Stochastic Variational Inequalities and Applications to Random Vibrations and Mechanical Structures

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Inéquations variationnelles stochastiques et applications aux vibrations de structures mécaniques

THÈSE

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Contents

Chapter 4
Asymptotic analysis of stochastic variational inequalities with vanishing jumps
submitted to Asymptotic Analysis

Chapter 5
Degenerate Dirichlet problems related to the ergodic theory for an elasto-plastic oscillator excited by a filtered white noise
submitted to IMA Journal of Applied Mathematics

Introduction et présentation des travaux

Cette thèse traite des inéquations variationnelles stochastiques (IVS) et de leurs applications aux vibrations de structures mécaniques.

Ce travail a été initié par Alain Bensoussan dans le cadre d'une collaboration avec Laurent Borsoi et Cyril Feau (Commissariat à l'Énergie Atomique et aux Énergies Alternatives, CEA). Ces derniers lui ont présenté le modèle de l'oscillateur élastique-parfaitement-plastique (EPP) excité par un bruit blanc qu'ils utilisent pour traduire les comportements globaux de structures mécaniques. Depuis quelques décennies, une importante littérature en sciences de l'ingénieur s'est développée autour de ce sujet. Récemment, Alain Bensoussan et Janos Turi ont découvert un domaine riche et varié pour l'application des mathématiques mettant en jeu de nouveaux processus stochastiques, une nouvelle classe d'équations aux dérivées partielles (EDPs) avec des conditions non-locales, ainsi que de nouvelles méthodes numériques pour leur résolution grâce au contact d'Olivier Pironneau. De plus, la théorie mathématique s'avère concrètement applicable à l'étude de la fatigue des matériaux et à l'étude du risque de défaillance des structures mécaniques.

La collaboration avec le CEA s'est remarquablement bien établie et la matérialisation de nos résultats en outils pour les ingénieurs s'est progressivement développée tout en dépassant nos barrières culturelles respectives. En effet, l'ambition commune d'aboutir aux applications s'est traduite par une forte volonté de proposer un cadre mathématique bien établi et pertinent aux yeux de l'ingénieur. Nous avons remarqué que l'on ne pouvait pas simplement utiliser les résultats mathématiques généraux pour traiter les questions d'applications à la fatigue ou à la fiabilité des structures. Un traitement mathématique intermédiaire s'est alors avéré nécessaire pour relier les résultats mathématiques généraux aux applications. Au contact de Stéphane Menozzi, Maître de conférences à l'université Paris 7, et de Hector Jasso-Fuentes, Assistant-professor à CINVESTAV, Mexico, d'importantes questions sur la nature probabiliste de la modélisation ont été résolues. Nous nous sommes notamment intéréssés à l'alternance des régimes élastique et plastique. Nous nous sommes rendus compte que l'abstraction mathématique du bruit blanc a des conséquences importantes qui peuvent ne pas être réalistes pour les applications. Pour rendre compte du phénomène physique, il a fallu alors étudier ces artefacts mathématiques.

La structure du manuscrit se décompose selon plusieurs axes de recherche :

- Le premier chapitre concerne l'étude d'un algorithme numérique déterministe alternatif à la méthode de Monte-Carlo pour le calcul de la mesure invariante de la vitesse et de la composante élastique de l'oscillateur EPP excité par un bruit blanc. Dans un chapitre reporté en annexe à la fin du manuscrit, nous appliquons cet algorithme à l'étude de la composante élastique dans le cadre d'une étude empirique. Basé sur l'expérimentation numérique, nous proposons un critère qui peut être utile à l'ingénieur pour estimer les fréquences et les statistiques de la déformation plastique.

- Le second chapitre est consacré à l'analyse des EDPs associées aux séquences qui, dans la trajectoire, contiennent une seule phase élastique et une seule phase plastique; nous les appelons *cycles courts*. Dans ce contexte, nous donnons une nouvelle preuve de l'existence et de l'unicité d'un état d'équilibre invariant en loi pour la dynamique du processus, vers lequel le système converge en temps grand : la propriété ergodique. De plus, nous proposons une caractérisation analytique de la mesure invariante.

- Ensuite, dans le troisième chapitre, la notion de *cycles longs* indépendants est introduite pour décrire la variance de la déformation totale. Cette dernière est une grandeur importante pour évaluer le risque de défaillance d'une structure mécanique. Par ailleurs, les ingénieurs ont découvert par des moyens empiriques qu'elle croît linéairement avec le temps. Notre contribution concerne la preuve de ce résultat et la caractérisation rigoureuse du coefficient de croissance linéaire.

- Dans le quatrième chapitre, nous étudions un modèle comportant des sauts aux instants de transition de l'état plastique vers l'état élastique. Dans ce cadre, la séparation temporelle des deux états est bien établie. Lorsque la taille du saut tend vers 0, nous justifions la convergence du modèle à sauts vers celui sans saut sur un intervalle de temps fini.

- Le cinquième chapitre élargit nos investigations sur les IVS à la propriété ergodique de l'oscillateur EPP excité par un bruit blanc filtré. Nous montrons la propriéte ergodique en suivant une démarche analogue au cas où l'excitation est un bruit blanc standard. Cela étend la méthode proposée par Bensoussan et Turi au cas de la dimension supérieure. Les conditions de bord non-locales exprimées sous forme d'équations différentielles en dimension 1 sont remplacées par des équations elliptiques en dimension 2.

1 Motivations de l'ingénieur : Analyse du risque de défaillance

Les vibrations aléatoires constituent l'un des risques majeurs de défaillance pour des structures mécaniques telles que des bâtiments, des ponts ou des centrales nucléaires. Ainsi, par exemple, le dimensionnement des installations nucléaires est conditionné, entre autres, par la prise en compte du danger sismique. L'action sismique est un mouvement vibratoire du sol issu de la propagation d'une perturbation ayant pris naissance à l'intérieur de l'écorce terrestre. A cause de son caractère imprévisible, il correspond à un phénomène aléatoire. En génie parasismique, un indicateur pour la prédiction de défaillance est établi par des méthodes issues des sciences de l'ingénieur provenant de calculs non-linéaires en mécanique probabiliste.

Les méthodes probabilistes en mécanique permettent de déterminer la probabilité de défaillance en fonction du temps en caractérisant les statistiques de la réponse d'une structure soumise à une excitation aléatoire. Dans la plupart des cas, il n'existe pas d'expression explicite et les méthodes numériques s'imposent pour calculer les grandeurs qui intéressent l'ingénieur. Pour les structures composées d'un grand nombre de degrés de libertés (d.d.l.), une telle caractérisation nécessite la réalisation de calculs dynamiques temporels non-linéaires en nombre important. Cela rend la méthode coûteuse en temps.

Cependant, pour une classe de structures mécaniques répondant principalement sur leur premier mode de vibration, l'étude peut se porter vers une modélisation élémentaire de type oscillateur à un d.d.l. Dans ce contexte, il s'agit de modèles aussi bien simples que représentatifs du comportement élastique-plastique. Par exemple, les tronçons de tuyauteries font partie des installations industrielles qui rentrent dans cette classe (voir Figures 1,2).

Figure 1: ***Vibration d'un tronçon de tuyauterie :*** *Illustration de la vibration d'un tronçon de ligne de tuyauterie sur une table d'expérimentation du CEA (Laboratoire d'Études de Mécanique SIsmique,EMSI). La structure est installée sur un plateau qui simule l'action sismique. La réponse de la structure à cette sollicitation présente une alternance entre les phases de déformation élastique et les phases de déformation plastique. Ici, la déformation permanente se manifeste aux coudées basses de la tuyauterie.*

En conséquence, une importante littérature en sciences de l'ingénieur s'est développée depuis quelques décennies sur l'étude des oscillateurs non-linéaires excités par un bruit blanc et leurs applications (voir [10, 13, 14, 15, 16, 28, 19, 24, 26, 12, 29, 30, 32, 34, 33, 35]).

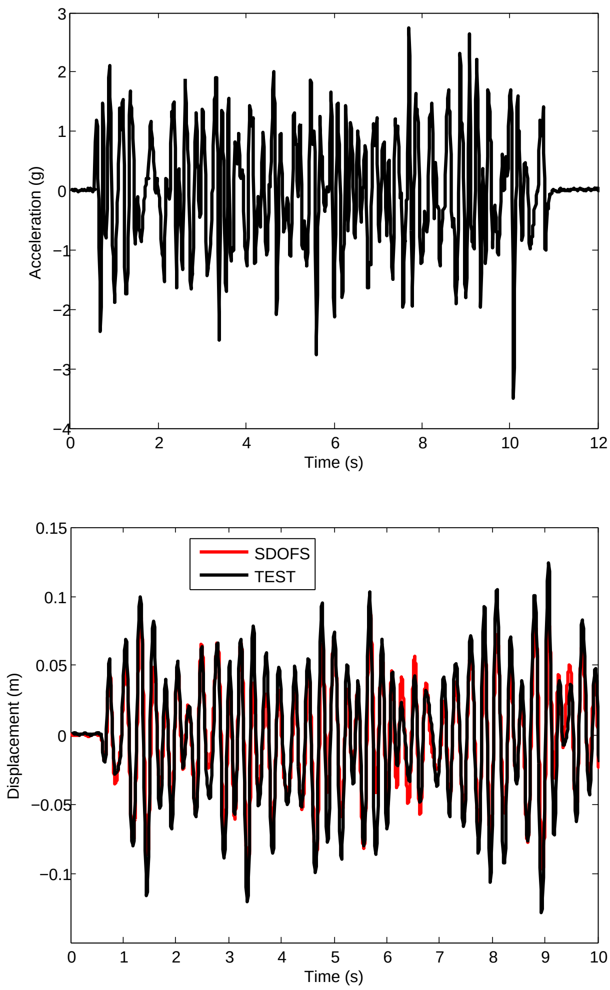


Figure 2: ***Excitation sismique / déplacement élasto-plastique : Comparaison entre le test expérimental et la réponse du modèle à une dimension :*** *en haut, un signal d'excitation sismique et en bas la superposition des deux réponses à ce signal fournies par un test expérimental (déplacement de la masse située en tête de la tuyauterie, Figure 1) et par la réponse d'un oscillateur non-linéaire.*

1.1 Le modèle de l'oscillateur EPP excité par un bruit blanc

La thèse est essentiellement consacrée aux IVS et à leurs applications au modèle de l'oscillateur EPP à un degré de liberté sous bruit blanc.

La dynamique de l'oscillateur élasto-plastique a été formulée comme un processus à mémoire. Cela est dû à l'apparition aléatoire et récurrente de phases non-linéaires (phases plastiques) pendant lesquelles l'effort subi par la structure dépasse une limite en élasticité provoquant une déformation permanente (ou déformation plastique).

En notant le déplacement élasto-plastique de l'oscillateur par $x(t)$, on étudie le problème :

$$\ddot{x} + c_0\dot{x} + \mathbf{F}(\{x(s), 0 \leqslant s \leqslant t\}) = \dot{w} \tag{1}$$

avec les conditions initiales de déplacement et de vitesse

$$x(0) = x \quad , \quad \dot{x}(0) = y.$$

Ici, $c_0 > 0$ est un coefficient d'amortissement, w est un processus de Wiener et $\mathbf{F}_t := \mathbf{F}(\{x(s), 0 \leqslant s \leqslant t\})$ est une fonctionnelle non-linéaire qui dépend de toute la trajectoire $\{x(s), 0 \leqslant s \leqslant t\}$. Le processus de déformation plastique noté par $\Delta(t)$ au temps t se déduit du couple $(x(t), \mathbf{F}_t)$. Nous considérons une force de type EPP

$$\mathbf{F}_t = \left\{ \begin{array}{rll} kY & si & x(t) = Y + \Delta(t), \\ k(x(t) - \Delta(t)) & si & x(t) \in] - Y + \Delta(t), Y + \Delta(t)[, \\ -kY & si & x(t) = -Y + \Delta(t), \end{array} \right. \tag{2}$$

où $k > 0$ est un coefficient de raideur, Y est la limite élasto-plastique et $\Delta(t) = \int_0^t \mathbf{1}_{\{|F_s|=kY\}} dx(s)$. Typiquement, la force F_t alterne entre les phases élastiques ($|F_s| < kY$) et plastiques ($|F_s| = kY$). Ainsi, D. Karnopp et T.D. Scharton [26] avaient naturellement proposé une séparation entre les états élastique et plastique. Ils ont introduit la variable "fictive" $z(t) := x(t) - \Delta(t)$ et ont remarqué le simple fait qu'entre deux phases plastiques $z(t)$ se comporte comme un oscillateur linéaire. Ainsi, $x(t)$ se décompose en $x(t) = z(t) + \Delta(t)$, où on appellera $z(t)$ la composante élastique et $\Delta(t)$ la composante plastique de $x(t)$. Pendant une phase élastique, $\Delta(t)$ reste constant et $\dot{x}(t) = \dot{z}(t)$. Ainsi, la variable $z(t)$ satisfait l'équation d'un oscillateur linéaire dont les conditions intiales sont déterminées par la phase plastique précédente :

$$\ddot{z} + c_0 \dot{z} + kz = \dot{w}. \tag{3}$$

Dans ce cas, la déformation totale $x(t)$ est élastique puisqu'elle est portée par $z(t)$. Pendant une phase plastique, la variable $z(t)$ reste constante ($z(t) = Y$ ou $z(t) = -Y$) et $\dot{x}(t) = \dot{\Delta}(t)$. Ainsi, la vitesse de $x(t)$, $y(t) := \dot{x}(t)$ est un processus d'Ornstein-Uhlenbeck et satisfait l'équation dont la condition initiale et le coefficient de dérive sont déterminés par la vitesse à la sortie de la phase élastique précédente :

$$\dot{y} + c_0 y \pm kY = \dot{w}. \tag{4}$$

Dans ce cas, la déformation totale $x(t)$ est plastique puisqu'elle est portée par $\Delta(t)$. Finalement, comme les instants de transition de phase ne sont pas connus à l'avance, la résolution temporelle de l'équation (1)-(2) s'obtient en considérant (3)-(4) dans le cadre de l'alternance des phases élastique et plastique.

