle_m} \frac{\widetilde{f}_{\beta,n}((1-\lambda^{\alpha,\beta}_{m,n})\xi)}{\langle(1-\lambda^{\alpha,\beta}_{m,n})\xi\rangle_n} ds. \tag{5.2.1}$$

Given the heuristic approach we made, it is natural that this equation approximates correctly the dynamics of the original equation. So as to prove this, we are going to proceed in two steps:

1. first of all, we have to show that the solutions of the resonant system exist for a time at least as long as the solutions of the original system (we are even going to get a longer existence time).
2. then we have to prove that the solutions of the resonant system are a good approximation (in a sense explained in section 5.4) of the solutions of the original one, as long as the former exists.

5.3 Long-time existence for the resonant system

The resonant system being simpler than the original equation, it seems reasonable to find an existence time at least as good as $1/\varepsilon^{\frac{4}{3}}$ where ε is the size of the initial data.

Our target is the same as in the proof of Theorem 4.1.10 : we are going to prove contraction estimates for the operator $\widetilde{\mathrm{Res}}_{\pm,p}(f,f)$. In order not to be too redundant, we are going to gather all the technical details in one proposition.
Since the Duhamel formula is symmetric in n and m, we are going to prove a contraction estimate for this "half Duhamel formula":

$$\begin{aligned}\widetilde{f}_{\pm,p}(t,\xi) =&\widetilde{f}_{\pm,p}(0,\xi)\\ &+\int_0^t \sum_{\substack{m,n\in\mathbb{Z}\\ \alpha,\beta\in\{\pm1\}\\ m\leq n\\ m\neq n \text{ or } \alpha\neq-\beta\\ (C)\text{ satisfied}}} \mathcal{M}(m,n,p)\frac{C}{\sqrt{s|\partial_\eta^2\phi(\xi,\eta_0)|}}\frac{\widetilde{f}_{\alpha,m}(\lambda_{m,n}^{\alpha,\beta}\xi)}{\langle\lambda_{m,n}^{\alpha,\beta}\xi\rangle_m}\frac{\widetilde{f}_{\beta,n}((1-\lambda_{m,n}^{\alpha,\beta})\xi)}{\langle(1-\lambda_{m,n}^{\alpha,\beta})\xi\rangle_n}ds.\end{aligned} \tag{5.3.1}$$

Remark 5.3.1. *This equation is different and much simpler to deal with than the Duhamel formula* (4.2.2) *for the original equation* (4.1.1)*. In fact, it does not involve any integration in ξ: then estimating the integral term will be based on Young's inequality instead of Hölder-like inequalities.*

We are going to give a few useful bounds for $\lambda = \lambda_{m,n}^{\alpha,\beta} := \frac{1}{1+\alpha\beta\sqrt{\frac{n+1}{m+1}}}$.

1. If $\alpha\beta = 1$, then λ and $1-\lambda$ are bounded by 1. Otherwise, since $m \neq n$, the maximum of λ is reached for $|n-m| = 1$. This leads to the bound
$$|\lambda| \lesssim \min(m,n). \tag{5.3.2}$$
2. Since we are in the case $m \leq n$, we also have the bound
$$\frac{1}{1-\lambda} = 1 + \alpha\beta\sqrt{\frac{m+1}{n+1}} \leq 2. \tag{5.3.3}$$

5.3.1 A preliminary bilinear estimate

Proposition 5.3.2. *Let a, b, c, and d be four integers, m and n two integers with $m < n$. Then if we define $S = S(a,b,c,d) = H^{a\wedge b}(\langle x\rangle^{A(c,d)})$, we have*

$$\left\| (\lambda\xi)^a((1-\lambda)\xi)^b \lambda^{e(c,d)} \frac{\partial_\xi^c \widetilde{f}_{\alpha,m}(\lambda\xi)}{\langle \lambda\xi\rangle_m} \frac{\partial_\xi^d \widetilde{f}_{\beta,n}((1-\lambda)\xi)}{\langle (1-\lambda)\xi\rangle_n} \right\|_{L^2_\xi} \lesssim \frac{1}{\sqrt{mn}} \left\| f_m \right\|_S \left\| f_n \right\|_S ,$$

where

$$a \wedge b = \max(a,b), \;\; e(c,d) = \begin{cases} 0 & \text{if} \quad c \leq d, \\ \frac{1}{2} & \text{if} \quad c > d, \end{cases} \quad \text{and } A(c,d) = \begin{cases} \max(c,d) & \text{if} \quad c \neq d, \\ c+1 & \text{if} \quad c = d. \end{cases}$$

For example, for $a = 1$, $b = N$, $c = d = 0$, we have

$$\left\| (\lambda\xi)((1-\lambda)\xi)^N \frac{\widetilde{f}_{\alpha,m}(\lambda\xi)}{\langle \lambda\xi\rangle_m} \frac{\widetilde{f}_{\beta,n}((1-\lambda)\xi)}{\langle (1-\lambda)\xi\rangle_n} \right\|_{L^2_\xi} \lesssim \frac{1}{\sqrt{mn}} \left\| f_m \right\|_{H^N(\langle x\rangle)} \left\| f_n \right\|_{H^N(\langle x\rangle)} .$$

If $a = 3/2$, $b = N$, $c = 1$, $d = 0$, we have

$$\left\| (\lambda\xi)^{3/2}((1-\lambda)\xi)^N \lambda^{1/2} \frac{\partial_\xi \widetilde{f}_{\alpha,m}(\lambda\xi)}{\langle \lambda\xi\rangle_m} \frac{\widetilde{f}_{\beta,n}((1-\lambda)\xi)}{\langle (1-\lambda)\xi\rangle_n} \right\|_{L^2_\xi} \lesssim \frac{1}{\sqrt{mn}} \left\| f_m \right\|_{H^N(\langle x\rangle)} \left\| f_n \right\|_{H^N(\langle x\rangle)} .$$

Proof :
Let us only prove two cases: $a > b$, $c > d$ and $a > b$, $c < d$.
If $a > b$ and $c > d$, then

$$\begin{aligned}
&\left\| (\lambda\xi)^a((1-\lambda)\xi)^b \sqrt{\lambda} \frac{\partial_\xi^c \widetilde{f}_{\alpha,m}(\lambda\xi)}{\langle \lambda\xi\rangle_m} \frac{\partial_\xi^d \widetilde{f}_{\beta,n}((1-\lambda)\xi)}{\langle (1-\lambda)\xi\rangle_n} \right\|_{L^2_\xi} \\
&\leq \left\| (\lambda\xi)^a \sqrt{\lambda} \frac{\partial_\xi^c \widetilde{f}_{\alpha,m}(\lambda\xi)}{\langle \lambda\xi\rangle_m} \right\|_{L^2_\xi} \left\| ((1-\lambda)\xi)^b \frac{\partial_\xi^d \widetilde{f}_{\beta,n}((1-\lambda)\xi)}{\langle (1-\lambda)\xi\rangle_n} \right\|_{L^\infty_\xi} \\
&\leq \left\| \xi^a \frac{\partial_\xi^c \widetilde{f}_{\alpha,m}(\xi)}{\langle \xi\rangle_m} \right\|_{L^2_\xi} \left\| \xi^b \frac{\partial_\xi^d \widetilde{f}_{\beta,n}(\xi)}{\langle \xi\rangle_n} \right\|_{L^\infty_\xi} ,
\end{aligned}$$

by a change of variable (dilation). Then, since $\frac{1}{\langle\xi\rangle_p}$ is bounded by $\frac{1}{\sqrt{p}}$,

$$\begin{aligned}
&\left\| (\lambda\xi)^a((1-\lambda)\xi)^b \sqrt{\lambda} \frac{\partial_\xi^c \widetilde{f}_{\alpha,m}(\lambda\xi)}{\langle \lambda\xi\rangle_m} \frac{\partial_\xi^d \widetilde{f}_{\beta,n}((1-\lambda)\xi)}{\langle (1-\lambda)\xi\rangle_n} \right\|_{L^2_\xi} \\
&\leq \frac{1}{\sqrt{mn}} \left\| \xi^a \partial_\xi^c \widetilde{f}_{\alpha,m}(\xi) \right\|_{L^2_\xi} \left\| \xi^b \partial_\xi^d \widetilde{f}_{\beta,n}(\xi) \right\|_{L^\infty_\xi} .
\end{aligned}$$

Then by L^2 continuity of the Fourier transform,

$$\begin{aligned}
\left\| \xi^a \partial_\xi^c \widetilde{f}_{\alpha,m}(\xi) \right\|_{L^2_\xi} &\lesssim \left\| D^a \left(x^c f_{\alpha,m} \right) \right\|_{L^2_x} \\
&\lesssim \left\| f_{\alpha,m} \right\|_{H^a(\langle x\rangle^c)} .
\end{aligned}$$

By $L^1 \to L^\infty$ continuity of the Fourier transform,

$$\begin{aligned}\left\|\xi^b \partial_\xi^d \widetilde{f}_{\beta,n}(\xi)\right\|_{L^\infty_\xi} &\lesssim \left\|D^b x^d f_{\beta,n}\right\|_{L^1_x} \\ &\lesssim \sqrt{\|D^b x^d f_{\beta,n}\|_{L^2_x} \|x D^b x^d f_{\beta,n}\|_{L^2_x}},\end{aligned}$$

by Proposition A.0.1. Then,

$$\left\|D^b x^d f_{\beta,n}\right\|_{L^2_x} \lesssim \|f_{\beta,n}\|_{H^b(\langle x\rangle^d)} \lesssim \|f_{\beta,n}\|_{H^a(\langle x\rangle^c)}.$$

Moreover,

$$xD^b = D^b x - bD^{b-1}.$$

Hence,

$$\begin{aligned}\left\|x D^b x^d f_{\beta,n}\right\|_{L^2_x} &\lesssim \|f_{\beta,n}\|_{H^b(\langle x\rangle^{d+1})} \\ &\lesssim \|f_{\beta,n}\|_{H^a(\langle x\rangle^c)}.\end{aligned}$$

This proves the theorem in the case $a > b$ and $c > d$.
In the case $a > b$, $c < d$, then

$$\begin{aligned}&\left\|(\lambda\xi)^a((1-\lambda)\xi)^b \frac{\partial_\xi^c \widetilde{f}_{\alpha,m}(\lambda\xi)}{\langle\lambda\xi\rangle_m} \frac{\partial_\xi^d \widetilde{f}_{\beta,n}((1-\lambda)\xi)}{\langle(1-\lambda)\xi\rangle_n}\right\|_{L^2_\xi} \\ &\leq \left\|(\lambda\xi)^a \frac{\partial_\xi^c \widetilde{f}_{\alpha,m}(\lambda\xi)}{\langle\lambda\xi\rangle_m}\right\|_{L^\infty_\xi} \left\|((1-\lambda)\xi)^b \frac{\partial_\xi^d \widetilde{f}_{\beta,n}((1-\lambda)\xi)}{\langle(1-\lambda)\xi\rangle_n}\right\|_{L^2_\xi} \\ &\leq \left\|\xi^a \frac{\partial_\xi^c \widetilde{f}_{\alpha,m}(\xi)}{\langle\xi\rangle_m}\right\|_{L^\infty_\xi} \frac{1}{\sqrt{1-\lambda}} \left\|\xi^b \frac{\partial_\xi^d \widetilde{f}_{\beta,n}(\xi)}{\langle\xi\rangle_n}\right\|_{L^2_\xi},\end{aligned}$$

by a change of variables. Then using the bound (5.3.3) reduces the problem to the previous case.
Hence gathering (5.3.4), (5.3.5) and (5.3.6) prove Proposition 5.3.3. ■

5.3.2 Bilinear estimates for (5.3.1).

We are going to use Proposition 5.3.2 to prove the following one:

Proposition 5.3.3. *Let $N \geq \frac{3}{2}$. Then, if*

$$I(\xi) := \frac{1}{\sqrt{|\partial_\eta^2 \phi(\xi,\eta_0)|}} \frac{\widetilde{f}_{\alpha,m}(\lambda\xi)}{\langle\lambda\xi\rangle_m} \frac{\widetilde{f}_{\beta,n}((1-\lambda)\xi)}{\langle(1-\lambda)\xi\rangle_n},$$

we have the following bounds

$$\left\|\mathcal{F}^{-1}(I)\right\|_{H^N(\langle x\rangle)} \lesssim m^{\frac{3}{2}} \frac{1}{\sqrt{mn}} \|f_{\alpha,m}\|_{H^N(\langle x\rangle)} \|f_{\beta,n}\|_{H^N(\langle x\rangle)},$$

$$\left\|\mathcal{F}^{-1}(I)\right\|_{H^N(\langle x\rangle^2)} \lesssim m^{\frac{9}{2}} \frac{1}{\sqrt{mn}} \|f_{\alpha,m}\|_{H^N(\langle x\rangle^2)} \|f_{\beta,n}\|_{H^N(\langle x\rangle^2)}.$$

Proof :
We are going to prove the estimate for the weight $\langle x\rangle^2$, the weight $\langle x\rangle$ being dealt with similarly. We are simply going to estimate the L^2 norms of $|\xi|^N I$, $|\xi|^N\partial_\xi I$ and $|\xi|^N\partial_\xi^2 I$.

1. We first recall that

$$\partial_\eta^2\phi^{\alpha\beta}_{m,n,p}(\xi,\eta)=\frac{2m+2}{(\eta^2+2m+2)^{\frac{3}{2}}}+\varepsilon\frac{2n+2}{((\xi-\eta)^2+2n+2)^{\frac{3}{2}}}.$$

where $\varepsilon=\alpha\beta$. Then, if $\eta=\eta_0:=\frac{\xi}{1+\varepsilon\sqrt{\frac{n+1}{m+1}}}$, a calculation shows that

$$\partial_\eta^2\phi^{\alpha\beta}_{m,n,p}(\xi,\eta_0(\xi))=\frac{2m+2}{\lambda\,(\lambda^2\xi^2+2m+2)^{\frac{3}{2}}}.$$

Hence

$$\frac{1}{\sqrt{|\partial_\eta^2\phi|}}\lesssim\frac{\lambda\langle\lambda\xi\rangle_m^{\frac{3}{2}}}{2m+2}\lesssim\langle\lambda\xi\rangle_m^{\frac{3}{2}},$$

by the bound (5.3.2) on λ. Then

$$|\xi|^N|I|\lesssim|\xi|^N\langle\lambda\xi\rangle_m^{\frac{3}{2}}\left|\frac{\widetilde{f}_{\alpha,m}(\lambda\xi)}{\langle\lambda\xi\rangle_m}\right|\left|\frac{\widetilde{f}_{\beta,n}((1-\lambda)\xi)}{\langle(1-\lambda)\xi\rangle_n}\right|.$$

If we write $|\xi|^N=\frac{1}{(1-\lambda)^N}|(1-\lambda)\xi|^N$, by the bound (5.3.3) we have

$$|\xi|^N|I|\lesssim\left(m^{\frac{3}{4}}+(\lambda\xi)^a\right)((1-\lambda)\xi)^b\left|\frac{\partial_\xi^c\widetilde{f}_{\alpha,m}(\lambda\xi)}{\langle\lambda\xi\rangle_m}\frac{\partial_\xi^d\widetilde{f}_{\beta,n}((1-\lambda)\xi)}{\langle(1-\lambda)\xi\rangle_n}\right|,$$

with

$$a=\frac{3}{2},\ b=N,\ c=0,\ d=0.$$

Then we are in the framework of Proposition 5.3.2, and we conclude by

$$\left\||\xi|^N I\right\|_{L^2}\lesssim\frac{m^{\frac{3}{4}}}{\sqrt{mn}}\left\|f_{\alpha,m}\right\|_{H^N}\left\|f_{\beta,n}\right\|_{H^N}. \tag{5.3.4}$$

2. Then we have to do the same for the weighted norms, i.e. for the ξ derivative of I. We can write

$$\partial_\xi\left(\frac{1}{\sqrt{|\partial_\eta^2\phi(\xi,\eta_0)|}}\frac{\widetilde{f}_{\alpha,m}(\lambda\xi)}{\langle\lambda\xi\rangle_m}\frac{\widetilde{f}_{\beta,n}((1-\lambda)\xi)}{\langle(1-\lambda)\xi\rangle_n}\right)=I_1+I_2+I_3+I_4+I_5,$$

where

$$I_1 := \partial_\xi \left(\frac{1}{\sqrt{|\partial_\eta^2 \phi(\xi, \eta_0)|}} \right) \frac{\widetilde{f}_{\alpha,m}(\lambda\xi)}{\langle \lambda\xi \rangle_m} \frac{\widetilde{f}_{\beta,n}((1-\lambda)\xi)}{\langle (1-\lambda)\xi \rangle_n},$$

$$I_2 := \frac{1}{\sqrt{|\partial_\eta^2 \phi(\xi, \eta_0)|}} \lambda \frac{\partial_\xi \widetilde{f}_{\alpha,m}(\lambda\xi)}{\langle \lambda\xi \rangle_m} \frac{\widetilde{f}_{\beta,n}((1-\lambda)\xi)}{\langle (1-\lambda)\xi \rangle_n},$$

$$I_3 := \frac{1}{\sqrt{|\partial_\eta^2 \phi(\xi, \eta_0)|}} \lambda^2 \xi \frac{\widetilde{f}_{\alpha,m}(\lambda\xi)}{\langle \lambda\xi \rangle_m^3} \frac{\widetilde{f}_{\beta,n}((1-\lambda)\xi)}{\langle (1-\lambda)\xi \rangle_n},$$

$$I_4 := \frac{1}{\sqrt{|\partial_\eta^2 \phi(\xi, \eta_0)|}} \frac{\widetilde{f}_{\alpha,m}(\lambda\xi)}{\langle \lambda\xi \rangle_m} (1-\lambda) \frac{\partial_\xi \widetilde{f}_{\beta,n}((1-\lambda)\xi)}{\langle (1-\lambda)\xi \rangle_n},$$

$$I_5 := \frac{1}{\sqrt{|\partial_\eta^2 \phi(\xi, \eta_0)|}} \frac{\widetilde{f}_{\alpha,m}(\lambda\xi)}{\langle \lambda\xi \rangle_m} (1-\lambda)^2 \xi \frac{\widetilde{f}_{\beta,n}((1-\lambda)\xi)}{\langle (1-\lambda)\xi \rangle_n^3}.$$

The expression of $\partial_\xi \left(\frac{1}{\sqrt{|\partial_\eta^2 \phi(\xi,\eta_0)|}} \right)$ is

$$\partial_\xi \left(\frac{1}{\sqrt{|\partial_\eta^2 \phi(\xi, \eta_0)|}} \right) = \frac{\lambda^{\frac{3}{2}} (\lambda\xi)}{\sqrt{2m+2}((\lambda\xi)^2 + 2m + 2)^{\frac{1}{4}}}.$$

This given, we can write that for every j,

$$|\xi|^N I_j \lesssim m^{\gamma_j} (\lambda\xi)^{a_j} ((1-\lambda)\xi)^{b_j} \lambda^{e(c_j,d_j)} \frac{\partial_\xi^{c_j} \widetilde{f}_{\alpha,m}(\lambda\xi)}{\langle \lambda\xi \rangle_m} \frac{\partial_\xi^{d_j} \widetilde{f}_{\beta,n}((1-\lambda)\xi)}{\langle (1-\lambda)\xi \rangle_n}.$$

The values of the coefficients are summed up in this array.

j	a_j	b_j	c_j	d_j	γ_j
1	1	N	0	0	3/4
2	3/2	N	1	0	3/4
3	1/2	N	0	0	3/2
4	3/2	N	0	1	3/2
5	3/2	N	0	0	3/2

This implies, by Proposition 5.3.2,

$$\left\| |\xi|^N \partial_\xi I \right\|_{L^2} \lesssim \frac{m^{\frac{3}{2}}}{\sqrt{mn}} \|f_{\alpha,m}\|_{H^N(\langle x_1 \rangle)} \|f_{\beta,n}\|_{H^N(\langle x_1 \rangle)}. \tag{5.3.5}$$

3. Dealing with the weight x^2 is very similar: we have to compute

$$\partial_\xi \left(\frac{\lambda^{\frac{3}{2}} (\lambda\xi)}{\sqrt{2m+2}((\lambda\xi)^2 + m + 1)^{\frac{1}{4}}} \right) = \frac{\lambda^{\frac{9}{2}} \frac{\xi^2}{2} + 2\lambda^{\frac{5}{2}}(m+1)}{\sqrt{2m+2}((\lambda\xi)^2 + 2m + 2)^{\frac{5}{4}}}.$$

This quantity can be bounded as follows:

$$\left| \partial_\xi \left(\frac{\lambda^{\frac{3}{2}} (\lambda\xi)}{\sqrt{2m+2}((\lambda\xi)^2 + m + 1)^{\frac{1}{4}}} \right) \right| \lesssim \lambda^{\frac{15}{4}} + \lambda^{\frac{7}{4}}.$$

When applying ∂_ξ to I_j we will get a sum of five terms $I_{j,1}, \dots, I_{j,5}$. We state that for all $1 \le j \le 5$ and $1 \le k \le 5$, for all $N \in \mathbb{N}$,

$$|\xi|^N I_{j,k} \lesssim m^{\gamma_{j,k}} (\lambda\xi)^{a_{j,k}} ((1-\lambda)\xi)^{b_{j,k}} \lambda^{e(c_{j,k},d_{j,k})} \frac{\partial_\xi^{c_{j,k}} \widetilde{f}_{\alpha,m}(\lambda\xi)}{\langle \lambda\xi\rangle_m} \frac{\partial_\xi^{d_{j,k}} \widetilde{f}_{\beta,n}((1-\lambda)\xi)}{\langle (1-\lambda)\xi\rangle_n},$$

with the parameters satisfying the following bounds:

$$\begin{aligned} a_{j,k} &\le \frac{3}{2}, \\ b_{j,k} &= N, \\ c_{j,k} + d_{j,k} &\le 2, \\ \gamma_{j,k} &\le \frac{9}{2}. \end{aligned}$$

Hence we obtain

$$\left\| |\xi|^N \partial_\xi I \right\|_{L^2} \lesssim \frac{m^{\frac{9}{2}}}{\sqrt{mn}} \left\| f_{\alpha,m} \right\|_{H^N(\langle x_1\rangle)} \left\| f_{\beta,n} \right\|_{H^N(\langle x_1\rangle)}. \tag{5.3.6}$$

Hence gathering (5.3.4), (5.3.5) and (5.3.6) we prove Proposition 5.3.3. ■

5.3.3 From the estimate to the theorem

Now we are going to prove Theorem 5.1.1 (in the case of a weight equal to $\langle x_1\rangle^2$). More precisely we are going to prove the following proposition which leads to the theorem by a contraction argument.

Proposition 5.3.4. *Let $N \ge \frac{3}{2}$, $M > 6$. Then if $f_0 \in \mathcal{H}^M H^N(\langle x_1\rangle^2)$, then for all t,*

$$\left\| f(t) \right\|_{\mathcal{H}^M H^N(\langle x_1\rangle^2)} \lesssim \left\| f_0 \right\|_{\mathcal{H}^M H^N(\langle x_1\rangle^2)} + \sqrt{t} \left\| f(t) \right\|^2_{\mathcal{H}^M H^N(\langle x_1\rangle^2)}.$$

Proof :
The proof only consists in summing over m and n the inequality proven in Proposition 5.3.3. We first write

$$\begin{aligned} &\left\| \widetilde{f}_{p,\pm} \right\|_{H^N(\langle x_1\rangle^2)} \\ &\lesssim \left\| \int_0^t \langle\xi\rangle^N \sum_{\substack{m,n\in\mathbb{Z} \\ \alpha,\beta\in\{\pm1\} \\ m\neq n \text{ or } \alpha\neq-\beta \\ m\le n \\ (C) \text{ satisfied}}} \mathcal{M}(m,n,p) \frac{1}{\sqrt{s|\partial_\eta^2\phi(\xi,\eta_0)|}} \frac{\widetilde{f}_{\alpha,m}(\lambda^{\alpha,\beta}_{m,n}\xi)}{\langle \lambda^{\alpha,\beta}_{m,n}\xi\rangle_m} \frac{\widetilde{f}_{\beta,n}((1-\lambda^{\alpha,\beta}_{m,n})\xi)}{\langle (1-\lambda^{\alpha,\beta}_{m,n})\xi\rangle_n} ds \right\| \\ &\lesssim \int_0^t \sum_{\substack{m,n\in\mathbb{Z} \\ \alpha,\beta\in\{\pm1\} \\ m\neq n \text{ or } \alpha\neq-\beta \\ m\le n \\ (C) \text{ satisfied}}} \mathcal{M}(m,n,p) \left\| \langle\xi\rangle^N \frac{1}{\sqrt{s|\partial_\eta^2\phi(\xi,\eta_0)|}} \frac{\widetilde{f}_{\alpha,m}(\lambda^{\alpha,\beta}_{m,n}\xi)}{\langle \lambda^{\alpha,\beta}_{m,n}\xi\rangle_m} \frac{\widetilde{f}_{\beta,n}((1-\lambda^{\alpha,\beta}_{m,n})\xi)}{\langle (1-\lambda^{\alpha,\beta}_{m,n})\xi\rangle_n} \right\|_{L^2_\xi} ds. \end{aligned}$$

Then by Proposition 5.3.3,

$$\left\|\widetilde{f}_{p,\pm}\right\|_{H^N(\langle x_1\rangle^2)} \lesssim \int_0^t \sum_{\substack{m,n\in\mathbb{Z}\\ \alpha,\beta\in\{\pm1\}\\ m\neq n \text{ or } \alpha\neq-\beta\\ m\leq n\\ (C) \text{ satisfied}}} \mathcal{M}(m,n,p)\frac{1}{\sqrt{s}}m^{\frac{9}{2}}\frac{1}{\sqrt{mn}}\left\|f_{\alpha,m}\right\|_{H^N(\langle x_1\rangle^2)}\left\|f_{\beta,n}\right\|_{H^N(\langle x_1\rangle^2)}ds.$$

Then, using $\|f_{\alpha,m}\|_{H^N(\langle x_1\rangle^2)} \leq m^{-M}\|f\|_{\mathcal{H}^M H^N(\langle x_1\rangle^2)}$ and integrating lead to

$$p^M\left\|\widetilde{f}_{p,\pm}\right\|_{H^N(\langle x_1\rangle^2)} \lesssim \sqrt{t}\,\|f\|^2_{\mathcal{H}^M H^N(\langle x_1\rangle^2)}\,p^M \sum_{\substack{m,n\in\mathbb{Z}\\ \alpha,\beta\in\{\pm1\}\\ m\neq n \text{ or } \alpha\neq-\beta\\ m\leq n\\ (C) \text{ satisfied}}} \mathcal{M}(m,n,p)m^{4-M}n^{\frac{1}{2}-M}.$$

Since $M>6$, we are in the framework of the half resummation Theorem C.1.1-(3): there exists a sequence $(u_p(t))_{p\in\mathbb{N}}$ in ℓ^2 such that

$$p^M\left\|\widetilde{f}_{p,\pm}(t)\right\|_{H^N(\langle x_1\rangle^2)} \lesssim \sqrt{t}\,\|f(t)\|^2_{\mathcal{H}^M H^N(\langle x_1\rangle^2)}\,u_p(t).$$

This proves Proposition 5.3.4. ■

5.4 Validity of the approximation

Let f be a solution to the initial system with initial data f_0. Let g be a solution to the resonant system with the same initial data. Our aim is to estimate the difference $h := f-g$. We are going to prove the approximation theorem 5.1.2.

