Results in perturbative quantum supergravity from string theory

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Results in perturbative quantum supergravity from string theory.

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Zvi Bern	Examinateur
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Michael Green	Examinateur (Président du jury)
Boris Pioline	Examinateur
David Skinner	Rapporteur
Dimitrios Tsimpis	Rapporteur
Pierre Vanhove	Directeur de thèse

A Anne-Marie, Marie-Hélène et Audrey,
à qui je dois tout.

Contents

Personnal publications

This manuscript is based on the first four publications and preprints [PT1, PT2, PT3, PT4] below. The content of [PT5] has not been covered in this text.

[PT1] P. Tourkine and P. Vanhove, "An R^4 non-renormalisation theorem in $\mathcal{N} = 4$ supergravity," Class. Quant. Grav. **29**, 115006 (2012) [arXiv:1202.3692 [hep-th]].

[PT2] P. Tourkine and P. Vanhove, "One-loop four-graviton amplitudes in $\mathcal{N} = 4$ supergravity models," Phys. Rev. D **87**, no. 4, 045001 (2013) [arXiv:1208.1255 [hep-th]].

[PT3] P. Tourkine, "Tropical Amplitudes," arXiv:1309.3551 [hep-th].

[PT4] A. Ochirov and P. Tourkine, "BCJ duality and double copy in the closed string sector," JHEP **1405**, 136 (2014) [arXiv:1312.1326 [hep-th]].

[PT5] N. E. J. Bjerrum-Bohr, P. H. Damgaard, P. Tourkine and P. Vanhove, "Scattering Equations and String Theory Amplitudes," arXiv:1403.4553 [hep-th].

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[1] Ou heureuse, selon que l'on travaille dans le BTP.

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[2]Je suspecte qu'Alexandre n'y est pas pour rien

[3]Eussé-je eu le courage de procéder ainsi, j'eus probablement ajouté à ceci une petite section appelée "Vannes Gratuites", à l'attention du Dr. Lazarescu.

Chapter 1

Introduction

Throughout this text we use the standard system of notations where $\hbar = c = 1$ and our space-time signature is $(-, +, \ldots, +)$. Our kinematic Mandelstam invariants for two-to-two scattering are defined by $s = -(k_1 + k_2)^2$, $t = -(k_1 + k_4)^2$ and $u = -(k_1 + k_3)^2$ where all particles are incoming particles with light-like momentum k_i. The bosonic sector of the heterotic string will always be the left-moving (anti-holomorphic) sector.

1.1 The UV question in quantum gravity.

Quantum gravity is one of the most challenging conundrums in modern physics. Conceptually, this theory is the missing link between quantum field theories that describe particles physics and Einstein's General Relativity that describes the dynamics of space-time. Einstein's equations relate the two realms as

$$\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\text{space-time}} = \underbrace{16\pi G_N T_{\mu\nu}}_{\text{matter, energy}} \,, \tag{1.1.1}$$

then how could one be quantum and not the other ? The issue is that a naive quantization process leads quickly to inconsistencies, as we expose below.

The quantum nature of space-time is supposed to manifest itself at the Planck energy mass-scale, $M_{\rm Pl} = 10^{19}$ GeV. Needless to say, this energy scale is far away from the reach of modern colliders. Quantum gravity effects are more likely to be detected in primordial cosmology experiments in the following decades, as the Big-Bang offers a more direct observational window to high energies.

Violation of unitarity One of the basic issues with a naive quantization of gravity is that it causes unitarity violations, already at the classical-level, as a consequence of the structure of gravitational interactions. These are described by the Einstein-Hilbert action coupled to matter

$$S_{\rm EH+matter} = \frac{1}{2\kappa_D^2} \int {\rm d}^D x \sqrt{-g} (R + \mathcal{L}_{\rm matt}(\phi, \psi, A_\mu)) \,, \tag{1.1.2}$$

where R is the scalar Ricci curvature, D is the space-time dimension and $\mathcal{L}_{\rm matt}$ is a given matter Lagrangian. By expanding this action around a flat background $g^{\mu\nu} = \eta^{\mu\nu} + \kappa_D h^{\mu\nu}$,

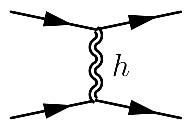


Figure 1.1: Graviton exchange violates unitary at tree-level.

where $h^{\mu\nu}$ is the spin-2 graviton field, standard manipulations [1] yield the Lagrangian of matter coupled to gravitons:

$$\begin{aligned} S_{EH} = \int \mathrm{d}^D x \bigg(&\frac{1}{2}\partial h\, \partial h + \kappa_D c_0 h \partial h\, \partial h + O(\kappa_D h \partial h \partial h) + \dots \\ &+ \frac{1}{2\kappa_D^2}\mathcal{L}_{\mathrm{mat}}(\phi, \psi, A_\mu) + \frac{h}{2\kappa_D}\mathcal{L}_{\mathrm{mat}}(\phi, \psi, A_\mu) + \dots \bigg) \, . \end{aligned} \tag{1.1.3}$$

The structure of this action indicates that gravitons couple to (massless) fields with a two-derivative interaction. Consequently, a single graviton exchange between massless fields such as the one depicted in figure 1.1 has an amplitude proportional to the dimensionless ratio E^2/κ_D where E is the energy of the process. Eventually, unitarity is violated for processes such that $E \gg \kappa_D^2$. At loop level, this classical breakdown of unitarity transfers directly to ultraviolet divergences of the theory. For instance, the amplitude of the double graviton-exchange depicted in fig. 1.2 contains an intermediate sum over states which induces a divergent behavior in the UV as

$$\frac{E^2}{\kappa_D^2} \int^{\Lambda} \mathrm{d}\tilde{E}\, \tilde{E} \sim \frac{E^2 \Lambda^2}{\kappa_D^2} \tag{1.1.4}$$

where Λ is a UV momentum cut-off. Alternatively these issues can be seen as the consequence of the positive mass dimension $\kappa_D = M_{\mathrm{Pl}}^{D-2}$ of the gravity coupling constant, which makes the theory non-renormalizable for $D > 2$. The first divergence of pure Einstein gravity occurs at the two-loop order [2, 3] and is followed by an infinite series of divergences which should be removed, thereby introducing an infinite amount of arbitrariness in the choice of the counterterms and making the quantum theory un-predictive or ill-defined.

Although this manuscript exclusively deals with the perturbative regime of quantum gravity and string theory, it is worth mentioning here that quantum gravity also seems to violate unitarity in black holes physics. On the one hand, the no-hair theorem states that classical black holes should be solely characterized by their mass, spin, and charge. Therefore, Hawking radiation has to be thermal and can not radiate away the information that fells in the black hole. On the other hand, when the black hole has completely evaporated, this information is "lost", which is impossible in a unitary quantum mechanical evolution. This goes under the name of the "information paradox".

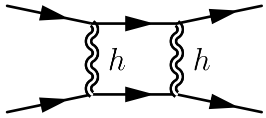


Figure 1.2: One-loop double graviton exchange diverge in the UV.

There exists nowadays two main paradigms to remedy these issues. The first possibility is to assume that gravity is truly a fundamental theory that should be quantized non-pertubatively and that the previously mentioned issues are artifacts of the perturbative quantization process. This is the point of view adopted by the Loop Quantum Gravity theory and by a somehow related Asymptotic Safety program.

The other option, that we follow and on which are based supergravity and string theory, postulates that general relativity is a low energy effective field theory of a more fundamental theory. Therefore, possibly drastic modifications of the law of physics at the quantum gravity scale are expected to happen.

UV divergences and effective field theories. The UV behavior of quantum gravity is of central importance in the effective field theory paradigm, where the presence of UV divergences signals a wrong identification of the microscopic degrees of freedom of the theory at high energy. In the language of effective actions, UV divergences correspond to local operators, called counterterms, that should be added to the effective action. They parametrize the ignorance about the high energy theory. These operators have to obey the symmetries of the theory, which in gravity include in particular diffeomorphism invariance. This constrains the counterterms to be expressed as tensorial combinations of the Riemann tensor $R_{\mu\nu\alpha\beta}$ or derivatives thereof. For a n-graviton scattering, they are of the form

$$\nabla^m R^n\,, \quad m = 0, 1, 2, \dots \tag{1.1.5}$$

where ∇ is the covariant derivative. These have mass dimension

$$[\nabla^m R^n] = M^{m+2n}\,. \tag{1.1.6}$$

In order to understand more precisely what kind of divergences these counterterm may cancel, let us look back at the structure of the action (1.1.3). The n-graviton vertex always carries at least two derivatives, therefore the most divergent graph at leading order in the gravity coupling constant is made of 3-valent vertices which bring two powers of loop momentum each. Using Euler's relation for a connected graph with L loops, V vertices and I internal edges legs:

$$V - I + L = 1 \tag{1.1.7}$$

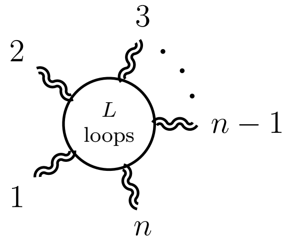
we obtain the *naive* superficial UV behavior of a L-loop n-graviton amplitude

$$\mathcal{M}_n^{L-\text{loop}} = \int \mathrm{d}^D\ell \frac{\ell^{2V}}{\ell^{2I}} \sim \int^\Lambda \frac{\mathrm{d}\ell}{\ell}\, \ell^{L(D-2)+2} \sim \Lambda^{L(D-2)+2}\,. \tag{1.1.8}$$

However, we know that a divergence should be canceled by a counterterm of the form $\nabla^m R^n$. In other words, diffeomorphism invariance of the theory implies that divergent integrals have to factor out a term which can be canceled by such operators, and if $\mathcal{M}_n^{L-\text{loop}}$ in (1.1.8) diverges, the power-counting (1.1.6) indicate that there exists a m a d a n such that

$$\mathcal{M}_n^{L-\text{loop}} = \nabla^m R^n \int^\Lambda \frac{\mathrm{d}\ell}{\ell}\, \ell^{L(D-2)+2-m-2n}\,. \tag{1.1.9}$$

From this we read the actual superficial degree of divergence of such an amplitude, as depicted in eq. (1.1.10). A priori, all of these operators appear in the effective action, where they allow for an infinite number of UV divergences.



$$\sim \quad \nabla^m R^n \times \Lambda^{L(D-2)+2-m-2n} \tag{1.1.10}$$

1.2 Supergravities

Supergravity theories are proposals for modifications of gravity at high energy with a enhanced UV behavior. These theories are conceptually close to Kaluza-Klein theories, where geometry in higher dimension provides matter and various interactions in lower dimensions. They differ from these in that the "higher" dimensions of supergravity theories include fermionic dimensions. Hence, the four-dimensional space-time is immersed in a higher-dimensional *superspace*.

From the field-theory viewpoint, the geometry of this superspace obeys a new local symmetry, called supersymmetry. This symmetry is characterized by a certain number of real anti-commuting charges called *supercharges*, from 4 in four dimensions to 32 for the maximal extension defined in any dimension up to $D = 11$ [4–6].[4] For definiteness, we shall refer to the number of four-dimensional supercharges $\mathcal{N}$ of the theory only when we talk about the four-dimensional theory. Consequently, $\mathcal{N} = 1$ supergravity is the minimal supergravity in four dimensions and has 4 real supercharges, while $\mathcal{N} = 8$ is maximal supergravity in four dimensions. Half-maximal supergravity or $\mathcal{N} = 4$ in four dimensions is the subject a full chapter of this manuscript, chap. 3. There, we distinguish between $(2, 2)$ and $(4, 0)$ type of constructions, depending on what string theory model the theory arises from.

These theories, some of which are listed in tab. 1.1, have much a richer spectrum than Einstein gravity and seemingly look more complicated. However, at the level of scattering amplitudes, the number of symmetries help to reduce the complexity of the theories. Part of the discussion in this manuscript is focused on understanding these simplifications from the string theory perspective.

Among these extended supergravity theories, maximal supergravity have held a favorite position as the most promising candidate for a four-dimensional UV complete point-like theory of quantum gravity. It was however understood that an R^4 counterterm did respect the *linearized* maximal supersymmetry, very likely indicating a 3-loop divergence [8–12] in four dimensions. Despite this belief, curious similarities between the UV behavior of maximal super-Yang-Mills (SYM) and maximal supergravity were observed in particular in [13, 14]. Since maximal SYM is UV finite in four dimensions [15], this suggested that $\mathcal{N} = 8$ might indeed be a UV finite theory. We recall that L-loop amplitudes in maximal SYM are UV finite in dimensions D such that [13, 16, 17]

$$D < D_c = 4 + 6/L\,, \tag{1.2.1}$$

where D_c is called the critical UV dimension and is defined here to be the smallest

[4] Adding more supercharges to a gravity theory forces to include an infinite tower of higher spin excitations

			$s=2$	$s=3/2$	$s=1$	$s=1/2$	$s=0$
$\mathcal{N}=0$	$s_{\max}=2$		1				
	$s_{\max}=2$	($\mathcal{N}=0$, YM)			1		
$\mathcal{N}=1$	$s_{\max}=2$		1	1			
	$s_{\max}=3/2$			1	1		
	$s_{\max}=1$				1	1	
	$s_{\max}=1/2$					1	1
$\mathcal{N}=2$	$s_{\max}=2$		1	1			
	$s_{\max}=3/2$			1	1		
	$s_{\max}=1$	($\mathcal{N}=2$, vect.)			1	1	
	$s_{\max}=1/2$	($\mathcal{N}=2$, hyper.)				1	1
$\mathcal{N}=4$	$s_{\max}=2$	($\mathcal{N}=4$, grav.)	1	4	6	4	1+1
	$s_{\max}=3/2$			1	4	6	4+4
	$s_{\max}=1$	($\mathcal{N}=4$, matt.)			1	4	6
$\mathcal{N}=6$	$s_{\max}=2$		1	6	15+1	20+6	15+15
	$s_{\max}=3/2$	($\mathcal{N}=6$, matt.)		1	6	15	20
$\mathcal{N}=8$	$s_{\max}=2$	($\mathcal{N}=8$)	1	8	28	56	70

Table 1.1: Partly reproduced after the textbook on supergravity theories [7]. Spin content of massless supersymmetry representations with maximal spin $s_{\max} \leq 2$ in four dimensions. The first line with $\mathcal{N}=0$ corresponds to pure Einstein gravity. The supermultiplet denominations within the parentheses correspond to notations used throughout the text.

space-time dimension for which the theory diverges logarithmically at L loops.

In ref. [18], Green *et al.* predicted, using non-renormalization theorems in string theory, that the UV behavior of maximal supergravity might be softer than previously expected and that the three-loop divergence at could actually vanish. This issue was definitely settled by the explicit computation of Bern *et al.* in [19, 20], that was followed by a similar result at four loops [21]. Nowadays, the most elaborate predictions based on string theory non-renormalization theorems [18, 22, 23], supersymmetry [24–26] and duality symmetry analysis [27–32] predict that the critical behavior should change abruptly at five loops due to an allowed $\nabla^8 R^4$ counterterm, according to

$$\begin{aligned} D < D_c &= 4 + 6/L\,, \quad L < 5\,, \\ D < D_c &= 2 + 14/L\,, \quad L \geq 5\,, \end{aligned} \tag{1.2.2}$$

This critical behavior predicts that maximal supergravity may diverge at seven loops in four dimensions.

Despite the important progress made in the last decade in the field of scattering amplitude computations (see chapter 4 for a short review), the 7-loop order is still out of reach. Nonetheless, already the analysis of the UV behavior at five loops may indicate if the current string theory understanding is correct or needs to be deepened. If the critical dimension of $\mathcal{N}=8$ is strictly the same as the one of $\mathcal{N}=4$ SYM, the five-loop divergence should occur in $D = 26/5$, corresponding to a $\nabla^{10} R^4$ counterterm. On the

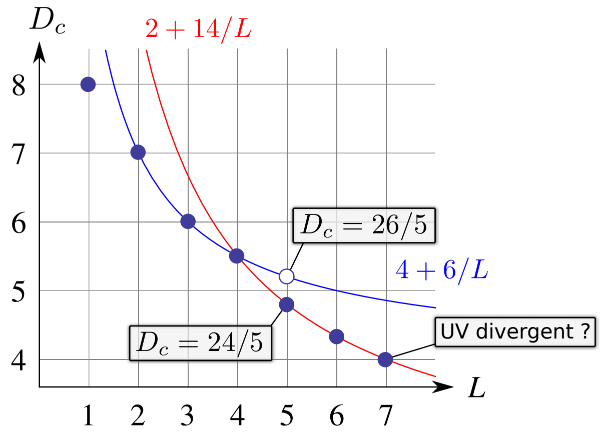


Figure 1.3: Critical UV behavior of maximal supergravity. ●: UV divergences predicted by string theory. ○ : 5-loop possible UV behavior indicating that $\mathcal{N}=8$ might be UV finite.

contrary, according to the previous predictions, the $\nabla^8 R^4$ counterterm is expected to cause a divergence in $D_c = 24/5$ at five loops.

The importance of the five-loop explicit computation is therefore crucial. As a matter of fact, this computation has been started for several years by the group of [21]. The first approach to the computation relied on the use of the "Bern-Carrasco-Johansson" duality [33] applied to the so-called double-copy construction of gravity amplitudes [34]. Despite important successes at three and four loops [34, 35], the prescription seems to work less efficiently at five-loop and for the moment has not been implemented [36]. In addition to the intrinsic interest of a first principle explanation of this BCJ duality, the five-loop problem acted as a motivation for the analysis of [PT4] which we describe in chap. 4, on first steps towards a string theoretic understanding of the BCJ duality at one-loop.

Another way to test the accuracy of string theory predictions is to study theories with reduced supersymmetry. This allows to trigger more divergent theories, thereby more accessible by explicit computations. In that perspective, half-maximal supergravity is the most natural candidate whose UV behavior should be investigated. This theory has a richer structure than maximal supergravity and can be realized in the low energy limit of various kind of string models, some of which we describe in chapter 3. The theory admits couplings to maximally SYM matter multiplets [37], which render the theory UV divergent already at one-loop for amplitudes with external matter fields [38]. The first section, sec. 3.1 of chap. 3 is dedicated to a review of the analysis given in [PT2] of graviton amplitudes at one-loop in several type of string theory models providing $\mathcal{N}=4$ supergravity, in heterotic string and orbifolds of type II string.

The following section, sec. 3.2 deals directly with the UV behavior of pure half-maximal supergravity. It was shown in [39–41] that R^4 is a half-BPS one-loop exact operator in heterotic string toroidal compactifications, and confirmed later in [42] by using the explicit two-loop computation of [43–49]. We review the analysis of [PT1] based on the use of the "Chaudhuri-Hockney-Lykken" [50–52] orbifolds of the heterotic string to show a non-renormalization theorem for the R^4 counterterm in pure half-maximal supergravity. This analysis provides a worldsheet supersymmetry argument for the origin of the vanishing of the R^4 3-loop logarithmic divergence in pure half-maximal supergravity originally observed in [53]. However, an additional element enters the analysis in this theory, due to the presence of a $U(1)$ anomaly [54] whose implication in the UV behavior of the theory is still unclear.

There are two lessons to draw from the previous discussion. First, it appears that string theory is a good tool to understand the UV behavior of supergravity theories. Second, supergravities do not seem to be drastic enough modifications of gravity to ensure a proper quantum behavior. Therefore, the same reason for which string theory is an efficient tool also indicates it as an empirical necessary UV completion for supergravity theories.

1.3 String theory

String theory has an even richer history than maximal supergravity, which we do not intend to recapitulate completely here.[5] It was born almost half-a-century ago as a model to describe strong interactions with the Veneziano amplitude [56], that was soon after supplemented by a proposal from Virasoro [57], which we reproduce here:

$$M^{\mathrm{Vir}}(s,t,u)=\frac{\Gamma(-1-\alpha's/4)\Gamma(-1-\alpha't/4)\Gamma(-1-\alpha'u/4)}{\Gamma(-2-\alpha's/4-\alpha't/4)\Gamma(-2-\alpha't/4-\alpha'u/4)\Gamma(-2-\alpha'u/4-\alpha's/4)}\,, \tag{1.3.1}$$

The variables s and t and u are the usual kinematic Mandelstam invariants, respectively defined by $-(k_1+k_2)^2$, $-(k_1+k_4)^2$ and $-(k_1+k_3)^2$ and α' was called the Regge slope. Later it was understood that these amplitudes describe the interactions and scattering of open and closed relativistic bosonic strings of size $\ell_s=\sqrt{\alpha'}$ and tension $T=(2\pi\alpha')^{-1}$. Quantization and Lorentz invariance imposed that they propagate in a target 26-dimensional space-time and that their spectrum contains an infinite tower higher spin excitations with quantized masses given by

$$m^2_{\mathrm{closed}}=\frac{4n}{\alpha'}\,,\quad m^2_{\mathrm{open}}=\frac{n}{\alpha'}\,,\qquad n=-1,0,1,\dots,+\infty \tag{1.3.2}$$

and maximal spins $J_{\mathrm{max}}=\alpha'm^2+1$. Both theories contained a tachyonic state (at $n=-1$) and the massless excitations of the closed string always contained a graviton. Later, the theory was extended to a theory of supersymmetric strings living in a 10-dimensional target space-time, where the tachyon was automatically projected out via the so-called "Gliozzi-Sherck-Olive" (GSO) projection [58]. This theory was shown to possess maximal supergravity in its massless spectrum in [59], making it a UV completion thereof.

Let us try to give a flavor of how string theory cures the structural problems of perturbative quantum gravity, namely unitarity violation and UV incompleteness. Firstly, the amplitude (1.3.1) has a high energy behavior now compatible with unitarity. In particular, in the hard scattering limit ($s,t\to+\infty$, fixed angle), this amplitude exhibits an exponentially soft behavior:

$$M^{\mathrm{Vir}}(s,t)\sim\exp\left(-\frac{\alpha'}{2}(s\ln s+t\ln t+u\ln u)\right) \tag{1.3.3}$$

[5] See [55] for a detailed historical perspective and list of references.

which can be seen as a restoration of the unitarity due to the infinite tower of massive states smoothing the interaction.

In order to comment on UV divergences, we need first to say a word on the quantization of string theory. String theory scattering amplitudes are computed in a first quantized formalism, as Feynman path integrals over the trajectories of the string in space-time. These trajectories draw a worldsheet, and the quantization process reduce the sum over trajectories to a finite dimensional integral over the moduli space of Riemann surfaces. The genus thereof, denoted by the letter g in this text, is related to the number of times the strings have split or joined during their evolution, and indicate the loop order of the interaction. One of the most notable features of string theory first-

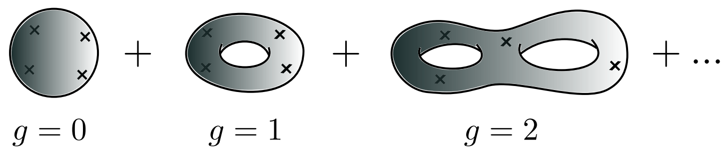


Figure 1.4: Perturbative expansion of string theory (four-point scattering exmaple).

quantized amplitudes is the compactness of the expressions. This is firstly a consequence of the fact that there is a single string graph at each order in perturbation theory; this is considerably simpler than the sum of Feynman graphs in quantum field theory. In addition, the computations of the string theory integrands are based on powerful conformal field theory (CFT) techniques which also simplify drastically the computations and give rise to a superior organization of the amplitude, in particular making manifest some cancellations not easily visible by other means. On the other hand, the integral over the moduli space of Riemann surfaces is most of the time impossible to carry, and despite the compactness of final answers, intermediate steps of computation can be fastidious.

Physically, the mathematical reduction from all trajectories to Riemann surfaces is a consequence of string theory not being simply a theory of an infinite tower of interacting states; the latter wouldn't be a UV complete theory, as recalled in [60, sec. 7.3]. String theory has an additional, crucial, physical feature: it gives a minimal length to space-time phenomena, the string length $\sqrt{\alpha'}$. In loop amplitudes, this implies that the ultraviolet region is simply absent from the phase space of string theory ! As a consequence, string theory is a UV complete theory.

In contrast, the theory has an infrared (IR) region, which is precisely the one of interest for us in this text, as it describes the regime in which the strings effectively behave as particles. We shall alternatively refer to this regime as the $\alpha' \to 0$ limit[6], low energy limit or the field theory limit of string theory. One of the objectives of this text is to discuss some of the techniques known in the literature concerning this limit in the context of string theory amplitudes. It is not surprising that if the advantages of string theory amplitudes motivate the use of such procedures, its drawbacks should be encountered along the way. There are basically two classes of physical objects that can be extracted out of string theory amplitudes; field theory amplitudes – with their UV divergences – and low energy effective actions. The present text mostly describes the first type of computations.

[6]In strict rigor, it is rather defined as a limit where the energy E of the scattered states is much smaller than the string scale $\alpha' E \ll 1$.

In chap. 2 we discuss the general procedure to extract field theory amplitudes from string theory amplitudes. These techniques were pioneered by Green and Schwarz in [59], and their use culminated at one-loop with the development by Bern and Kosower of a set of rules to write one-loop gauge theory n-gluon amplitudes in [61–64], as the $\alpha' \to 0$ limits of string theory amplitudes. The reason why such a procedure is efficient is because of the compactness of string amplitudes. The technical difficulties that are faced in general ($g \geq 2$) involve firstly the geometry of the moduli space of Riemann surfaces, which should be described correctly in order to reproduce the various graph topologies in the low energy. Another class of difficulties come precisely from the degenerations of the CFT on higher genus Riemann surfaces. In [PT3], we argued that tropical geometry, a somewhat recent branch of mathematics, helps to solve these issues.

Another remarkable feature of string theory is that it provides a framework where it is possible to carry the exact computation of some coefficient of the operators in low energy effective action. Let us here simply mention that the automorphic form program [22, 23, 65–71] led to the *exact non-perturbative* computation of R^4, $\nabla^4 R^4$ and $\nabla^6 R^4$ couplings in the type II string effective action in various dimensions. These exhibit directly non-renormalization properties, since they receive only a finite number of loop corrections. For instance, the essence of the previous prediction on the critical UV behavior of maximal supergravity follows from the fact that R^4 is not perturbatively renormalized beyond one-loop, $\nabla^4 R^4$ beyond two loops, $\nabla^6 R^4$ beyond three loops. The coupling corresponding to $\nabla^8 R^4$ has not been computed yet, but it is expected to receive contributions through all loop orders by different type of arguments mentioned above [24–32]. The counterpart of these computations in string theory amplitudes corresponds to factorization of derivatives in the pure spinor formalism [17]. In [PT1] we presented an explication for the vanishing of the three-loop divergence of $\mathcal{N} = 4$ pure supergravity [53] due to a non-renormalization theorem in heterotic string orbifold models for the R^4 term at two-loops. The computation is based on the explicit factorization of two derivatives in the computation of D'Hoker and Phong at two loops [43–49].

Structure of the manuscript.

Below is a quick summary of the organization of this manuscript.

In chap. 2 we review the analysis of [PT3] on the low energy limit of string theory amplitudes and the connexion with tropical geometry. We discuss applications in sec. 2.3, where we provide a novel analysis on the low energy limit and in particular the graph structure of the four-graviton three-loop amplitude computed in [72].

In chap. 3 we cover the content of [PT1, PT2] on half-maximal supergravity amplitudes at one-loop and the UV divergences of this theory at higher loops. We provide novel piece of analysis on the worldline structure of these amplitudes at two-loop.

In chap. 4, we describe the arguments presented in [PT4] towards a string theoretic understanding of the BCJ duality at one-loop.

The final chapter contains open questions and future directions of research.

Chapter 2

The field theory limit of closed string amplitudes and tropical geometry

In the introduction, we motivated the study of string theory amplitudes as an efficient way to access field theory amplitudes. Physically, there is no doubt that the perturbative expansion of string theory reproduces the Feynman graph expansion of the low energy effective field theory in the point-like limit $\alpha' \to 0$ of the string. However, this procedure has not been applied beyond one-loop[7] and a lot of technical tools are missing at higher genus. In [PT3], the author initiated a program to develop these tools by using a previously unnoticed connexion between the $\alpha' \to 0$ limit of string theory amplitudes and a recent field of mathematics called tropical geometry.[8] This chapter is a review of this work. We also present in the last section some elements of a novel three-loop analysis.

2.1 Closed string theory amplitudes and their field theory limit.

Our intention here is not to provide an exhaustive recapitulation of the material present in the standard textbooks [60, 77–79] on the perturbative quantization of string theory, but rather to recall some essential facts about string perturbation theory in order to introduce some important notions for the discussion of this chapter.

2.1.1 Superstring theory amplitudes

Bosonic string path integral String theory scattering amplitudes or S-matrix elements are computed in a first quantized formalism. The coordinates X^μ of the string define an embedding of the two-dimensional manifold swept by the string – the worldsheet – in the target space-time in which it evolves. From this worldsheet viewpoint, the

[7] Except the recent work at two loops of [73] in the Schottky parametrization, continuing the older works [74, 75]

[8] Let us take the opportunity here to mention that the construction of [76] is actually the first time where tropical geometry was used (even before its "official" birth !) in physics, in a different context though. There, tropical varieties, called "grid diagrams", were defined as configurations of branes in the five-dimensional decompactification limit of four-dimensional $\mathcal{N} = 2$ gauge theories.

X^μ's are scalar fields, whose dynamics is governed by Polyakov action[9]

$$S_{\text{Polyakov}} = -\frac{1}{4\pi\alpha'} \int \mathrm{d}\sigma \mathrm{d}\tau \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) \tag{2.1.1}$$

where σ and τ are the worldsheet coordinates, g^{ab} is the worldsheet quantum metric and $G_{\mu\nu}(X)$ is the target space-time metric. Lorentz invariance and mathematical consistency allow for only two kind of space-time interactions between strings (open or closed): splitting and joining.

The quantum mechanical amplitude for a process including propagation with or without interactions is given by a path integral over worldsheets that connect initial and final asymptotic states, weighted by the string action,

$$\int \frac{\mathcal{D}X\,\mathcal{D}g}{V_{\text{Diff}\times\text{Weyl}}} \exp(-S) \tag{2.1.2}$$

The factor $V_{\text{Diff}\times\text{Weyl}}$ is the volume of the diffeomorphisms and gauge freedom on the worldsheet required to counterbalance the over-counting of the path-integral. At the g-th order in perturbation theory, for an n-point scattering, standard BRST procedure fixes this gauge redundancy and reduces the integration to a finite dimensional space of dimension $3g - 3 + n$: the moduli space of genus-g n-pointed Riemann surfaces $\mathcal{M}_{g,n}$. The outcome of the quantization of the bosonic string is well known; the theory should live in 26 dimensions, has no fermions and has a tachyon.

Superstring path-integral Extending the bosonic formulation to a supersymmetric one projects out the bosonic string tachyon by introducing fermions. Conceptually, this gives a heuristic motivation for the existence of fermions, as a necessity to produce a sensible quantum theory of strings. A similar situation happens for supergravity, where fermions soften the bad UV behavior of Einstein gravity. Three formulations of superstring theory are known; the Green-Schwarz [77, 80] and Berkovits pure spinor [81, 82] space-time supersymmetric formalisms, and the Ramond-Neveu-Schwarz worldsheet supersymmetric formalism.

The advantage of the first two is to implement the very appealing physical idea that the superstring should move in a "super-spacetime". In this case the bosonic coordinates X^μ are supplemented with fermionic ones θ^α and space-time supersymmetry is guaranteed from the start. The drawback of the Green-Schwarz formalism is the difficulty to gauge the so-called κ-symmetry, which the pure spinor formalism manages to do thanks to the introduction of a pure spinor ghost field. A lot of results have been obtained in this formalism, among which a recent three-loop four-graviton amplitude computation [72] which we discuss in this text. It should be noted that above genus five, the prescription to compute the pure spinor ghost path integral has to be changed [83], and that the impact of this change in explicit computations has not been cross-checked so far.

On the other hand, the Ramond-Neveu-Schwarz formulation has the advantage of mathematical robustness. The formulation is based on the extension of the usual worldsheet to a super-worldsheet, by supersymmetrizing the Polyakov action and adding superpartners to the X^μ scalars, the fermionic fields ψ^μ, and a superpartner to the metric field

[9]We skipped the traditional pedagogical introduction of the Nambu-Goto action with its square root that creates difficulties in the quantization.

g, the gravitino field χ_α:

$$S_{RNS} = -\frac{1}{8\pi}\int_\Sigma \mathrm{d}\sigma\mathrm{d}\tau\sqrt{g}\left(\frac{2}{\alpha'}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\mu + 2i\psi^\mu\sigma^\alpha\partial_\alpha\psi^\mu \right.$$
$$\left. - i\psi^\mu\sigma^\beta\sigma^\alpha\chi_\beta\left(\sqrt{\frac{2}{\alpha'}}\partial_\alpha X^\mu - \frac{i}{4}(\chi_\alpha\psi^\mu)\right)\right) \tag{2.1.3}$$

In this formalism, suitable gauging of the supergravity fields on the a genus-g n-pointed super-worldsheet induces an integration over $3g-3+n$ bosonic and $2g-2+n$ fermionic moduli which span the moduli space of genus-g n-pointed super-Riemann surfaces $\mathfrak{M}_{g,n}$ [84–86]. The amplitude is obtained by integrating a correlation function of vertex operators $V_1,\ldots,V_n$ corresponding to external scattered states as

$$\mathsf{A}^{(g,n)}_{\alpha'} = \int_{\mathfrak{M}_{g,n}} \mathrm{d}\mu_{SS}\,\langle V_1...V_n\mathcal{O}_1...\mathcal{O}_k\rangle \tag{2.1.4}$$

where $\mathrm{d}\mu_{SS}$ is the supermoduli space measure and $\mathcal{O}_1\ldots\mathcal{O}_k$ are a certain number of picture changing operators, required to saturate superghosts background charges. Until the recent series of papers [86–91], the procedure to compute such integrals was believed to rely on the existence of a global holomorphic section of $\mathfrak{M}_{g,n}$ [85, 92]. This would allow to integrate out the odd moduli first and reduce the integral to an integral over its bosonic base. Such a procedure is now known not to exist in the general case. In particular, for $g\geq 5$ it is known that $\mathfrak{M}_{g,0}$ is not holomorphically projected [93], while the question remains open for $g=3,4$.

Our case In [PT3], the author discussed the low energy limit of string amplitudes in the cases where they *can be* written as integrals over the ordinary bosonic moduli space $\mathcal{M}_{g,n}$. As a consequence of the non-projectedness issues, the discussion is restricted to genus $g\leq 4$ amplitudes. Note that in the Green-Schwarz and pure spinor formalisms, this "restricted" RNS set-up is the standard set-up and the question of the compatibility of the three formalisms in view of this discussion is open. In this context, the amplitudes take the generic form:

$$\mathsf{A}^{(g,n)}_{\alpha'} = \int_{\mathcal{M}_{g,n}} \mathrm{d}\mu_{\mathrm{bos}}\;\mathcal{W}_{g,n}\,\exp(\mathcal{Q}_{g,n})\,, \tag{2.1.5}$$

where $\mathrm{d}\mu_{\mathrm{bos}}$ is a $(3g-3+n)$-dimensional integration measure and the string integrand is written as a product of $\mathcal{W}_{g,n}$, which generically accounts for the kinematics of the scattering process, with $\exp(\mathcal{Q}_{g,n})$, the universal Koba-Nielsen factor. It comes from the plane-wave parts of the vertex operators[10]

$$\langle : e^{ik_1X(z_1,\bar z_1)} : \cdots : e^{ik_nX(z_n,\bar z_n)} :\rangle = \exp\left(\sum_{i<j} k_i\cdot k_j\langle X(z_i,\bar z_i)X(z_j,\bar z_j)\rangle\right) \tag{2.1.6}$$

and writes explicitly

$$\mathcal{Q}_{g,n} = \sum_{i<j} k_i\cdot k_j\,\mathcal{G}(z_i-z_j,\bar z_i-\bar z_j) \tag{2.1.7}$$

[10]: : denotes normal ordering.

in terms of the momenta k_i of the n scattered states and of the two-point function

$$\mathcal{G}(z-w,\bar{z}-\bar{w}) = \langle X(z,\bar{z})X(w,\bar{w})\rangle\,. \tag{2.1.8}$$

whose explicit expression given below in eq. (2.2.34) was determined in [85, 94]. We shall describe several type of $\mathcal{W}_{g,n}$, these are obtained from application of Wick's theorem and typically write as products of two-point correlators of the X and ψ fields, as well as of ghosts and superghosts fields.

2.1.2 The field theory limit.

How could one create a graph out of a closed Riemann surface ? The first thing one would have in mind is to stretch the surface to create very long and thin tubes. This actually does not produce graphs but degenerate Riemann surfaces with nodes. Nevertheless, it is a good start, and physically these stretched surfaces probe the IR region of string theory. To obtain a graph out of these tubes one still has to integrate out their cylindrical dependence. A good flavor of what is tropical geometry can be found in the survey [95], where the tropicalization procedure is presented as a way to "*forget the phases of complex numbers*". In the previous example, if σ and τ are respectively angular and longitudinal coordinates along the tube, $w=\tau+i\sigma$ can be conformally mapped to the plane via $w\to z=e^{iw}$, and we see that integrating out the cylindrical dependence of w amounts to integrating out the phase of z. Therefore tropical geometry describes how surfaces are turned into graphs by integrating out the phases of complex numbers.

The genus one bosonic string partition function is a handful example to illustrate the basic mechanism of the field theory limit in string theory, and point out where do come from the "phases" and "modulus" of complex numbers. It can be found in standard string theory textbooks mentioned before and writes

$$Z(\tau,\bar{\tau}) = \mathrm{Tr}\,\left(q^{L_0-1}\bar{q}^{\tilde{L}_0-1}\right)\,, \tag{2.1.9}$$

where the trace is performed over the Hilbert space of physical states of string theory. The parameter q is defined by $q=\exp(2i\pi\tau)$ where $\tau=\mathrm{Re}\,\tau+i\mathrm{Im}\,\tau$ is the modulus of the complex torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. This expression can be rewritten to make manifest "phases" and "modulus" as:

$$Z(\tau,\bar{\tau}) = \mathrm{Tr}\,e^{+2i\pi\mathrm{Re}\,\tau(L_0-\bar{L}_0)}e^{-2\pi\mathrm{Im}\,\tau(L_0+\bar{L}_0-2)}\,. \tag{2.1.10}$$

Thus the level-matching condition $(L_0-\bar{L}_0)=0$ is enforced by integration over the "phases" $\int \mathrm{dRe}\,\tau$ while the "moduli" cause a mass weighting. More precisely, the masses of the oscillator states are given by $m^2=\frac{4}{\alpha'}\left(\frac{N+\bar{N}}{2}-1\right)$ where N and $\bar{N}$ are the number operators for the left-moving and right-moving sector defined by $L_0=N+\alpha'p^2/4-1$ and $\bar{L}_0=\bar{N}+\alpha'p^2/4-1$. The lowest mass state has $N=\bar{N}=0$; it is the tachyon, whose mass is $m^2=-4/\alpha'$. Then come the massless states at $N=\bar{N}=1$ which constitute the gravity sector. For $N=\bar{N}\geq 2$ the states are massive with masses $m^2=4(N-1)/\alpha'$.

Thus in the region of modular parameter τ where $\mathrm{Im}\,\tau\sim 1/\alpha'$, the torus looks like a long and thin wire and one has $\mathrm{Im}\,\tau(N+\bar{N}-2)\sim m^2$. As $\alpha'\to 0$, the massive states with $N\geq 2$ give rise to exponentially suppressed contributions in the partition function; only the massless modes propagate.[11] Since all states are now massless, the level matching

[11]The tachyon state $N=\bar{N}=0$ creates an infrared divergence, that can simply be ignored here.

condition is trivial and may be integrated out; we classically recover a worldline loop of length

$$T = \alpha' \mathrm{Im}\, \tau \,, \tag{2.1.11}$$

as we explain in detail in sec. 2.2.3. In the range of complex moduli τ where $\mathrm{Im}\,\tau$ stays of order $O(1)$, the massive modes are not decoupled and dictate the UV behavior of the low energy theory. We will see later that these tori, that are well known to generate the insertion of higher order operators and counter-terms in the supergravity effective action, give rise to natural objects of tropical geometry. Although there is no trivial integration of the phase dependence in this case, one can think of these phases as phases of complex numbers of vanishingly small modulus which are integrated out as well. To summarize, the tropical nature of the $\alpha' \to 0$ of string theory is apparent if we decompose it in two steps:

Step 1 (Point-like limit) Send $\alpha' \to 0$ and distinguish between the contribution of massive states and massless states in the amplitudes,

Step 2 (Level matching) Integrate out the phases of the complex numbers that are not vanishingly small to get the contributions of the massless states, and integrate out the regions of phase space where the complex numbers are vanishingly small to get the contributions of massive states.

The technical implementation to higher genus of these well known ideas led the author in [PT3] to study the general formula

$$\lim_{\alpha' \to 0} \mathsf{A}^{(g,n)}_{\alpha'} = \int_{\mathcal{M}^{\mathrm{trop}}_{g,n}} \mathrm{d}\mu^{\mathrm{trop}} F_{g,n} \,, \tag{2.1.12}$$

which means that string theory amplitudes, written as integrals over the bosonic moduli space, are projected onto integrals over the tropical moduli space $\mathcal{M}^{\mathrm{trop}}_{g,n}$ in the $\alpha' \to 0$ limit. This was called the "tropical representation" of the field theory limit of the string theory amplitude. Later we describe in detail the tropical form of the integrand $F_{g,n}$ and the structure of tropical moduli space $\mathcal{M}^{\mathrm{trop}}_{g,n}$. Physically one can think of this space as the set of all Feynman diagrams at a particular loop-order, parametrized in terms of Schwinger proper times. Hence, the formula in eq. (2.1.12) is a compact and well-defined way to write the result of the field theory limiting procedure in string theory and quantify how strings worldsheets are degenerated to different kind of worldlines. Moreover, the amplitude is renormalized according to a particular renormalization scheme that we describe later.

Such a formula, very natural for the physicist, would by itself not be of great interest besides its curious link with a new branch of mathematics, if it did not enable us to extract new physics and do new computations. In [PT3], we managed to derive the form of the low-energy limit of the genus-two four-graviton amplitude in type II superstring written in [96]. We shall recall the essential step of the reasoning here and show at little cost that the form of the genus-three amplitude written in [72] is compatible with the explicit set of graphs found in [34, 35]. Before getting there, we would like to discuss some aspects of tropical geometry, which we will need to describe the field theory limits of the genus two and three amplitudes.

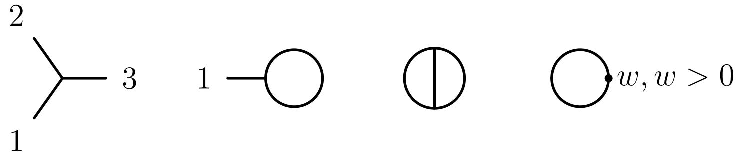


Figure 2.1: Examples of tropical graphs. From left to right: a 3-point tropical tree, a once-punctured graph of genus one, a genus-2 tropical graph, a graph of genus $1 + w$.

2.2 A few words about tropical geometry and its link with classical geometry

In this section, we introduce basic notions of tropical geometry of graphs, then recall the analogous classical notions for Riemann surfaces and finally come to the correspondence between classical and tropical geometry in the context of the $\alpha' \to 0$ limit of string theory amplitudes.

2.2.1 Tropical geometry

Tropical graphs From the viewpoint of particle physics, tropical graphs are Schwinger proper time parametrized graphs, where loops are allowed to degenerate to vertices with a weight indicating the number of degenerated loops. On these can be inserted operators or counterterms of the effective action of corresponding loop order, which regulate the high energy behavior of the theory. This is physically sensible since short proper times correspond to high energies.

A more formal definition is as follows. An (abstract) tropical graph is a connected graph with labeled legs, whose inner edges have a length and whose vertices are weighted. The external legs are called punctures. A *pure* tropical graph is a tropical graph that has only vertices of weight zero. Pure tropical graphs were first introduced in [97, 98], then later extended by [99, 100] to tropical graphs with weights, simply called tropical graphs here. Therefore a tropical graph Γ is a triple $\Gamma = (G, w, \ell)$ where G is a connected graph called the combinatorial type of Γ, ℓ and w are length and weight functions:

$$\begin{aligned} &\ell : E(G) \cup L(G) \to \mathbb{R}_+ \cup \{\infty\} \\ &w : V(G) \to \mathbb{Z}_+ \end{aligned} \tag{2.2.1}$$

In these two definitions, $E(G)$, $L(G)$ and $V(G)$ are respectively the sets of inner edges, legs and vertices of the graph. The total weight $|w|$ of a tropical graph is the sum of all the weights of its vertices $|w| = \sum_{V(G)} w(V)$. The genus $g(\Gamma)$ of a tropical graph $\Gamma = (G, w, \ell)$, is the number of loops $g(G)$ of G plus its total weight

$$g(\Gamma) = g(G) + |w| \,. \tag{2.2.2}$$

Hence the genus of a pure tropical graph is the number of loops of G in the usual sense. Moreover, every vertex of weight zero should have valence at least three (vertices with weight $w \geq 1$ may be of arbitrary non-zero valency). This automatically enforces a global stability condition for a given tropical graph of genus g and n punctures

$$2g - 2 + n \geq 1 \,, \tag{2.2.3}$$

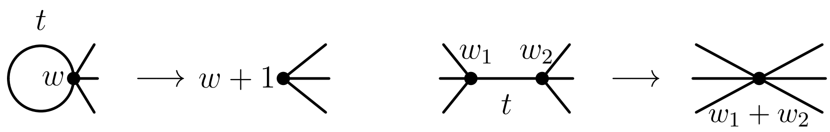


Figure 2.2: The genus of a graph is stable under degenerations $t \to 0$.

which is the exact analogues of the classical stability condition.[12] Vertices weights obey natural rules under degenerations as shown in the figure 2.2. Now it should be clear that the vertices weights *keep track* of degenerated loops. It is easily checked that the genus of a graph (2.2.2) and the stability criterion (2.2.3) are stable under specialization.

Tropical Jacobians In this paragraph, following closely [97], we introduce tropical analogues of the classical objects, such as abelian one-forms, period matrices and Jacobians. A slight subtlety absent in the classical case comes the fact that tropical graphs of identical genus may not have the same number of inner edges. For simplicity, here, we shall only deal with pure tropical graphs, while we mention in [PT3] how this is generalized following [99].

Tropical graphs support an homology basis and corresponding one-forms. Let Γ be a pure tropical graph of genus g and $(B_1, , \dots, B_g)$ be a canonical homology basis of Γ as in figure 2.3. The linear vector space of the g independent abelian one-forms ω_I^{trop} can be canonically defined by

$$\omega_I^{\text{trop}} = \begin{cases} 1 & \text{on } B_I\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.2.4}$$

These forms are *constant* on the edges of the graph. The period matrix K_{IJ} is defined as in the classical case by integration along B cycles,

$$\oint_{B_I} \omega_J^{\text{trop}} = K_{IJ}\,. \tag{2.2.5}$$

It is a $g \times g$ positive semi-definite real valued matrix. These abelian one-forms and period matrix were already used in [96, 101] where they were observed to be the exact analogs of the classical quantities. The Jacobian variety of Γ is a real torus defined by

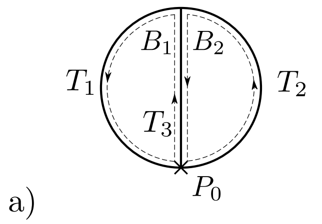
$$J(\Gamma) = \mathbb{R}^g / K\mathbb{Z}^g\,, \tag{2.2.6}$$

where $K\mathbb{Z}^g$ is the g-dimensional lattice defined by the g columns of the period matrix K.

The tropical version of the Abel-Jacobi map μ^{trop} of [97, 102] is defined by integration along a path γ between base point P_0 and end point P_1 of the vector of the abelian one-forms:

$$\mu_\gamma^{\text{trop}}(P_0, P_1) = \int_{P_0}^{P_1} (\omega_1^{\text{trop}}, \dots, \omega_g^{\text{trop}}) \mod K\mathbb{Z}^g\,. \tag{2.2.7}$$

[12] Strictly speaking, the local valency condition should be viewed as considering *classes* of abstract tropical graphs under the equivalence relation that contracts edges connected to 1-valent vertices of weight 0, and removes weight 0 bivalent vertices. Physically, on the worldline, this equivalence relation is perfectly sensible, since no interpretation of these 1- or 2- valent vertices of weight zero seem obvious in the absence of external classical sources.



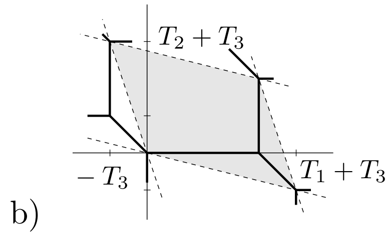


Figure 2.3: a) A $g = 2$ graph Γ with edges lengths T_1, T_2, T_3. b) The image of Γ (thick line) by the tropical Abel-Jacobi map in the Jacobian variety $J(\Gamma) = \mathbb{R}^2/K^{(2)}\mathbb{Z}^2$ (shaded area). Dashes indicate the $K^{(2)}\mathbb{Z}$ lattice.

Since changing γ by elements of the homology basis only results in the addition to the right hand side of elements of the lattice $K\mathbb{Z}^g$, μ^{trop} is well defined as a map in the Jacobian variety $J(\Gamma)$. Before we introduce the tropical moduli space, let us discuss two examples, taken from [97], in order to illustrate these notions.

Example 1. Let Γ be the genus two tropical graph of fig. 2.3 a) with canonical homology basis (B_1, B_2) as depicted. Using the definition (2.2.5), its period matrix is written:

$$K^{(2)} = \begin{pmatrix} T_1 + T_3 & -T_3 \\ -T_3 & T_2 + T_3 \end{pmatrix} . \tag{2.2.8}$$

Choosing P_0 as depicted, one can draw the image of Γ by the tropical Abel-Jacobi map in $J(\Gamma)$, as shown in the figure 2.3 b).

Example 2. The picture 2.4 below shows two inequivalent pure tropical graphs of genus two. The period matrix $K^{(2)}$ of the graph a) is given in (2.2.8), the period matrix of the graph b) is just $\text{Diag}(T_1, T_2)$. Thus, the Jacobian description is blind to such kind of "separating edges".

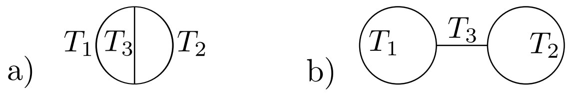


Figure 2.4: The period matrix is blind to the central length of the rightmost graph.

Tropical moduli space The moduli space $\mathcal{M}^{\text{trop}}(\Gamma)$ associated to a single tropical graph $\Gamma = (G, w, \ell)$ is the real cone spanned by the lengths of its inner edges modulo the discrete automorphism group of the graph [99]

$$\mathcal{M}^{\text{trop}}(\Gamma) = \mathbb{R}_+^{|E(G)|}/\text{Aut}(G) . \tag{2.2.9}$$

The moduli space of all genus-g, n-punctured tropical graphs is the space obtained from gluing all these cones together. This space is precisely the tropical moduli space introduced in [99, 100] denoted $\mathcal{M}_{g,n}^{\text{trop}}$ which enters the formula (2.1.12).

Below we describe a few examples of tropical moduli spaces. The moduli space of genus-0 tropical curves, $\mathcal{M}_{0,n}^{\text{trop}}$ is a well defined space that has the peculiar property of being itself a tropical variety of dimension $n-3$ [98, 103]. Because of the stability

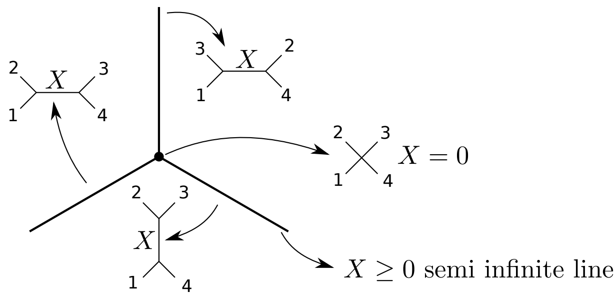


Figure 2.5: Tropical moduli space $\mathcal{M}_{0,4}^{\rm trop}$ (thick line). Each semi infinite line corresponds to one of three inequivalent graphs. The X coordinate on these gives the length of the inner edge of the graphs. The central point with $X = 0$ is common to the three branches.

condition (2.2.3) one should start with $n = 3$. The space $\mathcal{M}_{0,3}^{\rm trop}$ contains only one graph with no modulus (no inner length): the 3-pointed tropical curve. Hence $\mathcal{M}_{0,3}^{\rm trop}$ is just a one-point set. The space $\mathcal{M}_{0,4}^{\rm trop}$ has more structure; it has the topology of the three-punctured tropical curve and contains combinatorially distinct graphs which have at most one inner length, as shown in figure 2.5.

The space $\mathcal{M}_{0,5}^{\rm trop}$ is a two dimensional complex with an even richer structure. It is represented in figure 2.6. At genus one, $\mathcal{M}_{1,1}^{\rm trop}$ is still easily described. A genus one tropical graph with one leg is either a loop or a vertex of weight one. Hence, $\mathcal{M}_{1,1}^{\rm trop}$ is the half infinite line $\mathbb{R}_+$.

In general, Euler's relation gives that a given graph has at most $3g-3+n$ inner edges (and has exactly that number if and only if the graph is pure and possess only trivalent vertices). This implies that $\mathcal{M}_{g,n}^{\rm trop}$ is of "pure (real) dimension" $3g-3+n$, which means that some of its subsets are of lower dimension

$$\dim_{\mathbb{R}} \ \mathcal{M}_{g,n}^{\rm trop} \ \text{“} \leq \text{”} \ 3g-3+n\,. \tag{2.2.10}$$

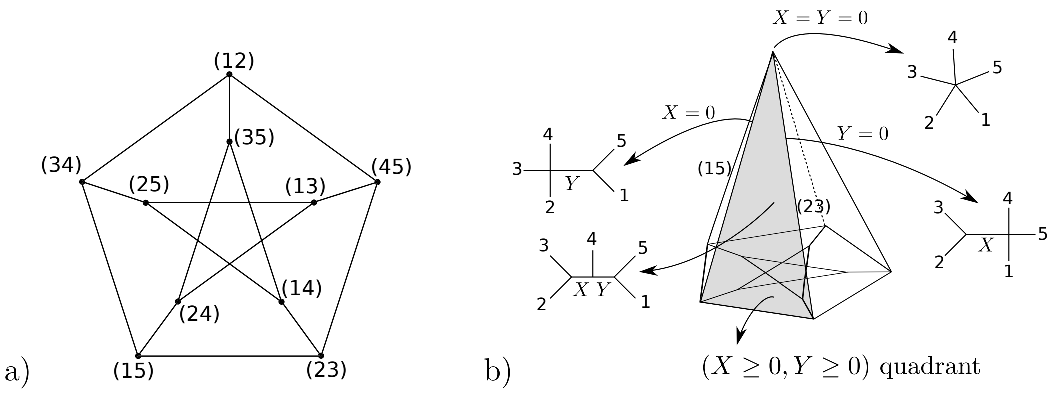


Figure 2.6: a) A slice of $\mathcal{M}_{0,5}$. The vertices (black dots) carry a two digits index, which corresponds to rays of $\mathcal{M}_{0,5}$, while edges corresponds to the 15 quadrants (one for each tree with 5 external legs and trivalent vertices). b) $\mathcal{M}_{0,5}$, with a specific quadrant in grey.

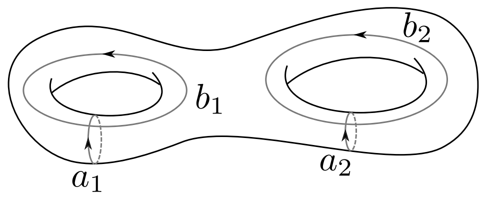


Figure 2.7: Canonical homology basis at $g = 2$.

2.2.2 Classical geometry of Riemann surfaces

We recall now classical facts about homology and Jacobian varieties of smooth Riemann surfaces. For a more elaborate introduction, we refer to the standard textbooks [104, 105]. Let Σ be a generic Riemann surface of genus g and let (a_I, b_J) $I, J = 1, \dots, g$ be a canonical homology basis on Σ with intersection $a_I \cap b_J = \delta_{IJ}$ and $a_I \cap a_J = b_I \cap b_J = 0$ as in figure 2.7. The abelian differential ω_I, $I = 1, \dots, g$ form a basis of holomorphic one-forms. They can be normalized along a-cycles so that their integral along the b-cycles defines the period matrix Ω_{IJ} of Σ:

$$2i\pi \oint_{a_I} \omega_J = \delta_{IJ} \,, \qquad 2i\pi \oint_{b_I} \omega_J = \Omega_{IJ} \,. \tag{2.2.11}$$

Note also Riemann's bilinear relations

$$\int_\Sigma \omega_I \wedge \bar{\omega}_J = -2i \, \mathrm{Im} \, \Omega_{IJ} \,. \tag{2.2.12}$$

The modular group $Sp(2g, \mathbb{Z})$ at genus g is spanned by the $2g \times 2g$ matrices of the form $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where A, B, C and D are $g \times g$ matrices with integer coefficients satisfying $AB^t = BA^t$, $CD^t = DC^t$ and $AD^t - BC^t = 1_g$. The $g \times g$ matrix 1_g is the identity matrix. For $g = 1$, the modular group reduces to $\mathrm{SL}(2, \mathbb{Z})$. The Siegel upper half-plane $\mathcal{H}_g$ is the set of symmetric $g \times g$ complex matrices with positive definite imaginary part

$$\mathcal{H}_g = \{\Omega \in \mathrm{Mat}(g \times g, \mathbb{C}) : \Omega^t = \Omega, \mathrm{Im}\,(\Omega) > 0\} \,. \tag{2.2.13}$$

The modular group $\mathrm{Sp}(2g, \mathbb{Z})$ acts on the Siegel upper half-plane by

$$\Omega \mapsto (A\Omega + B)(C\Omega + D)^{-1} \tag{2.2.14}$$

Period matrices of Riemann surfaces are elements of the Siegel upper half-plane and the action of the modular group on these is produced by Dehn twists of the surface along homology cycles. The Jacobian variety $J(\Sigma)$ of Σ with period matrix Ω is the complex torus

$$J(\Sigma) = \mathbb{C}^g / (\mathbb{Z}^g + \Omega \mathbb{Z}^g) \,. \tag{2.2.15}$$

The classical Abel-Jacobi map μ is defined by integration along a path C between two points (divisors) p_1 and p_2 on the surface of the holomorphic one-forms

$$\mu(p_1, p_2)_C = \int_{p_1}^{p_2} (\omega_1, \dots, \omega_g) \mod \mathbb{Z}^g + \Omega \mathbb{Z}^g \,. \tag{2.2.16}$$

As in the tropical case, the right hand side of (2.2.16) does not depend on the integration path as it is considered only modulo the Jacobian lattice. Note that apart for the very special case of genus one where $\mu(\Sigma_1) \cong \Sigma_1$, the image of a genus $g \geq 2$ Riemann surface Σ_g by μ is strictly included in $J(\Sigma_g)$, $\mu(\Sigma_g) \subsetneq J(\Sigma_g)$.

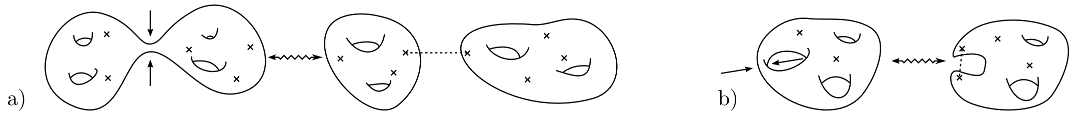


Figure 2.8: a) A separating degeneration. b) A non-separating degeneration. Dashes represent double points.

Classical Moduli $\mathcal{M}_{g,n}$ space and its Deligne-Mumford compactification $\overline{\mathcal{M}_{g,n}}$. Smooth Riemann surfaces of genus g with n punctures can be arranged in a moduli space denoted $\mathcal{M}_{g,n}$ of complex dimension is $3g-3+n$. The $3g-3+n$ complex parameters that span this space are called the moduli of the surface. This space is not compact, as surfaces can develop nodes when non-trivial homotopy cycles pinch and give rise to nodal surfaces with ordinary double points. The result of adding all such nodal curves to $\mathcal{M}_{g,n}$ is the well known Deligne-Mumford compactified moduli space of curves $\overline{\mathcal{M}_{g,n}}$ [106]. There exists two types of such degenerations. As depicted in fig. 2.8, a "separating" degeneration splits off the surface into a surface with two disconnected components that are linked by a double point, while a "non-separating" degeneration simply gives rise to a new surface with two points identified whose genus is reduced by one unit. Note that no degeneration may split off a surface that does not satisfy the stability criterion shared with tropical graphs, eq. (2.2.3). As a consequence, a maximally degenerated surface is composed of thrice punctured spheres.

These degenerations induce a stratification on $\overline{\mathcal{M}_{g,n}}$, characterized by the combinatorial structure of the nodal curves, represented by its "dual graph". It is obtained by making a line go through each pinched cycle and turning each non-degenerated component of genus $g \geq 0$ into a vertex of weight g. Finally, the legs of a dual graph are just what used to be punctures on the surface. Examples are provided in fig.2.9. The strata corresponding to maximally degenerated curves are the deepest ones. The stratum corresponding to the non-pinched curves, whose dual graphs are a vertex of weight g, is the most superficial one (it is the interior of $\overline{\mathcal{M}_{g,n}}$). We come back to this in section 2.2.3.

A surface where a node is developing locally looks like a neck whose coordinates x and y on each of its side obey the equation $xy = t$, where the complex number t of modulus $|t| < 1$ is a parameter measuring the deformation of the surface around the singularity in $\overline{\mathcal{M}_{g,n}}$. The surface is completely pinched when $t = 0$. After a conformal transformation, one sees that this surface is alternatively described by a long tube of length $-\ln|t|$ and the tropicalization procedure classically turn these tubes into edges. The exact relation in string theory involves a factor of α' such that for instance the length T of the worldloop coming from a torus is

$$T = -\alpha' \ln|q| \,, \tag{2.2.17}$$

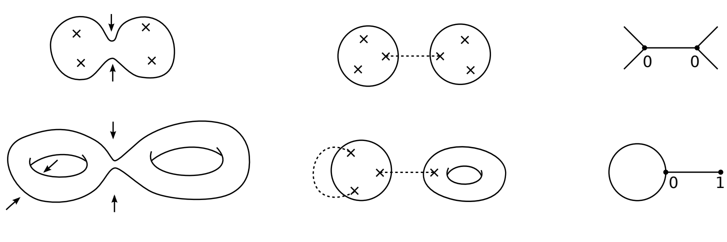


Figure 2.9: Degenerating surfaces, nodal curves and their dual graphs.

which coincides with (2.1.11).

2.2.3 From classical to tropical geometry

Moduli Spaces In [PT3] we outlined a construction of the tropicalization of $\mathcal{M}_{g,n}$ into $\mathcal{M}_{g,n}^{\text{trop}}$, which we later applied to string theory. Here we give a shortened version of this discussion based on more physical grounds. The starting point is the following question: "How can one commute the $\alpha' \to 0$ limit and the integration symbol in (2.1.5)"? Schematically, we wonder how to give sense to

$$\lim_{\alpha'\to 0}\left(\int_{\mathcal{M}_{g,n}} \mathcal{W}_{g,n}\, \exp(\mathcal{Q}_{g,n})\, \mathrm{d}\mu_{\text{bos}}\right) \stackrel{?}{=} \int_{\mathcal{M}_{g,n}} \lim_{\alpha'\to 0}\Big(\mathcal{W}_{g,n}\, \exp(\mathcal{Q}_{g,n})\, \mathrm{d}\mu_{\text{bos}}\Big)\,, \tag{2.2.18}$$

Such a procedure should treat well the integration domain, i.e. should not forget regions nor double counts others. If the integration domain were compact and the integrand a well behaved funcion, standard integration theorems would allow to simply commute the symbols. Here, we cannot replace $\mathcal{M}_{g,n}$ by its compactification $\overline{\mathcal{M}_{g,n}}$ precisely because the integrand has singularities at the boundary, which correspond to the IR singularities of string theory massless thresholds.

Hence, to deal with this integral we will follow the method pioneered at one-loop by Green and Schwarz in their work [59] where they showed that maximal supergravity and maximal SYM were the massless limits of type II and I strings, respectively. At generic loop order, their approach can be formulated by splitting the integral in the left-hand side of (2.2.18) into a sum of integrals over different regions where the limit can be safely taken. These regions are open sets of $\mathcal{M}_{g,n}$, such that

$$\mathcal{M}_{g,n} = \bigsqcup_{G} \mathcal{D}_G\,, \tag{2.2.19}$$

where each $\mathcal{D}_G$ contains the nodal curve with combinatorial type G. The point is that these dual graphs correspond to particular Feynman graphs, and the limit is obtained for each integral as

$$\int_{\mathcal{D}_G} \mathrm{d}\mu_{\text{bos}}\, \mathcal{W}_{g,n}\, \exp(\mathcal{Q}_{g,n}) = \int_{\mathcal{M}^{\text{trop}}(\Gamma)} \mathrm{d}\mu_{\text{trop}}\, W_{g,n}\, \exp(Q_{g,n}) + O(\alpha')\,. \tag{2.2.20}$$

In [PT3] we described in great detail the limiting integration measure and integrand and show that they coincide with the contribution of the Feynman graph corresponding to Γ.

Example at genus one Before we describe these technical points, let us come back to genus one and discuss what could be a decomposition of $\mathcal{M}_{1,1}$ like the one in (2.2.19). Genus one Riemann surfaces are complex tori[13] $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ parametrized by a complex parameter, τ with positive imaginary part, $\tau \in \mathcal{H}_1 = \{\tau \in \mathbb{C}, \operatorname{Im}(\tau) > 0\}$. Modding out by the action of the modular group $SL(2,\mathbb{Z})$ further restricts τ which eventually lies

[13] The complex tori is actually the Jacobian variety of the Riemann surface, but at genus one both are isomorphic. This property does not hold for higher genus curves.

in an $SL(2,\mathbb{Z})$ fundamental domain. A representative one that we will use is $\mathcal{F} = \{\tau \in \mathbb{C},\, |\tau| > 1,\, -1/2 \leq \operatorname{Re}\tau < 1/2,\, \operatorname{Im}\tau > 0\}$, depicted in the figure 2.10. Therefore

$$\mathcal{M}_{1,1} \cong \mathcal{F} \tag{2.2.21}$$

There is only one singularity in $\mathcal{M}_{1,1}$, the pinched torus, at $q = 0$. Topologically, it is a sphere with three punctures, two of which are connected by a double point. The dual graph G_1 of this surface is a single loop with one external leg, and the corresponding domain should be defined such that it is possible to integrate out the real part of τ (phase of q) independently of the value of $\operatorname{Im}\tau$. Therefore we see that if we define $\mathcal{D}_{G_1}$ in terms of an arbitrary number $L > 1$ such that $\mathcal{D}_{G_1} = \{\tau \in \mathcal{F}, \operatorname{Im}\tau > L\}$, we can define families of tori with $\operatorname{Im}\tau = T/\alpha'$ tropicalizing to a worldloop of length T, independently of $\operatorname{Re}\tau$. In this way, $\mathcal{F}$ is split in two parts, $\mathcal{F}^+(L) \equiv \mathcal{D}_{G_1}$ and a lower part $\mathcal{F}^-(L)$ defined by $\mathcal{F}^-(L) = \{\tau \in \mathcal{F},\, \operatorname{Im}\tau \leq L\}$:

$$\mathcal{F} = \mathcal{F}^+(L) \sqcup \mathcal{F}^-(L)\,. \tag{2.2.22}$$

The dual graph corresponding to $\mathcal{F}^-(L)$ is a weight one vertex with a single leg. To the knowledge of the author, this precise decomposition was first used by Green and Vanhove in [107], where $\mathcal{F}^+(L)$ and $\mathcal{F}^-(L)$ were respectively called $\mathcal{F}_L$ and $\mathcal{R}_L$. With this splitting, let us for definiteness introduce the following quantities

$$\mathsf{A}^{(1,4)}_{\alpha',+}(L) = \int_{\mathcal{F}^+(L)} \mathrm{d}\mu_{\text{bos}}\; \mathcal{W}_{g,n}\; \exp(\mathcal{Q}_{g,n}) \tag{2.2.23a}$$

$$\mathsf{A}^{(1,4)}_{\alpha',-}(L) = \int_{\mathcal{F}^-(L)} \mathrm{d}\mu_{\text{bos}}\; \mathcal{W}_{g,n}\; \exp(\mathcal{Q}_{g,n}) \tag{2.2.23b}$$

According to the previous property in (2.2.20), the $\alpha' \to 0$ limit of $\mathsf{A}^{(1,4)}_{\alpha',+}(L)$ gives the contribution of graphs with worldline loops of finite size $T = \operatorname{Im}\tau/\alpha'$. Therefore, the condition $\operatorname{Im}\tau > L$ gives the following field theory cut-off

$$T \geq T_{UV} = \alpha' L \tag{2.2.24}$$

In [107], the authors explained that since the total amplitude $\mathsf{A}^{(1,4)}_{\alpha',+}(L) + \mathsf{A}^{(1,4)}_{\alpha',-}(L)$ does not depend on L, any divergent term in $\mathsf{A}^{(1,4)}_{\alpha',+}(L)$ has to be canceled by a term coming from the field theory limit of $\mathsf{A}^{(1,4)}_{\alpha',-}(L)$. This supports the fact that these integrals produce

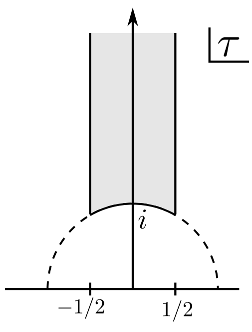


Figure 2.10: $SL(2,\mathbb{Z})$ fundamental domain of the torus.

counterterms in the effective action. In [PT3, section VI.B], we describe this cancellation in the trivial case of the quadratic R^4 one-loop 10-dimensional divergence of the four-graviton amplitudes in type II supergravity, while in this manuscript in appendix 3.A we present a one-loop computation for the R^4 logarithmic divergence in 8 dimensions for four-graviton amplitudes in heterotic string models. Several other examples are discussed in the original work [107]. As a consequence, the field theory limit of the sum of the two contributions is written as an integral over $\mathcal{M}_{1,1}^{\text{trop}}$ with a contact term inserted at $T=0$ and gives, as claimed below eq. (2.1.12), the *renormalized* field theory amplitude. Let us now come back to the integrand in the right-hand side of eq. (2.2.20).

Back to the tropical form of the integrand. The bosonic measure $\mathrm{d}\mu_{\text{bos}}$ is a $(3g-3+n)$-dimensional measure that can be traded for an integration over the period matrices for genus $1,2,3,4$.[14] In this way, the tropical limit of the measure is given by

$$\mathrm{d}\mu_{bos} = \frac{|\prod_{1\leq I<J\leq g}\mathrm{d}\Omega_{IJ}|^2}{|\det \operatorname{Im}\Omega|^5}\prod_{i=1}^{n}\mathrm{d}^2 z_i \rightarrow \mathrm{d}\mu_{\text{trop}} = \frac{\prod_{i\in\{edges\}}\mathrm{d}\ell(i)}{|\det K|^5} \tag{2.2.25}$$

where $\ell(i)$ is the length of the edge i. The interest of writing the measure explicitly in terms of period matrices is the appearance of the $\det\Omega^5$ factor, giving rise to $\det K$, an important element of Feynman graph as we explain below.

We also explained that the Koba-Nielsen factor descends to a tropical Koba-Nielsen factor, modulo a conjecture on a would-be "tropical" prime form which we mention below. Under this hypothesis, the bosonic propagator $\mathcal{G}$ descends to the worldline Green's function G^{trop} computed by Dai and Siegel in [101], and we have

$$\mathcal{Q}_{g,n} \underset{\alpha'\to 0}{\longrightarrow} Q_{g,n}^{\text{trop}} = -\sum_{i,j} k_i.k_j G^{\text{trop}}(Z_i,Z_j) + O(\alpha') \tag{2.2.26}$$

where details and explicit expressions can be found in [PT3, sec. V.A, eq(V.11)] and below in eq. (2.2.43).

To convince the reader that we are really describing Feynman graphs in this procedure, it is worth recalling the classical exponentiation procedure that leads to Schwinger proper time parametrized Feynman graphs. Starting from an arbitrary g-loop D-dimensional Feynman integral, we exponentiate the Feynman propagators D_i^2 and obtain

$$\begin{aligned}\int (\mathrm{d}^D p)^g \frac{n(p)}{\prod_i D_i^2} &= \int_0^\infty \prod_i \mathrm{d}a_i \int (\mathrm{d}^D p)^g\, n(p)\exp(-\sum a_i D_i^2)\\ &= \int_0^\infty \frac{\prod_i \mathrm{d}a_i}{(\det\tilde{K})^{D/2}}\left\langle n(a_i,\tilde{K})\right\rangle \exp(-Q_{g,n}^{\text{trop}})\end{aligned} \tag{2.2.27}$$

[14] We recall that the description of moduli of Riemann surfaces in terms of these of Jacobian varieties is called the classical Schottky problem (for a recent survey see [108]). Algebraically, the Jacobian varieties are characterized by $g(g+1)/2$ complex numbers that span period matrices, while Riemann surfaces have only $3g-3$ moduli. These numbers coincide for $g=1,2,3$ (for $g=1$ one should have $n=1$ because of conformal invariance on the torus), for $g=4$, $\mathcal{M}_4$ is a hypersurface in the moduli space of Jacobian variety of co-dimension one, but it is known that the zero locus of the so called "Schottky-Igusa" form furnishes a defining equation of this hypersurface. For $g\geq 5$ the problem is totally open. We describe later the Schottky-Igusa form, in chapter 3, sec. 3.2

where we used that $\sum a_i D_i^2 = 1/2\, {}^t p \cdot \tilde{K} \cdot p + {}^t A \cdot p + c(a_i)$ is a quadratic form which produces a determinant after completing the square to

$$\sum a_i D_i^2 = 1/2\, {}^t(p + \tilde{K}^{-1}A) \cdot \tilde{K} \cdot (p + \tilde{K}^{-1}A) - 1/2\, {}^t A \tilde{K}^{-1} A + c(a_i) \tag{2.2.28}$$

and D-dimensional Gaussian integration over a suitable Wick rotation of the shifted momentum $\tilde{p} = (p + \tilde{K}^{-1}A)$. The fact that the loop momentum constant part $-1/2\, {}^t A \tilde{K}^{-1} A + c(a_i)$ in the exponential equals $Q_{g,n}^{\mathrm{trop}}$ defined as in (2.2.26) is indirectly proven in [101]. The last line is the desired Schwinger proper time form of the Feynman graph, where the a_i correspond to the inner edges of the graph $\ell(i)$ in (2.2.25). In this way, it is then easy to show that $\tilde{K}$ is the period matrix K of the tropical graph as defined in (2.2.5). Finally, the bracket notation in (2.2.27) refers to the fact that $\langle n(a_i, K)\rangle$ is a Gaussian average of $n(p)$. The K-dependence comes from Gaussian integrating the terms with non-trivial loop momentum dependence in $n(p)$. The correspondence between the tropical form (2.2.20) with the measure (2.2.25) and this form is now, hopefully, clearer.[15] We use this procedure in the last chapter of this manuscript at one loop.

At this point, we have an almost complete description of the $\alpha' \to 0$ limit of string theory amplitudes between the tropical form of the integrand in (2.2.20) and the Feynman graph. The only missing point is also the most interesting one; the numerator of the Feynman graph $\langle n\rangle$, corresponding to $W_{g,n}$. Below we introduce a few technical elements necessary to tackle the tropical limit of the numerator $\mathcal{W}_{g,n}$ for $g \geq 2$.

Cohomology Thanks to the splitting of $\mathcal{M}_{g,n}$ (2.2.19), it is possible in each domain to safely define families of degenerating worldsheets, and show that their period matrices and one forms descend to their tropical analogues, as described in [PT3, sec. IV.C]. The one-forms, at a neck i_0 parametrized by a local coordinate t_0 around $t_0 = 0$, locally behave as on a very long tube:

$$\omega_I = \frac{c}{2i\pi} \frac{\mathrm{d}z}{z} + O(t_0)\,, \tag{2.2.29}$$

with $c = 1$ or 0 depending on if i belongs to the cycle b_i or not. As a consequence, the bilinear relation descends to

$$\int_\Sigma \omega_I \wedge \bar{\omega}_J = \boxed{-2i \operatorname{Im} \Omega_{IJ} \underset{\alpha' \to 0}{\sim} -2i\, \frac{K_{IJ}}{\alpha'} + O(1)}\,. \tag{2.2.30}$$

which indicates the fundamental scaling relation of the tropical limit. At one-loop, this is the relation (2.1.11), but at higher loop this gives non-trivial information on the behavior of the period matrices of the degenerating worldsheets.

Fourier-Jacobi expansions As a consequence, this provides a nice system of local coordinates in each patch $\mathcal{D}_G$ around the nodal curve G (at least for when G corresponds to deepest strata[16] of $\overline{\mathcal{M}_{g,n}}$), defined as

$$q_j = \exp(2i\pi\tau_j) \tag{2.2.31}$$

[15] The reader familiar with Symanzik polynomials may notice that the tropical Koba-Nielsen factor is the second Symanzik polynomial of the graph, while the $\det K$ of the proper time measure is the first Symanzik polynomial of the graph.

[16] We recall that we defined these strata before, as the ones corresponding to maximally degenerated curves made of tri-valent weight-0 vertices only.

for $j \in E(G)$ such that $\operatorname{Im}\tau_j = \ell(j)/\alpha'$. It is then possible to perform the so-called "Fourier-Jacobi" expansion of the various quantities defined on the worldsheet in terms of these q_j's. Generically a function F of the moduli of the worldsheet admits a Fourier-Jacobi expansion of the form (neglecting the the punctures for simplicity):

$$F = \sum_{n_i, m_j} F_{hol}^{(n_1,\dots,n_{3g-3})} q_1^{n_1} \dots q_{3g-3}^{n_{3g-3}} F_{anti-hol}^{(m_1,\dots,m_{3g-3})} (\bar{q}_1)^{m_1} \dots (\bar{q}_{3g-3})^{m_{3g-3}} \tag{2.2.32}$$

where at $g = 1$, it is understood that $3g-3$ should be replaced by 1. The general strategy to extract the tropical form of integrand $\mathcal{W}$ is to perform the Fourier-Jacobi expansion of the integrand (step 1 of sec. 2.1.2) then integrate the phase of the q_i's (step 2). The procedure tells us that it is safely possible to commute the integration and the Fourier Jacobi expansion. The outcome of this procedure is that higher order contributions in q_i vanish; only the constant terms stay and constitute the tropical form of the integrand.

At this point the reader might wonder why the phase integration is not simply redundant with the $q_j \to 0$ limit; since non-zero powers of q_j are projected out anyway, what is the point of the phase-integration of constant terms ? As a matter of fact, the integrands of string theory amplitudes do contain a partition function, whose Fourier-Jacobi expansion typically starts with *inverse* powers of q_j: q_j^{-1} for the bosonic sector of heterotic string and $q_j^{-1/2}$ for the NS sector of the superstring. Therefore, a term like $q_j(\bar{q}_j)^{-1}$ is not killed by the $q_j \to 0$ limit alone, while it is by the phase integration $\int \mathrm{d}(\operatorname{Re}\tau_j) q_j (\bar{q}_j)^{-1} = 0$; this is the level matching condition (step 2 of sec. 2.1.2). For maximal supergravity amplitudes, the tachyon is projected out of the spectrum by the GSO projection and the limit is easier to extract, but in general the inverse powers of the partition function do contribute via residue contributions of the form

$$\int \mathrm{dRe}\,\tau_1 \dots \mathrm{dRe}\,\tau_{3g-3} \frac{F}{q_1^{n_1} \dots q_{3g-3}^{n_{3g-3}} (\bar{q}_1)^{m_1} \dots (\bar{q}_{3g-3})^{m_{3g-3}}} = c_{n_1,\dots,m_{3g-3}} F^{\text{trop}}\,. \tag{2.2.33}$$

At one-loop, there is only one q and these techniques are perfectly well under control as we review in chap. 3. They led Bern and Kosower to develop the eponymous rules which allow to compute n-gluon amplitudes [61–64]. These were first derived from the low energy limit of heterotic string fermionic models, later understood from first principles in field theory [109] then extended to gravity amplitudes [64, 110]. See also the review [111] for an exhaustive account on this worldline formalism. At higher loop, such residue formulas are still not known, and are required to extract general amplitudes, as we discuss later in the chapter dedicated to half maximal supergravity amplitudes at two loops, 3.2. The basic building block of which these generic integrands are made of is the bosonic correlator $\mathcal{G}$ and its derivatives, to which we come now.

A tropical prime form ? In this paragraph, we describe the *first term* of the Fourier-Jacobi expansion of bosonic Green's function on Riemann surfaces. Its complete expression is [85, 94][17]

$$\mathcal{G}(z_1, z_2) = -\frac{\alpha'}{2} \ln\left(|E(z_1, z_2)|\right) - \frac{\alpha'}{2} \left(\int_{z_2}^{z_1} \omega_I \right) (\operatorname{Im}\Omega^{-1})^{IJ} \left(\int_{z_2}^{z_1} \omega_J \right) \tag{2.2.34}$$

[17] The normalization differs from the one used in [PT3] by the factor α' that we keep inside $\mathcal{G}$ here.

It is defined in terms of the prime form E, whose definition requires to introduce first the classical Riemann theta function:

$$\theta(\zeta|\Omega) = \sum_{n\in\mathbb{Z}^g} e^{i\pi n^t\Omega n}e^{2i\pi m^t\zeta} \tag{2.2.35}$$

where $\zeta \in J(\Sigma)$ and $\Omega \in \mathcal{H}_g$. Theta functions with characteristics are defined by

$$\theta\begin{bmatrix}\beta\\ \alpha\end{bmatrix}(\zeta|\Omega) = e^{i\pi\beta^t\Omega\beta+2i\pi\beta^t(\zeta+\alpha)}\theta(\zeta+\Omega\beta+\alpha|\Omega) \tag{2.2.36}$$

where α and β are g dimensional vectors of $\frac{1}{2}(\mathbb{Z}/2\mathbb{Z})^{2g}$ called the theta-characteristics. The prime form E is then defined by [85, 112, 113]

$$E : (x,y) \in \Sigma\times\Sigma \longrightarrow E(x,y|\Omega) = \frac{\theta\left[\begin{smallmatrix}\beta\\ \alpha\end{smallmatrix}\right](\mu(x,y)|\Omega)}{h_\kappa(x)h_\kappa(y)} \in \mathbb{C}\,, \tag{2.2.37}$$

with the requirement that $\kappa = \left[\begin{smallmatrix}\beta\\ \alpha\end{smallmatrix}\right] \in \frac{1}{2}(\mathbb{Z}/2\mathbb{Z})^{2g}$ should be a non-singular odd[18] theta-characteristics and h_κ the half-differentials defined on Σ by $h_\kappa(z) = \sqrt{\sum_{i=1}^g \omega_I(z)\partial_I\theta\left[\begin{smallmatrix}\beta\\ \alpha\end{smallmatrix}\right](0|\Omega)}$. Also, μ is the classical Abel-Jacobi map defined in (2.2.16). Defined in this way, the prime form is a differential form of weight $(-1/2,-1/2)$ which do not depend on the spin structure κ chosen. In a sense, it is the generalization of the map $x, y \in \mathbb{C}^2 \mapsto x-y$ to arbitrary Riemann surfaces. In particular, $E(x,y)$ vanishes only along the diagonal $x=y$ and locally behaves as

$$E(x,y) \underset{x\to y}{\sim} \frac{x-y}{\sqrt{dx}\sqrt{dy}}(1+O(x-y)^2) \tag{2.2.38}$$

It is multi-valued on $\Sigma\times\Sigma$ since it depends on the path of integration in the argument of the theta function. More precisely, it is invariant up to a sign if the path of integration is changed by a cycle a_I, but it picks up a multiplicative factor when changing the path of integration by a cycle b_J

$$E(x,y) \to \exp(-\Omega_{JJ}/2 - \int_x^y \omega_J)E(x,y)\,. \tag{2.2.39}$$

In $\mathcal{G}$, it is easily checked that the additional terms with holomorphic forms precisely cure this ambiguity.