1.2 Le calcul du risque de défaillance et la variance de la déformation totale

Pour calculer le risque de défaillance d'une structure mécanique modélisée par un oscillateur EPP, les ingénieurs disposent de formules explicites basées sur des tests empiriques. Ces formules reposent sur la connaissance de la variance $\sigma^2(x(t))$ de la déformation totale. Dans le chapitre 3, nous remarquerons que cette dernière a aussi le même taux de croissance que la variance de la déformation plastique puisque $x(t) = z(t) + \Delta(t)$ et que $z(t)$ est borné. Ainsi, beaucoup de travaux en sciences de l'ingénieur ont été consacrés à l'étude de $\sigma^2(x(t))$. En particulier, les ingénieurs ont observé, par des tests numériques et empiriques, que la variance de la déformation totale croît linéairement avec le temps (en temps grand) [11] :

$$\lim_{t\to\infty} \frac{\sigma^2(x(t))}{t} = \sigma^2 \tag{5}$$

où σ^2 est une constante positive qui dépend de Y. Pour approcher la valeur de σ^2, les ingénieurs ont développé des méthodes heuristiques. Ces dernières sont approximatives, mais en revanche, elles permettent d'estimer les statistiques de la déformation plastique de manière satisfaisante. Dans la thèse, nous apportons une preuve mathématique de ce que les ingénieurs ont observé et nous caractérisons le coefficient σ^2.

2 Une IVS modélisant l'oscillateur élasto-plastique

Les IVS ont déjà été introduites par Bensoussan et Lions [2] pour représenter des diffusions réfléchies dans des domaines convexes et les inéquations variationnelles aux dérivées partielles ont déjà été étudiées par Duvaut et Lions [18] pour les problèmes d'élasto-plasticité déterministes. Il n'est pas surprenant de voir réapparaître les inéquations dans la modélisation des vibrations aléatoires. Cet outil mathématique apporte le cadre exact pour étudier notamment la dynamique de l'oscillateur EPP et ses déformations.

2.1 L'avantage de l'inéquation

Les ingénieurs ont observé que l'alternance des phases reste délicate ([19, 20]). En effet, la transition de l'état plastique vers l'état élastique est affectée par le bruit blanc. En fait, une phase plastique débute lorsque $z(t)$ touche et est absorbé par Y (resp. $-Y$) avec une vitesse positive (resp. négative) $y(t) > 0$ (resp. $y(t) < 0$) i.e. $\operatorname{sign}(y(t))z(t) = Y$. Puis, elle s'arrête lorsque la vitesse change de signe. A ce moment, une phase élastique commence. Mais, la vitesse qui est affectée par le bruit blanc, change de signe une infinité de fois sur tout intervalle de temps. Souvent, cela provoque un retour rapide en phase plastique. A cause de ce phénomène, la suite des instants de transition de phases n'est pas clairement établie.

Alain Bensoussan et Janos Turi ont remarqué que le modèle utilisé dans la littérature sur l'oscillateur EPP est équivalent à une IVS. En effet, la relation entre la vitesse $y(t)$ et la composante élastique $z(t)$ est gouvernée par une inéquation variationnelle :

$$(\mathrm{d}z(t) - y(t)\mathrm{d}t)(\phi - z(t)) \geqslant 0, \quad \forall |\phi| \leqslant Y, \quad |z(t)| \leqslant Y.$$

Cette inéquation a pour avantage de clarifier la dynamique sans expliciter l'alternance entre les phases élastiques et plastiques.

2.2 Signification de la déformation plastique dans le cadre des diffusions réfléchies

L'équation (1)-(2) peut s'écrire dans le cadre des équations différentielles stochastiques (EDS) avec un processus de réflexion. Introduisons les notations

$$D := (-\infty, \infty) \times (-Y, Y), \quad D^- := (-\infty, 0) \times \{-Y\}, \quad D^+ := (0, \infty) \times \{Y\}.$$

Les processus $(y(t), z(t))$ et $\Delta(t)$ satisfont le problème suivant :

Problème 1. *Trouver un processus $(y(t), z(t))$ à valeur dans $\mathbb{R}^2$ tel que*

1. *$(y(t), z(t))$ adapté et continu,*

2. *il existe un processus scalaire $\Delta(t)$ continu, à variation bornée, adapté tel que $(y(t), z(t))$ vérifie*
$$dy(t) = -(c_0y(t) + kz(t))dt + dw(t), \quad z(t) = \int_0^t y(s)ds - \Delta(t). \tag{6}$$
3. *p.s.* $(y(t), z(t)) \in D \cup D^+ \cup D^-$,
4. *p.s.* $\int_{t_1}^{t_2} \mathbf{1}_D(y(s), z(s))d\Delta(s) = 0, \quad \forall t_1 < t_2.$

admet une unique solution.

Le processus de la déformation plastique $\Delta(t)$ joue le rôle du processus de réflexion. Ainsi, $(y(t), z(t))$ est l'unique solution de l'inéquation suivante :

$$\begin{cases} \mathrm{d}y(t) = -(c_0y(t) + kz(t))\mathrm{d}t + \mathrm{d}w(t), \\ (\mathrm{d}z(t) - y(t)\mathrm{d}t)(\phi - z(t)) \geqslant 0, \\ \forall |\phi| \leqslant Y, \quad |z(t)| \leqslant Y. \end{cases} \tag{$\mathcal{SVI}$}$$

La justification est analogue à la preuve de [2]. Il s'agit d'une découverte importante car elle clarifie dans un cadre mathématique solide la dynamique réelle de l'oscillateur élasto-plastique. Le processus $\Delta(t)$ n'apparaît pas explicitement dans l'inéquation ($\mathcal{SVI}$) mais il se déduit de la solution $(y(t), z(t))$ de la relation suivante

$$\Delta(t) = \int_0^t y(s)\mathrm{d}s - z(t).$$

2.3 Propriété ergodique du couple vitesse-composante élastique

A l'aide de la formule d'Itô pour les diffusions réfléchies, on peut calculer le générateur infinitésimal Λ de $(y(t), z(t))$. Pour toute fonction φ suffisamment régulière définie sur D, on a

$$\varphi(y(t), z(t)) - \varphi(y(0), z(0)) - \int_0^t L\varphi(y(s), z(s))\mathrm{d}s - \int_0^t R\varphi(y(s), z(s))\mathrm{d}\Delta(s)$$
$$= \int_0^t \nabla\varphi(y(s), z(s))\mathrm{d}w(s) \tag{7}$$

où les opérateurs L (diffusion) et R (réflexion) s'écrivent

$$L\varphi(y, z) = \frac{1}{2}\varphi_{yy} - (c_0y + kz)\varphi_y + y\varphi_z \quad \text{et} \quad R\varphi(y, z) = -\mathbf{1}_{\{\partial D\}}(y, z)\varphi_z(y, z).$$

Comme $\Delta(t) = \int_0^t \mathbf{1}_{\{sign(y(s))z(s)=Y\}} y(s)\mathrm{d}s$, l'égalité (7) devient :

$$\varphi(y(t), z(t)) - \varphi(y(0), z(0)) - \int_0^t \left\{L\varphi(y(s), z(s)) - \mathbf{1}_{\{sign(y(s))z(s)=Y\}} y(s)\varphi_z(y(s), z(s))\right\} \mathrm{d}s$$
$$= \int_0^t \varphi_y(y(s), z(s))\mathrm{d}w(s).$$

Alors par définition du générateur, on a

$$\Lambda\varphi(y_0, z_0) := \lim_{h\to 0} \frac{1}{h}\mathbb{E}\{\varphi(y(h), z(h)) - \varphi(y(0), z(0))|(y(0), z(0)) = (y_0, z_0)\}$$

satisfait

$$\Lambda\varphi = \begin{cases} -A\varphi(y,z) & \text{si} \quad (y,z) \in D, \\ -B_{+}\varphi(y,z) & \text{si} \quad (y,z) \in D^{+}, \\ -B_{-}\varphi(y,z) & \text{si} \quad (y,z) \in D^{-} \end{cases}$$

où

$$\begin{cases} A\varphi & := -\frac{1}{2}\varphi_{yy} + (c_0 y + kz)\varphi_y - y\varphi_z, \\ B_{+}\varphi & := -\frac{1}{2}\varphi_{yy} + (c_0 y + kY)\varphi_y, \\ B_{-}\varphi & := -\frac{1}{2}\varphi_{yy} + (c_0 y - kY)\varphi_y. \end{cases}$$

Le théorème suivant concerne le comportement en temps long de $(y(t), z(t))$ (voir [7]).

Théorème (Bensoussan-Turi,2007)**.** *Le couple $(y(t), z(t))$ est un processus de Markov ergodique. En conséquence, il existe une unique mesure invariante ν vers laquelle la loi du processus converge. De plus, ν admet une densité de probabilité m. On notera $\nu(f)$ l'application de la mesure invariante sur une fonction f bornée, ainsi*

$$\nu(f) = \int_D f(y,z)m(y,z)\,dy dz + \int_{D^+} f(y,Y)m(y,Y)\,dy + \int_{D^-} f(y,-Y)m(y,-Y)\,dy.$$

Par définition, la mesure invariante ν du processus $(y(t), z(t))$ est telle que

$$\nu(\Lambda\varphi) = 0, \quad \forall\varphi \text{ régulière}.$$

Pour m, cela signifie

$$\int_D A\varphi(y,z)m(y,z)\mathrm{d}y\mathrm{d}z + \int_{D^+} B_{+}\varphi(y,Y)m(y,Y)\mathrm{d}y + \int_{D^-} B_{-}\varphi(y,-Y)m(y,-Y)\mathrm{d}y = 0, \quad (\mathcal{M})$$

pour toute fonction φ régulière. Il s'agit de la formulation variationnelle ultra-faible de l'équation de la mesure invariante m. Le théorème suivant concerne une caractérisation de la distribution par dualité. (voir [8])

Théorème (Bensoussan-Turi,2010)**.** *Pour tout $\lambda > 0$ et pour toute fonction $f(y,z)$ bornée mesurable, il existe une solution unique $u_\lambda(y,z)$ continue et bornée sur $\bar{D}$ à l'équation suivante :*

$$\begin{cases} \lambda u_\lambda + A u_\lambda &= f(y,z) & \text{dans} \quad D, \\ \lambda u_\lambda + B_{+} u_\lambda &= f(y,Y) & \text{dans} \quad D^{+}, \\ \lambda u_\lambda + B_{-} u_\lambda &= f(y,-Y) & \text{dans} \quad D^{-} \end{cases} \qquad (P_\lambda)$$

avec les conditions de bord non-locales

$$u_\lambda(y,Y) \quad \textit{et} \quad u_\lambda(y,-Y) \quad \textit{sont continues}$$

telle que

$$\|u_\lambda\|_\infty \leqslant \frac{\|f\|_\infty}{\lambda} \quad \textit{et} \quad u_\lambda \quad \textit{est continue}$$

et

$$\lim_{\lambda\to 0} \lambda u_\lambda = \nu(f). \qquad (8)$$

3 Résultats de la thèse

Nous présentons maintenant en détail les chapitres de la thèse, nous résumons les résultats principaux en les replaçant dans leur contexte.

3.1 Synopsis du chapitre 1 : Analyse numérique de la mesure invariante de l'oscillateur élastique-parfaitement-plastique excité par un bruit blanc

Dans ce chapitre, on considère un algorithme numérique déterministe pour obtenir la solution numérique d'un oscillateur elasto-plastique excité par un bruit blanc. Puisque les simulations Monte-Carlo pour le processus sous-jacent sont trop lentes, une famille de solutions d'EDPs définissant la mesure invariante par dualité est étudiée comme alternative à la simulation probabiliste. Notre approche est adaptée parce que la régularité de la mesure invariante n'est pas suffisante pour employer une méthode des éléments finis usuelle. Cette difficulté est résolue à l'aide d'une méthode des éléments finis ultra-faible qui est développée et mise en oeuvre avec succès.