5.4.1 Duhamel formula for h

So as to clarify the notations, let us call D_1 and D_2 the two following bilinear forms:

1. the bilinear operator D_1 corresponds to the Duhamel formula for the original equation:

$$D_1(a,b) := \sum_{m,n}\sum_{\alpha,\beta=\pm1}\alpha\beta\mathcal{M}(n,m,p)\int_0^t\int_{\mathbb{R}} e^{\mp is\phi^{\alpha,\beta}_{m,n,p}}\frac{\widetilde{a}_{\alpha,m}(\eta)}{\langle\eta\rangle_m}\frac{\widetilde{b}_{\beta,n}(\xi-\eta)}{\langle\xi-\eta\rangle_n}d\eta ds,$$

with $\mathcal{M}(m,n,p)$ is the interaction term between three Hermite functions (4.1.4).

2. the bilinear operator D_2 corresponds to the resonant equation:

$$D_2(a,b) := \int_0^t \sum_{\substack{m,n\in\mathbb{Z}\\ \alpha,\beta\in\{\pm1\}\\ m\neq n \text{ or } \alpha\neq-\beta\\ (C) \text{ satisfied}}} \frac{C_{sp}}{\sqrt{t|\partial^2_\eta\phi(\xi,\eta_0)|}}\frac{\widetilde{a}_{\alpha,m}(\lambda^{\alpha,\beta}_{m,n}\xi)}{\langle\lambda^{\alpha,\beta}_{m,n}\xi\rangle_m}\frac{\widetilde{b}_{\beta,n}((1-\lambda^{\alpha,\beta}_{m,n})\xi)}{\langle(1-\lambda^{\alpha,\beta}_{m,n})\xi\rangle_n}ds,$$

with C_{sp} the constant occuring in the Stationary Phase Lemma (Proposition A.4.1) and $\lambda^{\alpha,\beta}_{m,n} = \left(1+\alpha\beta\sqrt{\frac{n+1}{m+1}}\right)^{-1}$.

The Duhamel formula for h can be written as follows:

$$\begin{aligned} h &= f - g \\ &= D_1(f,f) - D_2(g,g) \\ &= D_1(g+h, g+h) - D_2(g,g) \\ &= D_1(h,h) + 2D_1(g,h) + (D_1 - D_2)(g,g). \end{aligned}$$

Our aim is to estimate the $S_t^{M,N}$ norm of h in terms of t and ε (the size of g_0). So as to do it, we will first establish a differential inequality and then use Gronwall's Lemma.

Lemma 5.4.1. *Let $\mathcal{N}(t)$ and $\mathcal{M}(t)$ be the $\mathcal{H}^{M_0}L^2$ and $S_t^{M,N}$ norm of $h(t)$. Then*

$$\mathcal{N}(t) \lesssim \int_0^t \left(s^{\frac{1}{2}+\omega}\mathcal{N}(s)\mathcal{M}(s)^2 + s^{-\frac{1}{4}+\omega}\mathcal{N}(s)\mathcal{M}(s) + s^{-\frac{1}{4}+\omega}\varepsilon\mathcal{N}(s) + \langle s\rangle^{-1}\varepsilon^2 + s^{-\frac{1}{2}}\varepsilon^3 \right) ds.$$

Before going through the proof of the lemma, let us prove the approximation theorem.

Proof of Theorem 5.1.2 :
First of all, whenever $t \leq C\varepsilon^{-\frac{4}{3(1+\omega)}}$, we know by Theorems 4.1.10 and 5.1.1 that

$$\mathcal{M}(t) \leq \varepsilon.$$

Then the previous inequality can be rewritten as a Gronwall inequality

$$\mathcal{N}(t) \leq K\left(\int_0^t s^{\frac{1}{2}+\omega}\varepsilon^2\mathcal{N}(s) + s^{-\frac{1}{4}+\omega}\varepsilon\mathcal{N}(s) + s^{-\frac{1}{2}+\omega}\varepsilon^2 + s^{\omega}s^{\frac{1}{4}}\varepsilon^3 ds \right),$$

which can be simplified again, by using $\varepsilon \leq Ct^{-\frac{3(1+\omega)}{4}}$:

$$\begin{aligned} &s^{\frac{1}{2}+\omega}\varepsilon^2 \leq s^{\frac{1}{2}+\omega}Ct^{-\frac{3(1+\omega)}{4}}\varepsilon \leq Cs^{-\frac{1}{4}}s^{-\frac{\omega}{4}} \\ &s^{\omega}s^{\frac{1}{4}}\varepsilon^3 \leq s^{-\frac{1}{2}+\omega}\varepsilon^2, \end{aligned}$$

which leads to

$$\begin{aligned} \mathcal{N}(t) &\leq CK\int_0^t s^{-\frac{1}{4}+\omega}\varepsilon\mathcal{N}(s) + \langle s\rangle^{-1}\varepsilon^2 ds \\ &\leq CK\ln\langle t\rangle\varepsilon^2 + CK\int_0^t s^{-\frac{1}{4}+\omega}\varepsilon\mathcal{N}(s)ds. \end{aligned}$$

Then, Gronwall's lemma gives

$$\begin{aligned} \mathcal{N}(t) &\leq CK\ln\langle t\rangle\varepsilon^2 + \int_0^t CK\ln\langle s\rangle\varepsilon^2 CKs^{-\frac{1}{4}+\omega}\varepsilon\exp\left(CK\int_s^t r^{-\frac{1}{4}+\omega}\varepsilon dr\right)ds \\ &\leq CK\ln\langle t\rangle\varepsilon^2 + (CK)^2\varepsilon^3\int_0^t \ln\langle s\rangle s^{-\frac{1}{4}+2\omega}\exp\left(CK\varepsilon(t^{\frac{3}{4}+\omega} - s^{\frac{3}{4}+\omega})\right)ds. \end{aligned}$$

Whenever $t \leq C\varepsilon^{-\frac{4}{3+\omega}}$, we have

$$\begin{aligned} &CK\ln\langle t\rangle\varepsilon^2 \leq \frac{CK}{2}\ln(1 + C^2\varepsilon^{-\frac{8}{3+\omega}})\varepsilon^2 \lesssim \varepsilon^{\alpha}, \quad \forall\alpha < 2, \\ &\exp\left(CK\varepsilon(t^{\frac{3}{4}+\omega} - s^{\frac{3}{4}+\omega})\right) \leq \exp\left(CK\varepsilon^2\right), \\ &(CK)^2\varepsilon^3\int_0^t \ln\langle s\rangle s^{-\frac{1}{4}+2\omega}ds \leq (CK)^2\varepsilon^3 t^{\frac{3}{4}+2\omega} \leq (CK)^2\varepsilon^3\varepsilon^{-\frac{3+8\omega}{3+\omega}}. \end{aligned}$$

This proves that for all $\alpha < 2$, for all ω such that $3 - \frac{3+8\omega}{3+\omega} > \alpha$, i.e. $\omega < 3\frac{2-\alpha}{5+\alpha}$ there exists a $C(\alpha,\omega)$ such that, for ε small enough, for all $t \leq C(\alpha,\omega)\varepsilon^{-\frac{4}{3+\omega}}$, we have

$$\mathcal{N}(t) \leq \varepsilon^{\alpha}.$$

This proves the theorem. ■

Proof of Lemma 5.4.1 :
First of all, it has been proven in Theorem 4.2.5 that the operator D_1 satisfies the following inequality:

$$\begin{aligned} \|D_1(a,b)\|_{S_t^{M,N}} \lesssim \int_0^t s^{\omega}\langle s\rangle^{-\frac{1}{4}} \|a\|_{L^2} \|b\|_{S_s^{M,N}} + s^{\omega}\langle s\rangle^{\frac{1}{2}} \|a\|_{L^2} \|b\|_{S_s^{M,N}}^2 \, ds \\ + t^{\omega+\frac{1}{4}} \left(\|a(t)\|_{L^2} \|b(t)\|_{S_t^{M,N}} + \|a\|_{L^2} \|b\|_{S_1^{M,N}} \right) \end{aligned} \tag{5.4.1}$$

This allows to bound the terms $D_1(h,h)$ and $2D_1(g,h)$ involved in the Duhamel formula for h. The rest of the proof is devoted to the bounds for the remainder term, i.e.

$$D_1(g,g) - D_2(g,g).$$

In order to estimate the term $D_1(g,g) - D_2(g,g)$, we are going to write it according to the heuristics done in Section 5.2: we are writing

$$D_1(g,g) - D_2(g,g) = SI_{\pm,p}(g) + NR_{\pm,p}(g) + Osc_{\pm,p}(g),$$

where

1. $SI_{\pm,p}(g)$ is the *self-interacting* term, corresponding to the Hermite modes giving birth to a non space resonant mode for all $\xi \neq 0$:

$$SI_{\pm,p}(g) := \sum_{m} \sum_{\alpha=\pm1,\ \beta=-\alpha} \alpha\beta\mathcal{M}(m,m,p) \int_0^t \int_{\mathbb{R}} e^{\mp is\phi^{\alpha,\beta}_{m,m,p}} \frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m} \frac{\widetilde{g}_{\beta,m}(\xi-\eta)}{\langle\xi-\eta\rangle_m} d\eta ds.$$

2. $NR_{\pm,p}(g)$ corresponds to the stationary phase remainder:

$$NR_{\pm,p}(g) := \sum_{\substack{m,n\in\mathbb{N} \\ \alpha,\beta\in\{\pm1\} \\ m\neq n \text{ or } \alpha\neq-\beta}} \alpha\beta\mathcal{M}(n,m,p) NR^{\alpha,\beta}_{m,n}(t,\xi),$$

with

$$\begin{aligned} NR^{\alpha,\beta}_{m,n}(t,\xi) := \int_0^t \Bigg[\int_{\mathbb{R}} e^{\mp is\phi^{\alpha,\beta}_{m,n,p}} \frac{\widetilde{f}_{\alpha,m}(\eta)}{\langle\eta\rangle_m} \frac{\widetilde{f}_{\beta,n}(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta \\ - \frac{Ce^{\mp is\phi^{\alpha,\beta}_{m,n,p}(\xi,\eta_0(\xi))}}{\sqrt{t|\partial_\eta^2\phi(\xi,\eta_0)|}} \frac{\widetilde{f}_{\alpha,m}(\eta_0(\xi))}{\langle\eta_0(\xi)\rangle_m} \frac{\widetilde{f}_{\beta,n}(\xi-\eta_0(\xi))}{\langle\xi-\eta_0(\xi)\rangle_n} \Bigg] ds. \end{aligned}$$

3. $Osc_{\pm,p}(g)$ corresponds to the modes giving birth to time resonances, i.e.

$$Osc_{\pm,p}(g) := \sum_{\substack{m,n\in\mathbb{N} \\ \alpha,\beta\in\{\pm1\} \\ m\neq n \text{ or } \alpha\neq-\beta \\ (C) \text{ satisfied}}} \alpha\beta\mathcal{M}(n,m,p) \int_0^t \int_{\mathbb{R}} e^{\mp is\phi^{\alpha,\beta}_{m,n,p}} \frac{\widetilde{f}_{\alpha,m}(\eta)}{\langle\eta\rangle_m} \frac{\widetilde{f}_{\beta,n}(\xi-\eta)}{\langle\xi-\eta\rangle_n} d\eta ds.$$

The next three sections will be dedicated to bounding those three terms, and more precisely proving the following inequality

$$\|D_1(g,g) - D_2(g,g)\|_{L^2} \lesssim \int_0^t \left(\frac{1}{\langle s\rangle}\varepsilon^2 + \frac{1}{\sqrt{s}}\varepsilon^3\right) ds. \tag{5.4.2}$$

Then combining (5.4.1) and (5.4.2) give Lemma 5.4.1.

5.4.2 Estimates for the self-interaction remainder

We are going to prove the following lemma:

Lemma 5.4.2. *There exists a sequence $(u_p(s))_{p\in\mathbb{N}}$ in the unit ball of ℓ^2 such that*

$$p^{M_0} \|SI_{\pm,p}(g)\|_{L^2} \lesssim \int_0^t u_p(s)\frac{\varepsilon^2}{\langle s\rangle} ds.$$

Proof of Lemma 5.4.2 :
We want to estimate the L^2 norm of

$$\sum_m \sum_{\alpha=\pm 1,\ \beta=-\alpha} \alpha\beta\mathcal{M}(m,m,p)\int_0^t\int_{\mathbb{R}} e^{\mp is\phi^{\alpha,\beta}_{m,m,p}} \frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m}\frac{\widetilde{g}_{\beta,m}(\xi-\eta)}{\langle\xi-\eta\rangle_m} d\eta ds.$$

We know that in this case the quantity $\phi^{\alpha,\beta}_{m,m,p}(\xi,\eta_0(\xi))$ never vanishes except when $\xi = 0$. This is the reason why we will handle separately the zones around and outside the origin. Let χ be a smooth function, compactly supported, which is equal to 1 on $[-\frac{1}{2},\frac{1}{2}]$ and 0 outside $[-1,1]$. Then we write

$$\sum_m \sum_{\alpha=\pm 1,\ \beta=-\alpha} \alpha\beta\mathcal{M}(m,m,p)\int_0^t\int_{\mathbb{R}} e^{\mp is\phi^{\alpha,\beta}_{m,m,p}} \frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m}\frac{\widetilde{g}_{\beta,m}(\xi-\eta)}{\langle\xi-\eta\rangle_m} d\eta ds = SI_s + SI_l,$$

with SI_s corresponding to the small values of ξ

$$SI_s := \chi(\xi)\sum_m \sum_{\alpha=\pm 1,\ \beta=-\alpha} \alpha\beta\mathcal{M}(m,m,p)\int_0^t\int_{\mathbb{R}} e^{\mp is\phi^{\alpha,\beta}_{m,m,p}} \frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m}\frac{\widetilde{g}_{\beta,m}(\xi-\eta)}{\langle\xi-\eta\rangle_m} d\eta ds,$$

and SI_l corresponding to the large ones. We will use two different strategies for these integrals : SI_s will be bounded by using the time resonances method, SI_l by a stationary phase.

5.4.2.a Study of SI_s.

In the zone $|\xi| \le 1$, the phase $\phi^{\alpha,\beta}_{m,m,p}$ does not vanish and is easily bounded. In fact, since $\beta = -\alpha$,

$$\phi^{\alpha,\beta}_{m,m,p} = \sqrt{\xi^2+2p+2} + \alpha\left(\sqrt{\eta^2+2m+2} - \sqrt{(\xi-\eta)^2+2m+2}\right).$$

Since $|\xi| \le 1$,

$$\begin{aligned}\left|\sqrt{\eta^2+2m+2} - \sqrt{(\xi-\eta)^2+2m+2}\right| &\le |\xi|\sup_\eta \frac{|\eta|}{\sqrt{\eta^2+2m+2}}\\ &\le 1.\end{aligned}$$

Then

$$|\phi^{\alpha,\beta}_{m,m,p}| \geq \sqrt{2p+2} - 1 \geq 1.$$

Then it is possible to use the time resonances method, applied in page 106 for example. Writing $e^{\mp is\phi^{\alpha,\beta}_{m,m,p}} = \frac{1}{i\phi^{\alpha,\beta}_{m,m,p}} \partial_s e^{\mp is\phi^{\alpha,\beta}_{m,m,p}}$, we get, by integrating by parts,

$$SI_s = SI_s^1 + SI_s^2 + SI_s^3,$$

with

— SI_s^1 is the boundary term

$$SI_s^1 := \chi(\xi) \sum_m \sum_{\substack{\alpha=\pm 1 \\ \beta=-\alpha}} \alpha\beta\mathcal{M}(m,m,p) \int_{\mathbb{R}} \Bigg(\frac{e^{\mp it\phi^{\alpha,\beta}_{m,m,p}}}{\phi^{\alpha,\beta}_{m,m,p}} \frac{\widetilde{g}_{\alpha,m}(t,\eta)}{\langle\eta\rangle_m} \frac{\widetilde{g}_{\beta,m}(t,\xi-\eta)}{\langle\xi-\eta\rangle_m} \\ - \frac{1}{\phi^{\alpha,\beta}_{m,m,p}} \frac{\widetilde{g}_{\alpha,m}(0,\eta)}{\langle\eta\rangle_m} \frac{\widetilde{g}_{\beta,m}(0,\xi-\eta)}{\langle\xi-\eta\rangle_m} \Bigg) d\eta,$$

— SI_s^2 and SI_s^3 are the two terms involving time derivatives of g:

$$SI_s^2 := \chi(\xi) \sum_m \sum_{\substack{\alpha=\pm 1 \\ \beta=-\alpha}} \alpha\beta\mathcal{M}(m,m,p) \int_0^t \int_{\mathbb{R}} \frac{e^{\mp is\phi^{\alpha,\beta}_{m,m,p}}}{\phi^{\alpha,\beta}_{m,m,p}} \frac{\partial_s\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m} \frac{\widetilde{g}_{\beta,m}(\xi-\eta)}{\langle\xi-\eta\rangle_m} d\eta ds,$$

$$SI_s^3 := \chi(\xi) \sum_m \sum_{\substack{\alpha=\pm 1 \\ \beta=-\alpha}} \alpha\beta\mathcal{M}(m,m,p) \int_0^t \int_{\mathbb{R}} \frac{e^{\mp is\phi^{\alpha,\beta}_{m,m,p}}}{\phi^{\alpha,\beta}_{m,m,p}} \frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m} \frac{\partial_s\widetilde{g}_{\beta,m}(\xi-\eta)}{\langle\xi-\eta\rangle_m} d\eta ds.$$

Estimates for SI_s^1. Remarking that $\frac{1}{\phi^{\alpha,\beta}_{m,m,p}}$ is a Coifman-Meyer multiplier, we have the following bound:

$$\left\|SI_s^1\right\|_{L^2_\xi} \lesssim \sum_m m^{-2M-1}\mathcal{M}(m,m,p) \sum_{\alpha=-\beta=\pm 1} \alpha\beta \left(\|g_{\alpha,m}(t)\|_{L^\infty} \|g_{\beta,n}(t)\|_{L^2} \\ + \|g_{\alpha,m}(0)\|_{L^\infty} \|g_{\beta,n}(0)\|_{L^2} \right),$$

i.e.

$$\left\|SI_s^1\right\|_{L^2_\xi} \lesssim \sum_m m^{-2M-1}\mathcal{M}(m,m,p)\varepsilon^2.$$

Resumming in p is then not a problem since for all $\nu > 1/8$ and $\varpi < 1/24$,

$$\mathcal{M}(m,m,p) \leq C_K \frac{m^\nu}{p^\varpi} \frac{m^K}{p^K}.$$

Hence

$$\left\|SI_s^1\right\|_{L^2_\xi} \lesssim \varepsilon^2 u_p(t), \tag{5.4.3}$$

with $(u_p(t))_{p\in\mathbb{Z}}$ in the unit ball of ℓ^2.

Estimates for SI_s^2. Here we take advantage of the fact that $\partial_s \widetilde{g}_{\alpha,m}$ is quadratic in g: more precisely,

$$\partial_s g_{\alpha,m} = e^{is\langle D\rangle_m} \left(e^{-is\langle D\rangle_m} g_{\alpha,m}\right)^2,$$

which will lead to, using the dispersion inequality (A.1.1),

$$\left\|SI_s^2\right\|_{L^2_\xi} \lesssim \sum_m \sum_{\alpha=-\beta=\pm 1} \alpha\beta\mathcal{M}(m,m,p) \int_0^t \frac{1}{\langle s\rangle} \varepsilon^3 m^{-3M-1} ds. \tag{5.4.4}$$

Resumming is not a problem either. Moreover, the third integral, SI_s^3, can be bounded in the same way.

5.4.2.b Study of SI_l.

If $|\xi| \geq 1/2$ then $\partial_\eta \phi^{\alpha,\beta}_{m,m,p}$ never vanishes, so the stationary phase Lemma applies.
For ξ fixed and nonzero, the minimum of $\partial_\eta \phi(\xi,\eta)$ is reached in $\eta = \xi/2$ and equals

$$\frac{\xi/2}{(\xi^2/4 + 2m + 2)^{\frac{1}{2}}} \geq \frac{1}{4\sqrt{2m+2}}.$$

In order to be able to apply Proposition A.4.1 we need to localize in η: let $(\psi_j)_{j\in\mathbb{Z}}$ be a family of functions such that each ψ_j is supported in the annulus $2^j \leq |\eta| \leq 2^{j+1}$ and such that $\sum_j \psi_j = 1$. Let us write $SI_l^{j,p}(\xi)$ for the following integral:

$$SI_l^{j,p}(\xi) := (1-\chi(\xi)) \int_0^t \int_{\mathbb{R}} \psi_j(\eta) e^{\mp is\phi^{\alpha,\beta}_{m,m,p}} \frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m} \frac{\widetilde{g}_{\beta,m}(\xi-\eta)}{\langle\xi-\eta\rangle_m} d\eta ds.$$

Then Proposition A.4.1 applies with $F_j(\xi,\eta) := \psi_j(\eta) \frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m} \frac{\widetilde{g}_{\beta,m}(\xi-\eta)}{\langle\xi-\eta\rangle_m}$:

$$SI_l^{j,p}(\xi) \lesssim \int_0^t \frac{1}{\langle s\rangle} 2^{\frac{j}{2}} \sqrt{m} \left(\|F_j\|_{L^2_\eta}(\xi) + \|\partial_\eta F_j\|_{L^2_\eta}(\xi)\right) ds.$$

We want to take the L^2_ξ norm of $SI_l^{j,p}$. First we know that if f and g are two functions in L^2 such that for all ξ, $\eta \mapsto f(\eta)g(\xi-\eta)$ is in L^2, then

$$\|f(\eta)g(\xi-\eta)\|_{L^2_{\xi,\eta}} = \|f\|_{L^2} \|g\|_{L^2}.$$

Then, remaking that

$$|F_j| \leq \left|\psi_j(\eta) \frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m} \frac{\widetilde{g}_{\beta,m}(\xi-\eta)}{\langle\xi-\eta\rangle_m}\right| + \left|\psi_j(\eta) \frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m} \frac{\widetilde{g}_{\beta,m}(\xi-\eta)}{\langle\xi-\eta\rangle_m}\right|$$

implies

$$\|F_j\|_{L^2_{\xi,\eta}} \lesssim \left\|\psi_j(\eta) \frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m}\right\|_{L^2_\eta} \left\|\frac{\widetilde{g}_{\beta,m}(\eta)}{\langle\eta\rangle_m}\right\|_{L^2_\eta} + \left\|\psi_j(\eta) \frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m}\right\|_{L^2_\eta} \left\|\frac{\widetilde{g}_{\beta,m}(\eta)}{\langle\eta\rangle_m}\right\|_{L^2_\eta}.$$

Finally, since $\left|\psi_j(\eta)\frac{1}{\langle\eta\rangle_m}\right| \leq \min(1, 2^{-j})$, we obtain

$$
\begin{aligned}
\|F_j\|_{L^2_{\xi,\eta}} &\lesssim \min(1,2^{-j})\,\|g_{\alpha,m}\|_{L^2}\,\|g_{\beta,m}\|_{L^2}\\
&\lesssim \min(1,2^{-j})m^{-2M}\varepsilon^2.
\end{aligned}
$$

In the same fashion we obtain

$$
\left\|\|\partial_\eta F_j\|_{L^2_\eta}\right\|_{L^2_\xi} \lesssim \min(1,2^{-j})m^{-2M}\varepsilon^2,
$$

which leads to

$$
\left\|SI_l^{j,p}\right\|_{L^2_\xi} \lesssim \int_0^t \frac{1}{\langle s\rangle}2^{\frac{j}{2}}\sqrt{m}\min(1,2^{-j})m^{-2M}\varepsilon^2 ds.
$$

Since $2^{\frac{j}{2}}\sqrt{m}\min(1,2^{-j})$ is summable over $\mathbb{Z}$, we can sum $\left\|SI_l^{j,p}\right\|_{L^2_\xi}$ over $\mathbb{Z}$ and obtain

$$
\sum_{j\in\mathbb{Z}}\left\|SI_l^{j,p}\right\|_{L^2_\xi} \lesssim \int_0^t \frac{u_p(s)}{\langle s\rangle}\varepsilon^2 ds, \tag{5.4.5}
$$

with $(u_p(s))_{p\in\mathbb{Z}}$ in the unit ball of ℓ^2.
Finally, Inequalities (5.4.3), (5.4.4) and (5.4.5) end the proof of Lemma 5.4.2. ■

5.4.3 Estimates for the non-stationary remainder

Lemma 5.4.3. *THere exists a sequence $(u_p(s))_{p\in\mathbb{N}}$ such that*

$$
p^{M_0}\,\|NR_{\pm,p}(g)\|_{L^2} \lesssim a_p\frac{\varepsilon^2}{t^{\frac{3}{4}}},
$$

where $(a_p)_{p\in\mathbb{N}}$ is a sequence in the unit ball of ℓ^2.