In [PT3] was proposed a definition of a the tropical prime form as the result of the following limit:

$$E^{\text{trop}}(X,Y) := -\lim_{\alpha'\to 0}\left(\alpha' \ln\left|E(x_{\alpha'}, y_{\alpha'}|\Omega_{\alpha'})\right|\right) \tag{2.2.40}$$

where $\Omega_{\alpha'}$ are the period matrices of a family of curves tropicalizing as in (2.2.30), $x_{\alpha'}, y_{\alpha'}$ are two families of points on the surface whose image by the Abel-Jacobi map tropicalizes as in (2.2.29) and X and Y are the two limit points on the tropical graph. Inspired by

[18] Odd means that $2^{2n}\alpha\cdot\beta \equiv 1[2]$ and "non-singular" that $\theta[\kappa](\zeta|\Omega)$ vanishes exactly to first order at $\zeta = 0$. Even characteristics are these for which $2^{2n}\alpha\cdot\beta \equiv 0[2]$

[101], we made the conjecture that the tropical prime form defined in this way corresponds to the graph distance $d_\gamma(X,Y)$ between X and Y along a path γ:

$$E^{\mathrm{trop}}(X,Y) = d_\gamma(X,Y) \tag{2.2.41}$$

This object is also ill-defined on the graph since it depends on γ. To prove this conjecture, the first ingredient to use would be tropical theta functions with characteristics. Tropical theta functions without characteristics were introduced in [97] and it is easy to show directly that they arise in the limit of the classical ones;

$$\Theta^{\mathrm{trop}}(Z|K) = \lim_{\alpha'\to 0} -\alpha' \ln|\theta(\zeta_{\alpha'}|\Omega_{\alpha'})| \tag{2.2.42}$$

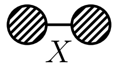
where $(\zeta_{\alpha'}) = \mu(x_{\alpha'}, y_{\alpha'})$ is sent to $Z = \mu^{\mathrm{trop}}(X,Y)$ as defined previously. So far, the author has not managed to prove this property in the case of tropical theta functions with characteristics, as defined in [97]; this is a crucial missing step. As this limit is not fully under control, it does not make sense to try to describe higher order corrections in the Fourier-Jacobi expansion of the prime form, which would enter residue formulas as (2.2.33).

The other class of terms in the right-hand side of (2.2.34) are easily dealt with by replacing the one-forms and period matrix by their tropical analogues, and, using (2.2.41) we obtain the $\alpha' \to 0$ limit of the bosonic correlator of (2.2.34) is the following quantity

$$G(Z_1,Z_2) = \lim_{\alpha'\to 0} \mathcal{G}(z_1,z_2) = -\frac{1}{2}E^{\mathrm{trop}}(Z_1,Z_2) - \frac{1}{2}\left(\int_{Z_2}^{Z_1}\omega^{\mathrm{trop}}\right) K^{-1} \left(\int_{Z_2}^{Z_1}\omega^{\mathrm{trop}}\right) \tag{2.2.43}$$

which is precisely the expression computed by Dai and Siegel in [101]. Note that it is now well defined on the graph.

A Remark On Contact Terms Before closing the section, let us clarify a point concerning contact-terms. In the usual perturbative expansion of quantum field theory, the Feynman rules include vertices of valency four in non-abelian gauge theories and arbitrarily high in gravity theories, to guarantee gauge invariance. What is referred to as "contact-term" in string theory is different. It is the vertex that results from integrating out the length of a separating edge in a one-particle-reducible graphs:

$\int \Big($  $\Big) \mathrm{d}X = c_0 \times$  (2.2.44)

In the tropicalization procedure, we do not perform these integrations. Therefore, higher valency vertices (of weight zero) are present in our considerations, but only as boundaries between domains in $\mathcal{M}_{g,n}^{\mathrm{trop}}$ of maximal codimension and should not carry any localized contribution in the integrands, unlike in Feynman rules where they carry a distinct structure compared to the lower valency vertices.

Furthermore, in string theory, these type of contributions only arise from configurations where two neighboring vertex operators V_i and V_j collide towards one another, $z_i - z_j \ll 1$. It can be shown in full generality that a contact term can arise only if the string integrand $\mathcal{W}_{g,n}$ contains a factor of $|\partial\mathcal{G}(z_i - z_j)|^2$. The basic argument is that in the region where the position of the punctures collide, the local behavior of E (2.2.38) grants that ∂G has a first order pole $\partial\mathcal{G} \sim 1/(z_i - z_j)$ and the change of variable[19] $z_i - z_j = e^{-X/\alpha'}e^{i\theta}$ can be used to take the limit of $\int \mathrm{d}^2 z_i |\partial\mathcal{G}(z_i - z_j)|^2 \exp\big(-k_i.k_j\alpha' \ln(z_i - z_j)\big)$ in the string amplitude to obtain the following integral over $\mathrm{d}X$, the length of the separating edge:

$$\int \mathrm{d}^2 z_i |\partial\mathcal{G}(z_i - z_j)|^2 e^{-2k_i.k_j\alpha' \ln|z_i - z_j|} = -\frac{2\pi}{\alpha'}\int \mathrm{d}X e^{-2Xk_i\cdot k_j} \tag{2.2.45}$$

after integrating the phase $\mathrm{d}\theta$. The crucial point here is that if $|\partial\mathcal{G}(z_i - z_j)|^2$ had not been in the integrand, either the local behavior would have failed or the phase integration would have killed the contribution.

The "Analytic" and the "Non-Analytic" domains.

For simplicity let us exclude the punctures of that discussion. The authors of [107] introduced the splitting (2.2.22) because it actually decomposes the string amplitude into its analytic and non-analytic parts, respectively obtained from the lower- and upper-domain integration. In [PT3] we proposed an extension of these "lower" and "upper" domains for higher genus. We defined the analytic and non-analytic domains in $\mathcal{M}_{g,n}$ by the requirement that the first should correspond to the more superficial stratum of $\overline{\mathcal{M}}_g$ and the second should correspond to the deepest strata of $\overline{\mathcal{M}}_g$ in the decomposition (2.2.19). These strata where defined in paragraph concerning the structure of $\overline{\mathcal{M}_{g,n}}$.

Therefore, the analytic domain is defined by removing all neighborhoods around the singularities of $\mathcal{M}_g$; it is a compact space. In this region, the string integrand has no singularity and the limit may be safely commuted with the integration, where the factor α' present in the definition of $\mathcal{Q}_{g,n}$ via $\mathcal{G}$ simply sends $\exp(\mathcal{Q}_{g,n})$ to 1. This reasoning justifies why in an important part of the literature, "taking the low energy limit" is often translated as getting rid of the Koba-Nielsen factor. This may be done only modulo these non-trivial geometric assumptions.

This also suggests that to compute the primary divergence of an amplitude, it is possible to compute the string integral over the analytic domain, as illustrated in the one-loop example of secs.2.2.3 and 3.A. Understanding the role of the precise form of the boundary of this domain is an open interesting question. Regarding the non-analytic domains, they provide the contribution of the pure tropical graphs, made of trivalent vertices only. Summed over, they give rise to the *unrenormalized* field theory amplitude, with all of its sub-divergences.

2.3 Extraction of supergravity amplitudes

The $\alpha' \to 0$ limit of tree-level amplitudes is sketched later in this text when we discuss a formula related to the BCJ duality at tree-level in sec. 4.2. One-loop amplitudes are also discussed in some detail in the following chapter 3. Here we discuss the somewhat non-trivial and interesting cases of $g = 2, 3$ four-graviton amplitudes in type II string theory.

[19] In [PT3] we discussed this tree-level behavior in detail

2.3.1 Two-loops field theory limit in maximal supergravity

The two-loop four-graviton type II amplitude in 10 dimensions has been computed explicitly in the RNS formalism by D'Hoker and Phong in [43–49] and later obtained in the pure spinor formalism in [114, 115]. The normalizations between the two results was carefully observed to match in [116]. We reproduce the RNS form here:

$$\mathsf{A}^{(2,4)}_{\alpha'} = \frac{t_8 t_8 R^4}{2^{12}\pi^4} \int_{\mathcal{F}_2} \frac{|\prod_{I\leq J} \mathrm{d}\Omega_{IJ}|^2}{(\det \operatorname{Im}\Omega)^5} \int_{\Sigma^4} |\mathcal{Y}_S|^2 \exp\big(\mathcal{Q}_{2,4}\big) \tag{2.3.1}$$

where $\int_{\Sigma^4}$ denotes integration over the surface Σ of the position of the four punctures and $t_8t_8R^4$ is the only supersymmetric invariant in maximal supergravity made of four powers of the Riemann tensor (see [77, Appendix 9.A]). The domain $\mathcal{F}_2$ is an $Sp(4,\mathbb{Z})$ fundamental domain, isomorphic to $\mathcal{M}_2$. The quantity $\mathcal{Y}_S$ arises from several contributions in the RNS computation and from fermionic zero-mode saturation in the pure spinor formalism. Its expression is given in terms of bilinears in the holomorphic one-forms $\Delta(z,w) = \omega_1(z)\omega_2(w) - \omega_1(w)\omega_2(z)$ as follows

$$3\mathcal{Y}_S = (k_1 - k_2)\cdot(k_3 - k_4)\,\Delta(z_1,z_2)\Delta(z_3,z_4) + (13)(24) + (14)(23)\,. \tag{2.3.2}$$

Thus, $|\mathcal{Y}_S|^2$ is a top-form on Σ^4. In [PT3], we checked the conjecture of [96] on the low energy limit of the string theory amplitude, starting from the field theory amplitude derived in [13] rewritten in a worldline language. This concerns only the non-analytic domain of the amplitude. The essence of the demonstration is to find the tropical form of $\mathcal{Y}_S$. As noted in the previous section, in amplitudes where maximal supersymmetry is not broken, the NS tachyons are projected out of the spectrum by the GSO projection, and there is no non-trivial residue to extract. The tropical form of $\mathcal{Y}_S$ is then immediately obtained:

$$3\mathcal{Y}_S \to 3Y_S = (k_1 - k_2)\cdot(k_3 - k_4)\,\Delta^{\mathrm{trop}}(12)\Delta^{\mathrm{trop}}(34) + (13)(24) + (14)(23)\,. \tag{2.3.3}$$

where Δ^{trop} descends from Δ by replacing ω by ω^{trop}. As explained in [PT3, section VI.C], it is not difficult to see that Y_S has the simple behavior summarized in table 2.1.

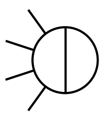
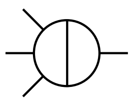
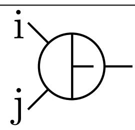
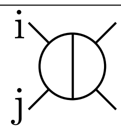
Graph				
Y_S	0	0	$-s_{ij}$	$-s_{ij}$

Table 2.1: Numerators for the two-loop four-graviton worldline graphs.

In total, the non-analytic part of the amplitude is written as

$$\mathsf{A}^{(2,4)}_{\mathrm{non-ana}}(L) = \mathcal{N} t_8 t_8 R^4 \int_{K_{22}>K_{11}\geq\alpha' L}^{\infty} \frac{\prod_{I\leq J}\mathrm{d}K_{IJ}}{(\det K)^5} \int_{\Gamma^4} Y_S^2 \exp\left(Q^{\mathrm{trop}}_{2,4}\right), \tag{2.3.4}$$

where $\mathcal{N}$ is a global normalization factor, $\int_{\Gamma^4}$ represents the integration of the positions of the four punctures on the graph and $\int_{K_{22}>K_{11}\geq\alpha' L}$ represents a possible choice for the

Figure 2.11: The two vacuum topologies at three loops: the Mercedes one and the ladder (hyperelliptic) one, endowed with the choice of a particular homology.

boundaries of the non-analytic domain described before.[20] This object coincides with the one derived in [96, eq. 2.12] from the two-loop field theory computation of [13], thus it is the two-loop unrenormalized four-graviton amplitude.

The other domains of $\mathcal{M}_2$ have been studied as well, but for the moment the author is missing some technology for genus-2 modular integrals, which hopefully would be resolved once the questions raised in [117] are answered. To be complete, we should also mention that the absence of $|\partial\mathcal{G}|^2$ terms forbids the appearance of contact-terms.

2.3.2 New results at three loops

Recently a four-graviton amplitude three-loop amplitude in type II superstring was proposed in [72] in the pure spinor formalism. This amplitude passes a very important consistency check by matching the S-duality prediction of [118] confirmed in [68] for the coefficient of the $\nabla^6 R^4$ in the effective action in ten dimensions after carefully matching normalizations.[21] Here we propose new results concerning the set of graphs that appear (or rather, the ones that do not) in the field theory limit of this amplitude in the non-analytic domains. There are two different vacuum topologies of genus 3 graphs, depicted in the figure 2.11. Let us reproduce the structure of this genus three amplitude. In our notations, up to a global normalization factor $\mathcal{N}_3$, it writes

$$\mathsf{A}^{(3,4)}_{\alpha'}(\epsilon_i, k_i) = \mathcal{N}_3 \int_{\mathcal{M}_3} \frac{|\prod_{I\leq J} \mathrm{d}\Omega_{IJ}|^2}{(\det \operatorname{Im}\Omega)^5} \int_{\Sigma^4} \left[\langle|\mathcal{F}|^2\rangle + \langle|\mathcal{T}|^2\rangle\right] \exp\left(\mathcal{Q}_{3,4}\right), \tag{2.3.5}$$

where $\int_{\Sigma^4}$ is again the integration of the positions of the four punctures. The integrand is a top form and $\mathcal{F}$ and $\mathcal{T}$ are correlation functions of the bosonic pure spinor ghosts $\lambda, \bar\lambda$, including kinematic invariants, polarization tensors, derivatives of the genus three Green's function and holomorphic one-forms $\omega_I(z_i), \bar\omega_J(z_j)$, where $I, J = 1, 2, 3$ and $i, j = 1, \ldots, 4$. The one-forms appear in objects generalizing the genus-two bilinears Δ defined by:

$$\Delta(z_i; z_j; z_k) = \epsilon^{IJK}\omega_I(z_i)\omega_J(z_j)\omega_K(z_k)\,, \tag{2.3.6a}$$

$$\Delta^\mu(z_i, z_j; z_k; z_l) = \epsilon^{IJK}(\Pi\omega)^\mu_I(z_i, z_j)\omega_J(z_k)\omega_K(z_l)\,, \tag{2.3.6b}$$

where $(\Pi\omega)^\mu_I := \Pi^\mu_I \omega_I(z_i)\omega_I(z_j)$ (no sum on I) and the index $\mu = 0, \ldots, 9$ is the target spacetime index. The quantity Π^μ_I is the zero mode part of the momentum Π^μ that flows

[20]Looking back at the explicit parametrization of K in (2.2.8), this contribution sets the length of both B_1 and B_2 loops to be greater than the cutoff scale.

[21]A subtle reasoning on the symmetries of genus-three surfaces led the authors of [72] to include a global factor of 1/3 *a posteriori*. A first-principle computation or a cross-check appears necessary to ensure the validity of this result.

through the cycle B_I. One-forms are also present in the derivatives of the Green's function, since $\partial_{z_i} G(z_i, z_j) = \sum_{I=1}^{3} \omega_I \partial_{\zeta_I} G(z_i, z_j)$ where ζ_I is the I-th component of $\mu(z_i, z_j)$. Finally, $\mathcal{F}$ is solely defined in terms of Δ and derivatives of the Green's function (not mentioning the tensorial structure involving polarization vectors, momenta and pure spinor ghosts) while $\mathcal{T}$ is only defined in terms of Δ^μ and does not contain derivatives of the Green's function.

This being said, what we want to show here is that in the tropical limit, $\mathcal{F}$ and $\mathcal{T}$ vanish *before integration* for both topologies of graphs in fig.2.11 where three or more particles are on the same edge of the graph, possibly via a tree-like contact-term. For the quantity, $\mathcal{F}$, this property follows from the antisymmetry of the tropical version of $\Delta(z_i; z_j; z_k)$, Δ^{trop}, defined by replacing the ω's by they tropical counterparts

$$\Delta^{\text{trop}}(i, j, k) = \epsilon^{IJK} \omega_I^{\text{trop}}(i) \omega_J^{\text{trop}}(j) \omega_K^{\text{trop}}(k) \tag{2.3.7}$$

Whenever two particles, for instance 1 and 2 are on the same edge, one has $\omega_I^{\text{trop}}(1) = \omega_I^{\text{trop}}(2)$ and $\Delta^{\text{trop}}(1, 2, i)$ vanishes by antisymmetry. Therefore, when three particles (or more) are on the same edge, any triplet of particles (i, j, k) necessarily involves two particles inserted on the same edge and Δ always vanishes. As regards $\mathcal{T}$, the vanishing follows from symmetry properties of its defining building blocks rather than on these of the Δ^μ's. We reproduce the definition of $\mathcal{T}$ given in eq. (3.26) in [72]:

$$\begin{aligned} \mathcal{T} = \quad & T^\mu_{1234} \Delta^\mu(z_1, z_2; z_3; z_4) + T^\mu_{1324} \Delta^\mu(z_1, z_3; z_2; z_4) + T^\mu_{1423} \Delta^\mu(z_1, z_4; z_2; z_3) \\ & + T^\mu_{2314} \Delta^\mu(z_2, z_3; z_1; z_4) + T^\mu_{2413} \Delta^\mu(z_2, z_4; z_1; z_3) + T^\mu_{3412} \Delta^\mu(z_3, z_4; z_1; z_2) \end{aligned} \tag{2.3.8}$$

where

$$T^\mu_{1234} = L^\mu_{1342} + L^\mu_{2341} + \frac{5}{2} S^\mu_{1234} \,. \tag{2.3.9}$$

We do not need any detail about L and S but their symmetry properties. The quantity L_{ijkl} is antisymmetric in $[ijk]$, which is enough to ensure vanishing of the L part in $\mathcal{T}$ when three particles are on the same edge of the graph. However S^μ_{ijkl} is only symmetric in (ij) and antisymmetric in $[kl]$ and we need additional identities. It is indeed possible to show the following relation

$$S^\mu_{1234} + S^\mu_{1324} + S^\mu_{2314} = 0 \tag{2.3.10}$$

from the properties of the defining constituents[22] of S^μ, and this identity eventually brings the desired results after a few manipulations of the indices i, j, k, l. The integrand does vanish in the more general case where *at least one B cycle is free of particles*, while it is not trivially zero in the other cases, we arrive at the aforementioned property. Before concluding, we shall also mention that the regions of the moduli space where vertex operators collide to one-another here provide non-vanishing contributions. The required $\partial \mathcal{G}$ terms are present in $\mathcal{F}$, which leaves room for contact-terms to arise in the field theory limit of (2.3.5).

The conclusion is the following; the tropical limit of the amplitude (2.3.5) describes the same set of 12 graphs as the one used in the computation of the four graviton three-loop amplitude in maximal supergravity of Bern *et. al.* in [34]. The complete extraction of the tropical form of the integrand would be a very interesting thing to do.

[22] The author is grateful to Carlos Mafra for a discussion and sharing results on that point.

Chapter 3

Half-Maximal Supergravity

In this chapter, we turn to the theory of half-maximal supergravity and its one- and two-loop amplitudes. We recall that this theory is interesting because of its UV behavior and because of its richer structure than $\mathcal{N}=8$ since it can be coupled to $\mathcal{N}=4$ SYM matter fields.

We review in sec. 3.1 the one-loop computation of [PT2], and focus in particular on the amplitudes computed in CHL orbifolds of the heterotic string. Then in sec. 3.2 we describe the two-loop analysis of [PT1] concerning the UV behavior of half-maximal supergravity. We also provide unpublished material on the genus-two partition function in CHL models and propose a genuine worldline description of the field theory limit of these two-loop amplitudes. Finally we present in appendix 3.A an example of computation of one-loop logarithmic divergence in the case of half-maximal supergravity four-graviton amplitudes $D=8$. This illustrates the discussion of the previous chapter on the cancellation of divergences between the analytic and non-analytic contributions at one-loop.

The one-loop analysis of this chapter does not require the technical material exposed in the previous section since the techniques involved were fully described already in the Bern-Kosower works [61–64]. In contrast, the analysis of the two-loop amplitude is what originally led the author to look for more advanced mathematical tools.[23]

3.1 String theory models and their one-loop amplitudes.

To start this chapter on gravity amplitudes on a concrete basis, we begin by recalling some details of the computation of one-loop amplitudes in string theory. At four-point, in heterotic or type II string, they write as a correlation function of a product of vertex operators

$$\mathcal{A}^{\text{1-loop}}_{\text{string}} = N \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\operatorname{Im}\tau^2} \int_{\mathcal{T}} \prod_{i=1}^{3} \frac{\mathrm{d}^2 z_i}{\operatorname{Im}\tau} \langle V_1(z_1)V_2(z_2)V_3(z_3)V_4(z_4)\rangle\,, \tag{3.1.1}$$

where the normalization constant N depends on the details of the string theory model. The domain of integration $\mathcal{F}$ has been defined in the previous chapter and the z_i belong

[23]The author would like to thank here the mathematician Samuel Grushevsky for suggesting him to look at tropical geometry.

to $\mathcal{T} = \{z \in \mathbb{C}, -1/2 < \operatorname{Re} z \leq 1/2, 0 < \operatorname{Im} z < \operatorname{Im} \tau\}$. One of the vertex operators is fixed to $z_4 = i \operatorname{Im} \tau$ by conformal invariance. The unintegrated vertex operators have a holomorphic part and an anti-holomorphic part:

$$V(z) = : V^{(L)}(z) V^{(R)}(\bar{z}) e^{ikX(z,\bar{z})} : , \tag{3.1.2}$$

where $V^{(L)}$ and $V^{(R)}$ are the chiral vertex operators for the left- and right-moving sectors.[24] In heterotic string, the anti-holomorphic chiral vertex operators for gravitons are the bosonic vertex operators, normalized as in [60]:

$$V^{(L)}_{(0)}(\bar{z}) = i\sqrt{\frac{2}{\alpha'}}\, \varepsilon_\mu \bar{\partial} X^\mu(\bar{z})\,, \tag{3.1.3}$$

while the right-moving are supersymmetric chiral vertex operators:

$$V^{(R)}_{(0)}(z) = \sqrt{\frac{2}{\alpha'}} \varepsilon_\mu(k) \left(i\partial X^\mu + \frac{\alpha'}{2}(k\cdot\psi)\psi^\mu \right). \tag{3.1.4}$$

Type II graviton vertex operators are obtained by choosing both chiral vertex operators to be supersymmetric vertex operators.

The periodicity conditions for the fermionic fields $\psi^\mu, \bar{\psi}^\nu$ upon transport along the a- and b-cycles, corresponding to the shifts $z \to z+1$ and $z \to z+\tau$, respectively, define *spin structures*, denoted by two integers $a, b \in \{0,1\}$ such that

$$\psi^\mu(z+1) = e^{i\pi a}\psi^\mu(z)\,, \qquad \psi^\mu(z+\tau) = e^{i\pi b}\psi^\mu(z)\,. \tag{3.1.5}$$

All of these sectors should be included for modular invariance of the string integrand. The GSO projection indicates relative signs between the corresponding partition functions. The partition function of a supersymmetric sector in the spin structure a, b writes

$$\mathcal{Z}^{ab}(\tau) \equiv \frac{\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](0|\tau)^4}{\eta^{12}(\tau)}\,, \tag{3.1.6}$$

where the Dedekind η function is defined by

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,, \tag{3.1.7}$$

and the theta functions with characteristics have been defined in (2.2.36). The GSO projection gives rise to supersymmetric cancellation identities on the worldsheet, which generically go under the name of “Riemann identities” [112, 113] of which we reproduce two below;

$$\sum_{\substack{a,b=0,1\\ ab=0}} (-1)^{a+b+ab} Z_{a,b}(\tau) = 0\,, \tag{3.1.8a}$$

$$\sum_{\substack{a,b=0,1\\ ab=0}} (-1)^{a+b+ab} Z_{a,b}(\tau) \prod_{i=1}^{4} S_{a,b}(z_i - z_{i+1}|\tau) \quad = -(2\pi)^4 \qquad (\text{with } z_5 \equiv z_1)\,, \tag{3.1.8b}$$

[24] The vertex operators V_i can all be chosen in the (0) superghost picture since the superghost background charge is zero on the torus.

The first identity ensures the vanishing of the string self-energy, as expected in supersymmetric theories. The second identity involves fermionic correlators $S_{a,b} = \langle \psi^\mu(z)\psi^\nu(w)\rangle_{a,b}$ in the spin structure a, b and is the consequence of supersymmetric simplifications on the worldsheet in the RNS formalism.[25] In amplitudes of maximally supersymmetric theories, these identities kill the terms in the correlator (3.1.1) with less than four bilinears of fermions $: \psi\psi :$. They produce the t_8F^4 tensor when there are exactly four of them. Details on these identities can be found in appendix A of [PT2].

In orbifold compactifications, the GSO boundary conditions can be mixed with target-space shifts and the fields X^μ and ψ^μ acquire non-trivial boundary conditions, mixing the standard spin structures with more general (g, h)-orbifold sectors [119, 120];

$$\begin{aligned} X^\mu(z+1) &= (-1)^{2h}\, X^\mu(z)\,, & \psi^\mu(z+1) &= -(-1)^{2a+2h}\psi^\mu(z)\,,\\ X^\mu(z+\tau) &= (-1)^{2g}\, X^\mu(z)\,, & \psi^\mu(z+\tau) &= -(-1)^{2b+2g}\,\psi^\mu(z)\,. \end{aligned} \tag{3.1.9}$$

The string theory four-graviton scattering amplitude is then computed using Wick's theorem as a sum in the various GSO/orbifold sectors in terms of the two points correlators $\langle X(z)X(w)\rangle$ and $\langle \psi(z)\psi(w)\rangle_{a,b}$. It assumes the following general form

$$\mathcal{A}^{\text{string}}_{\text{1-loop}} = N \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^{D/2-3}} \int_{\mathcal{T}} \prod_{i=1}^{3} \frac{\mathrm{d}^2 z_i}{\mathrm{Im}\,\tau} \left(\sum_{s,\tilde s} \mathcal{Z}^{s\tilde s} \left(\mathcal{W}^{(L)}_s(z)\, \overline{\mathcal{W}^{(R)}_{\tilde s}}(\bar z) + \mathcal{W}^{L-R}_{s,\tilde s}(z,\bar z)\right) e^{\mathcal{Q}} \right), \tag{3.1.10}$$

where we dropped the $(1, 4)$ index in the Koba-Nielsen factor $\mathcal{Q}_{1,4}$, $s = (a, b, g, h)$ and $\tilde s = (\tilde a, \tilde b, \tilde g, \tilde h)$ label the GSO and orbifold sectors of the theory with their corresponding partition function $\mathcal{Z}_{s,\tilde s}$ and conformal blocks $\mathcal{W}^{(L/R)}_{s,\tilde s}$, $\mathcal{W}^{L-R}_{s,\tilde s}(z,\bar z)$. Explicit expressions for these terms can be found in [PT2] for the heterotic and type II orbifold models. The term $\mathcal{W}^{L-R}_{s,\tilde s}(z,\bar z)$ contains contractions between left- and right- moving fields like

$$\langle \partial X(z_1,\bar z_1)\bar\partial X(z_2,\bar z_2)\rangle = -\alpha'\pi\delta^{(2)}(z_1-z_2) + \frac{\alpha'}{2\pi \mathrm{Im}\,\tau}\,. \tag{3.1.11}$$

In a generic compactification, the partition function contains a lattice sum corresponding to the Kaluza-Klein states and an oscillator part. For illustrative purposes, let us write the basic compactification lattice sum for a toroidal compactification to $D = 10-d$ dimensions

$$\Gamma_{d,d}(G,B) = \sum_{P_L,P_R} \bar q^{\frac{P_L^2}{2}} q^{\frac{P_R^2}{2}}\,, \tag{3.1.12}$$

where the momenta P_L and P_R span the Narain lattice of the compactification (see chapter 4.18.5 and appendix D of the textbook [79] for more details). In our toroidal compactifications, we will always be in a regime where the Kaluza-Klein states are decoupled. For this it is sufficient to choose the radii of compactification R_i to be of the order of the string-length $R_i \sim \sqrt{\alpha'}$. Therefore we will always set $\Gamma_{d,d}$ to 1 in the following.

The field theory limit is extracted by following the two-step procedure described in the previous section. Here we are interested only in the non-analytic part of the amplitude where the condition $\mathrm{Im}\,\tau \geq L$ gives the field theory cutoff $T \geq \alpha' L$. The heterotic

[25] In the space-time supersymmetric formalisms, there are no sums over spin structures since there are no worldsheet fermions and these simplifications occur from zero mode saturation.

string and type II orbifolds partition function respectively exhibit $1/q$ and $1/\sqrt{q}$ poles. When they hit the integrand, the amplitude picks up non-zero residues upon the phase integration as in (2.2.33). More precisely, in the heterotic string case, the following identities have been used in [PT2, eqs.(III.33), (III.38)]:[26]

$$\int_{-1/2}^{1/2} \mathrm{d}(\mathrm{Re}\,\tau)\mathrm{d}(\mathrm{Re}\,z)\frac{1}{\bar q}(\partial\mathcal{G}(z))^2) = (\alpha' i\pi)^2\,, \tag{3.1.13a}$$

$$\int_{-1/2}^{1/2} \mathrm{d}(\mathrm{Re}\,\tau)\prod_i \mathrm{d}(\mathrm{Re}\,z_i)\,\frac{1}{\bar q}\prod_j \partial\mathcal{G}(z_j - z_{j+1}) = (\alpha' i\pi)^4\,, \tag{3.1.13b}$$

They describe how derivatives of the propagator are eaten up by an inverse power of q. Note that other type of identities can be shown to produce vanishing contributions. Later we connect this to supersymmetric simplifications. Once all phases (real parts of τ and z_i's) are integrated out, the tropical variables corresponding to $\mathrm{Im}\,\tau$ and $\mathrm{Im}\,z$ are obtained by

$$\begin{aligned} \mathrm{Im}\,\tau &= T/\alpha'\,, & T &\in [\alpha' L; +\infty[\,,\\ \mathrm{Im}\,z_i &= Tu_i/\alpha'\,, & u_i &\in [0;1[\,. \end{aligned} \tag{3.1.14}$$

After repeated use of (3.1.13), we obtain the tropical form of $\mathcal{W}$ obtained by turning all $\partial\mathcal{G}$'s which have not been eaten-up in the process to derivatives of the worldline propagator (2.2.43) which writes explicitly at one-loop as

$$G(u_i,u_j) = T(|u_i-u_j| - (u_i-u_j)^2)\,. \tag{3.1.15}$$

Its derivatives with respect to the unscaled variables $t_i = Tu_i$ indicated by dots write

$$\begin{aligned} \dot G(u_i,u_j) &= \mathrm{sign}(u_i-u_j) - 2(u_i-u_j)\,,\\ \ddot G(u_i,u_j) &= \frac{2}{T}\left(\delta(u_i-u_j) - 1\right)\,, \end{aligned} \tag{3.1.16}$$

In supersymmetric sectors, the fermionic propagators left-over from Riemann identities are also subject to residues identities involving $1/\sqrt{q}$ poles. Normally the propagators escaping these two simplifications descend to their worldline analogues $G_F(u_i,u_j) = \mathrm{sign}(u_i-u_j)$. In the computations [PT2], these remaining terms eventually appeared in squares and disappeared of the final expressions. In conclusion, the field theory limit of our expressions can be recast as a worldline integrand W_X which schematically writes solely in terms of $\dot G$ and $\ddot G$ as $\sum_{n,m,i,j,k,l} C_{n,m}(\ddot G_{ij})^m(\dot G_{kl})^n$. The monomials satisfy the power counting

$$(\dot G)^n(\ddot G)^m \sim \frac{u_i^n}{T^m} \longleftrightarrow \ell^{n+2m}\,, \tag{3.1.17}$$

which can be proven by Gaussian integration of ℓ as explained in (2.2.27) and also in [121, 122]. Eventually, one obtains the following type of worldline integrals for the contribution of the multiplet X to the low energy limit of the string amplitudes

$$\mathcal{M}_X^{1-\text{loop}} = \frac{\pi^4 t_8 t_8 R^4}{4}\,\frac{\mu^{2\epsilon}}{\pi^{D/2}}\int_0^\infty \frac{\mathrm{d}T}{T^{D/2-3}}\int_0^1 \prod_{i=1}^3 \mathrm{d}u_i\, W_X\, e^{-\pi T\, Q^{\text{trop}}} \tag{3.1.18}$$

[26]These identities were obtained in [PT2] in a normalization where α' was set to 1/2.

where μ is an infrared mass scale and the factor $t_8 t_8 R^4$ encompasses the polarization dependence of these supersymmetric amplitudes. Moreover, we traded the hard cut-off $T \geq \alpha' L$ for dimensional regularization to non-integer dimension D. Now that this general discussion of the low energy limit of string theory one-loop amplitudes is complete, let us come to particular models.

3.1.1 CHL models in heterotic string

Chaudhuri-Hockney-Lykken models [50–52] are asymmetric $\mathbb{Z}_N$ orbifolds of the heterotic string compactified on a $T^5 \times S^1$ manifold that preserve all of the half-maximal supersymmetry.[27] They act geometrically by rotating N groups of ℓ bosonic fields $\bar{X}^a$ belonging to the internal T^{16} of the heterotic string or to the T^5 and produce an order-N shift on the S^1. More precisely, if we take a boson $\bar{X}^a$ of the $(p+1)$-th group $(p = 0, \dots, N-1)$ of ℓ bosons we have $a \in \{p\ell, p\ell+1, \dots, p\ell+(\ell-1)\}$ and for twists $g/2, h/2 \in \{0, 1/N, \dots, (N-1)/N\}$ we get

$$\bar{X}^a(\bar{z}+\bar{\tau}) = e^{i\pi g p/N} \bar{X}^a(\bar{z}), \qquad \bar{X}^a(\bar{z}+1) = e^{i\pi h p/N} \bar{X}^a(\bar{z}). \tag{3.1.19}$$

The massless spectrum is then composed of the half-maximal supergravity multiplet coupled to n_v maximal SYM matter multiplets. The number of matter vector multiplets is found to be

$$n_v = \frac{48}{N+1} - 2. \tag{3.1.20}$$

In [PT1, PT2], we restricted to prime N and considered the models with $N = 1, 2, 3, 5, 7$ displayed in the upper part of tab. 3.1. Here we also observe that it is in principle possible to formally define models with $N = 11$ as noted by [125, footnote 2], but also $N = 23$. This model would have $n_v = 0$, meaning that it would describe pure half-maximal supergravity. To achieve this in full rigor, one should actually compactify the theory further on a $T^6 \times S^1$ and $T^7 \times S^1$ to 3 and 2 dimensions, respectively. [28]

Finally, these models have the following moduli space:

$$\Gamma \backslash SU(1,1)/U(1) \times SO(6, n_v; \mathbb{Z}) \backslash SO(6, n_v)/SO(6) \times SO(n_v), \tag{3.1.21}$$

where the Γ's are discrete subgroups of $SL(2,\mathbb{Z})$ defined in appendix A.3 of [PT2]. The scalar manifold $SU(1,1)/U(1)$ is parametrized by the axion-dilaton in the $\mathcal{N} = 4$ gravity supermultiplet. The $U(1)$ duality symmetry is known to be an anomalous symmetry [54], whose intriguing implications in the UV behavior of the theory [129, 130] have not been clarified yet .

[27] There also exists type IIA duals [52, 123, 124].

[28] We did not make any additional comment on that point, as we already had a type II superstring compactification with $(4,0)$ supersymmetry that had $n_v = 0$. Here we note that CHL models also appear to be related to the Mathieu Moonshine program (see [126] and references therein), where in particular the order N of the orbifold should relate to the conjugacy class of the Mathieu group $\mathbb{M}_{24}$ via the duality with type II orbifolds of $K3$. To the understanding of the author, despite that an $N = 23$ model might exist, it has not been constructed yet. This putative model, similarly to the observation of [125] for the $N = 11$ one, should act non-geometrically, thus it would not be described by the previous geometric analysis (see also the review of [127] on the classical symmetries of the Mathieu group).

N	ℓ	k	n_v	Gauge group	
1	12	10	22	$U(1)^{22}$	
2	8	6	14	$U(1)^{14}$	
3	6	4	10	$U(1)^{10}$	
5	4	2	6	$U(1)^{6}$	
7	3	1	4	$U(1)^{4}$	
11	2	0	2	$U(1)^{2}$	Unphysical ?
23	1	-1	0	$\emptyset$	

Table 3.1: Adapted from [128]; CHL orbifolds geometry and massless spectrum.

In loop amplitudes, supergravity is realized by the following combination of the bosonic and supersymmetric sectors;

$$(1_{\mathbf{1}}, 1/2_{\mathbf{4}}, 0_{\mathbf{6}})_{\mathcal{N}=4,\text{vect.}} \times (1_{\mathbf{1}}, 1/2_{\mathbf{0}}, 0_{\mathbf{0}})_{\mathcal{N}=0} = (2_{\mathbf{1}}, 3/2_{\mathbf{4}}, 1_{\mathbf{6}}, 1/2_{\mathbf{4}}, 0_{\mathbf{1}+\bar{\mathbf{1}}})_{\mathcal{N}=4,\text{grav.}} \tag{3.1.22}$$

From the worldsheet point of view, the supersymmetric sector of the amplitude is left untouched by the orbifold and is computed as usual with Riemann identities which reduce the holomorphic integrand to the t_8F^4 tensor.

Hence, the non-trivial part of the computation concerns the bosonic sector. The orbifold partition function writes as a sum of the twisted partition functions in the orbifold twisted and untwisted sectors:

$$\mathcal{Z}^{(n_v)}_{(4,0)\text{het}}(\tau) = \frac{1}{N}\sum_{(g,h)} \mathcal{Z}^{h,g}_{(4,0)\text{het}}(\tau)\,. \tag{3.1.23}$$

At a generic point in the moduli space, Wilson lines give masses to the adjoint bosons of the $E_8 \times E_8$ or $SO(32)$ gauge group, and decouple the $(6, 24)$ Kaluza-Klein lattice sum. The low energy limit also decouples the massive states of the twisted sector ($h \neq 0$) of the orbifold. As a result, only the ℓ gauge bosons of the $U(1)^\ell$ group left invariant by the orbifold action stay in the massless spectrum. The untwisted ($g = h = 0$) partition function reduces to the bosonic string partition function

$$\mathcal{Z}^{0,0}_{(4,0)\text{het}}(\tau) = \mathcal{Z}_{\text{bos}} = \frac{1}{\bar\eta^{24}(\bar\tau)}\,, \tag{3.1.24}$$

and the partition functions describing the quantum fluctuations of the massless sectors of the theory with $g \neq 0$ are independent of g and write

$$\mathcal{Z}^{g,0}_{(4,0)\text{het}}(\tau) = \frac{1}{f_k(\bar\tau)}\,. \tag{3.1.25}$$

The modular form $f_k(\tau)$ has weight[29] $\ell = k + 2 = 24/(N + 1)$ and is defined by:

$$f_k(\tau) = (\eta(\tau)\eta(N\tau))^{k+2}\,. \tag{3.1.26}$$

In total, the low energy limit of the CHL partition function writes

$$\mathcal{Z}^{n_v}_{CHL} = \frac{1}{N}\left(\frac{1}{(\bar\eta(\bar\tau))^{24}} + \frac{N-1}{f_k(\bar\tau)}\right) = \frac{1}{\bar q} + (n_v + 2) + O(\bar q)\,, \tag{3.1.27}$$

[29] We recall that a modular form of weight w transforms as $f(\frac{az+b}{cz+d}) = (cz+d)^w f(z)$ for $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL(2,\mathbb{Z})$

where for the first time we encounter explicitly this $1/\bar{q}$ pole which was advertised.

At the next step of the computation, we need to write the conformal block $\overline{\mathcal{W}}^B$ coming from Wick contractions[30] of the bosonic chiral vertex operators. It is defined by (3.1.3)

$$\overline{\mathcal{W}}^B = \frac{\langle \prod_{j=1}^4 \epsilon^j \cdot \bar{\partial} X(z_j) e^{ik_j \cdot X(z_j)} \rangle}{\langle \prod_{j=1}^4 e^{ik_j \cdot X(z_j)} \rangle}\,, \tag{3.1.28}$$

which can be schematically rewritten as

$$\overline{\mathcal{W}}^B \sim \sum (\bar{\partial} \mathcal{G})^4\,. \tag{3.1.29}$$

The $\bar{\partial}\mathcal{G}$'s come from OPE's between the $\partial \bar{X}$ and the plane-waves, but also from integrating by parts the double derivatives created by $\bar{\partial} X \bar{\partial} X$ OPE's. The coefficients of the monomials are not indicated but carry the polarization dependence of the amplitude.

Putting everything together and using residue identities of the form given in (3.1.13), we find that the part of the integrand contributing to the low energy limit of the CHL amplitudes is given by

$$\mathcal{Z}_{(4,0)\text{het}} \overline{\mathcal{W}}^B e^{\mathcal{Q}} \to \underbrace{\left(\overline{\mathcal{W}}^B e^{\mathcal{Q}}\right)|_{\bar{q}}}_{\dot{G}^0, \dot{G}^2} + (n_v + 2) \underbrace{\left(\overline{\mathcal{W}}^B e^{\mathcal{Q}}\right)|_{\bar{q}^0}}_{\dot{G}^4} + O(\bar{q})\,. \tag{3.1.30}$$

This formula already exhibits the organization of the amplitude by the field theory limit. As indicated by the braces, the first terms give rise to worldline polynomials of degree $\dot{G}^0$ and $\dot{G}^2$, due to the $1/\bar{q}$ pole, while the second term is not reduced of full degree $\dot{G}^4$. Using the dictionary of (3.1.17), these respectively correspond to numerators with ℓ^0, ℓ^2 and ℓ^4 homogeneous polynomials in loop momentum.

Asymmetric orbifolds of type II superstrings In [PT2], we presented an analysis of the low energy limit of four-graviton amplitudes in the asymmetric orbifold of type II superstrings models with $(4, 0)$ supersymmetry of [131–133]. One of these models has the property that matter is totally decoupled [132, 133]. The physical and technical content of this analysis being highly redundant with the heterotic and symmetric orbifold cases, we shall skip it here.

3.1.2 Symmetric orbifolds of type II superstrings

Here we briefly discuss $(2, 2)$ models of four-dimensional $\mathcal{N} = 4$ supergravity. These models can be obtained from the compactification of type II string theory on symmetric orbifolds of $K_3 \times T^2$. The difference with the heterotic CHL models is that the scalar parametrizing the coset space $SU(1, 1)/U(1)$ that used to be the axio-dilaton S is now the Kähler modulus of the two-torus T^2 for the type IIA case or complex structure modulus for the type IIB case. The non-perturbative duality relation between these two classes of models is discussed in detail in [124, 131].

The way in which these models are constructed structurally forbids the possibility to decouple completely the matter states. Indeed, supersymmetry is realized by the tensor

[30] At four-point, supersymmetry in the right-moving sector does not allow for left-right contractions.

product between two $\mathcal{N}=2$ vector multiplet theories, which yield the $\mathcal{N}=4$ gravity multiplet plus two $\mathcal{N}=4$ matter vector states

$$(1_{\mathbf{1}}, 1/2_{\mathbf{2}}, 0_{\mathbf{2}})_{\mathcal{N}=2,\text{vect.}} \times (1_{\mathbf{1}}, 1/2_{\mathbf{2}}, 0_{\mathbf{2}})_{\mathcal{N}=2,\text{vect.}} = (2_{\mathbf{1}}, 3/2_{\mathbf{4}}, 1_{\mathbf{6}}, 1/2_{\mathbf{4}}, 0_{\mathbf{1}+\bar{\mathbf{1}}})_{\mathcal{N}=4,\text{grav.}} + \mathbf{2}\,(2_{\mathbf{0}}, 3/2_{\mathbf{0}}, 1_{\mathbf{1}}, 1/2_{\mathbf{4}}, 0_{\mathbf{6}})_{\mathcal{N}=4,\text{matt.}} \tag{3.1.31}$$

The same phenomenon arises when trying to construct pure gravity from pure Yang-Mills:

$$(1_{\mathbf{1}}, 1/2_{\mathbf{0}}, 0_{\mathbf{0}})_{\mathcal{N}=0,\text{YM}} \times (1_{\mathbf{1}}, 1/2_{\mathbf{0}}, 0_{\mathbf{0}})_{\mathcal{N}=0,\text{YM}} = (2_{\mathbf{1}}, 3/2_{\mathbf{0}}, 1_{\mathbf{2}}, 1/2_{\mathbf{0}}, 0_{\mathbf{1}})\,, \tag{3.1.32}$$

Therefore, if an $N=23$ CHL model was constructed, it would be interesting to understand the mechanism that decouples the matter fields and translate it in a type II symmetric duals. This might shed light on how to build pure gravity amplitude directly from Yang-Mills amplitudes [134].

Regarding the structure of the partition function, no novelties arise in this construction compared to the previous analysis. However, a new element enters the computation of the integrand where reduced supersymmetry on both sectors now leave enough room for Wick contractions between the holomorphic and anti-holomorphic sectors of the theory.

3.1.3 Worldline limit

The outcome of these three computations is first that the amplitudes computed in each model do match for identical n_v's. Second, the $\mathcal{N}=4$ supergravity coupled to n_v $\mathcal{N}=4$ vector supermultiplets field theory amplitude is always decomposed as follows;

$$\mathcal{M}^{\text{1-loop}}_{(\mathcal{N}=4,\text{grav})+n_v\,(\mathcal{N}=4\text{matt.})} = \mathcal{M}^{\text{1-loop}}_{\mathcal{N}=8,\text{grav}} - 4\,\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=6,\text{matt}} + (n_v+2)\,\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=4,\text{matt}}\,. \tag{3.1.33}$$

Explicit integrated expressions for the integrals can be found in [PT2, eqs. (IV.11), (IV.23),(IV.25)]. These match the known results of [135–137].

For ease, the computation was performed in a helicity configuration $(1^-, 2^-, 3^+, 4^+)$, called the MHV configuration.[31] We set as well the reference momenta q_i's of graviton $i=1,\ldots,4$ as follows, $q_1=q_2=k_3$ and $q_3=q_4=k_1$. In that fashion, the covariant quantities t_8F^4 and $t_8t_8R^4$ are written in the spinor helicity formalism[32] $2t_8F^4=\langle 12\rangle^2[34]^2$, and $4t_8t_8R^4=\langle 12\rangle^4[34]^4$, respectively. In this helicity configuration, no triangles or bubbles can be generated from neighboring vertex operators as in (2.2.45) in the symmetric construction.[33] We display below the integrands that were found:

$$W_{\mathcal{N}=8,\text{grav}} = 1\,, \tag{3.1.34a}$$

$$W_{\mathcal{N}=6,\text{matt}} = W_3\,, \tag{3.1.34b}$$

in both models. The matter contributions assume structurally different forms in the two models:

$$W^{\text{asym}}_{\mathcal{N}=4,\text{matt}} (= W^B) = W_1 + W_2\,, \tag{3.1.35a}$$

$$W^{\text{sym}}_{\mathcal{N}=4,\text{matt}} = W_3^2 + W_2/2\,, \tag{3.1.35b}$$

[31] At four points in supersymmetric theories, amplitudes with more + or − helicity states vanish.

[32] See [138] for an introduction to the Spinor-Helicity formalism.

[33] Supersymmetry discards them in the asymmetric models from the start.

$$\mathcal{M}^{\text{1-loop}}_{(\mathcal{N}=4,\text{grav})+n_v\,(\mathcal{N}=4\text{matt.})} = \quad + n_v$$

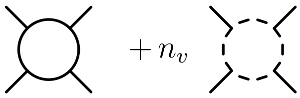


Figure 3.1: One loop worldline description of $\mathcal{N}=4$ gravity amplitudes coupled to matter fields. Straight lines indicate $\mathcal{N}=4$ gravity states, dashes indicate $\mathcal{N}=4$ matter states.

as a consequence of the different ways supersymmetry is realized in string theory, as apparent in eqs.(3.1.22), (3.1.31). Moreover, the factor W_2 in (3.1.35b) comes from the left-right mixing contractions allowed by half-maximal supersymmetry on both sectors of the symmetric orbifold. In the asymmetric models W_2 is simply present in double derivatives in W^B. The explicit worldline numerators write

$$\begin{aligned}
W_1 &= \frac{1}{16}(\dot G_{12}-\dot G_{14})(\dot G_{21}-\dot G_{24})(\dot G_{32}-\dot G_{34})(\dot G_{42}-\dot G_{43})\,,\\
W_2 &= -\frac{1}{u}(\dot G_{12}-\dot G_{14})(\dot G_{32}-\dot G_{34})\ddot G_{24}\,,\\
W_3 &= -\frac{1}{8}\Big((\dot G_{12}-\dot G_{14})(\dot G_{21}-\dot G_{24})+(\dot G_{32}-\dot G_{34})(\dot G_{42}-\dot G_{43})\Big)\,.
\end{aligned} \tag{3.1.36}$$

Notice that the $1/u$ factor in the definition of W_2 is dimensionally present since the double derivative $\ddot G_{24}$ in W_2 contains a $1/T$. Alternatively, integration by parts of the double derivative would bring down powers of $k_i\cdot k_j$ from the exponential of the tropical Koba-Nielsen factor (2.2.26) and trade $1/u\ddot G$ for terms like $s/u(\dot G)^2$.

Now that all the quantities entering the decomposition (3.1.33) are defined, we can look back at eq. (3.1.30). We confirm a posteriori the link between decreasing powers of $\dot G^n$ due to residue identities and the degree of supersymmetry of the multiplets running in the loop. This obeys the qualitative empirical power-counting in gravity amplitudes, which states that the maximal degree of loop momentum in a $(n=4)$-point one-loop amplitude should be related to the number of supersymmetries $\mathcal{N}$ by[34]

$$\ell^{2n-\mathcal{N}}\,. \tag{3.1.37}$$

Finally, we found interesting to associate to the expansion in eq. (3.1.30) a worldline description in terms of the $(\mathcal{N}\!=\!4)$ gravity and $(\mathcal{N}\!=\!4)$ matter multiplets, as depicted in fig. 3.1. This description extends to the two-loop analysis that we propose now.

3.2 Two loops

The techniques and results described in the previous sections are well under control and widely used since the 80's. In this section, we describe an attempt to push them at the second loop order, where almost nothing similar has been constructed so far. Our starting point is the two-loop heterotic string four-graviton amplitude of [43–49], adapted in CHL models. It assumes the general form in $D=10$:

$$\mathcal{M}^{(n_v)}_{4,2-loop} = \mathcal{N}_2\,\frac{t_8F^4}{64\pi^{14}}\int_{\mathcal{F}_2}\frac{|d^3\Omega|^2}{(\det\operatorname{Im}\Omega)^5}\,\mathcal{Z}_2^{(n_v)}\int\prod_{i=1}^{4}d^2\nu_i\,\overline{\mathcal{W}}^{(2)}\,\mathcal{Y}_s\,e^{\mathcal{Q}}\,, \tag{3.2.1}$$

[34] Another possibility for power-counting seems to be compatible: $\ell^{4s-\mathcal{N}}$.

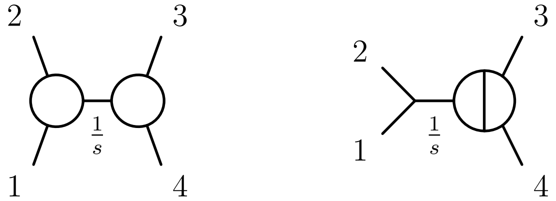


Figure 3.2: Potential pole singularities that would eat up the factorised ∂^2.

where $\mathcal{Z}_2^{(n_v)}$ is the full genus-two partition function of the model under consideration which contains an oscillator and a lattice part[35], $\mathcal{N}_2$ is a normalization constant and $\overline{\mathcal{W}}^{(2)}$ is defined as $\overline{\mathcal{W}}^{B}$ in (3.1.28) in terms of the genus two propagators and we dropped the index $(2,4)$ in $\mathcal{Q}$. The only difference between this amplitude and the heterotic one of [43–49] is that the chiral bosonic string partition function has been replaced with $\mathcal{Z}_2^{n_v}$ and the integration domain is now an $Sp(4,\mathbb{Z})$ fundamental domain (as in sec. 2.2.2).

In [PT1], we used this set-up to argue that there existed a non-renormalization theorem for the R^4 counterterm at two loops in *pure* half-maximal supergravity. The argument goes as follows. First, the $\mathcal{Y}_S$ term factors two derivatives out of the integral. Second, no $1/s_{ij}$ poles as in fig. 3.2 can appear to cancel this factorization in regions where $|z_i - z_j| \ll 1$. The reason for this is the absence of terms like $|\partial \mathcal{G}_{ij}|^2$ in the integrand of (3.2.1). Finally, the matter multiplet contributions, described solely by the partition function $\mathcal{Z}_2^{n_v}$ similarly to the one-loop case, do not prevent this factorization, therefore we may do as if there were none.

The bottom line of this non-renormalization theorem is a string theory explanation, based on worldsheet supersymmetry, for the cancellation of the 3-loop divergence of $\mathcal{N}=4$ pure supergravity in four dimensions [53]. Since R^4 is ruled out, the results of [25] on $\nabla^2 R^4$ being a full-superspace integral make this term a valid counter-term in $\mathcal{N}=4$ supergravity, which signals that a four-loop divergence should happen. This divergence has now been explicitly observed in [130], we shall come back on this result in the last chapter of this manuscript, chap. 5, where we discuss future directions of research.

In the rest of this chapter, we provide a novel analysis on the worldline structure of the low energy limit of the amplitude (3.2.1).

Worldline in the tropical limit

The amplitude (3.2.1) has a rather simple structure, in spite of the complexity of the RNS computation performed to derive it. In the supersymmetric sector, cancellations due to genus-two Riemann identities produced $\mathcal{Y}_S\, t_8 F^4$. Therefore, in analogy with the one-loop case, the essential step of the computation consists in understanding the partition function of the bosonic sector and its influence on the $\mathcal{W}^{(2)}$ when applying residue formulas.

The partition function The chiral genus-two partition function of the $G = E_8 \times E_8$ or $SO(32)$ bosonic sector of the heterotic string in ten dimensions writes as the product of the G lattice theta function Θ_G by the bosonic string partition function (quantum

[35] See explicit expressions in [67, 139] for the case of toroidal compactifications. More details on the twisted sectors of genus two string orbifolds and corresponding partition functions and propagators have been worked out in [140, 141] based on the classical references [142, 143].

oscillator part)

$$\mathcal{Z}^G_{g=2} = \frac{\overline{\Theta}_G}{\overline{\Phi}_{10}}\,, \tag{3.2.2}$$

where Φ_{10} is a cusp modular form of weight 12, known as the Igusa cusp form [144]. It is the analogue of the genus-one cusp form η^{24} and its explicit expression is given by the product of theta functions with even characteristics

$$\Phi_{10} = 2^{-12} \prod_{\delta\,\text{even}} (\theta[\delta](0,\Omega))^2\,. \tag{3.2.3}$$

The genus-two lattice theta functions for $E_8 \times E_8$ and $SO(32)$ have the following explicit expressions generalizing the one-loop ones (3.A.7)

$$\Theta_{E_8\times E_8}(\Omega) = \Big(\frac{1}{2}\sum_{\delta\ \text{even}} \left(\theta\,[\delta]\,(0|\Omega)\right)^8\Big)^2, \qquad \Theta_{SO(32)}(\Omega) = \frac{1}{2}\sum_{\delta\ \text{even}} \left(\theta\,[\delta]\,(0|\Omega)\right)^{16}\,, \tag{3.2.4}$$

Similarly to the one-loop case (3.A.8), it is possible to show the equality between these two objects

$$\Theta_{E_8\times E_8}(\Omega) = \Theta_{SO(32)}(\Omega)\,, \tag{3.2.5}$$

which ensures that the partition functions of the two heterotic strings are identical.[36]

Now that we have written down all explicit expressions, a `Mathematica` computation provides the first few terms of the Fourier-Jacobi expansion of these partition functions:

$$\Theta_{E_8\times E_8} = 1 + 480 \sum_{1\le i<j\le 3} q_i q_j + 26880\, q_1 q_2 q_3 + O(q_i^4)\,, \tag{3.2.6}$$

$$\frac{1}{\Phi_{10}} = \frac{1}{q_1 q_2 q_3} + 2\sum_{1\le i<j\le 3} \frac{1}{q_i q_j} + 24 \sum_{i=1}^{3} \frac{1}{q_i} + 0 + O(q_i)\,, \tag{3.2.7}$$

combine into:

$$\mathcal{Z}_2^{E_8\times E_8} = \frac{1}{\bar{q}_1 \bar{q}_2 \bar{q}_3} + 2\sum_{1\le i<j\le 3} \frac{1}{\bar{q}_i \bar{q}_j} + 504 \sum_{i=1}^{3} \frac{1}{\bar{q}_i} + 29760 + O(q_i)\,. \tag{3.2.8}$$

Before making any further comments, let us observe that when compactifying the theory on a d-dimensional torus we can introduce Wilson lines and break the heterotic gauge group to its Cartan subgroup $U(1)^{16}$. The partition function of this model is then simply equal to the quantum oscillator part

$$\mathcal{Z}_2^{U(1)^{16}} = \mathcal{Z}_2^{\text{bos}} \sim \frac{1}{\overline{\Phi}_{10}}\,. \tag{3.2.9}$$

[36] As a side comment, this identity is still valid at $g = 3$. At $g = 4$, the identity does not hold for all period matrices Ω, but only for the subset of these which precisely correspond to actual Riemann surfaces. We recall that at $g = 4$, the space of symmetric $g \times g$ matrices with positive definite imaginary parts, called $\mathcal{A}_4$, is 10-dimensional, while $\mathcal{M}_4$ is 9-dimensional. The Schottky problem consists in identifying the locus of $\mathcal{M}_g$ inside $\mathcal{A}_g$, which is solved in $g = 4$ since this locus is precisely the zero locus of the modular form defined by $\Theta_{E_8\times E_8} - \Theta_{SO(32)}$. For $g \geq 5$ no solution is known. The question of a connection between this five and the one of $\mathfrak{M}_5$ is, to the understanding of the author, an open question.

where it is understood that the previous identity is an equality when considering that a lattice partition function $\Gamma_{d,d}$ for the genus two amplitude as in eq. (3.1.12) reduces to unity due to a choice of vanishing radii of compactification $R_i \sim \sqrt{\alpha'}$ which causes both the Kaluza-Klein states and the $E_8 \times E_8$ or $SO(32)$ gauge bosons and higher mass modes to decouple. Therefore (3.2.9), is not the partition function of the full CFT but simply the quantum oscillator part, while the numerator, which should ensure a correct modular weight, has been decoupled.

We shall come back later on the form of the corresponding partition functions for the CHL models. For now, these two partition functions are sufficient to observe interesting consequences on the worldline limit.

Worldline limit The analysis of chap. 2 indicates the kind of residue formula analogous to (3.1.13) we should look for at two loops:

$$\int \prod_{i=1}^{3} \mathrm{d}(\mathrm{Re}\,\tau_i) \left(\frac{1}{q_1^{n_1} q_2^{n_2} q_3^{n_3}} \partial \mathcal{G}(z_{ij})^2 \right) = c_{n_1,n_2,n_3} \tag{3.2.10}$$

with n_1, n_2, n_3 being either 0 or 1, and similar kind of relations where $\partial \mathcal{G}(z_{ij})^2$ is replaced by a term of the form $\partial \mathcal{G}^4$. The author confesses his failure so far in deriving the values of the coefficients c_{n_1,n_2,n_3} from a direct computation, although he hopes that the tropical geometry program will help in this quest. One of the main issues in this computation was to obtain expressions independent on the odd spin-structure δ chosen to define $\mathcal{G}$ (see eq. (2.1.8)).

In the following, we simply assume that we are given such a set of identities. They render the extraction of the field theory limit of the amplitude (3.2.1) expressible in the following schematic worldline form, similarly to the one-loop case in eq. (3.1.30):

$$\lim_{q_1,q_2,q_3 \to 0} \int \prod_{i=1}^{3} \mathrm{d}(\mathrm{Re}\,\tau_i) \left(\mathcal{Z}_2^X \overline{\mathcal{W}}^{(2)} e^{\mathcal{Q}} \right) = \begin{cases} c_2 + c_1 \dot{G}^2 + 29760\, \dot{G}^4 & \text{if } X = E_8 \times E_8 \\ \tilde{c}_2 + \tilde{c}_1 \dot{G}^2 + 0 & \text{if } X = U(1)^{16} \end{cases} \tag{3.2.11}$$

where in particular the constant term present in (3.2.8) gives that the $E_8 \times E_8$ worldline integrand possesses a term of full degree $\dot{G}^4$, while the $U(1)^{16}$ integrand has only a $\dot{G}^2$.

Let us try to understand the implications of this remark. Following the dictionary (3.1.17), these integrands respectively correspond to loop momentum polynomials of maximum degree 4 and 2. However, the presence of a factorized $\nabla^2 R^4$ operator outside of the integrals does not allow for loop-momentum numerators of degree higher than two, as shown in the introduction in eq. (1.1.10). The situation is all the more puzzling that we already argued that no pole would transmute the ∂^2 to an ℓ^2, creating a total ℓ^4 in the numerators. Moreover, this implies that the $E_8 \times E_8$ integrand has a worse ultraviolet behavior than the $U(1)^{16}$ model, so the issue is definitely not innocent.

The solution to this apparent paradox is linked to the spectrum content and interactions of the $E_8 \times E_8$ (or $SO(32)$) model. This model indeed describes 16 abelian gauge bosons but also 480 *non-abelian* gauge bosons, which can create diagrams such as the rightmost one in fig.3.3. This diagram is not dressed with a κ_D^6 but with a $\kappa_D^4 g_{YM}^2$. Since the coupling constants are related via

$$2\kappa_D = \sqrt{\alpha'}\, g_{\mathrm{YM}} \tag{3.2.12}$$

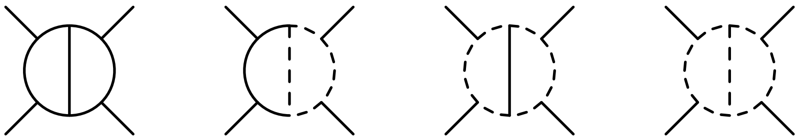


Figure 3.3: Two loop worldline diagrams $\mathcal{N}=4$ matter-coupled supergravity amplitudes. Plain lines are $\mathcal{N}=4$ gravity states, dashes are $\mathcal{N}=4$ YM matter states.

we now realize that an additional power of α' counterbalances the apparent superabundant ℓ^4 in the model with non-abelian gauge interactions. In addition, numerology indicates us that the numerical factor $29760 = 480 \times 496/8$ is related to the interactions of the non-abelian gauge bosons in one way or another. Therefore this divergence only arises in the mixed gravitational-Yang-Mills sector. This does not affect the discussion of the divergences in purely gravitational sector of $\mathcal{N}=4$ supergravity with or without vector-multiplets. This reasoning brushes aside the potential UV issue with the ℓ^4 term in the pure half-maximal supergravity amplitudes. In addition, it gives a heuristic argument on the form of the partition function for CHL models.

CHL models

In [125] were used the so called Siegel genus-two modular forms of weight k generalizing Φ_{10} as $\mathcal{N}=4$ CHL Dyon partition functions. We give below their Fourier-Jacobi expansion, as obtained from [125]:

$$\frac{1}{\Phi_k} = \frac{1}{q_1 q_2 q_3} + 2 \sum_{1\leq i<j\leq 3} \frac{1}{q_i q_j} + \frac{24}{N+1} \sum_{i=1}^{3} \frac{1}{q_i} + \frac{48N}{(N-1)(N+1)} + O(q_i)\,. \tag{3.2.13}$$

for $N = 2, 3, 5, 7$ with conjectural extension to $N = 11, 23$. See again (3.2.7) for $N = 1$. These forms are the analogues of the $f_k(\tau)$ defined in (3.1.26) at $g = 1$ and enter the computation of the genus two partition function, as we will see in an explicit example for $N = 2$ below.

The reasoning of the previous section indicates that the constant term of the partition function should vanish in the absence of non-abelian interactions in the massless spectrum, and that the dependence on the n_v should be linear in $n_v + 2$. This requirement and the knowledge of the Fourier-Jacobi expansion of the partition functions at $N = 1$ and $N = 2$ will be enough to prove that they should generally have the following Fourier-Jacobi expansion;

$$\mathcal{Z}_2^{CHL_N} = \frac{1}{\bar q_1 \bar q_2 \bar q_3} + 2 \sum_{1\leq i<j\leq 3} \frac{1}{\bar q_i \bar q_j} + (n_v+2) \sum_{i=1}^{3} \frac{1}{\bar q_i} + 0 + O(q_i)\,. \tag{3.2.14}$$

up to the lattice factor that reduce to one in the limit we are considering.

This relationship holds true for $N = 1$. Below we provide a short computation based on the derivation in [132] of the $N = 2$ CHL partition function in the context of dyon counting, after the classic reference [142]. The evaluation of the twisted quantum oscillator determinants is performed through the use of a double covering of the genus two surface by a Prym variety, and the dependence on the Prym period ultimately cancels and yield the following result for the partition function with a twist in a particular A cycle;

$$Z_{twisted} = \frac{1}{\Phi_6(\Omega)} + \frac{1}{16}\frac{1}{\Phi_6'(\Omega)} - \frac{1}{16}\frac{1}{\Phi_6''(\Omega)}. \tag{3.2.15}$$

where the theta function lattice (explicitly computed in [128]) have been replaced by 1's, since the gauge group is broken by Wilson lines, and the corresponding lattice partition function which also reduces to one have not been written. The Siegel modular forms Φ_6, Φ_6' and Φ_6'' are images of Φ_6 under modular transformations and their explicit expressions in terms of theta functions are given in [128], eqs. (4.32)-(4.34). The Fourier-Jacobi expansion of Φ_6 is given in (3.2.13), and we also computed explicitly the other two;

$$\begin{aligned} \Phi_6' &= \frac{16}{\sqrt{q_1}q_2\sqrt{q_3}} + \frac{128}{q_2} - 256 \\ \Phi_6'' &= \frac{16}{\sqrt{q_1}q_2\sqrt{q_3}} - \frac{128}{q_2} + 256 \end{aligned} \tag{3.2.16}$$

In total we obtain;

$$Z_{twisted} = \frac{1}{q_1q_2q_3} + \frac{2}{q_1q_3} + \frac{2}{q_2q_3} + \frac{2}{q_1q_2} + \frac{8}{q_1} + \frac{8}{q_3} + \frac{24}{q_2} + 0 + O(q_i) \tag{3.2.17}$$

As is, it is not symmetric under the exchange the q_i's together, which is required to ultimately yield the correct symmetry of the edges of the worldline graphs. Indeed, this partition function has been obtained for a particular twisted sector of the orbifold, along the A_2 cycle. Summing over all sectors, and including the untwisted one, with appropriate weight yields;

$$\frac{1}{4}\left(\frac{1}{\Phi_{10}} + (Z_{twisted} + (q_2 \leftrightarrow q_1) + (q_2 \leftrightarrow q_3))\right) \tag{3.2.18}$$

which has the Fourier-Jacobi expansion given in (3.2.14).

Finally, the worldline arguments developed before imply that we expect the dependence on n_v to be linear in these models, therefore having two points ($N = 1, 2$) is enough to show that the genus two partition function of the other models ($N \geq 3$) should have a Fourier-Jacobi expansion given by (3.2.14).

3.A Appendix on the one-loop divergence in $D = 8$ in CHL models

The section 3.1 was dedicated to the extraction of field theory amplitudes from the $\alpha' \to 0$ limit of the non-analytic part of string theory amplitudes, meaning that we focused on the part the moduli space restricted to the upper domain $\mathcal{F}^+(L)$ defined in eq. (2.2.22) and fig. 2.10.

In this section, we compute the 8-dimensional R^4 logarithmic divergence of these half-maximal supergravity amplitudes from both the non-analytic and analytic parts of the string theory amplitudes. As global normalizations between the two computation remain unfixed, relative normalizations between the contribution of the vectors multiplets and gravity multiplet agree. This section is intended to be a simple example of the reasoning of [107] described previously, supplementing the trivial computation given in [PT3] and the explicit examples given in the seminal paper. We expect that the $\ln(L)$ divergence coming from the integral over $\mathcal{F}^+(L)$ will be canceled by a term coming from $\mathcal{F}^-(L)$. The starting point is the four-graviton CHL amplitude

$$\mathcal{M}^{(n_v)}_{(4,0)het} = \mathcal{N} \left(\frac{\pi}{2}\right)^4 t_8 F^4 \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{(\operatorname{Im}\tau)^{D/2-3}} \int_{\mathcal{T}} \prod_{1\leq i<j\leq 4} \frac{\mathrm{d}^2 z_i}{\operatorname{Im}\tau}\, e^{\mathcal{Q}}\, \mathcal{Z}^{(n_v)}_{(4,0)het} \bar{\mathcal{W}}^B\,, \tag{3.A.1}$$

which we split into the sum of two integrals as in (2.2.23) that we denote $\mathcal{M}^{(n_v)}_{(4,0)het}(L,\pm)$.

3.A.1 Divergence in the non-analytic terms

The procedure described in the previous section produced explicit expressions for the D-dimensional worldline integrands of half-maximal supergravity scattering amplitudes descending from $\mathcal{M}^{(n_v)}_{(4,0)het}(L,+)$, given in (3.1.34), (3.1.35). All we have to do here is to extract the divergence piece of the corresponding integrals in eight dimensions.

The integration is most easily performed in dimensional regularization to $D = 8-2\epsilon$ dimensions, using the standard techniques described in [111, 145–147] The leading $1/\epsilon$ divergence of these integrals is found to be:

$$\begin{aligned}
\mathcal{M}^{spin\ 2}_{\mathcal{N}=8}\Big|_{D=8+2\epsilon,\, div} &= \frac{i}{(4\pi)^4}\langle 12\rangle^4[34]^4\left(\frac{1}{2\epsilon}\right)\\
\mathcal{M}^{spin\ 3/2}_{\mathcal{N}=6}\Big|_{D=8+2\epsilon,\, div} &= \frac{i}{(4\pi)^4}\langle 12\rangle^4[34]^4\left(\frac{1}{24\epsilon}\right)\\
\mathcal{M}^{spin\ 1}_{\mathcal{N}=4}\Big|_{D=8+2\epsilon,\, div} &= \frac{i}{(4\pi)^4}\langle 12\rangle^4[34]^4\left(\frac{1}{180\epsilon}\right)
\end{aligned} \tag{3.A.2}$$

where we expect the $1/\epsilon$ term to match the $\ln(\alpha' L)$ divergence. These divergences match the expressions of [148]. These of [149] are recovered as well after flipping a sign for the $\mathcal{N}=6$ spin-3/2 divergence. The divergence of the half-maximal supergravity multiplet is obtained from the decomposition (3.1.33) in $D = 8+2\epsilon$ with n_v vector multiplets:

$$\mathcal{M}^{n_v}_{\mathcal{N}=4}\Big|_{div} = \frac{i}{(4\pi)^4}\langle 12\rangle^4[34]^4\left(\frac{62+n_v}{180\,\epsilon}\right) \tag{3.A.3}$$

which matches eq (3.8) of [148] with the identitifaction $n_v = D_s - 4$. The normalizations are the ones of [PT4, eq. 5.16].

3.A.2 Divergence in the analytic terms

Let us now consider $\mathcal{M}^{(n_v)}_{(4,0)het}(L,-)$, defined by the integral (3.A.1) restricted to the region $\mathcal{F}^-(L)$. We already argued that since τ is of order $O(1)$, it is possible to safely take the $\alpha' \to 0$ limit of the string theory integrand, which results in dropping the Koba-Nielsen factor.[37] Following the classical reference [150, appendix A,B], the resulting integrals involve terms of the form[38]

$$\int_{\mathcal{T}} \prod_{i=1}^{3} \frac{\mathrm{d}^2 z_i}{\mathrm{Im}\,\tau} \left(\partial\mathcal{G}(z_{12})\right)^2 \left(\partial\mathcal{G}(z_{34})\right)^2 = \left(\frac{2\pi}{12}\hat{E}_2(\tau)\right)^4 , \tag{3.A.4a}$$

$$\int_{\mathcal{T}} \prod_{i=1}^{3} \frac{\mathrm{d}^2 z_i}{\mathrm{Im}\,\tau} \left(\partial\mathcal{G}(z_{12})\right) \left(\partial\mathcal{G}(z_{23})\right) \left(\partial\mathcal{G}(z_{34})\right) \left(\partial\mathcal{G}(z_{41})\right) = \frac{(2\pi)^4}{720} E_4(\tau) , \tag{3.A.4b}$$

where a global factor of α'^4 has not been displayed. Up to permutations of the indices, any other combination of propagators integrates to zero. The non-holomorphic Eisenstein series $\hat{E}_2$ writes

$$\hat{E}_2 = E_2 - \frac{3}{\pi \mathrm{Im}\,\tau} . \tag{3.A.5}$$

in term of the holomorphic Eisenstein series E_2, which together with E_4 write

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \frac{n q^n}{1-q^n} = 1 - 24q + O(q^2) , \tag{3.A.6a}$$

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n} = 1 + 240q + O(q^2) . \tag{3.A.6b}$$

Eisenstein series are also related to partition functions of toroidal lattice sums or lattice theta functions:

$$\begin{aligned} \Theta_{E_8\times E_8}(\tau) &= E_4(\tau)^2 = \frac{1}{2}(\theta_2(0,\tau)^8 + \theta_3(0,\tau)^8 + \theta_3(0,\tau)^8) , \\ \Theta_{SO(32)}(\tau) &= E_8(\tau) = \frac{1}{2}(\theta_2(0,\tau)^{16} + \theta_3(0,\tau)^{16} + \theta_3(0,\tau)^{16}) . \end{aligned} \tag{3.A.7}$$

The identity $E_4(\tau)^2 = E_8(\tau)$ ensures that the one-loop partition functions of the $E_8 \times E_8$ and $SO(32)$ heterotic string are identical:

$$\Theta_{E_8\times E_8} = \Theta_{SO(32)} . \tag{3.A.8}$$

Coming back to the amplitude and collecting the previous results, we obtain

$$\mathcal{M}^{(n_v)}_{(4,0)het}(L,-) = \mathcal{N} \left(\frac{\pi}{2}\right)^4 \int_{\mathcal{F}^-(L)} \frac{d^2\tau}{\mathrm{Im}\,\tau} \, \mathcal{A}(\mathcal{R},\tau) \tag{3.A.9}$$

where the reader should pay attention to the fact that we replaced D with $D = 8$, which explains the factor of $1/\mathrm{Im}\,\tau$ in the integrand. The quantity $\hat{\mathcal{A}}(\mathcal{R},\tau)$ is obtained from the heterotic string elliptic index of [150] by changing the bosonic string partition function $1/\eta(\tau)^{24}$ to the CHL partition function of eq. (3.1.27):

$$\hat{\mathcal{A}}(\mathcal{R},\tau) = \mathcal{Z}^{n_v}_{CHL} \left(\frac{1}{2^7 \cdot 3^2 \cdot 5} \, E_4 \, t_8 \,\mathrm{tr}\,\mathcal{R}^4 \; + \; \frac{1}{2^9 \cdot 3^2} \, \hat{E}_2^2 \, t_8 \,(\mathrm{tr}\,\mathcal{R}^2)^2 \right) . \tag{3.A.10}$$

[37] Actually one should here also make sure that no triangle like contribution may arise from colliding vertex operators, which is the case.

[38] The integrals involving double derivatives in $\mathcal{W}_2$ can always be turned into such kind of integrals after integration by parts.

where the normalization is adjusted so that the $t_8 \mathrm{tr}\, \mathcal{F}^4$ term has coefficient 1. The t_8, $\mathrm{tr}^{\,4}$ and $(\mathrm{tr}^{\,2})^2$ tensors are related by the following identity [151]:

$$t_8 t_8 R^4 = 24 t_8 \mathrm{tr}\, R^4 - 6 t_8 (\mathrm{tr}\, R^2)^2 \,. \tag{3.A.11}$$

The logic now is to compute the integral of eq. (3.A.9) and extract the $\ln L$ term. This could be done in full rigor by following the argument of [27] relating the coefficient of counterterms in the Einstein frame to the coefficient of the logarithm of the D-dimensional string coupling constant in the string frame. This coefficient has be exactly computed for integrals of the form of eq. (3.A.10) with a $\Gamma_{2,2}$ included and can be found in [152, Appendix E], [153], or by using the new methods developed in [139, 154–156] . The result of this procedure can be obtained by a shortcut where one attributes exclusively the coefficient of $\ln L$ in (3.A.9) to the logarithmic divergence created by the $1/\mathrm{Im}\,\tau$ term in the expansion of $\hat{\mathcal{A}}$. This term writes precisely

$$\hat{\mathcal{A}}(\mathcal{R},\tau)\Big|_{1/\mathrm{Im}\,\tau} = \frac{1}{\mathrm{Im}\,\tau}\left(\frac{1}{2^7\cdot 3^2\cdot 5}((n_v+2)+240)t_8 \mathrm{tr}\, \mathcal{R}^4 + \frac{1}{2^9\cdot 3^2}\;((n_v+2)-48)(\mathrm{tr}\, R^2)^2\right) . \tag{3.A.12}$$

Going to the MHV configuration thanks to the following identities

$$24 t_8 \mathrm{tr}\, R^4 = \frac{3}{8}\times [12]^4\langle 34\rangle^4\,, \qquad -6 t_8 (\mathrm{tr}\, R^2)^2 = -\frac{1}{8}\times [12]^4\langle 34\rangle^4\,. \tag{3.A.13}$$

gives the coefficient of $\ln L$

$$\mathcal{M}^{n_v}_{\mathcal{N}=4}\Big|_{div} = c_0'\langle 12\rangle^4[34]^4(62+n_v)\ln L \tag{3.A.14}$$

This result matches the one in (3.A.3) up to a global normalization constant which has not been fixed rigorously. An important consistency check that this example passes is that the relative coefficients between the vector multiplets and gravity contribution are identical in both approaches.

A similar computation is given in [PT3] for the case of the quadratic divergence of maximal supergravity in 10 dimensions, where exact matching is precisely observed. Moreover, several other examples are discussed in the original paper [107].

Chapter 4

BCJ double-copy in string theory

The domain of scattering amplitudes in quantum field theories is at the heart of high energy physics and bridges the gap between theory and collider experiments led nowadays at the Large Hadron Collider. It has been developing fast for the last twenty years, mostly pioneered by the work of Bern, Dixon and Kosower. For moderns reviews on scattering amplitudes, we refer to [138, 157]. In this context, gravitational scattering amplitudes are not directly related to precision physics[39] but rather to more conceptual aspects of the perturbative structure of quantum gravity. These can also serve as consistency checks for certain string theory computations.

The basic difficulty with gravity amplitudes is their complicated kinematical structure, partly due to the presence of arbitrarily high-valency vertices which make the number of diagrams grow very fast. The main idea to simplify these computations is to implement that some of the gravity (spin-2) degrees of freedom are described by the tensorial product of two Yang-Mills spin-1 fields. In string theory, this can be done very efficiently at tree-level, where the Kawai-Lewellen-Tye (KLT) relations [159] relate closed string amplitudes to a product of open strings amplitudes. The paradigm can be loosely formulated as

$$\text{“open} \times \text{open} = \text{closed”}\,. \tag{4.0.1}$$

The modern version of the KLT relations, known as the monodromy relations [160–163] led to the so-called "Momentum Kernel" construction of [164]. The latter relates closed-string amplitudes to open-string amplitudes at any multiplicity *via* the Momentum Kernel $\mathcal{S}_{\alpha'}$ as

$$\mathcal{M}^{\text{closed}}_{n,\text{tree}} = \mathcal{A}^{\text{open}}_{n,\text{tree}} \cdot \mathcal{S}_{\alpha'} \cdot \mathcal{A}^{\text{open}}_{n,\text{tree}}\,. \tag{4.0.2}$$

In the $\alpha' \to 0$ limit, this relation provides a similar relation between gravity and Yang-Mills amplitudes:

$$\mathcal{M}^{\text{gravity}}_{n,\text{tree}} = \mathcal{A}^{\text{YM}}_{n,\text{tree}} \cdot \mathcal{S} \cdot \mathcal{A}^{\text{YM}}_{n,\text{tree}} \tag{4.0.3}$$

where $\mathcal{S}$ is the field theory limit of $\mathcal{S}_{\alpha'}$. In this way, the computation of gravity amplitudes is considerably simplified, as it is reduced to that of gauge theory amplitudes, which is done with 3- and 4-valent vertices only.

At loop-level, the analytic structure of the S-matrix is not compatible with squaring. A $\ln(s)$ in a one-loop Yang-Mills amplitude does not indicate the presence of a $\ln(s)^2$

[39]Except the work [158] in which scattering amplitude methods are used to re-derive the first $\hbar$ correction to the Newtonian potential.

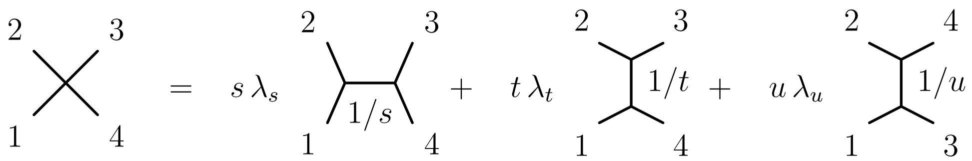


Figure 4.1: Generic ambiguities with blowing-up the contact-terms, with 2-parameters freedom determined by $\lambda_s+\lambda_t+\lambda_u=1$. The diagrams are dressed with $1/p^2$ propagators.

in any one-loop gravity amplitude – this would trivially violate unitarity of the theory. However, a squaring behavior similar to KLT was early observed at the level of the unitarity cuts of $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ amplitudes [13].

The Bern-Carrasco-Johansson duality and double-copy construction [33, 34] provide all-at-once a working algorithm to reduce gravity amplitudes to a cubic-graph expansion,[40] built from gauge theory amplitudes and working at loop level. These gauge theory amplitudes have to be written in a particular representation, satisfying the so-called BCJ duality. The analysis and discussion of current understanding of this construction in string theory is the subject this chapter.