Pour les applications, les ingénieurs sont intéressés par m, la mesure invariante du système, parce qu'elle décrit le régime asymptotique de l'oscillateur EPP pour les temps grands. La simulation numérique du système par une méthode de Monte-Carlo est immédiate à mettre en oeuvre, mais elle est lente. Dans ce chapitre, on étudie un algorithme numérique déterministe alternatif à la méthode de Monte-Carlo en résolvant une EDP pour m. Par ailleurs, le semi-groupe P_t associé à la solution $(y(t), z(t))$ de l'inéquation

$$\mathrm{d}y(t) = -(c_0 y(t) + kz(t))\mathrm{d}t + \mathrm{d}w(t), \quad (\mathrm{d}z(t) - y(t)\mathrm{d}t)(\phi - z(t)) \geqslant 0, \quad \forall|\phi| \leqslant Y, \quad |z(t)| \leqslant Y$$

avec la condition initiale $(y(0), z(0)) = (\eta, \zeta)$ satisfait $P_t(f)(\eta, \zeta) = \mathbb{E}[f(y(t), z(t))]$ pour toute fonction f suffisamment régulière. Plus précisement, on a

$$\mathbb{E}[f(y(t), z(t))] = \int_D f(y,z)p_t(y,z)\mathrm{d}y\mathrm{d}z + \int_{D^+} f(y,Y)p_t(y,Y)\mathrm{d}y + \int_{D^-} f(y,-Y)p_t(y,-Y)\mathrm{d}y$$

où p_t est la densité de $(y(t), z(t))$. Ainsi, la mesure invariante m peut être obtenue par le passage à la limite suivant : $m = \lim_{t\to\infty} p_t$. Rappelons que cette dernière est caractérisée par une formulation variationnelle ultra-faible : pour toute fonction φ continue bornée

$$\int_D m(y,z)A\varphi(y,z)\mathrm{d}y\mathrm{d}z + \int_{D^+} m(y,Y)B_+\varphi(y,Y)\mathrm{d}y + \int_{D^-} m(y,-Y)B_-\varphi(y,-Y)\mathrm{d}y = 0 \quad (9)$$

où

$$A\varphi := -\frac{1}{2}\varphi_{yy} + (c_0 y + kz)\varphi_y - y\varphi_z,$$

$$B_+\varphi := -\frac{1}{2}\varphi_{yy} + (c_0 y + kY)\varphi_y,$$

$$B_-\varphi := -\frac{1}{2}\varphi_{yy} + (c_0 y - kY)\varphi_y.$$

L'objectif de ce travail est de résoudre (9) et de comparer les résultats obtenus avec ceux de la méthode de Monte-Carlo standard. Le point clé est la résolution d'une EDP stationnaire avec des conditions de bord non-locales et avec une fonction f suffisamment régulière au second membre :

$$\left\{\begin{array}{rcll} \lambda u + Au & = & f(y,z) & \text{dans} \quad D, \\ \lambda u + B_+ u & = & f(y,Y) & \text{dans} \quad D^+, \\ \lambda u + B_- u & = & f(y,-Y) & \text{dans} \quad D^-. \end{array}\right. \tag{P_λ}$$

avec les conditions de bord non-locales

$$u_\lambda(y,Y) \quad \text{et} \quad u_\lambda(y,-Y) \quad \text{sont continues.}$$

De [8], on sait qu'il existe une solution unique $u_\lambda(y,z)$ au problème (P_λ) telle que

$$\|u_\lambda\|_\infty \leqslant \frac{\|f\|_\infty}{\lambda} \quad \text{et} \quad u_\lambda \quad \text{est continue.}$$

Nous justifions que cette formulation est très importante d'un point de vue numérique puisqu'elle permet également d'obtenir $\lim_{t\to\infty} \mathbb{E}[f(y(t),z(t))]$ sans résoudre un problème dépendant du temps. Nous verrons que le problème (P_λ) est compatible avec une méthode des éléments finis ultra-faible pour résoudre (9). Les méthodes ultra-faibles ont été utilisées théoriquement pour établir l'existence et l'unicité de solutions à certaines classes d'EDPs mais rarement numériquement, excepté dans [9]. En effet, nous prouvons que

$$\forall (\eta,\zeta) \in \bar{D}, \quad \lim_{\lambda\to 0} \lambda u_\lambda(\eta,\zeta) = \lim_{t\to\infty} \mathbb{E}[f(y(t),z(t))]$$

et puis que

$$\lim_{\lambda\to 0} \lambda u_\lambda(\eta,\zeta) = \int_D m(y,z) f(y,z) \mathrm{d}y\mathrm{d}z + \int_{D^+} m(y,Y) f(y,Y) \mathrm{d}y + \int_{D^-} m(y,-Y) f(y,-Y) \mathrm{d}y.$$

Cette dernière égalité est une caracterisation équivalente de m. Cette limite ne dépend pas de (η,ζ).

3.2 Synopsis du chapitre 2 : Une approche analytique pour la théorie ergodique des inéquations variationnelles stochastiques.

Dans ce chapitre, nous présentons une nouvelle caractérisation de l'unique mesure invariante. Dans ce contexte, nous montrons une relation liant des problèmes non-locaux et des problèmes locaux en introduisant la définition des cycles courts.

Cette caractérisation passe par l'étude des cycles courts définis ci-après.

Les cycles courts

Considérons $v_\lambda(y,z)$ la solution de

$$\lambda v_\lambda + A v_\lambda = f \quad \text{dans} \quad D, \quad \lambda v_\lambda + B_+ v_\lambda = f \quad \text{dans} \quad D^+, \quad \lambda v_\lambda + B_- v_\lambda = f \quad \text{dans} \quad D^- \tag{10}$$

avec les conditions de bord $v_\lambda(0^+,Y) = 0$ et $v_\lambda(0^-,-Y) = 0$. C'est un problème local. Notons que si f est symétrique (resp. antisymétrique) alors v_λ est aussi symétrique (resp. antisymétrique). Nous utilisons la notation $v_\lambda(y,z;f)$. Lorsque $\lambda \to 0$, on obtient que $v_\lambda \to v$ où

$$Av = f \quad \text{dans} \quad D, \quad B_+ v = f \quad \text{dans} \quad D^+, \quad B_- v = \quad \text{dans} \quad D^- \tag{P_v}$$

avec les conditions de bord locales $v(0^+, Y) = 0$ et $v(0^-, -Y) = 0$. Nous utilisons la notation $v(y, z; f)$ et on appelle $v(y, z; f)$ un cycle court. De plus, afin de présenter notre nouvelle formulation de la mesure invariante, nous introduisons $\pi^+(y, z)$ et $\pi^-(y, z)$ telles que

$$A\pi^+ = 0 \quad \text{dans} \quad D, \quad \pi^+ = 1 \quad \text{dans} \quad D^+, \quad \pi^+ = 0 \quad \text{dans} \quad D^- \tag{11}$$

et

$$A\pi^- = 0 \quad \text{dans} \quad D, \quad \pi^- = 0 \quad \text{dans} \quad D^+, \quad \pi^- = 1 \quad \text{dans} \quad D^-. \tag{12}$$

Nous avons $\pi^+ + \pi^- = 1$, ainsi on déduit l'existence et l'unicité de solutions bornées à (11) et (12). Une nouvelle formulation de la mesure invariante est donnée par :

Théorème 1 (Une nouvelle formulation de ν)**.** *Sous les hypothèses des théorèmes précédents, la mesure invariante vérifie la propriété suivante :*

$$\nu(f) = \frac{v(0^-, Y; f) + v(0^+, -Y; f)}{2v(0^+, Y; 1)}.$$

Définissons $\nu_\lambda(f) := \frac{v_\lambda(0^-,Y;f)+v_\lambda(0^+,-Y;f)}{2v_\lambda(0^-,Y;1)}$*, alors lorsque* $\lambda \to 0$*,*

$$u_\lambda(y, z; f) - \frac{\nu_\lambda(f)}{\lambda} \to u(y, z; f), \quad \nu_\lambda(f) \to \nu(f) \tag{13}$$

où u satisfait

$$Au = f - \nu(f) \quad \text{dans} \quad D, \quad B_+u = f - \nu(f) \quad \text{dans} \quad D^+, \quad B_-u = f - \nu(f) \quad \text{dans} \quad D^- \tag{14}$$

avec les conditions de bord non-locales

$$u(y, Y), \quad \textit{et} \quad u(y, -Y) \quad \textit{continues.}$$

Enfin, nous obtenons une représentation de u en relation avec les solutions de problèmes locaux :

$$u(y, z; f) = v(y, z; f) - \nu(f)v(y, z; 1) + \frac{\pi^+(y, z) - \pi^-(y, z)}{4\pi^-(0^-, Y)}(v(0^-, Y; f) - v(0^+, -Y; f)) \tag{15}$$

3.3 Synopsis du chapitre 3 : Comportement de la déformation plastique d'un oscillateur EPP excité par un bruit blanc

On étudie dans ce chapitre la variance d'un oscillateur EPP excité par un bruit blanc. De nombreux travaux en sciences de l'ingénieur, en partie expérimentaux, en partie numériques, ont montré qu'en temps long, la variance de la déformation totale croît linéairement avec le temps. Dans notre travail, nous démontrons ce résultat et nous caractérisons rigoureusement le coefficient de dérive. Notre étude repose sur l'inéquation variationnelle stochastique gouvernant la dynamique entre la vitesse de l'oscillateur et la force de rappel non-linéaire. Dans ce contexte, nous proposons une formulation simple et innovante de l'évolution du système en terme de temps d'arrêt afin d'identifier des séquences indépendantes dans la trajectoire : les cycles longs. Nous obtenons une formule probabiliste pour le coefficient σ^2*. Malheureusement, cette formule ne s'exprime pas sous la forme d'une EDP. Pour la calculer, il faut recourir à la simulation probabiliste. Les résultats des simulations numériques sont en accord avec notre prédiction théorique* *et avec les études empiriques auparavant réalisées par les ingénieurs.*

A travers ce chapitre, nous avons pour objectif de fournir une formulation mathématique rigoureuse (basée sur l'inéquation ($\mathcal{SVI}$)) des observations faites par les ingénieurs lors de leurs expérimentations sur (5). Dans la suite, on introduit les cycles longs qui permettent d'établir des séquences indépendantes dans la trajectoire.

Les cycles longs

On repère les portions de trajectoires indépendantes de la manière suivante : notons

$$\tau_0 := \inf\{t > 0, \quad y(t) = 0 \quad \text{et} \quad |z(t)| = Y\}$$

et $s := \text{sign}(z(\tau_0))$ qui identifie la première frontière touchée par le processus $(y(t), z(t))$. Definissons

$$\theta_0 := \inf\{t > \tau_0, \quad y(t) = 0 \quad \text{et} \quad z(t) = -sY\}.$$

D'une manière récurrente, pour $n \geqslant 0$, connaissant θ_n on peut définir

$$\begin{cases} \tau_{n+1} & := & \inf\{t > \theta_n, \quad y(t) = 0 \quad \text{et} \quad z(t) = sY\} \\ \theta_{n+1} & := & \inf\{t > \tau_{n+1}, \quad y(t) = 0 \quad \text{et} \quad z(t) = -sY\}. \end{cases}$$

D'après les définitions précédentes on peut définir le n-ième cycle long (resp. la première partie du cycle, la seconde partie du cycle) comme étant la portion de trajectoire délimitée par l'intervalle $[\tau_n, \tau_{n+1})$, (resp. $[\tau_n, \theta_{n+1})$ et $[\theta_{n+1}, \tau_{n+1})$). En effet, aux instants $\{\tau_n, n \geqslant 1\}$ le couple $(y(t), z(t))$ est dans le même état qu'à l'instant τ_0. Par ailleurs, il a deux types de cycles longs selon le signe de $s = \pm 1$. L'ensemble des temps d'arrêt $\{\tau_n, n \geqslant 0\}$ représente les temps d'occurrence des cycles longs.

Comme résultat principal, nous avons obtenu le théorème suivant :

Théorème 2 (Caractérisation de la variance de la déformation totale en terme des cycles long)**.** *Dans le contexte précédemment défini, on a montré que*

$$\lim_{t \to \infty} \frac{\sigma^2(x(t))}{t} = \frac{\mathbb{E}\left(\int_{\tau_0}^{\tau_1} y(t)\,dt\right)^2}{\mathbb{E}(\tau_1 - \tau_0)}.$$

Notre preuve est basée sur la résolution d'une classe d'EDPs reliées aux cycles longs et dont les conditions de bord sont non-locales. Cette formule peut se simplifier, cela sera expliqué en détail.