The proof of this lemma relies on the stationary phase Proposition A.4.1.

Proof of Lemma 5.4.3 :

The integral term we want to use Proposition A.4.1 on is the one occuring in Duhamel formula, that is to say:

$$
\begin{aligned}
D_1(g,g)-SI(g) &:= \sum_{\substack{m,n\in\mathbb{Z}\\ \alpha,\beta\in\{\pm1\}\\ m\neq n \text{ or } \alpha\neq-\beta\\ (C)\text{ satisfied}}} \int_0^t\int_{\mathbb{R}} e^{\mp is\phi^{\alpha,\beta}_{m,n,p}}\frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m}\frac{\widetilde{g}_{\beta,n}(\xi-\eta)}{\langle\xi-\eta\rangle_n}d\eta ds\\
&= \sum_{\substack{m,n\in\mathbb{Z}\\ \alpha,\beta\in\{\pm1\}\\ m\neq n \text{ or } \alpha\neq-\beta\\ (C)\text{ satisfied}}} I^{\alpha,\beta}_{m,n}(t,\xi).
\end{aligned}
$$

Here $\psi(\eta) := \phi^{\alpha,\beta}_{m,n,p}(\xi,\eta)$, the critical point is $\eta_0=\lambda^{\alpha,\beta}_{m,n}\xi$ and $F(\xi,\eta) := \frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m}\frac{\widetilde{g}_{\beta,n}(\xi-\eta)}{\langle\xi-\eta\rangle_n}$.
Let $\chi_\rho\in\mathcal{C}_0^\infty$ equal to zero on $B(0,\rho)^c$.
In order to apply Proposition A.4.1, we have to find

— either an lower bound for $|\psi''|$,

— or an upper bound for $\frac{\sqrt{|\psi(\eta)|}}{|\psi'(\eta)|}$.

Since in some cases (when $\alpha\beta = -1$) $\psi''(\eta) = \partial_\eta^2 \phi$ can vanish, it is better to try to bound directly

$$\frac{\sqrt{|\psi(\eta)|}}{|\psi'(\eta)|},$$

or rather

$$\frac{|\psi(\eta) - \psi(\eta_0)|}{\psi'(\eta)^2} = \frac{|\phi^{\alpha,\beta}_{m,n,p}(\xi,\eta) - \phi^{\alpha,\beta}_{m,n,p}(\xi, \lambda^{\alpha,\beta}_{m,n}\xi)|}{\left(\partial_\eta \phi^{\alpha,\beta}_{m,n,p}(\xi,\eta)\right)^2}.$$

The denominator vanishes at infinity or at $\lambda^{\alpha,\beta}_{m,n}\xi$.
Since $m \neq n$,

$$\partial_\eta^2 \phi^{\alpha,\beta}_{m,n}(\xi, \lambda^{\alpha,\beta}_{m,n}\xi) \neq 0.$$

Then $\frac{|\psi(\eta)-\psi(\eta_0)|}{\psi'(\eta)^2}$ is well-defined at the point $\lambda^{\alpha,\beta}_{m,n}\xi$ and

$$\frac{|\psi(\lambda^{\alpha,\beta}_{m,n}\xi) - \psi(\eta_0)|}{\psi'(\lambda^{\alpha,\beta}_{m,n}\xi)^2} = \frac{1}{|\partial_\eta^2 \phi^{\alpha,\beta}_{m,n,p}(\xi, \lambda^{\alpha,\beta}_{m,n}\xi)|}$$

Then, understanding the asymptotic behavior of $\frac{|\psi(\eta)-\psi(\eta_0)|}{\psi'(\eta)^2}$ will allow us to bound it (for sufficiently large values of ρ).

1. if $\alpha\beta = 1$, then

$$|\phi^{\alpha,\beta}_{m,n,p}(\xi,\eta) - \phi^{\alpha,\beta}_{m,n,p}(\xi, \lambda^{\alpha,\beta}_{m,n})| \sim_{\eta\to\infty} 2|\eta|$$

and

$$|\partial_\eta \phi^{\alpha,\beta}_{m,n,p}(\xi,\eta)| \to_{\eta\to\infty} 2,$$

hence

$$\frac{|\phi^{\alpha,\beta}_{m,n,p}(\xi,\eta) - \phi^{\alpha,\beta}_{m,n,p}(\xi, \lambda^{\alpha,\beta}_{m,n}\xi)|}{\left(\partial_\eta \phi^{\alpha,\beta}_{m,n,p}(\xi,\eta)\right)^2} \sim_{\eta\to\infty} \frac{\eta}{2}.$$

2. if $\alpha\beta = -1$, then

$$\begin{aligned}
&\lim_{\eta\to\infty} |\phi^{\alpha,\beta}_{m,n,p}(\xi,\eta) - \phi^{\alpha,\beta}_{m,n,p}(\xi, \lambda^{\alpha,\beta}_{m,n}\xi)| \\
&= \left| \alpha \sqrt{\left(\lambda^{\alpha,\beta}_{m,n}\xi\right)^2 + 2m + 2} + \beta \sqrt{\left((1-\lambda^{\alpha,\beta}_{m,n})\xi\right)^2 + 2n + 2} \right| \\
&= \frac{1}{|\lambda^{\alpha,\beta}_{m,n}|} \sqrt{\left(\lambda^{\alpha,\beta}_{m,n}\xi\right)^2 + 2m + 2}.
\end{aligned}$$

and

$$|\partial_\eta \phi^{\alpha,\beta}_{m,n,p}(\xi,\eta)| \sim_{\eta\to\infty} \frac{2|n-m|}{\eta^2}.$$

Hence

$$\begin{aligned}
\frac{|\phi^{\alpha,\beta}_{m,n,p}(\xi,\eta) - \phi^{\alpha,\beta}_{m,n,p}(\xi, \lambda^{\alpha,\beta}_{m,n}\xi)|}{\left(\partial_\eta \phi^{\alpha,\beta}_{m,n,p}(\xi,\eta)\right)^2} &\sim_{\eta\to\infty} \frac{\eta^4 \sqrt{\left(\lambda^{\alpha,\beta}_{m,n}\xi\right)^2 + 2m + 2}}{2|(m-n)^2 \lambda^{\alpha,\beta}_{m,n}|} \\
&\lesssim \eta^4 \sqrt{\xi^2 + 2(\sqrt{m+1} - \sqrt{n+1})^2}.
\end{aligned}$$

These bounds being established, it remains now to apply the stationary phase Proposition A.4.1, with

1. $\psi(\eta) := \phi^{\alpha,\beta}_{m,n,p}(\xi,\eta)$,
2. $F(\eta) := \frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m}\frac{\widetilde{g}_{\beta,n}(\xi-\eta)}{\langle\xi-\eta\rangle_n}$,
3. the critical point is $\eta_0 = \lambda^{\alpha,\beta}_{m,n}\xi$,
4. $M = 2$ since $\partial^3_\eta\phi = \frac{2(m+1)\eta}{(\eta^2+2m+2)^{5/2}} + \alpha\beta\frac{2(n+1)(\xi-\eta)}{((\xi-\eta)^2+2n+2)^{5/2}}$,
5. the bound m is given by

$$\begin{cases} \frac{1}{\rho} & \text{if} \quad \alpha\beta = 1, \\ \frac{1}{\rho^4\sqrt{\xi^2+2(\sqrt{m+1}-\sqrt{n+1})^2}} & \text{if} \quad \alpha\beta = -1. \end{cases}$$

This leads to

$$I^{\alpha,\beta}_{m,n}(t,\xi) = \frac{Ce^{it\phi(\lambda^{\alpha,\beta}_{m,n}\xi)}}{\partial^2_\eta\phi^{\alpha,\beta}_{m,n,p}(\xi,\lambda^{\alpha,\beta}_{m,n}\xi)\sqrt{t}}\chi(\lambda^{\alpha,\beta}_{m,n}\xi)\frac{\widetilde{g}_{\alpha,m}(\lambda^{\alpha,\beta}_{m,n}\xi)}{\langle\lambda^{\alpha,\beta}_{m,n}\xi\rangle_m}\frac{\widetilde{g}_{\beta,n}((1-\lambda^{\alpha,\beta}_{m,n})\xi)}{\langle(1-\lambda^{\alpha,\beta}_{m,n})\xi\rangle_n} + NR^{\alpha,\beta}_{m,n}(t,\xi),$$

with $NR^{\alpha,\beta}_{m,n}(t,\xi)$ bounded as follows:

1. if $\alpha\beta = 1$,

$$NR \lesssim \frac{1}{t^{\frac{3}{4}}}\left(\rho^{\frac{7}{4}}\left\|\frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m}\frac{\widetilde{g}_{\beta,n}(\xi-\eta)}{\langle\xi-\eta\rangle_n}\right\|_{L^2_\eta} + \rho^{\frac{3}{4}}\left\|\partial_\eta\left(\frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m}\frac{\widetilde{g}_{\beta,n}(\xi-\eta)}{\langle\xi-\eta\rangle_n}\right)\right\|_{L^2_\eta}\right),$$

2. if $\alpha\beta = -1$,

$$\begin{aligned} NR \lesssim & \frac{1}{t^{\frac{3}{4}}}\Bigg(\rho^7\left(\xi^2+2(\sqrt{m+1}-\sqrt{n+1})^2\right)^{\frac{7}{8}}\left\|\frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m}\frac{\widetilde{g}_{\beta,n}(\xi-\eta)}{\langle\xi-\eta\rangle_n}\right\|_{L^2_\eta} \\ & + \rho^3\left(\xi^2+2(\sqrt{m+1}-\sqrt{n+1})^2\right)^{\frac{3}{8}}\left\|\partial_\eta\left(\frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m}\frac{\widetilde{g}_{\beta,n}(\xi-\eta)}{\langle\xi-\eta\rangle_n}\right)\right\|_{L^2_\eta}\Bigg), \end{aligned}$$

Remark that the bound found in the case $\alpha\beta = -1$ is bigger than the one in the case $\alpha\beta = 1$. Moreover if we define χ on an annulus instead of a ball we have

$$\rho^k\left\|\partial_\eta\left(\frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m}\frac{\widetilde{g}_{\beta,n}(\xi-\eta)}{\langle\xi-\eta\rangle_n}\right)\right\|_{L^2_\eta} \sim \left\||\eta|^k\partial_\eta\left(\frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m}\frac{\widetilde{g}_{\beta,n}(\xi-\eta)}{\langle\xi-\eta\rangle_n}\right)\right\|_{L^2_\eta}.$$

Finally we obtain, in the case $\alpha\beta = -1$:

$$\begin{aligned} NR^{\alpha,\beta}_{m,n}(t,\xi) \lesssim & \frac{1}{t^{\frac{3}{4}}}\Bigg(\left(\xi^2+2(\sqrt{m+1}-\sqrt{n+1})^2\right)^{\frac{7}{8}}\left\||\eta|^7\frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m}\frac{\widetilde{g}_{\beta,n}(\xi-\eta)}{\langle\xi-\eta\rangle_n}\right\|_{L^2_\eta} \\ & + \left(\xi^2+2(\sqrt{m+1}-\sqrt{n+1})^2\right)^{\frac{3}{8}}\left\||\eta|^3\partial_\eta\left(\frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle\eta\rangle_m}\frac{\widetilde{g}_{\beta,n}(\xi-\eta)}{\langle\xi-\eta\rangle_n}\right)\right\|_{L^2_\eta}\Bigg). \end{aligned}$$

This asymptotical bound is also valid for the case $\alpha\beta = 1$.
We are focusing on the first term of the sum, the first one being even easiest. We can wrtie

$$\begin{aligned}
&\left\| \left(\xi^2 + 2(\sqrt{m+1} - \sqrt{n+1})^2\right)^{\frac{7}{8}} \left\| |\eta|^7 \partial_\eta \left(\frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle \eta \rangle_m} \frac{\widetilde{g}_{\beta,n}(\xi-\eta)}{\langle \xi - \eta \rangle_n} \right) \right\|_{L^2_\eta} \right\|_{L^2_\xi} \\
&\lesssim \left\| \left(\eta^2 + 2(\sqrt{m+1} - \sqrt{n+1})^2\right)^{\frac{7}{8}} |\eta|^7 \partial_\eta \left(\frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle \eta \rangle_m} \frac{\widetilde{g}_{\beta,n}(\xi-\eta)}{\langle \xi - \eta \rangle_n} \right) \right\|_{L^2_{\eta,\xi}(\mathbb{R}^2)} \\
&+ \left\| \left((\xi-\eta)^2 + 2(\sqrt{m+1} - \sqrt{n+1})^2\right)^{\frac{7}{8}} |\eta|^7 \partial_\eta \left(\frac{\widetilde{g}_{\alpha,m}(\eta)}{\langle \eta \rangle_m} \frac{\widetilde{g}_{\beta,n}(\xi-\eta)}{\langle \xi - \eta \rangle_n} \right) \right\|_{L^2_{\eta,\xi}(\mathbb{R}^2)}.
\end{aligned}$$

We only focus on the first term, which is the "worst" in terms of cost of derivatives. By sub-linearity, we are reduced to bound

$$\begin{aligned}
&\left\| |\eta|^{7+\frac{7}{4}} \frac{\partial_\eta \widetilde{g}_{\alpha,m}(\eta)}{\langle \eta \rangle_m} \frac{\widetilde{g}_{\beta,n}(\xi-\eta)}{\langle \xi - \eta \rangle_n} \right\|_{L^2_{\eta,\xi}(\mathbb{R}^2)} \\
&+ \left\| (\sqrt{m+1} - \sqrt{n+1})^{\frac{7}{4}} |\eta|^7 \frac{\partial_\eta \widetilde{g}_{\alpha,m}(\eta)}{\langle \eta \rangle_m} \frac{\widetilde{g}_{\beta,n}(\xi-\eta)}{\langle \xi - \eta \rangle_n} \right\|_{L^2_{\eta,\xi}(\mathbb{R}^2)}.
\end{aligned}$$

First,

$$\begin{aligned}
\left\| |\eta|^{7+\frac{7}{4}} \frac{\partial_\eta \widetilde{g}_{\alpha,m}(\eta)}{\langle \eta \rangle_m} \frac{\widetilde{g}_{\beta,n}(\xi-\eta)}{\langle \xi - \eta \rangle_n} \right\|_{L^2_{\eta,\xi}(\mathbb{R}^2)} &\lesssim \left\| |\eta|^{8+\frac{3}{4}} \frac{\partial_\eta \widetilde{g}_{\alpha,m}(\eta)}{\langle \eta \rangle_m} \right\|_{L^2_\eta} \left\| \frac{\widetilde{g}_{\beta,n}(\eta)}{\langle \eta \rangle_n} \right\|_{L^2_\eta} \\
&\lesssim \frac{1}{\sqrt{mn}} \left\| |\eta|^{8+\frac{3}{4}} \partial_\eta \widetilde{g}_{\alpha,m}(\eta) \right\|_{L^2_\eta} \left\| \widetilde{g}_{\beta,n}(\eta) \right\|_{L^2_\eta}.
\end{aligned}$$

Then, since g is in $\Sigma_T^{M,N}$, with $N > 9 - 1/4$,

$$\left\| |\eta|^{N+3+\frac{3}{4}} \partial_\eta \widetilde{g}_{\alpha,m}(t,\eta) \right\|_{L^2_\eta} = \varepsilon m^{-M} a_m(t),$$

with $(a_m(t))_{m\in\mathbb{N}}$ in the unit ball of ℓ^2. There exists also $(b_n(t))_{n\in\mathbb{N}}$ in the unit ball of ℓ^2 such that

$$\left\| |\eta|^{8+\frac{3}{4}} \frac{\partial_\eta \widetilde{g}_{\alpha,m}(\eta)}{\langle \eta \rangle_m} \frac{\widetilde{g}_{\beta,n}(\xi-\eta)}{\langle \xi - \eta \rangle_n} \right\|_{L^2_{\eta,\xi}(\mathbb{R}^2)} \lesssim \frac{1}{\sqrt{mn}} m^{-M} n^{-M} a_m(t) b_n(t) \varepsilon^2.$$

The summation Theorem C.1.1 ends the proof of Lemma 5.4.3.
Similarly, we have

$$\left\| (\sqrt{m+1} - \sqrt{n+1})^{\frac{7}{4}} |\eta|^7 \frac{\partial_\eta \widetilde{g}_{\alpha,m}(\eta)}{\langle \eta \rangle_m} \frac{\widetilde{g}_{\beta,n}(\xi-\eta)}{\langle \xi - \eta \rangle_n} \right\|_{L^2_{\eta,\xi}(\mathbb{R}^2)} \lesssim \frac{\max(m,n)^{\frac{7}{8}}}{\sqrt{mn}} m^{-M} n^{-M} a_m(t) b_n(t) \varepsilon^2,$$

which also fits in the hypotheses of Theorem C.1.1, since we assumed $M > M_0 + \frac{1}{8}$. This ends the proof of Lemma 5.4.3. ■

5.4.4 Estimates for the oscillating term

The oscillating term is

$$\begin{aligned}&\mathrm{Osc}_{\pm,p}(g)(\xi)\\ &:= \int_0^t \sum_{\substack{m,n\in\mathbb{Z}\\ \alpha,\beta\in\{\pm 1\}\\ m\neq n \text{ or } \alpha\neq-\beta\\ (C) \text{ not satisfied}}} \frac{C_{sp}}{\sqrt{t|\partial_\eta^2\phi(\xi,\eta_0)|}} e^{\mp is\phi^{\alpha,\beta}_{m,n,p}(\xi,\lambda^{\alpha,\beta}_{m,n}\xi)} \frac{\widetilde{g}_{\alpha,m}(\lambda^{\alpha,\beta}_{m,n}\xi)}{\langle\lambda^{\alpha,\beta}_{m,n}\xi\rangle_m} \frac{\widetilde{g}_{\beta,n}((1-\lambda^{\alpha,\beta}_{m,n})\xi)}{\langle(1-\lambda^{\alpha,\beta}_{m,n})\xi\rangle_n} ds.\end{aligned}$$

Lemma 5.4.4. *The L^2 norm of oscillating term can be bounded as follows: there exists $(u_p(s))_{p\in\mathbb{N}}$ inthe unit ball of ℓ^2 such that*

$$p^{M_0}\left\|\mathrm{Osc}_{\pm,p}(g)(\xi)\right\|_{L^2_\xi} \lesssim \int_0^t u_p(s)\frac{1}{s^{3/2}}\varepsilon^2 + \frac{1}{\sqrt{s}}\varepsilon^3 ds.$$

We shall not write the full proof of this proposition. We only recall that if (C) is not satisfied, then it is proven in Appendix B that

$$|\phi^{\alpha,\beta}_{m,n,p}(\xi,\lambda^{\alpha,\beta}_{m,n}\xi)| \geq \frac{1}{2\sqrt{n+1}(\sqrt{n+1}+\sqrt{m+1})\sqrt{(\lambda^{\alpha,\beta}_{m,n}\xi)^2+2m+2}}.$$

Then a integration by parts in time is feasible, and leads to Lemma 5.4.4.
Lemmas 5.4.2, 5.4.3 and 5.4.4 finally give Lemma 5.4.1.
■

Theorem 5.1.2 is then proved.

5.5 Perspective: understanding the dynamics of the resonant system

One problem with the resonant system (5.1.3) is that its dynamics seems hard to understand. A possibility would be to study a resonant equation in a simpler case, say for the wave equation on $\mathbb{R}\times\mathcal{T}$ (for Klein-Gordon, since the quadratic system is non-resonant, it is not relevant). Using Section 3.6.2, we recall that for the wave equation (3.6.1), the space-time resonances condition writes

$$\begin{aligned}\alpha=-1,\beta=-1,(0<m<p \text{ or } p<m<0)\\ \text{or } \alpha=-1,\beta=1,(0<p<m \text{ or } m<p<0).\end{aligned}$$

This leads to the following resonant system

$$\begin{aligned}\widetilde{f}_{\pm}(t,\xi,p) =&\widetilde{f}_{\pm}(0,\xi,p)\\ &+\int_0^t \frac{1}{\sqrt{s}}\left(1+\left(\frac{\xi}{p}\right)^2\right)^{-1/4}\\ &\times\Bigg[\sum_{\substack{0<m<p\\ \text{or } p<m<0}} \frac{1}{\sqrt{|p-m|+|m|}}\frac{\widetilde{f}_-\left(s,m\frac{\xi}{p},m\right)}{\sqrt{|m|}}\frac{\widetilde{f}_-\left(s,(p-m)\frac{\xi}{p},p-m\right)}{\sqrt{|p-m|}}\\ &-\sum_{\substack{0<p<m\\ \text{or } m<p<0}} \frac{1}{\sqrt{|p-m|+|m|}}\frac{\widetilde{f}_-\left(s,m\frac{\xi}{p},m\right)}{\sqrt{|m|}}\frac{\widetilde{f}_+\left(s,(p-m)\frac{\xi}{p},p-m\right)}{\sqrt{|p-m|}}\Bigg]ds.\end{aligned}$$

If we make this equation continuous, replacing p by η and m by ζ, we obtain the following equations :

— if $\eta > 0$,

$$\begin{aligned}\widetilde{f}_{\pm}(t,\xi,\eta) =&\widetilde{f}_{\pm}(0,\xi,\eta)\\ &+\int_0^t \frac{1}{\sqrt{s}}\left(1+\left(\frac{\xi}{\eta}\right)^2\right)^{-1/4}\\ &\times\Bigg[\int_0^{\eta} \frac{1}{\sqrt{|\eta-\zeta|+|\zeta|}}\frac{\widetilde{f}_-\left(s,\zeta\frac{\xi}{\eta},\zeta\right)}{\sqrt{|\zeta|}}\frac{\widetilde{f}_-\left(s,(\eta-\zeta)\frac{\xi}{\eta},\eta-\zeta\right)}{\sqrt{|\eta-\zeta|}}d\zeta\\ &+\int_{\eta}^{\infty} \frac{1}{\sqrt{|\eta-\zeta|+|\zeta|}}\frac{\widetilde{f}_-\left(s,\zeta\frac{\xi}{\eta},\zeta\right)}{\sqrt{|\zeta|}}\frac{\widetilde{f}_+\left(s,(\eta-\zeta)\frac{\xi}{\eta},\eta-\zeta\right)}{\sqrt{|\eta-\zeta|}}d\zeta\Bigg]ds.\end{aligned}$$

— if $\eta < 0$,

$$\begin{aligned}\widetilde{f}_{\pm}(t,\xi,\eta) =&\widetilde{f}_{\pm}(0,\xi,\eta)\\ &+\int_0^t \frac{1}{\sqrt{s}}\left(1+\left(\frac{\xi}{\eta}\right)^2\right)^{-1/4}\\ &\times\Bigg[\int_{\eta}^{0} \frac{1}{\sqrt{|\eta-\zeta|+|\zeta|}}\frac{\widetilde{f}_-\left(s,\zeta\frac{\xi}{\eta},\zeta\right)}{\sqrt{|\zeta|}}\frac{\widetilde{f}_-\left(s,(\eta-\zeta)\frac{\xi}{\eta},\eta-\zeta\right)}{\sqrt{|\eta-\zeta|}}d\zeta\\ &+\int_{-\infty}^{\eta} \frac{1}{\sqrt{|\eta-\zeta|+|\zeta|}}\frac{\widetilde{f}_-\left(s,\zeta\frac{\xi}{\eta},\zeta\right)}{\sqrt{|\zeta|}}\frac{\widetilde{f}_+\left(s,(\eta-\zeta)\frac{\xi}{\eta},\eta-\zeta\right)}{\sqrt{|\eta-\zeta|}}d\zeta\Bigg]ds.\end{aligned}$$

We remark that in these formulae, two interacting waves (one at frequency $(\zeta\frac{\xi}{\eta},\zeta)$, the other one at frequency $((\eta-\zeta)\frac{\xi}{\eta},\eta-\zeta))$ give birth to a third wave at frequency (ξ,η), i.e. on the same straight line. It may be natural to consider, for a given λ,

$$g_{\lambda,\pm}(t,\alpha) := \widetilde{f}_{\pm}(t,\lambda\alpha,\alpha).$$

In order to establish a Duhamel formula for g, we just recall that if u is real,

$$\overline{\widetilde{u}(\xi,p)} = \widetilde{u}(-\xi,-p).$$

Then,with

$$\widetilde{u}_+(p,\xi) = \widetilde{u}(p,\xi) + i\sqrt{\xi^2+p^2}\widetilde{u}(p,\xi),$$
$$\widetilde{u}_-(p,\xi) = \widetilde{u}(p,\xi) - i\sqrt{\xi^2+p^2}\widetilde{u}(p,\xi).$$

we can write

$$\widetilde{u}_-(p,\xi) = \overline{\widetilde{u}_+(-p,-\xi)}.$$

Similarly,

$$\widetilde{f}_-(p,\xi) = \overline{\widetilde{f}_+(-p,-\xi)}.$$

Then if $g_{\lambda,\pm}(\alpha) := \widetilde{f}_\pm(\alpha,\lambda\alpha)$, we get

$$g_{\lambda,-}(\alpha) = \overline{g_{\lambda,+}(-\alpha)}.$$

Now write $g_{\lambda,+} = g_\lambda$. The resonant system writes
• if $\alpha > 0$:

$$\begin{aligned} g_\lambda(t,\alpha) =& g_\lambda(0,\alpha) \\ &+ \int_0^t \frac{1}{\sqrt{s}}\frac{1}{(1+\lambda)^{\frac{1}{4}}}\frac{1}{\sqrt{|\alpha|}}\left[\int_0^\alpha \frac{\overline{g_\lambda(-\sigma)}}{\sqrt{|\sigma|}}\frac{\overline{g_\lambda(\sigma-\alpha)}}{\sqrt{|\sigma-\alpha|}}d\sigma - \int_\alpha^\infty \frac{\overline{g_\lambda(-\sigma)}}{\sqrt{|\sigma|}}\frac{g_\lambda(\alpha-\sigma)}{\sqrt{|\sigma-\alpha|}}d\sigma\right]ds, \end{aligned}$$

• if $\alpha < 0$:

$$\begin{aligned} g_\lambda(t,\alpha) =& g_\lambda(0,\alpha) \\ &+ \int_0^t \frac{1}{\sqrt{s}}\frac{1}{(1+\lambda)^{\frac{1}{4}}}\frac{1}{\sqrt{|\alpha|}}\left[\int_\alpha^0 \frac{\overline{g_\lambda(-\sigma)}}{\sqrt{|\sigma|}}\frac{\overline{g_\lambda(\sigma-\alpha)}}{\sqrt{|\sigma-\alpha|}}d\sigma - \int_{-\infty}^\alpha \frac{\overline{g_\lambda(-\sigma)}}{\sqrt{|\sigma|}}\frac{g_\lambda(\alpha-\sigma)}{\sqrt{|\sigma-\alpha|}}d\sigma\right]ds. \end{aligned}$$

The hamiltonian for this equation is

$$\begin{aligned} H_\lambda :=& \frac{1}{\sqrt{s}(1+\lambda)^{\frac{1}{4}}} \\ &\times \mathrm{Im}\left(\int_{\alpha=-\infty}^0 \int_{\sigma=-\infty}^\alpha \frac{\overline{g_\lambda(\alpha)}}{\sqrt{|\alpha|}}\frac{\overline{g_\lambda(-\sigma)}}{\sqrt{|\sigma|}}\frac{g_\lambda(\alpha-\sigma)}{\sqrt{|\sigma-\alpha|}}d\sigma - \int_{\sigma=\alpha}^0 \frac{\overline{g_\lambda(\alpha)}}{\sqrt{|\alpha|}}\frac{\overline{g_\lambda(-\sigma)}}{\sqrt{|\sigma|}}\frac{\overline{g_\lambda(\sigma-\alpha)}}{\sqrt{|\sigma-\alpha|}}d\sigma d\alpha\right. \\ &\left.+ \int_{\alpha=0}^\infty \int_{\sigma=0}^\alpha \frac{\overline{g_\lambda(\alpha)}}{\sqrt{|\alpha|}}\frac{\overline{g_\lambda(-\sigma)}}{\sqrt{|\sigma|}}\frac{\overline{g_\lambda(\sigma-\alpha)}}{\sqrt{|\sigma-\alpha|}}d\sigma - \int_{\sigma=\alpha}^\infty \frac{\overline{g_\lambda(\alpha)}}{\sqrt{|\alpha|}}\frac{\overline{g_\lambda(-\sigma)}}{\sqrt{|\sigma|}}\frac{g_\lambda(\alpha-\sigma)}{\sqrt{|\sigma-\alpha|}}d\sigma d\alpha\right). \end{aligned}$$

This hamiltonian does not seem to present a lot of invariances, except under the transform

$$g_\lambda(\alpha) \mapsto e^{i\kappa\alpha}g_\lambda(\alpha),$$

for any parameter κ.