4.1 Review of the BCJ duality and double-copy.

The BCJ duality between color and kinematics in gauge theory amplitudes is defined in tree and loop amplitudes written in terms of cubic-graphs only. This reduction induces a first level of ambiguity when the quartic contact-terms are blown-up to cubic vertices by multiplying and dividing by momentum invariants, as shown in fig. 4.1. In this way, gauge theory amplitudes write

$$\mathcal{A}_n^L = i^L g^{n+2L-2} \sum_{\text{cubic graphs } \Gamma_i} \int \prod_{j=1}^{L} \frac{\mathrm{d}^d \ell_j}{(2\pi)^d} \frac{1}{S_i} \frac{c_i\, n_i(\ell)}{D_i(\ell)}\,, \tag{4.1.1}$$

where the sum runs over distinct non-isomorphic cubic graphs. The denominator $D_i(\ell)$ is the product of the Feynman propagators of the graph and the integral is performed over L independent D-dimensional loop momenta. Finally, the symmetry factors $1/S_i$ remove over counts from summing over the different configurations of the external legs. The c_i are the color factors of the graph obtained by dressing each vertex with the structure constants of the gauge group $\tilde{f}^{abc}$ defined by

$$\tilde{f}^{abc} = i\sqrt{2} f^{abc} = \operatorname{tr}\left([T^a, T^b] T^c\right). \tag{4.1.2}$$

The n_i's are the kinematic numerators of the graph. This representation of the amplitude satisfies the BCJ duality if the Jacobi relations of the color factors are also satisfied by the corresponding kinematic numerators;

$$c_i - c_j = c_k \quad \Rightarrow \quad n_i - n_j = n_k\,, \tag{4.1.3}$$

as depicted in fig. 4.2. Let us emphasize that this property is really not restricted to tree-level four-point diagrams, but should hold for any situation where the graphs of fig. 4.2

[40]Cubic graphs are graphs made of trivalent vertices only.

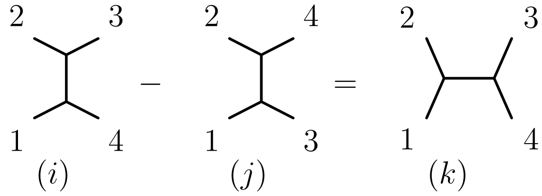


Figure 4.2: Jacobi identity for the color or numerator factors.

are embedded in a bigger graph. An example is show in fig. 4.3 at one-loop. Note that the loop momentum dependence should be traced with care and the "external legs" of the central edge on which the Jacobi relation is being applied should keep their momentum constant.

Such representations do not trivially follow from blowing-up the contact-terms randomly, but rather necessitate an important reshuffling of the amplitude. This is possible thanks to an additional freedom that possess BCJ representations, called "generalized gauge invariance". This freedom corresponds to the fact that a set of BCJ numerators $\{n_i\}$ can be deformed by any set of quantities that leave the Jacobi relations (4.1.3) invariant. If one defines $n'_m = n_m + \Delta_m$ for $m = i, j, k$, the numerators n'_m obey (4.1.3) as long as $\Delta_i - \Delta_j = \Delta_k$. This freedom can be used to reduce the non-locality of the BCJ numerators.

Once a BCJ duality satisfying representation is found, the double-copy procedure prescripts to replace the color factors c_i in (4.1.1) by another set of kinematic numerators $\tilde{n}_i$ to obtain the gravity amplitude:

$$\mathcal{M}_n^{L-\text{loop}} = i^{L+1}\left(\frac{\kappa}{2}\right)^{n+2L-2} \sum_{\text{cubic graphs } \Gamma_i} \int \prod_{j=1}^{L} \frac{\mathrm{d}^d \ell_j}{(2\pi)^d} \frac{1}{S_i} \frac{n_i(\ell)\, \tilde{n}_i(\ell)}{D_i(\ell)} \,. \tag{4.1.4}$$

Due to generalized gauge invariance of the first set of numerators $\{n_i\}$, the set $\{\tilde{n}_i\}$ does not need to be in a BCJ representation [165]. The duality has been demonstrated to hold classically by construction of a non-local Lagrangian [165]. In [166], it was observed to be more restrictive than the strict KLT relations, and later understood in open string theory by Mafra, Schlotterer and Stieberger in [167] by means of worldsheet integration by part (IBP) techniques. Part of our work [PT4] heavily relies on this "Mafra-Schlotterer-Stieberger" (MSS) construction to which we come back in sec. 4.2.

The BCJ duality was successfully applied in the hunt for UV divergences of supergravity theories, at three and four loops in $\mathcal{N} = 8$ [34, 35]. In half-maximal supergravity, the vanishing of the three-loop R^4 divergence in $D = 4$ was observed in a direct computation [53] and the four-loop logarithmic divergence created by the $\nabla^2 R^4$ in $D = 4$ explicitly

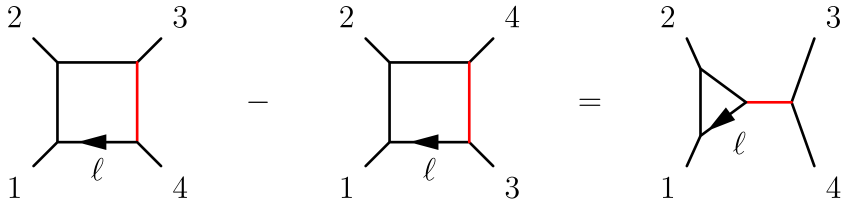


Figure 4.3: Sample Jacobi identity for one-loop numerators

Left-moving CFT	Right-moving CFT	Low-energy limit	Closed string theory
Spacetime CFT	Color CFT	Gauge theory	Heterotic
Spacetime CFT	Spacetime CFT	Gravity theory	Type II, (Heterotic)
Color CFT	Color CFT	Cubic color scalar	Bosonic

Table 4.1: Different string theories generate various field theories in the low-energy limit

determined [130]. More broadly, it was also applied to compute $\mathcal{N} \geq 4$ supergravity amplitudes at various loop orders [137, 148, 168, 169] and even for pure Yang-Mills and pure gravity theories at one and two loops [170].

The existence of BCJ satisfying representations at any loop order is an open question. In particular, at five loops in $\mathcal{N} = 4$, no BCJ representation has yet been found, despite tenacious efforts [36].[41] At one loop there exist constructive methods to build some class of BCJ numerators in $\mathcal{N} = 4$ SYM [171] and orbifolds thereof [172, 173]. Nevertheless, the generic method to find BCJ representations for numerators is to use an ansatz for the numerators, which is solved by matching the cuts of the amplitude [172, 174]. The free-parameters that remain (if any) after satisfying all the constraints are a subset of the full generalized gauge invariance. In [PT4], we also studied some aspects of the string theory viewpoint on the ansatz approach.

4.2 Tree-level string theory understanding of BCJ

We already described that the KLT relations in string theory relate open to closed strings amplitudes. However, this does not directly relate color to kinematics at the integrand level. In [PT4] we argued that this can be done by slightly modifying the paradigm of (4.0.1) to the following purely closed-string one:

$$\text{“left-moving sector} \times \text{right-moving sector} = \text{closed”}\,. \tag{4.2.1}$$

which means that instead of focusing on an definite string theory, we consider as a freedom the possibility to plug different CFT's in both sectors of the closed string. These are tied together by the low energy limit and realize various theories, as illustrated in Table 4.1, where “Color CFT” and “Spacetime CFT” refer to the respective target-space chiral polarizations and momenta of the scattered states.

A gauge theory is realized by the closed string when one of the chiral sectors of the external states is polarized in an internal color space, this is the basic mechanism of the heterosis [175]. The use of heterotic string in this context was first described in [162] where it was realized that it sets color and kinematics on the same footing. In this sense our work descends from these ideas. A gravity theory is realized when both the left- and right-moving polarizations of the gravitons have their target space in Minkowski spacetime, as it can be done both in heterotic and type II string.[42]

[41] We recall that 5-loop in $\mathcal{N} = 8$ is crucial to understand the UV behavior of the theory, see again fig. 1.3.

[42] Neither in the paper nor in this text have we described the gravity sector of the heterotic string, as it is always non-symmetric. Instead, we focused on the symmetric orbifolds of the type II string described in chapter 3 to obtain symmetric realizations of half-maximal supergravity.

The last line of the table, mostly shown as a curiosity, deserves a comment. As we mention later, this cubic scalar theory is the result of compactifying the bosonic string on a $(T^{16} \times \mathbb{R}^{1,9}) \times (T^{16} \times \mathbb{R}^{1,9})$ background where the T^{16} is the internal torus of the heterotic string. At tree-level, the bosonic string tachyon can be decoupled by hand, and the remaining massless states bi-polarized in the T^{16} give rise to these cubic color scalar interactions. At loop-level the tachyon cannot be easily decoupled and the construction probably cannot be given much sense.[43]

Our starting point for the following analysis is the open string construction of MSS [176]. We recall that MSS have shown how a worldsheet IBP procedure in the open string leads to a particular representation of the open string integrand in terms of $(n-2)!$ conformal blocks. From this representation, it is explained how to extract the BCJ numerators for gauge theory amplitudes at any multiplicity. In [PT4], we argued that this construction can be recast in the closed string sector and gives rise to a somewhat stronger result, where we get all-at-once the Jacobi identities of MSS but also the double-copy form of the gravity amplitudes. Our reasoning was mostly supported by an explicit five-point example that we worked out explicitly. We outlined a n-point proof of the systematics of the result.

Below we give a more detailed account on this systematics. Regarding the *material* available in the literature[44], we shall base our reasoning on the fact that type I and II string amplitudes are known at n-point in the pure spinor formalism [176, 178–180] and their field theory limits have been extensively studied in [176, 181] as well as their α' expansion in [181–184]. Hence we start with an n-point closed string theory amplitude, written as:

$$\mathcal{A}_n^{\text{string}} = |z_{1,n-1}z_{n-1,n}z_{n,1}|^2 \left\langle V_1(z_1)V_{n-1}(z_{n-1})V_n(z_n) \int \prod_{i=2}^{n-2} \mathrm{d}^2 z_i V_2(z_2)\dots V_{n-2}(z_{n-2}) \right\rangle . \tag{4.2.2}$$

A global normalization $g_c^{n-2} 8\pi/\alpha'$, where g_c is the closed string coupling constant, has been omitted. The factor $|z_{1,n-1}z_{n-1,n}z_{n,1}|^2$ comes from gauging the $SL(2,\mathbb{C})$ conformal invariance of the sphere by fixing the positions of 3 vertex operators, here z_1, z_{n-1} and z_n. The unintegrated vertex operators have a holomorphic part and an anti-holomorphic part:

$$V(z) = : V^{(L)}(z)V^{(R)}(\bar z)e^{ikX(z,\bar z)} : , \tag{4.2.3}$$

as already described in the beginning of chap 3. The anti-holomorphic part of the heterotic gauge-boson vertex operators are made of a current algebra

$$V^{(R)}(\bar z) = T^a\, J_a(\bar z) . \tag{4.2.4}$$

The currents satisfy the following operator product expansion (OPE):

$$J^a(\bar z)J^b(0) = \frac{\delta^{ab}}{\bar z^2} + i f^{abc}\frac{J^c(\bar z)}{\bar z} + \dots , \tag{4.2.5}$$

[43] Anyway the cubic scalar theory is possible to deal with by standard techniques.

[44] *Addendum*: After the first version of this manuscript was written, Schlotterer and Mafra proposed in [177] a formalism for describing the systematics of the tree combinatorics based on "multi-particle vertex operators", which can be used for the present problem.

where the f^{abc}'s are defined in (4.1.2).

On the sphere, there is a $(+2)/(+2,+2)$ superghost background charge in heretoric/type II string that needs to be canceled, therefore we need to provide expressions for the kinematic parts of the vertex operators in the (-1) picture:

$$V^{(L)}_{(-1)}(z) = \varepsilon_\mu(k)\, e^{-\phi}\psi^\mu \,. \tag{4.2.6}$$

The complete vertex operators for gluons or gravitons (4.2.3) are obtained by plugging together the pieces that we described, following tab. 4.1.

The essential point of the discussion is that the correlation function (4.2.2) can be split off as a product of a holomorphic and of an anti-holomorphic correlator thanks to the "canceled propagator argument". As explained in the classical reference [60, sec. 6.6], the argument is an analytic continuation which makes sure that Wick contractions between holomorphic and anti-holomorphic operators

$$\langle \partial X(z,\bar z)\bar\partial X(w,\bar w)\rangle = -\alpha'\pi\delta^{(2)}(z-w)\,, \tag{4.2.7}$$

provide only vanishing contributions at tree-level. At loop-level, the left- and right-moving sectors couple via the zero-mode of the $X(z,\bar z)$ field and, as we saw (3.1.35b), the left-right contractions are necessary to produce correct amplitudes. These terms are the subject of the one-loop analysis of [PT4], which we review in sec. 4.3.

Kinematic sector We start with the open-string kinematic correlator of MSS, who proved that it can decomposed in terms of $(n-2)!$ basis elements

$$\langle V_1(z_1)\dots V_n(z_n)\rangle = \sum_{\sigma\in S_{n-2}} \frac{\tilde{\mathcal{K}}_\sigma}{z_{1,\sigma(2)}z_{\sigma(2),\sigma(3)}\cdots z_{\sigma(n-2),\sigma(n)}z_{\sigma(n),n-1}z_{n-1,1}}, \tag{4.2.8}$$

where S_{n-2} is the set of permutations of $n-2$ elements and $\tilde{\mathcal{K}}_\sigma$ are kinematical objects whose explicit expression do not concern us here.[45] The procedure used to reach this representation solely relies on worldsheet IBP's and fraction-by-part identities.

As emphasized above, the use of the canceled propagator argument grants us that performing the same IBP's on the chiral heterotic string correlator does not yield contact terms due to ∂ derivatives hitting $\bar\partial\mathcal{G}$ for instance.[46] Therefore, one can legitimately consider that the formula written above in eq. (4.2.8) is also the expression of the chiral closed string kinematical correlator in full generality.

Planar color sector Following [185], we write the *planar sector*[47] of the n-point correlator for the color currents from the basic OPE (4.2.5) as follows:

$$\langle J^{a_1}(z_1)\dots J^{a_N}(z_N)\rangle\Big|_{\text{planar}} = -2^{n-3}\sum_{\sigma\in S_{n-2}} \frac{f^{a_1a_{\sigma(2)}c_1}f^{c_1a_{\sigma(3)}c_2}\dots f^{c_{n-3}a_{\sigma(n)}a_{n-1}}}{z_{1,\sigma(2)}z_{\sigma(2),\sigma(3)}\cdots z_{\sigma(n-2),\sigma(n)}z_{\sigma(n),n-1}z_{n-1,1}}, \tag{4.2.9}$$

[45]It was determined in terms of the $(n-2)$ elements $\mathcal{K}^l_\sigma$ of [167, eq. (3.5)] for $l=1,\dots,n-2$ and $\sigma\in S_{n-3}$. Here we implicitly relabeled these in terms of the elements of S_{n-2}.

[46]In the open string the IBP's on the boundary of the disk yield contact-terms which are also discarded by use of the canceled propagator argument.

[47]We decouple by hand the gravitational sector which creates non planar-corrections in heterotic string vacua.

Pay attention to the special ordering of the last terms of the denominator; it is designed so that one obtains directly the $(n-2)!$ element of the MSS basis. The low-energy limit of these correlators was thoroughly described in [185], and proven to reproduce the color ordering usually produced by color ordering along the boundary of the open string disk.

Low energy limit Now we need to describe how the two sectors of the closed string are tied together by the field theory limit. In [PT4] we carried the explicit procedure at five points and gave details on how the 5-punctured sphere degenerates into thrice punctured spheres connected by long tubes. Here, we rather focus on the similarities between the field theory limit in open string and the one in closed string, at tree-level.

This procedure is by now well understood and the limit can be described by the following rule [181]. In the open string, a given gauge theory cubic diagram Γ receives contributions from the color-ordered amplitude $\mathcal{A}_n^{\text{open}}(1,\sigma(2),\dots,\sigma(n-2),n,\sigma(n-1))$ only from the integrals

$$\mathcal{I}^O_{\rho,\sigma} = \int_{z_1=0<z_{\sigma(2)}<\cdots<z_{\sigma(n-1)}=1} \prod_{i=2}^{n-1} \mathrm{d}z_i \prod_{i<j} (z_{ij})^{-\alpha' k_i.k_j} \frac{1}{z_{1,\rho(2)} z_{\rho(2),\rho(3)} \cdots z_{\rho(n-2),\rho(n)} z_{\rho(n),n-1} z_{n-1,1}} \tag{4.2.10}$$

where the ordering of ρ and σ are compatible with the cubic graph G under consideration, see the section 4 of [181] for details and precise meaning of the compatibility condition. We can then write:

$$\mathcal{I}^O_{\rho,\sigma} = \sum_{\Gamma|(\rho\wedge\sigma)} \frac{1}{s_\Gamma} + O(\alpha') \tag{4.2.11}$$

where the summation is performed over the set of cubic graphs Γ compatible with both σ and ρ and s_Γ is the product of kinematic invariants associated to the pole channels of Γ.

In closed string, we first consider a heterotic gauge-boson amplitude. The latter has to match the result obtained from the field theory limit of the open string amplitude. Therefore, if we select a particular color-ordering σ for the open-string, we can identify the corresponding terms in the heterotic string color correlator (4.2.9). Actually, only one of them does, precisely the one with the permutation σ. This is actually sufficient to see that the mechanism that describes the low energy limit of closed string amplitudes has to be the following one: the field theory limit of the integrals

$$\begin{aligned}\mathcal{I}^C_{\sigma,\rho} = \int \prod_{i=2}^{n-1} \mathrm{d}^2 z_i \prod_{i<j} |z_{ij}|^{-2\alpha' k_i.k_j} \Big(& \frac{1}{z_{1,\rho(2)} z_{\rho(2),\rho(3)} \cdots z_{\rho(n-2),\rho(n)} z_{\rho(n),n-1} z_{n-1,1}} \times \\ & \frac{1}{\bar z_{1,\sigma(2)} \bar z_{\sigma(2),\sigma(3)} \cdots \bar z_{\sigma(n-2),\sigma(n)} \bar z_{\sigma(n),n-1} \bar z_{n-1,1}} \Big)\end{aligned} \tag{4.2.12}$$

contribute to the set of cubic diagrams which are compatible (in the sense mentioned above) with ρ and σ. This gives the same formula as for the open string

$$\mathcal{I}^C_{\rho,\sigma} = \sum_{\Gamma|(\rho\wedge\sigma)} \frac{1}{s_\Gamma} + O(\alpha') \tag{4.2.13}$$

up to factors of 2π created by phase integration of the z_{ij}'s. A direct proof in the sense of [181] would require to work out the complete combinatorics. This could be done, though it does not appear necessary as we are simply describing generic features of these amplitudes.

The formula (4.2.13) can now be applied to more general amplitudes, as long as their chiral correlators are recast in the MSS representation:

$$\langle V_1^{(L)}(z_1)\dots V_n^{(L)}(z_n)\rangle = \sum_{\sigma\in S_{n-2}} \frac{a^L_\sigma}{z_{1,\sigma(2)}z_{\sigma(2),\sigma(3)}\cdots z_{\sigma(n-2),\sigma(n)}z_{\sigma(n),n-1}z_{n-1,1}} \tag{4.2.14a}$$

$$\langle V_1^{(R)}(z_1)\dots V_n^{(R)}(z_n)\rangle = \sum_{\sigma\in S_{n-2}} \frac{a^R_\sigma}{\bar z_{1,\sigma(2)}\bar z_{\sigma(2),\sigma(3)}\cdots \bar z_{\sigma(n-2),\sigma(n)}\bar z_{\sigma(n),n-1}\bar z_{n-1,1}} \tag{4.2.14b}$$

In these formulas, the $a^{(L/R)}$ variables are independent of the z_i and carry color or kinematical information, they write as tensorial products between the group structure constants f^{abc} or polarization ε_i and momenta k_j of the external states. The total contribution to a given graph Γ of the low energy limit of closed string amplitude made of these chiral correlators is found to be given by the following sum

$$\begin{aligned}\frac{N_\Gamma}{s_\Gamma} &= \lim_{\alpha'\to 0} \sum_{\rho,\sigma\in\{\sigma_1,\dots\sigma_p\}} \mathcal{I}^C_{\rho,\sigma} a^{(L)}_\rho a^{(R)}_\sigma \\ &= \frac{1}{s_\Gamma}\underbrace{\left(\sum_{\rho\in\{\sigma_1,\dots\sigma_p\}} a^{(L)}_\rho\right)} \times \underbrace{\left(\sum_{\sigma\in\{\sigma_1,\dots\sigma_p\}} a^{(R)}_\sigma\right)} \\ &= \frac{1}{s_\Gamma}\Big(\quad n^{(L)}_\Gamma \quad\times\quad n^{(R)}_\Gamma \quad\Big)\,,\end{aligned} \tag{4.2.15}$$

where $\{\sigma_1,\dots\sigma_p\}$ is the set of permutations compatible with Γ. We see that the numerator of the graph N_Γ splits as a product of two numerators corresponding to each sector of the theory. Summing over all cubic graphs produces the total n-point field theory amplitude as:

$$\mathcal{A}_n^{\text{tree}}(L,R) = \sum_{\substack{\text{cubic graphs}\\ \Gamma_i}} \frac{n^{(L)}_{\Gamma_i} n^{(R)}_{\Gamma_i}}{s_{\Gamma_i}}\,, \tag{4.2.16}$$

where a global normalization factor of $(g_{YM})^{n-2}$ or $(\kappa_D/2)^{n-2}$ should be included according to what L and R vertex operators were chosen.[48]

This formula have been written without referring to the actual theories plugged in the left-moving and in the right-moving sector of the closed string, hence we have the possibility to choose the string theory we want, following the table 4.1. Therefore, $\mathcal{A}_n^{\text{tree}}(L,R)$ could be either a gauge theory amplitude, if for instance, we had been doing the computation in heterotic string where the ($L=$ col) and $n^{(L)}_{\Gamma_i} = c_{\Gamma_i}$ are color factors while ($R=$ kin) so that $n^{(R)}_{\Gamma_i} = n_{\Gamma_i}$ are kinematic factors. It could as well be a gravity amplitude if we had been doing the computation in type II where ($L=R=$ kin) so that both $n^{(L)}_{\Gamma_i}$ and $n^{(R)}_{\Gamma_i}$ are kinematic numerators. Another possibility would be to choose both $n^{(L)}_{\Gamma_i}$

[48]We recall that $g_c = \kappa_D/2\pi = (\sqrt{\alpha'}/4\pi)g_{YM}$. The appearance of $(2\pi)^{n-2}$ factors is compensated in the final result in the field theory limit by phase integrations for the tropicalized z_i.

and $n^{(R)}_{\Gamma_i}$ to be color factors, in which case $\mathcal{A}^{\text{tree}}_n(\text{col},\text{col})$ corresponds to the scattering amplitude between n color cubic scalars. Note also that these relations do not depend on supersymmetry nor on spacetime dimension.

Finally, let us note that recently, Cachazo, He and Yuan proposed a new prescription to compute scalar, vector and gravity amplitudes at tree-level [186–188]. This prescription was elucidated from first principles by Mason and Skinner in [189], where a holomorphic worldsheet sigma model for the so-called "Ambitwistor strings" was demonstrated to produce the CHY prescription at tree-level.[49] The CHY prescription at tree-level is naturally a closed string type of construction, although there are no right movers, and the way by which color and kinematics are generated is very similar to the one that we reviewed in this section; the authors of [189] built "type II" and "heterotic" Ambitwistor string sigma models. In [190], formulas for type II Ambitwistor n-graviton and heterotic Ambistwistor string n-gluon amplitudes have been proposed. The properties of the gravity amplitude form a very interesting problem on its own. It is also important to understand to what extent the heterotic Ambitwistor string have to or can be engineered in order to produce *pure* $\mathcal{N}=4$ Yang-Mills amplitudes at one-loop, where the couplings to $\mathcal{N}=4$ gravity are suppressed. These are a traditional issue encountered in Witten's twistor string [191].

4.3 Towards a string theoretic understanding in loop amplitudes

At loop-level, the form of the amplitudes integrand depends on the spectrum of the theory. We already emphasized the simplicity of maximally supersymmetric Yang-Mills and gravity theories. This simplicity here turns out to be a problem in the sense that the one- and two-loop four-gluon and four-graviton amplitudes are *too* simple to obtain non-trivial insight on a stringy origin of the BCJ duality. The box numerators reduce to 1 at one-loop in SYM and maximal supergravity [59] and they are given by s, t, u and s^2, t^2, u^2 at two loops (result of [13, 96] which we discussed in eq. (2.3.1)). In addition, there are no triangles and the Jacobi identities 4.3 are satisfied without requiring any special loop momentum identities besides the trivial $1-1=0$ and $s-s=0$.

To increase the complexity of the amplitudes, it is necessary to introduce a non-trivial dependence in the loop momentum. Considering the empirical power counting of eq. (3.1.37), this could be achieved in two ways; either by increasing the number of external particles, or by decreasing the level of supersymmetry. Five-point amplitudes in $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ supergravity were recently discussed in [192] in open and closed string. The appearance of left-right mixing terms was observed to be crucial in the squaring behavior of the open string integrand to the closed string one. These terms are central in our one-loop analysis as well.

In [PT4], we investigated the reduction of supersymmetry and studied four-graviton amplitudes obtained from the symmetric product of two $\mathcal{N}=2$ SYM copies. We already discussed in eq. (3.1.31) that these constructions structurally produce matter couplings in the gravity theory. Both in our string theory construction and in our direct BCJ

[49] Although the preprint [PT5] deals with this issues, as already emphasized it is not the intention of the author to discuss it in this text.

construction, the contribution of the ($\mathcal{N}=2$) vector-multiplet running in the loop is realized as

$$\mathcal{A}^{1-\text{loop}}_{(\mathcal{N}=2 \text{ vect.})} = \mathcal{A}^{1-\text{loop}}_{(\mathcal{N}=4 \text{ vect.})} - 2\,\mathcal{A}^{1-\text{loop}}_{(\mathcal{N}=2 \text{ hyper})} \tag{4.3.1}$$

in analogy with the similar relation in $\mathcal{N}=4$ gravity in eq. 3.1.33. It can be seen in tab. 1.1, that these identities are coherent with respect to the spectrum of the multiplets. This implies that the non-trivial loop momentum part of the integrands is described by the following product

$$(\mathcal{N}=2 \text{ hyper}) \times (\mathcal{N}=2 \text{ hyper}) = (\mathcal{N}=4 \text{ matter}) \tag{4.3.2}$$

which is therefore the important sector of the four-graviton amplitude on which we will focus from now on. Each of the hyper-multiplet copies will carry an ℓ^2 dependence in the loop momentum, respectively an $\dot{G}^2$ in the worldline integrand.

4.3.1 BCJ ansatz for ($\mathcal{N}=2$) hyper multiplets.

The ansatz that we used to find a BCJ satisfying representation of $\mathcal{N}=2$ gauge theory amplitudes is described in great detail in [PT4, sec.4]. The first constraint that we apply is our choice to start with two master boxes, corresponding to the (s,t) and (t,u) channels, the (s,u) channel being obtained from the (s,t) box by the exchange of the legs $3 \leftrightarrow 4$.

The second physical requirement was to stick as much as possible to our string theoretic construction which in particular has no triangle nor bubble integrals in the field theory limit. Since the Jacobi identities between boxes force triangles to be present, the best we could do was to require all bubbles to vanish. To our surprise, this turned out to be sufficient to force the triangles to vanish at the integrated level, despite a non-trivial loop-momentum numerator structure.

In total, after solving all the D-dimensional unitarity cuts constraints on the ansatz, only two free coefficients remain from the original definition of the ansatz, called α and β in the paper. They parametrize residual generalized gauge invariance in our representation. The total number of diagrams is therefore 9; three boxes and six triangles. Their explicit expressions may be found in [PT4, eqs. (4.20)-(4.21)]. As expected from power counting, the box numerators of these $\mathcal{N}=2$ gauge theory amplitudes have degree $(4-\mathcal{N})=2$ in the loop momentum. In addition, we provide in [PT4, appendix C] a short explicit computation for the vanishing of a particular gauge theory triangle after integration. An important additional feature of our ansatz, generally present in other ansatzes as well [172], is that it requires the inclusion of parity-odd terms $i\epsilon_{\mu\nu\rho\sigma}k_1^\mu k_2^\nu k_3^\rho \ell^\sigma$ for consistency. In gauge theory amplitudes, they vanish due to Lorentz invariance since the vector $i\epsilon_{\mu\nu\rho\sigma}k_1^\mu k_2^\nu k_3^\rho$ is orthogonal to any of the momenta of the scattered states k_i^σ. Combined with the triangles, these terms are invisible to the string theory amplitude because they vanish when the loop momentum is integrated. An essential feature of the BCJ double-copy is that these terms do contribute to the gravity amplitudes after squaring.

4.3.2 String theoretic intuition

We proposed in [PT4] a possible origin for this mechanism in string theory. Our physical intuition is based on the fact that in string theory gravity amplitudes possess additional terms coming from Wick contractions between the left- and right-moving sectors. Furthermore, these left-right moving contractions are absent in gauge theory amplitudes in heterotic string because the two CFT's (color and kinematical) have different target spaces and do not communicate. Therefore we naturally expect that these additional terms in BCJ and worldline gravity amplitudes have to be related, this is indeed what was shown in [PT4].

For illustrative purposes, we display below the form of the one-loop amplitudes in gauge theory and gravity as obtained from the generic four-point string theory amplitude in eq. (3.1.10) with the vertex operators described along the text:

$$\mathcal{A}^{\text{1-loop}}_{\text{gauge}} = \int_0^\infty \frac{\mathrm{d}T}{T^{d/2-3}} \int_0^1 \mathrm{d}^3u \cdot \left(W^{(L,\,\text{kin})}\,W^{(R,\,\text{col})}\right) \cdot e^{-TQ}\,, \tag{4.3.3a}$$

$$\mathcal{M}^{\text{1-loop}}_{\text{gravity}} = \int_0^\infty \frac{\mathrm{d}T}{T^{d/2-3}} \int_0^1 \mathrm{d}^3u \cdot \left(W^{(L,\,\text{kin})}\,W^{(R,\,\text{kin})} + W^{(L-R,\,\text{kin})}\right) \cdot e^{-TQ}\,. \tag{4.3.3b}$$

where the transparent abbreviations col and kin follow from the terminology used in the previous section. The form of the gravity amplitude has been discussed before, where the kinematic numerators $W^{(\cdot,\text{kin})}$ were described as polynomials in $\dot{G}$ and $\ddot{G}$. On the other hand, the form of the gauge theory worldline amplitude deserves a comment. The presence of a current algebra in the left-moving sector of the gauge boson heterotic-string CFT not only prevents mixed contractions, but also produces color ordered amplitudes, so that $W^{(R,\,\text{col})}$ writes

$$W^{(R,\,\text{col})} = \sum_{\sigma\in S_{n-1}} \operatorname{Tr}\left(T^{a_{\sigma(1)}}\dots T^{a_{\sigma(n-1)}}T^{a_n}\right) H(u_{\sigma(1)} < \dots < u_{\sigma(n-1)} < u_n)\,, \tag{4.3.4}$$

where H is a boolean Heaviside step function. This was demonstrated by Bern and Kosower in [61–64], where they proved that $1/\bar{q}$ residue identities tie particular combinations of the color factors to a given ordering of the external legs along the loop.[50]

It should be recalled now that the left-right mixing contractions present in the worldline integrand $W^{(L-R,\,\text{kin})}$ descend from string theory contractions such as $\langle \partial X(z,\bar{z}) \bar{\partial} X(w,\bar{w})\rangle$ as in eq. (3.1.11). In the field theory limit, they solely provide a $1/T$ factor, since the $\delta^{(2)}$-function drops out of the amplitude by the canceled propagator argument just like at tree-level:

$$\langle \partial X(z,\bar{w}) \bar{\partial} X(z,\bar{w})\rangle \underset{\alpha'\to 0}{\longrightarrow} -\frac{2}{T} \tag{4.3.5}$$

up to a global factor of α'^2 required for dimensionality. We give below the explicit worldline numerators for the $(\mathcal{N}=2)$ hyper multiplet[51] and also recall the form of the symmetric worldline integrand for the $(\mathcal{N}=4)$ matter multiplets

$$\begin{aligned} W_{(\mathcal{N}=2),\ \text{hyper}} &= W_3\,, \\ W_{(\mathcal{N}=4),\ \text{matt.}} &= W_3^2 + 1/2 W_2\,, \end{aligned} \tag{4.3.6}$$

where the worldline integrands W_2 and W_3 were defined in (3.1.36).

[50] In open string gauge theory amplitudes, color-ordering naturally follows from ordering of the external states along the boundary of the annulus.

[51] Similar computations as these performed in the gravity amplitudes can be performed to derive this polynomial in $\mathcal{N}=2$ orbifolds of the heterotic string, in which case one should make sure to decouple the gravitational multiplets by hand. Another possibility is to use $W_{\mathcal{N}=4,\text{vect}} = 1$ in the identity $(\mathcal{N}=2,\text{hyper}) \times (\mathcal{N}=4,\text{vector}) = (\mathcal{N}=6,\text{spin-3/2})$ to obtain $W_{\mathcal{N}=2,\text{hyper}} = W_{\mathcal{N}=6,\text{spin-3/2}}$.

4.3.3 Comparing the integrands

In [PT4], we carried the comparison of the integrands coming from the BCJ construction to the worldline one by turning the loop momentum representation to a Schwinger proper time representation.[52] This procedure was already sketched in chap. 2, eq. (2.2.27), when we needed to illustrate the generic form of a worldline integrand in terms of more common Feynman graphs quantities. We defined $\langle n\rangle$ to be the result of loop-momentum Gaussian-integration of a given numerator $n(\ell)$ after exponentiating the propagators. For a detailed account at one-loop, the reader is referred to the section 6.1 of [PT4]. For definiteness, let us simply reproduce here the defining equation for the bracket notation $\langle n\rangle$:

$$\int \frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{n(\ell)}{\ell^2(\ell-k_1)^2\dots(\ell-\sum_{i=1}^{n-1}k_i)^2}=\frac{(-1)^n i}{(4\pi)^{D/2}}\int_0^\infty\frac{\mathrm{d}T}{T^{D/2-(n-1)}}\int\prod_{i=1}^{n-1}\mathrm{d}u_i\langle\mathbf{n}\rangle e^{-TQ}\,. \tag{4.3.7}$$

which appears in boldface for readability in this equation. The left-hand side of this formula is a n-leg ($n=3,4$ here) one-loop Feynman integral in momentum space while the right-hand side is its Schwinger proper time representation. We recall that the u_i parameters are rescaled (see eq. (3.1.14)) so that they belong to $[0,1]$. Their ordering along the worldloop corresponds to the ordering indicated by the Feynman propagators in the left-hand side.

Gauge theory The first step of the analysis is to compare the gauge theory box integrand $\langle n_{\mathrm{box}}\rangle$ obtained from the BCJ procedure to the string based numerator W_3.[53] We observe matching of the two quantities only up to a new total derivative that we call δW_3:

$$\langle n_{\mathrm{box}}\rangle = W_3+\delta W_3\,. \tag{4.3.8}$$

This δW_3 integrates separately to zero in each color ordered sector of the amplitude. Moreover, it is sensitive to the subset of generalized gauge invariance left-over from solving the unitarity cut-constraints for the ansatz as it depends on α and β. A natural interpretation for this term is that it carries some information from the BCJ representation to the string integrand and indicates that the correct BCJ representation in string theory is not W_3 but $W_3+\delta W_3$.

From our experience of the MSS procedure at tree-level, we would expect the addition of this total derivative term to be the result of worldsheet integration by part. However, in [PT4] we argued that this is not the case; $W_3+\delta W_3$ cannot be the result of any chain of IBP's. The argument is based on a rewriting δW_3 as a worldline polynomial in the derivatives of the Green's function,[54] followed by the observation that this polynomial cannot be integrated away because of the presence of squares $\dot G_{ij}^2$ not paired with the required $\ddot G_{ij}$ which would make them originating from $\partial_i(\dot G_{ij}e^{-TQ})$.[55] The reason why

[52]In principle, it would have been desirable to perform the inverse procedure. However we faced technical obstacles in doing so, because of the quadratic nature of the gauge theory loop-momentum polynomials. Furthermore, the absence of triangles in string theory was also a severe issue to match the BCJ loop momentum triangles.

[53]We recall that the gauge theory triangle integrand vanish once the loop momentum is integrated, in other words we have $\langle n_{\mathrm{tri}}\rangle=0$ for all BCJ triangles.

[54]The complete expression may be found in appendix D of [PT4].

[55]See the discussion above and below (6.24) in [PT4].

there are no room for such terms as $\ddot{G}$ in δW_3 is related to the form of our box numerators, whose quadratic part in the loop-momentum turns out to be traceless. Ultimately, this is again a consequence of our restriction to discard bubble integrals in our gauge theory ansatz.

The first conclusion of this gauge theory analysis is that the BCJ representation is visible at the integrand level in string theory, as shows the necessity to select a particular representation. The second conclusion is that, contrary to the intuition from the MSS procedure, there seem to exist particular BCJ representations which cannot be reached directly from string theory, or at least not with solely "naive" IBP's.

Gravity At the gravity level, we compare the BCJ double-copy and string-based integrated results. They give schematically:

$$\int \sum \langle n_{\text{box}}^2 \rangle + \sum \langle n_{\text{tri}}^2 \rangle = \int W_3^2 + 1/2 W_2 \,. \tag{4.3.9}$$

The physical intuition that we have been following so far tells us that loop momentum total derivatives in the BCJ representation in gauge theory, which contribute after squaring, should match the new left-right mixing term W_2 arising in the string-based gravity amplitude. Therefore, we expect the triangles $\langle n_{\text{tri}}^2 \rangle$ and the parity-odd terms present in $\langle n_{\text{box}}^2 \rangle$ and $\langle n_{\text{tri}}^2 \rangle$ to be related to W_2. To understand this relation, it is necessary to use our knowledge gained in the analysis of the gauge theory integrands to first relate $\langle n_{\text{box}}^2 \rangle$ to W_3^2. Since we already argued that no IBP procedure may transform W_3 to $\langle n_{\text{box}} \rangle$, the best we can do is to introduce and remove by hand δW_3 in (4.3.9), which transforms W_3 to $W_3 + \delta W_3 = \langle n_{\text{box}} \rangle$ while the W_2 is modified to turn $W_2 \to W_2 + \delta W_2$ with

$$\delta W_2 = -2(2\delta W_3 W_3 + W_3^2) \,. \tag{4.3.10}$$

Contrary to δW_3, this new term *is not* a total derivative. This is expected, since its integral does not vanish. In total we obtain

$$\boxed{\int W_2 + \delta W_2 = \int \sum \langle n_{\text{tri}}^2 \rangle + \left(\langle n_{\text{box}}^2 \rangle - \langle n_{\text{box}} \rangle^2 \right))} \tag{4.3.11}$$

An interesting combination, $(\langle n_{\text{box}}^2 \rangle - \langle n_{\text{box}} \rangle^2))$, appears in the right-hand side of the previous equation. This term is computed in detail in subsection 6.3.2 of [PT4], by Gaussian integration of the loop momentum. In particular it contains contribution coming from the parity-odd terms and other total derivatives. However, its appearance is more generic than this and actually signals the non-commutativity of the squaring operation in loop momentum space and in Schwinger proper time space. Therefore, any string theory procedure supposed to explain the origin of the BCJ double-copy should elucidate the nature of these "square-correcting terms".

The difficulties caused by the non-IBP nature of δW_3 and δW_2 prevented us from pushing the quantitative analysis much further. However, in our conclusive remarks below we provide a qualitative statement based on the fact that the square-correcting terms are always of order $1/T$ at least (this can be proven by direct Gaussian integration).

Before, let us make one more comment. So far we did not describe the worldline properties of δW_2 and δW_3, besides explaining that we could rewrite δW_3 as a polynomial of in the derivatives of the worldline propagator. This implies that the same can be done for δW_2. By doing so, we mean that these polynomials, δW_2 and δW_3, are well defined worldline quantities and we are implicitly pretending that they descend from certain string theoretic ancestors, obtained by turning the G's for $\mathcal{G}$'s. However, nothing grants us from the start that the corresponding $\delta\mathcal{W}_2$ and $\delta\mathcal{W}_3$ would not produce triangles or bubbles in the field theory limit due to vertex operator colliding as in eq. (2.2.45). This would spoil a correct worldline interpretation for these corrections. Hence we had to carefully check this criterion for both polynomials, which they turn out to pass; in [PT4], this property was referred to as the *string-ancestor-gives-no-triangles* criterion. The conclusion of this paragraph gives strength to interpreting the δW's as "stringy" reactions to the BCJ change of parametrization in gauge an gravity amplitudes.

Conclusive remarks We can now conclude. The formula eq. (4.3.11) illustrates that the modified left-right moving contractions, $W_2 + \delta W_2$, are related to two terms in field theory; the BCJ triangles squared and the square-correcting terms.

Noting that the square correcting terms do contain in particular the squares of the parity-odd terms, we are lead to our first conclusion, which confirms our physical intuition; the left-right mixing contractions in string theory, modified by the BCJ representation, account for the need to include total derivatives in the BCJ representation of gauge theory amplitudes.

The second important conclusion is linked to the change of representation that we found, which we argued to be a non-IBP type of modification. At tree-level, the MSS paradigm consists in performing integrations by parts on the gauge theory integrands to put them in a particular representation (see eq. (4.2.8)). At one-loop, integrations-by-part produce additional left-right mixing contractions when ∂ derivatives hit $\bar{\partial}\mathcal{G}$ terms, which eventually give rise to worldline terms with $1/T$ factors (see eq. (4.3.5)). In view of our previous comment on the $1/T$ order of the square-correcting terms, it is natural to expect that these terms actually indicate missing worldsheet IBP's in the term $W_2 + \delta W_2$. Therefore, we face a paradox; on the one hand, no IBP can be done to produce the δW's, on the other hand the final result seem to lack IBP's.

A possible way out might lie in the definition of the ansatz itself. More precisely, the issues might be caused by a mutual incompatibility of the gauge choice in string theory producing the worldline integrand W_3 and forbidding triangle/bubble-like contributions with the choice of an ansatz constrained by discarding all bubbles, thereby producing BCJ triangles as total derivatives only. Put differently, the absence of triangle contributions in the string-based computation that lead us to consequently restrict the full generalized gauge invariance is possibly not the most natural thing to do from string theory viewpoint on the BCJ double-copy. Then what have we learned ? It seems that string theory is not compatible with certain too stringent restrictions of generalized gauge invariance. A more general quantitative analysis of this issue will certainly give interesting results on which of BCJ-ansatzes are natural from string theory and which are not, hopefully helping to find new ansatzes.

Chapter 5

Outlook

One of the aims of this manuscript was to draw a coherent scheme in the work of the author during his years of PhD. Their remain open questions after these works, in addition to the new ones that were raised. I would like to describe a few of them now. As they are related to several chapters at the same time, there is not point anymore in sectioning the text according to the previous chapters.

Role of the $U(1)$ anomaly We emphasized in the text that half-maximal supergravity has a $U(1)$ anomalous symmetry of the axio-dilaton parametrizing the coset $SU(1,1)/U(1)$ [54]. The direct computation of the four-loop $\nabla^2 R^4$ divergence in $D = 4-2\epsilon$ dimensions of [130] using the double-copy $(\mathcal{N}=0)\times(\mathcal{N}=4)$ also shows traces of this anomalous behavior, according to the authors of this work. Let us reproduce the amplitude here in order to recapitulate their reasoning:

$$\mathcal{M}_{n_v}^{\text{four-loop}}\Big|_{\text{div.}} = \frac{(\kappa_D/2)^{10}}{(4\pi)^8}\frac{(n_v+2)}{2304}\left[\frac{6(n_v+2)n_v}{\epsilon^2}+\frac{(n_v+2)(3n_v+4)-96(22-n_v)\zeta_3}{\epsilon}\right]\mathcal{T}\,,$$

where $\mathcal{T}$ encodes the polarization dependence of the amplitude in a covariant manner. The $(n_v+2)\zeta_3$ contribution is the important term here. On one hand, (n_v+2) was argued to be typical of anomalous one-loop amplitudes [129], on the other hand ζ_3 is a 3-loop object, therefore the four-loop divergence carried by $(n_v+2)\zeta_3$ does seem to be caused by the anomaly. It would be really interesting to investigate this issue further, below we describe possible topics of research related to it.

Extract exactly the coupling of R^4 and $\nabla^2 R^4$ in the CHL heterotic string action ? A computation that would shed light in this direction is to determine the exact value of the R^4 and $\nabla^2 R^4$ couplings in the effective action of CHL models. The program in $\mathcal{N}=8$ led to major advances both in physics and in mathematics, and it is very reasonable to expect that the similar program in $\mathcal{N}=4$ would imply the discovery of new kind of automorphic forms for orthogonal groups.

What is the significance of the $N = 23$ CHL orbifold ? This point is more speculative. We mentioned that an $N=23$ CHL model would decouple totally the matter fields, hence producing pure half-maximal supergravity from the start. The Mathieu moonshine program seems to indicate that there may exist such a model, as a consequence of a putative $\mathbb{M}_{24}$ fundamental symmetry of ... something. At the moment, it is not clear what theory the Mathieu group $\mathbb{M}_{24}$ could be a symmetry group of. It is known however that it cannot be the symmetry group of K_3 sigma models, preventing naive interpretations of this sort [127]. Maybe uncovering deeper aspects of these connexions may lead to powerful group theoretical arguments on the low energy effective action of pure half-maximal supergravity ?

Build some $4 \leq \mathcal{N} < 8$ orbifolds models in pure spinor superstring and extract non-renormalization theorems via zero-mode counting ? Another option to understand the role of the $U(1)$ anomaly, suggested by the authors of [130], would be to perform similar type of analysis in $\mathcal{N} \geq 5$ supergravities, where the anomalous symmetry is not present. From the superstring point of a view, such an analysis would most easily be performed by constructing asymmetric orbifolds models in the pure spinor superstring and perform systematically the zero-mode counting in the lines of [17].

Extract exactly the three-loop four-graviton amplitude in type II ? Going to the tropical limit program now, a very important computation to do is to extract explicitly the worldline numerators for the three-loop computation in type II of [72]. In addition to the intrinsic interest of such a computation, it may help to understand the apparent paradox between the supermoduli space non-projectedness issues in the RNS formalism and the bosonic moduli space integration of pure spinor formalism.

Extract exactly the two-loop four-graviton amplitude in CHL models of heterotic string ? The genus two case is really the turning point in terms of the technical machinery involved in extracting the tropical limit of string amplitudes formulated as integrals over $\mathcal{M}_{g,n}$. Therefore, developing the tropical limit technology enough to be able to extract the complete form of the worldline integrand of the two-loop heterotic string amplitude would settle the last subtleties with this aspect of the $\alpha' \to 0$ limit (at least in the non-analytic domains).

Towards a super-tropical geometry ? The analysis of [86–91] has shown that the non-projectedness of $\mathfrak{M}_{g,n}$ implies that the IR behavior of RNS superstring theory is naturally described by means of super-dual-graphs which characterize the holomorphic degenerations of the super-Riemann surfaces. They basically account for what kind of states, NSNS, RNS, NSR and RR are exchanged through the pinching necks. The corresponding super-dual-graphs in type II for instance are then built out of the following vertices and their weighted n-point generalization. It would be interesting and certainly

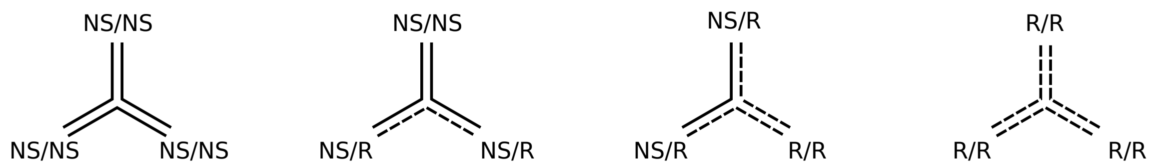


Figure 5.1: “Superworldline Feynman rules” in type II.

helpful to formulate in more details this super worldline picture for arbitrary RNS string theory amplitudes. Comparison with the pure spinor *worldline* formalism of [26] may then help to understand the connexions between the various perturbative formalisms in string theory.

Double-copy; find a constructive way at one-loop ? The question of understanding the nature of the generalized gauge invariance in string theory is conceptually important, as it may be used as a guideline for the direct ansatz approaches. Another result that hopefully may follow from a string theory analysis would be a procedure to derive BCJ numerators at loop level from first principles, in the lines of the tree-level MSS construction.

Is there any string theoretic understanding of the difficulties at five loop ? In the paradigm where we consider string theory as a natural framework where to understand the BCJ duality, it would be natural to assume that the supermoduli space discussion of [86–91] may have an impact on the BCJ duality, for instance by involving variations of the Jacobi identities ? A way to probe this statement would be to identify an amplitude in the RNS formalism that has to involve the super-graph picture in the low energy limit, and investigate if there are signs of a breakdown or alteration of the duality or of the double-copy.

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An R^4 non-renormalisation theorem in $\mathcal{N} = 4$ supergravity

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Abstract

We consider the four-graviton amplitudes in CHL constructions providing four-dimensional $\mathcal{N} = 4$ models with various numbers of vector multiplets. We show that in these models the two-loop amplitude has a prefactor of $\partial^2 R^4$. This implies a non-renormalisation theorem for the R^4 term, which forbids the appearance of a three-loop ultraviolet divergence in four dimensions in the four-graviton amplitude. We connect the special nature of the R^4 term to the $U(1)$ anomaly of pure $\mathcal{N} = 4$ supergravity.

I. INTRODUCTION

$\mathcal{N}=4$ supergravity in four dimensions has sixteen real supercharges and $SU(4)$ for R-symmetry group. The gravity supermutiplet is composed of a spin 2 graviton and two spin 0 real scalars in the singlet representation of $SU(4)$, four spin 3/2 gravitini and four spin 1/2 fermions in the fundamental representation $\mathbf{4}$ of $SU(4)$, and six spin 1 gravi-photons in the $\mathbf{6}$ of $SU(4)$. The only matter multiplet is the vector multiplet composed of one spin 1 vector which is $SU(4)$ singlet, four spin 1/2 fermions transforming in the fundamental of $SU(4)$, and six spin 0 real scalars transforming in the $\mathbf{6}$ of $SU(4)$. The vector multiplets may be carrying non-Abelian gauge group from a $\mathcal{N}=4$ super-Yang-Mills theory.

Pure $\mathcal{N}=4$ supergravity contains only the gravity supermultiplet and the two real scalars can be assembled into a complex axion-dilaton scalar S parametrizing the coset space $SU(1,1)/U(1)$. This multiplet can be coupled to n_v vector multiplets, whose scalar fields parametrize the coset space $SO(6,n_v)/SO(6)\times SO(n_v)$ [1].

$\mathcal{N}=4$ supergravity theories can be obtained by consistent dimensional reduction of $\mathcal{N}=1$ supergravity in $D=10$, or from various string theory models. For instance the reduction of the $\mathcal{N}=8$ gravity super-multiplet leads to $\mathcal{N}=4$ gravity super-multiplet, four spin 3/2 $\mathcal{N}=4$ super-multiplets, and six vector multiplets

$$\begin{aligned}(2_{\mathbf{1}},3/2_{\mathbf{8}},1_{\mathbf{28}},1/2_{\mathbf{56}},0_{\mathbf{70}})_{\mathcal{N}=8} &= (2_{\mathbf{1}},3/2_{\mathbf{4}},1_{\mathbf{6}},1/2_{\mathbf{4}},0_{\mathbf{1+1}})_{\mathcal{N}=4}\\ &\oplus\ \mathbf{4}\,(3/2_{\mathbf{1}},1_{\mathbf{4}},1/2_{\mathbf{6+1}},0_{\mathbf{4+\bar{4}}})_{\mathcal{N}=4}\\ &\oplus\ \mathbf{6}\,(1_{\mathbf{1}},1/2_{\mathbf{4}},0_{\mathbf{6}})_{\mathcal{N}=4}\,.\end{aligned} \tag{I.1}$$

Removing the four spin 3/2 $\mathcal{N}=4$ supermultiplets leads to $\mathcal{N}=4$ supergravity coupled to $n_v=6$ vector multiplets.

In order to disentangle the contributions from the vector multiplets and the gravity supermultiplets, we will use CHL models [2–4] that allow to construct $\mathcal{N}=4$ four dimensional heterotic string with gauge groups of reduced rank. In this paper we work at a generic point of the moduli space in the presence of (diagonal) Wilson lines where the gauge group is Abelian.

Various CHL compactifications in four dimensions can obtained by considering $\mathbb{Z}_N$ orbifold [3, 5, 6] of the heterotic string on $T^5\times S^1$. The orbifold acts on the current algebra and the right-moving compactified modes of the string (world-sheet supersymmetry is on the left moving sector) together with an order N shift along the S^1 direction. This leads to four-dimensional $\mathcal{N}=4$ models with $n_v = 48/(N+1) - 2$ vector multiplets at a generic point of the moduli space. Models with $(n_v, N) \in \{(22,1),(14,2),(10,3),(6,5),(4,7)\}$ have been constructed. No no-go theorem are known ruling out the $n_v = 0$ case although it will probably not arise from an asymmetric orbifold construction.[1]

It was shown in [7–9] that $t_8\mathrm{tr}(R^4)$ and $t_8\mathrm{tr}(R^2)^2$ are half-BPS statured couplings of the heterotic string, receiving contributions only from the short multiplet of the $\mathcal{N}=4$ super-algebra, with no perturbative corrections beyond one-loop. These non-renormalisation theorems were confirmed in [10] using the explicit evaluation of the genus-two four-graviton heterotic amplitude derived in [11–13]. For the CHL models, the following fact is crucially important: the orbifold action does not alter the left moving supersymmetric sector of the theory. Hence, the fermionic zero mode saturation will happen in the same manner as it does for the toroidally compactified heterotic string, as we show in this paper.

Therefore we prove that the genus-two four-graviton amplitude in CHL models satisfy the same non-renormalisation theorems, due to the factorization at the integrand level of the mass dimension ten $\partial^2 R^4$ operator in each kinematic channel. By taking the field theory limit of this amplitude in four dimensions, no reduction of derivative is found for generic numbers of vector multiplets n_v. Since this result is independent of n_v, we conclude that this rules out the appearance of a R^4 ultraviolet counter-term at three-loop order in four dimensional pure $\mathcal{N}=4$ supergravity as well. Consequently, the four-graviton scattering amplitude is ultraviolet finite at three loops in four dimensions.

The paper is organized as follows. In section II we give the form of the one- and two-loop four-graviton amplitude in orbifold CHL models. Then, in section III we evaluate their field theory limit in four dimensions. This gives us the scattering amplitude of four gravitons in $\mathcal{N}=4$ supergravity coupled to n_v vector multiplets. In section IV we discuss the implication of these results for the ultraviolet properties of pure $\mathcal{N}=4$ supergravity.

Note: As this paper was being finalized, the preprint [14] appeared on the arXiv. In this work the absence of three-loop divergence in the four-graviton amplitude in four dimensions is obtained by a direct field theory computation.

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II. ONE- AND TWO-LOOP AMPLITUDES IN CHL MODELS

Our conventions are that the left-moving sector of the heterotic string is the supersymmetric sector, while the right-moving contains the current algebra.

We evaluate the four-graviton amplitude in four dimensional CHL heterotic string models. We show that the fermionic zero mode saturation is model independent and similar to the toroidal compactification.

A. The one-loop amplitude in string theory

The expression of the one-loop four-graviton amplitude in CHL models in $D = 10 - d$ dimensions is an immediate extension of the amplitude derived in [15]

$$\mathcal{M}^{(n_v)}_{4,1-loop} = \mathcal{N}_1 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{2-\frac{d}{2}}} \mathcal{Z}^{(n_v)}_1 \int_{\mathcal{T}} \prod_{1\leq i<j\leq 4} \frac{d^2\nu_i}{\tau_2} \mathcal{W}^{(1)} \, e^{-\sum_{1\leq i<j\leq 4} 2\alpha' k_i\cdot k_j P(\nu_{ij})} \,, \tag{II.1}$$

where $\mathcal{N}_1$ is a constant of normalisation, $\mathcal{F} := \{\tau = \tau_1 + i\tau_2, |\tau| \geq 1, |\tau_1| \leq \frac{1}{2}, \tau_2 > 0\}$ is a fundamental domain for $SL(2,\mathbb{Z})$ and the domain of integration $\mathcal{T}$ is defined as $\mathcal{T} := \{\nu = \nu^1 + i\nu^2; |\nu^1| \leq \frac{1}{2}, 0 \leq \nu^2 \leq \tau_2\}$. $\mathcal{Z}^{(n_v)}_1$ is the genus-one partition function of the CHL model.

The polarisation of the rth graviton is factorized as $h^{(r)}_{\mu\nu} = \epsilon^{(r)}_\mu \tilde\epsilon^{(r)}_\nu$. We introduce the notation $t_8F^4 := t_8^{\mu_1\cdots\mu_8} \prod_{r=1}^4 k^{(r)}_{\mu_{2r-1}} \epsilon^{(r)}_{\mu_{2r}}$. The quantity $\mathcal{W}^{(1)}$ arises from the contractions of the right-moving part of the graviton vertex operator

$$\mathcal{W}^{(1)} := t_8F^4 \, \frac{\langle \prod_{j=1}^4 \tilde\epsilon^j \cdot \bar\partial X(z_j) e^{ik_j\cdot x(z_j)}\rangle}{\langle \prod_{j=1}^4 e^{ik_j\cdot x(z_j)}\rangle} = t_8F^4 \prod_{r=1}^4 \tilde\epsilon^{(r)}_{\nu_r} \, t^{\nu_1\cdots\nu_4}_{4;1} \,, \tag{II.2}$$

with $\hat t^{\nu_1\cdots\nu_4}_{4;1}$ the quantity evaluated in [15]

$$\hat t^{\nu_1\cdots\nu_4}_{4;1} := Q_1^{\nu_1}\cdots Q_4^{\nu_4} + \frac{1}{2\alpha'}\,(Q_1^{\nu_1}Q_2^{\nu_2}\delta^{\nu_3\nu_4}T(\nu_{34}) + perms) + \frac{1}{4\alpha'^2}\,(\delta^{\nu_1\nu_2}\delta^{\nu_3\nu_4}T(\nu_{12})T(\nu_{34}) + perms) \,, \tag{II.3}$$

where

$$Q^\mu_I := \sum_{r=1}^4 k^{(r)\mu}\,\bar\partial P(\nu_{Ir}|\tau); \qquad T(\nu) := \bar\partial^2_\nu P(\nu|\tau) \,. \tag{II.4}$$

We follow the notations and conventions of [16, 17]. The genus one propagator is given by

$$P(\nu|\tau) := -\frac{1}{4}\,\log\left|\frac{\theta_1(\nu|\tau)}{\theta_1'(0|\tau)}\right|^2 + \frac{\pi(\nu^2)^2}{2\tau_2}\,. \qquad \text{(II.5)}$$

In the $\alpha' \to 0$ limit relevant for the field theory analysis in section III, with all the radii of compactification scaling like $\sqrt{\alpha'}$, the mass of the Kaluza-Klein excitations and winding modes go to infinity and the genus-one partition function $\mathcal{Z}_1^{(n_v)}$ has the following expansion in $\bar q = \exp(-2i\pi\bar\tau)$

$$\mathcal{Z}_1^{(n_v)} = \frac{1}{\bar q} + c^1_{n_v} + O(\bar q)\,. \qquad \text{(II.6)}$$

The $1/\bar q$ contribution is the "tachyonic" pole, $c^1_{n_v}$ depends on the number of vector multiplets and higher orders in $\bar q$ coming from to massive string states do not contribute in the field theory limit.

B. The two-loop amplitude in string theory

By applying the techniques for evaluating heterotic string two-loop amplitudes of [10–13], we obtain that the four-graviton amplitudes in the CHL models are given by

$$\mathcal{M}^{(n_v)}_{4,2-loop} = \mathcal{N}_2 \int \frac{|d^3\Omega|^2}{(\det \Im\mathrm{m}\Omega)^{5-\frac{d}{2}}}\, \mathcal{Z}_2^{(n_v)} \int \prod_{i=1}^4 d^2\nu_i\, \mathcal{W}^{(2)}\, \mathcal{Y}_s\, e^{-\sum_{1\le i<j\le 4} 2\alpha' k^i\cdot k^j P(\nu_{ij})} \qquad \text{(II.7)}$$

where $\mathcal{N}_2$ is a normalization constant, $\mathcal{Z}_2^{(n_v)}(\Omega,\bar\Omega)$ is the genus-two partition function and

$$\mathcal{W}^{(2)} := t_8 F^4\, \frac{\langle \prod_{j=1}^4 \epsilon^j\cdot\bar\partial X(z_j) e^{ik_j\cdot x(z_j)}\rangle}{\langle \prod_{j=1}^4 e^{ik_j\cdot x(z_j)}\rangle} = t_8 F^4 \prod_{i=1}^4 \tilde\epsilon_i^{\nu_i}\, t_{4;2}^{\nu_1\cdot\nu_4}\,. \qquad \text{(II.8)}$$

The tensor $t_{4;2}^{\nu_1\cdot\nu_4}$ is the genus-two equivalent of the genus-one tensor given in (II.3)

$$t_{4;2}^{\nu_1\cdots\nu_4} = Q_1^{\nu_1}\cdots Q_4^{\nu_4} + \frac{1}{2\alpha'} Q_1^{\nu_1} Q_2^{\nu_2} T(\nu_{34})\delta^{\nu_3\nu_4} + \frac{1}{4(\alpha')^2}\,\delta^{\nu_1\nu_2}\delta^{\nu_3\nu_4} T(\nu_{12})T(\nu_{34}) + perms\,, \qquad \text{(II.9)}$$

this time expressed in terms of the genus-two bosonic propagator

$$P(\nu_1-\nu_2|\Omega) := -\log|E(\nu_1,\nu_2|\Omega)|^2 + 2\pi(\Im\mathrm{m}\Omega)^{-1}_{IJ}\,(\Im\mathrm{m}\int_{\nu_1}^{\nu_2}\omega_I)(\Im\mathrm{m}\int_{\nu_1}^{\nu_2}\omega_J)\,, \qquad \text{(II.10)}$$

where $E(\nu)$ is the genus-two prime form, Ω is the period matrix and ω_I with $I=1,2$ are the holomorphic abelian differentials. We refer to [13, Appendix A] for the main properties of these objects.

The $\mathcal{Y}_S$ quantity, arising from several contributions in the RNS formalism and from the fermionic zero modes in the pure spinor formalism [18, 19], is given by

$$3\mathcal{Y}_S = (k_1 - k_2) \cdot (k_3 - k_4)\, \Delta_{12}\Delta_{34} + (13)(24) + (14)(23)\,, \tag{II.11}$$

with

$$\Delta(z, w) = \omega_1(z)\omega_2(w) - \omega_1(w)\omega_2(z)\,. \tag{II.12}$$

Using the identity $\Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{42} + \Delta_{14}\Delta_{23} = 0$ we have the equivalent form $\mathcal{Y}_S = -3\,(s\Delta_{14}\Delta_{23} - t\Delta_{12}\Delta_{34})$, where $s = (k_1 + k_2)^2$, $t = (k_1 + k_4)^2$ and $u = (k_1 + k_3)^2$.

We use a parametrisation of the period matrix reflecting the symmetries of the field theory vacuum two-loop diagram considered in the next section

$$\Omega := \begin{pmatrix} \tau_1 + \tau_3 & \tau_3 \\ \tau_3 & \tau_2 + \tau_3 \end{pmatrix}\,. \tag{II.13}$$

With this parametrisation the expression for $\mathcal{Z}_2^{(n_v)}(\Omega, \bar\Omega)$ is completely symmetric in the variables $q_I = \exp(2i\pi\tau_I)$ with $I = 1, 2, 3$.

In the limit relevant for the field theory analysis in section III, the partition function of the CHL model has the following $\bar q_i$-expansion [20]

$$\mathcal{Z}_2^{(n_v)} = \frac{1}{\bar q_1 \bar q_2 \bar q_3} + a_{n_v} \sum_{1\leq i<j\leq 3} \frac{1}{\bar q_i \bar q_j} + b_{n_v} \sum_{1\leq i\leq 3} \frac{1}{\bar q_i} + c_{n_v} + O(q_i)\,. \tag{II.14}$$

III. THE FIELD THEORY LIMIT

In this section we extract the field theory limit of the string theory amplitudes compactified to four dimensions. We consider the low-energy limit $\alpha' \to 0$ with the radii of the torus scaling like $\sqrt{\alpha'}$ so that all the massive Kaluza-Klein states, winding states and excited string states decouple.

In order to simplify the analysis we make the following choice of polarisations $(1^{++}, 2^{++}, 3^{--}, 4^{--})$ and of reference momenta[2] $q_1 = q_2 = k_3$ and $q_3 = q_4 = k_1$, such that $2t_8F^4 = \langle k_1\, k_2\rangle^2\, [k_3\, k_4]^2$, and $4t_8t_8R^4 = \langle k_1\, k_2\rangle^4\, [k_3\, k_4]^4$. With these choices the expression for $\mathcal{W}^{(g)}$ reduces to

[2] Our conventions are that a null vector $k^2 = 0$ is parametrized by $k_{\alpha\dot\alpha} = k_\alpha \bar k_{\dot\alpha}$. The spin 1 polarisations of positive and negative helicities are given by $\epsilon^+(k, q)_{\alpha\dot\alpha} := \frac{q_\alpha \bar k_{\dot\alpha}}{\sqrt{2}\,\langle q\, k\rangle}$, $\epsilon^-(k, q)_{\alpha\dot\alpha} := -\frac{k_\alpha \bar q_{\dot\alpha}}{\sqrt{2}\,[q\, k]}$, where q is a reference momentum. One finds that $t_8F^{(1)+}\cdots F^{(4)+} = t_8F^{(1)-}\cdots F^{(4)-} = 0$ and $t_8F^{(1)-}F^{(2)-}F^{(3)+}F^{(4)+} = \frac{1}{16}\langle k_1\, k_2\rangle^2\, [k_3\, k_4]^2$

$$
\begin{aligned}
\mathcal{W}^{(g)} &= t_8t_8R^4\,(\bar\partial P(\nu_{12})-\bar\partial P(\nu_{14}))(\bar\partial P(\nu_{21})-\bar\partial P(\nu_{24}))(\bar\partial P(\nu_{32})-\bar\partial P(\nu_{34}))(\bar\partial P(\nu_{42})-\bar\partial P(\nu_{43}))\\
&\quad+\frac{t_8t_8R^4}{u}\,\bar\partial^2 P(\nu_{24})(\bar\partial P(\nu_{12})-\bar\partial P(\nu_{14}))(\bar\partial P(\nu_{32})-\bar\partial P(\nu_{34}))\,, \qquad \text{(III.1)}
\end{aligned}
$$

where $s=(k_1+k_2)^2$, $t=(k_1+k_4)^2$ and $u=(k_1+k_3)^2$. We introduce the notation $\mathcal{W}^{(g)}=t_8t_8R^4\,(\mathcal{W}_1^{(g)}+u^{-1}\mathcal{W}_2^{(g)})$.

The main result of this section is that the one-loop amplitudes factorizes a $t_8t_8R^4$ and that the two-loop amplitudes factorizes a $\partial^2t_8t_8R^4$ term. A more detailed analysis will be given in the work [20].

A. The one-loop amplitude in field theory

In the field theory limit $\alpha'\to 0$ and $\tau_2\to\infty$ with $t=\alpha'\tau_2$ fixed, we define $\nu^2=\tau_2\,\omega$ for $\nu=\nu^1+i\nu^2$.

Because of the $1/\bar q$ pole in the partition function (II.6) the integration over τ_1 yields two contributions

$$
\int_{-\frac12}^{\frac12} d\tau_1\,\mathcal{Z}_1^{(n_v)}F(\tau,\bar\tau)=F_1+c^1_{n_v}\,F_0\,, \qquad \text{(III.2)}
$$

where $F(\tau,\bar\tau)=F_0+\bar qF_1+c.c.+O(\bar q^2)$ represents the integrand of the one-loop amplitude.

The bosonic propagator can be split in an asymptotic value for $\tau_2\to\infty$ (the field theory limit) and a correction [16]

$$
P(\nu|\tau)=P^\infty(\nu|\tau)+\hat P(\nu|\tau) \qquad \text{(III.3)}
$$

that write:

$$
\begin{aligned}
P^\infty(\nu|\tau)&=\frac{\pi(\nu^2)^2}{2\tau_2}-\frac14\ln\left|\frac{\sin(\pi\nu)}{\pi}\right|^2\\
\hat P(\nu|\tau)&=-\sum_{m\geq1}\left(\frac{q^m}{1-q^m}\frac{\sin^2(m\pi\nu)}{m}+c.c.\right)+C(\tau), \qquad \text{(III.4)}
\end{aligned}
$$

where $q=\exp(2i\pi\tau)$ and $C(\tau)$ is a zero mode contribution which drops out of the amplitude due to the momentum conservation [16].

We decompose the asymptotic propagator $P^\infty(\nu|\tau)=\frac{\pi}{2}\,\tau_2\,P^{FT}(\omega)+\delta_s(\nu)$ into a piece that will dominate in the field theory limit

$$P^{FT}(\omega) = \omega^2 - |\omega|\,, \tag{III.5}$$

and a contribution $\delta_s(\nu)$ from the massive string modes [16, appendix A]

$$\delta_s(\nu) := \sum_{m\neq 0} \frac{1}{4|m|}\, e^{2i\pi m\nu^1 - 2\pi|m\nu^2|}\,. \tag{III.6}$$

The expression for Q_I^μ and T in (II.4) become

$$\begin{aligned} Q_I^\mu &= Q_I^{FT\,\mu} + \delta Q_I^\mu - \pi \sum_{r=1}^{4} k^{(r)\mu}\, \sin(2\pi\bar\nu_{Ir})\,\bar q + o(\bar q^2) \\ T(\bar\nu) &= T^{FT}(\omega) + \delta T(\bar\nu) + 2\pi\cos(2\pi\bar\nu)\,\bar q + o(\bar q^2)\,, \end{aligned} \tag{III.7}$$

where

$$Q_I^{FT\,\mu} := -\frac{\pi}{2}\,(2K^\mu + q_I^\mu) \tag{III.8}$$

$$K^\mu := \sum_{r=1}^{4} k^{(r)\mu}\,\omega_r \tag{III.9}$$

$$q_I^\mu := \sum_{r=1}^{4} k^{(r)\mu}\,\mathrm{sign}(\omega_I - \omega_r) \tag{III.10}$$

$$T^{FT}(\omega) \;=\; \frac{\pi\alpha'}{t}(1 - \delta(\omega))\,, \tag{III.11}$$

and

$$\begin{aligned} \delta Q_I^\mu(\bar\nu) &= \sum_{r=1}^{4} k^{(r)\mu}\,\bar\partial\delta_s(\bar\nu_{Ir}) = -\frac{i\pi}{2}\sum_{r=1}^{4}\mathrm{sign}(\nu_{Ir}^2)k^{(r)\mu}\sum_{m\geq 1} e^{-\mathrm{sign}(\nu_{Ir}^2)\,2i\pi m\bar\nu_{Ir}} \\ \delta T(\nu) &= \bar\partial^2\delta_s(\bar\nu) = -\pi^2\sum_{m\geq 1} m\, e^{-\mathrm{sign}(\nu_{Ir}^2)\,2i\pi m\bar\nu_{Ir}}\,. \end{aligned} \tag{III.12}$$

We introduce the notation

$$Q^{(1)}(\omega) := \sum_{1\leq i<j\leq 4} k_i\cdot k_j\, P^{FT}(\omega_{ij})\,, \tag{III.13}$$

such that $\partial_{\omega_i}Q^{(1)} = k_i\cdot Q_i^{FT}$.

In the field theory limit $\alpha' \to 0$ the integrand of the string amplitude in (II.1) becomes

$$M_{4;1}^{(n_v)} = N_1\, t_8 t_8 R^4 \int_0^\infty \frac{d\tau_2}{\tau_2^{2-\frac{d}{2}}} \int_{\Delta_\omega} \prod_{i=1}^{3} d\omega_i\ e^{t\, Q^{(1)}(\omega)} \times \tag{III.14}$$

$$\times \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1\, \frac{1 + c^1_{n_v}\bar q + o(\bar q^2)}{\bar q}\, (\mathcal{W}_1^{(1)} + \frac{1}{u}\mathcal{W}_2^{(1)})\ \times$$

$$\times \exp\left(\sum_{1\le i<j\le 4} 2\alpha'\, k_i\cdot k_j\ \left(\delta_s(\nu_{ij}) - \sum_{m\ge 1} \bar q\ \sin^2(\pi\bar\nu_{ij}) + O(\bar q) \right)\right)\,,$$

here N_1 is a constant of normalisation. The domain of integration $\Delta_\omega = [0,1]^3$ is decomposed into three regions $\Delta_w = \Delta_{(s,t)} \cup \Delta_{(s,u)} \cup \Delta_{(t,u)}$ given by the union of the (s,t), (s,u) and (t,u) domains. In the $\Delta_{(s,t)}$ domain the integration is performed over $0 \le \omega_1 \le \omega_2 \le \omega_3 \le 1$ where $Q^{(1)}(\omega) = -s\omega_1(\omega_3 - \omega_2) - t(\omega_2 - \omega_1)(1-\omega_3)$ with equivalent formulas obtained by permuting the external legs labels in the (t,u) and (s,u) regions (see [16] for details).

The leading contribution to the amplitude is given by

$$M_{4;1}^{(n_v)} = N_1\, t_8 t_8 R^4 \int_0^\infty \frac{d\tau_2}{\tau_2^{2-\frac{d}{2}}} \int_{\Delta_\omega} \prod_{i=1}^{3} d\omega_i\ e^{t\, Q^{(1)}(\omega)} \times \tag{III.15}$$

$$\times \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1 \left(\left(\mathcal{W}_1^{(1)} + \frac{1}{u}\mathcal{W}_2^{(1)}\right)\Big|_0 (c^1_{n_v} - \sum_{1\le i<j\le 4} 2\alpha'\, k_i\cdot k_j\ \sin^2(\pi\bar\nu_{ij})) + \left(\mathcal{W}_1^{(1)} + \frac{1}{u}\mathcal{W}_2^{(1)}\right)\Big|_1 \right)\,,$$

where $(\mathcal{W}_1^{(1)} + \frac{1}{u}\mathcal{W}_2^{(1)})|_0$ and $(\mathcal{W}_1^{(1)} + \frac{1}{u}\mathcal{W}_2^{(1)})|_1$ are respectively the zeroth and first order in the $\bar q$ expansion of $\mathcal{W}_i^{(1)}$.

Performing the integrations over the ν_i^1 variables leads to the following structure for the amplitude reflecting the decomposition in (I.1)

$$M_{4;1}^{(n_v)} = N_1 \frac{\pi^4}{4}\left(c^1_{n_v}\, M_{4;1}^{\mathcal{N}=4\ \mathrm{matter}} + M_{4;1}^{\mathcal{N}=8} - 4 M_{4;1}^{\mathcal{N}=4\ spin\ \frac{3}{2}} \right)\,. \tag{III.16}$$

The contribution from the $\mathcal{N}=8$ supergravity multiplet is given by the quantity evaluated in [21]

$$M_{4;1}^{\mathcal{N}=8} = t_8 t_8 R^4 \int_{\Delta_\omega} d^3\omega\, \Gamma\left(2+\epsilon\right)\, (Q^{(1)})^{-2-\epsilon}\,, \tag{III.17}$$

where we have specified the dimension $D = 4 - 2\epsilon$ and $Q^{(1)}$ is defined in (III.13). The contribution from the $\mathcal{N}=4$ matter fields vector multiplets is

$$M_{4;1}^{\mathcal{N}=4\ matter} = t_8 t_8 R^4\, \frac{\pi^4}{16} \int_{\Delta_\omega} d^3\omega \Big[\Gamma\left(1+\epsilon\right)\, (Q^{(1)})^{-1-\epsilon}\, W_2^{(1)} + \Gamma\left(2+\epsilon\right)\, (Q^{(1)})^{-2-\epsilon}\, W_1^{(1)}\Big] \tag{III.18}$$

where $W_i^{(1)}$ with $i=1,2$ are the field theory limits of the $\mathcal{W}_i^{(1)}$'s

$$\begin{aligned} W_2^{(1)} &= \frac{1}{u}\, (2\omega_2 - 1 + \mathrm{sign}(\omega_3-\omega_2))(2\omega_2 - 1 + \mathrm{sign}(\omega_1-\omega_2))\,(1-\delta(\omega_{24}))\\ W_1^{(1)} &= 2(\omega_2-\omega_3)(\mathrm{sign}(\omega_1-\omega_2) + 2\omega_2 - 1)\,\times\\ &\quad\times\,(\mathrm{sign}(\omega_2-\omega_1) + 2\omega_1 - 1)(\mathrm{sign}(\omega_3-\omega_2) + 2\omega_2 - 1)\,. \end{aligned} \tag{III.19}$$

Finally, the $\mathcal{N}=6$ spin $3/2$ gravitino multiplet running in the loop gives

$$M_{4;1}^{\mathcal{N}=6\ \mathrm{spin}\ \frac{3}{2}} = t_8 t_8 R^4 \int_{\Delta_\omega} d^3\omega\, \Gamma\left(2+\epsilon\right) \tilde W_2^{(1)}\, (Q^{(1)})^{-2-\epsilon}, \tag{III.20}$$

where

$$\begin{aligned} \tilde W_2^{(1)} &= (\mathrm{sign}(\omega_1-\omega_2) + 2\omega_2 - 1)(\mathrm{sign}(\omega_2-\omega_1) + 2\omega_1 - 1)\\ &\quad + (\mathrm{sign}(\omega_3-\omega_2) + 2\omega_2 - 1)(\omega_3-\omega_2)\,. \end{aligned} \tag{III.21}$$

The $\mathcal{N}=6$ spin $3/2$ supermultiplet is the sum of a $\mathcal{N}=4$ spin $3/2$ supermultiplet and two $\mathcal{N}=4$ spin 1 supermultiplet.

Using the dictionary given in [22, 23], we recognize that the amplitudes in (III.18) and (III.20) are combinations of scalar box integral functions $I_4^{(D=4-2\epsilon)}[\ell^n]$ evaluated in $D=4-2\epsilon$ with $n=4,2,0$ powers of loop momentum and $I_4^{(D=6-2\epsilon)}[\ell^n]$ with $n=2,0$ powers of loop momentum evaluated in $D=6-2\epsilon$ dimensions. The $\mathcal{N}=8$ supergravity part in (III.17) is only given by a scalar box amplitude function $I_4^{(D=4-2\epsilon)}[1]$ evaluated in $D=4-2\epsilon$ dimensions.

Those amplitudes are free of ultraviolet divergences but exhibit rational terms, in agreement with the analysis of [24–27]. This was not obvious from the start, since superficial power counting indicates a logarithmic divergence. More generally, in $\mathcal{N}=4$ supergravity models coupled to vector multiplets amplitudes with external vector multiplets are ultraviolet divergent at one-loop [28][3].

[3] We would like thank K.S. Stelle and Mike Duff for a discussion about this.

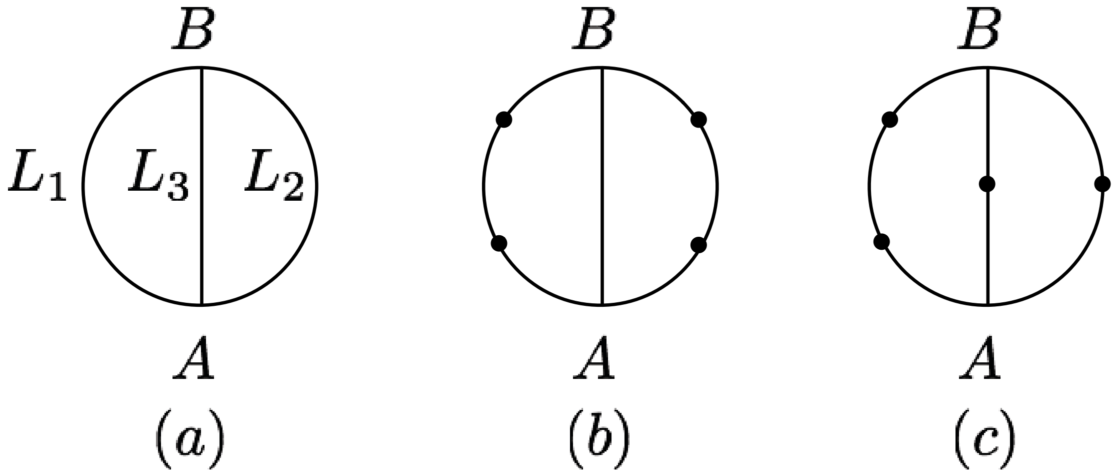

FIG. 1. *Parametrisation of the two-loop diagram in field theory. Figure (a) is the vacuum diagram and the definition of the proper times, and figures (b) and (c) the two configurations contributing to the four-point amplitude.*

B. The two-loop amplitude in field theory

We will follow the notations of [29, section 2.1] where the two-loop four-graviton amplitude in $\mathcal{N} = 8$ supergravity was presented in the world-line formalism. In the field theory limit $\alpha' \to 0$ the imaginary part of the genus-two period matrix Ω becomes the period matrix $K := \alpha' \Im\mathrm{m}\Omega$ of the two-loop graph in figure 1

$$K := \begin{pmatrix} L_1 + L_3 & L_3 \\ L_3 & L_2 + L_3 \end{pmatrix} . \tag{III.22}$$

We set $L_i = \alpha' \tau_i$ and $\Delta = \det K = L_1 L_2 + L_1 L_3 + L_2 L_3$. The position of a point on the line $l = 1, 2, 3$ of length L_l will be denoted by $t^{(l)}$. We choose the point A to be the origin of the coordinate system, i.e. $t^{(l)} = 0$ means the point is located at position A, and $t^{(l)} = L_l$ on the lth line means the point is located at position B.

It is convenient to introduce the rank two vectors $v_i = t_i^{(l_i)}\, u^{(l_i)}$ where

$$u^{(1)} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad u^{(2)} := \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad u^{(3)} := \begin{pmatrix} -1 \\ -1 \end{pmatrix} . \tag{III.23}$$

The v_i are the field theory degenerate form of the Abel map of a point on the Riemann surface to its divisor. The vectors $u^{(i)}$ are the degenerate form of the integrals of the holomorphic one-forms ω_I. If the integrations on each line are oriented from A to B, the integration element on line i is $du^{l_i} = dt_i\, u^{(l_i)}$. The canonical homology basis (A_i, B_i) of the genus two Riemann surface degenerates to $(0, b_i)$, with $b_i = L_i \cup \bar{L}_3$. $\bar{L}_3$ means that we circulate on the middle line from B to A. With these definitions we can reconstruct the period matrix (III.22) from

$$\begin{aligned}
\oint_{b_1} du \cdot u^{(1)} &= \int_0^{L_1} dt_1 + \int_0^{L_3} dt_3 = L_1 + L_3 \\
\oint_{b_2} du \cdot u^{(2)} &= \int_0^{L_2} dt_1 + \int_0^{L_3} dt_3 = L_2 + L_3 \\
\oint_{b_1} du \cdot u^{(2)} &= \int_0^{L_3} dt_3 = L_3 \\
\oint_{b_2} du \cdot u^{(1)} &= \int_0^{L_3} dt_3 = L_3 \,,
\end{aligned} \tag{III.24}$$

in agreement with the corresponding relations on the Riemann surface $\oint_{B_I} \omega_J = \Omega_{IJ}$. In the field theory limit, $\mathcal{Y}_S$ (II.11) becomes

$$3Y_S = (k_1 - k_2) \cdot (k_3 - k_4)\, \Delta_{12}^{FT} \Delta_{34}^{FT} + (13)(24) + (14)(23) \tag{III.25}$$

where

$$\Delta_{ij}^{FT} = \epsilon^{IJ} u_I^{(l_i)} u_J^{(l_j)} \,. \tag{III.26}$$

Notice that $\Delta_{ij}^{FT} = 0$ when the point i and j are on the same line (i.e. $l_i = l_j$). Therefore Y_S vanishes if three points are on the same line, and the only non-vanishing configurations are the one depicted in figure 1(b)-(c).