3.4 Synopsis du chapitre 4 : Les inéquations variationnelles stochastiques comportant des sauts tendant vers 0.

L'IVS modélisant l'oscillateur EPP a pour avantage de fournir un cadre mathématique solide à la relation entre la vitesse et la force de rappel non-linéaire. Cependant, elle ne permet pas de faire la distinction explicite entre les phases élastiques et les phases plastiques. De plus, en raison du bruit blanc, la trajectoire présente de petites et nombreuses phases élastiques (voir [19, 21]). Dans ce chapitre, nous introduisons des sauts aux instants de transition de l'état plastique vers *l'état élastique. Nous prouvons que la solution converge sur tout intervalle de temps fini vers la solution du problème sans saut, lorsque la taille du saut tend vers* 0.

En terme de la dynamique de $(y(t), z(t))$, une déformation plastique commence lorsque $z(t)$ atteint et est absorbé par Y (resp. $-Y$) avec une vitesse positive (resp. négative), $y(t) > 0$. (resp. $y(t) < 0$), c'est à dire lorsque $\text{sign}(y(t))z(t) = Y$. Ensuite, le comportement plastique se termine lorsque la vitesse change de signe. A ce moment là, le comportement élastique est réactivé. Cependant, la vitesse qui subit l'effet du bruit blanc, change de signe un nombre infini de fois sur tout intervalle de temps. Souvent, cela conduit à un retour rapide en comportement plastique, en un temps très court. Nous appelons ce phénomène *le phasage micro-élastique.* Nous avons observé dans une étude empirique [21] que ce phénomène joue un rôle important sur la fréquence et les statistiques de la déformation plastique. En effet, la fréquence d'occurence, les statistiques (durée ou valeur absolue de la déformation plastique) et la suite des temps d'entrée (resp. de sortie) en phase plastique ne sont pas bien définis. Dans ce travail, nous introduisons les IVS soumises à des sauts aux instants de transition entre les phases plastiques $\{|z(t)| = Y\}$ et les phases élastiques $\{|z(t)| < Y\}$.

Définition du modèle à sauts

Pour $\epsilon > 0$, on introduit des sauts de taille ϵ dans la solution de l'IVS $(\mathcal{SVI})$ apparaîssant dans la seconde composante $z(t)$. Ainsi il est naturel de noter le nouveau processus par $(y^\epsilon(t), z^\epsilon(t))$. L'évolution du système est décrite par la procédure suivante : on commence par définir $\tau_0^\epsilon := 0$ et $(y_0^\epsilon(t), z_0^\epsilon(t))$ solution de l'IVS $(\mathcal{SVI})$, avec les conditions initiales :

$$y_0^\epsilon(0) = y \quad \text{et} \quad z_0^\epsilon(0) = z,$$

ensuite, on définit

$$\tau_1^\epsilon := \inf\{t > 0, \quad y_0^\epsilon(t) = 0 \quad \text{et} \quad |z_0^\epsilon(t)| = Y\}.$$

Pour $t \geqslant \tau_1^\epsilon$, considérons $(y_1^\epsilon(t), z_1^\epsilon(t))$ solution de l'IVS $(\mathcal{SVI})$ avec les conditions initiales:

$$y_1^\epsilon(\tau_1^\epsilon) = 0 \quad \text{et} \quad z_1^\epsilon(\tau_1^\epsilon) = \text{sign}(z_0^\epsilon(\tau_1^\epsilon))\,(Y - \epsilon)\,,$$

similairement, on définit

$$\tau_2^\epsilon := \inf\{t > \tau_1^\epsilon, \quad y_1^\epsilon(t) = 0 \quad \text{et} \quad |z_1^\epsilon(t)| = Y\}.$$

D'une manière récurrente, connaissant τ_n^ϵ, $y_n^\epsilon(t)$, et $z_n^\epsilon(t)$, on définit

$$\tau_{n+1}^\epsilon := \inf\{t > \tau_n^\epsilon, \quad y_n^\epsilon(t) = 0 \quad \text{et} \quad |z_n^\epsilon(t)| = Y\},$$

et $(y_{n+1}^\epsilon(t), z_{n+1}^\epsilon(t))$ solution de l'IVS $(\mathcal{SVI})$ avec les conditions initiales :

$$y_{n+1}^\epsilon(\tau_{n+1}^\epsilon) = 0 \quad \text{et} \quad z_{n+1}^\epsilon(\tau_{n+1}^\epsilon) = \text{sign}(z_n^\epsilon(\tau_{n+1}^\epsilon))\,(Y - \epsilon)\,.$$

On définit ensuite le processus $(y^\epsilon(t), z^\epsilon(t))$ sur chaque intervalle de temps $[\tau_n^\epsilon, \tau_{n+1}^\epsilon)$ de la façon suivante :

$$\dot{y}^\epsilon(t) = -(c_0 y^\epsilon(t) + k z^\epsilon(t)) + \dot{w}(t) \quad ; \quad (\dot{z}^\epsilon(t) - y^\epsilon(t))(\phi - z^\epsilon(t)) \geqslant 0 \quad ; \quad \forall\, |\phi| \leqslant Y \quad ; \quad |z^\epsilon(t)| \leqslant Y \tag{16}$$

avec les conditions de sauts :

$$y^\epsilon(\tau_n^\epsilon-) = 0, \qquad z^\epsilon(\tau_n^\epsilon-) = z_{n-1}^\epsilon(\tau_n^\epsilon),$$

et

$$y^\epsilon(\tau_n^\epsilon) = 0, \qquad z^\epsilon(\tau_n^\epsilon) = \mathrm{sign}(z_{n-1}^\epsilon(\tau_n^\epsilon))(Y-\epsilon).$$

Remarque 1. *Par construction, le processus* $(y^\epsilon(t), z^\epsilon(t))$ *est continu à droite et a une limite à gauche (càdlàg). En particulier, pour chaque temps* $T > 0$ *fixé, le nombre de sauts apparaissant dans l'intervalle* $(0, T]$ *est fini p.s.*

Nous prouvons que la solution $(y^\epsilon(t), z^\epsilon(t))$ converge vers $(y(t), z(t))$ sur tout intervalle fini, lorsque ϵ tend vers 0 au sens du résultat suivant :

Résultat de convergence

Théorème 3. *Supposons* $k > X_+(c_0) := \frac{1}{2}\left(-\frac{c_0}{3} + c_0\sqrt{\frac{1}{9} + 4\frac{c_0}{6}}\right)$. *Pour* $T > 0$, *les processus* $(y(t), z(t))$ *et* $(y^\epsilon(t), z^\epsilon(t))$, *satisfaisant* $(\mathcal{SVI})$ *et* (16) *respectivement, vérifient la propriété de convergence suivante :*

$$\frac{1}{\epsilon}\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\left\{|y(t)-y^\epsilon(t)|^2 + k\,|z(t)-z^\epsilon(t)|^2\right\}\right] \to 0 \quad \textit{lorsque} \quad \epsilon \to 0.$$

3.5 Synopsis du chapitre 5 : Résolution des problèmes de Dirichlet dégénérés associés à la propriété ergodique d'un oscillateur élasto-plastique excité par un bruit blanc filtré

On définit une IVS pour modéliser un oscillateur élasto-plastique excité par un bruit blanc filtré. Nous prouvons la propriété ergodique du processus sous-jacent et nous caractérisons sa mesure invariante. On étend la méthode de A.Bensoussan et J. Turi ([7]) avec une difficulté supplémentaire due à l'accroissement de la dimension. Les conditions de bord non-locales exprimées sous forme d'équations différentielles en dimension 1 sont remplacées par des équations elliptiques en dimension 2. Dans ce contexte, la méthode de Khasminskii ([27]) conduit à l'étude de problèmes de Dirichlet dégénérés avec des conditions de bord non-locales exprimées sous forme d'EDPs.

Les oscillateurs non-linéaires soumis à des vibrations aléatoires sont des modèles simples et utiles pour prédire la réponse d'une structure mécanique sollicitée au delà de sa limite en élasticité. Lorsque l'excitation est un bruit blanc, une importante littérature s'est constituée sur ce sujet (voir [14, 15, 16, 28, 19, 20, 24, 26]). Dans le travail mentionné de A. Bensoussan et J. Turi, la réponse d'un oscillateur élasto-plastique parfait excité par un bruit blanc peut être modélisée par une IVS. Dans ce nouveau contexte mathématique, ils ont prouvé l'existence et l'unicité d'un état d'équilibre invariant en loi pour la dynamique du processus, vers lequel le système converge en temps grand : la propriété ergodique. De plus, les résultats dans [7] fournissent un cadre rigoureux pour accéder aux grandeurs qui intéressent l'ingénieur ([19, 20, 26]). Cependant, pour les ingénieurs, le choix du bruit blanc n'est pas suffisamment réaliste. Dans ce papier, nous proposons de considérer une excitation dans un sens qui peut être plus réaliste. Notre modèle généralise [7] en choisissant un processus d'Ornstein-Uhlenbeck réfléchi. En conséquence, en comparant notre modèle avec l'oscillateur EPP excité par un bruit blanc, un troisième processus apparaît dans l'inéquation variationnelle.

Considérons $w(t)$ and $\tilde{w}(t)$, deux processus de Wiener indépendants et $x(t)$ un processus d'Ornstein-Ulhenbeck réfléchi

$$\mathrm{d}x(t) = -\alpha x(t)dt + \mathrm{d}w(t) + 1_{\{x(t)=-L\}}\mathrm{d}\xi^1(t) - 1_{\{x(t)=L\}}\mathrm{d}\xi^2(t).$$

Ici, le processus $\xi := \{(\xi^1(t), \xi^2(t)), t \geqslant 0\}$ contraint $x(t)$ à prendre ses valeurs dans $[-L, L]$. Dans ce modèle, l'excitation est donnée par

$$-\beta x(t)dt + \mathrm{d}\tilde{w}(t). \tag{17}$$

L'inéquation variationnelle stochastique est constituée par

$$\begin{cases} \mathrm{d}x(t) = -\alpha x(t)\mathrm{d}t + \mathrm{d}w(t) + 1_{\{x(t)=-L\}}\mathrm{d}\xi^1_t - 1_{\{x(t)=L\}}\mathrm{d}\xi^2_t, \\ \mathrm{d}y(t) = -(\beta x(t) + c_0 y(t) + kz(t))dt + d\tilde{w}(t), \\ (\mathrm{d}z(t) - y(t)dt)(\zeta - z(t)) \geqslant 0, \\ |\zeta| \leqslant Y, \\ |z(t)| \leqslant Y. \end{cases} \tag{18}$$

Si $\beta \neq 0$, $x(t)$ apparaît dans la dynamique de $y(t)$. Dans ce cas nous appellerons ce modèle <u>cas 2d</u>. Sinon $\beta = 0$, $x(t)$ n'est plus impliqué dans le modèle et dans ce cas, le couple $(y(t), z(t))$ satisfait le problème de l'oscillateur EPP excité par un bruit blanc de [7], qui sera appelé <u>cas 1d</u>.

Notation 1. *Introduisons les opérateurs*

$$Au := \frac{1}{2}u_{yy} + \frac{1}{2}u_{xx} - \alpha x u_x - (\beta x + c_0 y + kz)u_y + yu_z,$$

$$B_+u := \frac{1}{2}u_{yy} + \frac{1}{2}u_{xx} - \alpha x u_x - (\beta x + c_0 y + kY)u_y,$$

$$B_-u := \frac{1}{2}u_{yy} + \frac{1}{2}u_{xx} - \alpha x u_x - (\beta x + c_0 y - kY)u_y.$$

Le générateur infinitésimal Λ de $(x(t), y(t), z(t))$ est donné par :

$$\Lambda : \ \phi \ \mapsto \begin{cases} A\phi & \text{si } z \in]-Y, Y[, \\ B_\pm \phi & \text{si } z = \pm Y, \pm y > 0. \end{cases}$$

Notation 2.

$\mathcal{O} := (-L, L)\times\mathbb{R}\times(-Y, Y)$; $\quad \mathcal{O}^+ := (-L, L)\times(0, +\infty)\times\{Y\}$; $\quad \mathcal{O}^- := (-L, L)\times(-\infty, 0)\times\{-Y\}$.