Even if its study does not seem straigtforward, understanding this resonant system should be easier than understanding Equation (5.1.3): obtaining information on it (conserved quantities, long-times dynamics, effect of the slope λ on the behavior) can help us in the general understanding of resonant systems for dispersive equations.

Appendix A

Some harmonic analysis tools

Contents

In this section we are going to present the main harmonic analysis tools used in this manuscript. We start by giving some results on linear Fourier multiplier. Then we focus on the bilinear case. Finally we end with some properties of Hermite functions.

Before studying Fourier multipliers, we state this useful proposition on Sobolev spaces.

Lemma A.0.1. *Let f be in H^k, such that xf is in H^k. Then f is in $W^{k,1}$ and*

$$\|f\|_{W^{k,1}} \leq 4\sqrt{\|f\|_{H^k} \, \|\langle x\rangle f\|_{H^k}}\,.$$

Proof :
Let A be a positive real number. By Cauchy-Schwarz inequality,

$$\begin{aligned}
\int_{x\in\mathbb{R}} |D^k f(x)|dx &= \int_{|x|\leq A} |D^k f(x)|dx + \int_{|x|\geq A} |D^k f(x)|dx \\
&= \int_{x\in\mathbb{R}} \mathbf{1}_{|x|\leq A}|D^k f(x)|dx + \int_{x\in\mathbb{R}} \mathbf{1}_{|x|\geq A}\frac{1}{|x|}|x||D^k f(x)|dx \\
&\leq 2A\,\|f\|_{H^k} + \frac{2}{A}\left\|xD^k f\right\|_{L^2} \\
&\leq 2A\,\|f\|_{H^k} + \frac{2}{A}\,\|\langle x\rangle f\|_{H^k}\,.
\end{aligned}$$

Then optimizing over A gives $A = \sqrt{\frac{\|xf\|_{H^k}}{\|f\|_{H^k}}}$. This ends the proof. ■

A.1 Linear Fourier multiplier estimates

A.1.1 Dispersive estimates

First write down the dispersive estimate for Klein-Gordon equation in dimension 1 which can be found in [33].

Proposition A.1.1. *Let u_0 be a function in the Sobolev space $W^{3/2,1}(\mathbb{R})$. Then for all t, $e^{-it\langle D\rangle}u_0$ is in $L^\infty(\mathbb{R})$ and the following inequalities hold for all $t > 0$.*

$$\left\|e^{-it\langle D\rangle}u_0\right\|_{L^\infty} \lesssim \frac{1}{\sqrt{\langle t\rangle}}\,\|u_0\|_{W^{3/2,1}(\mathbb{R})}\,. \tag{A.1.1}$$

This allows us to formulate a dispersive estimate for a more general symbol.

Proposition A.1.2. *Let k be a positive real number, u_0 be a function in the Sobolev space $W^{3/2,1}(\mathbb{R})$. Then for all t, $e^{-it\sqrt{-D^2+k}}u_0$ is in $L^\infty(\mathbb{R})$ and the following inequalities hold for all $t > 0$.*

$$\left\|e^{-it\sqrt{-\Delta+k}}u_0\right\|_{L^\infty} \lesssim \frac{k^{\frac{1}{4}}}{\sqrt{\langle t\rangle}}\,\|u_0\|_{W^{3/2,1}(\mathbb{R})}\,. \tag{A.1.2}$$

Proof :
Notice that if u is a solution of

$$\left\{\begin{array}{rcl} \partial_t^2 u - \Delta u + ku &=& 0, \\ u(0,x) &=& u_0(x), \end{array}\right.$$

then $v : (t,x) \mapsto u\left(\frac{t}{\sqrt{k}}, \frac{x}{\sqrt{k}}\right)$ is a solution of

$$\begin{cases} \partial_t^2 v - \Delta v + v &= 0, \\ v(0,x) &= v_0(x) = u_0\left(\frac{x}{\sqrt{k}}\right). \end{cases}$$

Then, using the dispersive inequality (A.1.1) for Klein-Gordon in dimension one, there exists C such that

$$\left\| e^{-it\sqrt{-\Delta+1}} v_0 \right\|_{L^\infty} \leq C \frac{1}{\sqrt{t}} \|v_0\|_{W^{3/2,1}}.$$

This leads to

$$\begin{aligned} \|u(t,\cdot)\|_{L^\infty} &= \left\| v(\sqrt{k}t, \sqrt{k}\cdot) \right\|_{L^\infty} \\ &= \left\| v(\sqrt{k}t, \cdot) \right\|_{L^\infty}, \text{ since we take a } L^\infty \text{ norm.} \\ &\leq C \frac{1}{k^{\frac{1}{4}}\sqrt{t}} \|v_0\|_{W^{3/2,1}}, \text{ by the dispersive estimate.} \end{aligned}$$

Then $\|v_0\|_{L^1} = \left\| u_0\left(\frac{\cdot}{\sqrt{k}}\right)\right\|_{L^1} = \sqrt{k}\,\|u_0\|_{L^1}$. Moreover,

$$\begin{aligned} \int_{\mathbb{R}} \left| |D|^{3/2} v_0(x) \right| dx &= \int_{\mathbb{R}} \frac{1}{k^{\frac{3}{4}}} \left| \left(|D|^{3/2} u_0\right)\left(\frac{x}{\sqrt{k}}\right) \right| dx \\ &= \frac{1}{k^{\frac{3}{4}}} \sqrt{k} \left\| |D|^{\frac{3}{2}} u_0 \right\|_{L^1}. \end{aligned}$$

Since k is an integer, $\sqrt{k} \geq \frac{1}{k^{\frac{1}{4}}}$, and we get the following estimate.

$$\begin{aligned} \|u(t,\cdot)\|_{L^\infty} &\leq C \frac{1}{k^{\frac{1}{4}}\sqrt{t}} \sqrt{k}\, \|u_0\|_{W^{3/2,1}} \\ &\leq C \frac{k^{\frac{1}{4}}}{\sqrt{t}} \|u_0\|_{W^{3/2,1}}. \end{aligned}$$

Proposition A.1.2 is proved. ■

Proposition A.1.3. *(Global dispersive estimate) Let f in $\Sigma_t^{N,M}$ for $M \geq 5$ and $N \geq \frac{3}{2}$. Then for all t, $e^{\pm it\sqrt{-\Delta+x_2^2+1}} f$ is in $L^\infty(\mathbb{R}^2)$ and*

$$\left\| e^{\pm it\sqrt{-\Delta+x_2^2+1}} f(t) \right\|_{L^\infty(\mathbb{R}^2)} \lesssim \langle t \rangle^{-\frac{1}{4}} \|f(t)\|_{S_t^{N,M}}. \tag{A.1.3}$$

Proof :
First of all, in dimension 1, $\mathcal{H}^1(\mathbb{R}) \subset H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ by Sobolev's embedding.
This implies in particular that

$$\left\| e^{\pm it\sqrt{-\Delta+x_2^2+1}} f(t) \right\|_{L^\infty(\mathbb{R}^2)} \lesssim \left\| e^{\pm it\sqrt{-\Delta+x_2^2+1}} f(t) \right\|_{L^\infty_{x_1}(\mathcal{H}^1_{x_2})}. \tag{A.1.4}$$

Then, recalling that $g_p(x_1) = (g(x_1, .), \psi_p)_{L^2}$, we can rewrite the right-hand side of (A.1.4) as

$$\begin{aligned}\left\| e^{\pm it\sqrt{-\Delta+x_2^2+1}} f(t) \right\|_{L^\infty_{x_1}(\mathcal{H}^1_{x_2})} &\leq \left(\sum_{p\geq 0} p^2 \left\| \left(e^{\pm it\sqrt{-\Delta+x_2^2+1}} f(t) \right)_p \right\|^2_{L^\infty_{x_1}} \right)^{\frac{1}{2}} \\ &= \left(\sum_{p\geq 0} p^2 \left\| e^{\pm i\sqrt{-\Delta_{x_1}+2p+2}} f_p(t) \right\|^2_{L^\infty_{x_1}} \right)^{\frac{1}{2}} .\end{aligned}$$

By Proposition $A.1.1$ and Lemma $A.0.1$,

$$\left\| e^{\pm it\sqrt{-\Delta_{x_1}+2p+2}} f_p(t) \right\|_{L^\infty_{x_1}} \lesssim \frac{p^{\frac{1}{4}}}{t^{\frac{1}{4}}} \sqrt{\|f_p(t)\|_{H^{\frac{3}{2}}} \ \|\langle x\rangle f_p(t)\|_{H^{\frac{3}{2}}}} .$$

Resumming over p leads to

$$\begin{aligned}\sum_{p\geq 0} p^2 \left\| e^{\pm i\sqrt{-\Delta_{x_1}+2p+2}} f_p(t) \right\|^2_{L^\infty_{x_1}} &\lesssim \frac{1}{\sqrt{t}} \sum_{p\geq 0} p^{2+\frac{1}{2}} \|f_p(t)\|_{H^{\frac{3}{2}}} \ \|\langle x\rangle f_p(t)\|_{H^{\frac{3}{2}}} \\ &\lesssim \frac{1}{\sqrt{t}} \left(\sum_{p\geq 0} p^5 \|f_p(t)\|^2_{H^{\frac{3}{2}}} \right)^{\frac{1}{2}} \left(\sum_{p\geq 0} \|\langle x\rangle f_p(t)\|^2_{H^{\frac{3}{2}}} \right)^{\frac{1}{2}} .\end{aligned}$$

Each factor can be bounded by the norm $\|f(t)\|_{S^{N,M}_t}$ (since $M \geq 5$):

$$\begin{aligned}\sum_{p\geq 0} p^5 \|f_p(t)\|^2_{H^{\frac{3}{2}}} &\lesssim \sum_{p\geq 0} p^M \|f_p\|^2_{H^N} \\ &\lesssim \|f(t)\|^2_{S^{N,M}_t} ,\end{aligned}$$

and

$$\begin{aligned}\sum_{p\geq 0} \|\langle x\rangle f_p(t)\|^2_{H^{\frac{3}{2}}} &\lesssim \sum_{p\geq 0} p^M \|\langle x\rangle f_p(t)\|^2_{H^{\frac{3}{2}}} \\ &\lesssim \|f(t)\|^2_{S^{N,M}_t} .\end{aligned}$$

Hence the quantity

$$\sum_{p\geq 0} p^2 \left\| e^{\pm it\sqrt{-\Delta_{x_1}+2p+2}} f_p(t) \right\|^2_{L^\infty_{x_1}}$$

is bounded, so is $\left\| e^{\pm it\sqrt{-\Delta+x_2^2+1}} f(t) \right\|_{L^\infty_{x_1}(\mathcal{H}^1_{x_2})}$:

$$\begin{aligned}\left\| e^{\pm it\sqrt{-\Delta+x_2^2+1}} f(t) \right\|_{L^\infty_{x_1}(\mathcal{H}^1_{x_2})} &\lesssim \left(\sum_{p\geq 0} p^2 \left\| e^{\pm it\sqrt{-\Delta_{x_1}+2p+2}} f_p(t) \right\|^2_{L^\infty_{x_1}} \right)^{\frac{1}{2}} \\ &\lesssim t^{-\frac{1}{4}} \|f(t)\|_{S^{N,M}_t} .\end{aligned}$$

This concludes the proof. ■

A.1.2 Other Fourier multiplier estimates.

Given the nature of the Duhamel formula, we will have to deal with multipliers of the form $\frac{1}{\langle\eta\rangle_n}$. We are gathering all these multiplier estimates in one proposition.

Proposition A.1.4. *The following multiplier estimates hold.*

— *There exists a constant C such that the following inequality holds for all $a = 0, 1$, $1 \leq q \leq \infty$, all f in L^q and all positive real number λ.*

$$\left\|\frac{f}{\sqrt{D^2+\lambda}}\right\|_{L^q(\langle x\rangle^a)} \leq C\left(\frac{1}{\sqrt{\lambda}} + \frac{1}{\lambda^{\frac{1+a}{2}}}\right)\|f\|_{L^q(\langle x\rangle^a)}. \tag{A.1.4-a}$$

— *There exists a constant C such that the following inequality holds for all $N > 0$ and f in $\mathcal{H}^N$,*

$$\left\|\frac{f}{\sqrt{-\Delta + x_2^2 + 1}}\right\|_{\mathcal{H}^N} \leq C\|f\|_{\mathcal{H}^N}. \tag{A.1.4-b}$$

— *There exists a constant C such that the following inequalities hold for all f in L^q, $1 < q < \infty$, and all positive real number λ:*

$$\left\|\frac{D}{\sqrt{D^2+\lambda}}f\right\|_{L^q} \leq C\|f\|_{L^q}. \tag{A.1.4-c}$$

— *More generally, for all $a > 1$, $1 < q < \infty$, all f in L^q and all positive real number λ, we have the following estimate.*

$$\left\|\left(\frac{|D|}{\sqrt{D^2+\lambda}}\right)^a f\right\|_{L^q} \leq C_a\|f\|_{L^q}. \tag{A.1.4-d}$$

— *For all M, N integers, for all f in H^N,*

$$\left\||D|^M(1-\theta_R(D))f\right\|_{L^2} \leq \frac{1}{R^{N-M}}\|f\|_{H^N}, \tag{A.1.4-e}$$

where $\theta_R(\xi)$ is localizing in the zone $|\xi| \leq R$.

Proof of Proposition A.1.4 :

Proof of (A.1.4-a). The Fourier multiplier $\frac{1}{\sqrt{D^2+\lambda}}$ can be rewritten as follows:

$$\begin{aligned}\left\|\frac{f}{\sqrt{D^2+\lambda}}\right\|_{L^p} := \left\|\mathcal{F}^{-1}\left(\frac{\hat{f}(\xi)}{\sqrt{\xi^2+\lambda}}\right)\right\|_{L^p} &= \left\|f \star \mathcal{F}^{-1}\left(\frac{1}{\sqrt{\xi^2+\lambda}}\right)\right\|_{L^p} \\ &\leq \|f\|_{L^p}\left\|\mathcal{F}^{-1}\left(\frac{1}{\sqrt{\xi^2+\lambda}}\right)\right\|_{L^1},\end{aligned}$$

by Young's inequality.
Then it remains to estimate

$$\left\|\mathcal{F}^{-1}\left(\frac{1}{\sqrt{\xi^2+\lambda}}\right)\right\|_{L^1}. \tag{A.1.5}$$

We define a function Φ by

$$\Phi(\xi) := \frac{1}{\sqrt{\xi^2+1}},$$

such that (A.1.5) can be expressed as follows:

$$\mathcal{F}^{-1}\left(\frac{1}{\sqrt{\xi^2+\lambda}}\right) = \frac{1}{\sqrt{\lambda}}\mathcal{F}^{-1}\left(\frac{1}{\sqrt{\left(\frac{\xi}{\sqrt{\lambda}}\right)^2+1}}\right) = \frac{1}{\sqrt{\lambda}}\sqrt{\lambda}\mathcal{F}^{-1}(\Phi)\left(\sqrt{\lambda}x\right).$$

By a change of variable,

$$\left\|\mathcal{F}^{-1}(\Phi)\left(\sqrt{\lambda}x\right)\right\|_{L^1_x} = \frac{1}{\sqrt{\lambda}}\left\|\mathcal{F}^{-1}\Phi\right\|_{L^1}.$$

Since Φ and $\partial_\xi^2\Phi$ are in $L^2(\mathbb{R})$, $x \mapsto (1+x^2)\mathcal{F}^{-1}\Phi$ is in $L^2(\mathbb{R})$ which implies that $\mathcal{F}^{-1}(\Phi)$ is in L^1. This leads to (A.1.4-a):

$$\begin{aligned}\left\|\frac{f}{\sqrt{D^2+\lambda}}\right\|_{L^p} &\le \|f\|_{L^p}\left\|\mathcal{F}^{-1}\left(\frac{1}{\sqrt{\xi^2+\lambda}}\right)\right\|_{L^1} \\ &\le \frac{1}{\sqrt{\lambda}}\left\|\mathcal{F}^{-1}(\Phi)\right\|_{L^1}\|f\|_{L^p} \\ &\le \frac{C}{\sqrt{\lambda}}\|f\|_{L^p},\end{aligned}$$

where $C := \left\|\mathcal{F}^{-1}(\Phi)\right\|_{L^1}$ is independent of λ, p and f.
The weighted case can be treated in the same way, if we write that

$$\begin{aligned}\left\|\frac{f}{\sqrt{D^2+\lambda}}\right\|_{L^p(\langle x\rangle)} &:= \left\|\mathcal{F}^{-1}\left(\partial_\xi\left(\frac{\hat{f}(\xi)}{\sqrt{\xi^2+\lambda}}\right)\right)\right\|_{L^p} \\ &\le \left\|\mathcal{F}^{-1}\left(\frac{\partial_\xi\hat{f}(\xi)}{\sqrt{\xi^2+\lambda}}\right)\right\|_{L^p} + \left\|\mathcal{F}^{-1}\left(\frac{-\hat{f}(\xi)\xi}{(\xi^2+\lambda)^{3/2}}\right)\right\|_{L^p},\end{aligned}$$

and if we bound the L^1 norm of $\mathcal{F}^{-1}\left(\frac{\xi}{(\xi^2+\lambda)^{3/2}}\right)$.

Proof of (A.1.4-b). As in Proposition A.1.3, we can extend the previous proposition for the operator $\sqrt{-\Delta+x_2^2+1}$: we will not give more details.

Proof of (A.1.4-c) and (A.1.4-d). Let us only prove (A.1.4-d). We will use the Marcinkiewicz' multiplier theorem (as stated in [14]).

Theorem A.1.5. *Le m be a bounded real function such that its variation is uniformly bounded on each dyadic interval in $\mathbb{R}$, i.e. such that*

$$\exists B, \forall j\in\mathbb{Z}, \int_{-2^{j+1}}^{-2^j}\left|\frac{dm}{d\xi}\right|d\xi \le B \text{ and } \int_{2^j}^{2^{j+1}}\left|\frac{dm}{d\xi}\right|d\xi \le B.$$

Then m defines a multiplier operator T_m which maps L^p to L^p, where

$$T_m(f)(x) := \int_{\mathbb{R}} e^{ix\xi}m(\xi)\hat{f}(\xi)d\xi,$$

and its norm depends only on B.

Here the symbol m is $m(\xi) := \left(\frac{|\xi|}{\sqrt{\xi^2+1}}\right)^a$. Then

$$\frac{dm}{d\xi} = a\frac{\xi}{|\xi|(\xi^2+1)^{\frac{3}{2}}}\left(\frac{\xi}{\sqrt{\xi^2+1}}\right)^{a-1}.$$

The symbol m is bounded and if j is an integer,

$$\begin{aligned}\int_{2^j}^{2^{j+1}}\left|\frac{dm}{d\xi}\right|d\xi &= \int_{2^j}^{2^{j+1}}\left|a\frac{1}{(\xi^2+1)^{\frac{3}{2}}}\left(\frac{\xi}{\sqrt{\xi^2+1}}\right)^{a-1}\right|d\xi\\ &\leq a2^j\frac{1}{(2^{2j}+1)^{\frac{3}{2}}}\\ &\leq a.\end{aligned}$$

Hence m is of bounded variation and by Marcinkiewicz' theorem A.1.5, the proposition is proved for $\lambda = 1$.
Then write

$$\begin{aligned}\left(\frac{D}{\sqrt{D^2+\lambda}}f\right)(x) &= \int_{\mathbb{R}} e^{ix\xi}\frac{\xi}{\sqrt{\xi^2+\lambda}}\hat{f}(\xi)d\xi\\ &= \int_{\mathbb{R}} e^{ix\xi}\frac{\frac{\xi}{\sqrt{\lambda}}}{\sqrt{\left(\frac{\xi}{\sqrt{\lambda}}\right)^2+1}}\hat{f}(\xi)d\xi\\ &= \sqrt{\lambda}\int_{\mathbb{R}} e^{ix\sqrt{\lambda}\eta}\frac{\eta}{\sqrt{\eta^2+1}}\hat{f}(\sqrt{\lambda}\eta)d\eta\\ &= \sqrt{\lambda}\left(\frac{D}{\sqrt{D^2+1}}g_\lambda\right)(\sqrt{\lambda}x),\end{aligned}$$

where $\widehat{g_\lambda}(\xi) := \hat{f}(\sqrt{\lambda}\xi)$. By Marcinkiewicz'theorem, we can write

$$\left\|\frac{D}{\sqrt{D^2+1}}g_\lambda\right\|_{L^p} \leq C\left\|g_\lambda\right\|_{L^p}.$$

Moreover, a change of variable allows us to write

$$\left\|\frac{D}{\sqrt{D^2+\lambda}}f\right\|_{L^p} = \sqrt{\lambda}\lambda^{-\frac{1}{2p}}\left\|\frac{D}{\sqrt{D^2+1}}g_\lambda\right\|_{L^p}. \tag{A.1.6}$$

Then, since $\widehat{g_\lambda}(\xi) := \hat{f}(\sqrt{\lambda}\xi)$, $g_\lambda(x) = \frac{1}{\sqrt{\lambda}}f\left(\frac{x}{\sqrt{\lambda}}\right)$, and so

$$\left\|g_\lambda\right\|_{L^p} = \frac{1}{\lambda}\lambda^{\frac{1}{2p}}\left\|f\right\|_{L^p}. \tag{A.1.7}$$

Equations (A.1.6) and (A.1.7) get us to the conclusion:

$$\left\|\frac{D}{\sqrt{D^2+\lambda}}f\right\|_{L^p} = \sqrt{\lambda}\lambda^{-\frac{1}{2p}}\left\|\frac{D}{\sqrt{D^2+1}}g_\lambda\right\|_{L^p} \leq C\sqrt{\lambda}\lambda^{-\frac{1}{2p}}\left\|g_\lambda\right\|_{L^p} = C\left\|f\right\|_{L^p}.$$

This proves (A.1.4-d).