In the field theory limit the leading contribution to Y_S is given by

$$Y_S = \begin{cases} s & \text{for } l_1 = l_2 \text{ or } l_3 = l_4 \\ t & \text{for } l_1 = l_4 \text{ or } l_3 = l_2 \\ u & \text{for } l_1 = l_3 \text{ or } l_2 = l_4 \,. \end{cases} \tag{III.27}$$

The bosonic propagator in (II.10) becomes

$$P_2^{FT}(v_i - v_j) := -\frac{1}{2}\, d(v_i - v_j) + \frac{1}{2}\, (v_i - v_j)^T K^{-1} (v_i - v_j) \,, \tag{III.28}$$

where $d(v_i - v_j)$ is given by $|t_i^{(l_i)} - t_j^{(l_j)}|$ if the two points are on the same line $l_i = l_j$ or $t_i^{(l_i)} + t_j^{(l_j)}$ is the two point are on different lines $l_i \neq l_j$.

We find that

$$\partial_{ij} P_2^{FT}(v_i - v_j) = (u_i - u_j)^T K^{-1}(v_i - v_j) + \begin{cases} \mathrm{sign}(t_i^{(l_i)} - t_j^{(l_j)}) & \text{if } l_i = l_j \\ 0 & \text{otherwise} \end{cases} , \tag{III.29}$$

and

$$\partial_{ij}^2 P_2^{FT}(v_i - v_j) = (u_i - u_j)^T K^{-1}(u_i - u_j) + \begin{cases} 2\delta(t_i^{(l_i)} - t_j^{(l_j)}) & \text{if } l_i = l_j \\ 0 & \text{otherwise} \end{cases} . \tag{III.30}$$

We define the quantity

$$Q^{(2)} = \sum_{1\leq i<j\leq 4} k_i \cdot k_j \, P_2^{FT}(v_i - v_j) \,. \tag{III.31}$$

In this limit the expansion of CHL model partition function $\mathcal{Z}_2^{(n_v)}$ is given by in (II.14) where $O(q_i)$ do not contribute to the field theory limit. The integration over the real part of the components of the period matrix projects the integrand in the following way

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} d^3\Re\mathrm{e}\Omega \, \mathcal{Z}_2^{(n_v)} F(\Omega, \bar\Omega) \; = c_{n_v} F_0 + F_{123} + a_{n_v} \left(F_{12} + F_{13} + F_{23}\right) + b_{n_v} \left(F_1 + F_2 + F_3\right) , \tag{III.32}$$

where $F(\Omega, \bar\Omega) = F_0 + \sum_{i=1}^3 \bar q_i F_i + \sum_{1\leq i<j\leq 3} \bar q_i \bar q_j F_{ij} + \bar q_1 \bar q_2 \bar q_3 \, F_{123} + c.c. + O(q_i \bar q_i)$ represents the integrand of the two-loop amplitude.

When performing the field theory limit the integral takes the form[4]

$$M_{4;2}^{(n_v)} = N_2 \, t_8 t_8 R^4 \int_0^\infty \frac{d^3 L_i}{\Delta^{2+\epsilon}} \oint d^4 t_i \, Y_S \, [W_1^{(2)} + W_2^{(2)}] \, e^{Q^{(2)}} \,. \tag{III.33}$$

The contribution of $W_1^{(2)}$ yields two kinds of two-loop double-box integrals evaluated in $D = 4 - 2\epsilon$; $I^{(D=4-2\epsilon)}_{double-box}[\ell^n]$ with $n = 4, 2, 0$ powers of loop momentum and $s/u \, I^{(D=4-2\epsilon)}_{double-box}[\ell^m]$ with $m = 2, 0$ powers of loop momentum. Those integrals are multiplied by and overall factor $s \times t_8 t_8 R^4$, $t \times t_8 t_8 R^4$ or $u \times t_8 t_8 R^4$ depending on the channel according to the decomposition of Y_S in (III.27).

The contribution of $W_2^{(2)}$ yields two-loop double-box integrals evaluated in $D = 6 - 2\epsilon$; $I^{(D=6-2\epsilon)}_{double-box}[\ell^n]$ with $n = 2, 0$ powers of loop momentum multiplied by $\frac{s}{u} \times t_8 t_8 R^4$ or $\frac{t}{u} \times t_8 t_8 R^4$ or $t_8 t_8 R^4$ depending on the channel according to the decomposition of Y_S in (III.27). We

[4] A detailed analysis of these integrals will be given in [20].

therefore conclude that the field theory limit of the four-graviton two-loop amplitude of the CHL models with various number of vector multiplets factorizes a $\partial^2 R^4$ term in four dimensions.

We remark that as in the one-loop case, the two-loop amplitude is free of ultraviolet divergence, in agreement with the analysis of Grisaru [30].

IV. NON-RENORMALISATION THEOREMS

The analysis performed in this paper shows that the two-loop four-graviton amplitude in $\mathcal{N}=4$ pure supergravity factorizes a $\partial^2 R^4$ operator in each kinematical sector. This result for the R^4 term holds point wise in the moduli space of the string theory amplitude. In the pure spinor formalism this is a direct consequence of the fermionic zero mode saturation in the two-loop amplitude. At higher-loop since there will be at least the same number of fermionic zero modes to saturate, this implies that higher-loop four-graviton amplitudes will factorize (at least) two powers of external momenta on a R^4 term.[5] This is in agreement with the half-BPS nature of the R^4 term in $\mathcal{N}=4$ models. We are then lead to the following non-renormalisation theorem: the R^4 term will not receive any perturbative corrections beyond one-loop in the four-graviton amplitudes.

Since the structure of the amplitude is the same in any dimension, a four-graviton L-loop amplitude with $L \geq 2$ in D dimensions would have at worst the following enhanced superficial ultraviolet behaviour $\Lambda^{(D-2)L-8}\,\partial^2 R^4$ instead of $\Lambda^{(D-2)L-6}\,R^4$, expected from supersymetry arguments [32]. This forbids the appearance of a three-loop ultraviolet divergence in four dimensions in the four-graviton amplitude and delays it to four loops.

However, a fully supersymmetric R^4 three-loop ultraviolet counter-terms in four dimensions has been constructed in [32], so one can wonder why no divergence occur. We provide here a few arguments that could explain why the R^4 term is a protected operator in $\mathcal{N}=4$ pure supergravity.

It was argued in [7–9] that R^4 is a half-BPS protected operator and does not receive perturbative corrections beyond one-loop in heterotic string compactifications. These non-renormalisation theorems were confirmed in [10] using the explicit evaluation of the genus-

[5] It is tempting to conjecture that the higher-loop string amplitudes will have a form similar to the two-loop amplitude in (II.7) involving a generalisation of $\mathcal{Y}_s$ in (II.11), maybe given by the ansatz proposed in [31, eq. (1.3)].

two four-graviton heterotic amplitude derived in [11–13]. In $D = 4$ dimensions the CHL models with $4 \leq n_v \leq 22$ vector multiplets obtained by an asymmetric orbifold construction satisfy the same non-renormalisation theorems. For these models the moduli space is $SU(1,1)/U(1) \times SO(6,n_v)/SO(6) \times SO(n_v)$. Since the axion-dilaton parametrizes the $SU(1,1)/U(1)$ factor it is natural to conjecture that this moduli space will stay factorized and that one can decouple the contributions from the vector multiplets. If one can set to zero all the vector multiplets, this analysis shows the existence of the R^4 non-renormalisation theorem in the pure $\mathcal{N} = 4$ supergravity case.

It was shown in [32] that the $SU(1,1)$-invariant superspace volume vanishes and the R^4 super-invariant was constructed as an harmonic superspace integral over 3/4 of the full superspace. The structure of the amplitudes analyzed in this paper and the absence of three-loop divergence point to the fact that this partial superspace integral is an F-term.

The existence of an off-shell formulation for $\mathcal{N} = 4$ conformal supergravity and linearized $\mathcal{N} = 4$ supergravity with six vector multiplets [33–35] makes this F-term nature plausible in the Poincaré pure supergravity.

What makes the $\mathcal{N} = 4$ supergravity case special compared to the other $5 \leq \mathcal{N} \leq 8$ cases is the anomalous $U(1)$ symmetry [36]. Therefore even without the existence of an off-shell formalism, this anomaly could make the R^4 term special and be the reason why it turns out to be ruled out as a possible counter-term in four-graviton amplitude in four dimensions. Because of the $U(1)$-anomaly, full superspace integrals of functions of the axion-dilaton superfield $\mathbb{S} = S + \cdots$ are allowed [32]

$$I = \kappa_{(4)}^4 \int d^4x d^{16}\theta \, E(x,\theta) \, F(\mathbb{S}) = \kappa_{(4)}^4 \int d^4x \sqrt{-g} \, f(S) \, R^4 + \text{susy completion} \,, \qquad \text{(IV.1)}$$

suggesting a three-loop divergence in the higher-point field theory amplitudes with four gravitons and scalar fields. Since one can write full superspace for $\partial^2 R^4$ in terms of the gravitino $\int d^{16}\theta \, E(x,\theta)(\chi\bar{\chi})^2$, one should expect a four-loop divergence in the four-graviton amplitude in four dimensions.

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One-loop four-graviton amplitudes in $\mathcal{N} = 4$ supergravity models

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Abstract

We evaluate in great detail one-loop four-graviton field theory amplitudes in pure $\mathcal{N} = 4$ $D = 4$ supergravity. The expressions are obtained by taking the field theory limits of $(4, 0)$ and $(2, 2)$ space-time supersymmetric string theory models. For each model we extract the contributions of the spin-1 and spin-2 $\mathcal{N} = 4$ supermultiplets running in the loop. We show that all of those constructions lead to the same four-dimensional result for the four-graviton amplitudes in pure supergravity even though they come from different string theory models.

I. INTRODUCTION

The role of supersymmetry in perturbative supergravity still leaves room for surprises. The construction of candidate counter-terms for ultraviolet (UV) divergences in extended four-dimensional supergravity theories does not forbid some particular amplitudes to have an improved UV behaviour. For instance, the four-graviton three-loop amplitude in $\mathcal{N} = 4$ supergravity turns out to be UV finite [1, 2], despite the construction of a candidate counter-term [3]. (Some early discussion of the three-loop divergence in $\mathcal{N} = 4$ has appeared in [4], and recent alternative arguments have been given in [5].)

The UV behaviour of extended supergravity theories is constrained in string theory by non-renormalisation theorems that give rise in the field theory limit to supersymmetric protection for potential counter-terms. In maximal supergravity, the absence of divergences until six loops in four dimensions [6–8] is indeed a consequence of the supersymmetric protection for $\frac{1}{2}$-, $\frac{1}{4}$- and $\frac{1}{8}$-BPS operators in string [9, 10] or field theory [11, 12]. In half-maximal supergravity, it was shown recently [2] that the absence of three-loop divergence in the four-graviton amplitude in four dimensions is a consequence of the protection of the $\frac{1}{2}$-BPS R^4 coupling from perturbative quantum corrections beyond one loop in heterotic models. We refer to [13–15] for a discussion of the non-renormalisation theorems in heterotic string.

Maximal supergravity is unique in any dimension, and corresponds to the massless sector of type II string theory compactified on a torus. Duality symmetries relate different phases of the theory and strongly constrain its UV behaviour [10, 12, 16–19].

On the contrary, half-maximal supergravity (coupled to vector multiplets) is not unique and can be obtained in the low-energy limit of $(4, 0)$ string theory models—with all the space-time supersymmetries coming from the world-sheet left-moving sector—or $(2, 2)$ string theory models—with the space-time supersymmetries originating both from the world-sheet left-moving and right-moving sectors. The two constructions give rise to different low-energy supergravity theories with a different identification of the moduli.

In this work we analyze the properties of the four-graviton amplitude at one loop in pure $\mathcal{N} = 4$ supergravity in four dimensions. We compute the genus one string theory amplitude in different models and extract its field theory limit. This method has been pioneered by [20]. It has then been developed intensively for gauge theory amplitudes by [21, 22], and then applied to gravity amplitudes in [23, 24]. In this work we will follow more closely the formulation given in [25].

We consider three classes of four-dimensional string models. The first class, on which was based the analysis in [2], are heterotic string models. They have $(4,0)$ supersymmetry and $4 \leq n_v \leq 22$ vector multiplets. The models of the second class also carry $(4,0)$ supersymmetry; they are type II asymmetric orbifolds. We will study a model with $n_v = 0$ (the Dabholkar-Harvey construction, see [26]) and a model with $n_v = 6$. The third class is composed of type II symmetric orbifolds with $(2,2)$ supersymmetry. For a given number of vector multiplets, the $(4,0)$ models are related to one another by strong-weak S-duality and related to $(2,2)$ models by U-duality [27, 28]. Several tests of the duality relations between orbifold models have been given in [29].

The string theory constructions generically contain matter vector multiplets. By comparing models with $n_v \neq 0$ vector multiplets to a model where $n_v = 0$, we directly check that one can simply subtract these contributions and extract the pure $\mathcal{N} = 4$ supergravity contributions in four dimensions.

We shall show that the four-graviton amplitudes extracted from the $(4,0)$ string models match that obtained in [24, 30–35]. We however note that all of those constructions are based on a $(4,0)$ construction, while our analysis covers both the $(4,0)$ and a $(2,2)$ models. The four-graviton amplitudes are expressed in a supersymmetric decomposition into $\mathcal{N} = 4\,s$ spin-s supermultiplets with $s = 1, \frac{3}{2}, 2$, as in [24, 30–35]. The $\mathcal{N} = 8$ and $\mathcal{N} = 6$ supermultiplets have the same integrands in all the models, while the contributions of the $\mathcal{N} = 4$ multiplets have different integrands. Despite the absence of obvious relation between the integrands of the two models, the amplitudes turn out to be equal after integration in all the string theory models. In a nutshell, we find that the four-graviton one-loop field theory amplitudes in the $(2,2)$ construction are identical to the $(4,0)$ ones.

The paper is organized as follows. For each model we evaluate the one-loop four-graviton string theory amplitudes in section II. In section III we compare the expressions that we obtained and check that they are compatible with our expectations from string dualities. We then extract and evaluate the field theory limit in the regime $\alpha' \to 0$ of those string amplitudes in section IV. This gives us the field theory four-graviton one-loop amplitudes for pure $\mathcal{N} = 4$ supergravity. Section V contains our conclusions. Finally, Appendices A and B contain details about our conventions and the properties of the conformal field theory (CFT) building blocks of our string theory models.

II. ONE-LOOP STRING THEORY AMPLITUDES IN $(4,0)$ AND $(2,2)$ MODELS

In this section, we compute the one-loop four-graviton amplitudes in four-dimensional $\mathcal{N}=4$ $(4,0)$ and $(2,2)$ string theory models. Their massless spectrum contains an $\mathcal{N}=4$ supergravity multiplet coupled to n_v $\mathcal{N}=4$ vector multiplets. Since the heterotic string is made of the tensor product of a left-moving superstring by a right-moving bosonic string, it only gives rise to $(4,0)$ models. However, type II compactifications provide the freedom to build $(4,0)$ and $(2,2)$ models [36].

A. Heterotic CHL models

We evaluate the one-loop four-graviton amplitudes in heterotic string CHL models in four dimensions [37–39]. Their low-energy limits are $(4,0)$ supergravity models with $4 \leq n_v \leq 22$ vector supermultiplets matter fields. We first comment on the moduli space of the model, then write the string theory one-loop amplitude and finally compute the CHL partition function. This allows us to extract the massless states contribution to the integrand of the field theory limit.

These models have the following moduli space:

$$\Gamma\backslash SU(1,1)/U(1)\times SO(6,n_v;\mathbb{Z})\backslash SO(6,n_v)/SO(6)\times SO(n_v)\,, \tag{II.1}$$

where n_v is the number of vector multiplets, and Γ is a discrete subgroup of $SL(2,\mathbb{Z})$. For instance, $\Gamma=SL(2,\mathbb{Z})$ for $n_v=22$ and $\Gamma=\Gamma_1(N)$ for the $\mathbb{Z}_N$ CHL $(4,0)$ orbifold. (We refer to Appendix A 3 for a definition of the congruence subgroups of $SL(2,\mathbb{Z})$.) The scalar manifold $SU(1,1)/U(1)$ is parametrized by the axion-dilaton in the $\mathcal{N}=4$ gravity supermultiplet.

The generic structure of the amplitude has been described in [2]. We will use the same notations and conventions. The four-graviton amplitude takes the following form[1]

$$\mathcal{M}^{(n_v)}_{(4,0)het}=\mathcal{N}\left(\frac{\pi}{2}\right)^4 t_8F^4\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^{\frac{D-6}{2}}}\int_{\mathcal{T}}\prod_{1\leq i<j\leq 4}\frac{d^2\nu_i}{\tau_2}\,e^{\mathcal{Q}}\,\mathcal{Z}^{(n_v)}_{(4,0)het}\bar{\mathcal{W}}^B\,, \tag{II.2}$$

[1] The t_8 tensor defined in [40, appendix 9.A] is given by $t_8F^4 = 4\mathrm{tr}(F^{(1)}F^{(2)}F^{(3)}F^{(4)}) - \mathrm{tr}(F^{(1)}F^{(2)})\mathrm{tr}(F^{(3)}F^{(4)}) + perms(2,3,4)$, where the traces are taken over the Lorentz indices. Setting the coupling constant to one, $t_8F^4 = stA^{tree}(1,2,3,4)$ where $A^{tree}(1,2,3,4)$ is the color stripped ordered tree amplitude between four gluons.

where $D = 10 - d$, and $\mathcal{N}$ is the normalization constant of the amplitude. The domains of integration are $\mathcal{F} = \{\tau = \tau_1 + i\tau_2; |\tau_1| \leq \frac{1}{2}, |\tau|^2 \geq 1, \tau_2 > 0\}$ and $\mathcal{T} := \{\nu = \nu_1 + i\nu_2; |\nu_1| \leq \frac{1}{2}, 0 \leq \nu_2 \leq \tau_2\}$. Then,

$$\bar{\mathcal{W}}^B := \frac{\langle \prod_{j=1}^4 \tilde{\epsilon}^j \cdot \bar{\partial} X(\nu_j) e^{ik_j \cdot X(\nu_j)} \rangle}{(2\alpha')^4 \langle \prod_{j=1}^4 e^{ik_j \cdot X(\nu_j)} \rangle} \tag{II.3}$$

is the kinematical factor coming from the Wick contractions of the bosonic vertex operators and the plane-wave part is given by $\langle \prod_{j=1}^4 e^{ik_j \cdot X(\nu_j)} \rangle = \exp(\mathcal{Q})$ with

$$\mathcal{Q} = \sum_{1 \leq i < j \leq 4} 2\alpha' k_i \cdot k_j \mathcal{P}(\nu_{ij}) \,, \tag{II.4}$$

where we have made use of the notation $\nu_{ij} := \nu_i - \nu_j$. Using the result of [41] with our normalizations we explicitly write

$$\bar{\mathcal{W}}^B = \prod_{r=1}^4 \tilde{\epsilon}_r \cdot \bar{\mathcal{Q}}_r + \frac{1}{2\alpha'} \left(\tilde{\epsilon}_1 \cdot \bar{\mathcal{Q}}_1 \, \tilde{\epsilon}_2 \cdot \bar{\mathcal{Q}}_2 \, \tilde{\epsilon}_3 \cdot \tilde{\epsilon}_4 \bar{\mathcal{T}}(\nu_{34}) + perms\right) + \frac{1}{4\alpha'^2} \left(\tilde{\epsilon}_1 \cdot \tilde{\epsilon}_2 \, \tilde{\epsilon}_3 \cdot \tilde{\epsilon}_4 \, \bar{\mathcal{T}}(\nu_{12}) \bar{\mathcal{T}}(\nu_{34}) + perms\right), \tag{II.5}$$

where we have introduced

$$\mathcal{Q}_I^\mu := \sum_{r=1}^4 k_r^\mu \, \partial \mathcal{P}(\nu_{Ir}|\tau); \qquad \mathcal{T}(\nu) := \partial_\nu^2 \mathcal{P}(\nu|\tau) \,, \tag{II.6}$$

with $\mathcal{P}(z)$ the genus one bosonic propagator. We refer to Appendix A 2 for definitions and conventions.

The CHL models studied in this work are asymmetric $\mathbb{Z}_N$ orbifolds of the bosonic sector (in our case the right-moving sector) of the heterotic string compactified on $T^5 \times S^1$. Geometrically, the orbifold rotates N groups of ℓ bosonic fields $\bar{X}^a$ belonging either to the internal T^{16} or to the T^5 and acts as an order N shift on the S^1. More precisely, if we take a boson $\bar{X}^a$ of the $(p+1)$-th group ($p = 0, \ldots, N-1$) of ℓ bosons, we have $a \in \{p\ell, p\ell+1, \ldots, p\ell+(\ell-1)\}$ and for twists $g/2, h/2 \in \{0, 1/N, \ldots, (N-1)/N\}$ we get

$$\begin{aligned} \bar{X}^a(z+\tau) &= e^{i\pi gp/N} \bar{X}^a(z) \,, \\ \bar{X}^a(z+1) &= e^{i\pi hp/N} \bar{X}^a(z) \,. \end{aligned} \tag{II.7}$$

We will consider models with $(N, n_v, \ell) \in \{(1, 22, 16), (2, 14, 8), (3, 10, 6), (5, 6, 4), (7, 4, 3)\}$. It is in principle possible to build models with $(N, n_v, \ell) = (11, 2, 2)$ and $(N, n_v, \ell) = (23, 0, 1)$ and thus decouple totally the matter fields, but it is then required to compactify the theory on a seven- and eight-dimensional torus respectively. We will not comment about it further, since we have anyway a type II superstring compactification with $(4,0)$ supersymmetry that already has $n_v = 0$ that we discuss in section II B 2. This issue could have been important, but it appears that at one loop in the field theory limit there are no problem to decouple the vector multiplets to obtain pure $\mathcal{N} = 4$ supergravity. The partition function of the right-moving CFT is given by

$$\mathcal{Z}^{(n_v)}_{(4,0)het}(\tau) = \frac{1}{|G|} \sum_{(g,h)} \mathcal{Z}^{h,g}_{(4,0)het}(\tau) \,, \tag{II.8}$$

where $|G|$ is the order of the orbifold group *i.e.* $|G| = N$. The twisted conformal blocks $\mathcal{Z}^{g,h}_{het}$ are a product of the oscillator and zero mode part

$$\mathcal{Z}^{h,g}_{(4,0)het} = \mathcal{Z}^{h,g}_{osc} \times \mathcal{Z}^{h,g}_{latt} \,. \tag{II.9}$$

In the field theory limit only the massless states from the $h = 0$ sector will contribute and we are left with:

$$\mathcal{Z}^{(n_v)}_{(4,0)het}(\tau) \to \frac{1}{N} \mathcal{Z}^{0,0}_{(4,0)het}(\tau) + \frac{1}{N} \sum_{\{g\}} \mathcal{Z}^{0,g}_{(4,0)het}(\tau) \,. \tag{II.10}$$

The untwisted partition function ($g = h = 0$) with generic diagonal Wilson lines A, as required by modular invariance, is

$$\mathcal{Z}^{0,0}_{(4,0)het}(\tau) := \frac{\Gamma_{(6,24)}(G,A)}{\bar\eta^{24}(\bar\tau)} \,, \tag{II.11}$$

where $\Gamma_{(6,24)}(G,A)$ is the lattice sum for the Narain lattice $\Gamma^{(5,5)} \oplus \Gamma^{(1,1)} \oplus \Gamma_{E_8\times E_8}$ with Wilson lines [42]. It drops out in the field theory limit where the radii of compactification $R \sim \sqrt{\alpha'}$ are sent to zero and we are left with the part coming from the oscillators

$$\mathcal{Z}^{0,0}_{(4,0)het}(\tau) \to \frac{1}{\bar\eta^{24}(\bar\tau)} \,. \tag{II.12}$$

At a generic point in the moduli space, the 480 gauge bosons of the adjoint representation of $E_8 \times E_8$ get masses due to Wilson lines, and only the ℓ gauge bosons of the $U(1)^\ell$ group left invariant by the orbifold action [43, 44] stay in the matter massless spectrum.

The oscillator part is computed to be

$$\mathcal{Z}^{h,g}_{osc} = \sum_{\{g,h\}} \prod_{p=0}^{N-1} \left(\mathcal{Z}^{h\times p, g\times p}_{X}\right)^\ell \,, \tag{II.13}$$

where the twisted bosonic chiral blocks $\mathcal{Z}^{h,g}_X$ are given in Appendix A. For $h = 0$, $\mathcal{Z}^{0,g}_{osc}$ is independent of g when N is prime and it can be computed explicitly. It is the inverse of the unique cusp form $f_k(\tau) = (\eta(\tau)\eta(N\tau))^{k+2}$ for $\Gamma_1(N)$ of modular weight $\ell = k+2 = 24/(N+1)$ with $n_v = 2\ell - 2$ as determined in [43, 44]. Then (II.10) writes

$$\mathcal{Z}_{(4,0)het} \to \frac{1}{N}\left(\frac{1}{(\bar\eta(\bar\tau))^{24}} + \frac{N-1}{f_k(\bar\tau)}\right) . \tag{II.14}$$

To conclude this section, we write the part of the integrand of (II.2) that will contribute in the field theory limit. When $\alpha' \to 0$, the region of the fundamental domain of integration $\mathcal{F}$ of interest is the large τ_2 region, such that $t = \alpha'\tau_2$ stays constant. Then, the objects that we have introduced admit an expansion in the variable $q = e^{2i\pi\tau} \to 0$. We find

$$\mathcal{Z}_{(4,0)het} \to \frac{1}{\bar q} + 2 + n_v + o(\bar q) . \tag{II.15}$$

Putting everything together and using the expansions given in (A.16), we find that the integrand in (II.2) is given by

$$\mathcal{Z}_{(4,0)het}\mathcal{W}^B e^{\mathcal{Q}} \to e^{\pi\alpha'\tau_2 Q}\left(\left(\mathcal{W}^B e^{\mathcal{Q}}\right)|_{\bar q} + (n_v+2)\left(\mathcal{W}^B e^{\mathcal{Q}}\right)|_{\bar q^0} + o(\bar q)\right) . \tag{II.16}$$

Order $\bar q$ coefficients are present because of the $1/\bar q$ chiral tachyonic pole in the non-supersymmetric sector of the theory. Since the integral over τ_1 of $\bar q^{-1}\left(\mathcal{W}^B e^{\mathcal{Q}}\right)|_{\bar q^0}$ vanishes, as a consequence of the level matching condition, we did not write it. We introduce $\mathcal{A}$, the massless sector contribution to the field theory limit of the amplitude at the leading order in α', for later use in sections III and IV

$$\mathcal{A}^{(n_v)}_{(4,0)het} = \frac{1}{2}\left(\frac{\pi}{2}\right)^4 t_8F^4\left(\bar{\mathcal{W}}^B|_{\bar q}(1+\alpha'\delta Q) + \bar{\mathcal{W}}^B|_{\bar q^0}\mathcal{Q}|_{\bar q} + (n_v+2)\bar{\mathcal{W}}^B|_{\bar q^0}\right) , \tag{II.17}$$

where we have made use of the notations for the $\bar q$ expansion

$$\begin{aligned}\bar{\mathcal{W}}^B &= \bar{\mathcal{W}}^B|_{q^0} + \bar q\,\bar{\mathcal{W}}^B|_q + o(\bar q^2) ,\\ \mathcal{Q} &= -\pi\,\alpha'\tau_2\,Q + \alpha'\delta Q + q\,\mathcal{Q}|_q + \bar q\,\mathcal{Q}|_{\bar q} + o(|q|^2) .\end{aligned} \tag{II.18}$$

B. Type II asymmetric orbifold

In this section we consider type II string theory on two different kinds of asymmetric orbifolds. They lead to $(4,0)$ models with a moduli space given in (II.1), where the axion-dilaton parametrizes the $SU(1,1)/U(1)$ factor. The first one is a $\mathbb{Z}_2$ orbifold with $n_v = 6$. The others are the Dabholkar-Harvey models [26, 45]; they have $n_v = 0$ vector multiplet.

First, we give a general formula for the treatment of those asymmetric orbifolds. We then study in detail the partition function of two particular models and extract the contribution of massless states to the integrand of the field theory limit of their amplitudes. A generic expression for the scattering amplitude of four gravitons at one loop in type IIA and IIB superstring is:

$$\begin{aligned}\mathcal{M}^{(n_v)}_{(4,0)II} = \mathcal{N} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{\frac{D-6}{2}}} \int_{\mathcal{T}} \prod_{1\leq i<j\leq 4} \frac{d^2\nu_i}{\tau_2}\, e^{\mathcal{Q}} \, \times \\ \times \frac{1}{2} \sum_{a,b=0,1} (-1)^{a+b+ab} \mathcal{Z}_{a,b} \mathcal{W}_{a,b} \times \\ \times \frac{1}{2|G|} \sum_{\substack{\bar a,\bar b=0,1\\ g,h}} (-1)^{\bar a+\bar b+\mu \bar a \bar b} (-1)^{C(\bar a,\bar b,g,h)}\, \bar{\mathcal{Z}}^{h,g}_{\bar a,\bar b} \tilde{\mathcal{W}}_{\bar a,\bar b}\,,\end{aligned} \tag{II.19}$$

where $\mathcal{N}$ is the same normalization factor as for the heterotic string amplitude and $C(\bar a,\bar b,g,h)$ is a model-dependent phase factor determined by modular invariance and discussed below. We have introduced the chiral partition functions in the (a,b)-spin structure

$$\mathcal{Z}_{a,b} = \frac{\theta\left[{a \atop b}\right](0|\tau)^4}{\eta(\tau)^{12}}\,; \qquad \mathcal{Z}_{1,1} = 0\,. \tag{II.20}$$

The value of μ determines the chirality of the theory: $\mu = 0$ for type IIA and $\mu = 1$ for type IIB. The partition function in a twisted sector (h,g) of the orbifold is denoted $\bar{\mathcal{Z}}^{h,g}_{\bar a,\bar b}$. Notice that the four-dimensional fermions are not twisted, so the vanishing of their partition function in the $(a,b) = (1,1)$ sector holds for a (g,h)-twisted sector: $\bar{\mathcal{Z}}^{h,g}_{1,1} = 0$. This is fully consistent with the fact that due to the lack of fermionic zero modes, this amplitude does not receive any contributions from the odd/odd, odd/even or even/odd spin structures. We use the holomorphic factorization of the $(0,0)$-ghost picture graviton vector operators as

$$V^{(0,0)} = \int d^2z\, :\epsilon^{(i)}\cdot V(z)\,\tilde\epsilon^{(i)}\cdot \bar V(\bar z)\, e^{ik\cdot X(z,\bar z)}:\,, \tag{II.21}$$

with

$$\epsilon^{(i)}\cdot V(z) = \epsilon^{(i)}\cdot\partial X - i\frac{F^{(i)}_{\mu\nu}}{2} :\psi^\mu\psi^\nu:; \qquad \tilde\epsilon^{(i)}\cdot\bar V(\bar z) = \tilde\epsilon^{(i)}\cdot\bar\partial X + i\frac{\tilde F^{(i)}_{\mu\nu}}{2} :\bar\psi^\mu\bar\psi^\nu:\,, \tag{II.22}$$

where we have introduced the field strengths $F^{(i)}_{\mu\nu} = \epsilon^{(i)}_\mu k_{i\,\nu} - \epsilon^{(i)}_\nu k_{i\,\mu}$ and $\tilde F^{(i)}_{\mu\nu} = \tilde\epsilon^{(i)}_\mu k_{i\,\nu} - \tilde\epsilon^{(i)}_\nu k_{i\,\mu}$.

The correlators of the vertex operators in the (a,b)-spin structure are given by $\mathcal{W}_{a,b}$ and $\bar{\mathcal{W}}_{\bar a,\bar b}$ defined by, respectively,

$$\mathcal{W}_{a,b} = \frac{\langle\prod_{j=1}^4 \epsilon^{(j)}\cdot V(z_j)\, e^{ik_j\cdot X(z_j)}\rangle_{a,b}}{(2\alpha')^4\langle\prod_{j=1}^4 e^{ik_j\cdot X(z_j)}\rangle}\,, \qquad \bar{\mathcal{W}}_{\bar a,\bar b} = \frac{\langle\prod_{j=1}^4 \tilde\epsilon^{(j)}\cdot \bar V(\bar z_j)\, e^{ik_j\cdot X(z_j)}\rangle_{\bar a,\bar b}}{(2\alpha')^4\langle\prod_{j=1}^4 e^{ik_j\cdot X(\bar z_j)}\rangle}\,. \tag{II.23}$$

We decompose the $\mathcal{W}_{a,b}$ into one part that depends on the spin structure (a,b), denoted $\mathcal{W}^F_{a,b}$, and another independent of the spin structure $\mathcal{W}^B$:

$$\mathcal{W}_{a,b} = \mathcal{W}^F_{a,b} + \mathcal{W}^B \,, \tag{II.24}$$

this last term being identical to the one given in (II.3). The spin structure-dependent part is given by the following fermionic Wick's contractions:

$$\mathcal{W}^F_{a,b} = \mathcal{S}_{4;a,b} + \mathcal{S}_{2;a,b} \,, \tag{II.25}$$

where $\mathcal{S}_{n;a,b}$ arise from Wick contracting n pairs of world-sheet fermions. Note that the contractions involving three pairs of fermion turn out to vanish in all the type II models by symmetry. We introduce the notation $\sum_{\{(i,\cdots),(j,\cdots)\}=\{1,2,3,4\}}\cdots$ for the sum over the ordered partitions of $\{1,2,3,4\}$ into two sets where the partitions $\{(1,2,3),1\}$ and $\{(1,3,2),1\}$ are considered to be independent. In that manner, the two terms in (II.25) can be written explicitly:

$$\begin{aligned}
\mathcal{S}_{4;a,b} &= \frac{1}{2^{10}} \sum_{\{(i,j),(k,l)\}=\{1,2,3,4\}} S_{a,b}(z_{ij})S_{a,b}(z_{ji})S_{a,b}(z_{kl})S_{a,b}(z_{lk})\,\mathrm{tr}(F^{(i)}F^{(j)})\,\mathrm{tr}(F^{(k)}F^{(l)}) \\
&\quad - \frac{1}{2^{8}} \sum_{\{(i,j,k,l)\}=\{1,2,3,4\}} S_{a,b}(z_{ij})S_{a,b}(z_{jk})S_{a,b}(z_{kl})S_{a,b}(z_{li})\,\mathrm{tr}(F^{(i)}F^{(j)}F^{(k)}F^{(l)}) \qquad \text{(II.26)} \\
\mathcal{S}_{2;a,b} &= -\frac{1}{2^{5}} \sum_{\{(i,j),(k,l)\}=\{1,2,3,4\}} S_{a,b}(z_{ij})S_{a,b}(z_{ji})\,\mathrm{tr}(F^{(i)}F^{(j)})\,(\epsilon^{(k)}\cdot\mathcal{Q}_k\,\epsilon^{(l)}\cdot\mathcal{Q}_l + \frac{1}{2\alpha'}\epsilon^{(k)}\cdot\epsilon^{(l)}\,\mathcal{T}(z_{kl}))\,.
\end{aligned}$$

Because the orbifold action only affects the right-moving fermionic zero modes, the left movers are untouched and Riemann's identities imply (see Appendix A 2 for details)

$$\sum_{\substack{a,b=0,1\\ ab=0}} (-1)^{a+b+ab}\mathcal{Z}_{a,b}\,\mathcal{W}_{a,b} = \left(\frac{\pi}{2}\right)^4 t_8F^4\,. \tag{II.27}$$

Notice that a contribution with less than four fermionic contractions vanishes. We now rewrite (II.19):

$$\begin{aligned}
\mathcal{M}^{(6)}_{(4,0)II} = -\mathcal{N}\,\frac{1}{2}\left(\frac{\pi}{2}\right)^4 t_8F^4 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{\frac{D-6}{2}}} \int_{\mathcal{T}} \prod_{1\leq i<j\leq 4} \frac{d^2\nu_i}{\tau_2}\, e^{\mathcal{Q}} \;\times \qquad \text{(II.28)} \\
\times\,\frac{1}{2|G|} \sum_{\substack{\bar a,\bar b=0,1\\ g,h}} (-1)^{\bar a+\bar b+\mu\bar a\bar b}(-1)^{C(\bar a,\bar b,g,h)}\,\bar{\mathcal{Z}}^{h,g}_{\bar a,\bar b}\bar{\mathcal{W}}_{\bar a,\bar b}\,.
\end{aligned}$$

For the class of asymmetric $\mathbb{Z}_N$ orbifolds with n_v vector multiplets studied here, the partition function $\mathcal{Z}^{(asym)}_{a,b} = |G|^{-1}\sum_{g,h}(-1)^{C(a,b,g,h)}\mathcal{Z}^{g,h}_{a,b}$ has the following low-energy expansion:

$$\mathcal{Z}^{(asym)}_{0,0} = \frac{1}{\sqrt{q}} + n_v + 2 + o(q)\,; \qquad \mathcal{Z}^{(asym)}_{0,1} = \frac{1}{\sqrt{q}} - (n_v + 2) + o(q)\,; \qquad \mathcal{Z}^{(asym)}_{1,0} = 0 + o(q)\,. \tag{II.29}$$

Because the four-dimensional fermionic zero modes are not saturated we have $\mathcal{Z}^{asym}_{1,1} = 0$.

Since in those constructions no massless mode arises in the twisted $h \neq 0$ sector, this sector decouples. Hence, at $o(\bar q)$ one has the following relation:

$$\sum_{\bar a,\bar b=0,1} (-1)^{a+b+ab}\bar{\mathcal{Z}}^{(asym)}_{\bar a,\bar b}\bar{\mathcal{W}}_{\bar a,\bar b} \to (\bar{\mathcal{W}}_{0,0} - \bar{\mathcal{W}}_{0,1})|_{\sqrt{q}} + (n_v+2)(\bar{\mathcal{W}}_{0,0} + \bar{\mathcal{W}}_{0,1})|_{q^0}\,. \tag{II.30}$$

The contribution of massless states to the field theory amplitude is given by

$$\mathcal{A}^{(n_v)}_{(4,0)II} = \frac{1}{4}\left(\frac{\pi}{2}\right)^4 t_8F^4\,\left((\bar{\mathcal{W}}_{0,0} - \bar{\mathcal{W}}_{0,1})|_{\sqrt{q}} + (n_v+2)(\bar{\mathcal{W}}_{0,0} + \bar{\mathcal{W}}_{0,1})|_{q^0}\right)\,. \tag{II.31}$$

Using the Riemann identity (II.27) we can rewrite this expression in the following form

$$\mathcal{A}^{(n_v)}_{(4,0)II} = \frac{1}{4}\left(\frac{\pi}{2}\right)^4 t_8F^4\,\left(\left(\frac{\pi}{2}\right)^4 t_8\tilde F^4 + (n_v-6)(\bar{\mathcal{W}}_{0,0} + \bar{\mathcal{W}}_{0,1})|_{q^0} + 16\bar{\mathcal{W}}_{1,0}|_{q^0}\right)\,. \tag{II.32}$$

Higher powers of $\bar q$ in $\mathcal{W}_{a,b}$ or in $\mathcal{Q}$ are suppressed in the field theory limit that we discuss in section IV.

At this level, this expression is not identical to the one derived in the heterotic construction (II.17). The type II and heterotic $(4,0)$ string models with n_v vector multiplets are dual to each other under the transformation $S \to -1/S$ where S is the axion-dilaton scalar in the $\mathcal{N}=4$ supergravity multiplet. We will see in section III that for the four-graviton amplitudes we obtain the same answer after integrating out the real parts of the positions of the vertex operators.

We now illustrate this analysis on the examples of the asymmetric orbifold with six or zero vector multiplets.

1. *Example: A model with six vector multiplets*

Let us compute the partition function of the asymmetric orbifold obtained by the action of the right-moving fermion counting operator $(-1)^{F_R}$ and a $\mathbb{Z}_2$ action on the torus T^6 [29, 46]. The effect of the $(-1)^{F_R}$ orbifold is to project out the sixteen vector multiplets arising from the R/R sector, while preserving supersymmetry on the right-moving sector. The moduli space of the theory is given by (II.1) with $n_v = 6$ and $\Gamma = \Gamma(2)$ (see [29] for instance).

The partition function for the $(4,0)$ CFT $\mathbb{Z}_2$ asymmetric orbifold model $\mathcal{Z}^{(asym),(n_v=6)}_{a,b} = \frac{1}{2}\sum_{g,h} \mathcal{Z}^{g,h}_{a,b}$ with

$$\mathcal{Z}^{h,g}_{a,b}(w) := (-1)^{ag+bh+gh} \mathcal{Z}_{a,b} \Gamma_{(4,4)}\, \Gamma^{w}_{(2,2)} \begin{bmatrix} h \\ g \end{bmatrix} , \tag{II.33}$$

where the shifted lattice sum $\Gamma^{w}_{(2,2)} \begin{bmatrix} h \\ g \end{bmatrix}$ is given in [29] and recalled in Appendix B. The chiral blocks $\mathcal{Z}_{a,b}$ have been defined in (II.20) and $\Gamma_{(4,4)}$ is the lattice sum of the T^4. Using the fact that $\Gamma^{w}_{(2,2)} \begin{bmatrix} h \\ g \end{bmatrix}$ reduces to 0 for $h = 1$, to 1 for $h = 0$ and that $\Gamma_{(4,4)} \to 1$ in the field theory limit, we see that the partition function is unchanged in the sectors $(a, b) = (0, 0)$ and $(0, 1)$ while for the $(a, b) = (1, 0)$ sector, the $(-1)^{ag}$ in (II.33) cancels the partition function when summing over g. One obtains the following result :

$$\mathcal{Z}^{(asym),(n_v=6)}_{0,0} = \mathcal{Z}_{0,0}\,; \qquad \mathcal{Z}^{(asym),(n_v=6)}_{0,1} = \mathcal{Z}_{0,1}\,; \qquad \mathcal{Z}^{(asym),(n_v=6)}_{1,0} = 0\,. \tag{II.34}$$

Using (A.6), one checks directly that it corresponds to (II.29) with $n_v = 6$.

2. Example: Models with zero vector multiplet

Now we consider the type II asymmetric orbifold models with zero vector multiplets constructed in [26] and discussed in [45].

Those models are compactifications of the type II superstring on a six-dimensional torus with an appropriate choice for the value of the metric G_{ij} and B-field B_{ij}. The Narain lattice is given by $\Gamma^{DH} = \{p_L, p_R; p_L, p_R \in \Lambda_W(\mathfrak{g}), p_L - p_R \in \Lambda_R(\mathfrak{g})\}$ where $\Lambda_R(\mathfrak{g})$ is the root lattice of a simply laced semi-simple Lie algebra $\mathfrak{g}$, and $\Lambda_W(\mathfrak{g})$ is the weight lattice.

The asymmetric orbifold action is given by $|p_L, p_R\rangle \to e^{2i\pi p_L\cdot v_L}\, |p_L, g_R p_R\rangle$ where g_R is an element of the Weyl group of $\mathfrak{g}$ and v_L is a shift vector appropriately chosen to avoid any massless states in the twisted sector [26, 45]. With such a choice of shift vector and because the asymmetric orbifold action leaves p_L invariant, we have $(4, 0)$ model of four-dimensional supergravity with no vector multiplets.

The partition function is given by

$$\mathcal{Z}^{asym}_{\bar a,\bar b} = \frac{\theta\left[\begin{smallmatrix}\bar a\\ \bar b\end{smallmatrix}\right]}{(\eta(\bar\tau))^3}\, \frac{1}{|G|} \sum_{\{g_j,h_j\}} \prod_{i=1}^{3} \mathcal{Z}^{h_j,g_j}_{\bar a,\bar b}\ , \tag{II.35}$$

where the sum runs over the sectors of the orbifold. For instance, in the $\mathbb{Z}_9$ model of Dabholkar and Harvey, one has $g_j \in j \times \{\frac{2}{9}, \frac{4}{9}, \frac{8}{9}\}$ with $j = 0, \ldots, 8$ and the same for h_j. The twisted conformal blocks are:

$$\mathcal{Z}^{h,g}_{\bar{a},\bar{b}} = \begin{cases} \left(\frac{\theta\left[\bar{a} \atop \bar{b}\right]}{\eta(\bar{\tau})}\right)^3 \times \left(\frac{1}{\eta(\bar{\tau})}\right)^6 & \text{if } (g,h) = (0,0) \mod 2, \\ e^{i\frac{\pi}{2}a(g-b)} 2\sin(\frac{\pi g}{2}) \frac{\theta\left[a+h \atop b+g\right]}{\theta\left[1+h \atop 1+g\right]} & \forall (g,h) \neq (0,0) \mod 2. \end{cases} \tag{II.36}$$

The phase in (II.19) is determined by modular invariance to be $C(\bar{a}, \bar{b}, g_R, h_R) = \sum_{i=1}^3 (\bar{a} g^i_R + \bar{b} h^i_R + g^i_R h^i_R)$.

In the field theory limit, we perform the low-energy expansion of this partition function and we find that it takes the following form for all of the models in [26, 45]:

$$\mathcal{Z}^{(asym),(n_v=0)}_{0,0} = \frac{1}{\sqrt{q}} + 2 + o(q)\,; \quad \mathcal{Z}^{(asym),(n_v=0)}_{0,1} = \frac{1}{\sqrt{q}} - 2 + o(q)\,; \quad \mathcal{Z}^{(asym),(n_v=0)}_{1,0} = 0 + o(q)\,, \tag{II.37}$$

which is (II.29) with $n_v = 0$ as expected.

C. Type II symmetric orbifold

In this section we consider $(2,2)$ models of four-dimensional $\mathcal{N} = 4$ supergravity. These models can be obtained from the compactification of type II string theory on symmetric orbifolds of $K_3 \times T^2$. The difference with the heterotic models considered in section II A is that the scalar parametrizing the coset space $SU(1,1)/U(1)$ that used to be the axio-dilaton S is now the Kähler modulus of the two-torus T^2 for the type IIA case or complex structure modulus for the type IIB case. The non-perturbative duality relation between these two models is discussed in detail in [29, 39].

Models with $n_v \geq 2$ have been constructed in [36]. The model with $n_v = 22$ is a $T^4/\mathbb{Z}_2 \times T^2$ orbifold, and the following models with $n_v \in \{14, 10\}$ are successive $\mathbb{Z}_2$ orbifolds of the first one. The model with $n_v = 6$ is a freely acting $\mathbb{Z}_2$ orbifold of the $T^4/\mathbb{Z}_2 \times T^2$ theory that simply projects out the sixteen vector multiplets of the R/R sector. The four-graviton amplitude can be effectively written in terms of the (g,h) sectors of the first $\mathbb{Z}_2$ orbifold of the T^4, and writes

$$\mathcal{M}^{(n_v)}_{(2,2)} = \mathcal{N} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{\frac{D-6}{2}}} \int_{\mathcal{T}} \prod_{1\leq i<j\leq 4} \frac{d^2\nu_i}{\tau_2}\, e^{\mathcal{Q}} \times \tag{II.38}$$

$$\times \frac{1}{4|G|} \sum_{h,g=0}^{1} \sum_{\substack{a,b=0,1\\ \bar a,\bar b=0,1}} (-1)^{a+b+ab}(-1)^{\bar a+\bar b+\bar a\bar b} \mathcal{Z}^{h,g,(n_v)}_{a,b}\, \bar{\mathcal{Z}}^{h,g,(n_v)}_{\bar a,\bar b}\, (\mathcal{W}_{a,b}\bar{\mathcal{W}}_{\bar a,\bar b} + \mathcal{W}_{a,b;\bar a,\bar b})\,,$$

where $\mathcal{N}$ is the same overall normalization as for the previous amplitudes and $\mathcal{Z}^{h,g,(n_v)}_{a,b}$ is defined in Appendix B. The term $\mathcal{W}_{a,b;\bar a,\bar b}$ is a mixed term made of contractions between holomorphic and anti-holomorphic fields. It does not appear in the $(4,0)$ constructions since the left/right contractions vanish due to the totally unbroken supersymmetry in the left-moving sector.

Two types of contributions arise from the mixed correlators

$$\mathcal{W}^1_{a,b;\bar a,\bar b} = \frac{\langle :\epsilon^{(i)}\cdot\partial X \tilde\epsilon^{(i)}\cdot\bar\partial X :: \epsilon^{(j)}\cdot\partial X\tilde\epsilon^{(j)}\cdot\bar\partial X : \prod_{r=1}^4 e^{ik_r\cdot X(z_r)}\rangle_{a,b;\bar a,\bar b}}{(2\alpha')^4\langle\prod_{j=1}^4 e^{ik_j\cdot X(z_j)}\rangle}; \tag{II.39}$$
$$\mathcal{W}^2_{a,b;\bar a,\bar b} = \frac{\langle\epsilon^{(i)}\cdot\partial X\,\epsilon^{(j)}\cdot\partial X\tilde\epsilon^{(k)}\cdot\bar\partial X\,\tilde\epsilon^{(l)}\cdot\bar\partial X\,\prod_{r=1}^4 e^{ik_r\cdot X(z_r)}\rangle_{a,b;\bar a,\bar b}}{(2\alpha')^4\langle\prod_{j=1}^4 e^{ik_j\cdot X(z_j)}\rangle}\,, \qquad (ij)\neq(kl)$$

with at least one operator product expansion (OPE) between a holomorphic and an anti-holomorphic operator. Explicitely, we find

$$\begin{aligned}
\mathcal{W}_{a,b;\bar a,\bar b} = &\sum_{\substack{\{i,j,k,l\}\in\{1,2,3,4\}\\(i,j)\neq(k,l)}} (S_{a,b}(\nu_{ij}))^2(\bar S_{\bar a,\bar b}(\bar\nu_{kl}))^2\ \times \mathrm{tr}(F^{(i)}F^{(j)})\mathrm{tr}(F^{(k)}F^{(l)})\\
&\times\Big(\epsilon^{(k)}\cdot\tilde\epsilon^{(i)}\hat{\mathcal{T}}(k,i)(\epsilon^{(l)}\cdot\tilde\epsilon^{(j)}\hat{\mathcal{T}}(l,j)+\epsilon^{(l)}\cdot\mathcal{Q}_l\,\tilde\epsilon^{(j)}\cdot\bar{\mathcal{Q}}_j)+(i\leftrightarrow j)\\
&+\epsilon^{(k)}\cdot\mathcal{Q}_k(\epsilon^{(l)}\cdot\tilde\epsilon^{(i)}\,\tilde\epsilon^{(j)}\cdot\bar{\mathcal{Q}}_j\hat{\mathcal{T}}(l,i)+\epsilon^{(l)}\cdot\tilde\epsilon^{(j)}\hat{\mathcal{T}}(l,j)\tilde\epsilon^{(i)}\cdot\bar{\mathcal{Q}}_i)\Big)\\
&+\sum_{\{i,j,k,l\}\in\{1,2,3,4\}} |S_{a,b}(\nu_{ij})|^4\times(\mathrm{tr}(F^{(i)}F^{(j)}))^2\\
&\times\Big(\epsilon^{(k)}\cdot\tilde\epsilon^{(l)}\hat{\mathcal{T}}(k,l)(\epsilon^{(l)}\cdot\tilde\epsilon^{(k)}\hat{\mathcal{T}}(l,k)+\epsilon^{(l)}\cdot\mathcal{Q}_l\,\tilde\epsilon^{(k)}\cdot\bar{\mathcal{Q}}_k)+(k\leftrightarrow l)\Big)
\end{aligned} \tag{II.40}$$

where

$$\hat{\mathcal{T}}(i,j) := \partial_{\nu_i}\bar\partial_{\bar\nu_j}\mathcal{P}(\nu_i-\nu_j|\tau) = \frac{\pi}{4}\left(\frac{1}{\tau_2}-\delta^{(2)}(\nu_i-\nu_j)\right)\,. \tag{II.41}$$

Forgetting about the lattice sum, which at any rate is equal to one in the field theory limit,

$$\mathcal{Z}^{h,g}_{a,b} = c_h\frac{(\theta\left[{a\atop b}\right](0|\tau))^2\theta\left[{a+h\atop b+g}\right](0|\tau)\theta\left[{a-h\atop b-g}\right](0|\tau)}{(\eta(\tau))^6(\theta\left[{1+h\atop 1+g}\right](0|\tau))^2} = c_h(-1)^{(a+h)g}\left(\frac{\theta\left[{a\atop b}\right](0|\tau)\theta\left[{a+h\atop b+g}\right](0|\tau)}{(\eta(\tau))^3\theta\left[{1+h\atop 1+g}\right](0\tau)}\right)^2\,, \tag{II.42}$$

where c_h is an effective number whose value depends on h in the following way: $c_0 = 1$ and $c_1 = \sqrt{n_v - 6}$. This number represents the successive halving of the number of twisted R/R states. We refer to Appendix B for details.

The sum over the spin structures in the untwisted sector $(g, h) = (0, 0)$ is once again performed using Riemann's identities:

$$\sum_{\substack{a,b=0,1\\ab=0}} (-1)^{a+b+ab} \mathcal{Z}^{0,0}_{a,b} \mathcal{W}_{a,b} = \left(\frac{\pi}{2}\right)^4 t_8 F^4 \,. \tag{II.43}$$

In the twisted sectors $(h, g) \neq (0, 0)$ we remark that $\mathcal{Z}^{0,1}_{0,1} = Z^{0,1}_{0,0}$, $\mathcal{Z}^{1,0}_{1,0} = \mathcal{Z}^{1,0}_{0,0}$, $\mathcal{Z}^{1,1}_{1,0} = \mathcal{Z}^{1,1}_{0,1}$, and $\mathcal{Z}^{1,0}_{0,1} = \mathcal{Z}^{0,1}_{1,0} = \mathcal{Z}^{1,1}_{0,0} = 0$, which gives for the chiral blocks in (II.38):

$$\begin{aligned}
\sum_{\substack{a,b=0,1\\ab=0}} (-1)^{a+b+ab} \mathcal{Z}^{0,1}_{a,b} \mathcal{W}_{a,b} &= \mathcal{Z}^{0,1}_{0,0} \,\left(\mathcal{W}_{0,0} - \mathcal{W}_{0,1}\right) ,\\
\sum_{\substack{a,b=0,1\\ab=0}} (-1)^{a+b+ab} \mathcal{Z}^{1,0}_{a,b} \mathcal{W}_{a,b} &= \mathcal{Z}^{1,0}_{0,0} \,\left(\mathcal{W}_{0,0} - \mathcal{W}_{1,0}\right) ,\\
\sum_{\substack{a,b=0,1\\ab=0}} (-1)^{a+b+ab} \mathcal{Z}^{1,1}_{a,b} \mathcal{W}_{a,b} &= \mathcal{Z}^{1,1}_{0,1} \,\left(\mathcal{W}_{0,1} - \mathcal{W}_{1,0}\right) .
\end{aligned} \tag{II.44}$$

Therefore the factorized terms in the correlator take the simplified form

$$\begin{aligned}
&\frac{1}{4|G|} \sum_{g,h} \sum_{\substack{a,b=0,1\\ \bar a,\bar b=0,1}} (-1)^{a+b+ab} (-1)^{\bar a+\bar b+\bar a\bar b} \mathcal{Z}^{h,g}_{a,b} \bar{\mathcal{Z}}^{h,g}_{\bar a,\bar b} \, \mathcal{W}_{a,b} \bar{\mathcal{W}}_{\bar a,\bar b}\\
&= \frac{1}{8} \left(\frac{\pi}{2}\right)^8 t_8 t_8 R^4 + \frac{1}{8} \,\left|\mathcal{Z}^{0,1}_{0,0} (\mathcal{W}_{0,0} - \mathcal{W}_{0,1})\right|^2 + \frac{1}{8} \,\left|\mathcal{Z}^{1,0}_{0,0} (\mathcal{W}_{0,0} - \mathcal{W}_{1,0})\right|^2 + \frac{1}{8} \,\left|\mathcal{Z}^{1,1}_{0,1} (\mathcal{W}_{0,1} - \mathcal{W}_{1,0})\right|^2 ,
\end{aligned} \tag{II.45}$$

where $t_8 t_8 R^4$ is the Lorentz scalar built from four powers of the Riemann tensor arising at the linearized level as the product $t_8 t_8 R^4 = t_8 F^4 \, t_8 \tilde{F}^4$. [2]

The mixed terms can be treated in the same way with the result

$$\begin{aligned}
&\frac{1}{4|G|} \sum_{g,h} \sum_{\substack{a,b=0,1\\ \bar a,\bar b=0,1}} (-1)^{a+b+ab} (-1)^{\bar a+\bar b+\bar a\bar b} \mathcal{Z}^{h,g}_{a,b} \bar{\mathcal{Z}}^{h,g}_{\bar a,\bar b} \, \mathcal{W}_{a,b;\bar a,\bar b}\\
&= \frac{1}{8} \,|\mathcal{Z}^{0,1}_{0,0}|^2 (\mathcal{W}_{0,0;0,0} - \mathcal{W}_{0,1;0;1}) + \frac{1}{8} \,|\mathcal{Z}^{1,0}_{0,0}|^2 (\mathcal{W}_{0,0;0,0} - \mathcal{W}_{1,0;1,0}) + \frac{1}{8} \,|\mathcal{Z}^{1,1}_{0,1}|^2 (\mathcal{W}_{0,1;0,1} - \mathcal{W}_{1,0;1,0}) ,
\end{aligned} \tag{II.46}$$

[2] This Lorentz scalar is the one obtained from the four-graviton tree amplitude $t_8 t_8 R^4 = stu M^{tree}(1, 2, 3, 4)$ setting Newton's constant to one.

Since the conformal blocks $\mathcal{Z}^{h,g}_{a,b}$ have the q expansion (see (A.3))

$$\mathcal{Z}^{0,1}_{0,0} = \frac{1}{\sqrt{q}} + 4\sqrt{q} + o(q)\,;\ \mathcal{Z}^{1,0}_{0,0} = 4\sqrt{n_v-6} + o(q)\,;\ \mathcal{Z}^{1,1}_{0,1} = 4\sqrt{n_v-6} + o(q)\,, \qquad \text{(II.47)}$$

the massless contribution to the integrand of (II.45) is given by

$$\begin{aligned}\mathcal{A}^{(n_v)}_{(2,2)} = \frac{1}{8}\left(\frac{\pi}{2}\right)^8 t_8 t_8 R^4 + \frac{1}{8}\left|\mathcal{W}_{0,0}|_{\sqrt{q}} - \mathcal{W}_{0,1}|_{\sqrt{q}}\right|^2 + \frac{1}{8}\left(\mathcal{W}_{0,0;0,0}|_{\sqrt{q}} - \mathcal{W}_{0,1;0,1}|_{\sqrt{q}}\right) \\ + 2(n_v-6)\left(\left|\mathcal{W}_{0,0}|_{q^0} - \mathcal{W}_{1,0}|_{q^0}\right|^2 + \left|\mathcal{W}_{0,1}|_{q^0} - \mathcal{W}_{1,0}|_{q^0}\right|^2\right) \\ + 2(n_v-6)\left(\mathcal{W}_{0,0;0,0}|_{q^0;\bar{q}^0} + \mathcal{W}_{0,1;0,1}|_{q^0;\bar{q}^0} - 2\mathcal{W}_{1,0;1,0}|_{q^0;\bar{q}^0}\right). \qquad \text{(II.48)}\end{aligned}$$

We notice that the bosonic piece $\mathcal{W}^B$ in $\mathcal{W}_{a,b}$ in (II.24) cancels in each term of the previous expression, due to the minus sign between the $\mathcal{W}_{a,b}$'s in the squares.

The integrand of the four-graviton amplitude takes a different form in the $(2,2)$ construction compared with the expression for the $(4,0)$ constructions in heterotic in (II.17) and asymmetric type II models in (II.32). We will show that after taking the field theory limit and performing the integrals the amplitudes will turn out to be the same.

As mentioned above, for a given number of vector multiplets the type II $(2,2)$ models are only non-perturbatively equivalent (U-duality) to the $(4,0)$ models. However, we will see that this non-perturbative duality does not affect the perturbative one-loop multi-graviton amplitudes. Nevertheless, we expect that both α' corrections to those amplitudes and amplitudes with external scalars and vectors should be model dependent.

In the next section, we analyze the relationships between the string theory models.

III. COMPARISON OF THE STRING MODELS

A. Massless spectrum

The spectrum of the type II superstring in ten dimensions is given by the following GSO sectors: the graviton G_{MN}, the B-field B_{MN}, and the dilaton Φ come from the NS/NS sector, the gravitini ψ^M, and the dilatini λ come from the R/NS and NS/R sectors, while the one-form C_M and three-form C_{MNP} come from the R/R sector in the type IIA string. The dimensional reduction of type II string on a torus preserves all of the thirty-two supercharges and leads to the $\mathcal{N}=8$ supergravity multiplet in four dimensions.

The reduction to $\mathcal{N}=4$ supersymmetry preserves sixteen supercharges and leads to the following content. The NS/NS sector contributes to the $\mathcal{N}=4$ supergravity multiplet and to six vector multiplets. The R/R sector contributes to the $\mathcal{N}=4$ spin-$\frac{3}{2}$ multiplet and to the vector multiplets with a multiplicity depending on the model.

In the partition function, the first Riemann vanishing identity

$$\sum_{a,b=0,1}(-1)^{a+b+ab}\mathcal{Z}_{a,b}=0\,, \tag{III.1}$$

reflects the action of the $\mathcal{N}=4$ supersymmetry inside the one-loop amplitudes in the following manner. The q expansion of this identity gives

$$\left(\frac{1}{\sqrt{q}}+8+o(\sqrt{q})\right)-\left(\frac{1}{\sqrt{q}}-8+o(\sqrt{q})\right)-(16+o(q))=0\,. \tag{III.2}$$

The first two terms are the expansion of $\mathcal{Z}_{0,0}$ and $\mathcal{Z}_{0,1}$ and the last one is the expansion of $\mathcal{Z}_{1,0}$. The cancellation of the $1/\sqrt{q}$ terms shows that the GSO projection eliminates the tachyon from the spectrum, and at the order q^0 the cancellation results in the matching between the bosonic and fermionic degrees of freedom.

In the amplitudes, chiral $\mathcal{N}=4$ supersymmetry implies the famous Riemann identities, stating that for $0\leq n\leq 3$ external legs, the one-loop n-point amplitude vanishes (see eq. (A.25)). At four points it gives:

$$\sum_{a,b=0,1}(-1)^{a+b+ab}\mathcal{Z}_{a,b}\mathcal{W}_{a,b}=\left(\frac{\pi}{2}\right)^4 t_8F^4\,. \tag{III.3}$$

In $\mathcal{W}_{a,b}$, see (II.25), the term independent of the spin structure $\mathcal{W}^B$ and the terms with less than four fermionic contractions $\mathcal{S}_{2;a,b}$ cancel in the previous identity. The cancellation of the tachyon yields at the first order in the q expansion of (III.3)

$$\mathcal{W}_{0,0}|_{q^0}-\mathcal{W}_{0,1}|_{q^0}=0\,. \tag{III.4}$$

The next term in the expansion gives an identity describing the propagation of the $\mathcal{N}=4$ super-Yang-Mills multiplet in the loop

$$8(\mathcal{W}_{0,0}|_{q^0}+\mathcal{W}_{0,1}|_{q^0}-2\mathcal{W}_{1,0}|_{q^0})+(\mathcal{W}_{0,0}|_{\sqrt{q}}-\mathcal{W}_{0,1}|_{\sqrt{q}})=\left(\frac{\pi}{2}\right)^4 t_8F^4\,. \tag{III.5}$$

In this equation, one should have vectors, spinors and scalars propagating according to the sector of the theory. In $\mathcal{W}_{a,b}$, $a=0$ is the NS sector, and $a=1$ is the Ramond sector. The scalars have already been identified in (III.4) and correspond to $\mathcal{W}_{0,0}|_{q^0}+\mathcal{W}_{0,1}|_{q^0}$. The vector, being a massless bosonic degree of freedom should then correspond to $\mathcal{W}_{0,0}|_{\sqrt{q}} - \mathcal{W}_{0,1}|_{\sqrt{q}}$. Finally, the fermions correspond to $\mathcal{W}_{1,0}|_{q^0}$. The factor of eight in front of the first term is the number of degrees of freedom of a vector in ten dimensions; one can check that the number of bosonic degrees of freedom matches the number of fermionic degrees of freedom.

B. Amplitudes and supersymmetry

In this section we discuss the relationships between the four-graviton amplitudes in the various $\mathcal{N} = 4$ supergravity models in the field theory limit. We apply the logic of the previous section about the spectrum of left or right movers to the tensor product spectrum and see that we can precisely identify the contributions to the amplitude, both in the $(4,0)$ and $(2,2)$ models. The complete evaluation of the amplitudes will be performed in section IV.

As mentioned above, the field theory limit is obtained by considering the large τ_2 region, and the integrand of the field theory amplitude is given by

$$A_X^{(n_v)} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1 \, \mathcal{A}_X^{(n_v)} \,, \tag{III.6}$$

where $X \in \{(4,0)het,\ (4,0)II,\ (2,2)\}$ indicates the model, as in (II.17), (II.32) or (II.48) respectively.

At one loop this quantity is the sum of the contribution from n_v $\mathcal{N} = 4$ vector (spin-1) supermultiplets running in the loop and the $\mathcal{N} = 4$ spin-2 supermultiplet

$$A_X^{(n_v)} = A_X^{spin-2} + n_v \, A_X^{spin-1} \,. \tag{III.7}$$

For the case of the type II asymmetric orbifold models with n_v vector multiplets we deduce from (II.32)

$$A_{(4,0)II}^{spin-1} = \frac{1}{4} \left(\frac{\pi}{2}\right)^4 t_8F^4 \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1 \, (\mathcal{W}_{0,0}|_{q^0} + \mathcal{W}_{0,1}|_{q^0}) \,. \tag{III.8}$$

Since t_8F^4 is the supersymmetric left-moving sector contribution (recall the supersymmetry identity in (III.3)), it corresponds to an $\mathcal{N} = 4$ vector multiplet and we recognize in (III.8) the product of this multiplet with the scalar from the right-moving sector:

$$(1_{\mathbf{1}}, 1/2_{\mathbf{4}}, 0_{\mathbf{6}})_{\mathcal{N}=4} = (1_{\mathbf{1}}, 1/2_{\mathbf{4}}, 0_{\mathbf{6}})_{\mathcal{N}=4} \otimes (0_{\mathbf{1}})_{\mathcal{N}=0} \,. \tag{III.9}$$

This agrees with the identification made in the previous subsection where $\mathcal{W}_{0,0}|_{q^0} + \mathcal{W}_{0,1}|_{q^0}$ was argued to be a scalar contribution.

The contribution from the $\mathcal{N} = 4$ supergravity multiplet running in the loop is given by

$$A^{spin-2}_{(4,0)II} = \frac{1}{4}\left(\frac{\pi}{2}\right)^4 t_8F^4 \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1 \left[2(\mathcal{W}_{0,0}|_{q^0} + \mathcal{W}_{0,1}|_{q^0}) + (\mathcal{W}_{0,0}|_{\sqrt{q}} - \mathcal{W}_{0,1}|_{\sqrt{q}})\right]. \quad \text{(III.10)}$$

The factor of 2 is the number of degrees of freedom of a vector in four dimensions. Since $\mathcal{Z}^{asym}_{1,0} = 0 + o(q)$ for the $(4,0)$ model asymmetric orbifold construction, the integrand of the four-graviton amplitude in (II.29) does not receive any contribution from the right-moving R sector. Stated differently, the absence of $\mathcal{W}_{1,0}$ implies that both R/R and NS/R sectors are projected out, leaving only the contribution from the NS/NS and R/NS. Thus, the four $\mathcal{N} = 4$ spin-$\frac{3}{2}$ supermultiplets and sixteen $\mathcal{N} = 4$ spin-1 supermultiplets are projected out, leaving at most six vector multiplets. This number is further reduced to zero in the Dabholkar-Harvey construction [26].

From (III.10) we recognize that the $\mathcal{N} = 4$ supergravity multiplet is obtained by the following tensor product

$$(2_{\mathbf{1}}, 3/2_{\mathbf{4}}, 1_{\mathbf{6}}, 1/2_{\mathbf{4}}, 0_{\mathbf{2}})_{\mathcal{N}=4} = (1_{\mathbf{1}}, 1/2_{\mathbf{4}}, 0_{\mathbf{6}})_{\mathcal{N}=4} \otimes (1_{\mathbf{1}})_{\mathcal{N}=0}\,. \quad \text{(III.11)}$$

The two real scalars arise from the trace part and the anti-symmetric part (after dualisation in four dimensions) of the tensorial product of the two vectors. Using the identification of $\mathcal{W}_{0,0}|_{q^0} + \mathcal{W}_{0,1}|_{q^0}$ with a scalar contribution and the equation (A.31) we can now identify $\mathcal{W}_{0,0}|_{\sqrt{q}} - \mathcal{W}_{0,0}|_{\sqrt{q}}$ with the contribution of a vector and two scalars. This confirms the identification of $\mathcal{W}_{1,0}|_{q^0}$ with a spin-$\frac{1}{2}$ contribution in the end of section III A.

Since

$$(3/2_{\mathbf{1}}, 1_{\mathbf{4}}, 1/2_{\mathbf{6+1}}, 0_{\mathbf{4+\bar{4}}})_{\mathcal{N}=4} = (1_{\mathbf{1}}, 1/2_{\mathbf{4}}, 0_{\mathbf{6}})_{\mathcal{N}=4} \otimes (1/2)_{\mathcal{N}=0}\,, \quad \text{(III.12)}$$

we see that removing the four spin $\frac{1}{2}$ (that is, the term $\mathcal{W}_{1,0}|_{q^0}$) of the right-moving massless spectrum of the string theory construction in asymmetric type II models removes the contribution from the massless spin $\frac{3}{2}$ to the amplitudes. For the asymmetric type II model, using (III.5), we can present the contribution from the $\mathcal{N} = 4$ supergravity multiplet in a form that reflects the decomposition of the $\mathcal{N} = 8$ supergravity multiplet into $\mathcal{N} = 4$

supermultiplets

$$\begin{aligned}(2_{\mathbf{1}},3/2_{\mathbf{8}},1_{\mathbf{28}},1/2_{\mathbf{56}},0_{\mathbf{70}})_{\mathcal{N}=8} &= (2_{\mathbf{1}},3/2_{\mathbf{4}},1_{\mathbf{6}},1/2_{\mathbf{4}},0_{\mathbf{1+1}})_{\mathcal{N}=4}\\ &\oplus\ \mathbf{4}\,(3/2_{\mathbf{1}},1_{\mathbf{4}},1/2_{\mathbf{6+1}},0_{\mathbf{4+\bar{4}}})_{\mathcal{N}=4}\\ &\oplus\ \mathbf{6}\,(1_{\mathbf{1}},1/2_{\mathbf{4}},0_{\mathbf{6}})_{\mathcal{N}=4}\,,\end{aligned} \tag{III.13}$$

as

$$A^{spin-2}_{(4,0)II} = A^{spin-2}_{\mathcal{N}=8} - 6\,A^{spin-1}_{(4,0)II} - 4\,A^{spin-\frac{3}{2}}_{(4,0)II}\,, \tag{III.14}$$

where we have introduced the $\mathcal{N}=8$ spin-2 supergravity contribution

$$A^{spin-2}_{\mathcal{N}=8} = \frac{1}{4}\left(\frac{\pi}{2}\right)^8 t_8t_8R^4\,, \tag{III.15}$$

and the $\mathcal{N}=4$ spin-$\frac{3}{2}$ supergravity contribution

$$A^{spin-\frac{3}{2}}_{(4,0)II} = -\left(\frac{\pi}{2}\right)^4 t_8F^4\int_{-\frac{1}{2}}^{\frac{1}{2}}\prod_{i=1}^4 d\nu_i^1\,\mathcal{W}_{1,0}|_{q^0}\,. \tag{III.16}$$

For the $(2,2)$ models the contribution of the massless states to the amplitude is given in (II.48). The contribution from a vector multiplet is

$$\begin{aligned}A^{spin-1}_{(2,2)} &= 2\int_{-\frac{1}{2}}^{\frac{1}{2}}\prod_{i=1}^4 d\nu_i^1\,\left(\Big|\mathcal{W}_{0,0}|_{q^0}-\mathcal{W}_{1,0}|_{q^0}\Big|^2+\Big|\mathcal{W}_{0,1}|_{q^0}-\mathcal{W}_{1,0}|_{q^0}\Big|^2\right)\\ &\quad+2\int_{-\frac{1}{2}}^{\frac{1}{2}}\prod_{i=1}^4 d\nu_i^1\,(\mathcal{W}_{0,0;0,0}|_{q^0;\bar{q}^0}+\mathcal{W}_{0,1;0,1}|_{q^0;\bar{q}^0}-2\mathcal{W}_{1,0;1,0}|_{q^0;\bar{q}^0})\,.\end{aligned} \tag{III.17}$$

Using that $\big|\mathcal{W}_{0,0}|_{q^0}-\mathcal{W}_{1,0}|_{q^0}\big|^2=\big|\mathcal{W}_{0,1}|_{q^0}-\mathcal{W}_{1,0}|_{q^0}\big|^2=\frac{1}{4}\big|\mathcal{W}_{0,0}|_{q^0}+\mathcal{W}_{0,1}|_{q^0}-2\mathcal{W}_{1,0}|_{q^0}\big|^2$ as a consequence of (III.4), we can rewrite this as

$$\begin{aligned}A^{spin-1}_{(2,2)} &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\prod_{1\leq i<j\leq 4} d\nu_i^1\,\big|\mathcal{W}_{0,0}|_{q^0}+\mathcal{W}_{0,1}|_{q^0}-2\mathcal{W}_{1,0}|_{q^0}\big|^2\\ &\quad+2\int_{-\frac{1}{2}}^{\frac{1}{2}}\prod_{i=1}^4 d\nu_i^1\,(\mathcal{W}_{0,0;0,0}|_{q^0;\bar{q}^0}+\mathcal{W}_{0,1;0,1}|_{q^0;\bar{q}^0}-2\mathcal{W}_{1,0;1,0}|_{q^0;\bar{q}^0})\,,\end{aligned} \tag{III.18}$$

showing that this spin-1 contribution in the $(2,2)$ models arises as the product of two $\mathcal{N}=2$ hypermultiplets $Q=(2\times 1/2_{\mathbf{1}},2\times 0_{\mathbf{2}})_{\mathcal{N}=2}$

$$2\times(1_{\mathbf{1}},1/2_{\mathbf{4}},0_{\mathbf{6}})_{\mathcal{N}=4} = (2\times 1/2_{\mathbf{1}},2\times 0_{\mathbf{2}})_{\mathcal{N}=2}\otimes(2\times 1/2_{\mathbf{1}},2\times 0_{\bar{\mathbf{2}}})_{\mathcal{N}=2}\,. \tag{III.19}$$

The contribution from the $\mathcal{N}=4$ supergravity multiplet running in the loop (obtained from (II.48) by setting $n_v=0$) can be presented in a form reflecting the decomposition in (III.13)

$$A^{spin-2}_{(2,2)} = A^{spin-2}_{\mathcal{N}=8} - 6A^{spin-1}_{(2,2)} - 4\,A^{spin-\frac{3}{2}}_{(2,2)}\,, \tag{III.20}$$

where $A^{spin-\frac{3}{2}}_{(2,2)}$ is given by

$$A^{spin-\frac{3}{2}}_{(2,2)} = -\frac{1}{8}\,A^{spin-2}_{\mathcal{N}=8} - \frac{1}{32}\int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{1\leq i<j\leq 4} d\nu_i^1\, \left|\mathcal{W}_{0,0}|_{\sqrt{q}} - \mathcal{W}_{0,1}|_{\sqrt{q}}\right|^2 . \tag{III.21}$$

C. Comparing of the string models

The integrands of the amplitudes in the two $(4,0)$ models in (II.17) and (II.32) and the $(2,2)$ models in (II.48) take a different form. In this section we show first the equality between the integrands of the $(4,0)$ models and then that any difference with the $(2,2)$ models can be attributed to the contribution of the vector multiplets.

The comparison is done in the field theory limit where $\tau_2\to+\infty$ and $\alpha'\to 0$ with $t=\alpha'\tau_2$ held fixed. The real parts of the ν_i variables are integrated over the range $-\frac{1}{2}\leq\nu_i^1\leq\frac{1}{2}$. In this limit the position of the vertex operators scale as $\nu_i=\nu_i^1+i\tau_2\omega_i$. The positions of the external legs on the loop are then denoted by $0\leq\omega_i\leq 1$ and are ordered according to the kinematical region under consideration. In this section we discuss the integration over the ν_i^1's only; the integration over the ω_i's will be performed in section IV.

1. Comparing the $(4,0)$ *models*

In the heterotic string amplitude (II.17), we can identify two distinct contributions; n_v vector multiplets and one $\mathcal{N}=4$ supergravity multiplet running in the loop. At the leading order in α', the contribution of the vector multiplets is given by:

$$A^{spin-1}_{(4,0)het} = \frac{1}{2}\left(\frac{\pi}{2}\right)^4 t_8F^4 \int_{-\frac{1}{2}}^{\frac{1}{2}}\prod_{i=1}^{4} d\nu_i^1\, \bar{\mathcal{W}}^B|_{\bar{q}^0}\,, \tag{III.22}$$

and the one of the supergravity multiplet by

$$A^{spin-2}_{(4,0)het} = \frac{1}{2}\left(\frac{\pi}{2}\right)^4 t_8F^4 \int_{-\frac{1}{2}}^{\frac{1}{2}}\prod_{i=1}^{4} d\nu_i^1\, \Big((\bar{\mathcal{W}}^B|_{\bar{q}}(1+\alpha'\delta Q) + \bar{\mathcal{W}}^B|_{\bar{q}^0}\,\mathcal{Q}|_{\bar{q}} + 2\bar{\mathcal{W}}^B|_{\bar{q}^0}\Big)\,. \tag{III.23}$$

The vector multiplet contributions take different forms in the heterotic construction in (III.22) and the type II models in (III.8). However using the expansion of the fermionic propagators given in Appendix A 2, it is not difficult to perform the integration over ν_i^1 in (III.8). We see that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{1\leq i<j\leq 4} d\nu_i^1\, (\mathcal{W}^F_{0,0}|_{\bar q^0} + \mathcal{W}^F_{0,1}|_{\bar q^0}) = 0\,. \tag{III.24}$$

Thus there only remains the bosonic part of $\mathcal{W}_{a,b}$, and we find that the contribution of the vector multiplet is the same in the heterotic and asymmetric orbifold constructions

$$A^{spin-1}_{(4,0)het} = A^{spin-1}_{(4,0)II}\,. \tag{III.25}$$

The case of the $\mathcal{N}=4$ super-graviton is a little more involved. In order to simplify the argument we make the following choice of helicity to deal with more manageable expressions: $(1^{++},2^{++},3^{--},4^{--})$. We set as well the reference momenta q_i's for graviton $i=1,\cdots,4$ as follows: $q_1=q_2=k_3$ and $q_3=q_4=k_1$. At four points in supersymmetric theories, amplitudes with more $+$ or $-$ helicity states vanish. In that manner the covariant quantities t_8F^4 and $t_8t_8R^4$ are written in the spinor helicity formalism[3] $2t_8F^4 = \langle k_1\, k_2\rangle^2\,[k_3\, k_4]^2$, and $4t_8t_8R^4 = \langle k_1\, k_2\rangle^4\,[k_3\, k_4]^4$, respectively. With this choice of gauge $\epsilon^{(1)}\cdot\epsilon^{(k)}=0$ for $k=2,3,4$, $\epsilon^{(3)}\cdot\epsilon^{(l)}=0$ with $l=2,4$ and only $\epsilon^{(2)}\cdot\epsilon^{(4)}\neq 0$. The same relationships hold for the scalar product between the right-moving $\tilde\epsilon$ polarizations and the left- and right-moving polarizations . We can now simplify the various kinematical factors $\mathcal{W}^B$ for the heterotic string and the $\mathcal{W}_{a,b}$'s for the type II models. We find $\bar{\mathcal{W}}^B = \frac{1}{2}t_8\tilde F^4\,\tilde{\mathcal{W}}^B$ where

$$\tilde{\mathcal{W}}^B = \tilde{\mathcal{W}}^B_1 + \frac{1}{\alpha' u}\tilde{\mathcal{W}}^B_2\,, \tag{III.26}$$

with

$$\begin{aligned}
\tilde{\mathcal{W}}^B_1 &= (\bar\partial\mathcal{P}(12)-\bar\partial\mathcal{P}(14))(\bar\partial\mathcal{P}(21)-\bar\partial\mathcal{P}(24))(\bar\partial\mathcal{P}(32)-\bar\partial\mathcal{P}(34))(\bar\partial\mathcal{P}(42)-\bar\partial\mathcal{P}(43))\,,\\
\tilde{\mathcal{W}}^B_2 &= \bar\partial^2\mathcal{P}(24)(\bar\partial\mathcal{P}(12)-\bar\partial\mathcal{P}(14))(\bar\partial\mathcal{P}(32)-\bar\partial\mathcal{P}(34))\,.
\end{aligned} \tag{III.27}$$

In these equations it is understood that $\mathcal{P}(ij)$ stands for $\mathcal{P}(\nu_i-\nu_j)$. We find as well that

[3] A null vector $k^2=0$ is parametrized as $k_{\alpha\dot\alpha}=k_\alpha\bar k_{\dot\alpha}$ where $\alpha,\dot\alpha=1,2$ are $SL(2,\mathbb{C})$ two-dimensional spinor indices. The positive and negative helicity polarization vectors are given by $\epsilon^+(k,q)_{\alpha\dot\alpha} := \frac{q_\alpha\bar k_{\dot\alpha}}{\sqrt2\,\langle q\,k\rangle}$ and $\epsilon^-(k,q)_{\alpha\dot\alpha} := -\frac{k_\alpha\bar q_{\dot\alpha}}{\sqrt2\,[q\,k]}$, respectively, where q is a massless reference momentum. The self-dual and anti-self-dual field strengths read $F^-_{\alpha\beta} := \sigma^{mn}_{\alpha\beta}F_{mn} = \frac{k_\alpha k_\beta}{\sqrt2}$ and $F^+_{\dot\alpha\dot\beta} := \bar\sigma^{mn}_{\dot\alpha\dot\beta}F_{mn} = -\frac{\bar k_{\dot\alpha}\bar k_{\dot\beta}}{\sqrt2}$, respectively.

$\mathcal{W}^F_{a,b} = \frac{1}{2} t_8 F^4 \tilde{\mathcal{W}}^F_{a,b}$ with $\tilde{\mathcal{W}}^F_{a,b} = \tilde{\mathcal{S}}_{4;a,b} + \tilde{\mathcal{S}}_{2;a,b}$, where

$$\begin{aligned}\tilde{\mathcal{S}}_{4;a,b} &= \frac{1}{2^8}\big(S_{a,b}(12)^2 S_{a,b}(34)^2 - S_{a,b}(1234) - S_{a,b}(1243) - S_{a,b}(1423)\big)\\ \tilde{\mathcal{S}}_{2;a,b} &= \frac{1}{2^4}(\partial\mathcal{P}(12) - \partial\mathcal{P}(14))(\partial\mathcal{P}(21) - \partial\mathcal{P}(24))(S_{a,b}(34))^2\\ &+ \frac{1}{2^4}(\partial\mathcal{P}(32) - \partial\mathcal{P}(34))(\partial\mathcal{P}(42) - \partial\mathcal{P}(43))(S_{a,b}(12))^2\,,\end{aligned} \tag{III.28}$$

where we have used a shorthand notation; $S_{a,b}(ij)$ stands for $S_{a,b}(z_i - z_j)$ while $S_{a,b}(ijkl)$ stands for $S_{a,b}(z_i - z_j)S_{a,b}(z_j - z_k)S_{a,b}(z_k - z_l)S_{a,b}(z_l - z_i)$. With that choice of helicity, we can immediately give a simplified expression for the contribution of a spin-1 supermultiplet in the $(4,0)$ models. We introduce the field theory limit of $\tilde{\mathcal{W}}^B$:

$$W^B := \lim_{\tau_2\to\infty}\left(\frac{2}{\pi}\right)^4 \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1\, \tilde{\mathcal{W}}^B|_{\bar{q}^0}\,. \tag{III.29}$$

In this limit, this quantity is given by $W^B = W_1 + W_2$ with

$$\begin{aligned}W_1 &= \frac{1}{16}(\partial P(12) - \partial P(14))(\partial P(21) - \partial P(24))(\partial P(32) - \partial P(34))(\partial P(42) - \partial P(43))\,,\\ W_2 &= \frac{1}{4\pi}\frac{1}{\alpha'\tau_2 u}\,\partial^2 P(24)(\partial P(12) - \partial P(14))(\partial P(32) - \partial P(34))\,,\end{aligned} \tag{III.30}$$

where $\partial^n P(\omega)$ is the n-th derivative of the field theory propagator (III.35) and where $\alpha'\tau_2$ is the proper time of the field one-loop amplitude. We can now rewrite (III.22) and find

$$A^{spin-1}_{(4,0)het} = \frac{1}{4}\left(\frac{\pi}{2}\right)^8 t_8 t_8 R^4\, W^B\,. \tag{III.31}$$

Let us come back to the comparison of the $\mathcal{N} = 4$ spin-2 multiplet contributions in the type II asymmetric orbifold model given in (III.10) and the heterotic one given in (III.23).