Notre résultat principal est le suivant :

Théorème 4. *Il existe une unique mesure de probabilité ν sur $\mathcal{O} \cup \mathcal{O}^- \cup \mathcal{O}^+$ satisfaisant*

$$\int_{\mathcal{O}} A\phi d\nu + \int_{\mathcal{O}^-} B_-\phi d\nu + \int_{\mathcal{O}^+} B_+\phi d\nu = 0, \quad \forall \phi \quad \text{régulière.}$$

De plus, ν a une densité de probabilité m qui satisfait

$$\int_{\mathcal{O}} m(x, y, z)\, dx\, dy\, dz + \int_{\mathcal{O}^+} m(x, y, Y)\, dx\, dy + \int_{\mathcal{O}^-} m(x, y, -Y)\, dx\, dy = 1,$$

où

- $\{m(x, y, z), \quad (x, y, z) \in \mathcal{O}\}$ *est la partie élastique,*
- $\{m(x, y, Y), \quad (x, y) \in \mathcal{O}^+\}$ *est la partie plastique positive,*
- *et* $\{m(x, y, -Y), \quad (x, y) \in \mathcal{O}^-\}$ *est la partie plastique négative.*

De plus, m satisfait au sens des distributions l'équation suivante :

$$\alpha \frac{\partial}{\partial x}[xm] + \frac{\partial}{\partial y}[(\beta x + c_0 y + kz)m] - y\frac{\partial m}{\partial z} + \frac{1}{2}\frac{\partial^2 m}{\partial x^2} + \frac{1}{2}\frac{\partial^2 m}{\partial y^2} = 0, \qquad \textit{dans } \mathcal{O},$$

et sur les bords

$$ym + \frac{\partial}{\partial x}[xm] + \frac{\partial}{\partial y}[(\beta x + c_0 y + kY)m] + \frac{1}{2}\frac{\partial^2 m}{\partial x^2} + \frac{1}{2}\frac{\partial^2 m}{\partial y^2} = 0, \qquad \textit{dans } \mathcal{O}^+,$$

$$-ym + \frac{\partial}{\partial x}[xm] + \frac{\partial}{\partial y}[(\beta x + c_0 y - kY)m] + \frac{1}{2}\frac{\partial^2 m}{\partial x^2} + \frac{1}{2}\frac{\partial^2 m}{\partial y^2} = 0, \qquad \textit{dans } \mathcal{O}^-,$$

$$m = 0, \qquad \textit{dans } (-L, L) \times (-\infty, 0) \times \{Y\} \cup (-L, L) \times (0, \infty) \times \{-Y\}.$$

La preuve est basée sur la résolution d'une suite de problèmes de Dirichlet intérieurs et extérieurs, qui sont intéressants en eux-mêmes. On met en parallèle les cas 1d et 2d, afin de faciliter la lecture. Dans le cas 1d, la variable x disparaît ($\beta = 0$), nous garderons la notation A, B_+, B_- pour les opérateurs définis plus haut sans la variable x.

Par ailleurs, notre étude présente un intérêt mathématique car elle généralise la méthode proposée par le premier auteur et J. Turi [7] au cas de la dimension supérieure. Les conditions de bord non-locales exprimées sous la forme d'équations différentielles en dimension 1 sont remplacées par des EDPs elliptiques en dimension 2. Dans le premier cas, il existe des solutions semi-explicites, ainsi les conditions de bord non-locales se réduisent à deux nombres inconnus. Dans le second cas, les solutions des équations elliptiques sur le bord n'admettent pas d'expression explicite et dépendent respectivement de deux fonctions inconnues définies sur $(-L, L)$.

Notons que le choix de l'excitation (17) est aussi motivé par deux considérations techniques :

- la première est d'imposer (grâce aux processus ξ^1 et ξ^2) à $x(t)$ d'évoluer dans l'ensemble compact $[-L, L]$. Ainsi, dans le cadre de notre preuve, un argument de compacité permet de montrer la propriété ergodique du triplet $(x(t), y(t), z(t))$. Notons que cela n'est pas gênant du point de vue des applications car si l'on choisit L suffisamment grand, alors le processus $x(t)$ est similaire à un processus d'Ornstein-Ulhenbeck.

- la deuxième concerne la décorrélation de $w(t)$ et $\tilde{w}(t)$. Dans notre démarche, basée sur les EDPs associées au triplet $(x(t), y(t), z(t))$, nous évitons l'apparition de termes de dérivées croisées dans le générateur infinitésimal Λ. Il s'agit d'une simplification du cas corrélé qui est techniquement plus complexe.

3.6 Synopsis du chapitre-annexe 6 : Une étude empirique de la déformation plastique

Dans ce chapitre, on s'intéresse aux propriétés statistiques de la déformation plastique de l'oscillateur EPP excité par un bruit blanc. Notre approche repose sur l'inéquation gouvernant l'évolution de la vitesse et de la force de rappel non-linéaire. Dans ce travail, nous étudions le phasage élastique au moyen de la mesure invariante de l'oscillateur EPP. D'abord, nous mettons en évidence par les simulations probabilistes le phénomène de phases micro-élastiques (aussi petites que nombreuses). La principale difficulté associée aux phases micro-élastiques concerne leur impact sur des grandeurs utiles à l'ingénieur comme la fréquence des déformations plastiques. En particulier, la fréquence des déformations plastiques ne peut pas être évaluée. Ensuite, nous présentons des approximations de la loi marginale de la composante élastique $z(t)$ en régime invariant et d'une expression analogue à la formule de Rice des franchissements de seuil. Ces quantités sont solutions d'EDPs. Les résultats numériques expérimentaux sur ces équations montrent que la composante élastique est fortement distribuée près du bord plastique. Finalement, un critère empirique qui peut être utile à l'ingénieur est fourni afin de ne pas prendre en compte les phases micro-élastiques et ainsi évaluer d'une façon réaliste les statistiques de la déformation plastique.

Rappelons qu'une déformation plastique commence lorsque $z(t)$ touche et est absorbé par Y (resp. Y) avec une vitesse positive (resp. négative) $y(t) > 0$ (resp. $y(t) < 0$); c'est à dire $\text{sign}(y(t))z(t) = Y$. Ensuite, le comportement plastique se termine lorsque la vitesse devient nulle. A ce moment, le comportement élastique est réactivé. Cependant, la vitesse, qui subit le bruit blanc, change de signe une infinité de fois pendant n'importe quel petit intervalle de temps. Souvent, cela mène à un rapide retour en phase plastique. On appelle ce phénomène *les phases micro-élastiques.* Elles jouent un rôle important dans la fréquence et les statistiques de la déformation. A cause de ce phénomène, les fréquences d'occurrence, les statistiques et la suite des temps d'entrée et de sortie en phase plastique ne sont pas bien définis.

Notre but est d'étudier les propriétés du comportement plastique pendant les intervalles de temps délimités par les instants d'entrée en phase plastique et les instants de franchissement de $Y - \epsilon$ (resp. $-Y + \epsilon$) par $z(t)$ avec une vitesse négative (resp. positive) pour un petit ϵ. Nous appelerons ces intervalles de temps *les phases plastiques élargies.*

Notre approche permet de relier rigoureusement le nombre de phases plastiques élargies au nombre de franchissements de $Y - \epsilon$ à vitesse négative et de $-Y + \epsilon$ à vitesse positive.

Pour expliquer la procédure, considérons $T > 0$, notons $\tau_0^\epsilon := 0$ et

$$\begin{aligned} \theta_{n+1}^\epsilon &:= \inf\{t > \tau_n^\epsilon, \quad |z(t)| = Y\}, \\ \tau_{n+1}^\epsilon &:= \inf\{t > \theta_{n+1}^\epsilon, \quad |z(t)| = Y - \epsilon\}, \quad \forall n \geqslant 1. \end{aligned} \tag{19}$$

On note également $N_T^\epsilon = \sum_{n\geqslant 0} \mathbf{1}_{\{\tau_n^\epsilon \leqslant T\}}$ le nombre de phases plastiques élargies jusqu'au temps T. Alors, pour toute fonction f mesurable telle que $f(y, z) = 0$ si $sign(y)z \neq Y$ et satisfaisant

$$\int_{D^-} |f(y, -Y)| m(y, -Y) \mathrm{d}y + \int_{D^+} |f(y, Y)| m(y, Y) \mathrm{d}y < \infty,$$

on a

$$\frac{1}{T}\int_0^T f(y(s),z(s))\mathrm{d}s = \frac{N_T^\epsilon}{T} \times \frac{1}{N_T^\epsilon}\sum_{n=1}^{N_T^\epsilon}\int_{\theta_n^\epsilon}^{\tau_n^\epsilon} f(y(s),z(s))\mathrm{d}s. \tag{20}$$

Par ailleurs, on peut définir séparement la fréquence de franchissements des seuils $\pm Y \mp \epsilon$ avec une vitesse négative ou positive par $z(t)$:

$$\nu(Y,\epsilon) := \lim_{T\to\infty} \frac{N_T^\epsilon}{T}$$

et la “statistique empirique” associée à la phase plastique élargie par

$$\Delta_f(Y,\epsilon) := \lim_{T\to\infty} \frac{1}{N_T^\epsilon}\sum_{n=1}^{N_T^\epsilon}\int_{\theta_n^\epsilon}^{\tau_n^\epsilon} f(y(s),z(s))\mathrm{d}s.$$

Rappelons que le comportement asymptotique a été étudié par Bensoussan et Turi dans [7]. Ils ont montré que $(y(t),z(t))$ est un processus de Markov ergodique satisfaisant l'inéquation variationnelle stochastique. Ainsi, il existe une unique mesure invariante, notée $m(y,z)$, et composée de

1. une composante élastique: $\{m(y,z), \quad |z| < Y\}$,
2. une composante plastique positive: $\{m(y,Y), \quad y > 0\}$,
3. une composante plastique négative: $\{m(y,-Y), \quad y < 0\}$.

En conséquence, lorsque T tend vers l'infini, (20) devient

$$\int_0^\infty f(y,Y)m(y,Y)\mathrm{d}y + \int_{-\infty}^0 f(y,-Y)m(y,-Y)\mathrm{d}y = \nu(Y,\epsilon)\Delta_f(Y,\epsilon). \tag{21}$$

La relation précédente est essentielle puisqu'elle établit que le produit de $\nu(Y,\epsilon)$ et $\Delta_f(Y,\epsilon)$ reste constant pour toute valeur de ϵ. Cependant, dans nos simulations probabilistes, nous observons que $\nu(Y,\epsilon)$ tend vers ∞ et que $\Delta_f(Y,\epsilon)$ tend vers 0. Dans ce travail, nous fournissons un critère empirique qui peut être utile à l'ingénieur pour calibrer ϵ. Le but est de calculer une fréquence et les statistiques de la déformation plastique qui ne prennent pas en compte les phases micro-élastiques. D'un point de vue empirique, nous avons observé une forte concentration de la distribution de $\lim_{t\to\infty} z(t)$ au voisinage des bords plastiques $\{z = \pm Y\}$. Celle-ci présente des points de minima qui sont identifiés, on les note $\pm(Y-\epsilon^\star)$. Notre approche repose sur les EDPs associées à la seconde marginale de $\{m(y,z), \quad |z| < Y\}$, qui est $s \to \int_{-\infty}^\infty m(y,s)dy$. Par ailleurs, les fréquences moyennes de franchissements de seuils $\pm(Y-\epsilon^\star)$ avec une vitesse positive ou négative sont calculées avec succès à l'aide d'une formule analogue à la formule de Rice [31]. Cette dernière est un outil très utilisé parmi les ingénieurs. Elle est rigoureusement établie pour les systèmes purement élastiques. Plus précisement, si l'on considère le cas : $Y = +\infty$ alors $z(t) = x(t)$ et le système (1) est décrit par le couple $(x(t),y(t))$ qui est la solution de l'EDS :

$$\mathrm{d}y(t) = -(c_0 y(t) + kx(t))\mathrm{d}t + \mathrm{d}w(t), \quad \mathrm{d}x(t) = y(t)\mathrm{d}t.$$

Le couple $(x(t),y(t))$ est un processus de Markov ergodique dont la mesure invariante $\bar{m}$ est donnée explicitement par [30] :

$$\bar{m}(x,y) = \frac{c_0\sqrt{k}}{\pi}\exp(-c_0kx^2)\exp(-c_0y^2).$$

Dans ce cadre, les fréquences moyennes de franchissements d'un seuil s par $x(t)$ à vitesse positive $y(t) > 0$ notée ν_s^+ ou à vitesse négative $y(t) < 0$ notée ν_s^- sont données par la formule de Rice :

$$\nu_s^+ = \int_0^{+\infty} y\bar{m}(y,s)\mathrm{d}y, \quad \nu_s^- = -\int_{-\infty}^0 y\bar{m}(y,s)\mathrm{d}y. \tag{22}$$

Dans le contexte de l'oscillateur EPP ($Y < +\infty$), une expression analogue de la formule de Rice pour le calcul de $\nu(Y,\epsilon)$ conduit à l'approximation suivante :

$$\nu(Y,\epsilon) \approx -\int_{-\infty}^0 ym(y, Y-\epsilon)\mathrm{d}y + \int_0^{+\infty} ym(y, -Y+\epsilon)\mathrm{d}y. \tag{23}$$

La preuve de la formule de Rice appliquée à la mesure m reste une question ouverte et malheureusement nous n'y répondons pas. Cependant, des tests numériques sur $\{m(y,z), \quad |z| < Y\}$ pour le calcul de $\nu(Y,\epsilon)$ ont fourni des résultats satisfaisants qui correspondent à la simulation probabiliste.

PUBLICATIONS

1. A. Bensoussan, L. Mertz, O. Pironneau, J. Turi, *An Ultra Weak Finite Element Method as an Alternative to a Monte Carlo Method for an Elasto-Plastic Problem with Noise*, SIAM J. Numer. Anal. , **47(5)** (2009), 3374–3396.