Proof of (A.1.4-e). First write, since Fourier transform is a L^2 isometry,

$$\begin{aligned}\left\| |D|^M (1-\theta_R(D)) f \right\|_{L^2}^2 &= \int_{\mathbb{R}} |\xi|^{2M} |(1-\theta_R(\xi))|^2 |\hat{f}|^2 d\xi \\ &= \int_{\mathbb{R}} \frac{1}{|\xi|^{N-M}} |\xi|^{2N} |(1-\theta_R(\xi))|^2 |\hat{f}|^2 d\xi.\end{aligned}$$

By definition of θ, $(1-\theta_R(\xi)) \neq 0$ if and only if $|\xi| \geq R$. Then

$$\int_{\mathbb{R}} \frac{1}{|\xi|^{2(N-M)}} |\xi|^{2N} |(1-\theta_R(\xi))|^2 |\hat{f}|^2 d\xi \leq \frac{1}{R^{2(N-M)}} \int_{\mathbb{R}} |\xi|^{2N} |\hat{f}|^2 d\xi \leq \frac{1}{R^{N-M}} \|f\|_{H^N}^2 .$$

This ends the proof of Proposition A.1.4. ■

A.1.3 Combination of dispersion and multiplier estimates

Proposition A.1.6. *Let n be an integer, f in B_s defined in 4.1.8 and $s > 0$. Then we have the following inequality.*

$$\left\| e^{is\langle D\rangle_n} \frac{f(s)}{\langle D\rangle_n} \right\|_{L^\infty} \lesssim s^{-\frac{1}{4}} n^{-\frac{1}{4}} \sqrt{\|f(s)\|_{H^{\frac{3}{2}}} \|f\|_{B_s}}. \tag{A.1.8}$$

Proof :
First, by Proposition A.1.4-a,

$$\left\| e^{is\langle D\rangle_n} \frac{f(s)}{\langle D\rangle_n} \right\|_{L^\infty} \lesssim \frac{1}{\sqrt{n}} \left\| e^{is\langle D\rangle_n} f(s) \right\|_{L^\infty} .$$

Then we can apply the dispersion estimate (A.1.1):

$$\left\| e^{is\langle D\rangle_n} \frac{f}{\langle D\rangle_n} \right\|_{L^\infty} \lesssim \frac{1}{\sqrt{n}} \frac{n^{\frac{1}{4}}}{\sqrt{s}} \|f\|_{W^{\frac{3}{2},1}} .$$

Finally, using lemma A.0.1 leads to

$$\left\| e^{is\langle D\rangle_n} \frac{f(s)}{\langle D\rangle_n} \right\|_{L^\infty} \lesssim \frac{n^{-\frac{1}{4}}}{\sqrt{s}} \sqrt{\|f(s)\|_{H^{\frac{3}{2}}} \|xf\|_{H^{\frac{3}{2}}}} .$$

Since $\|xf(s)\|_{H^{\frac{3}{2}}} \leq \sqrt{s} \|f\|_{B_s}$, we have proved the inequality. ■

A.2 Behavior with dilation operators

Definition A.2.1. *Let λ be a real number. We define the* Fourier dilation operator *of parameter λ, written $\mathcal{E}_\lambda$ by $\mathcal{E}_\lambda f := \mathcal{F}^{-1}\left(\xi \mapsto \hat{f}(\lambda\xi)\right)$.*

It is well known that $(\mathcal{E}_\lambda f)(x) = \frac{1}{\lambda} f\left(\frac{x}{\lambda}\right)$. The following lemma generalizes in the case where there is a Fourier multiplier.

Lemma A.2.2. *Let λ be a real number, p an integer and g the symbol of a Fourier multiplier. Then for all f we have*

$$\|\mathcal{E}_\lambda(g(D)f)\|_{L^p} = \lambda^{\frac{1}{p}-1} \|g(D)f\|_{L^p} .$$

Proof :

$$\begin{aligned}\mathcal{F}^{-1}\left(\xi \mapsto g(\lambda\xi)\hat{f}(\lambda\xi)\right)(x) &= \int_{\mathbb{R}} e^{-ix\xi} g(\lambda\xi)\hat{f}(\lambda\xi)d\xi \\ &= \frac{1}{\lambda}\int_{\mathbb{R}} e^{-ix\frac{\eta}{\lambda}} g(\eta)\hat{f}(\eta)d\eta \\ &= \frac{1}{\lambda}(g(D)f)\left(\frac{x}{\lambda}\right).\end{aligned}$$

The result is proven by taking the L^p norm of both sides. ■

We can also try to understand the behavior of bilinear multipliers with dilations.

Definition A.2.3. *Let $m(\xi,\eta)$ be a Fourier bilinear multiplier, λ be a real number. Define the following operators, T_m and T_m^λ.*

$$\begin{aligned}T_m(f,g) :=&\mathcal{F}^{-1}\left(\int_{\mathbb{R}} m(\xi,\eta)\hat{f}(\eta)\hat{g}(\xi-\eta)d\eta\right), \\ T_m^\lambda(f,g) :=&\mathcal{F}^{-1}\left(\int_{\mathbb{R}} m(\lambda\xi,\lambda\eta)\hat{f}(\eta)\hat{g}(\xi-\eta)d\eta\right).\end{aligned}$$

Proposition A.2.4. *For all $m(\xi,\eta)$, for all f and g in L^2, for all $\lambda \in \mathbb{R}$ we have the following equality.*

$$T_m(f,g) = \lambda\mathcal{E}_{\frac{1}{\lambda}}\left(T_m^\lambda(\mathcal{E}_\lambda f, \mathcal{E}_\lambda g)\right).$$

Then, for all $p > 1$, we have the L^p norm equality.

$$\|T_m(f,g)\|_{L^p} = \lambda^{2-\frac{1}{p}}\left\|T_m^\lambda(\mathcal{E}_\lambda f, \mathcal{E}_\lambda g)\right\|_{L^p}.$$

Proof :
First of all, the Fourier transform of $T_m(f,g)$ is

$$\mathcal{F}(T_m(f,g))(\xi) = \int_{\mathbb{R}} m(\xi,\eta)\hat{f}(\eta)\hat{g}(\xi-\eta)d\eta. \tag{A.2.1}$$

Considering the variables $\zeta = \frac{\xi}{\lambda}$ and $\nu = \frac{\eta}{\lambda}$, (A.2.1) can be rewritten as follows:

$$\begin{aligned}\mathcal{F}(T_m(f,g))(\xi) =& \int_{\mathbb{R}} m(\lambda\zeta,\lambda\nu)\hat{f}(\lambda\nu)\hat{g}(\lambda(\zeta-\nu))\lambda d\nu \\ =&\lambda\mathcal{F}(T_m(\mathcal{E}_\lambda f, \mathcal{E}_\lambda g))(\zeta) \\ =&\lambda\mathcal{F}(T_m(\mathcal{E}_\lambda f, \mathcal{E}_\lambda g))\left(\frac{\xi}{\lambda}\right).\end{aligned}$$

Then we can conclude that $T_m(f,g) = \lambda\mathcal{E}_{\frac{1}{\lambda}}\left(T_m^\lambda(\mathcal{E}_\lambda f, \mathcal{E}_\lambda g)\right)$.
This leads, thanks to Lemma A.2.2, to the equality on L^p norms and ends the proof.

$$\|T_m(f,g)\|_{L^p} = \lambda^{2-\frac{1}{p}}\left\|T_m^\lambda(\mathcal{E}_\lambda f, \mathcal{E}_\lambda g)\right\|_{L^p}.$$

Proposition A.2.4 is now proved. ■

A.3 Bilinear multiplier estimates

When dealing with the cutoffs defined in 4.2.9, a question arises: how can we keep "Hölder-like" estimates? More precisely, is it possible to have an inequality like

$$\|T_m(f,g)\|_{L^r} \le C\,\|f\|_{L^p}\,\|g\|_{L^q}\,,$$

with p,q,r satisfying some conditions (for example the Hölder condition $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$) ?
Answering for a general m is a very hard question, but some useful results are already known and work for the multipliers used in the three kinds of cut-offs.
This theory has been studied in the 60's by Coifman and Meyer [9], and the now known as Coifman-Meyer estimates will be very useful in our situation (see [42] for a proof).

Theorem A.3.1. *(Coifman-Meyer) Suppose that* $m \in L^\infty(\mathbb{R}^2)$ *is smooth away from the origin and satisfies*

$$|\partial_\xi^\alpha \partial_\eta^\beta m| \le \frac{C}{(|\xi|+|\eta|)^{\alpha+\beta}}, \tag{A.3.1}$$

for all $\alpha,\beta \le 3$ *(we say that* m *is a Coifman-Meyer symbol).*
Then A_m *is bounded from* $L^p \times L^q$ *to* L^r *where* $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$, $1\le p,q\le\infty$, $1\le r<\infty$.

Remark A.3.2. *Condition (A.3.1) is satisfied if* m *is* $\mathcal{C}^\infty$ *on a* (ξ,η) *sphere and homogeneous of degree* 0.

For example, the symbol used in the *high-low frequencies* cut-off satisfies Condition (A.3.1). Some other symbols with fail to satisfy the smoothness hypothesis of Coifman-Meyer's theorem: for example $\chi(\xi,\eta)\left|\frac{\xi}{\eta}\right|^a$ where $\chi(\xi,\eta)$ localizes around $|\xi-\eta|\le 2|\eta|$:

Lemma A.3.3. *Let* a *be a positive real number. Then the symbols*

$$\chi(\xi,\eta)\left|\frac{\xi}{\eta}\right|^a \ \textit{and}\ (1-\chi(\xi,\eta))\left|\frac{\xi}{\xi-\eta}\right|^a$$

satisfy Hölder-like estimates.

However the symbols used for the space resonant set cutoff will not fit in the conditions above. Another estimate, proven by Bernicot and Germain in [3] will be needed. First define the class $\mathcal{M}_\varepsilon^\Gamma$.

Definition A.3.4. *The scalar-valued symbol* m_ε *belongs to the class* $\mathcal{M}_\varepsilon^\Gamma$ *if*
- Γ *is a smooth curve in* $\mathbb{R}^2$.
- m_ε *is supported in* $B(0,1)$, *as well as in a neighborhood of size* ε *of* Γ.
- *The following inequality holds for sufficiently many indices* α *and* β $(\alpha,\beta=5)$.

$$\left|\partial_\xi^\alpha \partial_\eta^\beta m_\varepsilon(\xi,\eta)\right| \lesssim \varepsilon^{-\alpha-\beta}.$$

Theorem A.3.5. *[3] Consider* Γ *a compact and smooth curve. Let* $p,q,r\in[2,+\infty)$ *be exponents satisfying* $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1\ge 0$. *Then there exists a constant* $C=C(p,q,r)$ *such that for every* $\varepsilon>0$ *and symbols* $m_\varepsilon\in\mathcal{M}_\varepsilon^\Gamma$, *then*

$$\|T_{m_\varepsilon}(f,g)\|_{L^{r'}} \le C\varepsilon^{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1}\,\|f\|_{L^p}\,\|g\|_{L^q}\,.$$

As a consequence we have the following useful proposition.

Definition A.3.6. *The scalar-valued symbol m_ε belongs to the class $\mathcal{M}^{\Gamma}_{\varepsilon,M}$ if*
- *Γ is a smooth curve in $\mathbb{R}^2$.*
- *m_ε is supported in $B(0,1)$, as well as in a neighborhood of size $M\varepsilon$ of Γ.*
- *The following inequality holds for sufficiently many indices α and β.*

$$\left|\partial_\xi^\alpha \partial_\eta^\beta m_\varepsilon(\xi,\eta)\right| \lesssim \varepsilon^{-\alpha-\beta}.$$

Proposition A.3.7. *Consider Γ a compact and smooth curve. Let $p,q,r \in [2,+\infty)$ be exponents satisfying $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1 \geq 0$. Let M be a real number greater than 1. Then there exists a constant $C = C(p,q,r)$ such that for every $\varepsilon > 0$ and symbols $m_\varepsilon \in \mathcal{M}^{\Gamma}_{M,\varepsilon}$, then*

$$\|T_{m_\varepsilon}(f,g)\|_{L^{r'}} \leq CM\varepsilon^{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1} \|f\|_{L^p} \|g\|_{L^q}.$$

Proof :
Consider $[M]$ functions $\chi_1, ...\chi_{[M]}$ such that $\sum_j \chi_j = 1$ and $m_\varepsilon \chi_j$ is supported in a band of width $\leq \varepsilon$. Then $m_\varepsilon \chi_j$ belongs to $\mathcal{M}^{\Gamma}_{\varepsilon}$. Then it satisfies the Bernicot-Germain inequality and by the Minkowski inequality we have the proposition. ■

In the case of a straight line, this version of the theorem is very useful.

Theorem A.3.8. *Let ρ, ω, μ be real numbers, Γ a straight line and S be a symbol which satisfies the following properties.*
- *it is supported on a ball of radius ρ.*
- *it is supported in a band of width ω, around Γ.*
- *it satisfies the following estimate: for sufficiently many indices a, b,*

$$|\partial_\xi^a \partial_\eta^b \widetilde{S}| \lesssim \left(\frac{\rho}{\mu}\right)^{a+b}.$$

Then we have the following bilinear estimate.

$$\|T_S(f,g)\|_{L^{r'}} \lesssim \max\left(1, \frac{\omega}{\mu}\right) (\rho\omega)^{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1} \|f\|_{L^p} \|g\|_{L^q}.$$

Proof :
First we consider a new multiplier $\widetilde{S}(\xi,\eta) := S\left(\frac{\xi}{\rho}, \frac{\eta}{\rho}\right)$, and check that it satisfies Bernicot-Germain's theorem's hypothesis.
- Its support is of size ~ 1.
- Then the width of its support is ω/ρ.
- Moreover there is an estimate of the type $|\partial_\xi^a \partial_\xi^b \widetilde{S}| \lesssim \left(\frac{\rho}{\mu}\right)^{a+b}$.

If $\mu \geq \omega$ then we are in the framework of the Bernicot-Germain theorem A.3.5.
Otherwise the symbol $\widetilde{S}$ is in $\mathcal{M}^{\Gamma}_{\omega,\frac{\omega}{\mu}}$: we are in the framework of Bernicot-Germain's theorem's Corollary A.3.7. In all cases we have the following estimate:

$$\left\|T_{\widetilde{S}}(f,g)\right\|_{L^{r'}} \lesssim \max\left(1, \frac{\omega}{\mu}\right) \omega^{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1} \|f\|_{L^p} \|g\|_{L^q}.$$

Then use Lemma A.2.4 which describes the behavior of a bilinear multiplier with dilations. Thanks to this lemma we can write the following inequality.

$$\|T_S(f,g)\|_{L^{r'}} = \rho^{2-\frac{1}{r'}} \left\|T_{\widetilde{S}}(\mathcal{E}_\rho f, \mathcal{E}_\rho g)\right\|_{L^{r'}},$$

where $\mathcal{E}_\lambda f := \mathcal{F}^{-1}\left(\xi \mapsto \hat{f}(\lambda\xi)\right)$. Gather the two previous inequalities to get the following one:

$$\begin{aligned}\|T_S(f,g)\|_{L^{r'}} &= \rho^{2-\frac{1}{r'}} \left\|T_{\widetilde{S}}(\mathcal{E}_\rho f, \mathcal{E}_\rho g)\right\|_{L^{r'}} \\ &\lesssim \rho^{2-\frac{1}{r'}} \max\left(1, \frac{\omega}{\mu}\right) \omega^{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1} \|\mathcal{E}_\rho f\|_{L^p} \|\mathcal{E}_\rho g\|_{L^q}.\end{aligned}$$

Finally, using Lemma A.2.2 to estimate $\|\mathcal{E}_\rho f\|_{L^p}$, we get

$$\|T_S(f,g)\|_{L^{r'}} \lesssim \rho^{2-\frac{1}{r'}} \max\left(1, \frac{\omega}{\mu}\right) \omega^{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1} \rho^{\frac{1}{p}-1} \|f\|_{L^p} \rho^{\frac{1}{q}-1} \|g\|_{L^q}.$$

This leads to the desired inequality. ■

A.4 A L^2 stationary phase lemma

Proposition A.4.1. *Consider $\rho > 0$, $\chi \in \mathcal{C}_0^\infty$ equal to zero on $B(0,\rho)^c$, such that $|\chi'|$ is bounded by $1/\rho$, and ψ in $\mathcal{C}^\infty$. Let*

$$I = \int_{\mathbb{R}} e^{it\psi(x)} F(x)\chi dx. \tag{A.4.1}$$

1. *(non-stationary phase) Let m be a positive real number such that for all $x \in \operatorname{supp}(\chi)$, $|\psi'(x)| \geq m$. Then*
$$|I| \leq \frac{\sqrt{\rho}}{tm}\left(\|F\|_{L^2} + \|F'\|_{L^2}\right).$$
2. *(stationary phase) Let x_0 be the only point where $\psi'(x_0) = 0$. Let m and M two positive real numbers, such that for all x in $\operatorname{supp}(\chi)$,*
$$\begin{aligned}&\psi''(x) \geq m, \ |\psi'''(x)| \leq M, \\ \textit{or} \ &\left|\frac{\sqrt{\psi - \psi(x_0)}}{\psi'}\right| \lesssim \frac{1}{\sqrt{m}}, \ |\psi'''(x)| \leq M.\end{aligned}$$
Then
$$I = \frac{Ce^{it\psi(x_0)}}{\psi''(x_0)\sqrt{t}}\chi(x_0)F(x_0) + O\left(\frac{1}{t^{\frac{3}{4}}}\left(m^{-\frac{7}{4}}M\|F\|_{L^2} + m^{-\frac{3}{4}}\|F'\|_{L^2}\right)\right).$$
the constants being independent of χ and ψ.

Before proving Proposition A.4.1, we state two stationary phase lemmas, the first one being an idea and a proof of Germain-Pusateri-Rousset, the second being deduced by a change of variable:

Lemma A.4.2. *Consider $\chi \in \mathcal{C}^\infty$ such that $\chi = 0$ on $B(0,2)^c$ and $|\chi'| \leq 1$, then for all F such that F and F' are in L^2,*

$$\int_{\mathbb{R}} e^{ity^2} F(y)\chi(y)dy = \frac{C}{\sqrt{t}}\chi(0)F(0) + O\left(\frac{1}{t^{\frac{3}{4}}}\|F'\|_{L^2}\right).$$

Proof :
We are writing the proof given to the author by Germain, Pusateri and Rousset. Let us first decompose

$$\int_{\mathbb{R}} e^{ity^2} F(y)\chi(y)dy = \int_{\mathbb{R}} e^{ity^2}(F(y)-F(0))\chi(y)dy + \int_{\mathbb{R}} e^{ity^2} F(0)\chi(y)dy.$$

By the standard stationary phase lemma, the second term has the desired behavior: we now focus on bounding the first one.
Let Z be a cutoff function smooth, equal to 1 on $[-1,1]$ and to 0 outside $[-2,2]$. Let a be a parameter. Then

$$\begin{aligned}\left|\int_{\mathbb{R}} e^{ity^2}(F(y)-F(0))\chi(y)dy\right| \leq &\left|\int_{\mathbb{R}} Z\left(\frac{x}{a}\right) e^{ity^2}(F(y)-F(0))\chi(y)dy\right| \\ &+ \left|\int_{\mathbb{R}} \left(1-Z\left(\frac{y}{a}\right)\right) e^{ity^2}(F(y)-F(0))\chi(y)dy\right|.\end{aligned}$$

The first term can be written as

$$\begin{aligned}\left|\int_{\mathbb{R}} Z\left(\frac{y}{a}\right) e^{ity^2}(F(y)-F(0))\chi(y)dy\right| &= \left|\int_{\mathbb{R}} Z\left(\frac{y}{a}\right) e^{ity^2} y\left[\frac{1}{y}\int_0^y F(x)dx\right]\chi(y)dy\right| \\ &\lesssim \left(\int_{-2a}^{2a} y^2 dy\right)^{1/2}\left(\int_{\mathbb{R}}\left(\frac{1}{y}\int_0^y F(x)dx\right)^2 dy\right)^{1/2},\end{aligned}$$

by Cauchy-Schwarz' inequality. By Hardy's inequality,

$$\int_{\mathbb{R}}\left(\frac{1}{y}\int_0^y F'(x)dx\right)^2 dy \leq 4\left\|F'\right\|_{L^2}^2,$$

and we get

$$\left|\int_{\mathbb{R}} Z\left(\frac{y}{a}\right) e^{ity^2}(F(y)-F(0))\chi(y)dy\right| \lesssim a^{3/2}\left\|F'\right\|_{L^2}. \tag{A.4.2}$$

For the second one, we perform an integration by parts, using that

$$e^{ity^2} = \frac{1}{2ity}\partial_y e^{ity^2}.$$

$$\int_{\mathbb{R}}\left(1-Z\left(\frac{y}{a}\right)\right) e^{ity^2}(F(y)-F(0))\chi(y)dy = I_1+I_2+I_3+I_4,$$

with

$$\begin{aligned} I_1 &:= \int_{\mathbb{R}} \frac{1}{2ity}\frac{1}{a} Z'\left(\frac{y}{a}\right) e^{ity^2}(F(y)-F(0))\chi(y)dy, \\ I_2 &:= -\int_{\mathbb{R}} \frac{1}{2ity}\left(1-Z\left(\frac{y}{a}\right)\right) e^{ity^2} F'(y)\chi(y)dy, \\ I_3 &:= -\int_{\mathbb{R}} \frac{1}{2ity}\left(1-Z\left(\frac{y}{a}\right)\right) e^{ity^2}(F(y)-F(0))\chi'(y)dy, \\ I_4 &:= \int_{\mathbb{R}} \frac{1}{2ity^2}\left(1-Z\left(\frac{y}{a}\right)\right) e^{ity^2}(F(y)-F(0))\chi(y)dy. \end{aligned}$$

Now, by Cauchy-Schwarz' and Hardy's inequalities,

$$\begin{aligned}|I_1| &\lesssim \frac{1}{ta}\left(\int_{-2a}^{2a} 1dy\right)^{1/2}\left(\int_{\mathbb{R}}\left(\frac{1}{y}\int_0^y F'(x)dx\right)dy\right)^{1/2}\\ &\lesssim \frac{1}{t\sqrt{a}}\left\|F'\right\|_{L^2}.\end{aligned}$$

By Cauchy-Schwartz' inequality,

$$|I_2| \lesssim \frac{1}{t}\left(\int_{[-1,1]\setminus[-2a,2a]}\frac{1}{y^2}dy\right)^{1/2}\left\|F'\right\|_2 \lesssim \frac{1}{t\sqrt{a}}\left\|F'\right\|_2.$$

By Cauchy-Schwarz' and Hardy's inequalities and since the support of $(1-Z(y/r)\chi'(y))$is of constant size (for r small enough),

$$|I_3| \lesssim \frac{1}{t}\left\|F'\right\|_2 \lesssim \frac{1}{t\sqrt{a}}\left\|F'\right\|_2,$$

if $r < 1$.
Finally, we also obtain

$$\begin{aligned}|I_4| &\lesssim \frac{1}{ta}\left(\int_{[-1,1\setminus[-2a,2a]}\frac{1}{y}dy\right)^{1/2}\left(\int_{\mathbb{R}}\left(\frac{1}{y}\int_0^y F'(x)dx\right)dy\right)^{1/2}\\ &\lesssim \frac{1}{t\sqrt{a}}\left\|F'\right\|_{L^2}.\end{aligned}$$

Thus,

$$\left|\int_{\mathbb{R}}\left(1-Z\left(\frac{y}{a}\right)\right)e^{ity^2}(F(y)-F(0))\chi(y)dy\right| \lesssim \frac{1}{t\sqrt{a}}\left\|F'\right\|_{L^2}. \tag{A.4.3}$$

By (A.4.2) and (A.4.3), we finally obtain

$$\left|\int_{\mathbb{R}} e^{ity^2}(F(y)-F(0))\chi(y)dy\right| \lesssim a^{3/2}\left\|F'\right\|_2 + \frac{1}{t\sqrt{a}}\left\|F'\right\|_2,$$

and we optimize over a to end the proof of Lemma A.4.2. ■
Actually Lemma A.4.2 is invariant by dilation.

Lemma A.4.3. *Consider $r > 0$, $\chi \in \mathcal{C}^\infty$ such that $\chi = 0$ on $B(0,r)^c$ and $|\chi'| \leq 1/r$, then for all F such that F and F' are in L^2,*

$$\int_{\mathbb{R}} e^{ity^2}F(y)\chi(y)dy = \frac{C}{\sqrt{t}}\chi(0)F(0) + O\left(\frac{1}{t^{\frac{3}{4}}}\left\|F'\right\|_{L^2}\right).$$

Proof of Proposition A.4.1 :
For the non-stationary phase, writing $e^{is\psi(x)} = \frac{1}{is\psi'(x)}\frac{d}{dx}\left(e^{is\psi(x)}\right)$ directly leads to the desired estimate.

Now we have to study the case where the phase is stationary. We can assume $\psi(x_0) = 0$. Thanks to the proposition's hypothesis, we know that ψ' is increasing, vanishes at x_0. Consequently ψ is decreasing until x_0, then increasing. Let us consider the following change of variable

$$y = \operatorname{sign}(x)\sqrt{\psi(x)}.$$

Then (A.4.1) becomes

$$I = \int_{\mathbb{R}} e^{ity^2} \widetilde{F}(y)\widetilde{\chi}(y)dy, \tag{A.4.4}$$

with

$$\widetilde{F}(y) = \frac{F(\psi^{-1}(y^2))}{\psi'(\psi^{-1}(y^2))}|y|,$$
$$\widetilde{\chi}(y) = \chi(\psi^{-1}(y^2)).$$

In order to estimate (A.4.4), we are going to apply Lemma A.4.3.