We consider the following part of (III.23)

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1\, (\bar{\mathcal{W}}^B|_{\bar{q}}(1 + \alpha'\delta Q) + \bar{\mathcal{W}}^B|_{\bar{q}^0}\,\mathcal{Q}|_{\bar{q}})\,, \tag{III.32}$$

defined in the field theory limit for large τ_2.

The integral over the ν_i^1 will kill any term that have a non zero phase $e^{i\pi(a\nu_1^1 + b\nu_2^1 + c\nu_3^1 + d\nu_4^1)}$ where a, b, c, d are non-all-vanishing integers. In $\tilde{W}_1^B$ we have terms of the form $\partial\delta P(ij)\times \partial\mathcal{P}(ji)|_{\bar{q}} \times (\partial P(kl) - \partial P(k'l'))(\partial P(rs) - \partial P(r's'))$. Using the definition of $\delta\mathcal{P}(ij)$ given in (A.15) and the order $\bar{q}$ coefficient of the propagator in (A.14), we find that

$$\partial\delta P(ij) \times \partial\mathcal{P}(ji)|_{\bar{q}} = -\frac{i\pi^2}{2}\,\sin(2\pi\nu_{ij})\,\mathrm{sign}(\omega_{ij})e^{2i\pi\mathrm{sign}(\omega_{ij})\nu_{ij}}\,, \tag{III.33}$$

which integrates to $-\pi^2/4$. All such terms with $(ij) = (12)$ and $(ij) = (34)$ contribute in total to

$$\frac{1}{2}\left(\frac{\pi}{2}\right)^4 [(\partial P(12) - \partial P(14))(\partial P(21) - \partial P(24)) + (\partial P(32) - \partial P(34))(\partial P(42) - \partial P(43))] \,, \tag{III.34}$$

where $\partial P(ij)$ is for the derivative of the propagator in the field theory $\partial P(\omega_i - \omega_j)$ given by

$$\partial P(\omega) = 2\omega - \mathrm{sign}(\omega)\,. \tag{III.35}$$

The last contraction in $\tilde{W}_1^B$ for $(ij) = (24)$ leads to the same kind of contribution. However, they will actually be cancelled by terms coming from similar contractions in $\tilde{\mathcal{W}}_2^B|_{\bar{q}}$. More precisely, the non zero contractions involved yield

$$(\partial^2\mathcal{P}(24)e^{\mathcal{Q}})|_{\bar{q}} = -\frac{\alpha'\pi^2}{2}\left(\cos(2\pi\nu_{24})e^{2i\pi \mathrm{sign}(\omega_{24})\nu_{24}} - 2e^{2i\pi \mathrm{sign}(\omega_{24})\nu_{24}}\sin^2(\pi\nu_{24})\right)\,, \tag{III.36}$$

which integrates to $-\alpha'\pi^2/2$. The α' compensates the $1/\alpha'$ factor in (III.26) and this contribution precisely cancels the one from (III.33) with $(ij) = (24)$. Other types of terms with more phase factors from the propagator turn out to vanish after summation. In all, we get $-\pi^4 W_3/4$, where

$$W_3 = -\frac{1}{8}\Big((\partial P(12) - \partial P(14))(\partial P(21) - \partial P(24)) + (\partial P(32) - \partial P(34))(\partial P(42) - \partial P(43))\Big)\,, \tag{III.37}$$

Finally, let us look at the totally contracted terms of the form $\partial\delta P(ik)\partial\delta P(kl)\partial\delta P(lj) \times \partial\mathcal{P}(ij)|_{\bar{q}}$ that come from $\tilde{\mathcal{W}}_1^B|\bar{q}$. Those are the only terms of that type that survive the ν^1 integrations since they form closed chains in their arguments. They give the following terms

$$i\frac{\pi^4}{8}\sin(\pi\nu_{ij})\mathrm{sign}(\omega_{ik})\mathrm{sign}(\omega_{kl})\mathrm{sign}(\omega_{lj})e^{2i\pi\left(\mathrm{sign}(\omega_{ik})\nu_{ik} + \mathrm{sign}(\omega_{kl})\nu_{kl} + \mathrm{sign}(\omega_{lj})\nu_{lj}\right)}\,. \tag{III.38}$$

They integrate to $\pi^4/16$ if the vertex operators are ordered according to $0 \le \omega_i < \omega_k < \omega_l < \omega_j \le 1$ or in the reversed ordering. Hence, from $\tilde{\mathcal{W}}_1^B$ we will get one of the orderings we want in our polarization choice, namely the region (s,t). From $\tilde{\mathcal{W}}_2 e^{\mathcal{Q}}$, a similar computation yields the two other kinematical regions (s,u) and (t,u). In all we have a total integrated contribution of $\pi^4/16$. We collect all the different contributions that we have obtained, and (III.23) writes:

$$A^{spin-2}_{(4,0)het} = \frac{1}{4}\left(\frac{\pi}{2}\right)^8 t_8 t_8 R^4\,(1 - 4W_3 + 2\,W^B)\,, \tag{III.39}$$

where we used that $t_8 t_8 R^4 = t_8 F^4 t_8 \tilde{F}^4$ and (III.29) and (III.30).

We now turn to the spin-2 contribution in the type II asymmetric orbifold models given in (III.10). Using the q expansion detailed in Appendix A 2 c, we find that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1 \, (\tilde{\mathcal{W}}_{0,0}|_{\sqrt{q}} - \tilde{\mathcal{W}}_{0,1}|_{\sqrt{q}}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1 \, (\tilde{\mathcal{W}}^F_{0,0}|_{\sqrt{q}} - \tilde{\mathcal{W}}^F_{0,1}|_{\sqrt{q}}) = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1 \, \tilde{\mathcal{W}}^F_{0,0}|_{\sqrt{q}} \,. \tag{III.40}$$

We have then terms of the form $\tilde{\mathcal{S}}_{2;0,0}$ and $\tilde{\mathcal{S}}_{4;0,0}$. Their structure is similar to the terms in the heterotic case with, respectively, two and four bosonic propagators contracted. The bosonic propagators do not have a $\sqrt{q}$ piece and since $\tilde{\mathcal{S}}_{0,0}(12)^2|_{\sqrt{q}} = \tilde{\mathcal{S}}_{0,0}(34)^2|_{\sqrt{q}} = 4\pi^2$ we find that the terms in $\mathcal{S}_{2;0,0}$ give

$$2\int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1 \, \tilde{\mathcal{S}}_{2;0,0}|_{\sqrt{q}} = -4\left(\frac{\pi}{2}\right)^4 W_3 \,, \tag{III.41}$$

including the $1/2^4$ present in (II.26). The $\tilde{\mathcal{S}}_{4;0,0}$ terms have two different kind of contributions: double trace and single trace (see, respectively, first and second lines in (II.26)). In the spin structure $(0,0)$ the double trace always vanishes in $\tilde{\mathcal{S}}_{4;0,0}|_{\sqrt{q}}$ since

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1 \, \frac{\sin(\pi\nu_{ij})}{\sin^2(\pi\nu_{kl})\sin(\pi\nu_{ij})} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \prod_{i=1}^{4} d\nu_i^1 \, \frac{1}{\sin^2(\pi\nu_{kl})} = 0 \,. \tag{III.42}$$

However the single trace terms are treated in the same spirit as for the heterotic string. Only closed chains of sines contribute and are non zero only for specific ordering of the vertex operators. For instance,

$$-4\pi^4 \frac{\sin(\pi\nu_{ij})}{\sin(\pi\nu_{jk})\sin(\pi\nu_{kl})\sin(\pi\nu_{li})} \underset{\tau_2\to\infty}{\sim} -(2\pi)^4 \,, \tag{III.43}$$

for the ordering $0 \le \omega_j < \omega_l < \omega_k < \omega_i \le 1$. Summing all of the contributions from $\tilde{\mathcal{S}}_{4;0,0}$ gives a total factor of $-\pi^4/16$, including the normalization in (II.26). We can now collect all the terms to get

$$A^{spin-2}_{(4,0)II} = \frac{1}{4}\left(\frac{\pi}{2}\right)^8 t_8 t_8 R^4 \,(1 - 4W_3 + 2\,W^B) \,, \tag{III.44}$$

showing the equality with the heterotic expression

$$A^{spin-2}_{(4,0)het} = A^{spin-2}_{(4,0)II} \,. \tag{III.45}$$

We remark that the same computations give the contribution of the spin-$\frac{3}{2}$ multiplets in the two models, which are equal as well and write :

$$A^{spin-\frac{3}{2}}_{(4,0)het} = A^{spin-\frac{3}{2}}_{(4,0)II} = \frac{1}{4}\left(\frac{\pi}{2}\right)^8 t_8t_8R^4\,(W_3 - 2\,W^B)\,. \tag{III.46}$$

Thanks to those equalities for the spin-2, spin-$\frac{3}{2}$ and spin-1 in (III.25), from now we will use the notation $A^{spin-s}_{(4,0)}$ with $s = 1, \frac{3}{2}, 2$.

The perturbative equality between these two $(4,0)$ models is not surprising. For a given number of vector multiplets n_v the heterotic and asymmetric type II construction lead to two string theory $(4,0)$ models related by S-duality, $S \to -1/S$, where S is the axion-dilaton complex scalar in the $\mathcal{N} = 4$ supergravity multiplet. The perturbative expansion in these two models is defined around different points in the $SU(1,1)/U(1)$ moduli space. The action of $\mathcal{N} = 4$ supersymmetry implies that the one-loop amplitudes between gravitons, which are neutral under the $U(1)$ R-symmetry, are the same in the strong and weak coupling regimes.

2. *Comparing the* $(4,0)$ *and* $(2,2)$ *models*

In the case of the $(2,2)$ models, the contribution from the vector multiplets is given in (III.18). The string theory integrand is different from the one in (III.8) for the $(4,0)$ as it can be seen using the supersymmetric Riemann identity in (III.5). Let us first write the spin-1 contribution in the $(2,2)$ models. Performing the ν_i^1 integrations and the same kind of manipulations that we have done in the previous section, we can show that it is given by

$$A^{spin-1}_{(2,2)} = \frac{1}{4}\left(\frac{\pi}{2}\right)^8 t_8t_8R^4\,((W_3)^2 + \frac{1}{2}\,W_2)\,. \tag{III.47}$$

This is to be compared with (III.31). The expressions are clearly different, but will lead to the same amplitude. In the same manner, we find for the spin-$\frac{3}{2}$:

$$A^{spin-\frac{3}{2}}_{(2,2)} = \frac{1}{4}\left(\frac{\pi}{2}\right)^8 t_8t_8R^4\,(W_3 - 2((W_3)^2 + \frac{1}{2}\,W_2))\,. \tag{III.48}$$

This differs from (III.46) by a factor coming solely from the vector multiplets.

We now compare the spin-2 contributions in the $(4,0)$ model in (III.16) and the $(2,2)$ model in (III.21). Again, a similar computation to the one we have done gives the contribution of the spin-2 multiplet running in the loop for the $(2,2)$ model:

$$A^{spin-2}_{(2,2)} = \frac{1}{4}\left(\frac{\pi}{2}\right)^8 t_8t_8R^4\,(1 - 4W_3 + 2((W_3)^2 + \frac{1}{2}\,W_2))\,. \tag{III.49}$$

We compare this with (III.39) and (III.44) that we rewrite in the following form:

$$A^{spin-2}_{(4,0)II} = \frac{1}{4}\left(\frac{\pi}{2}\right)^8 t_8 t_8 R^4 \left(1 - 4W_3 + 2(W_1 + W_2)\right). \tag{III.50}$$

The difference between the two expressions originates again solely from the vector multiplet sector. Considering that the same relation holds for the contribution of the $\mathcal{N} = 4$ spin-$\frac{3}{2}$ multiplets, we deduce that this is coherent with the supersymmetric decomposition (III.13) that gives

$$A^{spin-2}_{(2,2)} = A^{spin-2}_{(4,0)} + 2\,(A^{spin-1}_{(2,2)} - A^{spin-1}_{(4,0)}). \tag{III.51}$$

The difference between the spin-2 amplitudes in the two models is completely accounted for by the different vector multiplet contributions. The string theory models are related by a U-duality exchanging the axion-dilaton scalar S of the gravity multiplet with a geometric modulus [27, 28, 36]. This transformation affects the coupling of the multiplet running in the loop, thus explaining the difference between the two string theory models. However at the supergravity level, the four graviton amplitudes that we compute are not sensitive to this fact and are equal in all models, as we will see now.

IV. FIELD THEORY ONE-LOOP AMPLITUDES IN $\mathcal{N} = 4$ SUPERGRAVITY

In this section we shall extract and compute the field theory limit $\alpha' \to 0$ of the one-loop string theory amplitudes studied in previous sections. We show some relations between loop momentum power counting and the spin or supersymmetry of the multiplet running in the loop.

As mentioned above, the region of the fundamental domain integration corresponding to the field theory amplitude is $\tau_2 \to \infty$, such that $t = \alpha' \tau_2$ is fixed. We then obtain a world-line integral of total proper time t. The method for extracting one-loop field theory amplitudes from string theory was pioneered in [20]. The general method that we apply consists in extracting the $o(q)^0$ terms in the integrand and taking the field theory limit and was developed extensively in [23, 24, 47]. Our approach will follow the formulation given in [25].

The generic form of the field theory four-graviton one-loop amplitude for $\mathcal{N} = 4$ supergravity with a spin-s ($s = 1, \frac{3}{2}, 2$) $\mathcal{N} = 4$ supermultiplet running is the loop is given by

$$M_X^{spin-s} = \left(\frac{4}{\pi}\right)^4 \frac{\mu^{2\epsilon}}{\pi^{\frac{D}{2}}} \int_0^\infty \frac{dt}{t^{\frac{D-6}{2}}} \int_{\Delta_\omega} \prod_{i=1}^{3} d\omega_i \, e^{-\pi\, t\, Q(\omega)} \times A_X^{spin-s} \,, \tag{IV.1}$$

where $D = 4 - 2\epsilon$ and X stands for the model, $X = (4,0)het$, $X = (4,0)II$ or $X = (2,2)$ while the respective amplitudes A_X^{spin-s} are given in sections III B and III C. We have set the overall normalization to unity.

The domain of integration $\Delta_\omega = [0,1]^3$ is decomposed into three regions $\Delta_w = \Delta_{(s,t)} \cup \Delta_{(s,u)} \cup \Delta_{(t,u)}$ given by the union of the (s,t), (s,u) and (t,u) domains. In the $\Delta_{(s,t)}$ domain the integration is performed over $0 \leq \omega_1 \leq \omega_2 \leq \omega_3 \leq 1$ where $Q(\omega) = -s\omega_1(\omega_3-\omega_2) - t(\omega_2-\omega_1)(1-\omega_3)$ with equivalent formulas obtained by permuting the external legs labels in the (t,u) and (s,u) regions (see [48] for details). We used that $s = -(k_1+k_2)^2$, $t = -(k_1+k_4)^2$ and $u = -(k_1+k_3)^2$ with our convention for the metric $(- + \cdots +)$.

We now turn to the evaluation of the amplitudes. The main properties of the bosonic and fermionic propagators are provided in Appendix A 2. We work with the helicity configuration detailed in the previous section. This choice of polarization makes the intermediate steps easier as the expressions are explicitly gauge invariant.

A. Supersymmetry in the loop

Before evaluating the amplitudes we discuss the action of supersymmetry on the structure of the one-loop amplitudes. An n-graviton amplitude in dimensional regularization with $D = 4 - 2\epsilon$ can generically be written in the following way:

$$M_{n;1} = \mu^{2\epsilon} \int \frac{d^D\ell}{(2\pi)^D} \frac{\mathfrak{N}(\epsilon_i, k_i; \ell)}{\ell^2(\ell-k_1)^2 \cdots (\ell - \sum_{i=1}^{n-1} k_i)^2} \,, \tag{IV.2}$$

where the numerator is a polynomial in the loop momentum ℓ with coefficients depending on the external momenta k_i and polarization of the gravitons ϵ_i. For ℓ large this numerator behaves as $\mathfrak{N}(\epsilon_i, k_i; \ell) \sim \ell^{2n}$ in non-supersymmetric theories. In an $\mathcal{N}$ extended supergravity theory, supersymmetric cancellations improve this behaviour, which becomes $\ell^{2n-\mathcal{N}}$ where $\mathcal{N}$ is the number of four-dimensional supercharges:

$$\mathfrak{N}^{\mathcal{N}}(\epsilon_i, k_i; \ell) \sim \ell^{2n-\mathcal{N}} \qquad \text{for} \qquad |\ell| \to \infty \,. \tag{IV.3}$$

The dictionary between the Feynman integral presentation given in (IV.2) and the structure of the field theory limit of the string theory amplitude states that the first derivative of a bosonic propagator counts as one power of loop momentum $\partial\mathcal{P} \sim \ell$, $\partial^2\mathcal{P} \sim \ell^2$ while fermionic propagators count for zero power of loop momentum $S_{a,b} \sim 1$. This dictionary was first established in [47] for gauge theory computation, and applied to supergravity amplitudes computations in [24] and more recently in [25].

With this dictionary we find that in the $(4,0)$ model the integrand of the amplitudes have the following behaviour

$$\begin{aligned} A^{spin-1}_{(4,0)} &\sim \ell^4\,,\\ A^{spin-\frac{3}{2}}_{(4,0)} &\sim \ell^2+\ell^4\,,\\ A^{spin-2}_{(4,0)} &\sim 1+\ell^2+\ell^4\,. \end{aligned} \tag{IV.4}$$

The spin-1 contribution to the four-graviton amplitude has four powers of loop momentum as required for an $\mathcal{N}=4$ amplitude according (IV.3). The $\mathcal{N}=4$ spin-$\frac{3}{2}$ supermultiplet contribution can be decomposed into an $\mathcal{N}=6$ spin-$\frac{3}{2}$ supermultiplet term with two powers of loop momentum, and an $\mathcal{N}=4$ spin-1 supermultiplet contribution with four powers of loop momentum. The spin-2 contribution has an $\mathcal{N}=8$ spin-2 piece with no powers of loop momentum, an $\mathcal{N}=6$ spin-$\frac{3}{2}$ piece with two powers of loop momentum and an $\mathcal{N}=4$ spin-1 piece with four powers of loop momentum.

For the $(2,2)$ construction we have the following behaviour

$$\begin{aligned} A^{spin-1}_{(2,2)} &\sim (\ell^2)^2\,,\\ A^{spin-\frac{3}{2}}_{(2,2)} &\sim \ell^2+(\ell^2)^2\,,\\ A^{spin-2}_{(2,2)} &\sim 1+\ell^2+(\ell^2)^2\,. \end{aligned} \tag{IV.5}$$

Although the superficial counting of the number of loop momenta is the same for each spin-$s=1,\frac{3}{2},2$ in the two models, the precise dependence on the loop momentum differs in the two models, as indicated by the symbolic notation ℓ^4 and $(\ell^2)^2$. This is a manifestation of the model dependence for the vector multiplet contributions. As we have seen in the previous section, the order four terms in the loop momentum in the spin-$\frac{3}{2}$ and spin-2 parts are due to the spin-1 part.

At the level of the string amplitude, the multiplets running in the loop (spin-2 and spin-1) are naturally decomposed under the $\mathcal{N}=4$ supersymmetry group. However, at the level of the amplitudes in field theory it is convenient to group the various blocks according to the number of powers of loop momentum in the numerator

$$A^{spin-s}_{\mathcal{N}=4s} \sim \ell^{4(2-s)}, \qquad s = 1, \frac{3}{2}, 2\,, \tag{IV.6}$$

which is the same as organizing the terms according to the supersymmetry of the corresponding $\mathcal{N} = 4s$ spin-$s = 1, \frac{3}{2}, 2$ supermultiplet. In this decomposition it is understood that for the two $\mathcal{N} = 4$ models the dependence in the loop momenta is not identical.

From these blocks, one can reconstruct the contribution of the spin-2 $\mathcal{N} = 4$ multiplet that we are concerned with using the following relations

$$M_X^{spin-2} = M^{spin-2}_{\mathcal{N}=8} - 4M^{spin-\frac{3}{2}}_{\mathcal{N}=6} + 2M^{spin-1}_X\,, \tag{IV.7}$$

where the index X refers to the type of model, $(4,0)$ or $(2,2)$.

This supersymmetric decomposition of the one-loop amplitude reproduces the one given in [24, 30–35].

We shall come now to the evaluation of those integrals. We will see that even though the spin-1 amplitudes have different integrands, *i.e.* different loop momentum dependence in the numerator of the Feynman integrals, they are equal after integration.

B. Model-dependent part : $\mathcal{N} = 4$ vector multiplet contribution

In this section we first compute the field theory amplitude with an $\mathcal{N} = 4$ vector multiplet running in the loop for the two models. This part of the amplitude is model dependent as far as concerns the integrands. However, the value of the integrals is the same in the different models. Then we provide an analysis of the IR and UV behaviour of these amplitudes.

1. *Evaluation of the field theory amplitude*

The contribution from the $\mathcal{N} = 4$ spin-1 vector supermultiplets in the $(4,0)$ models is

$$M^{spin-1}_{(4,0)} = \left(\frac{4}{\pi}\right)^4 \frac{\mu^{2\epsilon}}{\pi^{\frac{D}{2}}} \int_0^\infty \frac{dt}{t^{\frac{D-6}{2}}} \int_{\Delta_\omega} d^3\omega\; e^{-\pi\, t\, Q(\omega)} \times A^{spin-1}_{(4,0)}\,, \tag{IV.8}$$

where $A^{spin-1}_{(4,0)}$ is given in (III.31) for instance and Q defined in (A.17). Integrating over the proper time t and setting $D = 4 - 2\epsilon$, the amplitude reads

$$M^{spin-1}_{(4,0)} = t_8 t_8 R^4 \int_{\Delta_\omega} d^3\omega \big[\Gamma\left(1+\epsilon\right)\, Q^{-1-\epsilon}\, W_2 + \Gamma\left(2+\epsilon\right)\, Q^{-2-\epsilon}\, W_1\big]\,. \tag{IV.9}$$

The quantities W_1 and W_2 are given in (III.30), they have the following form in terms of the variables ω_i:

$$W_1 = \frac{1}{8}(\omega_2-\omega_3)(\text{sign}(\omega_1-\omega_2)+2\omega_2-1)(\text{sign}(\omega_2-\omega_1)+2\omega_1-1)(\text{sign}(\omega_3-\omega_2)+2\omega_2-1)$$
$$W_2 = -\frac{1}{4}\frac{1}{u}(2\omega_2-1+\text{sign}(\omega_3-\omega_2))(2\omega_2-1+\text{sign}(\omega_1-\omega_2))\,(1-\delta(\omega_{24}))\,. \qquad \text{(IV.10)}$$

Using the dictionary between the world-line propagators and the Feynman integral from the string-based rules [24, 25, 47], we recognize in the first term in (IV.9) a six-dimensional scalar box integral and in the second term four-dimensional scalar bubble integrals.[4] Evaluating the integrals with standard techniques, we find[5]

$$M^{spin-1}_{(4,0)} = \frac{t_8t_8R^4}{2s^4}\left(s^2 - s(u-t)\log\left(\frac{-t}{-u}\right) - tu(\log^2\left(\frac{-t}{-u}\right)+\pi^2)\right)\,. \qquad \text{(IV.11)}$$

The crossing symmetry of the amplitude has been broken by our choice of helicity configuration. However, it is still invariant under the exchange of the legs $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ which amounts to exchanging t and u. The same comment applies to all the field theory amplitudes evaluated in this paper. This result matches the one derived in [24, 30–34] and in particular [35, eq. (3.20)].

Now we turn to the amplitude in the $(2,2)$ models:

$$M^{spin-1}_{(2,2)} = \left(\frac{4}{\pi}\right)^4 \frac{\mu^{2\epsilon}}{\pi^{\frac{D}{2}}}\int_0^\infty \frac{dt}{t^{\frac{D-6}{2}}}\int_{\Delta_\omega} d^3\omega\; e^{-\pi\, t\, Q(\omega)} \times A^{spin-1}_{(2,2)}\,, \qquad \text{(IV.12)}$$

where $A^{spin-1}_{(2,2)}$ is defined in (III.18). After integrating over the proper time t, one gets

$$M^{spin-1}_{(2,2)} = t_8t_8R^4\int_{\Delta_\omega} d^3\omega\,[\Gamma\,(2+\epsilon)\;Q^{-2-\epsilon}\,(W_3)^2 + \frac{1}{2}\Gamma(1+\epsilon)\,Q^{-1-\epsilon}\,W_2]\,, \qquad \text{(IV.13)}$$

where W_3 defined in (III.37), is given in terms of the ω_i variables by

$$W_3 = -\frac{1}{8}(\text{sign}(\omega_1-\omega_2)+2\omega_2-1)(\text{sign}(\omega_2-\omega_1)+2\omega_1-1)$$
$$+\frac{1}{4}(\text{sign}(\omega_3-\omega_2)+2\omega_2-1)(\omega_3-\omega_2)\,. \qquad \text{(IV.14)}$$

There is no obvious relation between the integrand of this amplitude with the one for $(4,0)$ model in (IV.9). Expanding the square one can decompose this integral in three pieces that

[4] In [25, 49] it was wrongly claimed that $\mathcal{N}=4$ amplitudes do not have rational pieces. The argument in [25] was based on a naive application of the reduction formulas for $\mathcal{N}=8$ supergravity amplitudes to $\mathcal{N}=4$ amplitudes where boundary terms do not cancel anymore.

[5] The analytic continuation in the complex energy plane corresponds to the $+i\varepsilon$ prescription for the Feynman propagators $1/(\ell^2-m^2+i\varepsilon)$. We are using the notation that $\log(-s)=\log(-s-i\varepsilon)$ and that $\log(-s/-t) := \log((-s-i\varepsilon)/(-t-i\varepsilon))$.

are seen to be proportional to the $(4,0)$ vector multiplet contribution in (IV.11). A first contribution is

$$\frac{t_8t_8R^4}{2}\int_{\Delta_\omega} d^3\omega\,[\Gamma\,(2+\epsilon)\;Q^{-2-\epsilon}\,W_1+\Gamma(1+\epsilon)\,Q^{-1-\epsilon}\,W_2]=\frac{1}{2}\,M^{spin-1}_{(4,0)} \tag{IV.15}$$

and we have the additional contributions

$$\frac{t_8t_8R^4}{64}\int_{\Delta_\omega} d^3\omega\,\frac{\Gamma\,(2+\epsilon)}{Q^{2+\epsilon}}\,((\mathrm{sign}(\omega_1-\omega_2)+2\omega_2-1)(\mathrm{sign}(\omega_2-\omega_1)+2\omega_1-1))^2=\frac{1}{4}\,M^{spin-1}_{(4,0)} \tag{IV.16}$$

and

$$\frac{t_8t_8R^4}{64}\int_{\Delta_\omega} d^3\omega\,\frac{\Gamma\,(2+\epsilon)}{Q^{2+\epsilon}}\,((\mathrm{sign}(\omega_3-\omega_2)+2\omega_2-1)(\omega_3-\omega_2))^2=\frac{1}{4}\,M^{spin-1}_{(4,0)}\,. \tag{IV.17}$$

Performing all the integrations leads to

$$M^{spin-1}_{(2,2)}=M^{spin-1}_{(4,0)}\,. \tag{IV.18}$$

It is now clear that the vector multiplet contributions to the amplitude are equal in the two theories, $(4,0)$ and $(2,2)$. It would be interesting to see if this expression could be derived with the double-copy construction of [35].

In this one-loop amplitude there is no interaction between the vector multiplets. Since the coupling of individual vector multiplet to gravity is universal (see for instance the $\mathcal{N}=4$ Lagrangian given in [50, eq.(4.18)]), the four-graviton one-loop amplitude in pure $\mathcal{N}=4$ supergravity has to be independent of the model it comes from.

2. *IR and UV behaviour*

The graviton amplitudes with vector multiplets running in the loop in (IV.11) and (IV.18) are free of UV and IR divergences. The absence of IR divergence is expected, since no spin-2 state is running in the loop. The IR divergence occurs only when a graviton is exchanged between two soft graviton legs (see figure 1). This fact has already been noticed in [30].

This behaviour is easily understood by considering the soft graviton limit of the coupling between the graviton and a spin-$s\neq 2$ state. It occurs through the stress-energy tensor $V^{\mu\nu}(k,p)=T^{\mu\nu}(p-k,p)$ where k and p are, respectively, the momentum of the graviton and

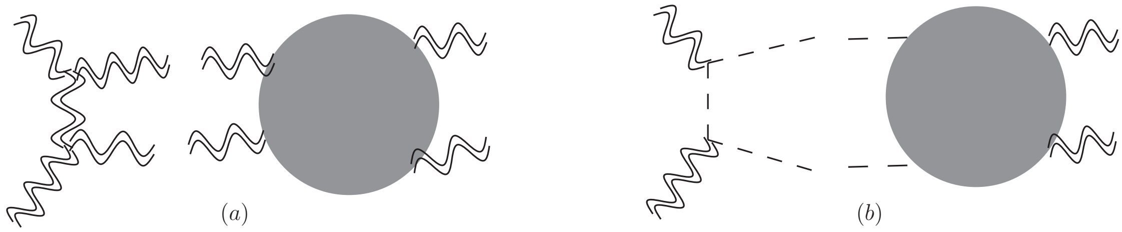

FIG. 1. Contribution to the IR divergences when two external gravitons (double wavy lines) become soft. If a graviton is exchanged as in (a) the amplitude presents an IR divergence. No IR divergences are found when another massless state of spin different from two is exchanged as in (b).

of the exchanged state. In the soft graviton limit the vertex behaves as $V^{\mu\nu}(p-k,p) \sim -k^\mu p^\nu$ for $p^\mu \sim 0$, and the amplitude behaves in the soft limit as

$$\int_{\ell\sim 0} \frac{d^4\ell}{\ell^2(\ell\cdot k_1)(\ell\cdot k_2)}\, T_{\mu\nu}(\ell-k_1,\ell)T^{\mu\nu}(\ell,\ell+k_2) \sim (k_1\cdot k_2)\int_{\ell\sim 0}\frac{d^4\ell}{\ell^2(\ell\cdot k_1)(\ell\cdot k_2)}\,\ell^2\,, \quad \text{(IV.19)}$$

which is finite for small values of the loop momentum $\ell \sim 0$. In the soft graviton limit, the three-graviton vertex behaves as $V^{\mu\nu}(k,p) \sim k^\mu k^\nu$ and the amplitude has a logarithmic divergence at $\ell \sim 0$:

$$(k_1\cdot k_2)^2 \int_{\ell\sim 0}\frac{d^4\ell}{\ell^2(\ell\cdot k_1)(\ell\cdot k_2)} = \infty\,. \quad \text{(IV.20)}$$

The absence of UV divergence is due to the fact that the R^2 one-loop counter-term is the Gauss-Bonnet term. It vanishes in the four-point amplitude since it is a total derivative [51].

C. Model-independent part

In this section we compute the field theory amplitudes with an $\mathcal{N}=8$ supergraviton and an $\mathcal{N}=6$ spin-$\frac{3}{2}$ supermultiplet running in the loop. These quantities are model independent in the sense that their integrands are the same in the different models.

1. *The $\mathcal{N}=6$ spin-$\frac{3}{2}$ supermultiplet contribution*

The integrand for the $\mathcal{N}=4$ spin-$\frac{3}{2}$ supermultiplet contribution is different in the two $(4,0)$ and $(2,2)$ constructions of the $\mathcal{N}=4$ supergravity models. As shown in equations (III.46) and. (III.48), this is accounted for by the contribution of the vector multiplets.

However, we exhibit an $\mathcal{N}=6$ spin-$\frac{3}{2}$ supermultiplet model-independent piece by adding two $\mathcal{N}=4$ vector multiplet contributions to the one of an $\mathcal{N}=4$ spin-$\frac{3}{2}$ supermultiplet

$$M_{\mathcal{N}=6}^{spin-\frac{3}{2}} = M_X^{spin-\frac{3}{2}} + 2M_X^{spin-1}\,. \tag{IV.21}$$

The amplitude with an $\mathcal{N}=6$ spin-$\frac{3}{2}$ multiplet running in the loop is

$$M_{\mathcal{N}=6}^{spin-\frac{3}{2}} = -\frac{t_8t_8R^4}{8}\int_{\Delta_\omega} d^3\omega\,\Gamma\left(2+\epsilon\right)\,W_3\,Q^{-2-\epsilon}, \tag{IV.22}$$

where W_3 is given in (IV.14). The integral is equal to the six-dimensional scalar box integral given in [35, eq. (3.16)] up to $o(\epsilon)$ terms. We evaluate it, and get

$$M_{\mathcal{N}=6}^{spin-\frac{3}{2}} = -\frac{t_8t_8R^4}{2s^2}\left(\log^2\left(\frac{-t}{-u}\right)+\pi^2\right)\,. \tag{IV.23}$$

This result is UV finite as expected from the superficial power counting of loop momentum in the numerator of the amplitude given in (IV.4). It is free of IR divergences because no graviton state is running in the loop (see the previous section). It matches the one derived in [24, 30–34] and in particular [35, eq. (3.17)].

2. The $\mathcal{N}=8$ spin-2 supermultiplet contribution

We now turn to the $\mathcal{N}=8$ spin-2 supermultiplet contribution in (IV.7). It has already been evaluated in [20, 52] and can be written as:

$$M_{\mathcal{N}=8}^{spin-2} = \frac{t_8t_8R^4}{4}\int_{\Delta_\omega} d^3\omega\,\Gamma\left(2+\epsilon\right)\,Q^{-2-\epsilon}\,. \tag{IV.24}$$

Performing the integrations we have

$$\begin{aligned} M_{\mathcal{N}=8}^{spin-2} = \frac{t_8t_8R^4}{4}\Bigg[\frac{2}{\epsilon}\left(\frac{\log\left(\frac{-t}{\mu^2}\right)}{su}+\frac{\log\left(\frac{-s}{\mu^2}\right)}{tu}+\frac{\log\left(\frac{-u}{\mu^2}\right)}{st}\right)+ \\ +\,2\left(\frac{\log\left(\frac{-s}{\mu^2}\right)\log\left(\frac{-t}{\mu^2}\right)}{st}+\frac{\log\left(\frac{-t}{\mu^2}\right)\log\left(\frac{-u}{\mu^2}\right)}{tu}+\frac{\log\left(\frac{-u}{\mu^2}\right)\log\left(\frac{-s}{\mu^2}\right)}{us}\right)\Bigg]\,, \end{aligned} \tag{IV.25}$$

where μ^2 is an IR mass scale. This amplitudes carries an ϵ pole signaling the IR divergence due to the graviton running in the loop.

Now we have all the blocks entering the expression for the $\mathcal{N}=4$ pure gravity amplitude in (IV.7).

V. CONCLUSION

In this work we have evaluated the four-graviton amplitude at one loop in $\mathcal{N}=4$ supergravity in four dimensions from the field theory limit of string theory constructions. The string theory approach includes $(4,0)$ models where all of the supersymmetry come from the left-moving sector of the theory, and $(2,2)$ models where the supersymmetry is split between the left- and right-moving sectors of the theory.

For each model the four-graviton one-loop amplitude is linearly dependent on the number of vector multiplets n_v. Thus we define the pure $\mathcal{N}=4$ supergravity amplitude by subtraction of these contributions. This matches the result obtained in the Dabholkar-Harvey construction of string theory models with no vector multiplets. We have seen that, except when gravitons are running in the loop, the one-loop amplitudes are free of IR divergences. In addition, all the amplitudes are UV finite because the R^2 candidate counter-term vanishes for these amplitudes. Amplitudes with external vector states are expected to be UV divergent [53].

Our results reproduce the ones obtained with the string-based rules in [24, 30] unitarity-based method in [31–34] and the double-copy approach of [35]. The structure of the string theory amplitudes of the $(4,0)$ and $(2,2)$ models takes a very different form. There could have been differences at the supergravity level due to the different nature of the couplings of the vector multiplet in the two theories as indicated by the relation between the two amplitudes in (III.51). However, the coupling to gravity is universal. The difference between the various $\mathcal{N}=4$ supergravity models are visible once interactions between vectors and scalars occur, as can be seen on structure of the $\mathcal{N}=4$ Lagrangian in [50], which is not the case in our amplitudes since they involve only external gravitons. Our computation provides a direct check of this fact.

The supergravity amplitudes studied in this paper are naturally organized as a sum of $\mathcal{N}=4s$ spin-$s=1,\frac{3}{2},2$ contributions, with a simple power counting dependence on the loop momentum $\ell^{4(2-s)}$. Such a decomposition has been already used in the string-based approach to supergravity amplitudes in [24]. Our analysis reproduces these results and shows that the $\mathcal{N}=4$ part of the four-graviton amplitude does not depend on whether one starts from $(4,0)$ or $(2,2)$ construction. We expect amplitudes with external scalars or vectors to take a different form in the two constructions.

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Appendix A: World-sheet CFT : chiral blocks, propagators.

In this Appendix we collect various results about the conformal blocks, fermionic and bosonic propagators at genus one, and their q expansions.

1. Bosonic and fermionic chiral blocks

▷ The genus one theta functions are defined to be

$$\theta\begin{bmatrix}a\\b\end{bmatrix}(z|\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\frac{a}{2})^2}\, e^{2i\pi(z+\frac{b}{2})(n+\frac{a}{2})}\,, \tag{A.1}$$

and Dedekind eta function:

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\,, \tag{A.2}$$

where $q=\exp(2i\pi\tau)$. Those functions have the following $q\to 0$ behaviour:

$$\begin{aligned}&\theta\begin{bmatrix}1\\1\end{bmatrix}(0,\tau)=0\,;\quad \theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)=-2q^{1/8}+o(q)\,;\quad \theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)=1+2\sqrt{q}+o(q)\,;\\ &\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)=1-2\sqrt{q}+o(q)\,;\quad \eta(\tau)=q^{1/24}+o(q)\,.\end{aligned} \tag{A.3}$$

▷ The partition function of eight world-sheet fermions in the (a,b)-spin structure, $\Psi(z+1)=-(-1)^{2a}\Psi(z)$ and $\Psi(z+\tau)=-(-1)^{2b}\Psi(z)$, and eight chiral bosons is

$$Z_{a,b}(\tau)\equiv\frac{\theta\left[{a\atop b}\right](0|\tau)^4}{\eta^{12}(\tau)}\,, \tag{A.4}$$

it has the following behaviour for $q\to 0$

$$\begin{aligned}Z_{1,1}&=0\,,\\ Z_{1,0}&=16+16^2q+o(q^2)\,,\\ Z_{0,0}&=\frac{1}{\sqrt{q}}+8+o(\sqrt{q})\,,\\ Z_{0,1}&=\frac{1}{\sqrt{q}}-8+o(\sqrt{q})\,.\end{aligned} \tag{A.5}$$

▷ The partition function of the twisted (X,Ψ) system in the (a,b)-spin structure is

$$\begin{aligned}X(z+1)&=(-1)^{2h}\,X(z); &\quad \Psi(z+1)&=-(-1)^{2a+2h}\Psi(z)\,,\\ X(z+\tau)&=(-1)^{2g}\,X(z); &\quad \Psi(z+\tau)&=-(-1)^{2b+2g}\,\Psi(z)\,.\end{aligned} \tag{A.6}$$

The twisted chiral blocks for a real boson are

$$\mathcal{Z}^{h,g}[X] = \left(i e^{-i\pi g} q^{-h^2/2} \frac{\eta(\tau)}{\theta \begin{bmatrix} 1+h \\ 1+g \end{bmatrix}} \right)^{1/2} . \tag{A.7}$$

The twisted chiral blocks for a Majorana or Weyl fermion are

$$\mathcal{Z}^{h,g}_{a,b}[\Psi] = \left(e^{-i\pi\, a(g+b)/2} q^{h^2/2} \frac{\theta \begin{bmatrix} a+h \\ b+g \end{bmatrix}}{\eta(\tau)} \right)^{1/2} . \tag{A.8}$$

The total partition function is given by

$$\mathcal{Z}^{h,g}_{a,b}[(X,\Psi)] = \mathcal{Z}^{h,g}[X]\mathcal{Z}^{h,g}_{a,b}[\Psi] = e^{i\frac{\pi}{4}(1+2g+a(g+b))} \sqrt{\frac{\theta \begin{bmatrix} a+h \\ b+g \end{bmatrix}}{\theta \begin{bmatrix} 1+h \\ 1+g \end{bmatrix}}} . \tag{A.9}$$

2. Bosonic and fermionic propagators

a. *Bosonic propagators*

Our convention for the bosonic propagator is

$$\langle x^\mu(\nu) x^\nu(0) \rangle_{one-loop} = 2\alpha' \, \eta^{\mu\nu} \, \mathcal{P}(\nu|\tau) \, , \tag{A.10}$$

with

$$\begin{aligned} \mathcal{P}(\nu|\tau) &= -\frac{1}{4} \ln \left| \frac{\theta \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right] (\nu|\tau)}{\partial_\nu \theta \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right] (0|\tau)} \right|^2 + \frac{\pi \nu_2^2}{2\tau_2} + C(\tau) \\ &= \frac{\pi \nu_2^2}{2\tau_2} - \frac{1}{4} \ln \left| \frac{\sin(\pi\nu)}{\pi} \right|^2 - \sum_{m\geq 1} \left(\frac{q^m}{1-q^m} \frac{\sin^2(m\pi\nu)}{m} + c.c. \right) + C(\tau), \end{aligned} \tag{A.11}$$

where $C(\tau)$ is a contribution of the zero modes (see e.g. [48]) that anyway drops out of the string amplitude because of momentum conservation so we will forget it in the following.

We have as well the expansions

$$\begin{aligned} \partial_\nu \mathcal{P}(\nu|\tau) &= \frac{\pi}{2i} \frac{\nu_2}{\tau_2} - \frac{\pi}{4} \frac{1}{\tan(\pi\nu)} - \pi \, q \, \sin(2\pi\nu) + o(q) \, , \\ \partial_\nu^2 \mathcal{P}(\nu|\tau) &= -\frac{\pi}{4\tau_2} + \frac{\pi^2}{4} \frac{1}{\sin^2(\pi\nu)} - 2\pi^2 \, q \cos(2\pi\nu) + o(q) \\ \partial_\nu \bar{\partial}_{\bar{\nu}} \mathcal{P}(\nu|\tau) &= \frac{\pi}{4} \left(\frac{1}{\tau_2} - \delta^{(2)}(\nu) \right) . \end{aligned} \tag{A.12}$$

leading to the following Fourier expansion with respect to ν_1

$$\partial_\nu \mathcal{P}(\nu|\tau) = \frac{\pi}{4i}\,(\frac{2\nu_2}{\tau_2} - \mathrm{sign}(\nu_2)) + i\frac{\pi}{4}\,\mathrm{sign}(\nu_2)\sum_{m\neq 0} e^{2i\pi m \mathrm{sign}(\nu_2)\nu} - \pi\, q\, \sin(2\pi\nu) + o(q)\,,$$

$$\partial_\nu^2 \mathcal{P}(\nu|\tau) = \frac{\pi}{4\tau_2}\,(\tau_2\delta(\nu_2) - 1) - \pi^2 \sum_{m\geq 1} m e^{2i\pi m \mathrm{sign}(\nu_2)\nu} - 2\pi^2\, q\, \cos(2\pi\nu) + o(q)\,. \quad \text{(A.13)}$$

Setting $\nu = \nu_1 + i\tau_2\omega$ we can rewrite these expressions in a form relevant for the field theory limit $\tau_2 \to \infty$ with $t = \alpha'\tau_2$ kept fixed. The bosonic propagator can be decomposed in an asymptotic value for $\tau_2 \to \infty$ (the field theory limit) and corrections originating from massive string modes

$$\mathcal{P}(\nu|\tau) = -\frac{\pi\, t}{2\alpha'}\, P(\omega) + \delta P(\nu) - q\,\sin^2(\pi\nu) - \bar{q}\,\sin^2(\pi\bar{\nu}) + o(q^2)\,, \quad \text{(A.14)}$$

and

$$P(\omega) = \omega^2 - |\omega|; \qquad \delta P(\nu) = \sum_{m\neq 0} \frac{1}{4|m|}\, e^{2i\pi m\nu_1 - 2\pi|m\nu_2|}\,. \quad \text{(A.15)}$$

The contribution δP corresponds to the effect of massive string states propagating between two external massless states. The quantity $\mathcal{Q}$ defined in (II.4) writes in this limit

$$\mathcal{Q} = -t\pi Q(\omega) + \alpha'\delta Q - 2\pi\alpha' \sum_{1\leq i<j\leq 4} k_i\cdot k_j\,(q\sin^2(\pi\nu_{ij}) + \bar{q}\sin^2(\pi\bar{\nu}_{ij})) + o(q^2)\,, \quad \text{(A.16)}$$

where

$$Q(\omega) = \sum_{1\leq i<j\leq 4} k_i\cdot k_j\, P(\omega_{ij})\,, \qquad \delta Q = 2\sum_{1\leq i<j\leq 4} k_i\cdot k_j \delta P(\nu_{ij})\,. \quad \text{(A.17)}$$

b. *Fermionic propagators*

Our normalization for the fermionic propagators in the (a,b)-spin structure is given by

$$\langle\psi^\mu(z)\psi^\nu(0)\rangle_{one-loop} = \frac{\alpha'}{2}\, S_{a,b}(z|\tau)\,. \quad \text{(A.18)}$$

▷ In the even spin structure fermionic propagators are

$$S_{a,b}(z|\tau) = \frac{\theta\begin{bmatrix}a\\b\end{bmatrix}(z|\tau)}{\theta\begin{bmatrix}a\\b\end{bmatrix}(0|\tau)}\,\frac{\partial_z\theta\begin{bmatrix}1\\1\end{bmatrix}(0|\tau)}{\theta\begin{bmatrix}1\\1\end{bmatrix}(z|\tau)}\,. \quad \text{(A.19)}$$

The odd spin structure propagator is

$$S_{1,1}(z|\tau) = \frac{\partial_z\theta\begin{bmatrix}1\\1\end{bmatrix}(z|\tau)}{\theta\begin{bmatrix}1\\1\end{bmatrix}(z|\tau)}\,, \quad \text{(A.20)}$$

and the fermionic propagator orthogonal to the zero modes is

$$\tilde{S}_{1,1}(z|\tau) = S_{1,1}(z|\tau) - 2i\pi\frac{z_2}{\tau_2} = -4\partial_z\mathcal{P}(z|\tau)\,. \tag{A.21}$$

The fermionic propagators have the following q expansion representation [54]

$$\begin{aligned}
S_{1,1}(z|\tau) &= \frac{\pi}{\tan(\pi z)} + 4\pi\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\,\sin(2n\pi z)\,,\\
S_{1,0}(z|\tau) &= \frac{\pi}{\tan(\pi z)} - 4\pi\sum_{n=1}^{\infty}\frac{q^n}{1+q^n}\,\sin(2n\pi z)\,,\\
S_{0,0}(z|\tau) &= \frac{\pi}{\sin(\pi z)} - 4\pi\sum_{n=1}^{\infty}\frac{q^{n-\frac{1}{2}}}{1+q^{n-\frac{1}{2}}}\,\sin((2n-1)\pi z)\,,\\
S_{0,1}(z|\tau) &= \frac{\pi}{\sin(\pi z)} + 4\pi\sum_{n=1}^{\infty}\frac{q^{n-\frac{1}{2}}}{1-q^{n-\frac{1}{2}}}\,\sin((2n-1)\pi z)\,.
\end{aligned} \tag{A.22}$$

▷ Riemann supersymmetric identities written in the text (II.27) derive from the following Riemann relation relation:

$$\sum_{a,b=0,1}(-1)^{a+b+ab}\prod_{i=1}^{4}\theta\begin{bmatrix}a\\b\end{bmatrix}(v_i) = -2\prod_{i=1}^{4}\theta_1(v'_i)\,, \tag{A.23}$$

with $v'_1 = \frac{1}{2}(-v_1+v_2+v_3+v_4)$ $v'_2 = \frac{1}{2}(v_1-v_2+v_3+v_4)$ $v'_3 = \frac{1}{2}(v_1+v_2-v_3+v_4)$, and $v'_4 = \frac{1}{2}(v_1+v_2+v_3-v_4)$. This identity can be written, in the form used in the main text, as vanishing identities

$$\sum_{\substack{a,b=0,1\\ab=0}}(-1)^{a+b+ab}Z_{a,b}(\tau) = 0\,, \tag{A.24}$$

$$\sum_{\substack{a,b=0,1\\ab=0}}(-1)^{a+b+ab}Z_{a,b}(\tau)\prod_{r=1}^{n}S_{a,b}(z) = 0 \quad 1\leq n\leq 3\,, \tag{A.25}$$

and the first non-vanishing one

$$\sum_{\substack{a,b=0,1\\ab=0}}(-1)^{a+b+ab}Z_{a,b}(\tau)\prod_{i=1}^{4}S_{a,b}(z_i|\tau) = -(2\pi)^4\,. \tag{A.26}$$

with $z_1+\cdots+z_4=0$ and where we used that $\partial_z\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0|\tau) = \pi\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](0|\tau)\theta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right](0|\tau)\theta\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right](0|\tau) = 2\pi\eta^3(\tau)$.

Two identities consequences of the Riemann relation in (A.23) are

$$\begin{aligned}
S^2_{0,0}(z) - S^2_{1,0}(z) &= \pi^2(\theta\begin{bmatrix}0\\1\end{bmatrix}(0|\tau))^4\left(\frac{\partial_z\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](z|\tau)}{\partial\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0|\tau)}\right)^2\\
S^2_{0,1}(z) - S^2_{1,0}(z) &= \pi^2(\theta\begin{bmatrix}0\\0\end{bmatrix}(0|\tau))^4\left(\frac{\partial_z\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](z|\tau)}{\partial\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0|\tau)}\right)^2\,.
\end{aligned} \tag{A.27}$$

c. q expansion

The q expansions of the fermionic propagators in the even spin structure are given by

$$\begin{aligned} S_{1,0}(z|\tau) &= \frac{\pi}{\tan(\pi z)} - 4\pi q \sin(2\pi z) + o(q^2)\,, \\ S_{0,0}(z|\tau) &= \frac{\pi}{\sin(\pi z)} - 4\pi\sqrt{q}\, \sin(\pi z) + o(q)\,, \\ S_{0,1}(z|\tau) &= \frac{\pi}{\sin(\pi z)} + 4\pi\sqrt{q}\, \sin(\pi z) + o(q)\,. \end{aligned} \tag{A.28}$$

Setting $\mathcal{S}^n_{a,b} = \prod_{i=1}^n S_{a,b}(z_i|\tau)$ we have the following expansion

$$\begin{aligned} S^n_{1,0} &= \prod_{i=1}^n \pi\cot(\pi z_i)\left(1 - 8q\sum_{i=1}^n \sin^2(\pi z_i)\right) + o(q^2)\,, \\ S^n_{0,0} &= \prod_{i=1}^n \pi(\sin(\pi z_i))^{-1}\left(1 - 4q\sum_{i=1}^n \sin^2(\pi z_i)\right) + o(q^2)\,, \\ S^n_{0,1} &= \prod_{i=1}^n \pi(\sin(\pi z_i))^{-1}\left(1 + 4q\sum_{i=1}^n \sin^2(\pi z_i)\right) + o(q^2)\,. \end{aligned} \tag{A.29}$$

Applying these identities with $n = 2$ and $n = 4$ we derive the following relations between the correlators $\mathcal{W}^F_{a,b}$ defined in (II.25)

$$\mathcal{W}^F_{0,0}|_{q^0} = \mathcal{W}^F_{0,1}|_{q^0}; \qquad \mathcal{W}^F_{0,0}|_{\sqrt{q}} = -\mathcal{W}^F_{0,1}|_{\sqrt{q}}\,. \tag{A.30}$$

Using the q expansion of the bosonic propagator, it is not difficult to realize that $\mathcal{W}^B|_{\sqrt{q}} = 0$, and we can promote the previous relation to the full correlator $\mathcal{W}_{a,b}$ defined in (II.23) (using the identities in (A.27))

$$\mathcal{W}_{0,0}|_{q^0} = \mathcal{W}_{0,1}|_{q^0}; \qquad \mathcal{W}_{0,0}|_{\sqrt{q}} = -\mathcal{W}_{0,1}|_{\sqrt{q}}\,. \tag{A.31}$$

Other useful relations are between the q expansion of the derivative bosonic propagator $\partial\mathcal{P}$ and the fermionic propagator $S_{1,0}$

$$\begin{aligned} \partial_\nu \mathcal{P}(\nu|\tau)|_{q^0} - \frac{\pi\nu_2}{2i\tau_2} &= -\frac{1}{4} S_{1,0}(\nu|\tau)|_{q^0} \\ \partial_\nu \mathcal{P}(\nu|\tau)|_{q^1} &= +\frac{1}{4} S_{1,0}(\nu|\tau)|_{q}\,. \end{aligned} \tag{A.32}$$

3. Congruence subgroups of $SL(2,\mathbb{Z})$

We denote by $SL(2,\mathbb{Z})$ the group of 2×2 matrix with integers entries of determinant 1. For any N integers we have the following subgroups of $SL(2,\mathbb{Z})$

$$\begin{aligned}\Gamma_0(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\} , \\ \Gamma_1(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\} , \\ \Gamma(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\} .\end{aligned} \tag{A.33}$$

They satisfy the properties that $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset SL(2,\mathbb{Z})$.

Appendix B: Chiral blocks for the type II orbifolds

We recall some essential facts from the construction of [29]. The shifted T^2 lattice sum writes

$$\Gamma^w_{(2,2)} \begin{bmatrix} h \\ g \end{bmatrix} := \sum_{P_L, p_R \in \Gamma_{(2,2)} + w\frac{h}{2}} e^{i\pi g l \cdot w} q^{\frac{P_L^2}{2}} \bar{q}^{\frac{P_R^2}{2}} , \tag{B.1}$$

where $\ell \cdot w = m_I b^I + a_I n^I$ where the shift vector $w = (a_I, b^I)$ is such that $w^2 = 2a\cdot b = 0$ and

$$\begin{aligned} P_L^2 &= \frac{|U(m_1 + a_1\frac{h}{2}) - (m_2 + a_2\frac{h}{2}) + T(n^1 + b^1\frac{h}{2}) + TU(n^2 + b^2\frac{h}{2})|^2}{2T_2U_2} , \\ P_L^2 - P_R^2 &= 2(m_I + a_I\frac{h}{2})(n^I + b^I\frac{h}{2}) . \end{aligned} \tag{B.2}$$

T and U are the moduli of the T^2. We recall the full expressions for the orbifold blocks :

$$\mathcal{Z}^{(22);h,g}_{a,b} := \begin{cases} \mathcal{Z}_{a,b} = (\text{II}.20) & (h,g) = (0,0) \\ 4(-1)^{(a+h)g} \left(\frac{\theta\left[\substack{a \\ b}\right](0|\tau)\theta\left[\substack{a+h \\ b+g}\right](0|\tau)}{\eta(\tau)^3 \theta\left[\substack{1+h \\ 1+g}\right]} \right)^2 \times \Gamma_{(2,2)}(T,U) & (h,g) \neq (0,0) , \end{cases} \tag{B.3}$$

$$\mathcal{Z}^{(14);h,g}_{a,b} = \frac{1}{2} \sum_{h',g'=0}^{1} \mathcal{Z}^{h,g}_{a,b} \begin{bmatrix} h' \\ g' \end{bmatrix} \Gamma^w_{(2,2)} \begin{bmatrix} h' \\ g' \end{bmatrix} , \tag{B.4}$$

$$\mathcal{Z}^{(10);h,g}_{a,b} = \frac{1}{2}\sum_{h_1,g_1=0}^{1}\frac{1}{2}\sum_{h_2,g_2=0}^{1}\mathcal{Z}^{h,g}_{a,b}\begin{bmatrix}h;h_1,h_2\\ g;g_1,g_2\end{bmatrix}\Gamma^{w_1,w_2}_{(2,2)}\begin{bmatrix}h_1,h_2\\ g_1,g_2\end{bmatrix}\,,\ \forall h,g\,. \tag{B.5}$$

For the $n_v = 6$ model, the orbifold acts differently and we get

$$\mathcal{Z}^{(6);h,g}_{a,b} = \frac{1}{2}\sum_{h',g'=0}^{1}(-1)^{hg'+gh'}\mathcal{Z}^{h,g}_{a,b}\Gamma^{w}_{(2,2)}\begin{bmatrix}h'\\ g'\end{bmatrix}\,. \tag{B.6}$$

In the previous expressions, the crucial point is that the shifted lattice sums $\Gamma^{w}_{(2,2)}\left[{h' \atop g'}\right]$ act as projectors on their untwisted $h'=0$ sector, while the g' sector is left free. We recall now the diagonal properties of the orbifold action (see [29] again) on the lattice sums:

$$\Gamma^{w_1,w_2}_{(2,2)}\begin{bmatrix}h,0\\ g,0\end{bmatrix} = \Gamma^{w_1}_{(2,2)}\begin{bmatrix}h\\ g\end{bmatrix},\ \Gamma^{w_1,w_2}_{(2,2)}\begin{bmatrix}0,h\\ 0,g\end{bmatrix} = \Gamma^{w_2}_{(2,2)}\begin{bmatrix}h\\ g\end{bmatrix},\ \Gamma^{w_1,w_2}_{(2,2)}\begin{bmatrix}h,h\\ g,g\end{bmatrix} = \Gamma^{w_{12}}_{(2,2)}\begin{bmatrix}h\\ g\end{bmatrix}\,, \tag{B.7}$$

The four-dimensional blocks $\mathcal{Z}^{h,g}_{a,b}$ have the following properties : $\mathcal{Z}^{h,g}_{a,b}\left[{0 \atop 0}\right] = \mathcal{Z}^{h,g}_{a,b}\left[{h \atop g}\right] = \mathcal{Z}^{h,g}_{a,b}$ (ordinary twist); $\mathcal{Z}^{0,0}_{a,b}\left[{h \atop g}\right]$ is a $(4,4)$ lattice sum with one shifted momentum and thus projects out the $h=0$ sector. Equivalent properties stand as well for the $n_v = 10$ model.

One has then in the field theory limit

$$\begin{aligned}
\mathcal{Z}^{(14);h,g}_{a,b} &\in \{\mathcal{Z}^{0,0}_{a,b},\, \mathcal{Z}^{0,1}_{a,b},\, \frac{1}{2}\mathcal{Z}^{1,0}_{a,b},\, \frac{1}{2}\mathcal{Z}^{1,1}_{a,b}\}\,,\\
\mathcal{Z}^{(10);h,g}_{a,b} &\in \{\mathcal{Z}^{0,0}_{a,b},\, \mathcal{Z}^{0,1}_{a,b},\, \frac{1}{4}\mathcal{Z}^{1,0}_{a,b},\, \frac{1}{4}\mathcal{Z}^{1,1}_{a,b}\}\,,\\
\mathcal{Z}^{(6);h,g}_{a,b} &\in \{\mathcal{Z}^{0,0}_{a,b},\, \mathcal{Z}^{0,1}_{a,b},\, 0,\, 0\}\,,
\end{aligned} \tag{B.8}$$

from where we easily deduce the effective definition given in (II.42) and the number c_h.

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Tropical Amplitudes

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Abstract

In this work, we argue that the point-like limit $\alpha' \to 0$ of closed string theory scattering amplitudes is a tropical limit. This constitutes an attempt to explain in a systematic way how Feynman graphs are obtained from string theory amplitudes. Our key result is to write the field theory amplitudes arising in the low energy limit of string theory amplitudes as integrals over a single object: the tropical moduli space. At the mathematical level, this limit is an implementation of the correspondence between the moduli space of Riemann surfaces and the tropical moduli space. As an example of non-trivial physical application, we derive the tropical form of the integrand of the genus two four-graviton type II string amplitude and match the direct field theory computations.

CONTENTS

I. INTRODUCTION

It is generally accepted that the low-energy limit[1] of string theory scattering amplitudes reproduces the usual quantum field theory perturbative expansion of the low-energy content of string theory. However an explicit general proof of that statement has not been given yet. Besides the intrinsic interest of such a proof, this problem is of great importance for several reasons.

Firstly, string inspired methods have already proved their efficiency at one-loop to compute scattering amplitudes in field theory [1–15] but also to obtain more general results about amplitudes [16–22]. One would like to be able to generalize these methods to higher loop order in a systematic way.

Secondly, it is very important to understand the general mechanisms by which string theory renormalizes supergravity theories. In particular, the question of the ultraviolet (UV) divergences of maximal supergravity in four dimensions continues to draw much attention [23–33] and string theory provides a well suited framework to analyze this issue [32–36].

The main contribution of this paper is an attempt to show in full generality in the number of loops and particles how the Feynman diagram perturbative expansion is reproduced by the $\alpha' \to 0$ limit of closed string theory amplitudes. The novelty in our approach is the central use of tropical geometry. This theory describes – amongst other things – how Riemann surfaces can be degenerated to a special kind of graphs called tropical graphs. For that reason, it provides a very natural framework to analyze the way in which string worldsheets degenerate to particle worldlines when the strings become point-like.

The first implication of considering the $\alpha' \to 0$ limit of string theory in the tropical sense is that worldlines of particles become tropical graphs. The mathematical definition thereof (see section III) is very similar to the one of worldline graphs: they are graphs whose edges have lengths – the Schwinger proper times of the worldline graph. The difference between worldline and tropical graphs comes from the fact that tropical graphs have weighted vertices. A weight one vertex in a tropical graph is the result of shrinking to a point a loop inside the graph, and more generally a weight w vertex is the result of shrinking to a point a sub-graph of genus w inside the graph. Therefore a genus g tropical graph does not necessarily have g loops, as some of its vertices may have non-zero weights.

From the point of view of the physical $\alpha' \to 0$ limit, these weighted vertices descend from regions

[1] In the text, we shall call indistinctly “low-energy”, “point-like”, “field theory”, “tropical” or “$\alpha' \to 0$” this limit. We recall that the Regge slope α' of the string is a positive quantity of mass dimension -2 related to the string length ℓ_s by $\alpha' = \ell_s^2$.

in the phase space of string theory where massive stringy modes propagate and regulate the UV behavior of the low-energy theory. In a nutshell, counter-terms to UV divergences are inserted on weighted vertices. A very simple implementation of these last two points can be found in section II, where it is explained what features of tropical geometry can be seen in the point-like limit of the one-loop bosonic string partition function.

Therefore the tropical geometry framework encompasses both of the aforementioned aspects of the $\alpha' \to 0$ limit: the structure of the graphs and the renormalization systematics. In this picture, what is maybe the most elegant object is the tropical moduli space of [37, 38] (defined in section III C). It is the space of all tropical graphs of genus g with n legs, denoted $\mathcal{M}_{g,n}^{\mathrm{trop}}$. Eventually, we write the field theory amplitudes obtained in the $\alpha' \to 0$ limit of string theory in a compact form as integrals over this single object $\mathcal{M}_{g,n}^{\mathrm{trop}}$ (theorem 4),

$$\lim_{\alpha' \to 0} \mathsf{A}_{\alpha'}^{(g,n)} = \int_{\mathcal{M}_{g,n}^{\mathrm{trop}}} \mathrm{d}\mu^{\mathrm{trop}} F_{g,n} \tag{I.1}$$

where $\mathsf{A}_{\alpha'}^{(g,n)}$ is a string theory amplitude and the right hand side is the renormalized field theory amplitude written in its "tropical representation". We call a field theory amplitude in a tropical representation a "tropical amplitude". In this representation induced by string theory, the integration measure $\mathrm{d}\mu^{\mathrm{trop}}$ is defined in terms of the graphs Schwinger proper times (lenghts of the inner edges). The integrand $F_{g,n}$ is a function of the Schwinger proper times and kinematical invariants that contains both the numerator and denominator of the Feynman diagram integral as detailed in the proposition 3.

Our starting point is then a g-loop n-point string theory amplitude. For simplicity, we work in a setting where it can be written as an integral over the ordinary bosonic moduli space of Riemann surfaces $\mathcal{M}_{g,n}$ (details on Riemann surfaces and their moduli space are recalled in section IV). The general strategy to extract tropical amplitudes is based decomposing the integration domain of the string amplitudes, $\mathcal{M}_{g,n}$, into various regions or cells. We shall use the so-called Kontsevich-Penner (KP) [39, 40] decomposition of $\mathcal{M}_{g,n}$, which has the essential property that each of the cells corresponds to a combinatorially distinct class of graphs.

Thus, as explained in section V, the crucial step of the tropicalization procedure consists in showing that the string theory integral when restricted to a particular cell corresponding to a graph G, provides in the $\alpha' \to 0$ limit the contribution of the Feynman diagram with topology G. This is basically the content of the conjecture 2, and we are able to prove how for a given topology, the graph together with its denominator structure is recovered. We also show in some cases how one can extract the numerator of the Feynman integral.

Let us insist here on the fact that the whole point of the procedure is that Feynman diagrams with counter-terms are naturally present, they come from cells labeled by graphs with weighted vertices. In particular, two kind of cells are most interesting; the ones that give the unrenormalized amplitude, and the one that gives the counter-term to the new g-loop primary divergence. Though understanding the systematics of the sub-divergences regularization is important, these two kind of cells indeed are crucial as concerns the two practical issues mentioned in the beginning. They generalize the analytic and non-analytic regions of [14] to higher loop order.

Summing up the various contributions arising from the KP decomposition, we obtain equation (I.1), or more precisely the theorem 4 stating that tropical amplitudes are indeed renormalized field theory amplitudes.

To ensure the consistency of this procedure, we apply it to the well known cases of tree and one-loop string theory amplitudes in section VI. We see how the existing computations of $\alpha' \to 0$ limits prove the previous statements. Although they are well understood computations, displaying their tropical nature is a necessary step in this work. To conclude the paper (section VI C), we apply the tropical technology to obtain a new result concerning the $\alpha' \to 0$ limit of the genus two four-graviton type II string amplitudes computed in [41–47]: we derive the tropical representation of the integrand of this amplitude. We find that it matches the integrand of [48], obtained from field theory methods and written in a Schwinger proper time parametrization. Performing the same procedure for the recent proposal for the genus three four-graviton amplitude in type II using pure spinors [49] would be a very interesting thing to do.

Besides the study of the $\alpha' \to 0$ limit of string amplitudes, our approach contributes to a new geometrical understanding of field theory amplitudes since these can be written as integrals over the tropical moduli space. The net effect of that is to give to Feynman parameters[2] a geometrical meaning; they are the rescaled lengths of the edges they are associated to. Therefore, the components of the Feynman integrands have also a geometrical origin. In particular, the first Symanzik polynomial[3] of the graph is seen to be the determinant of the period matrix of the associated tropical graph, while the second is given in terms of Green functions on the graph. These observations are not new, and indeed they can be traced back to the inspiring papers [48, 51].

Several restrictions to the scope of this work are detailed along the text. All of these are recalled in the discussion in section VII and presented as an invitation for further developments.

[2] That are nothing but rescaled Schwinger proper times.

[3] We recall that the first Symanzik polynomial of a graph comes out as the result of integrating out the g loop momenta in a Feynman diagram, while the second corresponds to the denominator of the Feynman integrand [50].

II. THE POINT-LIKE LIMIT OF STRING THEORY IS A TROPICAL LIMIT

How could one create a graph out of a Riemann surface (with no boundaries) ? The first thing one would have in mind is to stretch the surface to create very long and thin tubes. This actually does not produce graphs but degenerate Riemann surfaces with nodes. Nevertheless, it is a good start, and physically these stretched surfaces probe the infrared region of string theory. To obtain a graph out of these tubes one still have to integrate out the cylindrical dependence thereof.

A good flavor of what is tropical geometry can be found in the survey [52] where the tropicalization procedure is presented as a way to *forget the phase of complex numbers*. In the previous example, if σ and τ are respectively angular and longitudinal coordinates along the tube, $w = \tau + i\sigma$ can be conformally mapped to the plane via $w \to z = e^{iw}$ and indeed, integrating out the cylindrical dependence of w amounts to integrating out the phase of z.

Let us use the genus one bosonic string partition function to illustrate the basic mechanisms of the point-like limit of string theory and see the appearance of tropical geometry. It is a very good example to precisely identify where do come from the "phases" and "modulus" of complex numbers in string theory. It writes [53–55]

$$Z(\tau, \bar\tau) = \mathrm{Tr}\left(q^{L_0-1}\bar q^{\bar L_0-1}\right), \tag{II.1}$$

where the trace is performed over the Hilbert space of physical states of string theory. The parameter q is defined by $q := \exp(2i\pi\tau)$ where $\tau = \tau_1 + i\tau_2$ is the modulus of the complex torus. This expression can be rewritten to make manifest "phases" and "modulus" as:

$$Z(\tau, \bar\tau) = \mathrm{Tr}\, e^{+2i\pi\tau_1(L_0-\bar L_0)} e^{-2\pi\tau_2(L_0+\bar L_0-2)}\,. \tag{II.2}$$

Thus the level-matching condition $(L_0 - \bar L_0) = 0$ is enforced by integration over the "phases" $\int d\tau_1$ while the "moduli" cause a mass weighting. More precisely, the masses of the oscillator states are given by $m^2 = \frac{4}{\alpha'}\left(\frac{N+\bar N}{2} - 1\right)$ where N and $\bar N$ are the number operators for the left-moving and right-moving sector defined by $L_0 = N + \alpha' p^2/4 - 1$ and $\bar L_0 = \bar N + \alpha' p^2/4 - 1$. The lowest mass state has $N = \bar N = 0$; it is the tachyon, whose mass is $m^2 = -4/\alpha'$. Then come the massless states at $N = \bar N = 1$ which constitute the gravity sector. For $N = \bar N \geq 2$ the states are massive with masses $m^2 = 4N/\alpha'$.

Thus in the region of modular parameter τ where $\tau_2 \sim 1/\alpha'$, the torus looks like a long and thin wire and one has $\tau_2(N + \bar N - 2) \sim m^2$. As $\alpha' \to 0$, the massive states with $N \geq 2$ give rise to exponentially suppressed contributions in the partition function; only the massless modes propagate.[4] Since all states are now massless, the level matching condition is trivial and may be integrated out; one ends up with a worldline loop of length $\alpha'\tau_2$.

In the range of complex moduli τ where τ_2 stays of order $O(1)$, the massive modes are not decoupled and dictate the UV behavior of the theory. We will see later that these tori, that are well known to generate the insertion of higher order operators and counter-terms in the supergravity effective action give rise to natural objects of tropical geometry. Although there is no trivial integration of the phase dependence in this case, one can think of these phases as phases of complex numbers of vanishingly small modulus which are integrated out as well. To summarize, the tropical limit is performed in two steps:

Step 1 (Point-like limit) Send $\alpha' \to 0$ and distinguish between the contribution of massive states and massless states in the amplitudes,

Step 2 (Level matching) Integrate out the phases of the complex numbers that are not vanishingly small to get the contributions of the massless states, and integrate out the regions of phase space where the complex numbers are vanishingly small to get the contributions of massive states.

In the second part of the paper we shall revisit this well known technology that we just sketched in explicit amplitudes. Before that, let us introduce a few mathematical notions about tropical geometry and the tropicalization of Riemann surfaces.

III. TROPICAL GEOMETRY

Tropical geometry is a recent and very active field in mathematics. For an introduction to tropical geometry, see [52, 56–60] or for a more exhaustive bibliography see [37] and references therein. The basic objects, tropical varieties, can be either abstract [61] or defined as algebraic curves over certain spaces [58, 62]. Tropical varieties also arise as the result of a worst possible degeneration of the complex structure of complex varieties, called tropicalization [63, 64]. This point is particularly interesting for our study of the point-like limit of string theory where string worldsheets – Riemann surfaces – are degenerated to particle worldlines – tropical graphs.

[4] The tachyon state $N = \bar{N} = 0$ creates an infrared divergence, that can simply be ignored here.

A. Tropical Graphs

In this section we introduce basic definitions about tropical graphs. An (abstract) tropical graph is a connected graph with labeled legs, whose inner edges have a length and whose vertices are weighted. The external legs are called punctures or marked points and by convention they have infinite length. A pure tropical graph is a tropical graph that has only vertices of weight zero. Pure tropical graphs were first introduced in [58, 65] and then later extended by [37, 38] to tropical graphs with weights, simply called tropical graphs here.

A tropical graph Γ is then a triple $\Gamma = (G, w, \ell)$ where G is a connected graph called its combinatorial type, ℓ and w are length and weight functions on the edges and on the vertices

$$\ell : E(G) \cup L(G) \to \mathbb{R}_+ \cup \{\infty\} \tag{III.1}$$

$$w : V(G) \to \mathbb{Z}_+ \tag{III.2}$$

In these two equations, $E(G)$, $L(G)$ and $V(G)$ are respectively the sets of inner edges, legs and vertices of the graph. The total weight $|w|$ of a tropical graph is the sum of all the weights of the vertices $|w| = \sum_{V(G)} w(V)$. The genus $g(\Gamma)$ of a tropical graph $\Gamma = (G, w, \ell)$, is the number of loops $g(G)$ of G plus its total weight

$$g(\Gamma) = g(G) + |w| \,. \tag{III.3}$$

Hence the genus of a pure tropical graph is the number of loops of G in the usual sense.

Moreover every vertex of weight zero should have valence at least three (vertices with weight $w \geq 1$ may be of arbitrary non zero valency). This automatically enforces a global stability condition for a given tropical graph of genus g and n punctures. One should have

$$2g - 2 + n \geq 1 \,, \tag{III.4}$$

which is the exact analog of the classical stability condition [5]. Below in figure 1 are a few examples of tropical graphs.

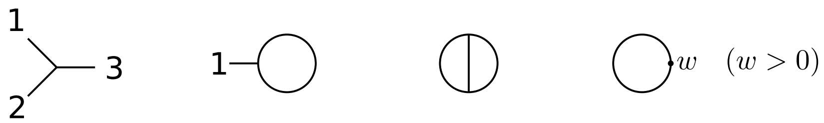


FIG. 1. From left to right: a three point tropical tree, a genus one tropical graph with a puncture, a genus two tropical graph, a graph of genus $1 + w$.

[5] Strictly speaking, the local valency condition should be viewed as considering *classes* of abstract tropical graphs under the equivalence relation that contracts edges connected to 1-valent vertices of weight 0, and removes weight 0 bivalent vertices. Physically, on the worldline, this equivalence relation is perfectly sensible, since no interpretation of these 1- or 2- valent vertices of weight zero seem obvious in the absence of external classical sources.

Vertices weights obey the rules under degenerations called specializations pictured in the figure 2.

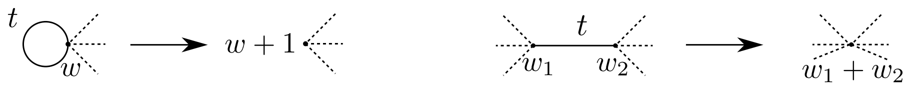


FIG. 2. The genus of a graph is stable under specializations.

This gives another way to interpret vertices weights; they *keep track* of degenerated loops. It is easily checked that the genus of a graph (III.3) and the stability criterion (III.4) are stable under specialization.

Finally, let us recall that a one-particle-irreducible (1PI) graph is a graph that can not be disconnected by cutting one inner edge. A graph that can is called one-particle-reducible (1PR).

Physically, we will interpret tropical graphs as being worldlines swept by the propagation of a particle in space-time. The lengths of its edges correspond to Schwinger proper times and a non zero weight on a vertex indicates the possible insertion of a counter-term in the graph. Indeed, loops with very short proper times probe the ultraviolet (UV) region of the theory and it is natural to insert counter-terms on loops of zero size to regulate the UV.

B. Homology, Forms and Jacobians Of Tropical Graphs

In this paragraph, following closely [58], we introduce basic notions of tropical geometry that are the exact analog of the classical one (that we shall recall later in section IV A), such as abelian one forms, period matrices and Jacobians. A little subtlety, absent in the classical case is linked to the fact that tropical graphs of identical genus may not have the same number of inner edges. For simplicity, here we shall only describe pure graphs and mention in the end how this could be generalized following [37].

Tropical graphs support an homology basis and corresponding one-forms. Let Γ be a pure tropical graph of genus g and $(B_1, , ..., B_g)$ be a canonical homology basis of Γ as in figure 3. The

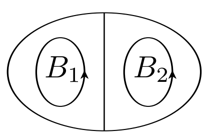


FIG. 3. Canonical homology basis, example at $g = 2$.

linear vector space of the g independent abelian one-forms ω_I^{trop} can be canonically defined by

$$\omega_I^{\text{trop}} = \begin{cases} 1 \text{ on } B_I\,, \\ 0 \text{ otherwise}\,. \end{cases} \tag{III.5}$$

These forms are *constant* on the edges of the graph. The period matrix K_{IJ} is defined as in the classical case by integration along B cycles,

$$\oint_{B_I} \omega_J^{\text{trop}} = K_{IJ} \,. \tag{III.6}$$

It is a $g \times g$ positive semi-definite real valued matrix. The abelian one forms and period matrix were already used in [48, 51] where they were observed to be the exact analogs of the classical quantities. The Jacobian of Γ is a real torus given by

$$J(\Gamma) = \mathbb{R}^g / K\mathbb{Z}^g \,. \tag{III.7}$$

Integration along a path γ between two end points P_1 and P_2 on the graphgives rise to the tropical version of the Abel-Jacobi map μ^{trop} [58, 61] defined by

$$\mu_\gamma^{\text{trop}}(P_0, P_1) = \int_{P_0}^{P_1} (\omega_1^{\text{trop}}, ..., \omega_g^{\text{trop}}) \mod K\mathbb{Z}^g \,. \tag{III.8}$$

Changing γ by elements of the homology basis results in adding to the integral in the right hand side some elements of the lattice $K\mathbb{Z}^g$. Thus μ^{trop} is well defined as a map in the Jacobian lattice.

Here are two observations taken from [58].

Example 1. Let Γ be the genus two tropical graph depicted in figure 4 a) with canonical homology basis as in figure 3. Its period matrix is

$$K^{(2)} = \begin{pmatrix} T_1 + T_3 & -T_3 \\ -T_3 & T_2 + T_3 \end{pmatrix} \,. \tag{III.9}$$

Choosing P_0 as depicted, one can draw the image of Γ by the tropical Abel-Jacobi map in $J(\Gamma)$, as shown in the figure 4 b).

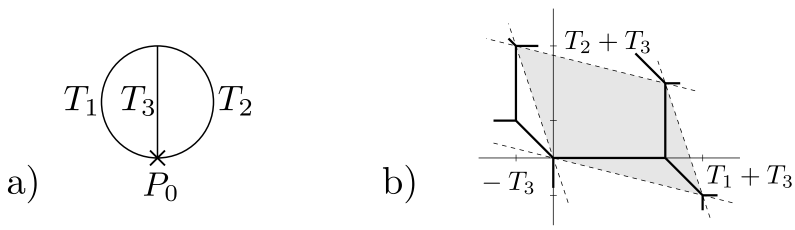


FIG. 4. a) A genus two graph Γ with the three lengths T_1, T_2, T_3 indicated. b) The image of Γ (thick line) by the tropical Abel-Jacobi map in the Jacobian variety $J^{\text{trop}}(\Gamma) = \mathbb{R}^2 / K^{(2)}\mathbb{Z}^2$ (shaded area).

Example 2. The picture 5 below shows two inequivalent pure tropical graphs of genus two. The period matrix $K^{(2)}$ of the 1PI graph a) is given in (III.9), the period matrix of the 1PR graph b) is just $\text{Diag}(T_1, T_2)$. Thus, the Jacobian description is blind to separating edges.

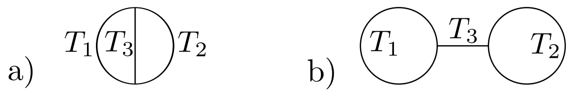


FIG. 5. The period matrix / Jacobian description is blind to the central length of the graph b).

The generalization of the previous discussion to the case of tropical graphs with weighted vertices depends on the approach one wants to use. In a simplistic approach, one simply "forgets" that there are weights on the vertices, and deal with a homology basis of size $g(G)$ instead of $g(\Gamma)$; in that case the same definitions can be applied straightforwardly. The issue with this approach is that the dimension of the Jacobian drops under specialization. For a more complete approach which cures this problem, see [37].

C. The Tropical Moduli Space

The moduli space $\mathcal{M}(G)$ associated to a single tropical graph $\Gamma = (G, w, \ell)$ is the cone spanned by all of the lengths of its inner edges modulo the discrete automorphism group of the graph [37]

$$\mathcal{M}(\Gamma) = \mathbb{R}_+^{|E(G)|}/\mathrm{Aut}(G)\,. \tag{III.10}$$

The moduli space of all genus g, n-punctured tropical graphs is the space obtained from gluing all these cones together. The existence of such an object is still an open issue in mathematics, as everybody do not agree on the definition of tropical graph. However we will argue that the moduli space of tropical graphs, or tropical moduli space introduced in [37, 38] denoted $\mathcal{M}_{g,n}^{\mathrm{trop}}$ is natural in the framework to study the point-like limit of string theory amplitudes. Therefore we shall focus on this space from now on.

Let us start this discussion with the moduli space of genus 0 tropical curves, $\mathcal{M}_{0,n}^{\mathrm{trop}}$. It is a well defined space that has the peculiar property of being itself a tropical variety (actually a tropical orbifold) of dimension $n-3$ [59, 65]. Because of the stability condition (III.4) one should start with $n = 3$. The space $\mathcal{M}_{0,3}^{\mathrm{trop}}$ contains only one graph with no modulus (no inner length): the 3-pointed tropical curve. Hence $\mathcal{M}_{0,3}^{\mathrm{trop}}$ is just a one point set. The space $\mathcal{M}_{0,4}$ has more structure; it has the structure of the three-punctured tropical curve and contains combinatorially distinct graphs which have at most one inner length, as shown below in figure 6.