2. A. Bensoussan, L. Mertz, *An analytic approach to the ergodic theory of stochastic variational inequalities*, submitted to Comptes Rendus Acad. Sci. Paris

3. A. Bensoussan, L. Mertz *Behavior of the plastic deformation of an elasto-perfectly-plastic oscillator with noise*, submitted to Comptes Rendus Acad. Sci. Paris

4. A. Bensoussan, H. Jasso-Fuentes, S. Menozzi, L. Mertz *Asymptotic analysis of stochastic variational inequalities modeling an elasto-plastic problem with vanishing jumps*, submitted to Asymptotic Analysis

5. A. Bensoussan, L. Mertz, *Degenerate Dirichlet problems related to the ergodic theory for an elasto-plastic oscillator excited by a filtered white noise*, submitted to IMA Journal of Applied Mathematics

6. C. Feau, L. Mertz *An empirical study on plastic deformations of an elasto-plastic problem with noise*, submitted to Probabilistic Engineering Mechanics

ORAL PRESENTATIONS IN INTERNATIONAL CONFERENCES

1. The 8th AIMS Conference on Dynamical Systems, Differential Equations and Applications Dresden , Germany, May 25 - 28, 2010 Special Session 29: Applied Analysis and Dynamics in Engineering and Sciences *An ultra weak finite element method as an alternative to a Monte Carlo for an elasto-plastic problem with noise*

2. SIAM Conference on Computational Science and Engineering 2011 Reno, Nevada, February 28-March 04 *A numerical application using the invariant distribution of an elasto-plastic oscillator to compute frequency of plastic deformations*

3. SIAM Conference on Control and its Applications 2011 Baltimore, Maryland, USA, July 25-27 *Stochastic variational inequalities and applications to random vibrations and mechanical structures*

VISITING SCHOLAR : PhD research done in collaboration with A. Bensoussan.

1. The University of Texas at Dallas: 2008, 2009, 2010 (14 months) (see [3], [4],[1])

2. The Hong Kong Polytechnic University: April 2011 (see [5], [6])

Chapter 1

Numerical analysis of the invariant measure for an elasto-plastic problem with noise

Ce chapitre fait l'objet d'un article publié dans SIAM Journal on Numerical Analysis, [3] en collaboration avec Alain Bensoussan, Olivier Pironneau et Janos Turi.

An efficient method for obtaining numerical solutions of an elasto-plastic oscillator with noise is considered. Since Monte-Carlo simulations for the underlying stochastic process are too slow, as an alternative, approximate solutions of the partial differential equation defining the invariant measure of the process are studied. The regularity of the solution of that partial differential equation is not sufficient to employ a "standard" finite element method. To overcome the difficulty, an ultra weak finite element method has been developed and successfully implemented.

1.1 Introduction

Modeling and simulation of elasto-plastic materials under random excitation has been studied by many authors (see e.g. [20], and the bibliography therein) providing important information on the resistance of structures to earthquakes and also on fatigue in general. To understand the problem the simplest is to consider a rod excited at one end by a random force and clamped at the other end. We are interested by the displacement $x(t)$ of the free end of the rod. The velocity at this end point is denoted by $y(t)$, and we have $\dot{x}(t) = y(t)$. Newton's law implies $\dot{y}(t) = -(c_0 y(t) + F(\{x(s), 0 \leqslant s \leqslant t\}) + f(t)$ where $-c_0 y(t)$ is a damping term, $-F(\{x(s), 0 \leqslant s \leqslant t\})$ is a nonlinear restoring force and $f(t)$ is the force applied at the free end of the rod. We assume that $f(t)$ is white noise. The conservation of forces written as a stochastic differential equation (SDE) is

$$\mathrm{d}y(t) = -(c_0 y(t) + F(\{x(s), 0 \leqslant s \leqslant t\}))\mathrm{d}t + \mathrm{d}w(t) \tag{1.1}$$

where $w(t)$ is a standard Wiener process.

Beyond a given threshold $|F(\{x(s), 0 \leqslant s \leqslant t\})| = kY$ for the nonlinear restoring force the material goes through plastic deformation. Introducing $\Delta(t)$, the total plastic yielding accumulated up to time t, we can define a new state variable $z(t)$ as $z(t) = x(t) - \Delta(t)$. It follows that in the plastic regime, $\dot{z}(t) = 0$, while $y(t)$ satisfies (1.1) where now the restoring force, $F(\{x(s), 0 \leqslant s \leqslant t\})$, is written as $F(\{X(s), 0 \leqslant s \leqslant t\}) = kz(t)$ for some constant k and $|z(t)| \leqslant Y$. This is the nonlinear single degree of freedom model of [20]; its mechanical analogy is a system containing a linear mass, dashpot and spring and Coulomb friction-slip joint studied in [26].

In [7] it is shown that (1.1) is equivalent to a stochastic variational inequality (SVI):

$$\begin{cases} \mathrm{d}y(t) = -(c_0 y(t) + kz(t))\mathrm{d}t + \mathrm{d}w(t), \\ (\mathrm{d}z(t) - y(t)\mathrm{d}t)(\zeta - z(t)) \geqslant 0, \\ |\zeta| \leqslant Y, \\ |z(t)| \leqslant Y. \end{cases} \tag{1.2}$$

This system is shown to be well posed for a given probability distribution ψ of the initial condition $(y(0), z(0))$. In [7] existence, uniqueness are proven for a given probability law for $(y(0), z(0))$; ergodicity is also shown. Hence, for the process $(y(t), z(t))$, there exists a unique invariant distribution ν on $D \cup D^+ \cup D^-$ where $D := \mathbb{R} \times (-Y, Y)$ is the elastic domain, $D^+ := (0, \infty) \times \{Y\}$ is the positive plastic domain and $D^- := (-\infty, 0) \times \{-Y\}$ is the negative plastic domain. The invariant distribution ν has a probability density function (pdf) m composed of three L^1 functions

1. an elastic part: $m(y, z)$ on D,
2. a positive plastic part: $m(y, Y)$ on D^+,
3. a negative plastic part: $m(y, -Y)$ on D^-

where $m(y, z), m(y, Y), m(y, -Y) \geqslant 0$ satisfy

$$\int_D m(y, z)\mathrm{d}y\mathrm{d}z + \int_{D^+} m(y, Y)\mathrm{d}y + \int_{D^-} m(y, -Y)\mathrm{d}y = 1.$$

Moreover, the law of the process $(y(t), z(t))$ converges to ν as t goes to ∞ : for all bounded measurable functions f,

$$\lim_{t\to\infty} \mathbb{E}[f(y(t), z(t))] = \int_D f(y, z)m(y, z)\mathrm{d}y\mathrm{d}z + \int_{D^+} f(y, Y)m(y, Y)\mathrm{d}y + \int_{D^-} f(y, -Y)m(y, -Y)\mathrm{d}y.$$

For applications, engineers are interested in m because that describes the asymptotic regime for large times. The numerical simulation of the system by a Monte-Carlo method is straightforward to implement but it is slow. The paper provides a deterministic numerical method, by solving a partial differential equation (PDE) for m. In addition, the semi-group P_t related to $(y(t), z(t))$ satisfies $P_t(f)(\eta, \zeta) = \mathbb{E}[f(y(t), z(t))]$ with $(y(0), z(0)) = (\eta, \zeta)$ for any bounded measurable function f. More precisely, we have

$$\mathbb{E}[f(y(t), z(t))] = \int_D f(y, z)p_t(y, z)\mathrm{d}y\mathrm{d}z + \int_{D^+} f(y, Y)p_t(y, Y)\mathrm{d}y + \int_{D^-} f(y, -Y)p_t(y, -Y)\mathrm{d}y.$$

where p_t is the pdf of $(y(t), z(t))$ and the invariant distribution m can be obtained by $m = \lim_{t\to\infty} p_t$. The latter is characterized by an ultra-weak variational formulation: for all smooth functions φ

$$\int_D m(y,z)A\varphi(y,z)\mathrm{d}y\mathrm{d}z + \int_{D^+} m(y,Y)B_+\varphi(y,Y)\mathrm{d}y + \int_{D^-} m(y,-Y)B_-\varphi(y,-Y)\mathrm{d}y = 0 \quad (1.3)$$

where

$$A\varphi := -\frac{1}{2}\varphi_{yy} + (c_0 y + kz)\varphi_y - y\varphi_z,$$

$$B_+\varphi := -\frac{1}{2}\varphi_{yy} + (c_0 y + kY)\varphi_y,$$

$$B_-\varphi := -\frac{1}{2}\varphi_{yy} + (c_0 y - kY)\varphi_y.$$

The purpose of the paper is to solve (1.3) and compare the results with a standard Monte-Carlo simulation. The key point is to solve a nonlocal PDE which does not depend on time with a function f sufficiently regular at the right hand side:

$$\begin{cases} \lambda u + Au &= f(y,z) \quad \text{in} \quad D, \\ \lambda u + B_+u &= f(y,Y) \quad \text{in} \quad D^+, \\ \lambda u + B_-u &= f(y,-Y) \quad \text{in} \quad D^-. \end{cases} \quad (P_\lambda)$$

with the nonlocal boundary conditions

$$u_\lambda(y,Y) \quad \text{and} \quad u_\lambda(y,-Y) \quad \text{continuous.}$$

From [8], we know that there exists a unique $u_\lambda(y,z)$ solution of (P_λ) such that

$$\|u_\lambda\|_\infty \leqslant \frac{\|f\|_\infty}{\lambda} \quad \text{and} \quad u_\lambda \quad \text{are continuous.}$$

In this work, we justify that this formulation is very relevant from a numerical point of view, since it allows also to obtain $\lim_{t\to\infty} \mathbb{E}[f(y(t), z(t))]$ in a way that does not require to solve a time dependent problem. We shall see that the problem (P_λ) is compatible with an ultra weak finite element method for solving (1.3). Indeed, ergodic theory has been proven in [7] for (1.2) which implies that

$$\forall(\eta,\zeta) \in \bar{D}, \quad \lim_{\lambda\to 0} \lambda u_\lambda(\eta,\zeta) = \lim_{t\to\infty} \mathbb{E}[f(y(t), z(t))]$$

and then

$$\lim_{\lambda\to 0} \lambda u_\lambda(\eta,\zeta) = \int_D m(y,z)f(y,z)\mathrm{d}y\mathrm{d}z + \int_{D^+} m(y,Y)f(y,Y)\mathrm{d}y + \int_{D^-} m(y,-Y)f(y,-Y)\mathrm{d}y,$$

which is an equivalent characterization of m. This limit does not depend on (η,ζ). The paper is organized as follows:

First, in Section 1 we provide a formal presentation of the PDE which characterize m. Then, we argue that we cannot solve this problem by a direct approach since the non-local condition $\int_D m(y,z)\mathrm{d}y\mathrm{d}z + \int_{D^+} m(y,Y)\mathrm{d}y + \int_{D^-} m(y,-Y)\mathrm{d}y = 1$ makes difficult to deal with the boundary conditions. Consequently, we prefer to treat the dual equation (P_λ) of the invariant distribution. Indeed, we show that this class of PDEs allows to recover an equivalent characterization of m as λ goes to 0.
In Section 2, we turn to the numerical solution of (P_λ) and an ultra-weak finite element method is derived to compute the invariant distribution m.
Then, in Section 3 we study a Monte-Carlo algorithm based on (1.1) which is reformulated in a mathematically more practical form with stopping times between elastic and plastic regimes. Trajectories are simulated using the Box-Muller formula and an explicit Euler time finite difference scheme. The probability density for $(y(T), z(T)) \in (y, y+\mathrm{d}y) \times (z, z+\mathrm{d}z)$ is computed and its limit $m(y,z)$ when T is large is the final product of these simulations. A comparison with the results obtained by the ultra-weak finite element method is provided. As usual in dimension 2, here Monte-Carlo method is slower than finite element method.

In conclusion we remark that the ultra-weak method is certainly harder to program compared with Monte-Carlo algorithms but it is faster and, most importantly, more precise even though it is hampered by the necessity to choose several important parameters such as the place at which the computational domain is truncated.