1. Support of $\widetilde{\chi}$. In our case, we know that χ has a support of size ρ. Then, if $|y| > r$, $\psi^{-1}(y^2) \geq \mu|y|$, where $\mu = \min_y \frac{d}{dy}\psi^{-1}(y^2)$. Let us calculate this derivative:

$$\frac{d}{dy}\psi^{-1}(y^2) = \frac{2y}{\psi'(\psi^{-1}(y))} = \frac{2\text{sign}(x)\sqrt{\psi(x)}}{\psi'(x)}. \tag{A.4.5}$$

Then (A.4.5) can be estimated thanks to the following lemma:

Lemma A.4.4. *For all $x \in \text{supp}(\chi)$,*

$$\left|\frac{\sqrt{\psi(x)}}{\psi'(x)}\right| \leq \frac{1}{\sqrt{m}}.$$

Proof :
The proof is rather elementary. Let us prove the inequality for x positive. Since ψ' is increasing and vanishes at 0, we get $\psi(x) \leq x \max_{0\leq y\leq x} \psi'(y) \leq x\psi'(x)$. Hence

$$\frac{\sqrt{\psi(x)}}{\psi'(x)} \leq \frac{\sqrt{x}}{\sqrt{\psi'(x)}}.$$

Then $\psi'(x) \geq x \min_{0\leq y\leq x} \psi''(x) \geq mx$, from which we deduce

$$\frac{\sqrt{\psi(x)}}{\psi'(x)} \leq \frac{1}{\sqrt{m}}.$$

■
From this we deduce that if $|y| \geq r$, then

$$\psi^{-1}(y^2) \geq \frac{r}{\sqrt{m}}.$$

For $R \geq \frac{\sqrt{m}\rho}{2}$, we get $\psi^{-1}(y^2) \geq \rho$ so $\chi(\psi^{-1}(y^2)) = 0$. Hence

$$\widetilde{\chi} \text{ has a support of size } \sim \sqrt{r}\rho.$$

2. Estimates on the derivative of $\widetilde{\chi}$. Differentiating $\widetilde{\chi}$ gives

$$\begin{aligned}\widetilde{\chi}'(y) &= \frac{d}{dy}\left(\psi^{-1}(y^2)\right)\chi'\left(\psi^{-1}(y^2)\right)\\ &= \frac{2\mathrm{sign}(x)\sqrt{\psi(x)}}{\psi'(x)}\chi'\left(\psi^{-1}(y^2)\right).\end{aligned}$$

Then,

$$\begin{aligned}\frac{2\mathrm{sign}(x)\sqrt{\psi(x)}}{\psi'(x)} &\leq \frac{1}{\sqrt{m}} \text{ by Lemma } A.4.4,\\ \chi'\left(\psi^{-1}(y^2)\right) &\leq \frac{1}{\rho} \text{ by hypothesis.}\end{aligned}$$

Thus $\widetilde{\chi}'(y) \leq 1/(\sqrt{m}\rho)$: $\widetilde{\chi}$ satisfies the hypothesis of Lemma A.4.3.

3. Use of Lemma A.4.3. Applying Lemma A.4.3 to I gives

$$I = \frac{C}{\sqrt{t}}\widetilde{\chi}(0)\widetilde{F}(0) + O\left(\frac{1}{t^{\frac{3}{4}}}\left\|\widetilde{F}'\right\|_{L^2}\right).$$

First of all,

$$\widetilde{\chi}(0) = \chi(x_0), \tag{A.4.6}$$

and

$$\widetilde{F}(0) = \frac{F(x_0)}{\sqrt{\psi'(x_0)}}. \tag{A.4.7}$$

Finally it remains to write $\left\|\widetilde{F}'\right\|_{L^2}$ in terms of $\|F\|_{L^2}$ and $\|F'\|_{L^2}$. In order to compute it, we write $g(y) := \psi^{-1}(y^2)$, so $\widetilde{F}'$ can be written as follows:

$$\widetilde{F}' = A + B, \tag{A.4.8}$$

with

$$\begin{aligned}A &:= g'F' \circ g\frac{\sqrt{\psi \circ g}}{\psi' \circ g},\\ B &:= g'F \circ g\frac{1}{\sqrt{\psi \circ g}}\frac{(\psi' \circ g)^2 - 2\psi \circ g \times \psi'' \circ g}{(\psi' \circ g)^2}.\end{aligned}$$

The L^2 norm of A is quite straigtforward to estimate:

$$\begin{aligned}\|A\|_{L^2}^2 &= \int_{\mathbb{R}} g'(y)^2 F'(g(y))^2\frac{\psi(g(y))}{\psi'(g(y))^2}dy\\ &=_{x=g(y)} \int_{\mathbb{R}} g'(g^{-1}(x))F'(x)^2\frac{\psi(x)}{\psi'(x)^2}dx.\end{aligned}$$

Since $g'(y) = \frac{2y}{\psi'(\psi^{-1}(y^2))}$ and $g^{-1}(x) = \mathrm{sign}(x)\sqrt{\psi(x)}$,

$$\|A\|_{L^2}^2 = \int_{\mathbb{R}} F'(x)^2\left(\frac{\sqrt{\psi(x)}}{|\psi'(x)|}\right)^3 dx,$$

hence

$$\|A\|_{L^2}^2 \leq \frac{1}{m^{\frac{3}{2}}} \|F'\|_{L^2}^2 . \tag{A.4.9}$$

Applying the same method for B leads to:

$$\|B\|_{L^2}^2 = \int_{\mathbb{R}} g'(y)^2 F(g(y))^2 \frac{1}{\psi(g(y))} \left(\frac{\psi'(g(y)^2 - 2\psi(g(y))\psi''(g(y))}{\psi'(g(y))^2} \right)^2 dy,$$

i.e.

$$\|B\|_{L^2}^2 = \int_{\mathbb{R}} F(x)^2 \frac{\left((\psi'(x))^2 - 2\psi(x)\psi''(x)\right)^2}{2(\psi'(x))^5\sqrt{\psi(x)}} dx. \tag{A.4.10}$$

We now focus on bounding

$$\frac{\left((\psi'(x))^2 - 2\psi(x)\psi''(x)\right)^2}{2(\psi'(x))^5\sqrt{\psi(x)}}$$

First of all,

$$\frac{d}{dx}\left((\psi'(x))^2 - 2\psi(x)\psi''(x)\right) = -2\psi(x)\psi'''(x).$$

Moreover, for all x,

$$\begin{aligned} |2\psi(x)\psi'''(x)| &\leq 2M|\psi| \\ &\leq 2M|\psi'(x)||x|. \end{aligned}$$

This leads to

$$\frac{\left((\psi'(x))^2 - 2\psi(x)\psi''(x)\right)^2}{2(\psi'(x))^5\sqrt{\psi(x)}} \leq 4M^2 \frac{|\psi'(x)|^2|x|^4}{|\psi'(x)|^5\sqrt{\psi(x)}} \leq 4M^2 \frac{|x|^4}{|\psi'(x)|^3\sqrt{\psi(x)}}.$$

Finally, $\psi(x) \geq m\frac{x^2}{2}$ and $|\psi'(x)| \geq mx$, hence

$$\left| \frac{(\psi'(x))^2 - 2\psi(x)\psi''(x)}{2(\psi'(x))^2\sqrt{\psi(x)}} \right| \leq 4m^{-\frac{7}{2}}M^2.$$

Finally, (A.4.10) becomes

$$\|B\|_{L^2}^2 \lesssim m^{-\frac{7}{2}}M^2 \|F\|_{L^2} . \tag{A.4.11}$$

Because of (A.4.9) and (A.4.11), (A.4.8) becomes

$$\left\|\widetilde{F}'\right\|_{L^2} \lesssim m^{-\frac{7}{4}}M \|F\|_{L^2} + m^{-\frac{3}{4}} \|F'\|_{L^2} . \tag{A.4.12}$$

Gathering (A.4.6), (A.4.7), and (A.4.12), Proposition A.4.1 is proven.
■

A.5 Interaction between Hermite functions

The integral

$$\mathcal{M}(m,n,p)=\int_{\mathbb{R}}\psi_m(x_2)\psi_n(x_2)\psi_p(x_2)dx_2$$

can be computed explicitly but the exact formula is not really helpful to get estimates.

Proposition A.5.1. *Let $\nu>1/8$ and $0\le\beta<1/24$, $\varepsilon>0$ and $0\le\theta\le 1$. Then for all $m\le n\le p$ and K integers, there exists C_K, C_ε and $C_{\varepsilon,\theta,K}$ three positive constants such that*

$$|\mathcal{M}(m,n,p)|\le C_K\frac{m^\nu}{p^\beta}\left(\frac{\sqrt{mn}}{\sqrt{mn}+p-n}\right)^K, \tag{A.5.1}$$

$$|\mathcal{M}(m,n,p)|\le C_\varepsilon \max(m,n,p)^{-\frac{1}{4}+\varepsilon}, \tag{A.5.2}$$

$$|\mathcal{M}(m,n,p)|\le C_{\varepsilon,\theta,K}\frac{m^{\theta\nu}}{p^{\theta\beta}}\left(\frac{\sqrt{mn}}{\sqrt{mn}+p-n}\right)^{\theta K}p^{-\frac{1}{4}+\frac{\theta}{4}+\theta\varepsilon} \tag{A.5.3}$$

Proof :
Inequality (A.5.1) is a result proven by Grébert, Imekraz and Paturel in [29] (Proposition 3.3).
We now prove (A.5.2). Suppose that $m\le n\le p$. The result comes from Hölder's inequality and the following lemma that can be found in [6].

Lemma A.5.2. *[6] There exists $C>0$ such that for all $m,n\in\mathbb{N}$*

$$\|\psi_n\psi_n\psi_p\|_{L^2}\le C\max(n,p)^{-\frac{1}{4}}\sqrt{\log(\min(n,p)+1)}.$$

Then write

$$\begin{aligned}\left|\int_{\mathbb{R}}\psi_m\psi_n\psi_p dx\right| &\le \|\psi_m\|_{L^2}\|\psi_n\psi_p\|_{L^2}\\ &\le C\max(n,p)^{-\frac{1}{4}}\sqrt{\log(\min(n,p)+1)}.\end{aligned}$$

Then the sequence $C\max(n,p)^{-\varepsilon}\sqrt{\log(\min(n,p)+1)}$ is bounded, say by C_ε. This concludes the proof of.
Interpolating between (A.5.1) and (A.5.2) gives (A.5.3). ■

Appendix B

Study of the phase

Contents

For the reader who does not want to go through the details, we are going to start by stating the main results of this Appendix, and then go through the proofs.

B.1 Main results

B.1.1 Statements

From now on, fix m, n, p and consider the phase

$$\phi(\xi,\eta) = \sqrt{\xi^2+2p+2} + \alpha\sqrt{\eta^2+2m+2} + \beta\sqrt{(\xi-\eta)^2+2n+2}.$$

We define $\varrho(m,j,\eta)$ by

$$\varrho(m,j,\eta) := \frac{\eta^2}{2^j m}. \tag{B.1.1}$$

First of all the resonant sets are described in the following theorem:

Theorem (4.2.3). *Let α and β be two elements of $\{-1,1\}$. Consider the phase*

$$\phi = \sqrt{\xi^2+2p+2} + \alpha\sqrt{\eta^2+2m+2} + \beta\sqrt{(\xi-\eta)^2+2n+2}.$$

Then the space resonant set is

$$\mathcal{S} = \left\{ \left(\left(1 + \frac{\beta}{\alpha}\sqrt{\frac{2n+2}{2m+2}} \right)\eta, \eta \right), \eta \in \mathbb{R} \right\}.$$

1. *In the case $(\alpha,\beta) = (1,1)$, there are no time resonances: $\mathcal{T} = \emptyset$.*
2. *Otherwise,*
   (a) *If $\alpha\beta p + \beta m < 0$ or $\alpha\beta p + \beta n < 0$, there are no time resonances.*
   (b) *Otherwise, there are space-time resonances if and only if the following condition is satisfied.*

$$\begin{cases} \alpha\beta p + \beta m + \alpha n > 0, \\ m^2+n^2+p^2-2mn-2pm-2pn-2m-2n-2p-3 = 0. \end{cases} \tag{$C_{\alpha,\beta}$}$$

In that case, the space coresonant set is equal to the space resonant set: $\widetilde{\mathcal{S}} = \mathcal{S}$.

Then the behavior of the phase is described in the following Lemma:

Lemma B.1.1. *The following asymptotics for the phase ϕ occur for all m, n, j, η.*

1. *(low-frequency asymptotics)*
   (a) *the equation of the level line $\partial_\eta \phi = 2^{-j}$ can be rewritten as*

$$\xi = \left(1 + \sqrt{\frac{n+1}{m+1}} \frac{1}{(1-2^{-2j})^{\frac{3}{2}}} \right) \eta \left(1 + r(j,m,n,\eta)\right) - \frac{2^{-j}\sqrt{2n+2}}{\sqrt{1-2^{-2j}}}, \tag{B.1.2}$$

where $r(j,m,n,\eta) \le \frac{1}{\sqrt{m}}(1-2^{-2j})\eta$.

   (b) *the equation of the level line $\partial_\xi \phi = 2^j$ can be rewritten as*

$$\eta = \left(1 - \sqrt{\frac{n+1}{p+1}} \frac{1}{(1-2^{-2j})^{\frac{3}{2}}} \right) \xi \left(1 + r_2\right) + \frac{2^{-j}\sqrt{2n+2}}{\sqrt{1-2^{-2j}}}, \tag{B.1.3}$$

where $r_2(\xi,p,j) \le \frac{1}{\sqrt{p}}(1-2^{-2j})\xi$.

(c) *in the asymptotic zone* $|\eta| << \sqrt{m}$, *the width of the band* $-2^{-j} \leq \partial_\eta \phi \leq -2^{-(j+1)}$ *is bounded by*

$$C2^{-j}\min(\sqrt{m}, \sqrt{n}), \tag{B.1.4}$$

where C *is a constant independent of* j, n *and* m.

2. *(high-frequency asymptotics with* ϱ *small – rho defined in B.1.1)*

(a) *the level line* $\partial_\eta\phi := -2^{-j}$ *can be rewritten as*

$$\xi = \eta + \sqrt{\frac{n+1}{m+1}}\eta - \sqrt{\frac{n+1}{m+1}}\sqrt{\eta^2 + 2m + 2\frac{2^{-j}\eta^2}{2m+2}} + \sqrt{\frac{n+1}{m+1}}\sqrt{\eta^2+2m+2r}, \tag{B.1.5}$$

where $r \lesssim \varrho^2$.

(b) *if* $|\eta| \gtrsim \sqrt{m}$ *and* $\varrho \leq \varrho_0$ *for* $\varrho_0 < 1$ *well-chosen, the width of the zone*

$$\left\{-2^{-j} \leq \partial_\eta\phi \leq -2^{-(j+1)},\ 2^k \leq |\eta| \leq 2^{k+1}\right\},$$

written $w_{\varrho<<1}(m,n,j,k)$ *is bounded as follows.*

$$w_{\varrho<<1}(m,n,j,k) \lesssim \frac{2^{3k}2^{-j}}{2m+2}. \tag{B.1.6}$$

3. *(high-frequency asymptotics with* ϱ *large)*

(a) *the equation for the level line* $\partial_\eta\phi = -2^j$ *can be rewritten*

$$\xi = \eta + \frac{\sqrt{2n+2}}{\sqrt{2^{1-j}-2^{-2j}}}(1 - 2^{-j} + r), \tag{B.1.7}$$

with $r \leq c\frac{1}{2^{1-j}-2^{-2j}}\frac{m+1}{\eta^2} \lesssim \frac{1}{2-2^{-j}}\frac{1}{\varrho}$.

(b) *under the hypothesis* $\varrho \geq \varrho_0$, ϱ_0 *being chosen in (B.1.6), the width of the band* $-2^{-j} \leq \partial_\eta\phi \leq -2^{-(j+1)}$ *is bounded by*

$$w_{\varrho\gtrsim 1}(m,n,j) \lesssim 2^{\frac{j}{2}}\sqrt{2n+2}. \tag{B.1.8}$$

Remark B.1.2. *Lemma B.1.1 can be understood as follows:*

— *in the the asmptotical regime* $|\eta| << \sqrt{m}$, *the level lines are almost straight lines.*

— *the two other asymptotical regimes correspond to different order of magnitude of the asymptotic parameter*

$$\varrho(m,j,\eta) := \frac{\eta^2}{2^j m}.$$

— *when this parameter is small, we can calculate the deviation with respect to the straight line of slope* $\Lambda_{m,n}$.

— *when the parameter is large, level lines are like straight lines of slope* 1.

When there are space-time resonances, we can compare the derivatives of the phase in both directions ξ and η:

Lemma B.1.3. *In the case where there are space-time resonances, for all* ξ, η *real numbers, we have*

$$|\partial_\xi\phi| \leq |\partial_\eta\phi|.$$

In the case where there are no space-time resonances we have the following lemma:

Lemma B.1.4. *Let $R > 0$. There exists a constant c such that for all (ξ, η) satisfying*

$$\sqrt{|\xi|^2 + |\eta|^2} \leq R,$$
$$\mathrm{dist}((\xi,\eta),\mathcal{S}) \leq \frac{c}{(\sqrt{n+1}+\sqrt{m+1})^2}\frac{1}{R},$$

then the modulus of the phase $|\phi|$ is bounded from below (up to a constant independent of R, m and n) by

$$\frac{1}{(\sqrt{n+1}+\sqrt{m+1})^2}\frac{1}{R}.$$

In the case where there are no time resonances, this lemma will also be useful:

Lemma B.1.5. *In the case where there are no time resonances (i.e. $n > p$ or $m > p$) then we have the following lower bound on phi.*

$$|\phi| \gtrsim \frac{1}{(\sqrt{n+1}+\sqrt{m+1})^2}\frac{1}{R}.$$

B.1.2 Organization of the proofs

Here we detail in which sections the different results are proven:
- Theorem 4.2.3 is proven in Section B.2.1,
- Lemma B.1.1 is proven in Section B.2.2:
  - (B.1.2) is proven in Lemma B.2.1, (B.1.3) in Lemma B.2.3 and (B.1.4) in Lemma B.2.2,
  - (B.1.5) is proven in Lemma B.2.4 and (B.1.6) in Lemma B.2.5,
  - (B.1.7) is proven in Lemma B.2.6 and (B.1.8) in Lemma B.2.8,
- Lemma B.1.3 is proven in Section B.2.3,
- Lemma B.1.4 is proven in Section B.2.4,
- Lemma B.1.5 is proven in Section B.2.5.

B.2 Proof of the main results

The aim of this section is to get a fine understanding of the behaviour of the phase:
- what are the resonant sets? (section B.2.1)
- what do the level lines of $\partial_\eta \phi$ and $\partial_\xi \phi$ look like, and what is the witdh of the band $\partial_\eta \phi = 2^{-j}$? (section B.2.2)
- how does $\partial_\xi \phi$ compare to $\partial_\eta \phi$? (section B.2.3)
- in the case where the space-time resonant set is empty, what is the distance between the space resonant set and the time resonant one? (section B.2.4)
- how does ϕ behave asymptotically? (section B.2.5)

B.2.1 Resonant sets

We are going to prove Theorem 4.2.3. We recall that we wrote C for the following condition.

$$m^2 + n^2 + p^2 - 2mn - 2pm - 2pn - 2m - 2n - 2p - 3 = 0. \tag{C}$$

Proof of Theorem 4.2.3 :

Space resonances. The cancellation condition $\partial_\eta \phi = 0$ writes

$$\frac{\eta}{\sqrt{\eta^2+2m+2}} = \frac{\beta}{\alpha}\frac{\xi-\eta}{\sqrt{(\xi-\eta)^2+2n+2}}.$$

The set of ξ, η satisfying this equation happens to be a straight line, more precisely the set

$$\mathcal{S} = \left\{ \left(\left(1 + \frac{\beta}{\alpha}\sqrt{\frac{2n+2}{2m+2}} \right) \eta, \eta \right), \eta \in \mathbb{R} \right\}.$$

So as to get easier-to-read formulas, write $\lambda := \frac{\beta}{\alpha}$ and $\Lambda^\lambda_{m,n} := 1 + \frac{\beta}{\alpha}\sqrt{\frac{n+1}{m+1}}$.

Space-time resonances. Replace ξ by $\left(1+\lambda\sqrt{\frac{2n+2}{2m+2}}\right)\eta$ in the time resonant condition $\phi(\xi,\eta)=0$.

$$\sqrt{\left(\Lambda^\lambda_{m,n}\right)^2 \left(\eta^2 + \frac{2p+2}{\left(\Lambda^\lambda_{m,n}\right)^2} \right)} + \alpha\sqrt{\eta^2+2m+2} + \beta\sqrt{\frac{2n+2}{2m+2}\eta^2+2n+2} = 0,$$

i.e.

$$\left|\Lambda^\lambda_{m,n}\right| \sqrt{\eta^2 + \frac{2p+2}{\left(\Lambda^\lambda_{m,n}\right)^2}} + \alpha\Lambda^\lambda_{m,n}\sqrt{\eta^2+2m+2} = 0,$$

hence the following equation:

$$\left|\Lambda^\lambda_{m,n}\right| \left(\sqrt{\eta^2 + \frac{2p+2}{\left(\Lambda^\lambda_{m,n}\right)^2}} + \alpha\frac{\Lambda^\lambda_{m,n}}{\left|\Lambda^\lambda_{m,n}\right|}\sqrt{\eta^2+2m+2} \right) = 0.$$

Co-space resonant set. An analogous proof shows that the zero set of $\partial_\xi\phi$ is the straight line

$$\frac{\eta}{\xi} = 1 + \beta\frac{2n+2}{2p+2}.$$

Thus, $\widetilde{\mathcal{S}} = \left\{ \left(\xi, \left(1+\beta\sqrt{\frac{n+1}{p+1}}\right)\xi\right), \xi \in \mathbb{R} \right\}$.

Analysis of the different cases.

1. In the case $\alpha=\beta=1$, ϕ is always strictly positive so there are no time resonances.
2. Case where $\alpha=\beta=-1$. Remark that the space-time resonances condition reads then

$$\sqrt{\eta^2 + \frac{2p+2}{\left(\Lambda^+_{m,n}\right)^2}} - \sqrt{\eta^2+2m+2} = 0.$$

We are going to prove that there are time resonances if, and only if $\partial_\xi\phi$ vanishes with $\partial_\eta\phi$, if and only if $p \geq m, n$.

(a) Case $p \leq m$. We are going to prove that $\phi(\xi,\eta)$ is negative for all ξ,η. Fix $\xi_0 \in \mathbb{R}$.

— since $p \leq m$, $\frac{2p+2}{\left(\Lambda^+_{m,n}\right)^2} < 2m+2$, i.e $\sqrt{\eta^2 + \frac{2p+2}{\left(\Lambda^+_{m,n}\right)^2}} - \sqrt{\eta^2+2m+2} < 0$ for all η. So we have the inequality

$$\phi\left(\xi_0, \eta_0\right) < 0,$$

where $\eta_0 = \frac{\xi}{\Lambda^+_{m,n}}$.

— Then recall that $\partial_\eta \phi = -\frac{\eta}{\sqrt{\eta^2+2m+2}} + \frac{\xi-\eta}{\sqrt{(\xi-\eta)^2+2n+2}}$ and

$$\partial^2_\eta \phi = -\frac{2m+2}{(\eta^2+2m+2)^{\frac{3}{2}}} - \frac{2n+2}{((\xi-\eta)^2+2n+2)^{\frac{3}{2}}} < 0.$$

Hence $\eta \mapsto \partial_\eta \phi(\xi_0, \eta)$ is decreasing and vanishes at $\eta = \xi/\Lambda^+_{m,n}$. This proves that $\eta \mapsto \phi(\xi_0, \eta)$ increases until η_0 and then decreases.

This means that for ξ fixed, $\phi(\xi,\eta) \leq \phi(\xi,\eta_0) < 0$ for all η. So $\phi(\xi,\eta) < 0$ for all ξ, η, i.e. there are no time resonances.

(b) In the case $p < n$, proceed the same way. Fix $\eta_0 \in \mathbb{R}$. Writing

$$\partial_\xi \phi(\xi, \eta_0) = \frac{\xi}{\sqrt{\xi^2+2p+2}} - \frac{\xi-\eta}{\sqrt{(\xi-\eta)^2+2n+2}},$$

we can study asymptotically the derivatives:

$$\begin{aligned}\frac{\xi}{\sqrt{\xi^2+2p+2}} &= \frac{\operatorname{sgn}(\xi)}{\sqrt{1+\frac{2p+2}{\xi^2}}} \\ &= \operatorname{sgn}(\xi)\left(1 - \frac{p+1}{\xi^2} + o_{|\xi|\to+\infty}\left(\frac{1}{\xi^2}\right)\right),\end{aligned}$$

and, similarly,

$$\frac{\xi-\eta}{\sqrt{(\xi-\eta)^2+2n+2}} = \operatorname{sgn}(\xi-\eta)\left(1 - \frac{n+1}{(\xi-\eta)^2} + o_{|\xi-\eta|\to+\infty}\left(\left(\frac{1}{\xi-\eta}\right)^2\right)\right).$$

Then since we assumed $p < n$

$$\begin{aligned}\partial_\xi \phi(\xi,\eta_0) < 0 \text{ for } \xi \to -\infty, \\ \partial_\xi \phi(\xi,\eta_0) > 0 \text{ for } \xi \to +\infty.\end{aligned}$$

Since $\xi \mapsto \partial_\xi(\xi, \eta_0)$ has only one zero, name it ξ_0, we deduce that ϕ is nonnegative on $[\xi_0, +\infty)$ and negative elsewhere.

So $\xi \mapsto \phi(\xi,\eta_0)$ is decreasing $(-\infty, \xi_0]$ and increasing on $[\xi_0, +\infty)$.

We assumed $p < n$: this implies that

$$\phi(\xi, \eta_0) < 0 \text{ for } \xi \text{ large enough.}$$

Given the variation of ϕ, we conclude that

$$\forall \xi \in \mathbb{R}, \ \phi(\xi, \eta_0) < 0.$$

Since η_0 has been chosen arbitrarily, this proves that ϕ is negative on $\mathbb{R}^2$. This proves that there are no time resonances.

(c) In the case $p > m, n$.

— **Study of the space-time resonant set.** Recall the space-time resonance condition

$$\sqrt{\eta^2 + \frac{2p+2}{\left(\Lambda^+_{m,n}\right)^2}} - \sqrt{\eta^2 + 2m + 2} = 0.$$

Studying this condition is equivalent to the analysis of

$$2p + 2 - (2m+2)\left(\Lambda^+_{m,n}\right)^2 = 0.$$

This implies the following equality

$$2p + 2 - (2m+2) - 4\sqrt{n+1}\sqrt{m+1} - (2n+2) = 0,$$

i.e

$$p - m - n - 1 = 2\sqrt{n+1}\sqrt{m+1}.$$

In the case $p > m + n$ (the only case which makes the equality possible), it is equivalent (by squaring) to the following equation.

$$\begin{aligned} &p^2 + m^2 + n^2 - 2pm - 2pn + 2mn - 2p + 2m + 2n + 1 \\ &= 4(n+1)(m+1) = 4mn + 4n + 4m + 4. \end{aligned}$$

Finally we get the following conditions: $p > m + n$ and

$$m^2 + n^2 + p^2 - 2mn - 2pm - 2pn - 2m - 2n - 2p - 3 = 0.$$

This is exactly the condition $(C_{\alpha,\beta})$.