The space $\mathcal{M}_{0,5}^{\mathrm{trop}}$ has an even richer structure. It is a two dimensional complex, and it can be embedded in $\mathbb{R}^3$. It is represented below in figure 7. Each of the fifteen faces of the complex is a two dimensional quadrant isomorphic to $\mathbb{R}_+^2$. The coordinates (X, Y) on the facet are the two

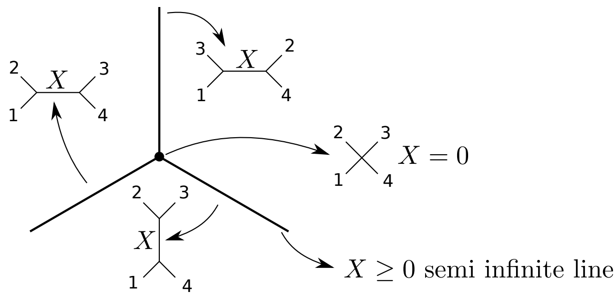


FIG. 6. Tropical moduli space $\mathcal{M}_{0,4}$ (thick line). Each semi infinite line corresponds to one of three inequivalent graphs. The X coordinate on these gives the length of the inner edge of the graphs. The central point with $X = 0$ is common to the three branches.

proper times of of the 15 combinatorially distinct graphs made of trivalent vertices. There are 10 rays, these are the number of distinct graphs with one trivalent and one four-valent vertex, they are labeled by a couple of indices which are the external legs attached to the trivalent vertex.

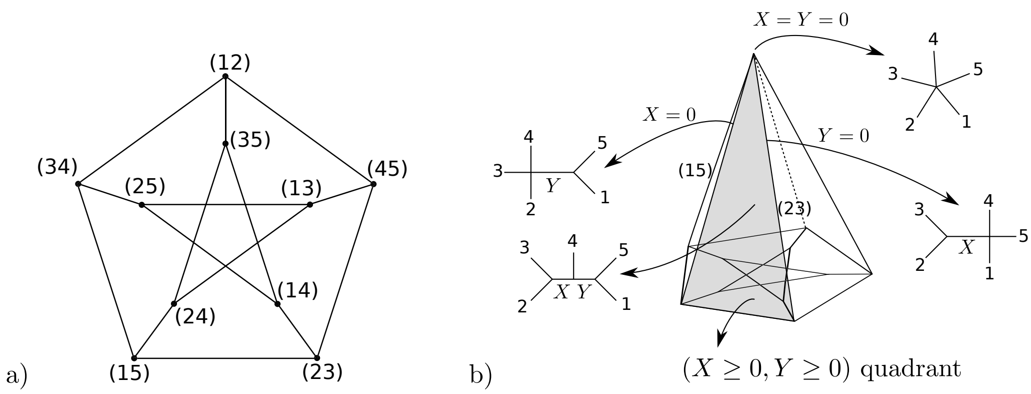


FIG. 7. a) A slice of the tropical moduli space $\mathcal{M}_{0,5}$. The vertices of the graph are indicated by black dots and carry a two digits index. They corresponds to rays of $\mathcal{M}_{0,5}$, while edges corresponds to quadrants. Note that there are 15 quadrants, one for each tree with 5 external legs and trivalent vertices. This graph is called the *Petersen graph.* b) The space $\mathcal{M}_{0,5}$, with a specific quadrant in grey.

At genus one, $\mathcal{M}_{1,1}^{\text{trop}}$ is still easy to draw. A genus one tropical graph with one leg is either a loop or a vertex of weight one. Hence, $\mathcal{M}_{1,1}^{\text{trop}}$ is the half infinite line $\mathbb{R}_+$ (see figure 8).

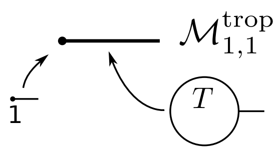


FIG. 8. The space $\mathcal{M}_{1,1}$ is the semi infinite line. The coordinate T on it is the length or Schwinger proper time of the loop. The weigth one vertex at $T = 0$ is a singular point; it corresponds the specialized loop.

In general, Euler's relation gives that a given graph has at most $3g-3+n$ inner edges (and has exactly that number if and only if the graph is pure and possess only trivalent vertices). This implies that $\mathcal{M}_{g,n}^{\text{trop}}$ is of "pure dimension" $3g-3+n$, which means that some of its subsets are of lower dimension. Moreover $\mathcal{M}_{g,n}^{\text{trop}}$ is a Haussdorff space [38] that has the structure of a "stacky fan" [66, 67].

The material presented in this section will appear soon as the result of special degenerations of Riemann surfaces and moduli spaces thereof, to which we now turn.

IV. CLASSICAL GEOMETRY AND THE LINK WITH THE TROPICAL WORLD

A. Classical Facts On Riemann Surfaces And Their Jacobians

We recall classical facts about homology and Jacobian varieties of smooth Riemann surfaces [68, 69]. Let Σ be a generic Riemann surface of genus g and let (a_I, b_J) $I, J = 1, ..., g$ be a canonical homology basis on Σ with intersection $a_I \cap b_J = \delta_{IJ}$ and $a_I \cap a_J = b_I \cap b_J = 0$ as in figure 9.

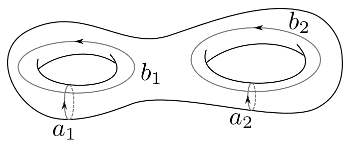


FIG. 9. Canonical homology basis, example for $g = 2$.

The abelian differential forms ω_I, $I = 1, ..., g$ are holomorphic 1-forms, they can be normalized along a cycles while their integral along the b cycles defines the period matrix Ω_{IJ} of Σ:

$$2i\pi \oint_{a_I} \omega_J = \delta_{IJ}\,, \qquad 2i\pi \oint_{b_I} \omega_J = \Omega_{IJ}\,. \tag{IV.1}$$

The modular group $Sp(2g,\mathbb{Z})$ at genus g is spanned by the $2g \times 2g$ matrices of the form $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where A, B, C and D are $g \times g$ matrices with integer coefficients satisfying $AB^t = BA^t$, $CD^t = DC^t$ and $AD^t - BC^t = 1_g$. The $g \times g$ matrix 1_g is the identity matrix. For $g = 1$, the modular group reduces to $\mathrm{SL}(2,\mathbb{Z})$. The Siegel upper half-plane $\mathcal{H}_g$ is the set of symmetric $g \times g$ complex matrices with positive definite imaginary part

$$\mathcal{H}_g = \{\Omega \in \mathrm{Mat}(g \times g, \mathbb{C}) : \Omega^t = \Omega, \mathrm{Im}\,(\Omega) > 0\}\,. \tag{IV.2}$$

The modular group $\mathrm{Sp}(2g,\mathbb{Z})$ acts on the Siegel upper half-plane by $\Omega \mapsto (A\Omega + B)(C\Omega + D)^{-1}$. Period matrices of Riemann surfaces are elements of the Siegel upper half-plane and the action of the modular group on these is produced by Dehn twists of the surface along homology cycles. The Jacobian variety $J(\Sigma)$ of Σ with period matrix Ω is the complex torus

$$J(\Sigma) = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)\,. \tag{IV.3}$$

Integration along a path C between two points p_1 and p_2 on the surface of the holomorphic one-forms define the classical Abel-Jacobi map μ:

$$\mu(p_1, p_2)_C = \int_{p_1}^{p_2} (\omega_1, ..., \omega_g) \mod \mathbb{Z}^g + \Omega\mathbb{Z}^g\,. \tag{IV.4}$$

As in the tropical case, the right hand side of (IV.4) does not depend on the integration path. Note that apart for the very special case of genus one where $\mu(\Sigma_1) \cong \Sigma_1$, the image of a genus $g \geq 2$ Σ_g Riemann surface by μ is strictly included in $J(\Sigma_g)$, $\mu(\Sigma_g) \subsetneq J(\Sigma_g)$.

B. Riemann Surfaces And Their Moduli Spaces $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}_{g,n}}$

Smooth Riemann surfaces of genus g with n punctures can be arranged in a moduli space denoted $\mathcal{M}_{g,n}$ of complex dimension is $3g-3+n$ spanned by $3g-3+n$ complex parameters called the moduli of the surface. This space is not compact, as surfaces can develop nodes when non trivial homotopy cycles pinch and give rise to surfaces with ordinary double points. The result of adding all such nodal curves to $\mathcal{M}_{g,n}$ is the well known Deligne-Mumford compactified moduli space of curves $\overline{\mathcal{M}_{g,n}}$ [70]. There exists two types of such degenerations, called separating and non-separating degenerations (see figure 10 below).

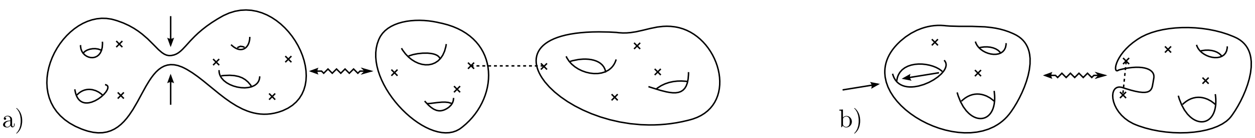


FIG. 10. a) A separating degeneration. b) A non-separating degeneration. Dashes represent double points.

A separating degeneration splits off the surface into a surface with two disconnected components that are linked by a double point, while a non separating degeneration simply give rise to a new surface with two points identified whose genus is reduced by one unit. Note that no degeneration may split off a surface that does not satisfy the stability criterion shared with tropical graphs (III.4). As a consequence, a maximally degenerated surface is composed of thrice punctures spheres.

Those degenerations induce a stratification on $\overline{\mathcal{M}_{g,n}}$, characterized by the combinatorial structure of the nodal curves. Indeed, one can associate to each degenerate surface a dual graph obtained by making a line go through each pinched cycle and turning each non degenerated component of genus $g \geq 0$ into a vertex of weight g. Finally, the legs of a dual graph are just what used to be punctures on the surface. Below we give a few examples of dual graphs of nodal Riemann surfaces.

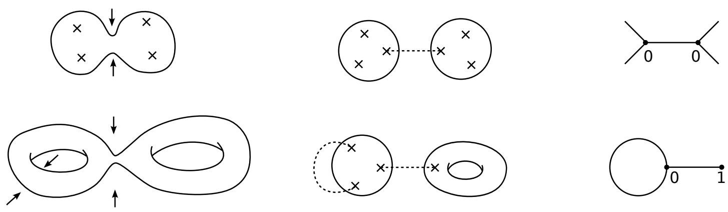


FIG. 11. On the leftmost column are degenerating surfaces, in the center the nodal version of these surfaces and in the rightmost column are dual graphs of the later.

A surface where a node is developing locally looks like a neck whose coordinates x and y on each of its side obey the following equation

$$xy = t\,, \tag{IV.5}$$

where the complex number t of modulus $|t| < 1$ is a parameter measuring the deformation of the surface around the singularity in $\overline{\mathcal{M}_{g,n}}$. The surface is completely pinched when $t = 0$. After a conformal transformation, one sees that this surface is alternatively described by a long tube of length $-\ln|t|$ and the goal of the tropicalization procedure is to turn these tubes into actual lines.

C. Tropicalizing $\mathcal{M}_{g,n}$

In the recent papers [63, 64], the authors presented a very sophisticated study of the links between $\mathcal{M}_{g,n}^{\text{trop}}$ and $\overline{\mathcal{M}_{g,n}}$ by making use of Berkovich analytic spaces. It would be interesting to recast their analysis in the context of string theory amplitudes.

To understand how string theory amplitudes are tropicalized in the $\alpha' \to 0$ limit, we will have to split the integration domain of string theory into a disjoint union of domains, so that each of them gives rise to a combinatorially distinct set of tropical graphs. We shall use the so called Kontsevich-Penner (KP) decomposition[6] of $\overline{\mathcal{M}_{g,n}}$ [39, 40]. In this approach, $\mathcal{M}_{g,n}$ is divided into a disjoint union of cells, in the center of which lies a nodal curve of $\overline{\mathcal{M}_{g,n}}$ with dual graph G labeling the cell:

$$\mathcal{M}_{g,n} = \bigsqcup_{G} \mathcal{D}_G \tag{IV.6}$$

[6] The author is grateful to Samuel Grushevsky for pointing out the existence of this construction.

where $\sqcup$ symbolizes disjoint union. In each KP cell $\mathcal{D}_G$, we have to find local coordinates – like t in (IV.5) – on $\mathcal{M}_{g,n}$ to parametrize the degenerating surfaces. Let us ignore what happens for the punctures in the following discussion for simplicity. Close to the singularity of the KP cell, the surface is developing a certain number N of narrow necks or long tubes, as many as there are inner edges in G. Each of these is parametrized by a complex parameter t_i for $i = 1, ..., N$ which are indeed local coordinates. The tropical graph is obtained in the cell by integrating out the phase dependence of the t_i's. The lengths T_i of its edges are then given by

$$T_i := -\frac{\alpha'}{2\pi} \ln |t_i| \,. \tag{IV.7}$$

Hence to obtain edges of finite size, one requires that the t_i's should have a particular tropical scaling, depending on α' and dictated by (IV.7):

$$t_i = \exp(2i\pi(\phi + iT_i/\alpha')) \to 0 \tag{IV.8}$$

where ϕ is just a phase. However we have not finished yet, since we have not dealt with the parts of the surface that did not undergo a degeneration. Since they keep their t_i of order $O(1)$, the tropical procedure requires to integrate out these regions of $\mathcal{M}_{g,n}$ to create a weighted vertex. Alternatively, keeping t_i fixed in (IV.7) corresponds to sending T_i to zero which is consistent with the definition of weighted vertices as the result of specialized loops.

Two specific kind of cells are of particular physical interest, that we call the analytic cell and the non-analytic cells. This terminology introduced by [14] in the context of the analyticity properties of one-loop string theory amplitudes refers in our case to the corresponding cells of the moduli space over which integration is performed. The analytic cell corresponds to the less deep strata of $\overline{\mathcal{M}_{g,n}}$ which tropicalizes to the tropical curve which is the n-valent weight-g vertex. The non-analytic cells correspond to the deepest strata of $\overline{\mathcal{M}_{g,n}}$ and give rise to pure tropical graphs made of trivalent vertices only.

We now compare classical and tropical objects. From the definitions of previous sections, we observe three facts:

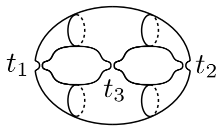
(i) When going from surfaces to graphs, one half of the homology disappears: the a cycles pinch, in accord with the physical picture where a cycles are closed strings becoming point-like.

(ii) In particular, since the a cycles integration provide the real part of the Jacobian variety, the imaginary part of the period matrices $\mathrm{Im}\,\Omega$ of tropicalizing surfaces should be related somehow to the period matrix of the tropical graph K.

(iii) The classical holomorphic one-forms become one-forms that are constant on the edges.

Can we interpret these facts in terms of what we expect from tropicalization, and in particular can we see that phases of complex numbers have been integrated out ?

As concerns period matrices and Jacobians, let us first deal with these of 1PI graphs. Under a tropical degeneration as in (IV.7), a given element in the period matrix have to be a linear combination of logarithms of the t_i's (see [71] for details), as in the following example shown in the figure 12.



$$\Omega^{(2)}_{\alpha'} = \frac{1}{2i\pi}\begin{pmatrix} -\ln(t_1 t_3) & \ln(t_3) \\ \ln(t_3) & -\ln(t_2 t_3) \end{pmatrix} + O(\alpha', t_i)$$

FIG. 12. Degenerating Riemann surface parametrized by local coordinates t_1, t_2, t_3 and its period matrix.

Therefore one recovers immediately the period matrix (III.9) of the two-loop tropical graph 4. More precisely we obtain that $\Omega^{(2)}_{\alpha'} = iK^{(2)}/\alpha' + O(1)$. This procedure generalizes straightforwardly to other cases and we obtain that in a given KP cell, the tropicalizing families of curves defined by (IV.7) have period matrices that approach the one of the tropical graph K as

$$\mathrm{Re}\,\Omega_{\alpha'} = M_0 + O(\alpha', t_i)\,, \qquad \mathrm{Im}\,\Omega_{\alpha'} = K/\alpha' + M_1 + O(\alpha', t_i)\,, \tag{IV.9}$$

where M_0 and M_1 are constant matrices with real coefficients. Thus, at the leading order and up to a rescaling by α', one has a similar statement for Jacobians

$$\mathrm{Im}\, J(\Sigma_{\alpha'}) = \alpha' J(\Gamma) + O(\alpha'^2)\,. \tag{IV.10}$$

As concerns 1PR graphs, the only thing one should take care of is that the one-forms are blind to the separating edges. In a KP cell corresponding to a dual graph G where an edge e splits of G into two 1PI graphs G_1 and G_2, let t_e be a local coordinate parametrizing such a separating degeneration. The period matrix of the degenerating family of curves is

$$\Omega^{(t_e)} = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix} + O(t_e) \tag{IV.11}$$

which can be tropicalized further following the previous discussion and provide the same splitting for the period matrix of the corresponding tropical graphs

$$K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}\,. \tag{IV.12}$$

As concerns one-forms, at a neck i parametrized by t_i, they behave as on a long tube:

$$\omega_I = \frac{c}{2i\pi}\frac{\mathrm{d}z}{z} + O(t_i)\,, \qquad \text{(IV.13)}$$

where $c = 1$ or $c = 0$ depending on whether the cycle b_I contains the node i or not. The Abel-Jacobi map (IV.4) then reduces to

$$\int^z \omega_I = \frac{c}{2i\pi}\ln(z) \in J(\Gamma) \qquad \text{(IV.14)}$$

where it is now clear that the phase of z is mapped to real parts in $J(\Gamma)$ in the tropical limit. Moreover, considering the following tropicalizing family of points z on the tube i:

$$z_{\alpha'} = \exp(2i\pi(x + iY/\alpha')) \qquad \text{(IV.15)}$$

where $x \in [0; 2\pi[$ and Y is a positive real number, one sees that the normalization $\omega_I^{\text{trop}} = 1$ on the cycle B_I ensures the correct limit for the Abel-Jacobi map

$$\alpha' \int^z \omega_i = i \int^Y \omega_I^{\text{trop}} = iY + O(\alpha') \in \alpha' \mathrm{Im}\, J(\Sigma_{\alpha'}) \equiv J(\Gamma) \qquad \text{(IV.16)}$$

according to (IV.10).

As a last remark on Jacobian varieties, note that the Jacobian $J = \mathbb{R}^2/K\mathbb{Z}^2$ of a two-loop graph is an ordinary complex torus. Modular $Sp(2,\mathbb{Z})$ properties of two-loop Feynman diagrams are well known, and this observation have been already implemented for instance in [48] and in the very complete analysis [72] of the sunset graph. It would be interesting to generalize this observation to higher loop order.

D. The Tropical Prime Form

Let Σ be a Riemann surface of genus g with period matrix Ω. The classical Riemann theta function is defined on the Jacobian variety of Σ by

$$\theta(\zeta|\Omega) := \sum_{n\in\mathbb{Z}^g} e^{i\pi n^t \Omega n} e^{2i\pi m^t z} \qquad \text{(IV.17)}$$

where $\zeta \in J(\Sigma)$ and $\Omega \in \mathcal{H}_g$. Theta functions with characteristics are defined by

$$\theta\begin{bmatrix}\beta\\ \alpha\end{bmatrix}(\zeta|\Omega) := e^{i\pi\beta^t\Omega\beta + 2i\pi\beta^t(\zeta+\alpha)}\theta(\zeta + \Omega\beta + \alpha|\Omega) \qquad \text{(IV.18)}$$

where α and β are g dimensional vectors of $\frac{1}{2}(\mathbb{Z}/2\mathbb{Z})^{2g}$ called the theta characteristics. An object of central interest for us is the so called prime form [71, 73, 74] defined by

$$E : (x,y) \in \Sigma\times\Sigma \longrightarrow E(x,y|\Omega) = \frac{\theta\begin{bmatrix}\beta\\ \alpha\end{bmatrix}(\int_x^y(\omega_1, ..., \omega_g)|\Omega)}{h_\kappa(x)h_\kappa(y)} \in \mathbb{C}\,, \qquad \text{(IV.19)}$$

where $\kappa = \begin{bmatrix} \beta \\ \alpha \end{bmatrix} \in \frac{1}{2}(\mathbb{Z}/2\mathbb{Z})^{2g}$ is a non singular odd theta characteristic and h_κ are the half differentials defined on Σ by $h_\kappa(z) = \sqrt{\sum_{i=1}^{g} \omega_I(z)\partial_I\theta \begin{bmatrix} \beta \\ \alpha \end{bmatrix} (0|\Omega)}$. We shall omit the mention of Ω in $E(.,.|\Omega)$ when it is not necessary. Defined in this way, the prime form is a differential form of weight $(-1/2, -1/2)$ which do not depend on the spin structure κ chosen. In some sense, it is the generalization of the map $x, y \in \mathbb{C} \to x - y$ to arbitrary Riemann surfaces. In particular, $E(x, y)$ vanishes only along the diagonal $x = y$. It is multi-valued on $\Sigma \times \Sigma$ since it depends on the path of integration in the argument of the theta function. More precisely, it is invariant up to a sign if the path of integration is changed by a cycle a_I, but it picks up a multiplicative factor when changing the path of integration by a cycle b_J

$$E(x, y) \to \exp(-\Omega_{JJ}/2 - \int_x^y \omega_J)E(x, y)\,. \tag{IV.20}$$

However in the physical quantity of interest – the bosonic Green function (V.6)– this ambiguity will be cured, making the objects well defined on the surface. We define the tropical prime form to be the result of the following limit:

$$E^{\text{trop}}(X, Y) := -\lim_{\alpha'\to 0} \left(\alpha' \ln \left|E(x_{\alpha'}, y_{\alpha'}|\Omega_{\alpha'})\right|\right) \tag{IV.21}$$

where $\Omega_{\alpha'}$ are the period matrices of a family of curves tropicalizing as in (IV.9), $x_{\alpha'}, y_{\alpha'}$ are two families of points as in (IV.15) and X and Y are two points on the tropical graph. Inspired by [51], we make the following conjecture.

Conjecture 1 *The tropical prime form defined in this way corresponds at any loop order to the scalar s of [51, eq.13], which is the graph distance $d_\gamma(X, Y)$ between X and Y along a path γ:*

$$E^{\text{trop}}(X, Y) = d_\gamma(X, Y) \tag{IV.22}$$

This object is ill-defined on the graph as it depends on γ, but this ambiguity will be also cured. To prove this conjecture, the first ingredient to use would be tropical theta functions with characteristics. As concerns tropical theta functions without characteristics introduced in [58] it is possible to show the following

$$\Theta^{\text{trop}}(Z|K) = \lim_{\alpha'\to 0} -\alpha' \ln|\theta(\zeta_{\alpha'}|\Omega_{\alpha'})| \tag{IV.23}$$

where we have defined the following families of points in $(\zeta_{\alpha'})$ in the Jacobian variety of tropicalizing curves as in (IV.9)

$$\zeta_{\alpha'} = \theta + iZ/\alpha' \tag{IV.24}$$

where θ and Z are real valued $g \times 1$ vectors. It is natural to wonder how that limiting procedure may be extended to tropical theta functions with characteristics introduced in [58, 75].

V. STRING THEORY AMPLITUDES, TROPICAL AMPLITUDES AND THE TROPICAL LIMIT

In the previous sections, we introduced tropical graphs and discussed how they appear in the tropicalization of Riemann surfaces. We are now ready to introduce string theory amplitudes and state precisely the main point of this paper concerning their $\alpha' \to 0$ limit.

A. The Tropical Limit Of String Theory

We call $\mathsf{A}_{\alpha'}^{(g,n)}(X \to Y)$ a generic string theory scattering amplitude in the Ramond-Neveu-Schwarz (RNS) formalism, for a process where the initial state X scatters to the final state Y. In the text, we shall often omit to mention the actual scattering process when it is not necessary. The amplitude writes as an integral over the supermoduli space of super Riemann surfaces $\mathfrak{M}_{g,n}$ [74, 76, 77]:

$$\mathsf{A}_{\alpha'}^{(g,n)}(X \to Y) = \int_{\mathfrak{M}_{g,n}} \mathrm{d}\mu_{\mathrm{s}}\, \mathcal{F}_{g,n}^{(s)}(X \to Y) \tag{V.1}$$

where $\mathrm{d}\mu_{\mathrm{s}}$ is an integration measure over the supermoduli space, and $\mathcal{F}_{g,n}^{(s)}(X \to Y)$ is an integrand coming the computation of a worldsheet conformal field theory correlator specific to the process $(X \to Y)$ considered. The measure $\mathrm{d}\mu_{\mathrm{s}}$ contains $3g-3+n$ even variables and $2g-2+n$ odd ones.

Until the recent series of papers [77–80], the procedure to compute $\mathsf{A}_{\alpha'}^{(g,n)}$ was believed to rely on the existence of a global holomorphic section of $\mathfrak{M}_{g,n}$ [44, 74] which is now known not to exist in the general case [81]. In particular, for $g \geq 5$ it is known that $\mathfrak{M}_{g,0}$ is not holomorphically projected.

However it has been proved that the computation of genus two amplitudes in [44] yields a correct expression for the superstring measure using the fact that $\mathfrak{M}_{2,0}$ is holomorphically projected [80]. We recall that the superstring measure is the integrand of the vacuum amplitude ($n = 0$) written after an hypothetical holomorphic projection of the super integral to an integral over $\mathcal{M}_{g,n}$. It is a generalization to the superstring of the Belavin-Knizhnik bosonic string measure [82–84] and a lot of work has been done to characterize it at higher genus [84–93]. It is not currently known if $\mathfrak{M}_{3,0}$ and $\mathfrak{M}_{4,0}$ do admit holomorphic projections, nor meromorphic ones that could perhaps be well behaved enough to permit a first principle derivation of this projection as for genus two. Notwithstanding, it is possible to formulate the problem of the existence of the RNS superstring measure purely in mathematical terms, and this problem turns out to admit a *unique* solution for genera $2, 3$ and 4 [86–88]. For $g \geq 5$, the solution is no more unique, and it is not even known if it can be consistently defined for $g \geq 6$. In any case, the unicity of the mathematical solution gives hope that there might exists a procedure to bypass the non-projectedness issues for these two cases.

The case of more general n-point amplitudes is even more intricate. It is known that $\mathfrak{M}_{2,n}$ is not holomorphically projected for $n \geq 1$, however the genus two solution of [44] for $n = 1, 2, 3, 4$ passes so many consistency checks that it is definitely reasonable to expect that a subtle argument explains why the computation is correct. An approach might consists in trying to identify to what extent the computation of these n-point amplitudes rely on $\mathfrak{M}_{2,0}$ rather that on $\mathfrak{M}_{2,n}$. As the amplitude possesses no pole, the bosonic space spanned by the positions of punctures is exactly $\mathcal{M}_{2,0}$, which is known to be the result of a holomorphic projection from $\mathfrak{M}_{2,0}$. Very recently, an expression was proposed for the four-graviton genus three amplitude using pure spinor formalism [49] in which the authors argued that this amplitude is not affected by the subtleties previously mentioned. As for the $g = 2$ amplitude, this amplitude has no pole [32] (neither should have the $g = 4$) and consequently might not really be defined over $\mathfrak{M}_{3,4}$ but rather over $\mathfrak{M}_{3,0} \times \Sigma^4$. It would be of course really important to understand more precisely these issues.

Since in our paper we deal only with integrals over the reduced base, it is a first step towards a fully general treatment of the super case that would use an hypothetical tropical supergeometry and the theorem that we are going to state should strictly speaking be understood with a genus restriction $g < 5$. Nevertheless its essence will definitely be underlying a general approach and so will the machinery developed. Furthermore, as recently discussed in [94], there exist a few consistent bosonic string realizations of the superstring, for instance proposed in [95] as well as some topological string amplitudes, for which the theorem should apply more generally.

The form of the superstring amplitudes that we will work with, also obtainable from the Green-Schwarz and pure spinor formalisms is

$$\mathsf{A}_{\alpha'}^{(g,n)} = \int_{\mathcal{M}_{g,n}} \mathrm{d}\mu_{\mathrm{bos}}\ \mathcal{F}_{g,n}\,, \tag{V.2}$$

where the integrand $\mathcal{F}_{g,n}$ has lost its superscript (s), and we have not written explicitly a possible sum over the spin structures of the surface. In the following cases that we will encounter, we will start from an integrand where the sum have already have been carried out, so we will not be more precise about that. The bosonic measure $\mathrm{d}\mu_{\mathrm{bos}}$ is a $(3g-3+n)$-dimensional measure that can be traded for an integration over the period matrices for genus $1,2,3$[7] and then explicitly writes

$$\mathrm{d}\mu_{bos} = \frac{|\prod_{1\leq I<J\leq g} \mathrm{d}\Omega_{IJ}|^2}{|\det \operatorname{Im}\Omega|^5} \prod_{i=1}^{n} \mathrm{d}^2 z_i\,. \tag{V.3}$$

The inverse fifth power is half of the dimension of space-time; if we were to consider a space-time compactification to d dimensions, this 5 would become a $d/2$. The integrand $\mathcal{F}_{g,n}$ can be decomposed further by means of introducing the following quantities:

$$\mathcal{F}_{g,n} := \mathcal{W}_{g,n} \exp(-\mathcal{Q}_{g,n})\,. \tag{V.4}$$

The function $\mathcal{W}_{g,n}$ carries the information about the particular process being studied while the exponential factor $\exp(-\mathcal{Q}_{g,n})$ is the Koba-Nielsen factor. It is a universal factor present in any amplitude, it may be explicitly written

$$\mathcal{Q}_{g,n} = \alpha' \sum_{1\leq i<j\leq n} k_i.k_j\, G(z_i,z_j)\,. \tag{V.5}$$

Here G is the bosonic Green function computed in [74, 96]

$$G(z_1,z_2) = -\frac{1}{2}\ln\left(|E(z_1,z_2)|\right) - \frac{1}{2}\left(\int_{z_2}^{z_1}\omega_I\right)(\operatorname{Im}\Omega^{-1})^{IJ}\left(\int_{z_2}^{z_1}\omega_J\right) \tag{V.6}$$

where Ω is the period matrix of the surface to which belong the points z_1 and z_2. One can check that G is well defined on the surface, unlike the prime form; any change in $\ln|E|$ as in (IV.20) is exactly canceled by the new term added.

As we explained in section II, the first step of the tropicalisation procedure consists in identifying the contribution of the massless states and of the massives states in the limit. At the mathematical level, this corresponds to showing the following conjecture.

Conjecture 2 *There exists a KP decomposition $\mathcal{M}_{g,n} = \left(\bigsqcup_{i=1}^{N}\mathcal{D}_G\right)\sqcup\mathcal{D}_0$ as in* (IV.6) *such that in the limit $\alpha'\to 0$, the string theory amplitudes* (V.2) *one has the two following points:*

(i) Integrating over outer domain $\mathcal{D}_0$ produces only subleading contributions:

$$\int_{\mathcal{D}_0} \mathrm{d}\mu_{bos}\,\mathcal{F}_{g,n} = O(\alpha')\,. \tag{V.7}$$

(ii) In each KP cell $\mathcal{D}_G$, there exist a function $F_{g,n}$ defined over the moduli space of tropical graphs $\Gamma=(G,\ell,w)$ with combinatorial type G, $\mathcal{M}^{\mathrm{trop}}(\Gamma)$, such that:

$$\int_{\mathcal{D}_G} \mathrm{d}\mu_{\mathrm{bos}}\ \mathcal{F}_{g,n} = \int_{\mathcal{M}^{\mathrm{trop}}(\Gamma)} \mathrm{d}\mu_{\mathrm{trop}}\ F_{g,n} + O(\alpha')\,. \tag{V.8}$$

[7] For which there is no Schottky problem.

The measure writes

$$\mathrm{d}\mu_{\text{trop}} := \frac{\prod_{i\in E(G)} \mathrm{d}\ell(i)}{(\det K)^5}\,, \tag{V.9}$$

where K is the period matrix of Γ. Again, the inverse power five is half of the space time dimension. In particular in four dimensions, this would be a two.

Physically, the right hand side of (V.8) is the contribution of the Feynman diagrams of field theory in the tropical representation corresponding to the graph G. The structure of the integrand $F_{g,n}$ can be made more precise as it turns out to factorize in two terms

$$F_{g,n} = W_{g,n}\exp(-Q_{g,n}) \tag{V.10}$$

where $W_{g,n}$ and $Q_{g,n}$ descend from their string theory ancestors (V.4). Finding their explicit expressions gives the tropical representation of the integrand and is the second step of the tropicalization procedure. The extraction of $W_{g,n}$ is easy in the trivial cases of maximal supergravity four-graviton amplitudes discussed later for $g = 0, 1, 2$ but it is much more intricate in the general case. It requires a lot of technical input (Fourier-Jacobi expansion in higher genus) and this topic will not be covered in this paper. We refer to the aforementioned papers [9–11, 97] for details on this point in genus one.

On the contrary, the tropical representation of $Q_{g,n}$ is generic and can be explicitly extracted, assuming the property of the tropical prime form given in the conjecture 1. Using (IV.21) and the limit form of the holomorphic differentials derived in (IV.14), the Green function (V.6) give rise to a tropical Green function defined by

$$G^{\text{trop}}(Z_1, Z_2) := \lim_{\alpha'\to 0} \alpha' G(z_1, z_2) = -\frac{1}{2}E^{\text{trop}}(Z_1, Z_2) - \frac{1}{2}\left(\int_{Z_2}^{Z_1} \omega^{\text{trop}}\right) K^{-1} \left(\int_{Z_2}^{Z_1} \omega^{\text{trop}}\right) \tag{V.11}$$

where the limit is to be understood as in section IV D. Using the previous assumption G^{trop}, is immediately shown to be the Green function computed in [51, 98]. Contrary to the tropical prime form, G^{trop} is independent of the integration path, and this result holds for any kind of tropical graph, pure or not, 1PI or not. It follows from these definitions that the tropical representation of exponential factor in (V.4) is

$$\exp\left(-\alpha' \sum k_i.k_j G(z_i, z_j)\right) = \exp\left(\sum k_i.k_j G^{\text{trop}}(Z_i, Z_j)\right) + O(\alpha')\,. \tag{V.12}$$

We can now collect (V.9) and (V.12) to obtain the following proposition.

Proposition 3 *The tropical representation of* (V.8) *is*

$$\int \prod_{i\in E(G)} \mathrm{d}\ell(i) \frac{W_{g,n}\exp(Q_{g,n})}{(\det K)^5}\,. \tag{V.13}$$

In this form, $\det(K)$ *and* $\exp(Q_{g,n})$ *are respectively the first and second Symanzik polynomials*[8] *obtained from Feynman rules in field theory, and* $W_{g,n}$ *is a numerator for the integrand.*

This assertion is clear from the physical point of view (see [51, sec. V] for a proof concerning $\det K$ and [48] for a recasting of the second Symanzik polynomial in terms of the tropical Green function of [51]). A direct proof using graph theory for the first and second polynomial would however be interesting for more formal aspects of the study of Feynman diagrams[9]. Note also that in this representation, it is obvious that the first Symanzik polynomial does not depend on the positions of the punctures. Examples in genus one and two are given in the following section VI. Using the conjecture 2, we can now obtain the following result.

Theorem 4 (Tropicalization of String Theory Amplitudes) *The* $\alpha' \to 0$ *limit of string theory amplitudes is a tropical limit. The integration over* $\mathcal{M}_{g,n}$ *is mapped to an integration over* $\mathcal{M}_{g,n}^{\mathrm{trop}}$ *and we have*

$$\int_{\mathcal{M}_{g,n}} \mathrm{d}\mu_{\mathrm{bos}}\ \mathcal{F}_{g,n} = \int_{\mathcal{M}_{g,n}^{trop}} \mathrm{d}\mu_{\mathrm{trop}}\ F_{g,n} + O(\alpha')\,, \tag{V.14}$$

where

$$\int_{\mathcal{M}_{g,n}^{trop}} \mathrm{d}\mu_{\mathrm{trop}} := \sum_{\Gamma} \int_{\mathcal{M}(\Gamma)} \mathrm{d}\mu^{\mathrm{trop}}\,. \tag{V.15}$$

The discrete finite sum runs over all the combinatorially distinct graphs Γ *of genus* g *with* n *legs. Moreover the right hand side of* (V.14) *corresponds to the field theory amplitude renormalized in the scheme induced by string theory. This scheme is defined such that*

$$\mathsf{A}_{\mathrm{trop}}^{(g,n)} := \int_{\mathcal{M}_{g,n}^{trop}} \mathrm{d}\mu_{\mathrm{trop}}\ F_{g,n} \tag{V.16}$$

where $\mathsf{A}_{\mathrm{trop}}^{(g,n)}$ *is the field theory amplitude written in its tropical representation (in short tropical amplitude) obtained in the low energy limit.*

---

[8] There is a slight difference of normalization compared to the usual definition given for instance in the classical reference [50] where the first and second Symanzik polynomials, denoted $\mathcal{U}$ and $\mathcal{F}$, are related to ours by: $\mathcal{U} = \det K$, $\mathcal{F} = \exp(Q_{g,n})\det K$, and where also $\exp(Q_{g,n})$ should strictly speaking be replaced by the result of integrating out a global scale factor for the lengths of the edges of the graph to go from Schwinger proper times to Feynman parameters.

[9] The author is grateful to Francis Brown for a discussion on this point.

The demonstration of identities such as (V.14) will be done by applying the procedure that was sketched in the section II. In the first step, one has to find a KP decomposition where the outer region does not bring any contribution at the leading order. The distinction between the contributions of massive and massless states follows from that decomposition; massive states propagate on surfaces corresponding to weighted vertices of the dual graphs and massless states run into finite lengths inner edges. In the second step, in each cell the level matching condition is trivial which makes it possible to extract the tropical representation of $\mathcal{F}_{g,n}$ and in particular of $\mathcal{W}_{g,n}$ since the case of $\exp(\mathcal{Q}_{g,n})$ has already been dealt with.

We dedicate the next section VI to give proofs and physical applications of these statements for zero-, one- and two-loop amplitudes, but before that let us make three short comments.

B. Three Easy Pieces

1. *The Analytic Cell And The Non Analytic Cells*

For simplicity let us exclude the punctures of that discussion. The analytic and non-analytic cells have been defined in section IV C by the requirement that the first should correspond to the more superficial stratum of $\overline{\mathcal{M}}_g$ and the second should correspond to the deepest strata of $\overline{\mathcal{M}}_g$.

Thus, the analytic cell is defined by removing all neighborhoods around the singularities of $\mathcal{M}_g$; it is a compact space. Inside that cell, the string integrand has no singularity and the limit may be safely taken inside the integrand, where the factor α' present in the definition of $\mathcal{Q}_{g,n}$ simply sends $\exp(\mathcal{Q}_{g,n})$ to 1. This reasoning justifies why in an important part of the literature, "taking the low energy limit" is often translated as getting rid of the Koba-Nielsen factor.

This also suggests that to compute the primary divergence of an amplitude, it is sufficient to compute the string integral over the analytic cell, as illustrated in the one-loop example of section VI B. Understanding the role of the precise form of the boundary of this cell is a non trivial question, that needs to be solved for having a computationnal answer to this issue.

As concerns the non analytic cells, they provide the contribution of the pure tropical graphs, made of trivalent vertices only. Summed over, those give the unrenormalized field theory amplitude, with all of its sub-divergences. We shall give in section VI C a computation of a tropical integrand in genus two in this non-analytic cell.

2. *A Remark On Contact Terms*

Let us clarify an issue about contact-terms. In physics, Feynman rules naturally include vertices of valency four in non abelian gauge theories and arbitrarily high in gravity theories to guarantee gauge invariance. The way they arise from string theory is not the same. What is called "contact-term" in string theory is usually the vertex that is result of integrating out the length dependence of a separating edge in a 1PR graph. In the tropicalization procedure, we do not perform these integrations. Thus higher valency vertices (of weight zero) are present in our considerations, but

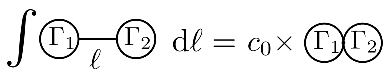


FIG. 13. Contact terms in string theory, Γ_1 and Γ_2 are two arbitrary tropical graphs and c_0 is the coefficient of the contact-term.

only as boundaries between cells in $\mathcal{M}_{g,n}^{\text{trop}}$ of maximal codimension and should not carry any localized contribution in the integrands, unlike in Feynman rules where they carry a distinct structure compared to the lower valency vertices.

3. *Hyperelliptic Surfaces And Graphs*

The very last item that we add to the tropical toolbox are hyperelliptic tropical graphs [66]. These are graphs endowed with an involution, just as in the classical case. The *assumption* that the KP cells corresponding to tropical graphs do contain only hyperelliptic surfaces and *vice versa*, may provide a way to compute field theory hyperelliptic diagrams from the hyperelliptic sector of string theory, way easier to handle than the full moduli space $\mathcal{M}_{g,n}$. Those graphs include the "ladder graphs", and all other graphs that carry the same reflexion property, as pictured below in figure 14.

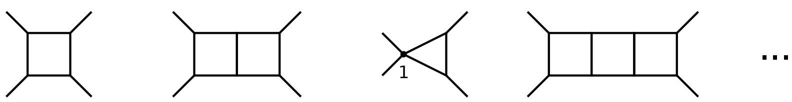


FIG. 14. Hyperelliptic tropical graphs with punctures.

VI. THE TROPICALIZATION IN ACTION

We now illustrate how the existing computations on the extraction of the low energy limit of string theory amplitudes can be nicely recast in the tropical framework for genus zero and one. This will prove the conjecture 2 in these cases. For genus two, we give the tropical representation of the four-graviton amplitude in the non-analytic cell of the KP decomposition.

A. Tree Level

As a warm-up, we begin our study by tree level string theory amplitudes. We wish to see explicitly how they generate all of the distinct tropical trees with their metric structure. We first look at the simplest example; the four-tachyon scattering in the bosonic string, then describe the case of four-graviton scattering in the type II superstring. The general case of n-particle scattering follows from the same method as the one exposed here, but the construction of the KP decomposition is technically more involved thus it will be treated elsewhere [99]. Later when are studied genus one amplitudes, we use this tree level KP decomposition to describe how diagrams with trees attached to loops are generated by the string theory amplitudes.

A closed string theory tree level n-point amplitude can be written in the general form[10]:

$$\mathsf{A}^{(0,n)}_{\alpha'} = g_c^{n-2}\frac{8\pi}{\alpha'}\int_{\mathcal{M}_{0,n}}\prod_{i=3}^{n-1}\mathrm{d}^2 z_i\,\langle (c\bar{c}V_1)(c\bar{c}V_2)V_3...V_{n-1}(c\bar{c}V_n)\rangle\,, \tag{VI.1}$$

where $\mathrm{d}^2 z := \mathrm{d}z\mathrm{d}\bar{z}$ and g_c is the string coupling constant. The vertex operators V_i corresponding to the external scattered states depend on the point z_i at which they are inserted, as well as the momentum k_i and possible polarization ϵ_i of the particles. The integration over the points $z_1,\ z_2$ and z_n has been suppressed and exchanged by the insertion of $c\bar{c}$ ghosts to account for the factorization of the infinite volume of the $SL(2,\mathbb{C})$ conformal group. Then, one has an integral over a set of $n-3$ distinct complex variables, which span the moduli space of n-punctured genus zero surfaces $\mathcal{M}_{0,n}$. The correlation function (VI.1) is computed using the two-point correlators on the sphere:

$$\langle X(z,\bar{z})X(w,\bar{w})\rangle = G(z,w) = -\alpha'\ln(|z-w|^2)\,,\qquad \langle c(z)c(w)\rangle = z-w\,, \tag{VI.2}$$

which gives for the ghost part

$$|\langle c(z_1)c(z_2)c(z_n)\rangle|^2 = |z_{12}z_{2n}z_{n1}|^2\,. \tag{VI.3}$$

[10] We follow the conventions of [53]

The correlation function (VI.1) can be written as in (V.2), by defining $\mathrm{d}\mu_{\mathrm{bos}} := \prod_{i=3}^{n-1} \mathrm{d}^2 z_i$ and

$$\mathcal{F}_{0,n} := g_c^{n-2}\frac{8\pi}{\alpha'}\mathcal{W}_{0,n}(z_{jk}^{-1}, \bar{z}_{lm}^{-1})\exp(\alpha'\mathcal{Q}_{0,n})\,, \tag{VI.4}$$

$$\mathcal{Q}_{0,n} := \alpha' \sum_{3\leq i<j\leq n-1} k_i.k_j \ln|z_i - z_j|\,, \tag{VI.5}$$

where $1 \leq j,k,l,m \leq n$ and $\mathcal{W}_{0,n} = 1$ for the scattering of n tachyons, while it is a rational function of the z_{jk} in the general case of NS–NS states scattering. Its coefficients are then made of powers of α', scalar products of polarization tensors and external momenta and include the gauge structure for gauge theory interactions.

Let us start with the scattering of four tachyons string states ϕ. The vertex operator of a tachyon with momentum k_i ($k_i^2 = -m_{\mathrm{tach}}^2 := 4/\alpha'$) is a plane wave $V_i = e^{ik.X(z_i,\bar{z}_i)}$. From (VI.1) we obtain

$$\mathsf{A}_{\alpha'}^{(0,4)}(\phi\phi\phi\phi) = g_{tach}^2\,|z_{12}z_{24}z_{41}|^2 \int \mathrm{d}^2 z_3\, e^{(\alpha' k_1.k_3 \ln|z_{13}z_{24}| + \alpha' k_2.k_3 \ln|z_{23}z_{14}| + \alpha' k_4.k_3 \ln|z_{12}z_{34}|)}\,, \tag{VI.6}$$

where we have introduced the tachyon cubic interaction coupling constant $g_{tach} := 8\pi g_c/\alpha'$ and kept z_1, z_2 and z_4 fixed but arbitrary. Momentum conservation imposes $k_1 + k_2 + k_3 + k_4 = 0$ and the Mandelstam kinematic invariants s,t,u are defined by $s = -(k_1+k_2)^2$, $t = -(k_1+k_4)^2$, $u = -(k_1+k_3)^2$. Their sum is the sum of the masses of the scattering particles $s+t+u = \sum_1^4 m_i^2$. The integral (VI.6) can be computed explicitly in a closed form and writes

$$\mathsf{A}_{\alpha'}^{(0,4)}(\phi\phi\phi\phi) = 2\pi g_{tach}^2 \frac{\Gamma(\alpha(s))\Gamma(\alpha(t))\Gamma(\alpha(u))}{\Gamma(\alpha(t)+\alpha(u))\Gamma(\alpha(u)+\alpha(s))\Gamma(\alpha(s)+\alpha(t))} \tag{VI.7}$$

where $\alpha(s) := -1 - s\,\alpha'/4$. It has poles in the tachyon's kinematic channels, for instance

$$\mathsf{A}_{\alpha'}^{(0,4)} \overset{s\to -4/\alpha'}{\sim} g_{tach}^2 \frac{1}{-s-4/\alpha'}\,. \tag{VI.8}$$

We want to recover these poles in the point-like limit using our tropical procedure. It is a well known fact that poles appear in string theory amplitudes from regions where vertex operators collide to one another. In particular at tree level, there are nothing but poles so the cells $\mathcal{D}$ of the KP decomposition in equation (IV.6) must correspond to these regions. At four points, only one coordinate is free and the cells are just open discs of radius ℓ centered around z_1 z_2 and z_4 called $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_4$ as shown in the picture 15 below. The KP decomposition is then

$$\mathcal{M}_{0,4} = (\mathcal{D}_1 \sqcup \mathcal{D}_2 \sqcup \mathcal{D}_4) \sqcup \mathcal{D}_0\,. \tag{VI.9}$$

We will see how the integrals over each domain respectively provide the u, t and s channel tachyon exchanges in this tropical language while the integral over $\mathcal{D}_0$ gives a subleading contribution.

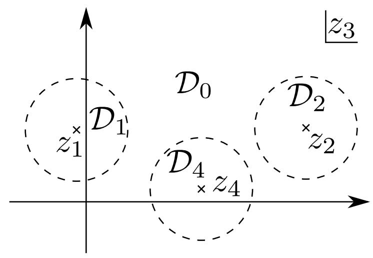


FIG. 15. The four KP cells split into three domains and the outer domain.

We focus first on the the integral over the cell $\mathcal{D}_1$. As the KP cells are disjoint, we have $|z_{21}| > \ell$ and $|z_{41}| > \ell$. Thus the terms $\alpha' k_2.k_3 \log|z_{32}z_{14}| + \alpha' k_4.k_3 \log|z_{34}z_{12}|$ in (VI.6) behave like

$$(-\alpha' k_1.k_3 - 4)\log|z_{12}z_{14}| + O(\alpha' z_{31}, \alpha' \bar{z}_{31}) \tag{VI.10}$$

which gives in the integral:

$$\int_{\mathcal{D}_1} \mathrm{d}^2 z_3 \frac{|z_{24}|^2}{|z_{12}z_{14}|^2}\, e^{\alpha' k_1.k_3 \log\left|\frac{z_{31}z_{24}}{z_{12}z_{14}}\right|} + O(\alpha')\,, \tag{VI.11}$$

where the $O(\alpha')$ terms account for massless and massive states exchanges between the tachyons. Note also that the phase of z_{31} is now trivial. We may now go to the tropical variable X as in (IV.15) that geometrically indicates how close the point z_3 is from z_1:

$$z_3 = z_1 + c\ \exp(-X/\alpha' + i\theta)\,, \tag{VI.12}$$

where c is a conformal factor given by $c = z_{24}/(z_{12}z_{14})$ and θ is a phase. In this variable, the closer z_3 is from z_1, the larger is X. The integration measure becomes $|c|^2 \mathrm{d}^2 z_3 = -\frac{2}{\alpha'}\, e^{-2X/\alpha'} dX\, d\theta$ and the radial integration domain is now $X \in [-\alpha' \ln \ell, +\infty[$. As nothing depends on θ, we can just integrate it out and forgetting the ℓ-dependent terms that are subleading, we get the following contribution to the amplitude

$$\mathsf{A}^{(0,4)}_{\alpha'}(\phi\phi\phi\phi)|_{u-\text{channel}} = g^2_{tach}\left(\int_0^\infty dX\ e^{-((k_1+k_3)^2 + m^2_{\text{tach}})X} + O(\alpha')\right) \tag{VI.13}$$

which is nothing but the exponentiated Feynman propagator of a scalar ϕ^3 theory with coupling constant g_{tach} and mass m_{tach}. In this form, the modulus X of the graph is the Schwinger proper time of the exchanged particle.

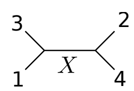


FIG. 16. X is the modulus of the tropical graph. The larger it is, the closer z_1 from z_3.

We can now repeat the same operations in the two other kinematic regions to obtain s and t channel exchanges. To conclude, one has to check that the integral over $\mathcal{D}_0$ does yield only $O(\alpha')$ contributions. In the pathological case of tachyon scattering, one has to slightly "cheat" and forget that m_{tach} is dependent of α' and rather consider it to be fixed. From that follows the desired result, i.e. the "proof" of the conjecture 2 and theorem 4 for $n = 4$ tachyons at $g = 0$ in the bosonic string

$$\mathsf{A}_{\alpha'}^{(0,4)}(\phi\phi\phi\phi) \to \mathsf{A}^{(0,4)}(\phi\phi\phi\phi) = \int_{\mathcal{M}_{0,4}^{\text{trop}}} \mathrm{d}\mu_{\text{trop}}\, F_{0,4} \tag{VI.14}$$

where the measure pulls back to regular integration measure dX on each edge, while $F_{0,4}$ is given by $F_{0,4} = \exp\left(-X((k_i + k_3)^2 + m_{\text{tach}}^2)\right)$ where $i = 1, 2, 4$ depending on the edge of $\mathcal{M}_{0,4}^{\text{trop}}$ considered on which X is the corresponding coordinate.

Let us now investigate the more realistic case of four-graviton scattering in superstring theory. The KP decomposition is unchanged; the qualitative difference with the scalar case will be the introduction of a non trivial $\mathcal{W}$. We will work in a representation of the integrands where all double poles have been integrated by parts, this can always been done [100, 101]. The tree level four-graviton (denoted h) amplitude writes

$$\mathsf{A}_{\alpha'}^{(0,4)}(hhhh) = \frac{8\pi g_c^2}{\alpha'} \langle c\bar{c}V_{(-1,-1)}(z_1) c\bar{c}V(z_2)_{(-1,-1)} V_{(0,0)}(z_3) c\bar{c}V_{(0,0)}(z_4)\rangle\,, \tag{VI.15}$$

where the graviton vertex operators in the $(-1,-1)$ and $(0,0)$ pictures writes

$$V_{(-1,-1)}(z) = \epsilon_{\mu\nu}(k) e^{-\phi-\bar{\phi}} \left(\psi^\mu \bar{\psi}^\nu\right) e^{ik.X(z,\bar{z})}\,, \tag{VI.16}$$

$$V_{(0,0)}(z) = \frac{2}{\alpha'}\epsilon_{\mu\nu}(k) \left(i\bar{\partial} X^\mu + \frac{\alpha'}{2} k.\bar{\psi}\bar{\psi}^\mu\right) \left(i\partial X^\mu + \frac{\alpha'}{2} k.\psi\psi^\mu\right) e^{ik.X(z,\bar{z})}\,, \tag{VI.17}$$

where $\epsilon_{\mu\nu} := \epsilon_\mu \tilde{\epsilon}_\nu$ is the polarization vector of the graviton and where the bosonized superconformal ghost's two point function is $\langle \phi(z)\phi(w)\rangle = -\ln(z-w)$ while the one of the fermions is $\psi^\mu(z)\psi^\nu(w) = \eta^{\mu\nu}/(z-w)$. The amplitude (VI.18) can be computed explicitly (see the classical reference [54]) and turns out to write

$$\mathsf{A}_{\alpha'}^{(0,4)}(hhhh) = \frac{8\pi g_c^2}{\alpha'} C(s,t,u)\, t_8 t_8 R^4 \tag{VI.18}$$

where the Weyl tensor R appears here linearized $R^{\mu\nu\rho\sigma} = F^{\mu\nu}\tilde{F}^{\rho\sigma}$. The tensors F and $\tilde{F}$ are on-shell linearized field strengths such that the graviton i with polarization $\epsilon_i^{\mu\nu} = \epsilon_i^\mu \tilde{\epsilon}_i^\nu$ and momentum k_i has $F_i^{\mu\nu} = \epsilon_i^{[\mu} k_i^{\nu]}$ and $\tilde{F}_i^{\rho\sigma} = \tilde{\epsilon}^{[\rho} k^{\sigma]}$. The quantities C and the tensor t_8 are defined in [54] in

(7.4.56) and (7.4.42) respectively, we reproduce them here:

$$C(s,t,u) := -\pi\frac{\Gamma(-\alpha's/4)\Gamma(-\alpha't/4)\Gamma(-\alpha'u/4)}{\Gamma(1+\alpha's/4)\Gamma(1+\alpha't/4)\Gamma(1+\alpha'u/4)}\,, \tag{VI.19}$$

$$\begin{aligned} t_8F^4 := &-st(\epsilon_1.\epsilon_3)(\epsilon_2.\epsilon_4) + 2t(\epsilon_2.k_1\,\epsilon_4.k_3\,\epsilon_3.\epsilon_1 + \epsilon_3.k_4\,\epsilon_1.k_2\,\epsilon_2.\epsilon_4 \\ &+\ \epsilon_2.k_4\,\epsilon_1.k_3\,\epsilon_3.\epsilon_4 + \epsilon_3.k_1\,\epsilon_4.k_2\,\epsilon_2.\epsilon_1) + (2\leftrightarrow 3) + (3\leftrightarrow 4) \end{aligned} \tag{VI.20}$$

Schematically, t_8F^4 writes as a polynomial in the kinematic invariants with coefficient made of scalar products between polarizations and momenta

$$t_8F^4 = C_s s + C_t t + C_u u + C_{st} st + C_{tu} tu + C_{us} us\,. \tag{VI.21}$$

Since $C(s,t,u) \sim 1/(\alpha'^3 stu)$, using multiple times the on-shell condition $s+t+u=0$, the amplitude (VI.18) can be written as

$$\mathsf{A}^{(0,4)}_{\alpha'} \sim \frac{A_s}{s} + \frac{A_t}{t} + \frac{A_u}{u} + A_0 + O(\alpha') \tag{VI.22}$$

where the A's are sums of terms like $C_s\overline{C}_t$, etc. As the tensorial structure of this object is rather complicated, we will only focus ourselves on one particular term; a contribution to A_u. In the correlation function (VI.15), such a contribution comes from the following term:

$$\begin{aligned} &-(\alpha'/2)^2(\epsilon_2.\epsilon_4)\frac{1}{z_{24}^2}(\epsilon_1.k_4)(\epsilon_3.k_2)\left(\left(\frac{1}{z_{14}}-\frac{1}{z_{13}}\right)\left(\frac{1}{z_{32}}-\frac{1}{z_{31}}\right)+\frac{1}{z_{13}^2}\right)\times \\ &(-1)(\alpha'/2)^2\,(\tilde\epsilon_2.\tilde\epsilon_4)\,\frac{1}{\bar z_{24}^2}(\tilde\epsilon_1.k_2)(\tilde\epsilon_3.k_4)\left(\left(\frac{1}{\bar z_{12}}-\frac{1}{\bar z_{13}}\right)\left(\frac{1}{\bar z_{34}}-\frac{1}{\bar z_{31}}\right)+\frac{1}{\bar z_{13}^2}\right) \end{aligned} \tag{VI.23}$$

where we have used the conservation of momentum $k_1+k_2+k_3+k_4=0$, the on-shell condition $\epsilon_i.k_i=0$, and the expression of P given in (VI.2). It is now straightforward to check that the term corresponding to $1/|z_{31}|^2$ in the previous expression is accompanied with a factor of $|z_{12}z_{24}z_{41}|^{-2}$ which cancels precisely the conformal factor from the $c\bar c$ ghosts integration (VI.3) and one ends up with the following integral

$$-\left(\frac{\alpha'}{2}\right)^3\int \mathrm{d}^2 z_{31}\frac{1}{|z_{31}|^2}e^{\alpha' k_1.k_3\ln|z_{31}|} + O(\alpha')\,, \tag{VI.24}$$

The phase dependence of the integral is either pushed to $O(\alpha')$ terms or canceled due to level matching in the vicinity of z_1. Thus, we can integrate it out and recast the integral in its tropical form using the same change of variables as in (VI.12) and one gets the following contribution to the amplitude (VI.15)

$$4\kappa_d^2\left(\int_0^\infty \mathrm{d}X e^{-uX} + O(\alpha')\right) \tag{VI.25}$$

where $\kappa_d := 2\pi g_c$ is the d dimensional coupling constant that appears in the Einstein-Hilbert action. Other terms are generated in the exact same manner, by combinations of various massless poles (even A_0, despite that it has no explicit pole structure).

The generalization to the n point case is more subtle since there are $n-3$ non fixed points spanning $\mathcal{M}_{0,n}$. The trees with edges of finite lengths will be generated by similar regions of the moduli space where the points z_i collides towards one another. The crucial step in the proof that we shall perform in [99] is to build an explicit KP decomposition that produces the correct combinatorics of collapsing for the $(2m-5)!!$ distinct trivalent trees.

B. One-Loop

The technical aspects of the extraction of the point-like limit of genus one open and closed string theory amplitudes are well understood, and in this section, we shall recast in the tropical framework the older result on the subject. We first focus on the four-graviton type II superstring amplitudes since we are ultimately interested in higher genus four-graviton amplitudes. That amplitude is a nice toy model to see how the tropical limiting procedure naturally generates the so called analytic and non analytic terms [14, 33, 35, 102] of the amplitudes along with the renormalization apparatus of counter-terms and give these a natural geometrical meaning. When this is done, we discuss the general n point case and make connection with the previous section on tree-level amplitudes and discuss what regions of the string theory moduli space integral give rise to trees attached to the loop, following the analysis of [6–10].

Let us first recall some facts about genus one Riemann surfaces. These are complex tori $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ parametrized by a single complex modulus τ in the genus one Siegel upper half-plane $\mathcal{H}_1 = \{\tau \in \mathbb{C},\, \mathrm{Im}\,(\tau) > 0\}$ [11]. Modding out by the action of the modular group $SL(2,\mathbb{Z})$ further restricts τ which eventually lies in an $SL(2,\mathbb{Z})$ fundamental domain. A representative one that we will use is $\mathcal{F} = \{\tau \in \mathbb{C},\, |\tau| > 1,\, -1/2 \geq \mathrm{Re}\,\tau < 1/2,\, \mathrm{Im}\,\tau > 0\}$, shown in the figure 17 below. We also recall that q is defined by $q := e^{2i\pi\tau}$. If we now include the three moduli associated to the four punctures at distinct positions $\zeta_i \in \mathcal{T}$, $i = 1, 2, 3$ where $\mathcal{T} = \{\zeta \in \mathbb{C}, -1/2 < \mathrm{Re}\,\zeta < 1/2,\, 0 \leq \mathrm{Im}\,\zeta < \tau_2\}$ and ζ_4 fixed at $\zeta_4 = \mathrm{Im}\,\tau$, we can describe completely the moduli space $\mathcal{M}_{1,4}$ over which our string theory amplitude (V.2) is being integrated

$$\mathsf{A}^{(1,4)}_{\alpha'} = \int_{\mathcal{M}_{1,4}} \mathrm{d}\mu_{\mathrm{bos}}\ \mathcal{F}_{1,4}\,. \tag{VI.26}$$

[11] The complex tori is actually the Jacobian variety of the surface, but at genus one both are isomorphic. This property does not hold for higher genus curves.

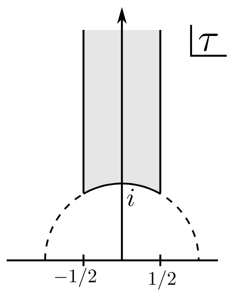


FIG. 17. An $SL(2,\mathbb{Z})$ fundamental domain for complex tori.

We start the analysis by the case of the four-graviton type II amplitude in 10 flat compact dimensions. This amplitude is known not to have pole because of supersymmetry, thus there will be no need to consider regions of the moduli space $\mathcal{M}_{1,4}$ which could give rise to one-loop diagrams with trees attached to the loop. This will be justified *a posteriori*. For this amplitude $\mathcal{F}_{1,4}$ is particularly simple since it is reduced to the Koba-Nielsen factor (V.5)

$$\mathcal{F}_{1,4} = (2\pi)^8 t_8 t_8 R^4 \exp\left(\alpha' \sum_{i<j} k_i.k_j G(\zeta_i - \zeta_j)\right) \tag{VI.27}$$

where $t_8t_8R^4$ has been defined in the four-graviton tree analysis and the measure is given by

$$\int_{\mathcal{M}_{1,4}} \mathrm{d}\mu_{\mathrm{bos}} = \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^5} \int_{\mathcal{T}} \prod_{i=1}^{3} \mathrm{d}^2\zeta_i\,. \tag{VI.28}$$

The one-loop bosonic propagator writes

$$G(\zeta_i,\zeta_j) = -\frac{1}{2}\ln\left|\frac{\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](\zeta_i-\zeta_j|\tau)}{\partial_\zeta\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](0|\tau)}\right|^2 + \frac{\pi(\mathrm{Im}\,(\zeta_i-\zeta_j))^2}{4\mathrm{Im}\,\tau}\,, \tag{VI.29}$$

We can now start the tropicalization procedure, following the section V A. First one focuses on the case of the torus alone, then punctures will be included. One wants to find a KP decomposition for $\mathcal{F}$. As q is a local coordinate on the moduli space around the nodal curve at infinity, one would want to use it as in the section IV C. We saw in (IV.7) that to obtain a loop of finite size T one had to set $|q| = \exp(-2\pi T/\alpha')$. This defines a family of tori parametrized by their modulus $\tau_{\alpha'}$:

$$\mathrm{Re}\ \tau_{\alpha'} = \tau_1 \in [-1/2;1/2[\,, \qquad \mathrm{Im}\ \tau_{\alpha'} = T/\alpha' \in [0;+\infty[\,. \tag{VI.30}$$

The trouble with the previous scaling is that for $\mathrm{Im}\,\tau_{\alpha'} < 1$, the real part can not be unrestricted in $\mathcal{F}$; there is a region where q is of order $O(1)$. Hence, to build the KP decomposition, we follow [14] and introduce by hand a parameter $L > 1$ to split the fundamental domain into an upper part, the non-analytic cell $\mathcal{F}^+(L)$ and a lower part, the analytic cell $\mathcal{F}^-(L)$[12] defined by

[12] Respectively called $\mathcal{F}_L$ and $\mathcal{R}_L$ in [14].

$\mathcal{F}^+(L) = \{\tau \in \mathcal{F}, \mathrm{Im}\,\tau > L\}$ and $\mathcal{F}^-(L) = \{\tau \in \mathcal{F}, \mathrm{Im}\,\tau \leq L\}$. Hence, the KP decomposition writes

$$\mathcal{F} = \mathcal{F}^+(L) \sqcup \mathcal{F}^-(L)\,. \tag{VI.31}$$

For any $T \geq \alpha' L$ we may now define the following families of complex tori in $\mathcal{F}^+(L)$

$$\mathrm{Re}\ \tau_{\alpha'} = \tau_1 \in [-1/2; 1/2[\,, \qquad \mathrm{Im}\ \tau_{\alpha'} = T/\alpha' \in [L; +\infty[\,. \tag{VI.32}$$

This accomplishes the step 1 of section II, in the sense that only massless states can propagate on long distances of order T/α' and to the contrary, nothing prevent massive states to propagate in the tori of the region $\mathcal{F}^-(L)$. After performing the step 2 they will give contributions located at the point of $\mathcal{M}_{1,4}^{\mathrm{trop}}$ where the loop is contracted to a point: counter-terms and higher order operators.

To achieve a full KP decomposition, one still has to deal with the positions of the punctures. Firstly, the splitting (VI.31) induces a similar decomposition of $\mathcal{M}_{1,4}$ into two domains depending on L, defined by the position of τ in $\mathcal{F}$

$$\mathcal{M}_{1,4} = \mathcal{M}_{1,4}^+(L) \sqcup \mathcal{M}_{1,4}^-(L) \tag{VI.33}$$

Then in $\mathcal{M}_{1,4}^-(L)$, there is nothing to do since the positions of the punctures can be integrated out completely. In $\mathcal{M}_{1,4}^+(L)$ however, it is well known since [1] (see also [103]) that to obtain a sensible result in the $\alpha' \to 0$ limit one should split the integration domain spanned the punctures into three regions, one for each inequivalent ordering of the graph. Hence $\mathcal{M}_{1,4}^+(L)$ is split further into three disjoint domains depending on τ, labeled by the three inequivalent permutations under reversal symmetry $\sigma \in \mathfrak{S}_3/\mathbb{Z}_2 = \{(123), (231), (312)\}$ defined by

$$\mathcal{D}_{(ijk)} := \mathcal{F}(L)^+ \times \{\zeta_i, \zeta_j, \zeta_k \,|\, 0 < \mathrm{Im}\,\zeta_i < \mathrm{Im}\,\zeta_j < \mathrm{Im}\,\zeta_k < \mathrm{Im}\,\tau\}\,. \tag{VI.34}$$

In all, we have the explicit KP decomposition

$$\mathcal{M}_{1,4} = \left(\bigsqcup_{\sigma} \mathcal{D}_\sigma\right) \sqcup \mathcal{M}_{1,4}^-(L) \tag{VI.35}$$

where σ runs over the set $\{(123), (231), (312)\}$. There is no complementary set $\mathcal{D}_0$ because the integrand vanishes by supersymmetry on the regions of the moduli space where a tree splits off from the torus. Thus there is no need to refine the decomposition to take into account vertex operators colliding to one another.

To show the conjecture 2, we focus on the two regions separately and find a tropical form of the integrand. Accordingly, we follow [14] and define the two different parts of the amplitude, respectively upper and lower, or non-analytic and analytic:

$$\mathsf{A}^{(1,4)}_{\alpha',+}(L) = \sum_{i=(s,t),(t,u),(u,s)} \int_{\mathcal{D}_i} \mathrm{d}\mu_{\mathrm{bos}}\, \mathcal{F}_{1,4}\,, \qquad \mathsf{A}^{(1,4)}_{\alpha',-}(L) = \int_{\mathcal{M}^-_{1,4}(L)} \mathrm{d}\mu_{\mathrm{trop}}\, \mathcal{F}_{1,4}\,. \tag{VI.36}$$

They should of course sum up to the complete amplitude.

In $\mathcal{M}^+_{1,4}(L)$, we have already seen that for any $T \geq \alpha' L$ one could define the family of complex tori (VI.32). As for the punctures, in $\mathcal{D}_{(ijk)}$ we define the following family of points:

$$\zeta_{i\alpha'} = \mathrm{Re}\,\zeta_i + iX_i/\alpha'\,, \quad \mathrm{Re}\,\zeta_i \in [0;2\pi[\,, \quad 0 < X_i < X_j < X_k < X_4 = T\,. \tag{VI.37}$$

This scaling is not different as the one introduced in (IV.15) and used for trees in (VI.12), provided that one remembers that ζ does belong to a Jacobian variety (the complex torus). Thus is linked to a point z on the surface by the Abel-Jacobi map. Considering that the the latter is reduced in the tropical limit to a logarithmic map (IV.14), we realize that $z = \exp(2i\pi\zeta)$ does satisfy (IV.15). We will now recall a classical computation to see how the prime form at genus one gives rise to a tropical prime form, which explains why the expression for the tropical Green function given in (V.11) is correct. The explicit expression given in (VI.29) has the following q-expansion:

$$G(\zeta_i - \zeta_j) = \frac{\pi(\mathrm{Im}\,(\zeta_i-\zeta_j))^2}{\mathrm{Im}\,\tau} - \frac{1}{2}\ln\left|\frac{\sin(\pi(\zeta_i-\zeta_j))}{\pi}\right|^2 - 2\sum_{m\geq 1}\left(\frac{q^m}{1-q^m}\frac{\sin^2(m\pi(\zeta_i-\zeta_j))}{m} + h.c.\right)\,, \tag{VI.38}$$

which, in terms of $\tau_{\alpha'}$, $\zeta_{i\alpha'}$ and $\zeta_{j\alpha'}$ becomes

$$\alpha' G(\zeta_{i\alpha'},\zeta_{j\alpha'}) = \frac{\pi}{T}(X_i-X_j)^2 - \frac{\alpha'}{2}\ln\left|e^{-\pi(X_i-X_j)/\alpha'}e^{i\pi\mathrm{Re}\,(\zeta_{ij})} - e^{\pi(X_i-X_j)/\alpha'}e^{-i\pi\mathrm{Re}\,(\zeta_{ij})}\right|^2 + O(\alpha') \tag{VI.39}$$

up to $O(q)$ terms and where ζ_{ij} stands for $\zeta_i - \zeta_j$. At leading order in α', the logarithm is equal to the absolute value of $X_i - X_j$ and one gets

$$\lim_{\alpha'\to 0}(\alpha' G(\zeta_{i\alpha'},\zeta_{j\alpha'})) = G^{\mathrm{trop}}(X_i - X_j) = \pi\left(-|X_i-X_j| + \frac{(X_i-X_j)^2}{T}\right) \tag{VI.40}$$

which is the well known worldline propagator on the circle derived in [104] though with a different normalization. By plugging that result in $\mathcal{F}_{1,4}$ one obtains

$$\mathcal{F}_{1,4} \to F_{1,4} = (2\pi)^8 t_8 t_8 R^4 \exp\left(-\sum k_i.k_j G^{\mathrm{trop}}(X_i-X_j)\right) + O(\alpha')\,, \tag{VI.41}$$

where nothing depends anymore on the phases $\mathrm{Re}\,\zeta_i$ or $\mathrm{Re}\,\tau$. We can integrate them out and the measure (VI.28) becomes

$$\mathrm{d}\mu_{\mathrm{bos}} \to \mathrm{d}\mu_{\mathrm{trop}} = \alpha'\frac{\mathrm{d}T}{T^5}\prod_{i=1}^{3}\mathrm{d}X_i \tag{VI.42}$$

over the integration domains

$$\mathcal{D}_{(ijk)} \to D_{(ijk)} := \{T \in [\alpha' L, +\infty\,[\,\} \times \{X_i, X_j, X_k \in [0; T[\ |\ 0 < X_i < X_j < X_k < T\}\,. \quad \text{(VI.43)}$$

For instance in the ordering 1234, the exponential factor writes explicitly $\mathcal{Q}_{1,4} = X_1(X_3 - X_2)s + (X_2 - X_1)(X_4 - X_3)t$ which can be recognized to be the second Symanzik polynomial of this graph. The first Symanzik polynomial is simply T.

Collecting all these results, one ends up with the following integral

$$\begin{aligned}\mathsf{A}^{(1,4)}_{\alpha',+}(L) \to \mathsf{A}^{(1,4)}_{+}(L) &= \sum_{\sigma} \int_{D_\sigma} \mathrm{d}\mu_{\mathrm{trop}}\, F_{1,4} \qquad \text{(VI.44)}\\ &= \alpha'(2\pi)^8 t_8 t_8 R^4 \Big(\int_{\alpha' L}^{\infty} \frac{\mathrm{d}T}{T^2} \int_0^T \frac{\mathrm{d}X_3}{T} \int_0^{X_3} \frac{\mathrm{d}X_2}{T} \int_0^{X_2} \frac{\mathrm{d}X_1}{T} e^{(-\sum k_i.k_j G^{\mathrm{trop}}(X_i - X_j))}\\ &\quad + 2 \text{ other orderings} \Big)\,,\end{aligned}$$

up to subleading corrections. This is nothing but the classical result of [1]; the splitting between the three domains $\mathcal{D}_\sigma$ produced the field theory four-graviton amplitude in a Schwinger proper time form with loop proper time T, written as a sum over the three inequivalent kinematical channels corresponding to the orderings. Now one could drop the restriction $T > \alpha' L$ and use dimensional regularization. However, to make the underlying tropical nature of the limit manifest, one should keep the hard UV cut-off $\alpha' L$. Then in 10 dimensions, this integral has a power behaved UV divergence given by

$$\mathsf{A}^{(1,4)}_{\alpha',+}\Big|_{leading\ div} = \alpha'(2\pi)^8 t_8 t_8 R^4 \left(\frac{1}{\alpha' L}\right) \qquad \text{(VI.45)}$$

As already observed in [14], the full amplitude $\mathsf{A}^{(1,4)}_{\alpha'}$ does not depend on L, thus any non vanishing term in $\mathsf{A}^{(1,4)}_{\alpha',+}$ that depends on L in the tropical limit should be canceled by the including contributions from the analytic cell. In particular, the divergence (VI.45) should be canceled by a counter-term coming from $\mathsf{A}^{(1,4)}_{\alpha',-}$.

The functions in the exponential factor are continuous on $\mathcal{M}^{-}_{1,4}(L)$, which is a compact space, thus according to the discussion in section V B 1 one may interchange the $\alpha' \to 0$ limit and the integration which amounts to setting the exponential factor to one. The entire integration over the ζ_i's is now trivial and one is left with an integral that can be computed straight away:

$$\mathsf{A}^{(1,4)}_{\alpha',-}(L) \to \mathsf{A}^{(1,4)}_{-}(L) = (2\pi)^8 t_8 t_8 R^4 \int_{\mathcal{F}_L} \frac{\mathrm{d}^2\tau}{\tau_2^2} + O(\alpha') = (2\pi)^8 t_8 t_8 R^4 \left(\frac{\pi}{3} - \frac{1}{L}\right) + O(\alpha')\,. \quad \text{(VI.46)}$$

This integral provides two physically distinct contributions; $\pi/3$ and $1/L$. The first is the so called analytic part of the amplitude. After going from the string frame to the Einstein frame, it is solely expressed in terms of gravitational quantities and is the leading order contribution of higher order operators in the effective action of supergravity. The second is the counter-term required to cancel the leading UV divergence (VI.45). From the tropical point of view, this integral may be thought of as a being localized at the singular point $T = 0$ of the tropical moduli space which corresponds to a graph with a vertex of weight one.

We can now sum up (VI.44) and (VI.46) to obtain the field theory amplitude written a as integral over the full tropical moduli space $\mathcal{M}_{1,4}^{\text{trop}}$. It is regularized by the inclusion of a counter-term the point $T = 0$ of the moduli space. Pictorially, the figure 18 summarizes this discussion.

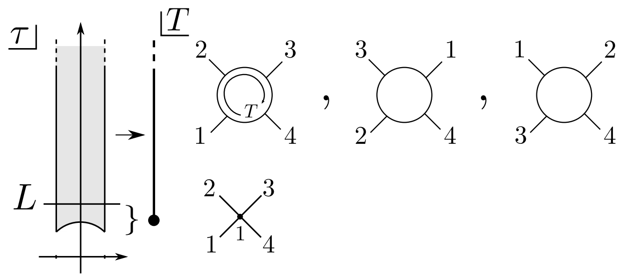


FIG. 18. Summary of the tropicalization of the four-graviton genus one amplitude in type II string.

In the general case, $\mathcal{W}_{1,n}$ acquires a possibly complicated structure and one often has to perform a Fourier-Jacobi expansion of $\mathcal{W}_{1,n} \exp(\mathcal{Q}_{1,n})$ in terms of q or $\sqrt{q}$ (see [7–9] and more recently for instance [97, 105] for heterotic string computations). Although these term would seem subleading – as they are exponentially suppressed when $\tau_2 \to \infty$ – the worldsheet realization of generic models with non maximal supersymmetry is based on altering the GSO projection which eventually results in the appearance of "poles" in $1/q$ and $1/\sqrt{q}$. In all consistent models these poles are automatically either compensated by terms of the Fourier-Jacobi expansion or killed by real part integration (step 2) of τ_1 via $\int_{-1/2}^{1/2} \mathrm{d}\tau_1 q^n \bar{q}^m = 0$ if $n \neq m$. In the bosonic string they are not, and this makes the theory inconsistent at loop level.

Let us explicit the general form of the KP decomposition for n point amplitudes from the detailed construction of [6–10]. There are now $(n-1)!/2$ domains $\mathcal{D}_\sigma$ for $\sigma \in \mathfrak{S}_{n-1}/\mathbb{Z}_2$ defined exactly as in (VI.34) that generate 1PI tropical graphs with orderings σ. In this previous analysis we did not have to deal with regions in the moduli space where points collide to one another because supersymmetry told us that no such regions may contribute in this four point example. However, in general one has to include them for both physical reasons (because we know that there are contact terms in generic amplitudes) and mathematical reasons (because the tropical moduli space do naturally include regions corresponding to separating degenerations in $\overline{\mathcal{M}_{1,n}}$).

Hence we refine the previous definition of the KP cells $\mathcal{D}_\sigma$ and define new cells $\hat{\mathcal{D}}_\sigma$ and $\hat{\mathcal{M}}^-(L)$ where the open discs $|\zeta_i - \zeta_j| < e^{-\ell\alpha'}$ have been cut out[13]. Then, the complementary set of the union of the previous domains in $\mathcal{M}^+(L)$ is made of domains of the form $\hat{\mathcal{D}}_\sigma$ where $\sigma \in \mathfrak{S}_{p-1}/\mathbb{Z}_2$ indicates the ordering of p points on the future loop while $n-p$ points are grouped into one or more discs of radius ℓ centered around one or more of the first p points.

To finish the description of the KP decomposition, one has to deal with these clusters of points. Locally, such a cluster of m points on a disc of radius ℓ looks like a sphere. Thus as in the tree level analysis, $\mathcal{M}_{1,n}$ is split it into $(2m-3)!!$ domains corresponding to the $(2m-3)!!$ combinatorially distinct trees. Note the shift $m \to m+1$ compared to the tree level case due to the fact that such trees with m external legs have one additional leg attached to the loop. At this point, one could basically conclude the by invoking the "Bern-Kosower" rules [6–10] which would yield the desired tropical form of the one-loop amplitude. The previous analysis is more a formal illustration of the tropical features of the limit rather than an explicit method for writing down one-loop field theory amplitudes. Let us then be brief, and consider for simplicity a cluster of two points, where ζ_j is treated like before (VI.37) and ζ_i collides to ζ_j according to

$$\zeta_{i\alpha'} = \zeta_j + e^{i\theta} e^{-X/\alpha'}, \quad \theta \in [0; 2\pi[, \quad X \in [\alpha'\ell, +\infty[\tag{VI.47}$$

where ζ_j is fixed, X is the future tropical length of the tree connecting legs i and j to the loop as in the tree level analysis and ℓ is an IR cut-off. In this simple example, there is no outer region $\mathcal{D}_0$ as in and the KP decomposition is built. As concerns the tropical form of the integrand and the proposition 3 one has to look at $\mathcal{F}_{1,n} = \mathcal{W}_{1,n} e^{\mathcal{Q}_{1,n}}$. For simplicity, we work in a representation of $\mathcal{W}_{1,n}$ where all double derivatives of the propagator have been integrated out by parts. Using the general short distance behavior of the propagator on a generic Riemann surface

$$G(z-w) = -1/2 \ln|z-w|^2 + O((z-w)^3)\,, \tag{VI.48}$$

one sees that $\mathcal{Q}_{1,n}$ gives a term $-Xk_i.k_j$ while any term of the form $G(\zeta_k - \zeta_i)$ is turned into a $G(\zeta_k - \zeta_j)$ at leading order in α':

$$\sum_{k<l} k_k.k_l G(kl) = -Xk_i.k_j + \sum_{k\neq i,j} (k_i + k_j).k_k G(jk) + \sum_{\substack{k<l \\ k,l\neq i,j}} k_k.k_l G(kl) + O(\alpha')\,. \tag{VI.49}$$

The factor $e^{-X\,k_i.k_j}$ provides a contact term via a pole in the amplitude if and only if $\mathcal{W}$ contains a factor of the form $|\partial G(ij)|^2 \sim e^{2X/\alpha'}$ exactly as in the tree level analysis. Then in $\mathcal{W}$ any ζ_i-dependent term is replaced by a ζ_j at the leading order in $O(\alpha')$.

[13] Note that ℓ have to be small enough compared to L so that $\hat{\mathcal{M}}^-(L)$ is non-empty. Typically $\ell \ll \sqrt{L/n\pi}$.

A similar tree analysis can be performed in the region $\mathcal{M}^{-}(L)$ where we have to include the contributions of poles. To conclude, we have shown that the existing results about the low energy limit of one-loop string theory amplitude show a complete correspondence between the string theory integration over $\mathcal{M}_{1,n}$ and its field theory point-like limit which can be expressed as an integral over the tropical moduli space $\mathcal{M}_{1,n}^{\text{trop}}$.

C. Two-Loop

Zero to four point two-loop amplitudes in type II and heterotic string have been worked out completely in [41–43, 45–47, 106] together with an N-point prescription. The four-graviton amplitude have also been derived using the pure spinor formalism [107] and shown in [108] to be equivalent to the RNS computation.

However no point-like limit have been explicitly extracted from these result. In [48], the four-graviton two-loop amplitude in maximal supergravity in field theory of [109] was written in a worldline form resembling the string theory integral. In this section, our goal is to prove rigorously that the tropical limit of the string theory integrand does match this result by making use of the tropical machinery that we have developed.