1.2 An equivalent characterization for the invariant measure

In this section, we give a formal presentation of the equation related to m and we argue that we cannot solve it by a direct approach. Testing m with (P_λ), we obtain on D

$$\begin{aligned}
&\lambda \int_D u_\lambda(y,z) m(y,z)\mathrm{d}y\mathrm{d}z - \int_D f(y,z) m(y,z)\mathrm{d}y\mathrm{d}z = \\
&\int_D \left\{ \frac{1}{2} u_{\lambda,yy} - (c_0 y + kz) u_{\lambda,y} + y u_{\lambda,z} \right\} m(y,z)\mathrm{d}y\mathrm{d}z
\end{aligned}$$

which means

$$\begin{aligned}
&\lambda \int_D u_\lambda(y,z) m(y,z)\mathrm{d}y\mathrm{d}z - \int_D f(y,z) m(y,z)\mathrm{d}y\mathrm{d}z = \\
&\int_D u_\lambda(y,z) \left\{ \frac{1}{2} m_{yy} + \frac{\partial}{\partial y}[(c_0 y + kz)m] - ym \right\} \mathrm{d}y\mathrm{d}z \\
&+ \int_{-\infty}^{\infty} y u_\lambda(y,Y) m(y,Y)\mathrm{d}y - \int_{-\infty}^{\infty} y u_\lambda(y,-Y) m(y,-Y)\mathrm{d}y.
\end{aligned}$$

Also, we have on D^+

$$\begin{aligned}
&\lambda \int_{D^+} u_\lambda(y,Y) m(y,Y)\mathrm{d}y - \int_{D^+} f(y,Y) m(y,Y)\mathrm{d}y = \\
&\int_{D^+} u_\lambda(y,Y) \left\{ \frac{1}{2} m_{yy} + \frac{\partial}{\partial y}[(c_0 y + kY)m] \right\} \mathrm{d}y \\
&+ u_\lambda(0^+,Y) \left\{ \frac{1}{2} m_y(0^+,Y) + kY m(0^+,Y) \right\} - \frac{1}{2} u_{\lambda,y}(0^+,Y) m(0^+,Y)
\end{aligned}$$

and on D^-

$$\begin{aligned}
&\lambda \int_{D^-} u_\lambda(y,-Y)m(y,-Y)\mathrm{d}y - \int_{D^-} f(y,-Y)m(y,-Y)\mathrm{d}y = \\
&\int_{D^-} u_\lambda(y,-Y)\left\{\frac{1}{2}m_{yy} + \frac{\partial}{\partial y}[(c_0 y - kY)m]\right\}\mathrm{d}y \\
&+ u_\lambda(0^-,-Y)\left\{kYm(0^-,-Y) - \frac{1}{2}m_y(0^-,-Y)\right\} + u_{\lambda,y}(0^-,-Y)\frac{1}{2}m(0^-,-Y)
\end{aligned}$$

Finally, collecting terms we obtain

$$\begin{aligned}
&\int_0^\infty u_\lambda(y,Y)\left\{ym(y,Y) + \frac{\partial}{\partial y}[(c_0 y + kY)m] + \frac{1}{2}m_{yy}\right\}\mathrm{d}y \\
&+ \int_{-\infty}^0 u_\lambda(y,-Y)\left\{-ym(y,-Y) + \frac{\partial}{\partial y}[(c_0 y - kY)m] + \frac{1}{2}m_{yy}\right\}\mathrm{d}y \\
&+ \int_{-\infty}^0 yu_\lambda(y,Y)m(y,Y)\mathrm{d}y - \int_0^\infty yu_\lambda(y,-Y)m(y,-Y)\mathrm{d}y \\
&+ \int_D u_\lambda(y,z)\left\{\frac{1}{2}m_{yy} + \frac{\partial}{\partial y}[(c_0 y + kz)m] - ym\right\}\mathrm{d}y\mathrm{d}z \\
&+u_\lambda(0^+,Y)\left\{\frac{1}{2}m_y(0^+,Y) + kYm(0^+,Y)\right\} - \frac{1}{2}u_{\lambda,y}(0^+,Y)m(0^+,Y) \\
&+u_\lambda(0^-,-Y)\left\{\frac{1}{2}m_y(0^-,-Y) - kYm(0^-,-Y)\right\} + \frac{1}{2}u_{\lambda,y}(0^-,-Y)m(0^-,-Y) \\
&= \\
&\lambda \int_D u_\lambda(y,z)m(y,z)\mathrm{d}y\mathrm{d}z - \int_D f(y,z)m(y,z)\mathrm{d}y\mathrm{d}z \\
&+\lambda \int_{D^+} u_\lambda(y,Y)m(y,Y)\mathrm{d}y - \int_{D^+} f(y,Y)m(y,Y)\mathrm{d}y \\
&+\lambda \int_{D^-} u_\lambda(y,-Y)m(y,-Y)\mathrm{d}y - \int_{D^-} f(y,-Y)m(y,-Y)\mathrm{d}y.
\end{aligned}$$

Now, using that we also have

$$\forall(\eta,\zeta), \quad u_\lambda(\eta,\zeta) \to \infty, \quad \lambda u_\lambda(\eta,\zeta) \to \nu(f), \quad u_{\lambda,y}(0^+,Y) \to 0, \quad u_{\lambda,y}(0^-,-Y) \to 0,$$

when λ goes to 0, we deduce a formal equation for m which is the following:

$$\left\{\begin{array}{rcl}
-ym_z + \frac{\partial}{\partial y}[m(c_0 y + kz)] + \dfrac{1}{2}m_{yy} & = & 0 \text{ in } D, \\
|y|m + \frac{\partial}{\partial y}[m(c_0 y + kz)] + \dfrac{1}{2}m_{yy} & = & 0 \text{ on } \{sign(y)z = Y\}, \\
m & = & 0 \text{ on } \{sign(y)z = -Y\}, \\
\frac{1}{2}m_y(0^+,Y) + kYm(0^+,Y) & = & 0, \\
\frac{1}{2}m_y(0^-,-Y) - kYm(0^-,-Y) & = & 0
\end{array}\right. \tag{1.4}$$

with

$$m \geqslant 0, \quad \int_D m(y,z)\mathrm{d}y\mathrm{d}z + \int_0^\infty m(y,Y)\mathrm{d}y + \int_{-\infty}^0 m(y,-Y)\mathrm{d}y = 1.$$

The last condition is essential since it express that m cannot be 0 (as a probability density function). Moreover, this express a non-local condition which makes difficult to treat the function on the boundaries. Consequently, we do not know how solve m by a direct approach.

Now, let us recover from the problem (P_λ) an equivalent characterization for the invariant distribution as λ goes to 0.

Proposition 1. *Consider u_λ the solution of (P_λ), then for all $(\eta, \zeta) \in \bar{D}$, we have*

$$\lim_{\lambda \to 0} \lambda u_\lambda(\eta, \zeta) = \int_D m(y,z) f(y,z)\, dy dz + \int_{D^-} m(y,-Y) f(y,-Y)\, dy + \int_{D^+} m(y,Y) f(y,Y)\, dy \tag{1.5}$$

Proof. The result is a direct consequence of the ergodic theory for $(y(t), z(t))$. □

Remark 1. *The formulation* (1.5) *is numerically relevant because we can compute*

$$\lim_{t \to \infty} \mathbb{E}[f(y(t), z(t))]$$

in a way which does not require to solve a time dependent problem.

We shall now show how to use (1.5) to compute m.

1.2.1 Computation of m

Assume that $m \in L^2$. Next, we consider $(g^i)_{i \in I}$ a set of independent piecewise continuous functions in $L^2(D) \cup L^2(D^-) \cup L^2(D^+)$ and we approximate $\{m(y,z), (y,z) \in D\}$ in $L^2(D)$, $\{m(y,Y), y > 0\}$ in $L^2(D^+)$, $\{m(y,-Y), y < 0\}$ in $L^2(D^-)$, by $\sum_{i \in I} m_i g^i$ where $\{m_i, i \in I\}$ is a sequence of real numbers. To each basis function g^i,$i \in I$, we associate the unique solution u^i of (P_λ) where g^i is at the right hand side. We shall build m by inverting (1.5); this means solving a linear system with matrix

$$\mathbf{A}_{i,j} = \int_D g^i(y,z) g^j(y,z) \mathrm{d}z\mathrm{d}y + \int_{-\infty}^0 g^i(y,-Y) g^j(y,-Y) \mathrm{d}y + \int_0^\infty g^i(y,Y) g^j(y,Y) \mathrm{d}y \tag{1.6}$$

Note that (1.5) is also $(\mathbf{A}m)_i = (\lim_{\lambda \to 0} \lambda u^i)$.

1.3 Numerical result with the equivalent characterization for the invariant measure

For the computations, we need to truncate D into D_L and add an artificial condition on $y = \pm L$. We denote

$$D_L^+ = \{0 < y < L, z = Y\} \text{ and } D_L^- = \{-L < y < 0, z = -Y\}.$$

We add a Neumann condition on the boundary $y = \pm L$:

$$\begin{cases} \lambda u + Au &= g & \text{in } D_L, \\ \lambda u + B_+ u &= g_+ & \text{in } D_L^+, \\ \lambda u + B_- u &= g_- & \text{in } D_L^-, \\ \partial_n u &= 0 & \text{on } -Y < z < Y, y = \pm L. \end{cases} \tag{1.7}$$

We found that the choice of L is critical, it needs to be small to reduce the computing time and large for precision. Next, the main difficulty was to deal with boundary conditions.

1.3.1 A superposition method for the boundary conditions

We observed that the conditions at $0 < y < L, z = \pm Y$ are 2 autonomous ODE in the y variable which could be solved separately in order to obtain non homogenous Dirichlet conditions for u. But, we needed to impose appropriate boundaries conditions at 0 and at $\pm L$.

It seemed reasonable to impose homogenous Neumann conditions at $(y, z) = (\pm L, \pm Y)$. But, the values of u – denoted by $u_\pm$ – at $y = 0, z = \pm Y$ are unknown, so we shall use the linearity of u with respect to these real numbers and dertermine them later.

Computation of $u_\pm$

In [8] it is shown that u_λ is continuous, so continuity at the points $(0, Y)$ and $(0, -Y)$ of u_λ are the equations which determine the unknown constants of $u_\pm$.
By linearity the solution of (1.7) is also a linear combination of the three following problems:

$$\left\{\begin{array}{rcll} \lambda u_0 + Au_0 &=& g & \text{in } D, \\ \lambda u_0 + B_+ u_0 &=& g_+ & \text{in } D_L^+, \\ \lambda u_0 + B_- u_0 &=& g_- & \text{in } D_L^-, \end{array}\right.$$
$$\text{with } u_{0,+} = 0, u_{0,-} = 0,$$

$$\left\{\begin{array}{rcll} \lambda u_1 + Au_1 &=& 0 & \text{in } D, \\ \lambda u_1 + B_+ u_1 &=& 0 & \text{in } D_L^+, \\ \lambda u_1 + B_- u_1 &=& 0 & \text{in } D_L^-, \end{array}\right.$$
$$\text{with } u_{1,+} = 1, u_{1,-} = 0,$$

$$\left\{\begin{array}{rcll} \lambda u_2 + Au_2 &=& 0 & \text{in } D, \\ \lambda u_2 + B_+ u_2 &=& 0 & \text{in } D_L^+, \\ \lambda u_2 + B_- u_2 &=& 0 & \text{in } D_L^-, \end{array}\right.$$
$$\text{with } u_{2,+} = 0, u_{2,-} = 1.$$

We must find α and β such that $u = u_0 + \alpha u_1 + \beta u_2$ is continuous in $(0, -Y)$ and $(0, Y)$, i.e.

$$\begin{array}{rcl} u_0(0^+, Y) + \alpha u_1(0^+, Y) + \beta u_2(0^+, Y) &=& u_0(0^-, Y) + \alpha u_1(0^-, Y) + \beta u_2(0^-, Y) \\ u_0(0^+, -Y) + \alpha u_1(0^+, -Y) + \beta u_2(0^+, -Y) &=& u_0(0^-, -Y) + \alpha u_1(0^-, -Y) + \beta u_2(0^-, -Y) \end{array}$$

Finally, we solve the following linear system.
$$\begin{pmatrix} u_1(0^+, Y) - u_1(0^-, Y) & u_2(0^+, Y) - u_2(0^-, Y) \\ u_1(-0^+, -Y) - u_1(0^-, -Y) & u_2(0^+, -Y) - u_2(0^-, -Y) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} u_0(0^-, Y) - u_0(0^+, Y) \\ u_0(0^-, -Y) - u_0(0^+, -Y) \end{pmatrix}$$

Test of Convergence of λu_λ to a constant value

We have verified numerically that λu_λ converges, to a constant value. Figure 1.1-1.2 corresponds to the following choice:

$$g(y, z) = \frac{1}{2\pi\sigma_z\sigma_y} e^{-\frac{1}{2}\frac{z^2}{\sigma_z^2}} e^{-\frac{1}{2}\frac{y^2}{\sigma_y^2}} \tag{1.8}$$

In figure 1.3-1.4, g is similar but non zero on $D^\pm$:

$$g(y, z) = \frac{1}{2\pi\sigma_z\sigma_y} e^{-\frac{1}{2}\frac{(z-Y)^2}{\sigma_z^2}} e^{-\frac{1}{2}\frac{y^2}{\sigma_y^2}} \tag{1.9}$$

where $\sigma_z = \sigma_y = \sqrt{1/200}$. Both plots show that indeed λu_λ tends to a constant when λ tends to zero.