— **Study of the space co-resonant set.** Here the space co-resonant set is the straight line $\eta = (1 - \sqrt{(n+1)/(p+1)})\xi$. We will prove that under the condition $(C_{\alpha,\beta})$ it is exactly the space resonant set. This is equivalent to proving that

$$\left(1 + \sqrt{\frac{n+1}{m+1}}\right) = \left(1 - \sqrt{\frac{n+1}{p+1}}\right)^{-1},$$

i.e.

$$\left(1 + \sqrt{\frac{n+1}{m+1}}\right)\left(1 - \sqrt{\frac{n+1}{p+1}}\right) = 1.$$

This can be written $1 + \sqrt{\frac{n+1}{m+1}} - \sqrt{\frac{n+1}{p+1}} - \frac{n+1}{\sqrt{m+1}\sqrt{p+1}} = 1$, i.e.

$$\sqrt{p+1} = \sqrt{m+1} + \sqrt{n+1}.$$

(here comes again the necessary condition $p > m+n$). This is equivalent to

$$p + 1 = m + 1 + 2\sqrt{m+1}\sqrt{n+1} + n + 1, \text{ i.e. } p - m - n - 1 = 2\sqrt{m+1}\sqrt{n+1}.$$

This is exactly the same calculation as the one to determine the space-time resonant set.

3. Case where $(\alpha, \beta) = (-, +)$. Then the space resonant set is $\xi = \left(1 - \sqrt{\frac{n+1}{m+1}}\right)\eta$. Then the space-time resonance condition writes

$$\sqrt{\eta^2 + \frac{2p+2}{\left(\Lambda^-_{m,n}\right)^2}} - \frac{\Lambda^-_{m,n}}{\left|\Lambda^-_{m,n}\right|}\sqrt{\eta^2 + 2m + 2}.$$

   (a) We immediately see that if $\Lambda^-_{m,n} < 0$, i.e. if $m < n$, then there are no space-time resonances. Be more precise and prove that there are no time resonances at all. First, remark that if $\Lambda^-_{m,n} < 0$, then it implies that for all η, $\phi((1-\Lambda^-_{m,n})\eta, \eta) > 0$, or, equivalently, for all ξ, $\phi(\xi, \xi/(1-\Lambda^-_{m,n})) > 0$).
   Fix $\xi_0 \in \mathbb{R}$ and study $\eta \mapsto \phi(\xi_0, \eta)$. For $\eta = \eta_0 := \frac{\xi_0}{1-\Lambda^-_{m,n}}$, $\phi(\xi_0, \eta_0) > 0$. Then remark that $\eta \mapsto \partial_\eta \phi(\xi_0, \eta)$ cancels at one point only which is η_0. Then the same asymptotics as in the case $(\alpha, \beta) = (-1, -1)$ can be done:

$$\frac{\eta}{\sqrt{\eta^2 + 2m + 2}} = \operatorname{sgn}(\eta)\left(1 - \frac{2m+2}{\eta^2} + o_{|\eta| \to +\infty}\left(\frac{1}{\eta^2}\right)\right),$$

$$\frac{\eta - \xi}{\sqrt{(\eta-\xi)^2 + 2n + 2}} = \operatorname{sgn}(\eta - \xi)\left(1 - \frac{2n+2}{(\eta-\xi)^2} + o_{|\eta-\xi| \to +\infty}\left(\left(\frac{1}{\eta - \xi}\right)^2\right)\right).$$

   Then $\partial_\eta \phi(\xi_0, \eta) < 0$ (resp. > 0) when η goes to $+\infty$ (resp. $-\infty$). So $\eta \mapsto \phi(\xi_0, \eta)$ is increasing until η_0 then decreasing. Then $\phi(\xi_0, \eta) > 0$ for $|\eta|$ large enough, which proves that $\phi(\xi_0, \eta) > 0$ for all η. Since ξ_0 is chosen arbitrarily, $\phi > 0$ on $\mathbb{R}^2$, so there are no time resonances.

   (b) Now prove that there are no time resonances when $m < p$. The proof is roughly the same as before: consider $\eta_0 \in \mathbb{R}$ and the function $\xi \mapsto \phi(\xi, \eta_0)$. It is positive at $\xi_0 := \Lambda^-_{m,n}\eta_0$, and its derivative is negative for $\xi < \xi_0$ and positive for $\xi > \xi_0$.

   (c) Then the remaining case is $m > p, n$.
   — The space-time resonance condition is still

$$\frac{2p+2}{(\Lambda^-_{m,n})^2} = 2m + 2.$$

   This implies $2p+2-(2m+2)(1-\sqrt{n+1}/\sqrt{m+1})^2 = 0$, which is equivalent to the following equation.

$$2p + 2 - 2m - 2 - 2n - 2 + 4\sqrt{n+1}\sqrt{m+1},$$

   i.e.

$$m + n - p + 1 = 2\sqrt{n+1}\sqrt{m+1}.$$

   From this equality, we have that m has to be greater than $p+n$. in fact, if we had $p \leq n+p$, we would have $m+n-p+1 \leq 2n+1$, i.e. $2\sqrt{n+1}\sqrt{m+1} \leq 2n+1$. Taking the square and dividing by $4(n+1)^2$ would lead to

$$\frac{m+1}{n+1} \leq \frac{(2n+1)^2}{4(n+1)^2} < 1,$$

   which is impossible because $m > n$. With this condition, the equality $m + n - p + 1 = 2\sqrt{n+1}\sqrt{m+1}$ is equivalent to

$$m^2 + n^2 + p^2 + 2mn - 2mp - 2pn + 2m + 2n - 2p + 1 = 4mn + 4m + 4n + 4.$$

   This is the condition (C).

— Now the space coresonant set is the straight line $\eta = \left(1 + \sqrt{\frac{n+1}{p+1}}\xi\right)\xi = \Lambda^+_{m,n}\xi$. So as to prove that is is the same line as $\mathcal{S} = \xi = \Lambda^-_{m,n}\eta$, it suffices to prove that

$$\left(1 - \sqrt{\frac{n+1}{m+1}}\right)\left(1 + \sqrt{\frac{n+1}{p+1}}\right) = 1.$$

This leads to the equation

$$(\sqrt{m+1} - \sqrt{n+1})(\sqrt{p+1} + \sqrt{n+1}) = \sqrt{m+1}\sqrt{p+1},$$

i.e. to the following one.

$$\sqrt{m+1}\sqrt{p+1} + \sqrt{m+1}\sqrt{n+1} - \sqrt{n+1}\sqrt{p+1} - n + 1 = \sqrt{m+1}\sqrt{p+1},$$

i.e. to $\sqrt{m+1} - \sqrt{p+1} - \sqrt{n+1} = 0$ whiche is equivalent to

4. The case $(\alpha, \beta) = (+, -)$ is similar to the previous one, it will be skipped.

■

For now on, we assume that $\alpha = \beta = -1$, the other cases being dealt with similarly.

B.2.2 Asymptotics for $\partial_\eta\phi$ and $\partial_\xi\phi$

Thanks to the central symmetry, let us focus only on the level lines under the space-time resonant set. This corresponds to negative values of $\partial_\eta\phi$ and positive ones of $\partial_\xi\phi$.
First notice that we have an explicit expression for level lines. The level line $\partial_\eta\phi := -2^{-j}$ is

$$\xi = \eta + \sqrt{\frac{(2n+2)}{1 - ([\eta]_m - 2^{-j})^2}([\eta]_m - 2^{-j})},$$

where $[\eta]_m := \frac{\eta}{\sqrt{\eta^2+2m+2}}$. Similarly, the explicit expression for the level line $\partial_\xi\phi = 2^{-j}$ is

$$\eta = \xi - \sqrt{\frac{2n+1}{1 - ([\xi]_p - 2^{-j})}([\xi]_p - 2^{-j})}.$$

B.2.2.a Low-frequency asymptotics

Here we are studying the asymptotics for "low" frequencies: we write a formula for the level line with a remainder term small if and only if $|\eta| << \sqrt{m}$.

Lemma B.2.1. *The equation of the level line $\partial_\eta\phi = 2^{-j}$ can be rewritten as*

$$\xi = \left(1 + \sqrt{\frac{n+1}{m+1}}\frac{1}{(1-2^{-2j})^{\frac{3}{2}}}\right)\eta\,(1 + r_1(j,m,n,\eta)) - \frac{2^{-j}\sqrt{2n+2}}{\sqrt{1-2^{-2j}}},$$

where $r_1(j,m,n,\eta) \leq \frac{1}{\sqrt{m}}(1-2^{-2j})\eta$.

Proof :
Taylor-Lagrange inequalitites give, for all x in $\mathbb{R}$,

$$\left|\frac{1}{\sqrt{1+x}} - 1\right| \leq |x|,$$
$$\left|\frac{1}{\sqrt{1+x}} + \frac{x}{2} - 1\right| \leq \frac{x^2}{2}.$$

Hence we can write

$$\begin{aligned}[\eta]_m &= \eta\left(\eta^2 + 2m + 2\right)^{-\frac{1}{2}} \\ &= \frac{\eta}{\sqrt{2m+2}}\left(1 + \frac{\eta^2}{2m+2}\right)^{-\frac{1}{2}} \\ &= \frac{\eta}{\sqrt{2m+2}}\left(1 + r(\eta, m)\right),\end{aligned}$$

where $|r| \leq \frac{\eta^2}{2m+2}$.
So $[\eta]_m = \frac{\eta}{\sqrt{2m+2}} + R(\eta, m)$ with $|R(\eta, m)| \leq \frac{\eta^3}{(2m+2)^{\frac{3}{2}}}$. This allows us to write

$$\begin{aligned}\frac{1}{\sqrt{1 - ([\eta]_m - 2^{-j})^2}} &= \frac{1}{\sqrt{1 - (\frac{\eta}{\sqrt{2m+2}} + R - 2^{-j})^2}} \\ &= \frac{1}{\sqrt{1-2^{-2j}}} \frac{1}{\sqrt{1 + 2^{1-j}\frac{\eta}{(1-2^{-2j})\sqrt{2m+2}} + 2^{1-j}R - 2R\frac{\eta}{\sqrt{2m+2}}}} \\ &= \frac{1}{\sqrt{1-2^{-2j}}}\left(1 - 2^{-j}\frac{\eta}{(1-2^{-2j})\sqrt{2m+2}} + R'\right),\end{aligned}$$

with $R' \lesssim \frac{\eta^2}{m}$. Hence

$$\begin{aligned}&\sqrt{\frac{2n+2}{1 - ([\eta]_m - 2^{-j})^2}}([\eta]_m - 2^{-j}) \\ &= \sqrt{2n+2}([\eta]_m - 2^{-j})\frac{1}{\sqrt{1-2^{-2j}}}\left(1 - 2^{-j}\frac{\eta}{(1-2^{-2j})\sqrt{2m+2}} + R'\right) \\ &= \sqrt{2n+2}\left(\frac{\eta}{\sqrt{2m+2}} + R(\eta, m) - 2^{-j}\right)\frac{1}{\sqrt{1-2^{-2j}}}\left(1 - 2^{-j}\frac{\eta}{(1-2^{-2j})\sqrt{2m+2}} + R'\right) \\ &= -\frac{2^{-j}\sqrt{2n+2}}{\sqrt{1-2^{-2j}}} + \frac{\sqrt{2n+2}}{\sqrt{1-2^{-2j}}}\left(\frac{1}{\sqrt{2m+2}} + 2^{-2j}\frac{1}{(1-2^{-2j})\sqrt{2m+2}}\right)\eta + R'' \\ &= -\frac{2^{-j}\sqrt{2n+2}}{\sqrt{1-2^{-2j}}} + \sqrt{\frac{n+1}{m+1}}\frac{\eta}{(1-2^{-2j})^{\frac{3}{2}}} + R'',\end{aligned}$$

with $R'' \lesssim \frac{\sqrt{n}}{m}\eta^2\frac{1}{\sqrt{1-2^{-2j}}}$. Hence Lemma B.2.1 is proved. ■

Lemma B.2.2. *In the asymptotic zone $|\eta| << \sqrt{m}$, the width of the band*

$$-2^{-j} \leq \partial_\eta\phi \leq -2^{-(j+1)}$$

is bounded by

$$C2^{-j}\min(\sqrt{m},\sqrt{n})$$

where C is a constant independent of j, n and m.

Proof :
Let us determine the width of the band $\partial_\eta\phi \sim -2^{-j}$. Take η in the ball of radius $\sqrt{\frac{m+1}{2}}$, so $[\eta]_m \leq \frac{1}{\sqrt{2}}$. The vertical width at a point η is given by

$$\begin{aligned}
vw(n,m,j,\eta) =& \eta + \sqrt{\frac{(2n+2)}{1-([\eta]_m - 2^{-(j+1)})^2}}([\eta]_m - 2^{-(j+1)}) \\
& - \eta - \sqrt{\frac{(2n+2)}{1-([\eta]_m - 2^{-j})^2}}([\eta]_m - 2^{-j}) \\
=& \sqrt{2n+2}\left(\frac{[\eta]_m - 2^{-(j+1)}}{\sqrt{1-([\eta]_m - 2^{-(j+1)})^2}} - \frac{[\eta]_m - 2^{-j}}{\sqrt{1-([\eta]_m - 2^{-j})^2}}\right).
\end{aligned}$$

Since $[\eta]_m \leq \frac{1}{\sqrt{2}}$, $1-[\eta]_m^2 \geq \frac{1}{2}$. Thanks to this lower bound, the following asyptotic can be done:

$$\begin{aligned}
(1-([\eta]_m - 2^{-j})^2)^{-\frac{1}{2}} &= (1-[\eta]_m^2 + 2^{-j+1}[\eta]_m - 2^{-2j})^{-\frac{1}{2}} \\
&= \frac{1+r}{sqrt 1 - [\eta]_m^2},
\end{aligned}$$

with

$$\begin{aligned}
|r(j,m,\eta)| &\lesssim \frac{2^{-j+1}[\eta]_m}{1-[\eta]_m^2} + \frac{2^{-2j}}{1-[\eta]_m^2} \\
&\lesssim 2^{-j}.
\end{aligned}$$

Then

$$\begin{aligned}
&|vw(n,m,j,\eta)| \\
&\lesssim \sqrt{2n+2}\left(\frac{[\eta]_m - 2^{-(j+1)}}{\sqrt{1-[\eta]_m^2}}(1+r(j+1,m,\eta)) - \frac{[\eta]_m - 2^{-j}}{\sqrt{1-[\eta]_m^2}}(1+r(j,m,\eta))\right) \\
&\lesssim \sqrt{2n+2}\left(\frac{2^{j+1}}{\sqrt{1-[\eta]_m^2}} + \frac{\left|[\eta]_m - 2^{-(j+1)}\right|}{\sqrt{1-[\eta]_m^2}}r(j+1,m,\eta) + \frac{\left|[\eta]_m - 2^{-j}\right|}{\sqrt{1-[\eta]_m^2}}r(j,m,\eta)\right) \\
&\lesssim 2^{-j}\sqrt{2n+2},
\end{aligned}$$

Since the slope of the level line in this asymptotic is, at first order, $\Lambda_{m,n}$, the width of the level zone $\partial_\eta\phi \sim -2^{-j}$ is

$$\frac{2^{-j}\sqrt{2n+2}}{\sqrt{1+\Lambda_{m,n}^2}} \lesssim 2^{-j}\min(\sqrt{m},\sqrt{n}).$$

This ends the proof of Lemma B.2.2. ■

We can also get the same result for $\partial_\xi\phi$.

Lemma B.2.3. *The equation of the level line $\partial_\xi \phi = 2^j$ can be rewritten as*

$$\eta = \left(1 - \sqrt{\frac{n+1}{p+1}} \frac{1}{(1-2^{-2j})^{\frac{3}{2}}}\right) \xi\left(1 + r_2\right) + \frac{2^{-j}\sqrt{2n+2}}{\sqrt{1-2^{-2j}}},$$

where $r_2 \leq \frac{1}{\sqrt{p}}(1-2^{-2j})\xi$.
This can be rewritten as

$$\partial_\xi \phi = \frac{\sqrt{1-(\partial_\xi \phi)^2}}{\sqrt{2n+2}} \left(\eta - \left(1 - \sqrt{\frac{n+1}{p+1}} \frac{1}{(1-(\partial_\xi \phi)^2)^{\frac{3}{2}}}\right) \xi\left(1 + r'\right)\right).$$

B.2.2.b Asymptotics for ϱ small

Lemma B.2.4. *The level line $\partial_\eta \phi := -2^{-j}$ can be rewritten as*

$$\xi = \eta + \sqrt{\frac{n+1}{m+1}}\eta - \sqrt{\frac{n+1}{m+1}}\sqrt{\eta^2+2m+2}\frac{2^{-j}\eta^2}{2m+2} + \sqrt{\frac{n+1}{m+1}}\sqrt{\eta^2+2m+2}r',$$

where $r' \lesssim \varrho^2$.

Proof :
First write

$$\begin{aligned}
1-([\eta]_m - 2^{-j})^2 &= 1 - [\eta]_m^2 + 2.[\eta]_m.2^{-j} - 2^{-2j} \\
&= \frac{2m+2}{\eta^2+2m+2} + 2.[\eta]_m.2^{-j} - 2^{-2j} \\
&= \frac{2m+2}{\eta^2+2m+2}\left(1 - 2^{-j}\frac{1}{m+1}\eta\sqrt{\eta^2+2m+2} + 2^{-2j}\frac{\eta^2+2m+2}{2m+2}\right)
\end{aligned}$$

It allows us to write

$$\frac{1}{\sqrt{1-([\eta]_m-2^{-j})^2}} = \sqrt{\frac{\eta^2+2m+2}{2m+2}} \frac{1}{\sqrt{1+2^{-j}\frac{1}{m+1}\eta\sqrt{\eta^2+2m+2} - 2^{-2j}\frac{\eta^2+2m+2}{2m+2}}}$$

Since we are not in the zone $\eta \gtrsim \sqrt{m}$, we have

$$2^{-j}\frac{1}{m+1}\eta\sqrt{\eta^2+2m+2} \lesssim \frac{\eta^2}{2^j m}.$$

Then we have

$$\begin{aligned}
&\left(1 + 2^{-j}\frac{1}{m+1}\eta\sqrt{\eta^2+2m+2} + 2^{-2j}\frac{\eta^2+2m+2}{2m+2}\right)^{-\frac{1}{2}} \\
&= 1 - 2^{-j}\frac{\eta\sqrt{\eta^2+2m+2}}{2m+2} - 2^{-2j-1}\frac{\eta^2+2m+2}{2m+2} + r,
\end{aligned}$$

where $r \lesssim \left(\frac{\eta^2}{2^j m}\right)^2$. In the zone

$$|\eta| \gtrsim \sqrt{m},$$

we have

$$2^{-j} \lesssim \frac{\eta^2 2^{-j}}{\sqrt{2m+2}}.$$

Hence,

$$\left(1+2^{-j}\frac{1}{m+1}\eta\sqrt{\eta^2+2m+2}+2^{-2j}\frac{\eta^2+2m+2}{2m+2}\right)^{-\frac{1}{2}}=1-2^{-j}\frac{\eta\sqrt{\eta^2+2m+2}}{2m+2}+r_1,$$

where $r_1 \lesssim \left(\frac{\eta^2}{2^j m}\right)^2$. Multiplying by $[\eta]_m - 2^{-j}$ gives:

$$\begin{aligned}&\left(1+2^{-j}\frac{1}{m+1}\eta\sqrt{\eta^2+2m+2}+2^{-2j}\frac{\eta^2+2m+2}{2m+2}\right)^{-\frac{1}{2}}\times\left([\eta]_m-2^{-j}\right)\\&=\frac{\eta}{\sqrt{\eta^2+2m+2}}-2^{-j}\frac{\eta^2}{2m+2}+r',\end{aligned}$$

where $r' \lesssim \left(\frac{\eta^2}{2^j m}\right)^2$. Finally,

$$\begin{aligned}&\sqrt{\frac{(2n+2)}{1-([\eta]_m-2^{-j})^2}}([\eta]_m-2^{-j})\\&=\sqrt{\frac{n+1}{m+1}}\sqrt{\eta^2+2m+2}\left(\frac{\eta}{\sqrt{\eta^2+2m+2}}-2^{-j}\frac{\eta^2}{2m+2}+r'\right)\\&=\sqrt{\frac{n+1}{m+1}}\eta-\sqrt{\frac{n+1}{m+1}}\sqrt{\eta^2+2m+2}\frac{2^{-j}\eta^2}{2m+2}+\sqrt{\frac{n+1}{m+1}}\sqrt{\eta^2+2m+2}r'.\end{aligned}$$

This calculation concludes the proof. ■

Lemma B.2.5. *If $|\eta| \gtrsim \sqrt{m}$ and $\varrho \le \varrho_0$ for $\varrho_0 < 1$ well-chosen, the width of the zone*

$$\left\{-2^{-j} \le \partial_\eta\phi \le -2^{-(j+1)},\ 2^k \le |\eta| \le 2^{k+1}\right\},$$

written $w_{\varrho<<1}(m,n,j,k)$ is bounded as follows.

$$w_{\varrho<<1}(m,n,j,k) \lesssim \frac{2^{3k}2^{-j}}{2m+2}.$$

Proof :
First, η being chosen, the vertical witdth of the band $-2^{-j} \le \partial_\eta\phi \le -2^{-(j+1)}$ is given by

$$\begin{aligned}vw(m,n,j,\eta)=&-\sqrt{\frac{n+1}{m+1}}\sqrt{\eta^2+2m+2}\frac{2^{-(j+1)}\eta^2}{2m+2}+\sqrt{\frac{n+1}{m+1}}\sqrt{\eta^2+2m+2}r'_{j+1}\\&+\sqrt{\frac{n+1}{m+1}}\sqrt{\eta^2+2m+2}\frac{2^{-j}\eta^2}{2m+2}-\sqrt{\frac{n+1}{m+1}}\sqrt{\eta^2+2m+2}r'_j\\=&\sqrt{\frac{n+1}{m+1}}\sqrt{\eta^2+2m+2}\left(-\frac{2^{-(j+1)}\eta^2}{2m+2}+r'_{j+1}+\frac{2^{-j}\eta^2}{2m+2}-r'_j\right)\\=&\sqrt{\frac{n+1}{m+1}}\sqrt{\eta^2+2m+2}\frac{\varrho}{2}\left(1+2\frac{r'_{j+1}-r'_j}{\varrho}\right).\end{aligned}$$

Given the hypothesis of the lemma, we bound $\sqrt{\eta^2+2m+2}$ by $|\eta|$ and $\frac{r'_{j+1}-r'_j}{\varrho}$ by ϱ, i.e. by $c'<1$ if ϱ_0 is correctly chosen. If we are in the zone $\left\{-2^{-j}\leq\partial_\eta\phi\leq-2^{-(j+1)},\ 2^k\leq|\eta|\leq 2^{k+1}\right\}$,

$$|vw_{\varrho<<1}(m,n,j,\eta)|\lesssim\sqrt{\frac{n+1}{m+1}}\frac{2^{3k}2^{-j}}{2m+2},$$

hence the width of this zone:

$$|w_{\varrho<<1}(m,n,j,\eta)|\lesssim\frac{2^{3k}2^{-j}}{2m+2}.$$

■

B.2.2.c Asymptotics for ϱ large

Lemma B.2.6. *The equation for the level line $\partial_\eta\phi=-2^j$ can be rewritten*

$$\xi=\eta+\frac{\sqrt{2n+2}}{\sqrt{2^{1-j}-2^{-2j}}}(1-2^{-j}+r),$$

with $r\leq c\frac{1}{2^{1-j}-2^{-2j}}\frac{m+1}{\eta^2}$.

Proof :
First write that $[\eta]_m=1-\frac{m+1}{\eta^2}+R$ where $R\leq\left(\frac{2m+2}{\eta^2}\right)^2$.
Then

$$\begin{aligned}1-([\eta]_m-2^{-j})^2=&1-[\eta]_m^2+2^{1-j}[\eta]_m-2^{-2j}\\
=&1-\left(1-\frac{m+1}{\eta^2}+R\right)^2+2^{1-j}\left(1-\frac{m+1}{\eta^2}+R\right)-2^{-2j}\\
=&-\left(\frac{m+1}{\eta^2}\right)^2-R^2+2\frac{m+1}{\eta^2}+2R\frac{m+1}{\eta^2}-2R+2^{1-j}\\
&-2^{1-j}\frac{m+1}{\eta^2}+2^{1-j}R-2^{-2j}\\
=&(2^{1-j}-2^{-2j})\Bigg(1+\frac{2-2^{1-j}}{2^{1-j}-2^{-2j}}\frac{m+1}{\eta^2}-\frac{1}{2^{1-j}-2^{-2j}}\left(\frac{2m+2}{\eta^2}\right)^2\\
&-\frac{2-2^{1-j}}{2^{1-j}-2^{-2j}}R+\frac{1}{2^{1-j}-2^{-2j}}2R\frac{m+1}{\eta^2}-\frac{1}{2^{1-j}-2^{-2j}}R^2\Bigg)\\
=&(2^{1-j}-2^{-2j})(1+R'),\end{aligned}$$

where $R'\leq C'\frac{1}{2^{1-j}-2^{-2j}}\frac{m+1}{\eta^2}$ Hence we write

$$\frac{1}{\sqrt{1-([\eta]_m-2^{-j})^2}}=\frac{1}{\sqrt{2^{1-j}-2^{-2j}}}\frac{1}{1+R''}=\frac{1}{\sqrt{2^{1-j}-2^{-2j}}}(1+R'''),$$

with $R''\leq C''\frac{1}{2^{1-j}-2^{-2j}}\frac{m+1}{\eta^2}$. This finally leads to

$$\xi=\eta+\frac{\sqrt{2n+2}}{\sqrt{2^{1-j}-2^{-2j}}}(1-2^{-j}+R'''),$$

with $R'''\leq C'''\frac{1}{2^{1-j}-2^{-2j}}\frac{m+1}{\eta^2}$. ■
Adapting Lemma B.1.7 gives the following one for $\partial_\xi\phi$:

Lemma B.2.7. *The equation for the level line $\partial_\xi \phi = 2^j$ can be rewritten*

$$\xi = \eta + \frac{\sqrt{2n+2}}{\sqrt{2^{1-j} - 2^{-2j}}}(1 - 2^{-j} + r'),$$

with $r' \leq c\frac{1}{2^{1-j}-2^{-2j}}\frac{p+1}{\xi^2}$.