Let us recall some facts about genus two Riemann surfaces. At genus two (and three), there is no Schottky problem, thus all Jacobian varieties correspond to a Riemann surface; the moduli space of genus two Riemann surfaces is in one to one correspondence with the moduli space of Jacobian varieties. We have already seen in section III that Jacobian varieties are defined by $g \times g$ matrices, elements of the Siegel upper half-plane $\mathcal{H}_g$. We obtain a fundamental domain $\mathcal{F}_2$ by modding out $\mathcal{H}_2$ by the action of the modular group $Sp(4,\mathbb{Z})$. This space has a complicated structure and can be defined by a certain number of inequalities similar to the one defining $\mathcal{F}$ at genus one, see for instance [110]. We choose a canonical homology basis (a_I, b_J) as in figure 9 with normalized holomorphic one forms (IV.1). The period matrix Ω is parametrized by three complex moduli τ_1, τ_2 and τ_3:

$$\Omega := \begin{pmatrix} \tau_1 + \tau_3 & -\tau_3 \\ -\tau_3 & \tau_2 + \tau_3 \end{pmatrix} . \tag{VI.50}$$

The two-loop amplitude in 10 dimensions then writes [44, 108, 111, 112]

$$\mathsf{A}_{\alpha'}^{(2,4)}(\epsilon_i, k_i) = \frac{t_8 t_8 R^4}{2^{12}\pi^4} \int_{\mathcal{F}_2} \frac{|\prod_{I\leq J} \mathrm{d}\Omega_{IJ}|^2}{(\det \operatorname{Im}\Omega)^5} \int_{\Sigma^4} |\mathcal{Y}_S|^2 \exp\Big(-\alpha' \sum_{i<j} k_i \cdot k_j \, G(z_i, z_j) \Big), \tag{VI.51}$$

where $\int_{\Sigma^4}$ denotes integration over the surface Σ of the position of the four punctures. The quantity $\mathcal{Y}_S$ arises from several contributions in the RNS computation and from fermionic zero modes in the pure spinor formalism [107, 108, 113]. It writes

$$3\mathcal{Y}_S = (k_1 - k_2) \cdot (k_3 - k_4)\, \Delta(z_1, z_2)\Delta(z_3, z_4) + (13)(24) + (14)(23)\,, \tag{VI.52}$$

with

$$\Delta(z, w) = \omega_1(z)\omega_2(w) - \omega_1(w)\omega_2(z)\,. \tag{VI.53}$$

Thus, $|\mathcal{Y}_S|^2$ is a top form for Σ^4. Note that integrating $\omega\bar{\omega}$ instead of $dzd\bar{z}$ is very similar to the one-loop case where we had $d\zeta d\bar{\zeta}$. Indeed, the Abel-Jacobi map μ defined in (IV.4) defines the coordinates ζ of the Jacobian variety by $\mu(z - z_0)_I = \zeta_I = \int_{z_0}^{z} \omega_I$, thus $d\zeta_I = \omega_I$. Hence, in the genus two amplitude we can identify a measure and an integrand as follows

$$\mathrm{d}\mu_{\mathrm{bos}} = \int_{\mathcal{F}_2} \frac{|\prod_{I\leq J} \mathrm{d}\Omega_{IJ}|^2}{(\det \mathrm{Im}\, \Omega)^5} \int_{\Sigma^4} |\mathcal{Y}_S|^2\,, \tag{VI.54}$$

$$\mathcal{F}_{2,4} = t_8 t_8 R^4 \exp\Big(- \alpha' \sum_{i<j} k_i \cdot k_j\, G(z_i, z_j)\Big), \tag{VI.55}$$

where the numerator factor $\mathcal{W}_{2,4}$ is again trivial.

We want to use our knowledge of the mechanisms of the tropical limit to be able to compute the point-like limit of these expressions the non-analytic KP cells. This region is of particular interest as it gives rise to the (unrenormalized) field theory result at two loops. There should be two types non-analytic KP cells giving rise to the two pure graphs of figure 5. However, we can discard right away 1PR diagrams, since there are no terms of the form $|\partial G|^2$ at leading order in α' in the integrand (VI.55) necessary for massless poles to appear. Therefore the only graph Γ that one can obtain in the non analytic region is the "sunset graph" a) of the figure 5.

The KP cell where we want to study the amplitude is precisely the one of the figure 12 where the period matrices of the degenerating curves are written in terms of the local coordinates t_1, t_2 and t_3, we recall it here:

$$\Omega^{(2)}_{\alpha'} = \frac{1}{2i\pi} \begin{pmatrix} -\ln(t_1 t_3) & \ln(t_3) \\ \ln(t_3) & -\ln(t_2 t_3) \end{pmatrix} + O(\alpha', t_i)\,. \tag{VI.56}$$

Thus, keeping the parameters $T_i := -\alpha'/(2\pi) \ln |t_i|$ fixed, the curves are collapsing on the tropical graph of figure 4 with inner lengths T_i and period matrix $K^{(2)} = \left(\begin{smallmatrix} T_1+T_3 & -T_3 \\ -T_3 & T_2+T_3 \end{smallmatrix}\right)$ defined in (III.9). Let us remark that it could also be possible to use the parameters $q_i = \exp(2i\pi\tau_i)$ defined from the period matrix representation (VI.50) as local parameters on $\mathcal{M}_2$ around the singularity corresponding to the graph of figure 4. Granted that we consider a family of Jacobians such that

$$\operatorname{Im}\tau_i = -T_i/\alpha' \,. \tag{VI.57}$$

for $i = 1, 2, 3$, at the first order in the q_i- or Fourier-Jacobi expansion one has $q_i = t_i + O(q_i^2, q_i q_j)$ (see [127, eq 4.6] for an explicit relation between the Schottky representation and the q_i parameters in the case of the genus two open string worldsheet).

It should be emphasized that this relation holds only here in genus two (and would possibly hold in genus three) because there is no Schottky problem. In higher genus, one should proceed as explained before[14].

The question of the exact boundaries of this KP cell is more intricate, let us try to constrain it. Reasoning backwards, one wants to be sure to exclude regions of the tropical moduli space where a loop degenerates to zero size, to avoid the regulation of any divergence by massive stringy modes (primary and sub divergences). This can be translated in terms of a UV cut-off $\alpha' L$ where $L > 0$ by the inequalities $T_1 + T_2 \geq \alpha' L$, $T_2 + T_3 \geq \alpha' L$ and $T_3 + T_1 \geq \alpha' L$. Therefore, it seems that the KP cell might be defined by translating these inequalities to the imaginary parts of the parameters τ_1, τ_2 and τ_3 using the scaling (IV.9) according to $\operatorname{Im}\tau_1 + \operatorname{Im}\tau_2 \geq L + O(\alpha')$, $\operatorname{Im}\tau_2 + \operatorname{Im}\tau_3 \geq L + O(\alpha')$ and $\operatorname{Im}\tau_3 + \operatorname{Im}\tau_1 \geq L + O(\alpha')$. it would be interesting to check precisely this point with explicit computations.

Let us now write the tropical limit of $\mathcal{Y}_S$. The tropical limit of the holomorphic one-forms (III.5) gives[15]

$$\Delta(z_i, z_j) \to \Delta^{\text{trop}}(ij) = \omega_1^{\text{trop}}(i)\omega_2^{\text{trop}}(j) - \omega_1^{\text{trop}}(j)\omega_2^{\text{trop}}(i) \tag{VI.58}$$

This tropical version of Δ can only take values in $\{-1, 0, 1\}$, depending on the positions of the legs i and j. More precisely, one has

$$\Delta^{\text{trop}}(ij) = \begin{cases} 0 & \text{if } (i,j) \in B_1 \text{ or } (i,j) \in B_2 \\ 1 & \text{if } i \in B_1 \text{ and } j \in B_2 \\ -1 & \text{if } i \in B_2 \text{ and } j \in B_1 \end{cases} \tag{VI.59}$$

Then the tropical form of $\mathcal{Y}_S$ is immediately obtained:

$$3\mathcal{Y}_S \to 3Y_S = (k_1 - k_2)\cdot(k_3 - k_4)\,\Delta^{\text{trop}}(12)\Delta^{\text{trop}}(34) + (13)(24) + (14)(23)\,. \tag{VI.60}$$

[14] The author is grateful to Samuel Grushevsky for a discussion on this point.
[15] Δ^{trop} was called Δ^{FT} in [105]

It is a function of the positions of the four vertex operators on the graph. Using the form of Δ^{trop} that we just derived, we first see that Y_s vanishes for any configuration where three or four points lie on the same edge of the graph. In all other cases, it is given by

$$Y_S = \left\{\begin{matrix} -s & \text{if } 1,2 \text{ or } 3,4 \\ -t & \text{if } 1,4 \text{ or } 2,3 \\ -u & \text{if } 1,3 \text{ or } 2,4 \end{matrix}\right\} \text{belong to the same edge of the graph.} \tag{VI.61}$$

Let us mention that $\det K = T_1T_2 + T_2T_3 + T_3T_1$ does not depend on the positions of the punctures and is easily seen to be the usual form of the first Symanzik polynomial of the sunset graph. This achieves the study of the tropicalization of the integration measure.

The last thing to do is to compute the tropical representation of (VI.55), which was already done in (V.5) where, upon the assumption (V.11) that the worldline Green function of [51] is recovered in the tropical limit the result was stated for arbitrary genus. We then obtain

$$\mathsf{A}^{(2,4)}_{non-ana}(L) = \mathcal{N} t_8 t_8 R^4 \int_{T_i+T_j\geq\alpha' L}^{\infty} \frac{\mathrm{d}T_1\mathrm{d}T_2\mathrm{d}T_3}{(\det K)^5} \int_{\Gamma^4} Y_S \exp\Big(-\sum_{i<j} k_i\cdot k_j\, G^{\text{trop}}(Z_i,Z_j)\Big), \tag{VI.62}$$

where $\mathcal{N}$ is a global normalization factor, $\int_{\Gamma^4}$ represents the integration of the positions of the four punctures on the graph and $\int_{T_i+T_j\geq\alpha' L}$ represents a possible choice for the boundaries of the tropicalized KP cell described before. This object coincides with the one derived in [48, eq. 2.12] from the two-loop field theory computation of [109], thus it is the two-loop unrenormalized four-graviton amplitude.

To continue the procedure and remove the sub divergences and the primary divergence (when there are some), one should include in the analysis the regions of the moduli space giving rise to tropical graphs with weighted vertices. For instance the region where $t_1, t_3 \sim O(1)$ would yield the insertion of a counter-term in the loop of the homology cycle B_1 to cancel a potential sub-divergence, etc. Those computations would illustrate the systematics of renormalization in the tropicalization procedure in the presence of subdivergences and one should match the field theory computations of [48, 114].

The extension of the previous computation to more complicated amplitudes, for instance the genus two four-graviton scattering in Heterotic string of this method is a more challenging task. It should be based, as explained in [105] on a Fourier-Jacobi expansion of the string integrand in the parameters q_i. Extracting precisely the tropical form of the integrand as we have done here would be a very interesting thing to do.

VII. DISCUSSION

The material presented in this paper is in the vein of the active and fast recent developments undergone by the domain of string theory scattering amplitudes, stimulated by the introduction of new mathematical structures for instance in the automorphic form program [33–36, 115–117] or the analysis of the structure of the supermoduli space [77–81, 118, 119] or the systematic study of multiple zeta values in the α' expansion of tree level string amplitudes [120–122] and a lot more [123–126]. These interactions between physics and mathematics have yielded significant advances in both domains and we hope that the present work might raise similar curiosity. As a matter of fact, several restrictions have been made to the scope of this work during the text, that call for further developments.

First of all, let us recap what we wanted to prove in this paper and what we did not prove. We have proved that the string theory integral, once split up according to the KP decomposition provides in KP each cell an integral that has the exact same structure as the expected Feynman integral, over the graph with the correct topology (upon assuming that the conjecture 1 that we did not prove). By structure, we mean "denominator" or equivalently, first and second Symanzik polynomials. However, we did not prove that the result of the integration matches automatically the result obtained from Feynman rules. This is a totally separate issue. However, we gave some details about how one should proceed to extract the tropical numerator (in particular in genus two), which, by assuming that string theory provides the correct answers, can be considered as a first step towards a map between string theory and field theory numerators.

More generally, it should be noted that we did not study at all the open string. In the works [12, 13, 127], some detailed analysis were performed concerning the field theory limit of open string amplitudes at the one- and two-loop level, using the so called Schottky parametrization of Riemann surfaces. An extension to the open string of our discussion will inevitably be inspired by these works.

Then, we considered only superstring theory amplitudes that can be written as integrals over the ordinary moduli space of Riemann surfaces. However, as recently emphasized in the series of papers [77–80, 118, 119] it is not possible in general to consistently reduce the original integrals over the full supermoduli space of super Riemann surfaces to integrals over $\mathcal{M}_{g,n}$. In particular for genera $g \geq 5$, this is certainly not true anymore, thus our procedure will have to be upgraded to describe these cases. The authors of [127] provided a very interesting analysis in that direction as they studied the field theory limit of a super Schottky parametrization for super Riemann surfaces, still in the open string setting. As concerns the closed string, a super analog of tropical geometry has not been studied yet, though it would be an extremely interesting subject.

Another point to elucidate is the role of the Feynman $i\epsilon$ prescription in the tropicalization procedure. This issue has been recently analyzed in [80] where a solution to this issue was explained, but using a different approach compared to ours. Understanding how to it translate systematically in the tropical language would definitely be very interesting.

As concerns mathematics, a practical understanding of the theory of tropical theta functions and their links with ordinary ones is a crucial element for the definition of the tropical prime form. In order to be able to compute more general tropical amplitudes, the tropicalization procedure should also be extended at the next-to-leading order in multi-loop amplitudes in the Fourier-Jacobi expansion (q_i- or t_i-expansion) of the various quantities in the tropical limit. For instance the $\alpha' \to 0$ limit of the four-graviton genus two heterotic string amplitude of [44] approached in [105] has not be pushed further because of those issues.

Last, but not least, an elaborate mathematical construction has been provided in [63, 64], aiming at giving a rigorous functorial description of the links between $\mathcal{M}_{g,n}$ and $\mathcal{M}_{g,n}^{\mathrm{trop}}$, using an underlying Berkovich analytic geometry. It would be very interesting to recast our construction, and in particular the use of the KP decomposition of $\mathcal{M}_{g,n}$, following these concepts.

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BCJ duality and double copy in the closed string sector

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Abstract: This paper is focused on the loop-level understanding of the Bern-Carrasco-Johansson double copy procedure that relates the integrands of gauge theory and gravity scattering amplitudes. At four points, the first non-trivial example of that construction is one-loop amplitudes in $\mathcal{N}=2$ super-Yang-Mills theory and the symmetric realization of $\mathcal{N}=4$ matter-coupled supergravity. Our approach is to use both field and string theory in parallel to analyze these amplitudes. The closed string provides a natural framework to analyze the BCJ construction, in which the left- and right-moving sectors separately create the color and kinematics at the integrand level. At tree level, in a five-point example, we show that the Mafra-Schlotterer-Stieberger procedure gives a new direct proof of the color-kinematics double copy. We outline the extension of that argument to n points. At loop level, the field-theoretic BCJ construction of $\mathcal{N}=2$ SYM amplitudes introduces new terms, unexpected from the string theory perspective. We discuss to what extent we can relate them to the terms coming from the interactions between left- and right-movers in the string-theoretic gravity construction.

Keywords: Supersymmetric gauge theory, Scattering Amplitudes, Superstrings and Heterotic Strings

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Contents

1 Introduction

The Bern-Carrasco-Johansson color-kinematics duality [1, 2] implements in a powerful and elegant way the relationship between gauge theory and gravity scattering amplitudes from tree level to high loop orders [3–16]. At tree level, this duality is usually perceived in terms of the celebrated Kawai-Lewellen-Tye relations [17], but a first-principle understanding at loop level is still missing.[1]

In this paper, we search for possible string-theoretic ingredients to understand the color-kinematics double copy in one-loop four-point amplitudes. The traditional "KLT" approach, based on the factorization of closed string *amplitudes* into open string ones: "open $\times$ open $=$ closed" at the *integral* level, does not carry over to loop level. Instead, one has to look for relations at the *integrand* level. In this paper, adopting the approach of [18, 19], we shall use the fact that the tensor product between the left- and right-moving sectors of the closed string, i.e.

"left-moving $\times$ right-moving $=$ closed",

relates color and kinematics at the worldsheet integrand level. It is illustrated in table 1, where "Color CFT" and "Spacetime CFT" refer to the respective target-space chiral polarizations and momenta of the scattered states. A gauge theory is realized by the closed string when one of the chiral sectors of the external states is polarized in an internal color space. This is the basic mechanism of the heterosis which gave rise to the beautiful heterotic string construction [20]. A gravity theory is realized when both the left- and right-moving polarizations of the gravitons have their target space in Minkowski spacetime, as it can be done both in heterotic and type II string. In the paper, we shall not describe the gravity sector of the heterotic string, as it is always non-symmetric. Instead, we will focus on symmetric orbifolds of the type II string to obtain, in particular, symmetric realizations of half-maximal ($\mathcal{N} = 4$ in four dimensions) supergravity.

In section 3, we review how the closed-string approach works at tree level with the five-particle example discussed in [18, 19]. We adapt to the closed string the Mafra-Schlotterer-Stieberger procedure [21], originally used to derive "BCJ" numerators in the open string. The mechanism, by which the MSS chiral block representation, in the field theory limit, produces the BCJ numerators in the heterotic string, works exactly in the same way in gravity. However, instead of mixing color and kinematics, it mixes kinematics with kinematics and results in a form of the amplitude where the double copy squaring prescription is manifest. We outline a n-point proof of this observation.

Then we thoroughly study the double copy construction in four-point one-loop amplitudes. First, we note that the BCJ construction is trivial both in field theory and string theory when one of the four-point gauge-theory copy corresponds to $\mathcal{N}=4$ SYM. Then we come to our main subject of study, $\mathcal{N}=2$ gauge theory and symmetric realizations of $\mathcal{N}=4$ gravity amplitudes in four dimensions. We study these theories both in field theory and string theory and compare them in great detail. The real advantage of the closed string in this perspective is that we have already at hand a technology for building field theory

[1] With the exception of one-loop amplitudes in the self-dual sector of Yang-Mills theory [16].

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Left-moving CFT	Right-moving CFT	Low-energy limit	Closed string theory
Spacetime CFT	Color CFT	Gauge theory	Heterotic
Spacetime CFT	Spacetime CFT	Gravity theory	Type II, (Heterotic)

Table 1. Different string theories generating field theories in the low-energy limit.

amplitudes from general string theory models, with various level of supersymmetry and gauge groups.

In section 4, we provide a BCJ construction of half-maximal supergravity coupled to matter fields as a double copy of $\mathcal{N} = 2$ SYM. Then in section 5, we give the string-based integrands and verify that they integrate to the same gauge theory and gravity amplitudes. Finally, we compare the two calculations in section 6 by transforming the field-theoretic loop-momentum expressions to the same worldline form as the string-based integrands, and try to relate the BCJ construction to the string-theoretic one.

Both of them contain box diagrams, but the field-theoretic BCJ construction of gauge theory amplitudes has additional triangles, which integrate to zero and are invisible in the string-theoretic derivation. Interestingly, at the integrand level, the comparison between the BCJ and the string-based boxes is possible only up to a new total derivative term, which we interpret as the messenger of the BCJ representation information in the string-based integrand. However, we argue that, against expectations, this change of representation cannot be obtained by integrations by part, and we suggest that this might be linked to our choice of the BCJ representation. Therefore, it provides non-trivial physical information on the various choices of BCJ ansatzes.

The square of the BCJ triangles later contributes to the gravity amplitude. String theory also produces a new term on the gravity side, which is due to left-right contractions. We manage to relate it to triangles squared and parity-odd terms squared, which is possible up to the presence of "square-correcting-terms", whose appearance we argue to be inevitable and of the same dimensional nature as the string-theoretic left-right contractions.

We believe that our work constitutes a step towards a string-theoretic understanding of the double copy construction at loop level in theories with reduced supersymmetry, although some facts remain unclarified. For instance, it seems that simple integration-by-part identities are not enough to obtain some BCJ representations (e.g. ours) from string theory.

2 Review of the BCJ construction

In this section, we briefly review the BCJ duality and the double copy construction in field theory, as well as the current string-theoretic understanding of these issues (see also the recent review [22, section 13]).

To begin with, consider a n-point L-loop color-dressed amplitude in gauge theory as a sum of Feynman diagrams. The color factors of graphs with quartic gluon vertices, written in terms of the structure constants f^{abc}, can be immediately understood as sums of cubic color diagrams. Their kinematic decorations can also be adjusted, in a non-unique way,

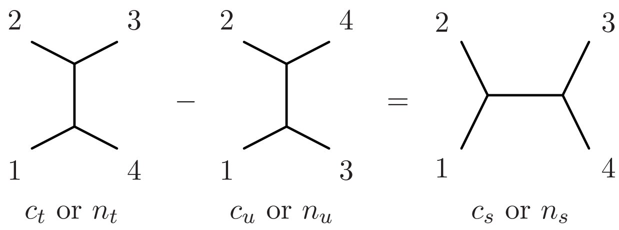

Figure 1. Basic Jacobi identity for the color factors.

so that their pole structure would correspond to that of trivalent diagrams. This can be achieved by multiplying and dividing terms by the denominators of missing propagators. Each four-point vertex can thus be interpreted as a s-, t- or u-channel tree, or a linear combination of those. By performing this ambiguous diagram-reabsorption procedure, one can represent the amplitude as a sum of cubic graphs only:

$$\mathcal{A}_n^L = i^L g^{n+2L-2} \sum_{\text{cubic graphs } \Gamma_i} \int \prod_{j=1}^{L} \frac{\mathrm{d}^d \ell_j}{(2\pi)^d} \frac{1}{S_i} \frac{c_i\, n_i(\ell)}{D_i(\ell)}\,, \tag{2.1}$$

where the denominators D_i, symmetry factors S_i and color factors c_i are understood in terms of the Feynman rules of the adjoint scalar ϕ^3-theory (without factors of i) and the numerators n_i generically lose their Feynman rules interpretation.

Note that the antisymmetry $f^{abc} = -f^{bac}$ and the Jacobi identity

$$f^{a_2 a_3 b} f^{b a_4 a_1} - f^{a_2 a_4 b} f^{b a_3 a_1} = f^{a_1 a_2 b} f^{b a_3 a_4}\,, \tag{2.2}$$

shown pictorially in figure 1 induces numerous algebraic relations among the color factors, such as the one depicted in figure 2.

We are now ready to introduce the main constraint of *the BCJ color-kinematics duality* [1, 2]: let the kinematic numerators n_i, defined so far very vaguely, satisfy the same algebraic identities as their corresponding color factors c_i:

$$\begin{aligned} c_i = -c_j \quad &\Leftrightarrow \quad n_i = -n_j\,, \\ c_i - c_j = c_k \quad &\Leftrightarrow \quad n_i - n_j = n_k\,. \end{aligned} \tag{2.3}$$

This reduces the freedom in the definition of $\{n_i\}$ substantially, but not entirely, to the so-called generalized gauge freedom. The numerators that obey the duality 2.3 are called the BCJ numerators. Note that even the basic Jacobi identity (2.2), obviously true for the four-point tree-level color factors, is much less trivial when written for the corresponding kinematic numerators.

Once imposed for gauge theory amplitudes, that duality results in *the BCJ double copy* construction for gravity amplitudes in the following form:[2]

$$\mathcal{M}_n^L = i^{L+1} \left(\frac{\kappa}{2}\right)^{n+2L-2} \sum_{\text{cubic graphs } \Gamma_i} \int \prod_{j=1}^{L} \frac{\mathrm{d}^d \ell_j}{(2\pi)^d} \frac{1}{S_i} \frac{n_i(\ell)\, \tilde{n}_i(\ell)}{D_i(\ell)}\,, \tag{2.4}$$

[2]In the rest of this paper, we omit trivial coupling constants by setting $g = 1$, $\kappa = 2$. At one loop, we can also rescale the numerators by a factor of $-i$ to completely eliminate the prefactors in (2.1) and (2.4).

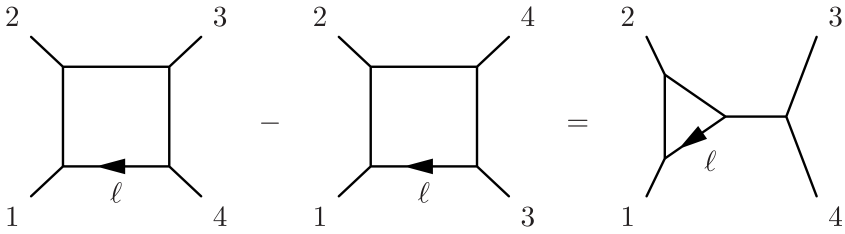

Figure 2. Sample Jacobi identity for one-loop numerators.

where only one of the numerator sets, $\{n_i\}$ or $\{\tilde{n}_i\}$, needs to obey the color-kinematics duality (2.3). In this way, gauge and gravity theories are related at the integrand level in loop momentum space. In this paper, we loosely refer to eqs. (2.1) and (2.4), related by the duality (2.3), as *the BCJ construction*.

A comment is due at loop level: the loop-momentum dependence of numerators $n_i(\ell)$ should be traced with care. For instance, in the kinematic Jacobi identity given in figure 2, one permutes the legs 3 and 4, but keeps the momentum ℓ fixed, because it is external to the permutation. Indeed, if one writes that identity for the respective color factors, the internal line ℓ will correspond to the color index outside of the basic Jacobi identity of figure 1. In general, the correct loop-level numerator identities correspond to those for the unsummed color factors in which the internal-line indices are left uncontracted.

Formulas (2.1) and (2.4) are a natural generalization of the original discovery at tree level [1]. The double copy for gravity (2.4) has been proven in [23] to hold to any loop order, if there exists a BCJ representation (2.1) for at least one of the gauge theory copies. Such representations were found in numerous calculations [2, 5–14, 24, 25] up to four loops in $\mathcal{N} = 4$ SYM [4]. A systematic way to find BCJ numerators is known for Yang-Mills theory at tree level [26], and in $\mathcal{N} = 4$ SYM at one loop [27]. Moreover, for a restricted class of amplitudes in the self-dual sectors of gauge theory and gravity, one can trace the Lagrangian origin of the infinite-dimensional kinematic Lie algebra [16, 28].

The string-theoretic understanding of the double copy at tree level dates back to the celebrated KLT relations [17] between tree-level amplitudes in open and closed string theory, later improved with the discovery of monodromy relations and the momentum kernel in [18, 19, 29–31]. In the field theory limit, these relations implement the fact that in amplitudes the degrees of freedom of a graviton can be split off into those of two gauge bosons. Recently, a new chiral block representation of the open-string integrands was introduced [21] to construct BCJ numerators at n points. All of this is applicable at tree level, whereas at loop level, the relationship between open and closed string *amplitudes* becomes obscure.

At the integrand level, five-point amplitudes were recently discussed in [32] in open and closed string. The authors of that work studied how the closed-string integrand is related to the square of the open-string integrand, and observed a detailed squaring behavior. They also discussed the appearance of left-right mixing terms in this context. These terms are central in our one-loop analysis, even though at the qualitative level, four-points amplitudes in $(\mathcal{N}{=}2) \times (\mathcal{N}{=}2)$ are more closely related to six-point ones in $(\mathcal{N}{=}4) \times (\mathcal{N}{=}4)$.

3 Review of tree level in string theory

In this section, we review RNS string amplitude calculations at tree level in order to perform explicitly a five-point heterotic and type II computation, as a warm-up exercise before going to the loop level. Type I and II string amplitudes are known at n points from the pure spinor formalism [33–36] and their field theory limits were extensively studied in [36, 37], as well as their α' expansion in [37–40]. As observed in [18, 19], the important point here is not to focus on the actual string theory amplitude, but rather to realize different field theory limits by plugging different CFT's in the left- and right-moving sectors of the string. In that context, an observation that we shall make is that the Mafra-Schlotterer-Stieberger open-string chiral block representation introduced in [21] to compute BCJ numerators can be used to construct directly gravity amplitudes and make the double copy property manifest. We perform this explicitly in the five-point case and briefly outline an n-point extension.

Let us start from the integral for the five-particle scattering amplitude:

$$\mathcal{A}_5^{\text{string}} = |z_{14}z_{45}z_{51}|^2 \int \mathrm{d}^2z_2\mathrm{d}^2z_3 \langle V_1(z_1)V_2(z_2)V_3(z_3)V_4(z_4)V_5(z_5)\rangle\,, \tag{3.1}$$

where $|z_{14}z_{45}z_{51}|^2$ is the classical $c\bar{c}$ ghost correlator, and we use the conformal gauge freedom to set $z_1 = 0, z_4 = 1, z_5 \to \infty$. The unintegrated vertex operators have a holomorphic and an anti-holomorphic part:

$$V(z) =: V^{(\mathrm{L})}(z)V^{(\mathrm{R})}(\bar{z})e^{ikX(z,\bar{z})} :, \tag{3.2}$$

where $V^{(\mathrm{L})}$ and $V^{(\mathrm{R})}$ are the chiral vertex operators for the left- and right-moving sectors.[3] The notation for superscripts (L) and (R) coincides with the one used in [19]. Now, depending on what CFT we plug in these two sectors, different theories in the low-energy limit can be realized, as summarized in table 1. The anti-holomorphic vertex operators for the color CFT are gauge currents

$$V^{(\mathrm{R})}(\bar{z}) = T^a J_a(\bar{z})\,, \tag{3.3}$$

where the T^a matrices are in the adjoint representation of the gauge group under consideration (for instance, $E_8 \times E_8$ or SO(32) in the heterotic string or more standard SU(N) groups, after proper gauge group breaking by compactification). The chiral vertex operators in the spacetime supersymmetric CFT have a superghost picture number, (-1) or (0), required to cancel the $(+2)$ background charge:

$$V^{(\mathrm{L})}_{(-1)}(z) = \varepsilon_\mu(k)\, e^{-\phi}\psi^\mu\,, \tag{3.4a}$$

$$V^{(\mathrm{L})}_{(0)}(z) = \sqrt{\frac{2}{\alpha'}}\varepsilon_\mu(k)\left(i\partial X^\mu + \frac{\alpha'}{2}(k\cdot\psi)\psi^\mu\right), \tag{3.4b}$$

where $\varepsilon_\mu(k)$ is the gluon polarization vector. Therefore, at tree level, exactly two vertex operators must be chosen in the (-1) picture.

[3] The chiral fields $X(z)$ and $X(\bar{z})$ are defined to contain half of the zero modes of the field $X(z,\bar{z}) = x_0 + X_L(z) + X_R(\bar{z}))$ so that $X(z) = x_0/2 + X_L(z)$ and $X(\bar{z}) = x_0/2 + X_R(\bar{z})$.

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The anti-holomorphic vertex operators are then obtained from the holomorphic ones by complex conjugation. The total vertex operators of gluons and gravitons are constructed as products of the chiral ones in accordance with table 1, and the polarization tensor of the graviton is defined by the symmetric traceless part of the product $\varepsilon_{\mu\nu}(k) = \varepsilon_\mu(k)\varepsilon_\nu(k)$.

The correlation function (3.1) can be also computed as a product of a holomorphic and an anti-holomorphic correlator thanks to the "canceled propagator argument". As explained in the classical reference [41, section 6.6], the argument is essentially an analytic continuation which makes sure that Wick contractions between holomorphic and anti-holomorphic operators

$$\langle \partial X(z,\bar z)\bar\partial X(w,\bar w)\rangle = -\alpha'\pi\delta^{(2)}(z-w)\,, \tag{3.5}$$

provide only vanishing contributions at tree level.[4]

Therefore, the chiral correlators can be dealt with separately. Our goal is to write them in the MSS chiral block representation [21], in which

$$\langle V_1^{(\mathrm{L})}V_2^{(\mathrm{L})}V_3^{(\mathrm{L})}V_4^{(\mathrm{L})}V_5^{(\mathrm{L})}\rangle = \left(\frac{a_1^{(\mathrm{L})}}{z_{12}z_{23}}+\frac{a_2^{(\mathrm{L})}}{z_{13}z_{23}}+\frac{a_3^{(\mathrm{L})}}{z_{12}z_{34}}+\frac{a_4^{(\mathrm{L})}}{z_{13}z_{24}}+\frac{a_5^{(\mathrm{L})}}{z_{23}z_{34}}+\frac{a_6^{(\mathrm{L})}}{z_{23}z_{24}}\right)\,, \tag{3.6a}$$

$$\langle V_1^{(\mathrm{R})}V_2^{(\mathrm{R})}V_3^{(\mathrm{R})}V_4^{(\mathrm{R})}V_5^{(\mathrm{R})}\rangle = \left(\frac{a_1^{(\mathrm{R})}}{\bar z_{12}\bar z_{23}}+\frac{a_2^{(\mathrm{R})}}{\bar z_{13}\bar z_{23}}+\frac{a_3^{(\mathrm{R})}}{\bar z_{12}\bar z_{34}}+\frac{a_4^{(\mathrm{R})}}{\bar z_{13}\bar z_{24}}+\frac{a_5^{(\mathrm{R})}}{\bar z_{23}\bar z_{34}}+\frac{a_6^{(\mathrm{R})}}{\bar z_{23}\bar z_{24}}\right)\,, \tag{3.6b}$$

where $a^{(\mathrm{L/R})}$ are independent of z_i and carry either color or kinematical information. Accordingly, they are constructed either from the structure constants f^{abc} of the gauge group or from momenta k_i and polarization vectors ε_i of the external states. The plane-wave correlator (known as the Koba-Nielsen factor) writes

$$\exp\left(-\sum_{i<j}k_i\cdot k_j\langle X(z_i,\bar z_i)X(z_j,\bar z_j)\rangle\right) = \prod_{i<j}|z_{ij}|^{\alpha' k_i\cdot k_j}\,, \tag{3.7}$$

where the bosonic correlator is normalized as follows:

$$\langle X^\mu(z,\bar z)X^\nu(w,\bar w)\rangle = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln(|z-w|^2)\,. \tag{3.8}$$

It was implicitly taken into account when writing eqs. (3.6), since we included all possible Wick contractions, including those of the form $\langle \partial X e^{ikX}\rangle$.

As we will see, taking the limit $\alpha'\to 0$ of eq. (3.1) will lead us to the BCJ construction for the field theory amplitudes. Note that if one derives $a^{(\mathrm{L/R})}$ in a completely covariant way, as is done in [21], one eventually obtains the BCJ numerators valid in any dimension. In this way, the whole BCJ construction can be regarded as a mere consequence of the worldsheet structure in the low-energy limit.

In the following, we review the case of a correlator of anti-holomorphic gauge currents, then we go to the supersymmetric kinematic sector.

[4] At one loop, a zero mode term modify the right hand side of eq. (3.5), see eq. (5.6). This brings non-vanishing contributions, whose analysis of the relationship with the BCJ construction is one of the aims of this paper.

3.1 Gauge current correlators

At level one, the current correlators of the Kac-Moody algebra of a given gauge group are built from the standard OPE's:

$$J^a(\bar{z})J^b(0) = \frac{\delta^{ab}}{\bar{z}^2} + if^{abc}\frac{J^c(\bar{z})}{\bar{z}} + \dots\,, \tag{3.9}$$

where the f^{abc}'s are the structure constants of the gauge group, defined by

$$[T^a, T^b] = if^{abc}T^c\,. \tag{3.10}$$

At four points, one can thus obtain the following correlator:

$$\langle J^{a_1}(\bar{z}_1)J^{a_2}(\bar{z}_2)J^{a_3}(\bar{z}_3)J^{a_4}(\bar{z}_4)\rangle = \frac{\delta^{a_1a_2}\delta^{a_3a_4}}{\bar{z}_{12}^2\bar{z}_{34}^2} - \frac{f^{a_1a_2b}f^{ba_3a_4}}{\bar{z}_{12}\bar{z}_{23}\bar{z}_{34}\bar{z}_{41}} + (2\leftrightarrow 3) + (2\leftrightarrow 4)\,, \tag{3.11}$$

where the conformal gauge is not fixed. In the low-energy limit of the heterotic string amplitude, the $\delta\delta$-terms in (3.11) produce the non-planar contribution of the gravity sector (singlet exchange), while the ff-terms result in the gluon exchange channel. In the following, we shall decouple these non-planar corrections by hand.

At five points, to obtain the correct MSS chiral blocks for the gauge current correlator, one only needs to repeatedly use the Jacobi identities (2.2). After fixing the conformal gauge by setting $z_1 = 0, z_4 = 1, z_5 \to \infty$, we get

$$\begin{aligned}
&\langle J^{a_1}(\bar{z}_1)J^{a_2}(\bar{z}_2)J^{a_3}(\bar{z}_3)J^{a_4}(\bar{z}_4)J^{a_5}(\bar{z}_5)\rangle \\
&\quad = \frac{f^{a_1a_2b}f^{ba_3c}f^{ca_4a_5}}{\bar{z}_{12}\bar{z}_{23}} - \frac{f^{a_1a_3b}f^{ba_2c}f^{ca_4a_5}}{\bar{z}_{13}\bar{z}_{23}} - \frac{f^{a_1a_2b}f^{ba_5c}f^{ca_3a_4}}{\bar{z}_{12}\bar{z}_{34}} \\
&\qquad - \frac{f^{a_1a_3b}f^{ba_5c}f^{ca_2a_4}}{\bar{z}_{13}\bar{z}_{24}} + \frac{f^{a_1a_5b}f^{ba_2c}f^{ca_4a_3}}{\bar{z}_{23}\bar{z}_{34}} - \frac{f^{a_1a_5b}f^{ba_3c}f^{ca_4a_2}}{\bar{z}_{23}\bar{z}_{24}} \\
&\qquad + \text{non-planar terms}\,,
\end{aligned} \tag{3.12}$$

Now we can immediately read off the following set of 6 color factors:

$$\begin{aligned}
a_1^{(\mathrm{R})} &= f^{a_1a_2b}f^{ba_3c}f^{ca_4a_5}\,, & a_2^{(\mathrm{R})} &= f^{a_1a_3b}f^{ba_2c}f^{ca_4a_5}\,, & a_3^{(\mathrm{R})} &= f^{a_1a_2b}f^{ba_5c}f^{ca_3a_4}\,, \\
a_4^{(\mathrm{R})} &= f^{a_1a_3b}f^{ba_5c}f^{ca_2a_4}\,, & a_5^{(\mathrm{R})} &= f^{a_1a_5b}f^{ba_2c}f^{ca_4a_3}\,, & a_6^{(\mathrm{R})} &= f^{a_1a_5b}f^{ba_3c}f^{ca_4a_2}\,.
\end{aligned} \tag{3.13}$$

It actually corresponds to the color decomposition into $(n-2)!$ terms uncovered in [42].

3.2 Kinematic CFT

Now let us compute the RNS 5-point left-moving correlator in the supersymmetric sector,

$$\langle V^{(\mathrm{L})}_{(-1)}(z_1)V^{(\mathrm{L})}_{(0)}(z_2)V^{(\mathrm{L})}_{(0)}(z_3)V^{(\mathrm{L})}_{(0)}(z_4)V^{(\mathrm{L})}_{(-1)}(z_5)\rangle\,, \tag{3.14}$$

where the chiral vertex operators for the kinematic CFT were defined in (3.4). In (3.14), we picked two vertex operators to carry ghost picture number (-1) in such a way that all double poles can be simply eliminated by a suitable gauge choice. The correlator (3.14) is computed using Wick's theorem along with the two-point function (3.8) and

$$\langle \psi^\mu(z)\psi^\nu(w)\rangle = \eta^{\mu\nu}/(z-w)\,, \tag{3.15a}$$

$$\langle \phi(z)\phi(w)\rangle = -\ln(z-w)\,. \tag{3.15b}$$

For a completely covariant calculation, we refer the reader to [21], whereas here for simplicity we restrict ourselves to the MHV amplitude $\mathcal{A}(1^+,2^-,3^-,4^+,5^+)$ with the following choice of reference momenta:

$$(q_1^{\mathrm{ref}},q_2^{\mathrm{ref}},q_3^{\mathrm{ref}},q_4^{\mathrm{ref}},q_5^{\mathrm{ref}}) = (k_2,k_1,k_1,k_2,k_2)\,. \tag{3.16}$$

In combination with the ghost picture number choice, this gauge choice eliminates a lot of terms and, in particular, all double poles. We end up with only ten terms of the form[5]

$$\frac{(\varepsilon_3\varepsilon_5)(\varepsilon_1k_3)(\varepsilon_2k_3)(\varepsilon_4k_1)}{8z_{13}z_{23}} - \frac{(\varepsilon_3\varepsilon_5)(\varepsilon_1k_3)(\varepsilon_2k_4)(\varepsilon_4k_3)}{8z_{13}z_{24}z_{34}} + \dots\,.$$

To reduce them to the six terms of the MSS chiral block representation, one could apply in this closed-string context the open string technology based on repeated worldsheet IBP's described in [35–37, 39]. However, the situation is greatly simplified here, since we have already eliminated all double poles. Thanks to that, we can proceed in a pedestrian way and only make use of partial fractions identities, such as

$$\frac{1}{z_{12}z_{24}z_{34}} = -\frac{z_{12}+z_{24}}{z_{12}z_{24}z_{34}}\,, \tag{3.17}$$

where we take into account that $z_{41}=1$. Our final result, similarly to the one in appendix D of [21], contains two vanishing and four non-vanishing coefficients. In the spinor-helicity formalism, they are

$$\begin{aligned}
&a_1^{(\mathrm{L})} = 0\,, && a_2^{(\mathrm{L})} = \frac{\langle 23\rangle^4\,[31]^2\,[54]}{\langle 24\rangle\langle 25\rangle\langle 12\rangle\,[21]}, && a_3^{(\mathrm{L})} = 0\,,\\
&a_4^{(\mathrm{L})} = \frac{\langle 23\rangle^3\,[31]\,[41]\,[54]}{\langle 12\rangle\langle 25\rangle\,[21]}\,, && a_5^{(\mathrm{L})} = \frac{\langle 23\rangle^3\,[51]^2\,[43]}{\langle 24\rangle\langle 12\rangle\,[21]}\,, && a_6^{(\mathrm{L})} = \frac{\langle 23\rangle^3\,[41]\,[43]\,[51]}{\langle 12\rangle\langle 25\rangle\,[21]}\,.
\end{aligned} \tag{3.18}$$

3.3 Low-energy limit

Before specializing to a particular theory (gauge theory or gravity), let us review the general low-energy limit mechanism at tree level. In the open string, very efficient procedures have been developed for extracting the low-energy limit of n-points amplitudes in a systematic way [36, 37]. The essential point, common to both open and closed string procedures, consists in the observation that a pole[6] in the channel $s_{ij}s_{kl}$ comes from integrating over the region of the moduli space where z_i and z_k collide to z_j and z_l, respectively, provided that the integrand contains a pole in the variables $z_{ij}z_{kl}$. In these regions, the closed string worldsheet looks like spheres connected by very long tubes (see figure 3), and we simply have to integrate out the angular coordinates along the tubes to obtain graph edges. This

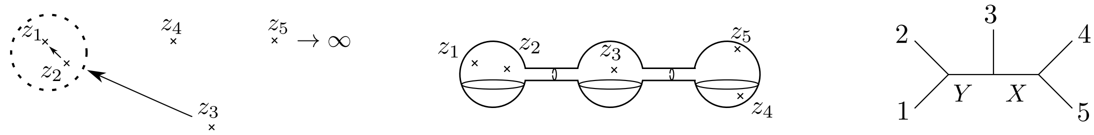

Figure 3. Five-point low-energy limit: the complex plane, the tubed worldsheet and the tropical (worldline) graph. X and Y are the lengths of the edges, or proper-time variables.

is the basic mechanism of the tropical limiting procedure reviewed in [43] (see section VI.A for four-tachyon and four-graviton examples).

A slight subtlety to take into account is that the s_{45}-channel pole, for instance, is not due to the pole $1/z_{45}$, as both z_4 and z_5 are fixed and $z_5 = \infty$ is already absent from the expressions. Rather, it is created when both $z_2, z_3 \to z_1$, i.e. it appears as a s_{123} pole. Moreover, the details of the double limit matter as well: suppose we want the kinematic pole $1/(s_{12}s_{123})$, it is generated by the *successive* limit $z_2 \to z_1$ and then z_3 to the cluster formed by z_1 and z_2. In other words, it arises from regions of the moduli space where $|z_{12}| \ll |z_{23}| \sim |z_{13}| \ll 1$. We can describe the geometry of the worldsheet in this limit by going to the following tropical variables:

$$z_{13} = -e^{i\theta} e^{-X/\alpha'}, \qquad z_{12} = z_{13} e^{i\phi} e^{-Y/\alpha'} = -e^{i(\theta+\phi)} e^{-(X+Y)/\alpha'}, \tag{3.19}$$

where X and Y are the lengths (or the Schwinger proper time variables) of the edges of the graph as pictured in figure 3. The phases θ and ϕ are the cylindrical coordinates along the tubes that need to be integrated out in order to recover a purely one-dimensional variety corresponding to a Feynman graph.

Accordingly, the integration measure produces a Jacobian:

$$\mathrm{d}^2 z_2 \to -\frac{1}{\alpha'} \mathrm{d}X \mathrm{d}\theta \, e^{-2X/\alpha'}, \quad \mathrm{d}^2 z_3 \to -\frac{1}{\alpha'} \mathrm{d}Y \mathrm{d}\phi \, e^{-2(X+Y)/\alpha'}. \tag{3.20}$$

Then one can check that the exponential factor transforms as follows:

$$e^{\alpha'(k_1 \cdot k_2) \ln |z_{12}| + \alpha'(k_1 \cdot k_3) \ln |z_{13}| + \alpha'(k_2 \cdot k_3) \ln |z_{23}|} = e^{-X s_{123}/2 - Y s_{12}/2} + O(\alpha'), \tag{3.21}$$

where the phase dependence in θ and ψ is trivial, granted that Y and X are greater than some UV cutoff of order α'. At this point, we have integrands of the form

$$\frac{1}{\alpha'^2} \mathrm{d}X \mathrm{d}Y \mathrm{d}\theta \mathrm{d}\phi \frac{e^{-2X/\alpha'} e^{-2(X+Y)/\alpha'}}{z_{ij} z_{kl} \bar{z}_{mn} \bar{z}_{pq}} e^{-X s_{123}/2 - Y s_{12}/2}. \tag{3.22}$$

To make them integrate to the expected double pole $1/s_{12}s_{123}$, we need the Jacobian to be compensated by the z_{ij} poles in such a way that we can integrate out the phases θ and ϕ.

[5] See the full expression in appendix B.

[6] We define Mandelstam kinematic invariants s_{ij} in the $(+,-,-,-)$ signature by $s_{ij} = (k_i + k_j)^2$.

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This is carried out by writing the amplitude (3.1) in the MSS chiral block representation:

$$\begin{aligned}\mathcal{A}_5^{\text{string}}(1^+,2^-,3^-,4^+,5^+;\alpha') = (2\alpha')^2 \int & \mathrm{d}z_2\,\mathrm{d}z_3 \prod_{i<j} |z_{ij}|^{\alpha' k_i\cdot k_j} \\ & \times \left(\frac{a_1^{(\mathrm{L})}}{z_{12}z_{23}} + \frac{a_2^{(\mathrm{L})}}{z_{13}z_{23}} + \frac{a_3^{(\mathrm{L})}}{z_{12}z_{34}} + \frac{a_4^{(\mathrm{L})}}{z_{13}z_{24}} + \frac{a_5^{(\mathrm{L})}}{z_{23}z_{34}} + \frac{a_6^{(\mathrm{L})}}{z_{23}z_{24}} \right) \\ & \times \left(\frac{a_1^{(\mathrm{R})}}{\bar z_{12}\bar z_{23}} + \frac{a_2^{(\mathrm{R})}}{\bar z_{13}\bar z_{23}} + \frac{a_3^{(\mathrm{R})}}{\bar z_{12}\bar z_{34}} + \frac{a_4^{(\mathrm{R})}}{\bar z_{13}\bar z_{24}} + \frac{a_5^{(\mathrm{R})}}{\bar z_{23}\bar z_{34}} + \frac{a_6^{(\mathrm{R})}}{\bar z_{23}\bar z_{24}} \right) . \end{aligned} \tag{3.23}$$

It is not difficult to convince oneself that the only terms that do not vanish in this particular limit, where $|z_{12}| \ll |z_{23}| \ll 1$, are exactly the products of $1/|z_{12}|^2$ with any of the following: $1/|z_{23}|^2$, $1/(z_{23}\bar z_{13})$, $1/(z_{13}\bar z_{23})$, or $1/|z_{13}|^2$, since $1/z_{13} = 1/z_{23} + O(e^{-Y/\alpha'})$. Any of these terms obviously cancel the Jacobian. Moreover, they do not vanish when the phase integration is performed. If instead one had a term like $1/(z_{12}\bar z_{12}^3) = e^{2X/\alpha'}e^{i\theta}$, it would cancel $e^{-2X/\alpha'}$ in the Jacobian but would vanish after integration over θ.

It is a characteristic feature of the MSS representation that only the terms with the correct weight are non-zero upon phase integration. That is why it is particularly suitable for the analysis of the low-energy limit of the closed string. In other words, the phase dependence is trivial by construction, which means that the level-matching is automatically satisfied. To sum up, to obtain a pole in $1/s_{12}s_{123}$, we have to pick up exactly two chiral blocks $1/z_{12}z_{23}$ and $1/\bar z_{12}\bar z_{23}$ in (3.23) which come with a factor of $a_1^{(L)}a_1^{(R)}$. Furthermore, it can be easily proven that any other region of the moduli space, where at least one of z_2 or z_3 stay at finite distance from other points, contributes only to subleading $O(\alpha')$ terms. In total, this region of the moduli space contributes the following pole to the amplitude:

$$\frac{a_1^{(\mathrm{L})}a_1^{(\mathrm{R})}}{s_{12}s_{123}}\,, \tag{3.24}$$

where one reads $n_1^{(\mathrm{L/R})} = a_1^{(\mathrm{L/R})}$. One can then repeat this operation in the other kinematic channels. For instance,[7] the region $z_{23} \ll z_{34} \sim z_{24} \ll 1$ receives non-zero contributions both from $1/(z_{23}z_{34})$ and $1/(z_{23}z_{24})$ (and their complex conjugates). This results in the following contribution to the low-energy limit of the amplitude:

$$\frac{(a_5^{(\mathrm{L})}+a_6^{(\mathrm{L})})(a_5^{(\mathrm{R})}+a_6^{(\mathrm{R})})}{s_{23}s_{234}}\,. \tag{3.25}$$

By repeating this operation in all other kinematic channels, one can generate the 15 combinatorially-distinct trivalent graphs of the low-energy limit and thus obtain the full

[7]The channels generated by z_2 and/or $z_3 \to z_5 = \infty$ are dealt with by introducing a $+$ sign in the exponential in (3.19). Then the pole is generated by a similar procedure.

field theory amplitude, valid in any dimension:

$$\mathcal{A}_5^{\text{tree}} = g^3 \left(\frac{n_1^{(\text{L})} n_1^{(\text{R})}}{s_{12}s_{45}} + \frac{n_2^{(\text{L})} n_2^{(\text{R})}}{s_{23}s_{51}} + \frac{n_3^{(\text{L})} n_3^{(\text{R})}}{s_{34}s_{12}} + \frac{n_4^{(\text{L})} n_4^{(\text{R})}}{s_{45}s_{23}} + \frac{n_5^{(\text{L})} n_5^{(\text{R})}}{s_{51}s_{34}} + \frac{n_6^{(\text{L})} n_6^{(\text{R})}}{s_{14}s_{25}} \right.$$
$$+ \frac{n_7^{(\text{L})} n_7^{(\text{R})}}{s_{32}s_{14}} + \frac{n_8^{(\text{L})} n_8^{(\text{R})}}{s_{25}s_{43}} + \frac{n_9^{(\text{L})} n_9^{(\text{R})}}{s_{13}s_{25}} + \frac{n_{10}^{(\text{L})} n_{10}^{(\text{R})}}{s_{42}s_{13}} + \frac{n_{11}^{(\text{L})} n_{11}^{(\text{R})}}{s_{51}s_{42}} + \frac{n_{12}^{(\text{L})} n_{12}^{(\text{R})}}{s_{12}s_{35}}$$
$$\left. + \frac{n_{13}^{(\text{L})} n_{13}^{(\text{R})}}{s_{35}s_{24}} + \frac{n_{14}^{(\text{L})} n_{14}^{(\text{R})}}{s_{14}s_{35}} + \frac{n_{15}^{(\text{L})} n_{15}^{(\text{R})}}{s_{13}s_{45}} \right), \tag{3.26}$$

in terms of the following numerators:

$$\begin{aligned}
n_1^{(\text{L/R})} &= a_1^{(\text{L/R})}, & n_2^{(\text{L/R})} &= a_5^{(\text{L/R})} + a_6^{(\text{L/R})}, \\
n_3^{(\text{L/R})} &= -a_3^{(\text{L/R})}, & n_4^{(\text{L/R})} &= a_1^{(\text{L/R})} + a_2^{(\text{L/R})}, \\
n_5^{(\text{L/R})} &= a_5^{(\text{L/R})}, & n_6^{(\text{L/R})} &= -a_2^{(\text{L/R})} - a_4^{(\text{L/R})} + a_3^{(\text{L/R})} + a_5^{(\text{L/R})}, \\
n_7^{(\text{L/R})} &= -a_1^{(\text{L/R})} - a_2^{(\text{L/R})} + a_5^{(\text{L/R})} + a_6^{(\text{L/R})}, & n_8^{(\text{L/R})} &= a_3^{(\text{L/R})} + a_5^{(\text{L/R})}, \\
n_9^{(\text{L/R})} &= -a_2^{(\text{L/R})} - a_4^{(\text{L/R})}, & n_{10}^{(\text{L/R})} &= -a_4^{(\text{L/R})}, \\
n_{11}^{(\text{L/R})} &= a_6^{(\text{L/R})}, & n_{12}^{(\text{L/R})} &= a_1^{(\text{L/R})} + a_3^{(\text{L/R})}, \\
n_{13}^{(\text{L/R})} &= a_4^{(\text{L/R})} + a_6^{(\text{L/R})}, & n_{14}^{(\text{L/R})} &= -a_4^{(\text{L/R})} - a_6^{(\text{L/R})} + a_1^{(\text{L/R})} + a_3^{(\text{L/R})}, \\
n_{15}^{(\text{L/R})} &= -a_2^{(\text{L/R})}.
\end{aligned} \tag{3.27}$$

It is now trivial to check that, by construction, n_i's satisfy the Jacobi identities, which we recall in appendix B. The linear relations (3.27) between n_i's and a_i's coincide with those derived for gauge theory amplitudes in [21], where covariant expressions for the kinematical numerators at any multiplicity were obtained.

The crucial point here is that we have not referred to the actual expressions of the n_i's derived in the previous sections but simply started from the MSS representation (3.23) for the string amplitude. Therefore, the final result (3.26) can be either a gauge theory or a gravity amplitude, depending on the type of string theory in which the low-energy limit is taken, as indicated in table 1. In heterotic string theory, if $n_i^{(\text{L})} = c_i$ are color factors and $n_i^{(\text{R})} = n_i$ are kinematic numerators, one obtains a gluon amplitude, whereas in type II string theory, where both $n_i^{(\text{L})}$ and $n_i^{(\text{R})}$ are kinematic numerators, one gets a graviton scattering amplitude.

Another option would be to choose both $n_i^{(\text{L})}$ and $n_i^{(\text{R})}$ to be color factors c_i, in which case (3.26) would correspond to the scattering amplitude of five color cubic scalars. From the perspective of the low-energy limit of string theory, this corresponds to compactifying both sectors of the bosonic string on the same torus as the one of the heterosis mechanism and then choosing external states bipolarized in the internal color space. This string theory, of course, suffers from all known inconsistencies typical for the bosonic string. However, at tree level, if one decouples by hand in both sectors the terms which create non-planar corrections in the heterotic string, the pathological terms disappear.

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Therefore, the formula (3.26) can be extended to produce the tree-level five-point amplitudes of the three theories: gravity, Yang-Mills and cubic scalar color. This is done by simply choosing different target-space polarizations for (L) and (R), as in table 1, to which, in view of the previous discussion, we could now add a new line for the cubic scalar color model.

The point of this demonstration was to illustrate the fact that the product of the left- and right-moving sectors produces in the low-energy limit the form of the amplitude in which the double copy construction is transparent and is not a peculiarity of gravity but rather of any of the three theories. This suggests that both the BCJ duality in gauge theory and the double copy construction of gravity follow from the inner structure of the closed string and its low-energy limit.

Furthermore, the MSS chiral block representation exists for n-point open string amplitudes [21, 35, 36], so to extend those considerations to any multiplicity, one would only need to rigorously prove that any open string pole channel corresponds to a closed string one and verify that level matching correctly ties the two sectors together. Then the MSS construction would produce the BCJ construction at any multiplicity, and this would constitute a string-theoretic proof that the BCJ representation of Yang-Mills amplitudes implies the double copy construction of gravity amplitudes at tree level. Finally, note that this procedure is different from the KLT approach [17] in that it relates the numerators of cubic diagrams in the various theories, rather than the amplitudes themselves. All of this motivates our study of the double copy construction at higher loops in the purely closed string sector.

We conclude this section by the observation that, in the recent works related to the "scattering equations" [44–50], there appeared new formulas for tree-level scattering amplitudes of cubic color scalars, gauge bosons and gravitons, in which color and kinematics play symmetric roles. It was also suggested that this approach might be generalizable to higher-spin amplitudes. Naturally, it would be interesting to make a direct connection between the scattering equations and the approach based on the low-energy limit of the closed string.

4 One loop in field theory

In this section, we turn to the study of the BCJ duality at one loop. Here and in the rest of this paper, we will deal only with amplitudes with the minimal number of physical external particles in supersymmetric theories — four.[8] At one loop, a color-dressed four-gluon amplitude can be represented as

$$\mathcal{A}^{\text{1-loop}}(1^-,2^-,3^+,4^+) = \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \Bigg\{ f^{ba_1c} f^{ca_2d} f^{da_3e} f^{ea_4b} \frac{n^{\text{box}}(1^-,2^-,3^+,4^+;\ell)}{\ell^2(\ell-k_1)^2(\ell-k_1-k_2)^2(\ell+k_4)^2} \\ + f^{ba_1c} f^{ca_2d} f^{da_4e} f^{ea_3b} \frac{n^{\text{box}}(1^-,2^-,4^+,3^+;\ell)}{\ell^2(\ell-k_1)^2(\ell-k_1-k_2)^2(\ell+k_3)^2}$$

[8] Higher multiplicity amplitudes in maximal SYM and supergravity have been addressed in the context of the BCJ duality in the upcoming paper [51, 52].

$$+ f^{ba_1c} f^{ca_4d} f^{da_2e} f^{ea_3b} \frac{n^{\text{box}}(1^-,4^+,2^-,3^+;\ell)}{\ell^2(\ell-k_1)^2(\ell-k_1-k_4)^2(\ell+k_3)^2}$$
$$+ \text{triangles, etc.}\Bigg\} . \tag{4.1}$$

Recall that the color factors can also be written in terms of color traces, for example:

$$\begin{aligned} c^{\text{box}}(1^-,2^-,3^+,4^+) &\equiv f^{ba_1c} f^{ca_2d} f^{da_3e} f^{ea_4b} \\ &= N_c \left(\text{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + \text{tr}(T^{a_4}T^{a_3}T^{a_2}T^{a_1})\right) + \text{double traces} . \end{aligned} \tag{4.2}$$

In this way, one can easily relate the color-kinematics representation (2.1) to the primitive amplitudes that are defined as the coefficients of the leading color traces [53].

4.1 Double copies of one $\mathcal{N}=4$ SYM

The maximally supersymmetric Yang-Mills theory has the simplest BCJ numerators. At four points, they are known up to four loops [2, 6, 54], and only at three loops they start to depend on loop momenta, in accordance with the string theory understanding [55–57]. For example, the one-loop amplitude is just a sum of three scalar boxes [58], which is consistent with the color-kinematic duality in the following way: the three master boxes written in (4.1) have the same trivial numerator $\langle 12\rangle^2 [34]^2 = istA^{\text{tree}}(1^-,2^-,3^+,4^+)$ (which we will always factorize henceforward), and all triangle numerators are equal to zero by the Jacobi identities.

Thanks to that particularly simple BCJ structure of $\mathcal{N}=4$ SYM, the double copy construction for $\mathcal{N}\geq 4$ supergravity amplitudes simplifies greatly [7]. Indeed, as the second gauge theory copy does not have to obey the BCJ duality, one can define its box numerators simply by taking its entire planar integrands and putting them in a sum over a common box denominator. Since the four-point $\mathcal{N}=4$ numerators are independent of the loop momentum, the integration acts solely on the integrands of the second Yang-Mills copy and thus produces its full primitive amplitudes:

$$\begin{aligned} \mathcal{M}^{\text{1-loop}}_{\mathcal{N}=4+N,\text{grav}}(1^-,2^-,3^+,4^+) = \langle 12\rangle^2 [34]^2 \Big\{ & A^{\text{1-loop}}_{\mathcal{N}=N,\text{vect}}(1^-,2^-,3^+,4^+) \\ & + A^{\text{1-loop}}_{\mathcal{N}=N,\text{vect}}(1^-,2^-,4^+,3^+) + A^{\text{1-loop}}_{\mathcal{N}=N,\text{vect}}(1^-,4^+,2^-,3^+) \Big\} . \end{aligned} \tag{4.3}$$

The $\mathcal{N}=8$ gravity amplitude is then simply given by the famous result of [58] in terms of the scalar box integrals I_4, recalled in appendix A:

$$\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=8,\text{grav}}(1^-,2^-,3^+,4^+) = \frac{i}{(4\pi)^{d/2}} \langle 12\rangle^4 [34]^4 \Big\{ I_4(s,t) + I_4(s,u) + I_4(t,u) \Big\} . \tag{4.4}$$

For a less trivial example, let us consider the case of the $\mathcal{N}=6$ gravity, for which the second copy is the contribution of a four-gluon scattering in $\mathcal{N}=2$ SYM. It is helpful to use the one-loop representation of the latter as

$$A^{\text{1-loop}}_{\mathcal{N}=2,\text{vect}} = A^{\text{1-loop}}_{\mathcal{N}=4,\text{vect}} - 2\, A^{\text{1-loop}}_{\mathcal{N}=2,\text{hyper}} , \tag{4.5}$$

where the last term is the gluon amplitude contribution from the $\mathcal{N}=2$ hyper-multiplet (or, equivalently, $\mathcal{N}=1$ chiral-multiplet in the adjoint representation) in the loop. This multiplet is composed of two scalars and one Majorana spinor, so its helicity content can be summarized as $\left(1_{\frac{1}{2}}, 2_0, 1_{-\frac{1}{2}}\right)$. If we use eq. (4.3) to "multiply" eq. (4.5) by $\mathcal{N}=4$ SYM, we obtain a similar expansion for the gravity amplitudes:

$$\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=6,\text{grav}} = \mathcal{M}^{\text{1-loop}}_{\mathcal{N}=8,\text{grav}} - 2\,\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=6,\text{matt}}\,, \tag{4.6}$$

where "$\mathcal{N}=6$ matter" corresponds to the formal multiplet which contains a spin-3/2 Majorana particle and can be represented as $\left(1_{\frac{3}{2}}, 6_1, 15_{\frac{1}{2}}, 20_0, 15_{-\frac{1}{2}}, 6_{-1}, 1_{-\frac{3}{2}}\right)$. Its contribution to the amplitude can be constructed through eq. (4.3) as $(\mathcal{N}=4)\times(\mathcal{N}=2$ hyper), where the second copy is also well known [59, 60] and is most easily expressed in terms of scalar integrals I_n:

$$A^{\text{1-loop}}_{\mathcal{N}=2,\text{hyper}}(1^-,2^-,3^+,4^+) = \frac{i}{(4\pi)^{d/2}}\,\langle 12\rangle^2\,[34]^2\left\{-\frac{1}{st}I_2(t)\right\}, \tag{4.7a}$$

$$\begin{aligned} A^{\text{1-loop}}_{\mathcal{N}=2,\text{hyper}}(1^-,4^+,2^-,3^+) = \frac{i}{(4\pi)^{d/2}}\,\langle 12\rangle^2\,[34]^2\bigg\{&\frac{tu}{2s^2}I_4(t,u) \\ &+\frac{t}{s^2}I_3(t)+\frac{u}{s^2}I_3(u)+\frac{1}{st}I_2(t)+\frac{1}{su}I_2(u)\bigg\}. \end{aligned} \tag{4.7b}$$

This lets us immediately write down the result from [7]:

$$\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=6,\text{matt}}(1^-,2^-,3^+,4^+) = \frac{i}{(4\pi)^{d/2}}\,\langle 12\rangle^4\,[34]^4\left\{\frac{tu}{2s^2}I_4(t,u)+\frac{t}{s^2}I_3(t)+\frac{u}{s^2}I_3(u)\right\}. \tag{4.8}$$

A comment is due here: here and below, we use the scalar integrals I_n recalled in appendix A, just as a way of writing down integrated expressions, so the scalar triangles in eq. (4.8) do not contradict with the no-triangle property of $\mathcal{N}=4$ SYM. As explained earlier, the BCJ double copy construction behind eq. (4.3), and its special case (4.8), contains only the box topology with all the scalar integrals in eq. (4.7) collected into non-scalar boxes.

Having thus computed $\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=6}(1^-,2^-,3^+,4^+)$ through the expansion (4.6), we can consider computing

$$\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=4,\text{grav}} = \mathcal{M}^{\text{1-loop}}_{\mathcal{N}=8,\text{grav}} - 4\,\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=6,\text{matt}} + 2\,\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=4,\text{matt}}\,, \tag{4.9}$$

where the $\mathcal{N}=4$ matter multiplet $\left(1_1, 4_{\frac{1}{2}}, 6_0, 4_{-\frac{1}{2}}, 1_{-1}\right)$ can be constructed either through eq. (4.3) as $(\mathcal{N}=4)\times(\mathcal{N}=0$ scalar), or as $(\mathcal{N}=2$ hyper$)^2$. In the former case, one only needs the full amplitudes from [60] to obtain the following result [7, 61]:

$$\begin{aligned} &\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=4,\text{matt}}(1^-,2^-,3^+,4^+) \\ &= \frac{-i}{(4\pi)^{d/2}}\,\langle 12\rangle^4\,[34]^4\bigg\{\frac{tu}{s^3}I_4^{d=6-2\epsilon}(t,u)-\frac{u}{s^3}I_2(t)-\frac{t}{s^3}I_2(u) \\ &\qquad -\frac{1}{s^2}\left(I_3^{d=6-2\epsilon}(t)+I_3^{d=6-2\epsilon}(u)\right)+\frac{\epsilon(t-u)}{s^3}\left(I_3^{d=6-2\epsilon}(t)-I_3^{d=6-2\epsilon}(u)\right) \\ &\qquad +\frac{\epsilon(1-\epsilon)}{s^2}\left(I_4^{d=8-2\epsilon}(s,t)+I_4^{d=8-2\epsilon}(s,u)+I_4^{d=8-2\epsilon}(t,u)\right)\bigg\}, \end{aligned} \tag{4.10}$$

which is valid to all orders in ϵ.

All of one-loop constructions with $\mathcal{N} > 4$, as we discuss in section 6, fit automatically in the string-theoretic picture of the BCJ double copy. This is due to fact that, just as the field-theoretic numerators are independent of the loop momentum, the $\mathcal{N} = 4$ string-based integrands do not depend on Schwinger proper times.

4.2 Double copy of one $\mathcal{N} = 2$ SYM

The second option to compute $\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=4,\text{matt}}(1^-,2^-,3^+,4^+)$ requires the BCJ representation for the $\mathcal{N}=2$ hyper-multiplet amplitude. The latter can also be used to construct gravity amplitudes with $\mathcal{N} < 4$ supersymmetries [12], such as ($\mathcal{N}=1$ gravity) = ($\mathcal{N}=1$ SYM) $\times$ (pure Yang-Mills). However, we will consider it mostly in the context of obtaining the BCJ numerators for $\mathcal{N}=2$ SYM:

$$n_{\mathcal{N}=2,\text{vect}}(\ell) = n_{\mathcal{N}=4,\text{vect}} - 2\, n_{\mathcal{N}=2,\text{hyper}}(\ell)\,, \tag{4.11}$$

whose double-copy square $\mathcal{N}=4$ supergravity coupled to two $\mathcal{N}=4$ matter multiplets:

$$\begin{aligned} \mathcal{M}^{\text{1-loop}}_{(\mathcal{N}=2)\times(\mathcal{N}=2),\text{grav}} &= \mathcal{M}^{\text{1-loop}}_{\mathcal{N}=8,\text{grav}} - 4\,\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=6,\text{matt}} + 4\,\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=4,\text{matt}} \\ &= \mathcal{M}^{\text{1-loop}}_{\mathcal{N}=4,\text{grav}} + 2\,\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=4,\text{matt}}\,. \end{aligned} \tag{4.12}$$

As a side comment, the problem of decoupling matter fields in this context is analogous to the more difficult issue of constructing pure gravity as a double copy of pure Yang-Mills [62].

Most importantly for the purposes of this paper, $\mathcal{A}^{\text{1-loop}}_{\mathcal{N}=2,\text{hyper}}$ is the simplest four-point amplitude with non-trivial loop-momentum dependence of the numerators, i.e. $O(\ell^2)$, which is already reflected in its non-BCJ form (4.7) by the fact that no rational part is present in the integrated amplitudes. The rest of this paper is mostly dedicated to studying both from the BCJ construction and field theory the double copy

$$(\mathcal{N}=4 \text{ matter}) = (\mathcal{N}=2 \text{ hyper})^2\,. \tag{4.13}$$

Here, the left-hand side stands for the contribution of vector matter multiplets running in a four-graviton loop in $\mathcal{N}=4$ supergravity, while the right-hand side indicates multiplets running in a four-gluon loop in SYM. In the rest of this section, we obtain the field-theoretic numerators for the latter amplitude contribution. In the literature [12, 59, 63, 64], it is also referred to as the contribution of the $\mathcal{N}=1$ chiral multiplet in the adjoint representation and is not to be confused with the $\mathcal{N}=1$ chiral multiplet in the fundamental representation, the calculation for which can be found in [62]. By calling the former multiplet $\mathcal{N}=2$ hyper, we avoid that ambiguity and keep the effective number of supersymmetries explicit.

4.3 Ansatz approach

The standard approach to finding kinematic numerators which satisfy Jacobi identities is through an ansatz [5, 12], as to our knowledge, there is no general constructive way of doing this, apart from the case of $\mathcal{N} = 4$ SYM at one loop [27]. Recently, however, there has been considerable progress [12, 13] in applying orbifold constructions to finding BCJ numerators.

In [12, 13, 64], several types of ansatz were used for one-loop four-point computations, starting from three independent master box numerators $n^{\text{box}}(1^-2^-3^+4^+)$, $n^{\text{box}}(1^-2^-4^+3^+)$ and $n^{\text{box}}(1^-4^+2^-3^+)$ from which all other cubic diagrams were constructed through Jacobi identities.

In comparison, our ansatz starts with two master boxes, $n^{\text{box}}(1^-,2^-,3^+,4^+)$ and $n^{\text{box}}(1^-,4^+,2^-,3^+)$, considered as independent while $n^{\text{box}}(1^-,2^-,4^+,3^+)$ is obtained from $n^{\text{box}}(1^-,2^-,3^+,4^+)$ by exchanging momenta k_3 and k_4.

From Feynman-rules power-counting, string theory and supersymmetry cancellations [59] we expect numerators to have at most two powers of the loop momentum. Moreover, the denominator of (4.7) contains s and s^2, but only t and u. Thus, it is natural to consider the following minimal ansatz:

$$\begin{aligned} n^{\text{box}}_{\mathcal{N}=2,\text{hyper}}(1^-,2^-,3^+,4^+;\ell) &= istA^{\text{tree}}(1^-,2^-,3^+,4^+) \qquad \text{(4.14a)} \\ &\times \frac{1}{s^2tu}\left\{P^{\text{(split-hel)}}_{4;2;1}\left(s,t;(\ell\cdot k_1),(\ell\cdot k_2),(\ell\cdot k_3);\ell^2\right) + P^{\text{(split-hel)}}_{2}(s,t)\,4i\epsilon(k_1,k_2,k_3,\ell)\right\}, \\ n^{\text{box}}_{\mathcal{N}=2,\text{hyper}}(1^-,4^+,2^-,3^+;\ell) &= istA^{\text{tree}}(1^-,2^-,3^+,4^+) \qquad \text{(4.14b)} \\ &\times \frac{1}{s^2tu}\left\{P^{\text{(alt-hel)}}_{4;2;1}\left(s,t;(\ell\cdot k_1),(\ell\cdot k_2),(\ell\cdot k_3);\ell^2\right) + P^{\text{(alt-hel)}}_{2}(s,t)\,4i\epsilon(k_1,k_2,k_3,\ell)\right\}. \end{aligned}$$

In eq. (4.14), $P_2(s,t)$ is a homogeneous polynomial of degree 2 and $P_{4;2;1}\left(s,t;\tau_1,\tau_2,\tau_3;\lambda\right)$ is a homogeneous polynomial of degree 4, but not greater than 2 for arguments τ_1, τ_2 and τ_3 and at most linear in the last argument λ. The 84 coefficients of these polynomials are the free parameters of the ansatz, that we shall determine from the kinematic Jacobi identities and cut constraints.

Following [12], we introduced in (4.14) parity-odd terms

$$\epsilon(k_1,k_2,k_3,\ell) = \epsilon_{\lambda\mu\nu\rho}k_1^\lambda k_2^\mu k_3^\nu \ell^\rho\,, \tag{4.15}$$

which integrate to zero in gauge theory but may contribute to gravity when squared in the double copy construction.

The first constraints on the coefficients of the ansatz come from imposing the obvious graph symmetries shown in figure 4 given by

$$\begin{aligned} &n^{\text{box}}(1^-,2^-,3^+,4^+;\ell) = n^{\text{box}}(2^-,1^-,4^+,3^+;k_1+k_2-\ell)\,, \\ &n^{\text{box}}(1^-,4^+,2^-,3^+;\ell) = n^{\text{box}}(1^-,3^+,2^-,4^+;k_1-\ell) = n^{\text{box}}(2^-,4^+,1^-,3^+;-\ell-k_3)\,, \end{aligned} \tag{4.16}$$

after which 45 coefficients remain unfixed.

Another set of constraints comes from the cuts. In particular, the quadruple cuts provide 10 more constraints on the master boxes alone. As we define triangle and bubble numerators through numerator Jacobi identities, such as the one shown in figure 2, 35 remaining parameters propagate to other numerators and then define the full one-loop integrand. Note that whenever there are multiple Jacobi identities defining one non-master numerator, the master graph symmetries (4.16) guarantee that they are equivalent.

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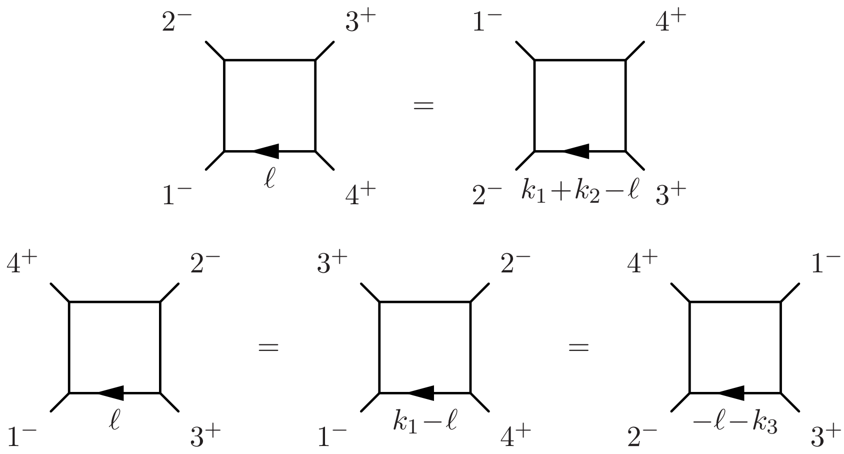


Figure 4. Box graph symmetries.

Double cuts are sensitive not only to boxes, but also to triangles and bubbles. Imposing them gives further 18 constraints. As a consistency check, we can impose double cuts without imposing quadruple cuts beforehand and in that case double cuts provide 28 conditions with the 10 quadruple-cut constraints being a subset of those. In any case, we are left with 17 free parameters after imposing all physical cuts.

For simplicity, we choose to impose another set of conditions: vanishing of all bubble numerators, including bubbles on external lines (otherwise rather ill-defined in the massless case). This is consistent with the absence of bubbles in our string-theoretic setup of section 5. Due to sufficiently high level of supersymmetry, that does not contradict the cuts and helps us eliminate 14 out of 17 free coefficients. Let us call 3 remaining free coefficients α, β and γ. For any values of those, we can check by direct computation that our solution integrates to (4.7), which is a consequence of the cut-constructibility of supersymmetric gauge theory amplitudes. However, there is still one missing condition, which we will find from the d-dimensional cuts in section 4.4.

4.4 Double copy and d-dimensional cuts

The double copy of the gluon amplitude with $\mathcal{N} = 2$ hyper multiplet in the loop naturally produces the graviton amplitude with $\mathcal{N} = 4$ matter multiplet in the loop, as in (4.13). First, we check that the gravity integrand satisfies all cuts.

So far we have been considering only four-dimensional cuts and cut-constructible gauge theory amplitudes for which it does not matter if during integration ℓ^2 term in the numerator is considered as 4- or $(4 - 2\epsilon)$-dimensional. After all, the difference will be just $\mu^2 = \ell_{(4)}^2 - \ell_{(d)}^2$ which integrates to $O(\epsilon)$. Note that we consider external momenta to be strictly four-dimensional, thus the scalar products with external momenta k_i like $\ell_{(d)} \cdot k_i = \ell_{(4)} \cdot k_i$ are four-dimensional automatically.

The issue is that now $\mathcal{N} = 4$ gravity amplitudes are not longer cut-constructible, so the fact that double copy satisfies all four-dimensional cuts is not enough to guarantee the right answer. This is reflected by the fact that the difference between $\ell_{(4)}^4$ and $\ell_{(d)}^4$ now integrates to $O(1)$ and produces rational terms. It seems natural to treat ℓ in (4.14) as

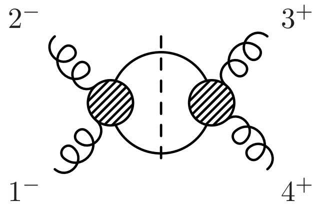


Figure 5. s-channel cut for $A^{\text{1-loop}}_{\mathcal{N}=2,\text{hyper}}(1^-,2^-,3^+,4^+)$.

strictly four-dimensional. Then our gravity solution integrates to

$$\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=4,\text{matt}}(1^-,2^-,3^+,4^+) = \langle 12\rangle^4 \, [34]^4 \frac{i r_\Gamma}{(4\pi)^{d/2}} \frac{1}{2s^4} \Bigg\{ -tu \left(\ln^2\!\left(\frac{-t}{-u}\right) + \pi^2 \right) + s(t-u)\ln\!\left(\frac{-t}{-u}\right) + s^2 \left(1 - \frac{1}{16}(3+2\gamma)^2\right) \Bigg\}, \tag{4.17}$$

where r_Γ the standard prefactor defined in (A.4). That coincides with the known answer from [61] and the truncated version of (4.10) [7], if $\gamma = -3/2$.

For the double copy to have predictive power beyond the cut-constructible cases, one should start with gauge theory numerators that satisfy all d-dimensional cuts. For $\mathcal{N}=2$ SYM, the difference should just be related to the μ^2 ambiguity mentioned above. As we already know that we miss only one extra condition, it suffices to consider the simplest cut sensitive to μ^2 terms, i.e. the s-channel cut for $A^{\text{1-loop}}_{\mathcal{N}=2,\text{hyper}}(1^-,2^-,3^+,4^+)$ that vanishes in four dimensions (figure 5).

We can either construct this cut from massive scalar and fermion amplitudes provided in [60], or simply use their final d-dimensional expression for this color-ordered amplitude:

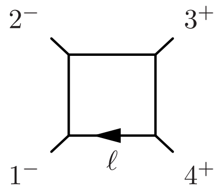
$$A^{\text{1-loop}}_{\mathcal{N}=2,\text{hyper}}(1^-,2^-,3^+,4^+) = \langle 12\rangle^2 \, [34]^2 \frac{i}{(4\pi)^{d/2}} \Bigg\{ -\frac{1}{st} I_2(t) + \frac{1}{s} I_4(s,t)[\mu^2] \Bigg\}. \tag{4.18}$$

Unifying all our gauge theory numerators into one box and making use of massive s-cut kinematics we retrieve the the following cut expression:

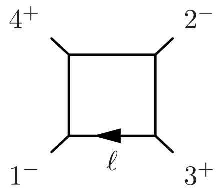
$$\delta_s A^{\text{1-loop}}_{\mathcal{N}=2,\text{hyper}}(1^-,2^-,3^+,4^+) = \langle 12\rangle^2 \, [34]^2 \frac{1}{s} \int d\text{LIPS} \frac{\mu^2 \left(1 - \frac{1}{4}(3+2\gamma)\right)}{((\ell-k_1)-\mu^2)((\ell+k_4)-\mu^2)}\,, \tag{4.19}$$

which coincides with the s-cut of (4.18) if $\gamma = -3/2$. Thus, we have reproduced the missing condition invisible to four-dimensional cuts.

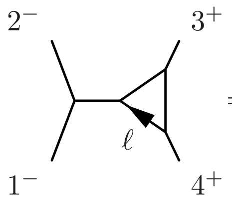
We preserve the remaining two-parameter freedom and write down the full set of numerators for the $\mathcal{N}=2$ hyper (or, equivalently, $\mathcal{N}=1$ chiral) multiplet amplitude as follows:



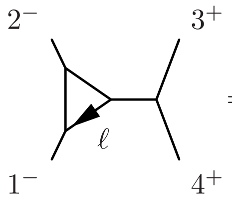
$$= \frac{\ell^2}{s} - \frac{(\ell(k_1+k_3))}{2u} + \frac{1}{s^2}[s(\ell k_1) - t(\ell k_2) + u(\ell k_3)]$$
$$- \frac{1}{s^2}\left[(\ell k_1)^2 + (\ell k_2)^2 + 6(\ell k_1)(\ell k_2) - t\frac{(\ell(k_1+k_3))^2}{u} - u\frac{(\ell(k_2+k_3))^2}{t}\right]$$
$$+ \frac{\beta}{s^2}\left[(\ell k_1)^2 + (\ell k_2)^2 - (\ell k_3)^2 - (\ell k_4)^2 + u(\ell(k_2+k_3))\right]$$
$$+ \frac{\alpha(s^2+tu)}{s^2 tu}(\ell(k_2+k_3))[2(\ell(k_1+k_3)) - u]\,, \tag{4.20a}$$



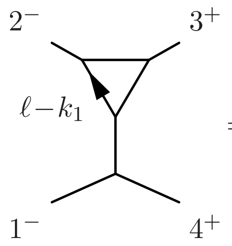
$$= \frac{\ell^2}{s} + \frac{(\ell(k_1+k_2))}{2s} - \frac{t(\ell(k_1+k_3))}{s^2} - \frac{2i\epsilon(k_1,k_2,k_3,\ell)}{s^2}$$
$$- \frac{1}{s^2}\left[(\ell k_1)^2 + (\ell k_2)^2 + 6(\ell k_1)(\ell k_2) - t\frac{(\ell(k_1+k_3))^2}{u} - u\frac{(\ell(k_2+k_3))^2}{t}\right]$$
$$+ \frac{\beta}{s^2}\left[(\ell k_1)^2 + (\ell k_2)^2 - (\ell k_3)^2 - (\ell k_4)^2 - t(\ell(k_1+k_3))\right]$$
$$+ \frac{\alpha(s^2+tu)}{s^2 tu}(\ell(k_1+k_3))[2(\ell(k_2+k_3)) + t]\,, \tag{4.20b}$$



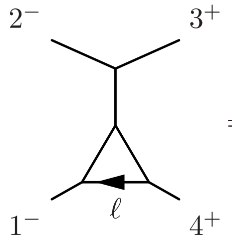
$$= \left(-\frac{1-\alpha+\beta}{s^2} - \frac{1-2\alpha}{2tu}\right)[s(\ell k_2) - t(\ell k_4) - u(\ell k_3)]\,, \tag{4.21a}$$



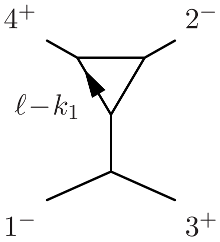
$$= \left(-\frac{1-\alpha+\beta}{s^2} + \frac{1+2\alpha}{2tu}\right)[s(\ell k_3) - t(\ell k_1) - u(\ell k_2)]\,, \tag{4.21b}$$



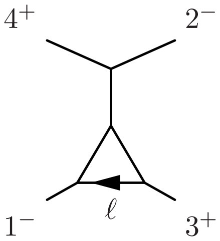
$$= \frac{1}{2su}[su - s(\ell k_3) + t(\ell k_1) - u(\ell k_2)] + \frac{2i\epsilon(k_1,k_2,k_3,\ell)}{s^2}\,, \tag{4.21c}$$



$$= -\frac{1}{2su}[s(\ell k_1) - t(\ell k_3) + u(\ell k_4)] - \frac{2i\epsilon(k_1,k_2,k_3,\ell)}{s^2}\,, \tag{4.21d}$$



$$= \frac{1}{2st}[st + s(\ell k_2) + t(\ell k_4) - u(\ell k_3)] - \frac{2i\epsilon(k_1, k_2, k_3, \ell)}{s^2}\,, \tag{4.21e}$$



$$= -\frac{1}{2st}[s(\ell k_3) + t(\ell k_1) - u(\ell k_2)] - \frac{2i\epsilon(k_1, k_2, k_3, \ell)}{s^2}\,, \tag{4.21f}$$

where for brevity we omitted the trivial kinematic prefactor $\langle 12 \rangle^2 [34]^2$.