Now the width of a level zone $-2^{-j} \leq \partial_\eta \phi \leq -2^{-(j+1)}$ is given by the following lemma:

Lemma B.2.8. *Under the hypothesis $\varrho \geq \varrho_0$, ϱ_0 being chosen in lemma B.1.6, the width of the band $-2^{-j} \leq \partial_\eta \phi \leq -2^{-(j+1)}$ is bounded by*

$$w_{\varrho \gtrsim 1}(m, n, j) \lesssim 2^{\frac{j}{2}} \sqrt{2n+2}.$$

Proof :
Take $|\eta| \gtrsim \sqrt{m}$. The vertical width vw of the band $-2^{-j} \leq \partial_\eta \phi \leq -2^{-(j+1)}$ is

$$vw_{\varrho >> 1} = \sqrt{2n+2}\left(\frac{2^{\frac{j+1}{2}}(1 - 2^{-(j+1)} + r_j)}{2 - 2^{-(j+1)}} - \frac{2^{\frac{j}{2}}(1 - 2^{-j} + r_j)}{2 - 2^{-j}}\right),$$

where $r_j \leq \frac{c}{2-2^j}\frac{1}{\varrho}$. Hence, since the slope of the level line in this zone is equal to 1 (at first order), we can directly bound the width of the zone $-2^{-j} \leq \partial_\eta \phi \leq -2^{-(j+1)}$:

$$w_{\varrho \gtrsim 1}(m, n, j) \lesssim 2^{\frac{j}{2}} \sqrt{2n+2}$$

Hence Lemma B.2.8 is proved. ■

B.2.3 Comparison between $\partial_\eta \phi$ and $\partial_\xi \phi$

Proof of Lemma B.1.3 :
We prove the inequality in the zone $\xi \leq \left(1 + \sqrt{\frac{n+1}{m+1}}\right)\eta$ (i.e. below the space-time resonant set). In this zone, $\partial_\eta \phi$ is negative and $\partial_\xi \phi$ is positive, which means that $|\partial_\xi \phi| = \frac{\xi}{\langle \xi \rangle_p} - \frac{\xi - \eta}{\langle \xi - \eta \rangle_n}$ and $|\partial_\eta \phi| = \frac{\eta}{\langle \eta \rangle_m} - \frac{\xi - \eta}{\langle \xi - \eta \rangle_n}$. Then the inequality to prove is

$$\frac{\xi}{\sqrt{\xi^2 + 2p + 2}} \leq \frac{\eta}{\sqrt{\eta^2 + 2m + 2}}.$$

Since $\xi \mapsto \frac{\xi}{\sqrt{\xi^2+2p+2}}$ is increasing, in the zone $\xi \leq \left(1 + \sqrt{\frac{n+1}{m+1}}\right)\eta$ we have

$$\begin{aligned}\frac{\xi}{\sqrt{\xi^2 + 2p + 2}} &\leq \frac{\left(1 + \sqrt{\frac{n+1}{m+1}}\right)\eta}{\sqrt{\left(1 + \sqrt{\frac{n+1}{m+1}}\right)^2 \eta^2 + 2p + 2}} \\ &\leq \frac{\eta}{\sqrt{\eta^2 + \frac{2p+2}{\left(1+\sqrt{\frac{n+1}{m+1}}\right)^2}}}.\end{aligned}$$

But thanks to the relation between m, n and p, $\frac{2p+2}{\left(1+\sqrt{\frac{n+1}{m+1}}\right)^2} = 2m + 2$ and the inequality is proved. ■

B.2.4 Distance between $\mathcal{S}$ and $\mathcal{T}$

Proof of Lemma B.1.4 :
Here we are in the case where $p \leq n+m$ or $p^2+m^2+n^2-2pm-2mn-2pn-2p-2m-2n-3 \neq 0$. The first step will be to evaluate $\phi(\Lambda_{m,n}\eta, \eta)$ (which is no longer equal to 0) and then to find the width of a neighborhood of the straight line $\xi - \Lambda_{m,n}\eta$ where ϕ remains different from 0.
So as to prove Proposition B.1.4, we will estimate ϕ on $\mathcal{S}$ and on a neigborhood of this set.

B.2.4.a Estimates for ϕ on $\mathcal{S}$

1. Suppose that $p \leq m+n$. Then

$$p - m - n - 1 < 0 < 2\sqrt{n+1}\sqrt{m+1}.$$

Then write a the negative integer such that

$$\begin{aligned}&2p + 2 - (2n+1) - 4\sqrt{n+1}\sqrt{m+1} - (2m+2) = a,\\ &\text{i.e } \ 2p+2 = (2m+2)(\Lambda_{m,n})^2 + a.\end{aligned}$$

This means that

$$\begin{aligned}\sqrt{\eta^2 + \frac{2p+2}{\Lambda_{m,n}}} - \sqrt{\eta^2+2m+2} &= \sqrt{\eta^2+2m+2+\frac{a}{(\Lambda_{m,n})^2}} - \sqrt{\eta^2+2m+2}\\ &= \sqrt{\eta^2+2m+2}\left(\sqrt{1+\frac{a}{(\eta^2+2m+2)(\Lambda_{m,n})^2}} - 1\right).\end{aligned}$$

Then we can write

$$\left|\sqrt{1+\frac{a}{(\eta^2+2m+2)(\Lambda_{m,n})^2}} - 1\right| \geq \frac{|a|}{(\eta^2+2m+2)(\Lambda_{m,n})^2},$$

since $a \leq (\eta^2+2m+2)(\Lambda_{m,n})^2$. At the end of the day we have the following lower bound for ϕ.

$$\begin{aligned}|\phi(\Lambda_{m,n}\eta, \eta)| &= \Lambda_{m,n}\left|\sqrt{\eta^2+\frac{2p+2}{\Lambda_{m,n}}} - \sqrt{\eta^2+2m+2}\right|\\ &\geq \frac{|a|}{\Lambda_{m,n}\sqrt{\eta^2+2m+2}}\\ &\geq \frac{1}{\Lambda_{m,n}\sqrt{\eta^2+2m+2}}.\end{aligned}$$

2. Suppose now that $p^2+m^2+n^2-2pm-2mn-2pn-2p-2m-2n-3 = b \neq 0$ (and that $p > n+m$). Then $|b| \geq 1$. This implies that

$$(p-n-m-1)^2 = 4(m+1)(n+1) + b.$$

Thus,

$$p - n - m - 1 = 2\sqrt{m+1}\sqrt{n+1}\sqrt{1+\frac{b}{4(m+1)(n+1)}}.$$

This means that $p-n-m-1 = 2\sqrt{m+1}\sqrt{n+1} + r$ with $r \geq \frac{b}{2\sqrt{m+1}\sqrt{n+1}}$. Now we can do as in 1. and write the following lower bound.

$$|\phi(\Lambda_{m,n}\eta, \eta)| \geq \frac{\frac{|b|}{2\sqrt{m+1}\sqrt{n+1}}}{\Lambda_{m,n}\sqrt{\eta^2+2m+2}}$$

Hence

$$|\phi(\Lambda_{m,n}\eta, \eta)| \geq \frac{1}{2\sqrt{n+1}(\sqrt{n+1}+\sqrt{m+1})\sqrt{\eta^2+2m+2}}. \tag{B.2.1}$$

B.2.4.b Behavior of ϕ around $\mathcal{S}$

In this section, we will consider a straight line at a distance μ of $\mathcal{S}$ and estimate ϕ on this straight line. The aim will be to find a $\mu_0 > 0$ such that for all ξ, η at a distance less than μ_0 from $\mathcal{S}$, $\phi(\xi, \eta) \neq 0$.
Let $\mu > 0$ and consider the straight line $\xi = \Lambda_{m,n}\eta + \mu$.

$$\begin{aligned}
\phi(\Lambda_{m,n}\eta+\mu, \eta) =& \sqrt{(\Lambda_{m,n}\eta+\mu)^2+2p+2} - \sqrt{\eta^2+2m+2} - \sqrt{\left(\sqrt{\frac{n+1}{m+1}}\eta+\mu\right)^2+2n+2} \\
=& \sqrt{(\Lambda_{m,n})^2\eta^2+2p+2}\sqrt{1+\frac{2\Lambda_{m,n}\eta\mu+\mu^2}{(\Lambda_{m,n})^2\eta^2+2p+2}} - \sqrt{\eta^2+2m+2} \\
&- \sqrt{\frac{n+1}{m+1}\eta^2+2n+2}\sqrt{1+\frac{2\sqrt{\frac{n+1}{m+1}}\eta\mu+\mu^2}{\frac{n+1}{m+1}\eta^2+2n+2}} \\
=& \sqrt{(\Lambda_{m,n})^2\eta^2+2p+2} - \sqrt{\eta^2+2m+2} - \sqrt{\frac{n+1}{m+1}\eta^2+2n+2} + r,
\end{aligned}$$

where

$$\begin{aligned}
r \lesssim& \frac{2\Lambda_{m,n}|\eta||\mu|}{\sqrt{(\Lambda_{m,n})^2\eta^2+2p+2}} + \frac{2\sqrt{\frac{n+1}{m+1}}|\eta||\mu|}{\sqrt{\frac{n+1}{m+1}\eta^2+2n+2}} \\
\lesssim& \frac{\Lambda_{m,n}|\eta||\mu|}{\sqrt{\frac{n+1}{m+1}\eta^2+2n+2}} \\
\lesssim& \sqrt{\frac{m+1}{n+1}}\frac{\Lambda_{m,n}|\eta||\mu|}{\sqrt{\eta^2+2m+2}},
\end{aligned}$$

since $p > n$. Then, given B.2.1, there exists a universal constant c such that if

$$\begin{aligned}
|\mu| \leq& c\sqrt{\frac{n+1}{m+1}}\frac{1}{\Lambda_{m,n}|\eta|}\frac{1}{2\sqrt{n+1}(\sqrt{n+1}+\sqrt{m+1})\sqrt{\eta^2+2m+2}} \\
=& c\frac{1}{|\eta|(\sqrt{n+1}+\sqrt{m+1})^2},
\end{aligned}$$

then $\phi(\Lambda_{m,n}\eta+\mu, \eta) \gtrsim \frac{1}{(\sqrt{n+1}+\sqrt{m+1})^2}\frac{1}{R}$.
The problem is that this bound is not uniform in η. However, if η is bounded it is possible to find a uniform estimate, and even to determine the width of a zone where ϕ does not

vanish.
First of all, if a, b are two real numbers, the distance between the straight line $y = ax$ and the straight line $y = ax + b$ is equal to $\frac{b}{\sqrt{a^2+1}}$.
If

$$K_{m,n} := \frac{1}{\sqrt{(\Lambda_{m,n})^2 + 1}},$$

then for $|\eta|^2 + |\xi|^2 \leq R$ (i.e. $\eta^2 \leq K_{m,n}R$), there is a band of non-cancellation of ϕ of width of order

$$\frac{K_{m,n}}{(\sqrt{n+1} + \sqrt{m+1})^2} \frac{1}{K_{m,n}R} = \frac{1}{(\sqrt{n+1} + \sqrt{m+1})^2} \frac{1}{R}.$$

This proves Lemma B.1.4■

B.2.5 Asymptotics for ϕ

Now we study the asymptotics for ϕ: our goal is to prove Lemma B.1.5.
Proof of Lemma B.1.5 :
So as to study this phase, try to understand it along a given direction. Let γ be a real number.

$$\begin{aligned}\phi(\gamma\eta, \eta) =& \sqrt{\gamma^2\eta^2 + 2p + 2} - \sqrt{\eta^2 + 2m + 2} - \sqrt{(\gamma - 1)^2\eta^2 + 2m + 2} \\ =& |\gamma|\sqrt{\eta^2 + \frac{2p+2}{\gamma^2}} - \sqrt{\eta^2 + 2m + 2} - |\gamma - 1|\sqrt{\eta^2 + \frac{2m+2}{(\gamma-1)^2}}.\end{aligned}$$

From this formula comes the following alternative: if $\gamma < 1$ then $\phi(\gamma\eta, \eta)$ goes to $-\infty$ when η goes to $\pm\infty$; if $\gamma \geq 1$ then $\phi(\gamma\eta, \eta)$ goes to 0 when η goes to $\pm\infty$. Hence the asymptotics have to be done in that zone. In the proof of Theorem 4.2.3 we proved that $|\phi|$ was minimal on the space resonant set.
Since we are in the case $p < m$ or $p < n$ we have $p < n + m + 1$ and the previous lower bound can be used.

$$\begin{aligned}|\phi| &\gtrsim \frac{1}{2\sqrt{n+1}(\sqrt{n+1} + \sqrt{m+1})\sqrt{\eta^2 + 2m + 2}} \\ &\gtrsim \frac{1}{2\sqrt{n+1}(\sqrt{n+1} + \sqrt{m+1})\sqrt{(K_{m,n}R)^2 + 2m + 2}}.\end{aligned}$$

■

Appendix C

Paraproduct for the Hermite expansions

Contents

C.1 Statement of the theorem

In this section we are going to prove the following theorem.

Theorem C.1.1. *Let $a > 0$, $\gamma > 0$, $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ two sequences in ℓ^2, $M > a+2$. Then there exists and a constant C_γ and a sequence $(u_p)_{p\in\mathbb{N}}$ in the unit ball of ℓ^2 such that for all integer p,*

1.
$$p^M \sum_{m,n} \mathcal{M}(m,n,p)\frac{\max(m,n)^a}{\sqrt{mn}} m^{-M} a_m n^{-M} b_n \leq C_\gamma p^{a-\frac{3}{4}+\gamma} u_p,$$

This inequality has two consequences:

2. *(bounded sums theorem) for all $R > 0$,*
$$p^M \sum_{m,n\leq R} \mathcal{M}(m,n,p)\frac{\max(m,n)^a}{\sqrt{mn}} m^{-M} a_m n^{-M} b_n \leq C_\gamma R^{a-\frac{3}{4}+\gamma} u_p,$$

3. *(half sums theorem) for all $M_0 > 2$,*
$$p^M \sum_{m<n} \mathcal{M}(m,n,p) n^{a-\frac{1}{2}} m^{-M_0} a_m n^{-M} b_n \leq C_\gamma p^{a-\frac{3}{4}+\gamma} u_p.$$

C.2 Proof

We are going to deal with three different cases, corresponding to different orders of magnitude of the input frequencies m, n and the output p and use Proposition A.5.1.

1. Section C.2.1. If $p > Cm$ and $p > Cm$, C large enough chosen later, we will use the fact that the interaction term $\mathcal{M}(m,n,p)$ becomes very small.
2. Section C.2.2. If $p \leq Cn$ and $p \leq Cm$, we will simply use that $m^a \lesssim p^a$ if $a < 0$. However three cases will have to be dealt with
   (a) the case $p \leq m \leq n$.
   (b) the case $m \leq p \leq n$.
   (c) the case $m \leq n \leq p$.
3. Section C.2.3. If $Cm \leq p \leq Cn$ (the case $Cn \leq p \leq Cm$ being dealt with similarly given the symmetry of the situation), then we will try to use the interaction term as a convolution one.

C.2.1 If $p >> m$ and $p >> n$.

Let C be a constant greater than 1, and consider the following "low-low->high" term S_{llh}

$$S_{llh}(m,n,p) := p^M \sum_{m\leq n\leq \frac{p}{C}} \mathcal{M}(m,n,p)\frac{\max(m,n)^a}{\sqrt{mn}} m^{-M} a_m n^{-M} b_n,$$

the term

$$\widetilde{S}_{llh} := p^M \sum_{n\leq m\leq \frac{p}{C}} \mathcal{M}(m,n,p)\frac{\max(m,n)^a}{\sqrt{mn}} m^{-M} a_m n^{-M} b_n,$$

being dealt with similarly. First, by Proposition A.5.1, if $K \in \mathbb{N}$, $\nu > 1/8$ and $\beta < 1/24$, we can write

$$\mathcal{M}(m,n,p) \le C_K \frac{m^\nu}{p^\beta}\left(\frac{\sqrt{mn}}{\sqrt{mn}+p-n}\right)^K.$$

Then, since $p > Cn$, we can bound $\frac{\sqrt{mn}}{\sqrt{mn}+p-n}$ by $\frac{\sqrt{mn}}{(1-\frac{1}{C})p}$ and write

$$\mathcal{M}(m,n,p) \le C_K \frac{m^\nu}{p^\beta} m^{\frac{K}{2}} n^{\frac{K}{2}} \left(\frac{C}{C-1}\right)^K p^{-K}.$$

Then collect the terms in m, n and p in the original sum to get

$$S_{llh}(p) \le C_K \left(\frac{C}{C-1}\right)^K p^{M-K-\beta}\left(\sum_m m^{\nu+\frac{K}{2}-\frac{1}{2}-M} a_m\right)\left(\sum_n n^{a+\frac{K}{2}-\frac{1}{2}-M} b_n\right).$$

If

$$M > \frac{K}{2} + \max(\nu, a),$$

i.e.

$$\left(m^{\nu+\frac{K}{2}-\frac{1}{2}-M}\right)_{m\in\mathbb{N}} \text{ and } \left(n^{a+\frac{K}{2}-\frac{1}{2}-M}\right)_{n\in\mathbb{N}} \text{ are in } \ell^2,$$

then both series in m and n converge. Moreover, if $M < K - \frac{1}{2}$, $(S_{llh}(p))_{p\in\mathbb{N}}$ is in ℓ^2.

C.2.2 If $p \le Cm$ and $p \le Cn$.

The zone $p \le Cm$ and $p \le Cn$ corresponds to a "low-low→low", "high-high→high" or a "high-high→low" interaction. We will deal with the term S_1 defined by

$$S_1(p) := p^M \sum_{\frac{p}{C}\le m \le n} \mathcal{M}(m,n,p)\frac{\max(m,n)^a}{\sqrt{mn}} m^{-M} a_m n^{-M} b_n,$$

the term

$$\widetilde{S}_1 := p^M \sum_{\frac{p}{C}\le n \le m} \mathcal{M}(m,n,p)\frac{\max(m,n)^a}{\sqrt{mn}} m^{-M} a_m n^{-M} b_n,$$

being dealt with similarly.

Here, we do not need to find a fine bound for the interaction term $\mathcal{M}(m,n,p)$, we will simply bound it by a constant $\mathcal{M}_0$. Hence, collecting the terms in m, n and p we get

$$S_1(p) \lesssim p^M \left(\sum_{\frac{p}{C}\le m} m^{-M-\frac{1}{2}}\right)\left(\sum_{\frac{p}{C}\le n} n^{a-M-\frac{1}{2}}\right).$$

Then, Hölder's inequality and a comparison series-integral give

$$\sum_{\frac{p}{C}\le m} m^{-M-\frac{1}{2}} a_m \lesssim \left(\sum_{\frac{p}{C}\le m} m^{-2M-1}\right)^{\frac{1}{2}} \lesssim C^{-M} p^{-M}$$

$$\sum_{\frac{p}{C}\le n} n^{a-M-\frac{1}{2}} b_n \lesssim \left(\sum_{\frac{p}{C}\le n} n^{2a-2M-1}\right)^{\frac{1}{2}} \lesssim C^{a-M} p^{a-M}.$$

Finally we get

$$S_1(p) \lesssim p^{a-M},$$

which is in ℓ^2 for M large enough.

C.2.3 If $p \geq Cm$ and $p \leq Cn$ or $p \geq Cn$ and $p \leq Cm$.

C.2.3.a Classical paraproduct [Theorem C.1.1-(1)]

First assume that $Cm \leq p \leq Cn$, the case $Cn \leq p \leq Cm$ being symmetric. Denote bu S_{lhh} the term

$$S_{lhh} := p^M \sum_{Cm \leq p \leq n} \mathcal{M}(m,n,p) \frac{\max(m,n)^a}{\sqrt{mn}} m^{-M} a_m n^{-M} b_n.$$

Then assume that $Cm \leq p \leq n$: the case $Cm \leq n \leq p \leq Cn$ is dealt with similarly, simply by multiplying by powers of C.

Let $\theta, \varepsilon > 0$, $\nu > 1/8$, $0 \leq \beta < 1/24$ and K integer, there exists a constant $C_{\varepsilon,\theta,K}$ and a sequence $(u_p)_{p\in\mathbb{N}}$ in ℓ^2 such that

$$\begin{aligned}\mathcal{M}(m,n,p) &\leq C_{\varepsilon,\theta,K} \frac{m^{\theta\nu}}{n^{\theta\beta}} \left(\frac{\sqrt{mp}}{\sqrt{mp}+n-p}\right)^{\theta K} n^{-\frac{1}{4}+\frac{\theta}{4}+\theta\varepsilon} u_p^{1-\theta} \\ &\leq C_{\varepsilon,\theta,K} \frac{m^{\theta\nu}}{n^{\theta\beta}} \left(\frac{\sqrt{mp}}{1+n-p}\right)^{\theta K} n^{-\frac{1}{4}+\frac{\theta}{4}+\theta\varepsilon} u_p^{1-\theta}.\end{aligned}$$

Then, when gathering the terms in m, n and p we obtain

$$S_{lhh} \leq C_{\varepsilon,\theta,K} u_p^{1-\theta} p^{M+\frac{\theta K}{2}} \left(\sum_{Cm\leq p} a_m m^{\theta\nu+\frac{\theta K}{2}-M-\frac{1}{2}}\right) \left(\sum_{p\leq n} \left(\frac{1}{1+n-p}\right)^{\theta K} b_n n^{a-M-\frac{1}{2}-\theta\beta-\frac{1}{4}+\frac{\theta}{4}+\theta\varepsilon}\right).$$

Then the sum

$$\sum_{Cm\leq p} m^{\theta\nu+\frac{\theta K}{2}-M-\frac{1}{2}}$$

is finite whenever M is large enough.
In order to bound

$$\sum_{p\leq n} \left(\frac{1}{1+n-p}\right)^{\theta K} n^{a-M-\frac{1}{2}-\theta\beta-\frac{1}{4}+\frac{\theta}{4}+\theta\varepsilon} b_n,$$

first we bound n^{-1} by p^{-1} and write

$$\sum_{p\leq n} \left(\frac{1}{1+n-p}\right)^{\theta K} b_n n^{a-M-\frac{1}{2}-\theta\beta-\frac{1}{4}+\frac{\theta}{4}+\theta\varepsilon} \leq p^{a-M-\frac{1}{2}-\theta\beta-\frac{1}{4}+\frac{\theta}{4}+\theta\varepsilon} \sum_{n\geq p} \left(\frac{1}{1+n-p}\right)^{\theta K} b_n.$$

Then b_n is in ℓ^2 and so is $\left(\frac{1}{1+n}\right)_{n\in\mathbb{N}}$, whenever $2K\theta = 1+\delta$, $\delta > 0$. Finally the following bound holds for S_{lhh}.

$$\begin{aligned}S_{lhh}(p) &\lesssim u_p^{1-\theta} p^{M+\frac{\theta K}{2}} p^{a-M-\frac{1}{2}-\theta\beta-\frac{1}{4}+\frac{\theta}{4}+\theta\varepsilon} \\ &\lesssim u_p^{1-\theta} p^{M+\frac{1}{4}+\frac{\delta}{2}} p^{a-M-\frac{1}{2}-\theta\beta-\frac{1}{4}+\frac{\theta}{4}+\theta\varepsilon} \\ &\lesssim u_p^{1-\theta} p^{a-\frac{1}{2}+\theta(\varepsilon+\frac{1}{4}-\beta)+\frac{\delta}{2}}.\end{aligned}$$

The remaining problem is that $\left(u_p^{1-\theta}\right)_{p\in\mathbb{N}}$ is not in ℓ^2. However, remark that for all $\alpha > 0$, $\left(u_p^{1-\theta} p^{-\frac{\theta}{2}(1+\alpha)}\right)_{p\in\mathbb{N}}$ is in ℓ^2: it can be checked by writing

$$\sum_p u_p^{2-2\theta} p^{-2\theta(1+\alpha)} \lesssim \left(\sum_p \left(u_p^{2-2\theta}\right)^{\frac{2}{2-2\theta}}\right)^{\frac{2-\theta}{2}} \left(\sum_p \left(p^{-2\frac{\theta}{2}(1+\alpha)}\right)\right)^{\frac{1}{\theta}},$$

which is finite. Finally, writing $v_p := u_p^{1-\theta} p^{-\frac{\theta}{2}(1+\alpha)}$, the following bound holds:

$$S_{lhh}(p) \lesssim v_p p^{a-\frac{1}{2}+\gamma},$$

with $(v_p)_{p\in\mathbb{N}} \in \ell^2$ and $\gamma = \theta(\frac{1+\alpha}{2} + \varepsilon + \frac{1}{4} - \beta) + \frac{\delta}{2}$.

C.2.3.b Case of a bounded sum [Theorem C.1.1-(2)]

In this case, we are summing over all m and n less than or equal to R. Since we are in the situation "$p \geq Cm$ and $p \leq Cn$ or $p \geq Cn$ and $p \leq Cm$" , p can be bounded by R and the bounded resummation theorem follows.

Theorem C.1.1-(3) is skipped: this ends the proof.

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