The numerators that we obtain are non-local, as they contain inverse powers of Mandelstam invariants on top of those already included in their denominators. This is a feature of using the spinor-helicity formalism for BCJ numerators [5, 12, 13, 65] and is understood to be due to the choice of helicity states for the external gluons. Indeed, the numerators given in [66] in terms of polarization vectors are local while gauge-dependent.

We first note that the box numerators (4.20) do not possess constant terms. Later, we will relate this to a similar absence of constant terms in the string-based integrand. Moreover, the triangles integrate to null contributions to gauge theory amplitudes (4.7). Nonetheless, they are necessary for the double copy construction of the gravity amplitude (4.17), where they turn out to integrate to purely six-dimensional scalar triangles $I_3^{d=6-2\epsilon}$. The easiest way to check these statements is to explicitly convert the triangle numerators (4.21) to the Feynman parameter space, as explained in appendix C. We will use both of these facts later in section 6.

Finally, there are conjugation relations that hold for the final amplitudes, but are not automatic for the integrand numerators:

$$\left(n^{\text{box}}_{\mathcal{N}=2,\text{hyper}}(1^-, 2^-, 3^+, 4^+; \ell)\right)^* = n^{\text{box}}_{\mathcal{N}=2,\text{hyper}}(4^-, 3^-, 2^+, 1^+; -\ell)\,, \tag{4.22a}$$

$$\left(n^{\text{box}}_{\mathcal{N}=2,\text{hyper}}(1^-, 4^+, 2^-, 3^+; \ell)\right)^* = n^{\text{box}}_{\mathcal{N}=2,\text{hyper}}(3^-, 2^+, 4^-, 1^+; -\ell)\,. \tag{4.22b}$$

Although they are not necessary for the integrated results to be correct, one might choose to enforce them at the integrand level, which would fix both remaining parameters to

$$\alpha = 0\,, \quad \beta = -1\,, \tag{4.23}$$

and thus produce the unambiguous bubble-free BCJ solution. However, leaving two parameters unfixed can have its advantages to discern analytically pure coincidences from systematic patterns at the integrand level.

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5 One loop in string theory

This section is mostly a review of a detailed calculation given in [67] in order to explain the string-theoretic origin of the worldline integrands of the $\mathcal{N}=2$ SYM and symmetric $\mathcal{N}=4$ supergravity, in heterotic string and type II string in $d=4-2\epsilon$ dimensions, respectively.

The reader not familiar with the worldline formalism may simply observe that the general formula (5.13) contains a contribution to the gravity amplitude which mixes the left- and the-right-moving sectors and thus makes it look structurally different from the double copy construction. Then the $\mathcal{N}=2$ gauge theory and the $\mathcal{N}=4$ gravity integrands are given in eqs. (5.19) and (5.26), respectively, in terms of the Schwinger proper-time variables. They are integrated according to (5.16). These are the only building blocks needed to go directly to section 6, where the link between the worldline formalism and the usual Feynman diagrams is described starting from the loop-momentum space.

5.1 Field theory amplitudes from string theory

A detailed set of rules known as the Bern-Kosower rules was developed in [53, 68–70] to compute gauge theory amplitudes from the field theory limit of fermionic models in heterotic string theory. It was later extended to asymmetric constructions of supergravity amplitudes in [61, 70] (see also the review [71] and the approach of [72–74] using the Schottky parametrization). One-loop amplitudes in the open string are known at any multiplicity in the pure spinor formalism [75].

Here we recall the general mechanism to extract the field theory limit of string amplitudes at one loop in the slightly different context of orbifold models of the heterotic and type II string. A general four-point closed-string theory amplitude writes

$$\mathcal{A}^{\text{string}}_{\text{1-loop}} = N \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2} \int_{\mathcal{T}} \prod_{i=1}^{3} \frac{\mathrm{d}^2 z_i}{\tau_2} \langle V_1(z_1)V_2(z_2)V_3(z_3)V_4(z_4)\rangle \,. \tag{5.1}$$

The integration domains are $\mathcal{F}=\{\tau=\tau_1+i\tau_2; |\tau_1|\leq \frac{1}{2}, |\tau|^2\geq 1, \tau_2>0\}$ and $\mathcal{T}=\{|\mathrm{Re}\, z|\leq \frac{1}{2},\, 0\leq \mathrm{Im}\, z\leq \tau_2\}$. The normalization constant N is different for heterotic and type II strings. We will omit it throughout this section except the final formula (5.16), where the normalization is restored. The z_i are the positions of the vertex operators in the complex torus $\mathcal{T}$, and z_4 has been set to $z_4=i\tau_2$ to fix the genus-one conformal invariance.

On the torus, the fermionic fields ψ^μ and $\bar\psi^\nu$ can have different boundary conditions when transported along the A- and B-cycles (which correspond to the shifts $z\to z+1$ and $z\to z+\tau$, respectively). These boundary conditions define spin structures, denoted by two integers $a,b\in\{0,1\}$ such that

$$\psi(z+1)=e^{i\pi a}\psi(z)\,, \qquad \psi(z+\tau)=e^{i\pi b}\psi(z)\,. \tag{5.2}$$

In an orbifold compactification, these boundary conditions can be mixed with target-space shifts and the fields X and ψ can acquire non-trivial boundary conditions, mixing the standard spin structures (or Gliozzi-Scherk-Olive sectors) with more general orbifold sectors [76, 77].

The vertex operator correlation function (5.1) is computed in each orbifold and GSO sector, using Wick's theorem with the two-point functions

$$\langle X(z)X(w)\rangle = \mathcal{G}(z-w,\tau)\,, \tag{5.3a}$$

$$\langle \psi(z)\psi(w)\rangle_{a,b} = S_{a,b}(z-w,\tau)\,, \tag{5.3b}$$

whose explicit expressions are not needed for the purposes of this review. They can be found, for instance, in [67, eqs. (A.10),(A.19)]. The vertex operators of the external gluons are in the (0) picture (see vertex operator obtained from (3.3)×(3.4b)) and the external gravitons ones are in the $(0,0)$ picture (see (3.4b)×(3.4b)).

The total correlation function can then be written in the following *schematic* form:

$$\mathcal{A}^{\text{string}}_{\text{1-loop}} = \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^{d/2-3}} \int_{\mathcal{T}} \prod_{i=1}^{3} \frac{\mathrm{d}^2 z_i}{\tau_2} \sum_{s,s'} \mathcal{Z}^{ss'} \left(\mathcal{W}^{(\mathrm{L})}_s(z)\, \overline{\mathcal{W}^{(\mathrm{R})}_{s'}}(\bar z) + \mathcal{W}^{(\text{L-R})}_{s,s'}(z,\bar z) \right) e^{\mathcal{Q}}\,. \tag{5.4}$$

where s and s' correspond to the various GSO and orbifold sectors of the theory with their corresponding conformal blocks, and $\mathcal{Z}^{ss'}$ is defined so that it contains the lattice factor $\Gamma_{10-d,10-d}$ or twistings thereof according to the orbifold sectors and background Wilson lines.[9] The exponent of the plane-wave factor $e^{\mathcal{Q}}$ writes explicitly

$$\mathcal{Q} = \sum_{i<j} k_i \cdot k_j\; \mathcal{G}(z_i - z_j)\,, \tag{5.5}$$

similarly to (3.7) at tree level. The partition function now intertwines the left- and right-moving CFT conformal blocks and gives rise not only to pure chiral contractions $\mathcal{W}^{(\mathrm{L})}_s \overline{\mathcal{W}^{(\mathrm{R})}_{s'}}$ but also to left-right mixed contractions $\mathcal{W}^{(\text{L-R})}_{s,s'}(z,\bar z)$. The latter comes from terms such as

$$\langle \partial X(z,\bar z)\bar\partial X(w,\bar w)\rangle = -\alpha'\pi\delta^{(2)}(z-w) + \frac{\alpha'}{2\pi \mathrm{Im}\,\tau}\,, \tag{5.6}$$

where the first term on the right-hand side gives a vanishing contribution due the canceled propagator argument in the same way as at tree level. The second term in eq. (5.6) was absent at tree level, see eq. (3.5), but now generates left-right mixed contractions in case the two sectors have coinciding target spaces, i.e. in gravity amplitudes. However, in gauge theory amplitudes in the heterotic string, the target spaces are different, and contractions like (5.6) do not occur.

The main computation that we use in this section was performed in great detail in [79], and the explicit expressions for partition functions, lattice factors and conformal blocks may be found in the introductory sections thereof.

The mechanism by which the string integrand descends to the worldline (or tropical) integrand is qualitatively the same as at tree level.[10] In particular, one considers families

[9] More details on these objects can be found in classical textbooks, for instance, [78, Chapters 9-10].

[10] However, it has to be adapted to the geometry of the genus-one worldsheet. In particular, phases of complex numbers on the sphere become real parts of coordinates on the complex torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. As explained in [43], this is due to the fact that the complex torus is the image by the Abel-Jacobi map of the actual Riemann surface. In the limit $\alpha'\to 0$, this map simplifies to a logarithmic map, and coordinates on the surface $z,\bar z$ are related to coordinates on the complex torus $\zeta,\bar\zeta$ by: $\zeta = -2i\pi\alpha' \ln z$. The same is true for the modular parameter τ, whose real and imaginary parts are linked to phases and modulus of q, respectively.

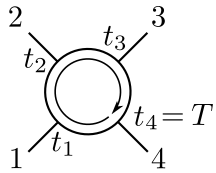

Figure 6. Generic four-point worldline graph in the ordering (1234).

of tori becoming thinner and thinner as $\alpha' \to 0$. On these very long tori, only massless states can propagate (massive states are projected out), so the level-matching condition of string theory associated to the cylindrical coordinate on the torus can be integrated out and the tori become worldloops. Quantitatively, one performs the following well-known change of variables:

$$\begin{aligned} \operatorname{Im}\tau \in \mathcal{F} \quad &\longrightarrow \quad T = \operatorname{Im}\tau/\alpha' \in [0; +\infty[\,, \\ \operatorname{Im} z_i \in \mathcal{T} \quad &\longrightarrow \quad t_i = z_i/\alpha' \in [0; T[\,, \end{aligned} \tag{5.7}$$

where T is the Schwinger proper time of the loop[11] and t_i are the proper times of the external legs along it (see figure 6).

We should also mention that to obtain a truly d-dimensional amplitude, one should not forget to decouple Kaluza-Klein modes of the compactified string by sending the radii R of compactification to zero, so that $R \sim \sqrt{\alpha'}$ (for instance, in this way one sets the untwisted lattice factor $\Gamma_{10-d,10-d}$ to 1). The field theory worldline amplitude is obtained after that — possibly lengthy — process of integrating out the real parts of τ and z's, and one is left with an integral of the form[12] [81]:

$$\mathcal{A}^{\text{1-loop}} = \int_0^\infty \frac{\mathrm{d}T}{T^{d/2-3}} \int_0^1 (\mathrm{d}u_i)^3 \sum_{s,s'} Z_{ss'} \left(W_s^{(\mathrm{L})} W_{s'}^{(\mathrm{R})} + W_{s,s'}^{(\text{L-R})} \right) e^{-TQ}\,, \tag{5.8}$$

where $u_i = t_i/T$ are rescaled proper times. As reviewed later in section 6, the exponential factor e^{-TQ} can also be regarded as a result of exponentiating the loop-momentum denominator of the corresponding Feynman diagram, with $Z_{ss'} \left(W_s^{(\mathrm{L})} W_{s'}^{(\mathrm{R})} + W_{s,s'}^{(\text{L-R})} \right)$ coming from its numerator. Formula (5.8) can be written in terms of derivatives of the worldline Green's function [82, 83] which descends from the worldsheet one and is defined by

$$G(u_i, u_j) = T(|u_i - u_j| - (u_i - u_j)^2)\,. \tag{5.9}$$

For example, eq. (5.5) becomes

$$Q = \sum_{i<j} k_i \cdot k_j \; G(u_i, u_j)\,. \tag{5.10}$$

---

[11] Strictly speaking, as originally observed in [80] and recently reviewed in [43], one should cut off the region $\mathcal{F}$ by a parameter L, so that the region of interest for us is actually the upper part $\operatorname{Im}\tau > L$ of $\mathcal{F}$, which in the field theory limit gives a hard Schwinger proper-time cutoff $T > \alpha' L$. Here we trade this cutoff for dimensional regularization with $d = 4 - 2\epsilon$.

[12] Note that the calligraphic letters $\mathcal{G}, \mathcal{Q}, \mathcal{W}, \mathcal{Z}$ refer to string-theoretic quantities, whereas the plain letters G, Q, W, Z refer to their worldline analogues.

The partition function factor $Z_{ss'}$, in the field theory limit, just induces a sum over multiplet decompositions, as in eqs. (4.5), (4.9) and (4.6), but does not change the qualitative nature of the objects.

Moreover, it is worth mentioning that the field theory limit of mixed contractions (5.6) produces only factors of $1/T$:

$$\langle \partial X(z,\bar{w}) \bar{\partial} X(z,\bar{w}) \rangle \underset{\alpha' \to 0}{\longrightarrow} -\frac{2}{T} \tag{5.11}$$

without further dependence on the positions t_i of the legs on the worldloop. Note that in general, factors of $1/T^k$ modify the overall factor $1/T^{d/2-(n-1)}$ and thus act as dimension shifts $d \to d+2k$.

Let us now discuss the differences between color and kinematics in the integrand of eq. (5.8). In heterotic string theory, the two sectors have different target spaces and do not communicate with each other. In particular, the right-moving sector is a color CFT: it is responsible for the color ordering in the field theory limit as demonstrated in the Bern-Kosower papers [53, 68–70], and its factor writes

$$W^{(\mathrm{R,\,color})} = \sum_{\mathfrak{S} \in S_{n-1}} \mathrm{tr}(T^{a_{\mathfrak{S}(1)}} \dots T^{a_{\mathfrak{S}(n-1)}} T^{a_n}) \Theta(u_{\mathfrak{S}(1)} < \dots < u_{\mathfrak{S}(n-1)} < u_n)\,, \tag{5.12}$$

where the sum runs over the set S_{n-1} of permutations of $(n-1)$ elements. It is multiplied by a $W^{(\mathrm{L,\,kin})}$ which contains the kinematical loop-momentum information.

In gravity, both sectors are identical, they carry kinematical information and can mix with each other. To sum up, we can write the following worldline formulas for gauge theory and gravity amplitudes:

$$\mathcal{A}^{\text{1-loop}}_{\text{gauge}} = \int_0^\infty \frac{\mathrm{d}T}{T^{d/2-3}} \int_0^1 \mathrm{d}^3 u \cdot \left(W^{(\mathrm{L,\,kin})}\, W^{(\mathrm{R,\,col})} \right) \cdot e^{-TQ}\,, \tag{5.13a}$$

$$\mathcal{M}^{\text{1-loop}}_{\text{gravity}} = \int_0^\infty \frac{\mathrm{d}T}{T^{d/2-3}} \int_0^1 \mathrm{d}^3 u \cdot \left(W^{(\mathrm{L,\,kin})}\, W^{(\mathrm{R,\,kin})} + W^{(\mathrm{L\text{-}R,\,kin})} \right) \cdot e^{-TQ}\,. \tag{5.13b}$$

Besides the fact that these formulas are not written in the loop-momentum space, the structure of the integrand of the gravity amplitude (5.13b) is different from the double-copy one in eq. (2.4): it has non-squared terms that come from left-right contractions. This paper is devoted to analysis of their role from the double copy point of view, in the case of the four-point one-loop amplitude in $(\mathcal{N}=2) \times (\mathcal{N}=2)$ gravity.

The kinematic correlators W^{kin} are always expressed as polynomials in the derivatives of the worldline Green's function G:

$$\begin{aligned} \dot{G}(u_i,u_j) &= \mathrm{sign}(u_i-u_j) - 2(u_i-u_j)\,, \\ \ddot{G}(u_i,u_j) &= \frac{2}{T}\left(\delta(u_i-u_j)-1\right)\,, \end{aligned} \tag{5.14}$$

where the factors of T take into account the fact that the derivative is actually taken with respect to the unscaled variables t_i, where $\partial_{t_i} = T^{-1}\partial_{u_i}$.

To illustrate the link with the loop-momentum structure, let us recall the qualitative dictionary between the worldline power-counting and the loop-momentum one [79, 84, 85].

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A monomial of the form $(\dot{G})^n(\ddot{G})^m$ contributes to $O(\ell^{n+2m})$ terms in four-dimensional loop-momentum integrals:[13]

$$(\dot{G})^n(\ddot{G})^m \sim \ell^{n+2m}\,. \tag{5.15}$$

Later in section 6, we describe in more detail the converse relation, i.e. the quantitative link between the loop momentum and the Schwinger proper-time variables.

For definiteness, in order to have well-defined conventions for worldline integration, we define a theory-dependent worldline numerator W_X to be carrying only loop-momentum-like information:

$$\begin{aligned}\mathcal{A}_X^{\text{1-loop}} &= i\,\frac{2t_8F^4}{(4\pi)^{d/2}}\sum_{\mathfrak{S}\in S_3}\mathrm{tr}(T^{a_{\mathfrak{S}(1)}}T^{a_{\mathfrak{S}(2)}}T^{a_{\mathfrak{S}(3)}}T^{a_4}) &(5.16\text{a})\\ &\qquad\times\int_0^\infty\frac{\mathrm{d}T}{T^{d/2-3}}\int_0^1\mathrm{d}u_{\mathfrak{S}(1)}\int_0^{u_{\mathfrak{S}(1)}}\mathrm{d}u_{\mathfrak{S}(2)}\int_0^{u_{\mathfrak{S}(2)}}\mathrm{d}u_{\mathfrak{S}(3)}\cdot W_X\cdot e^{-TQ}\,,\\ \mathcal{M}_X^{\text{1-loop}} &= i\,\frac{4t_8t_8R^4}{(4\pi)^{d/2}}\int_0^\infty\frac{\mathrm{d}T}{T^{d/2-3}}\int_0^1\mathrm{d}^3u\cdot W_X\cdot e^{-TQ}\,. &(5.16\text{b})\end{aligned}$$

In (5.16a), the sum runs over six orderings $\mathfrak{S}\in S_3$, three out of which, $(123), (231), (312)$, are inequivalent and correspond to the three kinematic channels (s,t), (t,u) and (u,s) Moreover, the tensorial dependence on the polarization vectors is factored out of the integrals. The field strength $F^{\mu\nu}$ is the linearized field strength defined by $F^{\mu\nu}=\varepsilon^\mu k^\nu - k^\mu\varepsilon^\nu$ and $R^{\mu\nu\rho\sigma}=F^{\mu\nu}F^{\rho\sigma}$. The tensor t_8 is defined in [86, appendix 9.A] in such a way that $t_8F^4=4\mathrm{tr}(F^{(1)}F^{(2)}F^{(3)}F^{(4)})-\mathrm{tr}(F^{(1)}F^{(2)})\mathrm{tr}(F^{(3)}F^{(4)})+\text{perms}\,(2,3,4)$, where the traces are taken over the Lorentz indices. In the spinor-helicity formalism, we find

$$2t_8F^4 = \langle 12\rangle^2\,[34]^2\,, \tag{5.17a}$$

$$4t_8t_8R^4 = \langle 12\rangle^4\,[34]^4\,. \tag{5.17b}$$

The compactness of the expressions (5.16) is characteristic to the worldline formalism. In particular, the single function W_X determines the whole gauge theory amplitude in all of its kinematic channels.

Note that, contrary to the tree-level case, where integrations by parts have to be performed to ensure the vanishing of tachyon poles, at one loop, the field theory limit can be computed without integrating out the double derivatives.[14]

5.2 $\mathcal{N}=2$ SYM amplitudes from string theory

In this section, we provide the string-theoretic integrands for the scattering amplitudes of four gauge bosons in $\mathcal{N}=2$ SYM in heterotic string. Starting from the class of $\mathcal{N}=2$

[13]Therefore, qualitatively, double derivatives count as squares of simple derivatives. At one loop, an easy way to see this is to integrate by parts: when the second derivative ∂_{u_i} of $\ddot{G}(u_{ij})$ hits the exponential e^{-TQ}, a linear combination of $\dot{G}$ comes down (see definition of Q in eq. (5.10)) and produces $\dot{G}^2$. In the non-trivial cases where one does not just have a single $\ddot{G}$ as a monomial, it was proven [53, 68–70] that it is always possible to integrate out all double derivatives after a finite number of integrations by parts. Another possibility is to observe that the factor $1/T$ present in $\ddot{G}$ produces a dimension shift $d\to d+2$ in the worldline integrands, which in terms of loop momentum schematically corresponds to adding ℓ^2 to the numerator of the d-dimensional integrand.

[14]At least when there are no triangles.

four-dimensional heterotic orbifold compactifications constructed in [87, 88] and following the recipe of the previous section, detailed computations have been given in [67] based on the previously explained method. We shall not repeat them here but simply state the result. First of all, we recall that the expansion (4.5) of the $\mathcal{N}=2$ gluon amplitude into a sum of the $\mathcal{N}=4$ amplitude with that of the $\mathcal{N}=2$ hyper-multiplet. The corresponding worldline numerators for the color-ordered amplitudes of eq. (5.16) are:

$$W_{\mathcal{N}=4,\text{vect}} = 1\,, \qquad W_{\mathcal{N}=2,\text{hyper}} = W_3\,, \tag{5.18}$$

and, according to eq. (4.5), combine into $W_{\mathcal{N}=2,\text{vect}}$ as follows:

$$W_{\mathcal{N}=2,\text{vect}} = 1 - 2W_3\,. \tag{5.19}$$

The polynomial W_3, derived originally in the symmetric $\mathcal{N}=4$ supergravity construction of [67], is defined by

$$W_3 = -\frac{1}{8}\Big((\dot{G}_{12}-\dot{G}_{14})(\dot{G}_{21}-\dot{G}_{24})+(\dot{G}_{32}-\dot{G}_{34})(\dot{G}_{42}-\dot{G}_{43})\Big)\,. \tag{5.20}$$

where we introduce the shorthand notation $G_{ij} = G(u_i,u_j)$ and, accordingly, $\dot{G}_{ij}$ are defined in eq. (5.14). In spinor-helicity formalism, for the gauge choice

$$(q_1^{\text{ref}}, q_2^{\text{ref}}, q_3^{\text{ref}}, q_4^{\text{ref}}) = (k_3, k_3, k_1, k_1)\,, \tag{5.21}$$

it writes explicitly as follows:

$$\begin{aligned} W_3 = &-\frac{1}{8}(\text{sign}(u_1-u_2)+2u_2-1)(\text{sign}(u_2-u_1)+2u_1-1) \\ &+\frac{1}{4}(\text{sign}(u_3-u_2)+2u_2-1)(u_3-u_2)\,. \end{aligned} \tag{5.22}$$

It is of the form $\dot{G}^2$, so according to the dictionary (5.15), it corresponds to four-dimensional box numerators with two powers of loop momentum. This statement is coherent with the results of the field-theoretic calculation of section 4, namely, with the box numerators (4.20). Moreover, it obviously has no constant term originating from $(\text{sign}(u_{ij}))^2$, which is consistent with its absence in the loop-momentum expressions. We also checked that this worldline numerator integrates to the correct field theory amplitudes (4.7).

5.3 Absence of triangles

A direct application of the Bern-Kosower formalism immediately rules out the possibility of having worldline triangles in the field theory limit, however it is worth recalling the basic procedure to show this.

On the torus, trees attached to loops are produced by vertex operators colliding to one another, exactly as at tree level. For instance, consider an s_{12}-channel pole, as drawn in figure 7. It originates from a region of the worldsheet moduli space where $z_{12} \ll 1$. Locally, the worldsheet looks like a sphere, and in particular the short distance behavior of the torus propagator is as on the sphere:

$$\mathcal{G}(z_{12}) = -\frac{\alpha'}{2}\ln|z_{12}|^2 + O(z_{12})\,, \qquad S_{a,b}(z_{12}) = \frac{1}{z_{12}} + O(z_{12})\,. \tag{5.23}$$

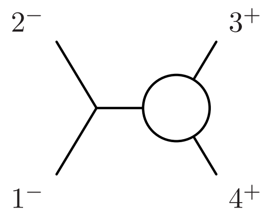

Figure 7. s_{12}-channel "would-be" worldline triangle.

Repeating the same reasoning as at tree level, a pole will be generated if and only if a term like $1/|z_{12}|^2$ is present in the numerator factor $\mathcal{W}^{(\mathrm{L})}\mathcal{Z}\overline{\mathcal{W}^{(\mathrm{R})}}$. In the gauge current sector, this requires a term like $S_{a,b}(\bar{z}_{12})$ that comes along with a single or double trace, like $\mathrm{tr}(\ldots T^{a_1}T^{a_2}\ldots)$ or $\mathrm{tr}(T^{a_1}T^{a_2})\mathrm{tr}(T^{a_3}T^{a_4})$, which causes no trouble. However, in the supersymmetric sector, this term has to be a $\partial\mathcal{G}(z_{12})$ which amounts to extraction from $\mathcal{W}_3$ of the following term:

$$\partial\mathcal{G}(z_{12})(\partial\mathcal{G}(z_{14})-\partial\mathcal{G}(z_{24}))\simeq z_{12}\partial\mathcal{G}(z_{12})\partial^2\mathcal{G}(z_{24})+O(z_{12})^2\,, \tag{5.24}$$

which obviously does not provide the expected $1/z_{12}$ behavior. Note that $(\partial\mathcal{G}(z_{12}))^2$ does not work either, as it is killed by the phase integration.

It is not difficult to check that no triangles are generated in the other channels, and this is independent of the gauge choice. As we shall explain later in the comparison section, our BCJ triangles (4.21) are invisible in the worldline formulation, which is consistent with the previous observation.

We could also try to observe Jacobi identities on W_3 directly on the worldline. A possible natural way to do so is to consider the following difference: $W_3\big|_{u_1<u_2}-W_3\big|_{u_2<u_1}$ and try to associate it to a BCJ triangle. This quantity, when it is non-zero, can be found to be proportional to u_i-u_j. This definitely vanishes when considering a triangle-like configuration with coinciding points $u_i\to u_j$.

5.4 $(2,2)$ $\mathcal{N}=4$ supergravity amplitudes from string theory

The four-graviton amplitudes in $(2,2)$ string theory models have been studied in [67] using type II symmetric orbifold constructions of [89]. Here we shall not recall the computation but only describe the structure of the numerator $\mathcal{W}^{(L)}\mathcal{Z}\overline{\mathcal{W}^{(R)}}$. In the symmetric $(2,2)$ constructions, both the left-moving and the right-moving sectors of the type II string have the half-maximal supersymmetry. Therefore this leaves room for internal left-right contractions in addition to the usual chiral correlators when applying Wick's theorem to compute the conformal blocks. Schematically, the integrand can be written as follows:

$$(1-2\mathcal{W}_3)(1-2\overline{\mathcal{W}}_3)+2\mathcal{W}_2\,, \tag{5.25}$$

where the partition function has explicitly produced a sum over the orbifold sectors to give 1 and $-2\mathcal{W}_3$.

After taking the field theory limit, one obtains the following worldline numerators for $\mathcal{N}=4$ supergravity coupled to two $\mathcal{N}=4$ vector multiplets:

$$W_{\mathcal{N}=4,\text{ grav + 2 vect}}=(1-2W_3)^2+2W_2\,, \tag{5.26}$$

where W_3 is the same as in eq. (5.20) and the polynomial W_2 writes

$$W_2 = -(\dot{G}_{12} - \dot{G}_{14})(\dot{G}_{32} - \dot{G}_{34})\ddot{G}_{24}\,, \tag{5.27}$$

in the gauge choice (5.21), its explicit expression is

$$W_2 = -\frac{1}{4T}\,\frac{1}{u}\,(2u_2 - 1 + \mathrm{sign}(u_3 - u_2))(2u_2 - 1 + \mathrm{sign}(u_1 - u_2))\,. \tag{5.28}$$

According to the dictionary (5.15), in the field-theoretic interpretation, W_3^2 corresponds to a four-dimensional box numerator of degree four in the loop momentum, whereas W_2 can be interpreted as a degree-two box numerator in six dimensions, due to its dimension-shifting factor $1/T$ characteristic of the left-right-mixed contractions, see eq. (5.11). Following the supersymmetry decomposition (4.9), we can rewrite eq. (5.26) as

$$W_{\mathcal{N}=4,\text{ grav }+\text{ 2 vect}} = W_{\mathcal{N}=8,\text{grav}} - 4W_{\mathcal{N}=6,\text{matt}} + 4W_{\mathcal{N}=4,\text{matt}}\,, \tag{5.29}$$

where the integrands are given by:

$$W_{\mathcal{N}=8,\text{grav}} = 1\,, \tag{5.30a}$$
$$W_{\mathcal{N}=6,\text{matt}} = W_3\,, \tag{5.30b}$$
$$W_{\mathcal{N}=4,\text{matt}} = W_3^2 + W_2/2\,. \tag{5.30c}$$

These numerators respectively integrate to the following expressions:

$$\begin{aligned}\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=8,\text{grav}} = \frac{t_8 t_8 R^4}{4}\Bigg\{\frac{2}{\epsilon}\left[\frac{1}{su}\ln\left(\frac{-t}{\mu^2}\right) + \frac{1}{tu}\ln\left(\frac{-s}{\mu^2}\right) + \frac{1}{st}\ln\left(\frac{-u}{\mu^2}\right)\right] \\ + 2\left[\frac{1}{st}\ln\left(\frac{-s}{\mu^2}\right)\ln\left(\frac{-t}{\mu^2}\right) + \frac{1}{tu}\ln\left(\frac{-t}{\mu^2}\right)\ln\left(\frac{-u}{\mu^2}\right) \\ + \frac{1}{us}\ln\left(\frac{-u}{\mu^2}\right)\ln\left(\frac{-s}{\mu^2}\right)\right]\Bigg\}\,,\end{aligned} \tag{5.31}$$

$$\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=6,\text{matt}} = -\frac{t_8 t_8 R^4}{2s^2}\left(\ln^2\left(\frac{-t}{-u}\right) + \pi^2\right)\,, \tag{5.32}$$

$$\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=4,\text{matt}} = \frac{t_8 t_8 R^4}{2s^4}\left[s^2 + s(t-u)\ln\left(\frac{-t}{-u}\right) - tu\left(\ln^2\left(\frac{-t}{-u}\right) + \pi^2\right)\right]\,, \tag{5.33}$$

which match to the field theory amplitudes from section 4 (μ being an infrared mass scale).

6 Comparison of the approaches

In this section, we compare the field-theoretic and the string-based constructions for gauge theory and gravity amplitudes. We start with the simplest cases of section 4.1 in which the BCJ construction contains at least one $\mathcal{N}=4$ gauge theory copy.

Looking at the string-based representations for $\mathcal{N}>4$ supergravity amplitudes in eqs. (5.30a) and (5.30b), one sees that they do verify the double copy prescription, because the $\mathcal{N}=4$ Yang Mills numerator $W_{\mathcal{N}=4,\text{vect}}$ is simply 1. Therefore, regardless of the details of how we interpret the worldline integrand in terms of the loop momentum, the the double copy prescription (2.4) is immediately deduced from the following relations which express the gravity worldline integrands as products of gauge theory ones:

$$W_{\mathcal{N}=8,\text{grav}} = W_{\mathcal{N}=4,\text{vect}} \times W_{\mathcal{N}=4,\text{vect}}\,, \tag{6.1a}$$
$$W_{\mathcal{N}=6,\text{matt}} = W_{\mathcal{N}=4,\text{vect}} \times W_{\mathcal{N}=2,\text{hyper}}\,. \tag{6.1b}$$

These $\mathcal{N} > 4$ cases match directly to their field-theoretic construction described in section 4.1. Unfortunately, they do not allow us to say anything about the string-theoretic origin of kinematic Jacobi identities, as there are no triangles in both approaches, therefore they require only the trivial identity $1 - 1 = 0$.

We can also derive, without referring to the full string-theoretic construction, the form of the $\mathcal{N} = 6$ supergravity amplitude, simply by using its supersymmetry decomposition (4.6):

$$W_{\mathcal{N}=6,\text{grav}} = W_{\mathcal{N}=4,\text{vect}} \times \left(W_{\mathcal{N}=4,\text{vect}} - 2\,W_{\mathcal{N}=2,\text{hyper}}\right)\,, \tag{6.2}$$

which, according to eq. (5.19), can be rewritten as

$$W_{\mathcal{N}=6,\text{grav}} = W_{\mathcal{N}=4,\text{vect}} \times W_{\mathcal{N}=2,\text{vect}}\,. \tag{6.3}$$

The first really interesting case at four points is the symmetric construction of $\mathcal{N} = 4$ gravity with two vector multiplets, whose string-based numerator was given in eq (5.26). This numerator is almost the square of (5.19), up to the term W_2 which came from the contractions between left-movers and right-movers. Due to the supersymmetry expansion (4.9), the same holds for the string-based numerator of $\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=4,\text{matt}}$. In the following sections, we will compare the integrands of that amplitude coming from string and field theory, and see that the situation is quite subtle.

The aim of the following discussion (and, to a large extent, of the paper) is to provide a convincing illustration that the presence of total derivatives imposed by the BCJ representation of gauge theory integrands in order to obtain the correct gravity integrals has a simple physical meaning from the point of view of closed string theory.

As we have already explained, in the heterotic string construction of Yang-Mills amplitudes, the left- and right-moving sector do not communicate to each other as they have different target spaces. However, in gravity amplitudes, the two sectors mix due to left-right contractions.

Our physical observation is that these two aspects are related. To show this, we will go through a rather technical procedure in order to compare loop-momentum and Schwinger proper-time expressions, to finally write the equality (6.37) of the schematic form

$$\int \text{left-right contractions} = \int (\text{BCJ total derivatives})^2 + (\dots)\,. \tag{6.4}$$

We shall start by the gauge theory analysis and see that, despite the absence of left-right contractions, the string theory integrand is not completely blind to the BCJ representation and has to be corrected so as to match it at the integrand level, see eq. (6.21).

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On the gravity side, the essential technical difficulty that we will face is the following: in the two approaches, the squaring is performed in terms of different variables, and a square of an expression in loop momentum space does not exactly correspond to a square in the Schwinger proper-time space. This induces the presence of "square-correcting terms", the terms contained in $(\dots)$ on the right-hand side of eq. (6.4).

6.1 Going from loop momenta to Schwinger proper times

In principle, there are two ways to to compare loop-momentum expressions to worldline ones: one can either transform the loop-momentum into Schwinger proper times, or the converse. We faced technical obstacles in doing the latter, mostly because of the quadratic nature of the gauge theory loop-momentum polynomials, so in the present analysis we shall express loop-momentum numerators in terms of Schwinger proper-time variables.

We use the standard exponentiation procedure [90, 91][15] which we review here. First of all, let us consider the scalar box:

$$I[1] = \int \frac{d^d\ell_{(d)}}{(2\pi)^d} \frac{1}{\ell_{(d)}^2(\ell_{(d)}-k_1)^2(\ell_{(d)}-k_1-k_2)^2(\ell_{(d)}+k_4)^2}\,. \tag{6.5}$$

We exponentiate the four propagators using

$$\frac{1}{D_i^2} = \int_0^\infty dx_i \exp(-x_i D_i^2)\,, \tag{6.6}$$

and obtain

$$I[1] = \int \frac{d^d\ell_{(d)}}{(2\pi)^d} \int_0^\infty \prod_{i=1}^4 dx_i \exp\left(-\ell_{(d)}^2 \sum_{i=1}^4 x_i + 2\ell_{(d)}\cdot(x_2k_1 + x_3(k_1+k_2) - x_4k_4) - x_3(k_1+k_2)^2\right), \tag{6.7}$$

after expanding the squares. Then we introduce the loop proper time $T = \sum_i x_i$ and rescale the x_i's by defining the standard Feynman parameters

$$a_i = \frac{x_i}{T}\,, \tag{6.8}$$

which gives:

$$I[1] = \int_0^\infty dT\, T^3 \int \frac{d^d\ell_{(d)}}{(2\pi)^d} \int_0^1 \prod_{i=1}^4 da_i\, \delta\left(1 - \sum_{i=1}^4 a_i\right) \exp\left(-T(\ell_{(d)}-K)^2 - TQ\right)\,. \tag{6.9}$$

In this expression, the scalar Q is the second Symanzik polynomial and is given by

$$Q = -a_1a_3s - a_2a_4t\,, \tag{6.10}$$

while the shift vector $K = (x_2k_1 + x_3(k_1+k_2) - x_4k_4)$ defines the shifted loop momentum

$$\tilde{\ell}_{(d)} = \ell_{(d)} - K\,. \tag{6.11}$$

Of course, the expressions for Q and K change with the ordering in this parametrization.

[15] See also [84, 85] for an n-point review of the procedure in connection with the worldline formalism.

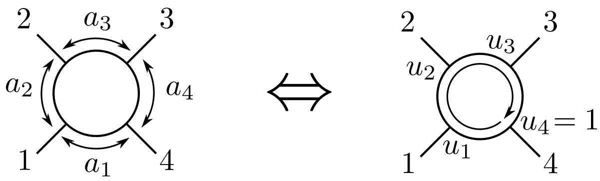

Figure 8. The u_i's are unambiguously defined for all orderings.

If we go to the worldline proper times t_i, or rather their rescaled versions

$$u_i = \frac{t_i}{T}\,, \tag{6.12}$$

defined as sums of the Feynman parameters, as pictured in figure 8, one obtains a parametrization valid for any ordering of the legs, in which the vector K writes [84, 85]

$$K^\mu = -\sum u_i {k_i}^\mu\,. \tag{6.13}$$

The scalar Q also has an invariant form in these worldline parameters, already given in (5.10). Finally, the Gaussian integral in $\tilde{\ell}_{(d)}$ is straightforward to perform, and we are left with:

$$I[1] = \frac{i}{(4\pi)^{d/2}} \int_0^\infty \frac{\mathrm{d}T}{T^{d/2-3}} \int \prod_{i=1}^{3} \mathrm{d}u_i \, \exp\left(-TQ\right)\,. \tag{6.14}$$

In (6.14), the integration domain $\{0 < u_1 < u_2 < u_3 < 1\}$, gives the box (6.5) ordered as (k_1, k_2, k_3, k_4), whereas the two other orderings are given by the integration domains $\{0 < u_2 < u_3 < u_1 < 1\}$ and $\{0 < u_3 < u_1 < u_2 < 1\}$.

6.2 Comparison of gauge theory integrands

Now we can repeat the same procedure for a box integral $I[n(\ell)]$ with a non-trivial numerator. Our BCJ box numerators (4.20) are quadratic in the four-dimensional loop momentum[16] and can be schematically written as:

$$n_{\text{box}}^{(\mathfrak{S})}(\ell) = A_{\mu\nu}\ell^\mu\ell^\nu + B_\mu^{(\mathfrak{S})}\ell^\mu\,, \tag{6.15}$$

where the label $\mathfrak{S}$ refers to one of the inequivalent orderings $\{(123), (231), (312)\}$. One can verify that the quadratic form $A_{\mu\nu}$ does not depend on the ordering. Note that we did not write the constant term in eq. (6.15), because there are none in our master BCJ boxes (4.20). The exponentiation produces an expression which depends both on Schwinger proper times and the shifted loop momentum:

$$n_{\text{box}}^{(\mathfrak{S})}(\tilde{\ell} + K) = A_{\mu\nu}\tilde{\ell}^\mu\tilde{\ell}^\nu + (2A_{\mu\nu}K^\nu + B_\mu^{(\mathfrak{S})})\tilde{\ell}^\mu + A_{\mu\nu}K^\mu K^\nu + B_\mu^{(\mathfrak{S})}K^\mu\,. \tag{6.16}$$

The linear term in $\tilde{\ell}$ integrates to zero in the gauge theory amplitude, but produces a non-vanishing contribution when squared in the gravity amplitude. Lorentz invariance projects $A_{\mu\nu}\tilde{\ell}^\mu\tilde{\ell}^\nu$ on its trace, which turns out to vanish in our ansatz:

$$\mathrm{tr}A = 0\,. \tag{6.17}$$

[16] We recall that the d- and four- dimensional loop momenta are related by $\ell_{(d)}^2 = \ell^2 - \mu^2$.

	Field theory	Worldline
Parameters	$\ell_{(d)}$	u_i, T
Numerator	$n(\ell)$	$W_X, \langle n\rangle$
Denominator	$\frac{1}{D(\ell)}$	e^{-TQ}
Power counting	$O(\ell^k)$	$O(u_i^k)$

Table 2. Basic ingredients of the loop integrand expressions in field theory and the field theory limit of string theory.

Then we define $\langle n_{\text{box}}^{(\mathfrak{S})}\rangle$ to be the result of the Gaussian integration over $\tilde{\ell}$:

$$\langle n_{\text{box}}^{(\mathfrak{S})}\rangle = A_{\mu\nu}K^\mu K^\nu + B_\mu^{(\mathfrak{S})}K^\mu \,. \tag{6.18}$$

Note that here and below, for definiteness and normalization, we use the bracket notation $\langle \ldots \rangle$ for integrand numerators in terms of the rescaled Schwinger proper times u_i so that $I[n]$ can be written in any integration-parameter space:

$$\int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{n(\ell)}{\ell_{(d)}^2(\ell_{(d)}-k_1)^2 \ldots (\ell_{(d)} - \sum_{i=1}^{n-1} k_i)^2} = \frac{(-1)^n i}{(4\pi)^{\frac{d}{2}}} \int_0^\infty \frac{\mathrm{d}T}{T^{\frac{d}{2}-(n-1)}} \int \prod_{i=1}^{n-1} \mathrm{d}u_i \, \langle n\rangle \, e^{-TQ} \,, \tag{6.19}$$

where the integration domain in u_i corresponds to the momentum ordering in the denominator. From the previous reasoning, it is easy to show the following dictionary: a polynomial $n(\ell)$ of degree k in the loop momentum is converted to a polynomial $\langle n\rangle$ of the same degree in Schwinger proper times:

$$n(\ell) = O(\ell^k) \quad \Rightarrow \quad \langle n\rangle = O\left(\frac{u^{k-2p}}{T^p}\right) , \quad p \geq 0 \tag{6.20}$$

where the inverse powers of T^p correspond to terms of the form $\tilde{\ell}^{2p}$ and both consistently act as dimension shifts, as it can be seen on the standard replacement rules given later in eq. (6.34). This is consistent with (5.15). We can recast the previous procedure in table 2, to summarize the common points between the worldline formalism and usual Feynman diagrams.

We apply this method to the BCJ box numerators, in order to compare them to the string-based numerator W_3. These two quantities have equal integrated contributions, as was noted before. However, at the integrand level, they turn out not to be equal. We denote their difference by δW_3:

$$\langle n_{\text{box}}^{(\mathfrak{S})}\rangle = W_3 + \delta W_3 \,. \tag{6.21}$$

By definition, δW_3 integrates to zero separately in each sector.

Making contact with the tree-level analysis, where the integrands had to be put in the particular MSS representation in string theory to ensure the manifest BCJ duality, one can wonder if this term δW_3 has a similar meaning at one loop. We note that the information that it carries, of order ℓ^2, is not trivial and is sensitive to the BCJ solution, since the quadratic terms in the box numerators (4.20) are fixed to comply with the kinematic Jacobi identities.

Therefore, δW_3 seems to be a messenger of the BCJ representation information and indicate a specific worldline representation of the string integrand. In order to be more precise on this statement, let us first rewrite δW_3 in terms of worldline quantities, i.e. as a polynomial in the worldline Green's functions. As it is of order u_i^2, it has to come from a polynomial with at most binomials of the form $\dot{G}_{ij}\dot{G}_{kl}$. By a brute-force ansatz, we have expressed δW_3 as a function of all possible quantities of that sort. Imposing the defining relation (6.21) in the three sectors results in a three-parameter space of possibilities for δW_3 (see the full expression (D.1) in the appendix). All consistency checks were performed on this numerator. At this level, the parameters α and β of the BCJ numerators (4.20), (4.21) are still free. It turns out that they create a non-locality in the worldline integrand, of the form tu/s^2. To cancel it, one has to enforce the condition

$$1-\alpha+\beta=0\,, \tag{6.22}$$

consistent with the choice (4.23). Below we provide a representative element of the family of δW_3's that we obtained from our ansatz:

$$\begin{aligned}\delta W_3=\frac{1}{2}\big(2\dot{G}_{12}^2-2\dot{G}_{34}^2-(\dot{G}_{23}-3\dot{G}_{14})(\dot{G}_{13}-\dot{G}_{24})+\dot{G}_{12}(\dot{G}_{23}-\dot{G}_{13}+3\dot{G}_{24}-3\dot{G}_{14})\\+\dot{G}_{34}(3\dot{G}_{14}-3\dot{G}_{13}-\dot{G}_{23}+\dot{G}_{24})\big)\,.\end{aligned} \tag{6.23}$$

In order to safely interpret δW_3 as a natural string-based object, it is important to verify that its string ancestor would not create any triangles in the field theory limit. We will refer to this condition as the "*string-ancestor-gives-no-triangles*" criterion. This is not a trivial property, and it can be used to rule out some terms as possible worldline objects (see, for example, the discussion in appendix E). In the present case, it was explicitly checked that the full form of δW_3 given in appendix D satisfies this property, following the procedure recalled in section 5.3.

Now that we have expressed δW_3 in this way, let us look back at what is the essence of the tree-level MSS approach. It is based on the fact that the correct tree-level form of the integrand is reached after a series of integration by parts.[17] One might hope that the worldline numerator defined by $W_3+\delta W_3$ is actually a result of application of a chain of integration by parts. Unfortunately, we have not found any sensible way in which the worldline numerator $W_3+\delta W_3$ could be obtained from W_3 by such a process. The reason for this is the presence of squares in δW_3, of the form $\dot{G}_{ij}^2$, which are not possible to eliminate by adjusting the free parameters of eq. (D.1). These terms are problematic for basically the same reason as at tree level, where, to integrate them out by parts, you always need a double derivative and a double pole to combine together. This can be seen at one loop by inspecting the following identity:

$$\partial_1\left(\dot{G}_{12}e^{-TQ}\right)=\ddot{G}_{12}+\dot{G}_{12}\left(k_1\cdot k_2\,\dot{G}_{12}+k_1\cdot k_3\,\dot{G}_{13}+k_1\cdot k_4\,\dot{G}_{14}\right)\,, \tag{6.24}$$

[17] In section 3 we did not have to perform any due to a sufficiently restrictive gauge choice.

where we see that the square $\dot{G}_{12}^2$ always goes in pair with the double derivative $\ddot{G}_{12}$. A similar equation with ∂_1 replaced with ∂_2 does not help, as the relative signs between the double derivative and the square are unchanged. This kind of identities show that, in the absence of double derivatives in δW_3, W_3 and $(W_3 + \delta W_3)$ are not related by a chain of integration by parts. The reason why we cannot include these double derivatives in our ansatz for δW_3 is because they would show up as $1/T$ terms in eq. (6.18) which is impossible in view of the tracelessness of $A_{\mu\nu}$, eq. (6.17).

Therefore, the introduction of δW_3 in the string integrand to make it change representation, although not changing the integrated result and satisfying this "string-ancestor-gives-no-triangles" property, appears to be a non-IBP process, in contrast with the MSS procedure. It would be interesting to understand if this property is just an artifact of our setup, or if it is more generally a sign that string theory does not obey the full generalized gauge invariance of the BCJ representation.

Finally, we note that δW_3 is not directly related to the BCJ triangles. Recall that they are defined through the BCJ color-kinematics duality and are crucial for the double copy construction of gravity. But in section 5.3, we saw that there are no triangles in our string-theoretic construction. So even though we find total derivatives both on the field theory side:[18]

$$\sum n_{\text{box}} + \sum n_{\text{tri}}, \tag{6.25}$$

and on the worldline side in the BCJ inspired form:

$$W_3 + \delta W_3 \,, \tag{6.26}$$

where the BCJ triangles and δW_3 integrate to zero, they cannot be made equal by going to proper-time variables, as[19]

$$\langle n_{\text{tri}} \rangle = 0 \,. \tag{6.27}$$

In addition to that, δW_3 is truly as a box integrand.

In any case, the important point for the next section is that both δW_3 and the BCJ triangles contribute to the gravity amplitude when squared. We will try to relate them to the new term W_2 that appears in gravity from left-right mixing terms.

6.3 Comparison of gravity integrands

The goal of this final section is to dissect the BCJ gravity numerators obtained by squaring the gauge theory ones in order to perform a thorough comparison with the string-based result. In particular, we wish to illustrate that the role of the left-right contractions is to provide the terms corresponding to the squares of the total derivatives in the loop momentum space (the BCJ triangles and the parity-odd terms).

[18] In eq. (6.25), we omitted the denominators for notational ease.
[19] See appendix C for more details on eq. (6.27).

6.3.1 String-based BCJ representation of the gravity integrand

At the level of integrals, we can schematically equate the gravity amplitude obtained from the two approaches:

$$\int \sum \langle n_{\text{box}}^2 \rangle + \sum \langle n_{\text{tri}}^2 \rangle = \int W_3^2 + W_2/2 \,, \tag{6.28}$$

where we omitted the integration measures and the factors of $\exp(-TQ)$. In order to relate the left-right contractions in W_2 to the triangles $\langle n_{\text{tri}}^2 \rangle$, we first need to consider the relationship between the squares W_3^2 and $\langle n_{\text{box}}^2 \rangle$ via $\langle n_{\text{box}} \rangle^2$, using the result of the previous section. From eq. (6.21), we know that at the gauge theory level, the integrands match up to a total derivative δW_3. Therefore, let us introduce by hand this term in the string-based gravity integrand:

$$W_3^2 + W_2/2 = \langle n_{\text{box}} \rangle^2 + W_2/2 - (2W_3 \delta W_3 + \delta W_3{}^2) \,. \tag{6.29}$$

The cost for this non-IBP change of parametrization is the introduction of a correction to W_2, that we call δW_2, in the string-based integrand:

$$\delta W_2 = -2(2W_3 \delta W_3 + \delta W_3^2) \,. \tag{6.30}$$

Note that this term *is not* a total derivative. The meaning of this correcting term is that, when we change W_3 to $\langle n_{\text{box}} \rangle$, we also have to modify W_2. In this sense, it is induced by the Jacobi relations in the gauge theory numerators n_{box}. Moreover, had we managed to do only integration by parts on W_3^2, W_2 would have received corrections due to the left-right contractions appearing in the process. These would show up as factors of $1/T$, as already explained below eq. (5.11).

Again, to be complete in the interpretation of δW_2 as a proper worldline object, we should make sure that it obeys the "string-ancestor-gives-no-triangles" criterion, as we did for δW_3. Since we have a symmetric construction for the gravity amplitude, it is natural to assume that both sectors would contribute equally to this string-theoretic correction:

$$\delta \mathcal{W}_2 = -2 \left(\mathcal{W}_3 \overline{\delta \mathcal{W}_3} + \overline{\mathcal{W}_3} \delta \mathcal{W}_3 + |\delta \mathcal{W}_3|^2 \right) \,. \tag{6.31}$$

Following the analysis of section 5.3, it is easy to convince oneself that since neither $\mathcal{W}_3$ nor $\delta \mathcal{W}_3$ gave any triangles in gauge theory, any combination thereof will not either.

Therefore, it seems legitimate to interpret δW_2 as a string-based correction, and this lets us rewrite the worldline numerator of the gravity amplitude as

$$\int \sum \langle n_{\text{box}}^2 \rangle + \sum \langle n_{\text{tri}}^2 \rangle = \int \sum \langle n_{\text{box}} \rangle^2 + (W_2 + \delta W_2)/2 \,. \tag{6.32}$$

6.3.2 Loop momentum squares vs. worldline squares

The next step is to relate $\langle n_{\text{box}}^2 \rangle$ to $\langle n_{\text{box}} \rangle^2$. Let us first look at the gravity box numerator. As before, it can be written as a function of the shifted loop momentum $\tilde{\ell}$:

$$\begin{aligned} \left(n_{\text{box}}(\tilde{\ell} + K) \right)^2 = & \ \tilde{\ell}^\mu \tilde{\ell}^\nu \tilde{\ell}^\rho \tilde{\ell}^\sigma A_{\mu\nu} A_{\rho\sigma} \\ & + \tilde{\ell}^\mu \tilde{\ell}^\nu \left((2A_{\mu\rho} K^\rho + B_\mu)(2A_{\nu\sigma} K^\sigma + B_\nu) + 2A_{\mu\nu}(A_{\rho\sigma} K^\rho K^\sigma + B_\rho K^\rho) \right) \\ & + (A_{\rho\sigma} K^\rho K^\sigma + B_\rho K^\rho)^2 \,, \end{aligned} \tag{6.33}$$

where we omitted the terms odd in $\tilde{\ell}$ since they integrate to zero. Notice, however, that the terms of n_{box} linear in $\tilde{\ell}$, which used to be total derivatives in gauge theory, now contribute to the integral, with the $\epsilon(k_1, k_2, k_3, \ell)^2$ terms inside them. To obtain the proper-time integrand $\langle n_{\text{box}}^2 \rangle$, we go again through the exponentiation procedure of section 6.1, followed by a dimension shift [60], together with the standard tensor reduction:[20]

$$\tilde{\ell}^\mu \tilde{\ell}^\nu \to -\frac{\eta^{\mu\nu}}{2T}, \tag{6.34a}$$

$$\tilde{\ell}^\mu \tilde{\ell}^\nu \tilde{\ell}^\rho \tilde{\ell}^\sigma \to \frac{\eta^{\mu(\nu}\eta^{\rho\sigma)}}{4T^2}, \tag{6.34b}$$

where $\eta^{\mu(\nu}\eta^{\rho\sigma)}$ stands for $\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}$. We obtain:

$$\langle n_{\text{box}}^2 \rangle = \frac{\eta^{\mu(\nu}\eta^{\rho\sigma)} A_{\mu\nu} A_{\rho\sigma}}{4T^2} - \frac{(2A_{\mu\nu}K^\nu + B_\mu)^2}{2T} + (A_{\rho\sigma}K^\rho K^\sigma + B_\rho K^\rho)^2\,, \tag{6.35}$$

or, equivalently, using (6.18),

$$\langle n_{\text{box}}^2 \rangle - \langle n_{\text{box}} \rangle^2 = \frac{\eta^{\mu(\nu}\eta^{\rho\sigma)} A_{\mu\nu} A_{\rho\sigma}}{4T^2} - \frac{(2A_{\mu\nu}K^\nu + B_\mu)^2}{2T}\,. \tag{6.36}$$

This formula describes precisely how squaring in loop momentum space is different from squaring in the Schwinger parameter space, so we will call the terms on the right-hand side of (6.36) *square-correcting terms*. Note that the fact that there are only $1/T^k$ with $k > 0$ terms on the right-hand side of eq. (6.36) is not accidental, and would have hold even without the tracelessness of A, eq. (6.17). It can indeed be seen in full generality that squaring and the bracket operation do commute at the level of the $O(T^0)$ terms, while they do not commute at the level of the $1/T^k$. Below we connect this with the structural fact that left-right contractions naturally yield $1/T$ terms. In appendix E, we also provide another description of these terms based on a trick which lets us rewrite the $1/T^2$ terms as worldline quantities.

6.3.3 Final comparison

Using eq. (6.32), we rewrite the contribution of $W_2 + \delta W_2$ at the integrated level as follows:

$$\int \frac{1}{2}(W_2 + \delta W_2) = \int \sum \langle n_{\text{tri}}^2 \rangle + \frac{\eta^{\mu(\nu}\eta^{\rho\sigma)} A_{\mu\nu} A_{\rho\sigma}}{4T^2} - \sum_{\mathfrak{S}} \frac{(2A_{\mu\nu}K^\nu + B_\mu^{(\mathfrak{S})})^2}{2T} \tag{6.37}$$

In total, we have argued that the total contribution on the left-hand side is a modification of W_2 generated by the BCJ representation of the gauge theory numerators in a non-IBP-induced way. This was supported by the aforementioned "string-ancestor-gives-no-triangles" criterion satisfied by δW_2. We are now able to state the conclusive remarks on our interpretation of eq. (6.37). Its right-hand side is made of two parts, of different physical origin:

[20] Remember that the numerator loop momentum $\tilde{\ell}$ is strictly four-dimensional and integration is over the d-dimensional $\tilde{\ell}_{(d)}$.

– the squares of gauge theory BCJ triangles,

– the square-correcting terms.

Note that some of the latter come from the contributions of the gauge theory integrand which were linear in the loop momentum, including the parity-odd terms $\epsilon_{\mu\nu\rho\sigma}k_1^\nu k_2^\rho k_3^\sigma$ present in $B^{(\mathfrak{S})}$.

Formula (6.37) shows clearly the main observation of this paper: the squares of the total derivatives introduced into the gravity amplitude by the BCJ double copy construction physically come from the contractions between the left- and right-moving sectors in string theory. At a more technical level, the contribution of these contractions to the string-based integrand also had to be modified to take into account for the BCJ representation of the gauge theory amplitudes.

This being said, the presence of the square-correcting terms on the right-hand side deserves a comment. They contain the dimension-shifting factors of $1/T$, characteristic of the left-right contractions, as already mentioned. It is therefore not surprising, that the square-correcting terms show up on the right-hand side of eq. (6.37), since the left-hand side is the (BCJ modified) contribution of the left-right contractions.

More interestingly, this seems to suggest that it should be possible to absorb them into the left-right mixing terms systematically by doing IBP's at the string theory level. However, if one considers the worldline polynomials corresponding to $(2AK+B)/T$, they imply a string-theoretic ancestor of the form $\partial\bar{\partial}\mathcal{G}\times\sum\partial\mathcal{G}\overline{\partial\mathcal{G}}$ which eventually does not satisfy the "string-ancestor-gives-no-triangles" criterion.[21] Therefore, not all of the square-correcting terms possess a nice worldline interpretation, and this makes the situation not as simple as doing IBP's. This fact is to be connected with the impossibility to obtain the BCJ worldline gauge theory numerator $W_3+\delta W_3$ by integrating by parts W_3 in our setup.

Perhaps the main obstacle here is that the vanishing of the BCJ triangles after integration does not exactly correspond to the absence of string-based triangles before integration. All of this suggests that there might exist BCJ representations which cannot be obtained just by doing integrations by parts. The characterization thereof in terms of the subset of the generalized gauge invariance respected by string theory would be very interesting. For instance, it might be that our choice to put all the BCJ bubbles to zero, allowed by the generalized gauge invariance, is not sensible in string theory for this particular amplitude with the gauge choice (5.21).

Notwithstanding, we believe that the main observations of our paper concerning the observation that the BCJ representation *can be seen* in string theory and the physical origin of the squares of total derivatives and observation that the BCJ construction is related to the presence of left-right mixing terms in string theory holds very generally.

7 Discussion and outlook

In this paper, we have studied various aspects of the BCJ double copy construction. At tree level, we used the MSS chiral block representation both in heterotic and type II closed

[21] We have checked that the arising triangles are not the same as our BCJ triangles squared.

strings to rewrite the five-point field theory amplitudes in a way in which color factors can be freely interchanged with kinematic numerators to describe scattering amplitudes of cubic color scalars, gluons or gravitons. In this context, the Jacobi identities of [21] appear as consequences of the MSS representation and are on the same footing as the equivalence between color and kinematics. In particular, we did not have to use them to write down the final answer. Working out the n-point combinatorics in the lines of our five-point example would constitute a new direct proof of the color-kinematics duality at tree level.

At one loop, we performed a detailed analysis of four-point amplitudes in $\mathcal{N} = 4$ supergravity from the double copy of two $\mathcal{N}=2$ SYM theories, both in field theory and the worldline limit of string theory. This symmetric construction automatically requires adding two matter vectors multiplets to the gravity spectrum. Our choice of the BCJ ansatz for which the BCJ bubbles were all set to zero is an effective restriction of the full generalized gauge invariance of the BCJ numerators. We focused on the non-trivial loop-momentum structure of the BCJ gauge theory integrands, which we expressed as worldline quantities to make comparison with the string-based ones.

The major drawback of this procedure is that, in the process, one loses some of the information contained in the loop-momentum gauge theory numerators. For example, our BCJ gauge theory triangles turned out to vanish after integration in this procedure, so one could think that they are invisible for string theory. However, the box numerators match the string-based numerator up to a new term that we called δW_3. This term integrates to zero in each kinematic channel, thus guaranteeing matching between the two approaches. This total derivative δW_3 shifts the string-based integrand to the new representation $W_3 + \delta W_3$. We argued that this process is not IBP-induced, in the sense that $W_3 + \delta W_3$ cannot be obtained simply by integrating W_3 by parts. We gave a possible clue to interpret this puzzle, based on the fact that the restriction of the generalized gauge invariance might be incompatible with string theory in the gauge choice (5.21). It would be interesting to investigate this point further.

At the gravity level, we wanted to argue that the characteristic ingredients of the BCJ double copy procedure, namely the squares of terms required by the kinematic Jacobi identities, are generated in string theory by the left-right contractions.

The first observation is that, going to the non-IBP-induced representation $W_3 \to W_3 + \delta W_3$ in the string-based integrand induces a modification of the left-right mixing terms, $W_2 \to W_2 + \delta W_2$, which can be safely interpreted as a worldline quantity, because it obeys the "string-ancestor-gives-no-triangles" criterion.

Furthermore, the difference between squaring in loop momentum space and in Schwinger proper time space induces the square-correcting terms. We related them to $W_2 + \delta W_2$ and observed that they are of the same nature as the left-right mixing terms in string theory. Such terms are generically obtained from IBP's, which suggests that the right process (if it exists) in string theory to recover full BCJ construction makes use of worldsheet integration by part, just like in the MSS construction at tree level. However, these square-correcting terms do not obey the "string-ancestor-gives-no-triangles" property, which makes them ill-defined from the string-theoretic point of view. We suppose that the issues of the non-IBP nature of δW_2 and δW_3 might come from the incompatibility between our restriction of generalized gauge invariance and our string-based computation in the gauge (5.21).

In any case, this shows that string theory has something to say about generalized gauge invariance. We believe that this opens very interesting questions related to the process of finding BCJ numerators by the ansatz approach and to a possible origin of this generalized gauge invariance in string theory.

Finally, we present the bottom line of our paper in the formula (6.37): we identified a representation of the left-right mixing terms in which they are related to the squares of the BCJ triangles and the squares of parity-odd terms $(i\epsilon_{\mu\nu\rho\sigma}k_1^\mu k_2^\nu k_3^\rho \ell^\sigma)^2$. Besides the previous discussion on the nature of the square-correcting terms in the right-hand side of eq. (6.37), we believe this sheds some light on the a priori surprising fact that total derivatives in the BCJ representation of gauge theory amplitudes play such an important role. The physical reason is deeply related to the structure of the closed string: in the heterotic string, the left-moving sector does not communicate with the right-moving one in gluon amplitudes, while this happens naturally in gravity amplitudes in the type II string and generates new terms, invisible from the gauge theory perspective.

Concerning further developments, in addition to the open issues that we already mentioned, it would be natural to explore the possibility that the MSS chiral blocks might be generalized to loop level amplitudes and understand the role of δW_3 and generalized gauge invariance in this context. For that, one would have to account for the left-right mixing terms, generated naturally by worldsheet integration by parts, which must play a central role starting from one loop. Such an approach, if it exists, would amount to disentangling the two sectors in a certain sense, and a string-theoretic understanding of such a procedure would be definitely very interesting.

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A Integrals

We adopt the conventional definition [90, 91] for dimensionally-regularized massless scalar integrals:

$$I_n^d = (-1)^{n+1}(4\pi)^{\frac{d}{2}} i \int \frac{\mathrm{d}^d \ell_{(d)}}{(2\pi)^d} \frac{1}{\ell_{(d)}{}^2(\ell_{(d)}-k_1)^2 \dots (\ell_{(d)}+k_n)^2}\,, \tag{A.1}$$

where by default $d = 4 - 2\epsilon$. Here we give only the integrals, relevant for this paper, i.e. the zero-mass box:

$$I_4(s,t) = \frac{2r_\Gamma}{st}\left\{\frac{1}{\epsilon^2}\left((-s)^{-\epsilon} + (-t)^{-\epsilon}\right) - \frac{1}{2}\left(\ln^2\left(\frac{-s}{-t}\right) + \pi^2\right)\right\} + O(\epsilon)\,, \tag{A.2}$$

and the one-mass triangle:

$$I_3(t) = \frac{r_\Gamma}{\epsilon^2}\frac{(-t)^{-\epsilon}}{(-t)}\,, \tag{A.3}$$

where

$$r_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,. \tag{A.4}$$

We also encounter six-dimensional versions of these integrals:

$$I_4^{d=6-2\epsilon}(s,t) = -\frac{r_\Gamma}{2(1-2\epsilon)(s+t)}\left(\ln^2\left(\frac{-s}{-t}\right) + \pi^2\right) + O(\epsilon)\,, \tag{A.5}$$

$$I_3^{d=6-2\epsilon}(t) = \frac{r_\Gamma}{2\epsilon(1-\epsilon)(1-2\epsilon)}(-t)^{-\epsilon}\,, \tag{A.6}$$

as well as the eight-dimensional box:

$$I_4^{d=8-2\epsilon}(s,t) = \frac{r_\Gamma}{(3-2\epsilon)(s+t)}\left(\frac{st}{2}I_4^{d=6-2\epsilon}(s,t) + sI_3^{d=6-2\epsilon}(s) + tI_3^{d=6-2\epsilon}(t)\right) + O(\epsilon)\,. \tag{A.7}$$

B Five-point tree-level numerators

The chiral correlator (3.14) produces the following sum of ten terms in the MHV gauge choice of (3.16):

$$\begin{aligned}
&-\frac{(\varepsilon_3\varepsilon_4)(\varepsilon_1k_4)(\varepsilon_2k_3)(\varepsilon_5k_3)}{8z_{23}z_{34}} - \frac{(\varepsilon_3\varepsilon_4)(\varepsilon_1k_4)(\varepsilon_2k_4)(\varepsilon_5k_3)}{8z_{24}z_{34}} + \frac{(\varepsilon_3\varepsilon_5)(\varepsilon_1k_3)(\varepsilon_2k_3)(\varepsilon_4k_1)}{8z_{13}z_{23}}\\
&+\frac{(\varepsilon_3\varepsilon_5)(\varepsilon_1k_3)(\varepsilon_2k_4)(\varepsilon_4k_1)}{8z_{13}z_{24}} + \frac{(\varepsilon_3\varepsilon_5)(\varepsilon_1k_4)(\varepsilon_2k_3)(\varepsilon_4k_3)}{8z_{23}z_{34}} - \frac{(\varepsilon_3\varepsilon_4)(\varepsilon_1k_3)(\varepsilon_2k_3)(\varepsilon_5k_4)}{8z_{13}z_{23}z_{34}}\\
&-\frac{(\varepsilon_3\varepsilon_5)(\varepsilon_1k_3)(\varepsilon_2k_3)(\varepsilon_4k_3)}{8z_{13}z_{23}z_{34}} + \frac{(\varepsilon_3\varepsilon_5)(\varepsilon_1k_4)(\varepsilon_2k_4)(\varepsilon_4k_3)}{8z_{24}z_{34}} - \frac{(\varepsilon_3\varepsilon_4)(\varepsilon_1k_3)(\varepsilon_2k_4)(\varepsilon_5k_4)}{8z_{13}z_{24}z_{34}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \frac{(\varepsilon_3\varepsilon_5)(\varepsilon_1k_3)(\varepsilon_2k_4)(\varepsilon_4k_3)}{8z_{13}z_{24}z_{34}}\,.
\end{aligned} \tag{B.1}$$

This particular gauge choice killed all double poles. By using the partial fraction identities of the form (3.17) to obtain the MSS chiral block representation.

The Jacobi identities satisfied by the numerators of section 3 are:

$$\begin{aligned}
n_3^{(\mathrm{L/R})} - n_5^{(\mathrm{L/R})} + n_8^{(\mathrm{L/R})} &= 0,\\
n_3^{(\mathrm{L/R})} - n_1^{(\mathrm{L/R})} + n_{12}^{(\mathrm{L/R})} &= 0,\\
n_{10}^{(\mathrm{L/R})} - n_{11}^{(\mathrm{L/R})} + n_{13}^{(\mathrm{L/R})} &= 0,\\
n_4^{(\mathrm{L/R})} - n_2^{(\mathrm{L/R})} + n_7^{(\mathrm{L/R})} &= 0,
\end{aligned}$$

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$$
\begin{aligned}
n_4^{(\mathrm{L/R})} - n_1^{(\mathrm{L/R})} + n_{15}^{(\mathrm{L/R})} &= 0, \\
n_{10}^{(\mathrm{L/R})} - n_9^{(\mathrm{L/R})} + n_{15}^{(\mathrm{L/R})} &= 0, \\
n_8^{(\mathrm{L/R})} - n_6^{(\mathrm{L/R})} + n_9^{(\mathrm{L/R})} &= 0, \\
n_5^{(\mathrm{L/R})} - n_2^{(\mathrm{L/R})} + n_{11}^{(\mathrm{L/R})} &= 0, \\
(n_7^{(\mathrm{L/R})} - n_6^{(\mathrm{L/R})} + n_{14}^{(\mathrm{L/R})} &= 0)\,.
\end{aligned}
\tag{B.2}
$$

where the last one is a linear combination of the others.

The $(2n-5)!!$ color factors are obtained from the six ones of (3.12) plugged in (3.27) and give rise to the expected result:

$$
\begin{aligned}
c_1 &= f^{12b} f^{b3c} f^{c45}\,, & c_2 &= f^{23b} f^{b4c} f^{c51}\,, & c_3 &= f^{34b} f^{b5c} f^{c12}\,, \\
c_4 &= f^{45b} f^{b1c} f^{c23}\,, & c_5 &= f^{51b} f^{b2c} f^{c34}\,, & c_6 &= f^{14b} f^{b3c} f^{c25}\,, \\
c_7 &= f^{32b} f^{b5c} f^{c14}\,, & c_8 &= f^{25b} f^{b1c} f^{c43}\,, & c_9 &= f^{13b} f^{b4c} f^{c25}\,, \\
c_{10} &= f^{42b} f^{b5c} f^{c13}\,, & c_{11} &= f^{51b} f^{b3c} f^{c42}\,, & c_{12} &= f^{12b} f^{b4c} f^{c35}\,, \\
c_{13} &= f^{35b} f^{b1c} f^{c24}\,, & c_{14} &= f^{14b} f^{b2c} f^{c35}\,, & c_{15} &= f^{13b} f^{b2c} f^{c45}\,.
\end{aligned}
\tag{B.3}
$$

C Integrating the triangles

The BCJ triangle numerators (4.21) are linear in the loop momentum, so if we apply to them the exponentiation procedure of section 6.1, we get the following terms

$$
n_{\text{tri}}(\tilde{\ell} + K) \propto B_\mu(\tilde{\ell}^\mu + K^\mu) + C\,, \tag{C.1}
$$

where $K = -\sum u_i k_i$. The linear term linear integrates to zero by parity and the constant term $BK+C$ vanishes for each triangle numerator. For example, for the numerator (4.21c),

$$
B^\mu = -s k_3^\mu + t k_1^\mu - u k_2^\mu + \frac{4iu}{s} k_{1\mu_1} k_{2\mu_2} k_{3\mu_3} \epsilon^{\mu_1\mu_2\mu_3\mu}\,, \qquad C = su\,. \tag{C.2}
$$

So it can be easily checked that

$$
B_\mu K^\mu = u_1 su - u_4 su\,. \tag{C.3}
$$

Taking into account that $u_4 = 1$ and that the particular triangle (4.21c) is obtained from the worldline box parametrization by setting $u_1 = 0$, we indeed obtain $BK = -su = -C$.

Moreover, in the gravity amplitude, the triangle numerators squared become simply:

$$
n_{\text{tri}}^2(\tilde{\ell} + K) \propto B_\mu B_\nu \tilde{\ell}^\mu \tilde{\ell}^\nu\,. \tag{C.4}
$$

The standard tensor reduction transforms $\tilde{\ell}^\mu \tilde{\ell}^\nu$ to $\ell^2 \eta^{\mu\nu}/4$, which is known to induce a dimension shift [60] from $d = 4 - 2\epsilon$ to $d = 6 - 2\epsilon$. As a result, in the double copy construction the BCJ triangles produce six-dimensional scalar triangle integrals (A.6) with the coefficients (E.7).

D Explicit expression of δW_3

In section 6.2, we expressed δW_3 in terms of $\dot{G}$'s:

$$\begin{aligned}
\delta W_3 \;=\; & \frac{1}{2}\Big((1+2\alpha-2A_1-2A_2)(\dot{G}_{14}^2-\dot{G}_{23}^2)-2(1+A_1)\dot{G}_{12}^2-2(1-A_1)\dot{G}_{34}^2 \\
& \quad -2A_3(\dot{G}_{13}-\dot{G}_{14}-\dot{G}_{23}+\dot{G}_{24})(\dot{G}_{13}-\dot{G}_{14}+\dot{G}_{23}-\dot{G}_{24}+2\dot{G}_{34}) \\
& \quad +2(1-2\alpha+2A_1+A_2)(\dot{G}_{12}\dot{G}_{14}-\dot{G}_{12}\dot{G}_{24}+\dot{G}_{14}\dot{G}_{24}) \\
& \quad -2(2+2\alpha-2A_1-A_2)(\dot{G}_{23}\dot{G}_{34}-\dot{G}_{24}\dot{G}_{34}-\dot{G}_{23}\dot{G}_{24}) \\
& \quad +2(1-A_2)(\dot{G}_{13}\dot{G}_{34}-\dot{G}_{14}\dot{G}_{34}-\dot{G}_{13}\dot{G}_{14}) \\
& \quad -2A_2(\dot{G}_{12}\dot{G}_{13}-\dot{G}_{12}\dot{G}_{23}+\dot{G}_{13}\dot{G}_{23})\Big) \\
& -\frac{tu}{s^2}(1-\alpha+\beta)\Big(\dot{G}_{14}^2-\dot{G}_{23}^2-2\dot{G}_{12}\dot{G}_{14}+2\dot{G}_{12}\dot{G}_{24}-2\dot{G}_{14}\dot{G}_{24} \\
& \qquad +2\dot{G}_{23}\dot{G}_{24}-2\dot{G}_{23}\dot{G}_{34}+2\dot{G}_{24}\dot{G}_{34}\Big),
\end{aligned} \tag{D.1}$$

where α and β are the free parameters of the BCJ ansatz, and A_1, A_2 and A_3 are those from matching to a string-inspired ansatz.

E Trick to rewrite the square-correcting terms

In this appendix, we use a trick to partly rewrite the square-correcting terms (6.36) as string-based quantities. This section is mostly provided here for the interesting identity (E.9) which relates the BCJ triangles to the quadratic part of the box numerators.

First, we introduce a new element in the reduction technique. Recall that factors of $1/T^k$ modify the overall factor $1/T^{d/2-(n-1)}$ and thus act as dimension shifts $d \to d+2k$. Therefore, $(2A_{\mu\nu}K^\nu + B_\mu)^2/(2T)$ is the numerator of a six-dimensional worldline box.

However, we choose to treat the $1/T^2$ differently. Since $A_{\mu\nu}$ does not depend on the ordering, we can rewrite the $1/T^2$ square-correcting term as a full worldline integral

$$\frac{i}{(4\pi)^{\frac{d}{2}}}\frac{\eta^{\mu(\nu}\eta^{\rho\sigma)}A_{\mu\nu}A_{\rho\sigma}}{4}\int_0^\infty \frac{\mathrm{d}T}{T^{\frac{d}{2}-3}}\int \mathrm{d}^3u\frac{e^{-TQ}}{T^2}\,, \tag{E.1}$$

where the proper-time domain in u_i contains all three inequivalent box orderings. Now let us consider the second derivative of the worldline propagator

$$\ddot{G}_{ij} = \frac{2}{T}\left(\delta(u_{ij})-1\right)\,, \tag{E.2}$$

to obtain a useful identity valid for any i,j,k,l:

$$\frac{1}{T^2} = \frac{1}{4}\ddot{G}_{ij}\ddot{G}_{kl} + \frac{1}{T^2}\left(\delta(u_{ij})+\delta(u_{kl})\right) - \frac{1}{T^2}\delta(u_{ij})\delta(u_{kl})\,. \tag{E.3}$$

The factors of $1/T^2$ combine with delta-functions and thus properly change the number of external legs and dimensions, such that from the right-hand side of (E.3), we can read off the following integrals: a four-dimensional worldline box with numerator $\ddot{G}_{ij}\ddot{G}_{kl}$, two six-dimensional scalar triangles and a four-dimensional scalar bubble. Since we are free to choose indices i,j,k,l, we can as well use a linear combination of the three several choices, as long as we correctly average the sum. For instance, we can now create s-, t- and u-channel six-dimensional scalar triangles (along with four-dimensional scalar bubbles), if we choose $(i,j,k,l) \in \{(1,2,3,4),(1,4,2,3),(1,3,2,4)\}$ and sum over them with coefficients λ_s, λ_t and λ_u:

$$\int_0^\infty \frac{\mathrm{d}T}{T^{\frac{d}{2}-3}} \int \mathrm{d}^3u \frac{e^{-TQ}}{T^2} = \Bigg[\frac{1}{4} \int_0^\infty \frac{\mathrm{d}T}{T^{\frac{d}{2}-3}} \int \mathrm{d}^3u \left(\lambda_s \ddot{G}_{12}\ddot{G}_{34} + \lambda_t \ddot{G}_{14}\ddot{G}_{23} + \lambda_u \ddot{G}_{13}\ddot{G}_{24} \right) e^{-TQ} \\ + 2 \sum_{c=s,t,u} \lambda_c I_3^{6d}(c) - \frac{1}{2} \sum_{c=s,t,u} \lambda_c I_2^{4d}(c) \Bigg] \frac{1}{\sum_{c=s,t,u} \lambda_c}, \tag{E.4}$$

This expression is written at the integrated level, in order to be completely explicit with the subtle normalizations. In particular, we took into account that the scalar triangles coming from $\delta(u_{12})$ and $\delta(u_{34})$ depend only on s and have equal integrated contributions. We summed over the three orderings and used the fact that $\delta(u_{ij})$ generate factors of 1/2 due to always acting on the border of their proper-time domains.

To sum up, the term $\eta^{\mu(\nu}\eta^{\rho\sigma)} A_{\mu\nu} A_{\rho\sigma}/(4T^2)$ in (E.1) produces the three following scalar triangles:

$$\frac{i}{(4\pi)^{\frac{d}{2}}} \frac{\eta^{\mu(\nu}\eta^{\rho\sigma)} A_{\mu\nu} A_{\rho\sigma}}{4\sum_{c=s,t,u} \lambda_c} \sum_{c=s,t,u} (2\lambda_c) I_3^{6d}(c), \tag{E.5}$$

in addition to the four-dimensional non-scalar boxes and scalar bubbles.

At this point, let us recall that the BCJ gravity amplitude contains triangles that integrate to six-dimensional scalar triangles:

$$\sum \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{n_{\text{tri}}^2(\ell)}{D_{\text{tri}}(\ell)} = -\frac{i}{(4\pi)^{\frac{d}{2}}} \sum_{c=s,t,u} (\lambda_{c_1} + \lambda_{c_2}) I_3^{6d}(c), \tag{E.6}$$

with coefficients:

$$\begin{aligned} \lambda_{s_1} &= \frac{((1-\alpha)s^2 + 2(1-\alpha+\beta)tu)^2}{8s^4tu}, \\ \lambda_{s_2} &= \frac{((1-\alpha)s^2 - 2(1-\alpha+\beta)tu)^2}{8s^4tu}, \\ \lambda_{t_1} &= \lambda_{t_2} = \frac{s^2-4u^2}{8s^3u}, \\ \lambda_{u_1} &= \lambda_{u_2} = \frac{s^2-4t^2}{8s^3t}. \end{aligned} \tag{E.7}$$

Therefore, we can try to match the actual coefficient of the BCJ triangles in (E.6) by choosing different fudge factors λ_s, λ_t and λ_u in (E.5) for the triangles generated by the reduction of the $1/T^2$ term as follows: (E.6):

$$\lambda_c = \lambda_{c_1} + \lambda_{c_2}, \qquad c = s,t,u. \tag{E.8}$$

This choice implies that for any values of α and β

$$\frac{\eta^{\mu(\nu}\eta^{\rho\sigma)} A_{\mu\nu} A_{\rho\sigma}}{4\sum_{c=s,t,u} \lambda_c} = 1. \tag{E.9}$$

This lets us carefully relate the scalar-triangle contributions (E.5) coming from the square-correcting terms (E.1) to be equal to the (-2) times the BCJ triangles squared.[22]

This seeming coincidence deserves a few comments. We defined $A_{\mu\nu}$ as the coefficient of $\ell^\mu\ell^\nu$ in the BCJ box numerators (4.20), but in principle, we know that the boxes could have been made scalar in the scalar integral basis, as in (4.7). To comply with the kinematic Jacobi identities, the BCJ color-kinematics duality reintroduces ℓ^2 into the boxes by shuffling them with the scalar triangles and bubbles. In our final BCJ construction, we set bubble numerators to zero, so the information that was inside the original scalar triangles and bubbles was equally encoded in the dependence of the BCJ box and triangle numerators on the loop momentum. This is why the coincidence between the $A_{\mu\nu}$ and λ_c is not miraculous.

Finally, we can rewrite eq. (6.37) using our trick:

$$\int \frac{1}{2}(W_2+\delta W_2) = \int \bigg\{ -\sum \langle n_{\text{tri}}^2\rangle - \sum_{\mathfrak{S}} \frac{(2A_{\mu\nu}K^\nu + B_\mu^{(\mathfrak{S})})^2}{2T} \\ + \frac{1}{4}\Big(\lambda_s \ddot{G}_{12}\ddot{G}_{34} + \lambda_t \ddot{G}_{14}\ddot{G}_{23} + \lambda_u \ddot{G}_{13}\ddot{G}_{24}\Big) - \frac{1}{T^2}\left(\lambda_s\delta_{12}\delta_{34} + \lambda_t\delta_{14}\delta_{23} + \lambda_u\delta_{13}\delta_{24}\right)\bigg\}. \tag{E.10}$$

We could not apply the same trick to the $1/T$ square-correcting terms because they do not seem to have a nice string-theoretic interpretation with respect to the "string-ancestor-gives-no-triangles" criterion. More precisely, we expressed it as a worldline polynomial by the same ansatz method that we used to determine the expression of δW_3, and observed explicitly that this term does not satisfy this criterion, i.e. it creates triangles in the field theory limit. Moreover, we checked the non-trivial fact that the coefficients of these triangles cannot be made equal to these of the BCJ triangles.

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[22] Recall that the overall normalization of λ_c's was irrelevant in eq. (E.4), so the factor (-2) is fixed and prevents us from completely eliminating the triangles.

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