1.3.2 Computing m by an ultra-weak finite element method

Given a mesh of D_L generated with the software `freefem++`, we consider a family of gaussian function centered on each node (y_i, z_i) of the mesh.

$$g_i(y,z) = \frac{1}{2\pi\sigma_z\sigma_y} e^{-\frac{1}{2}(\frac{z-z_i}{\sigma_z})^2} e^{-\frac{1}{2}(\frac{y-y_i}{\sigma_y})^2}$$

Then we solve the problem (1.7) by a finite element method of degree one, also using `freefem++`. Finally we solve the linear system for m. The results are shown on figure 1.5,1.6 and the comparison with the Monte-Carlo method is given of Figure 1.10. We denote m_N the solution given by the ultra weak finite element method where N is the number of basis functions.

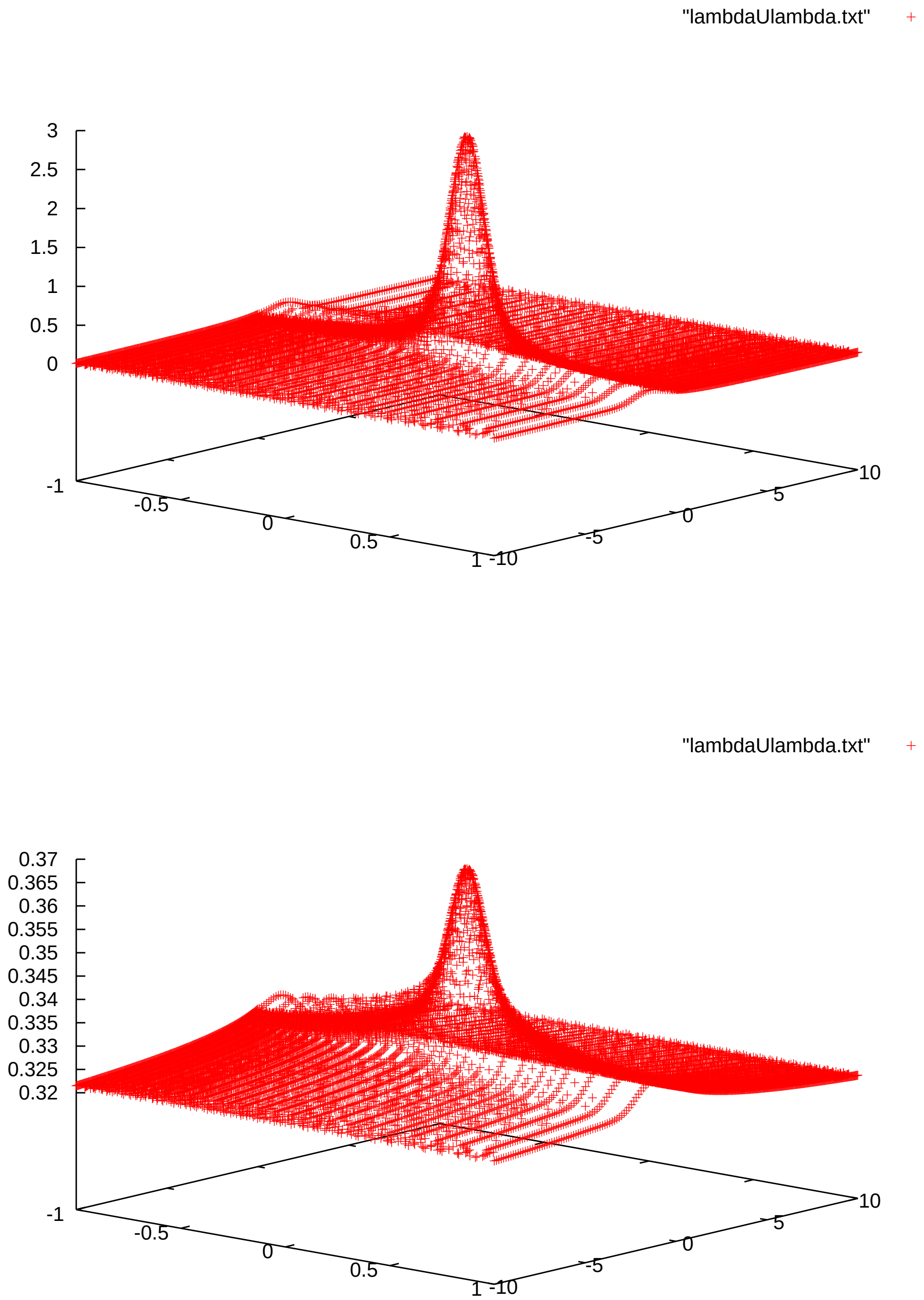
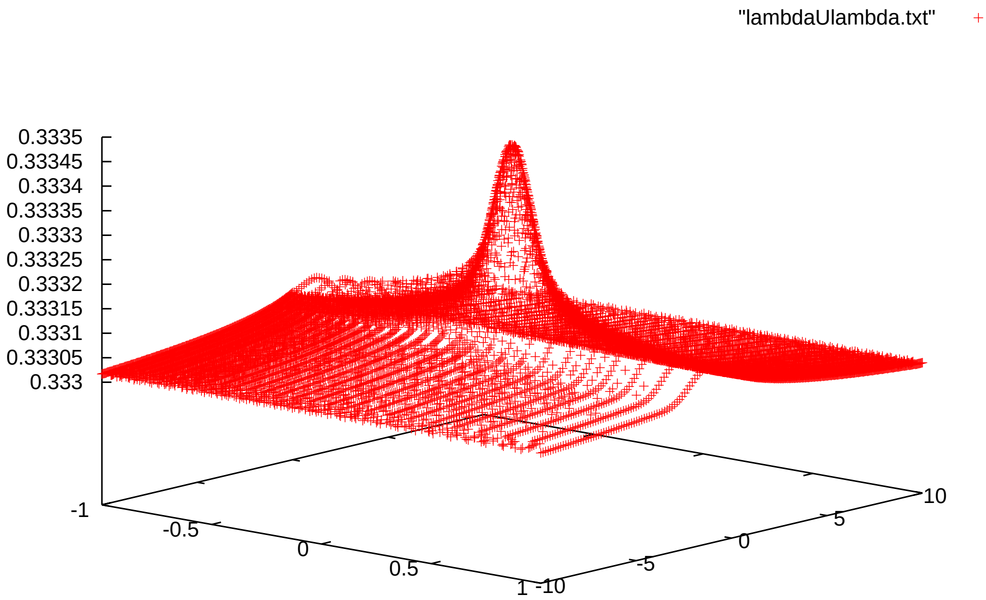


Figure 1.1: *From the top to the bottom:* λu_λ *for* $\lambda = 1.0$ *and* $\lambda = 10^{-1}$*, with* $g(y,z) = \frac{1}{2\pi\sigma_z\sigma_y} e^{-\frac{1}{2}\frac{z^2}{\sigma_z^2}} e^{-\frac{1}{2}\frac{y^2}{\sigma_y^2}}$ *and* $\sigma_z = \sigma_y = \sqrt{1/200}$.



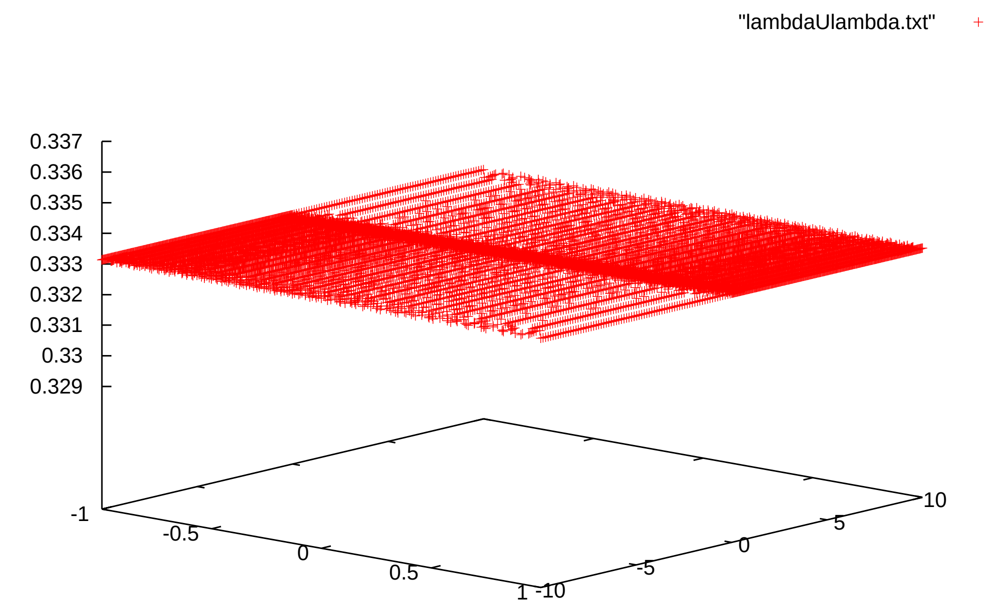
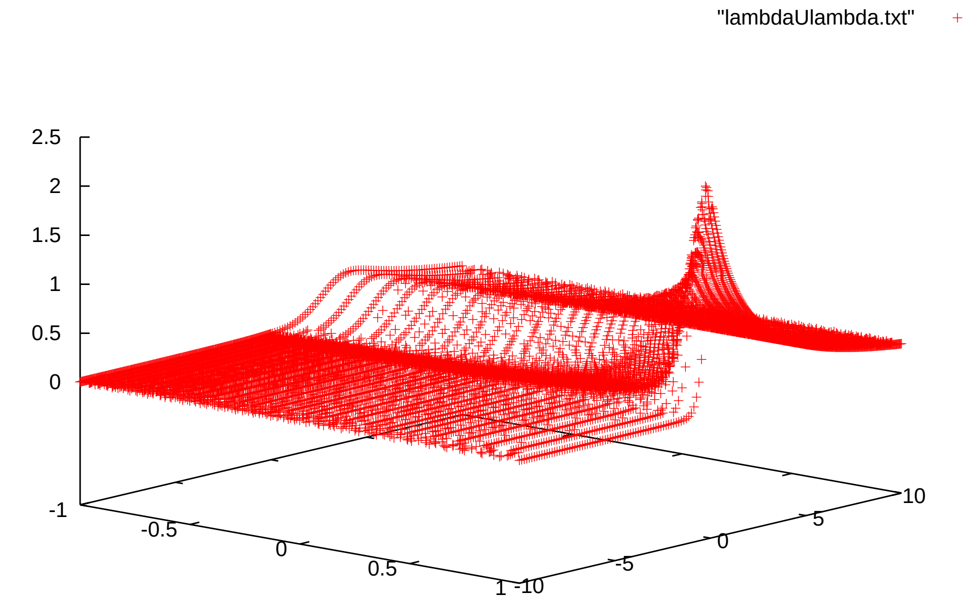


Figure 1.2: *From the top to the bottom:* λu_λ *for* $\lambda = 10^{-4}$ *and* $\lambda = 10^{-9}$*, with* $g(y,z) = \frac{1}{2\pi\sigma_z\sigma_y} e^{-\frac{1}{2}\frac{z^2}{\sigma_z^2}} e^{-\frac{1}{2}\frac{y^2}{\sigma_y^2}}$ *and* $\sigma_z = \sigma_y = \sqrt{1/200}$.



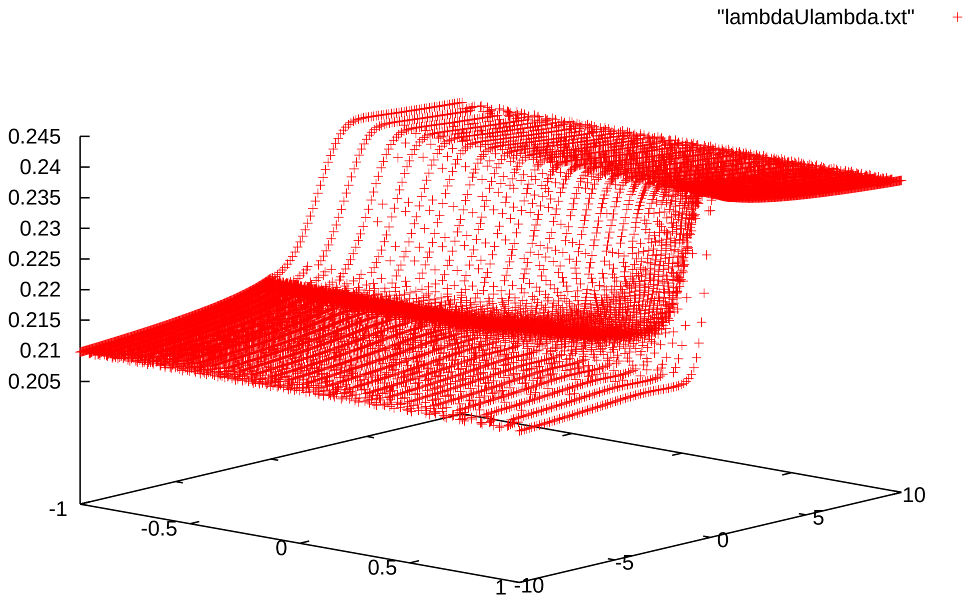
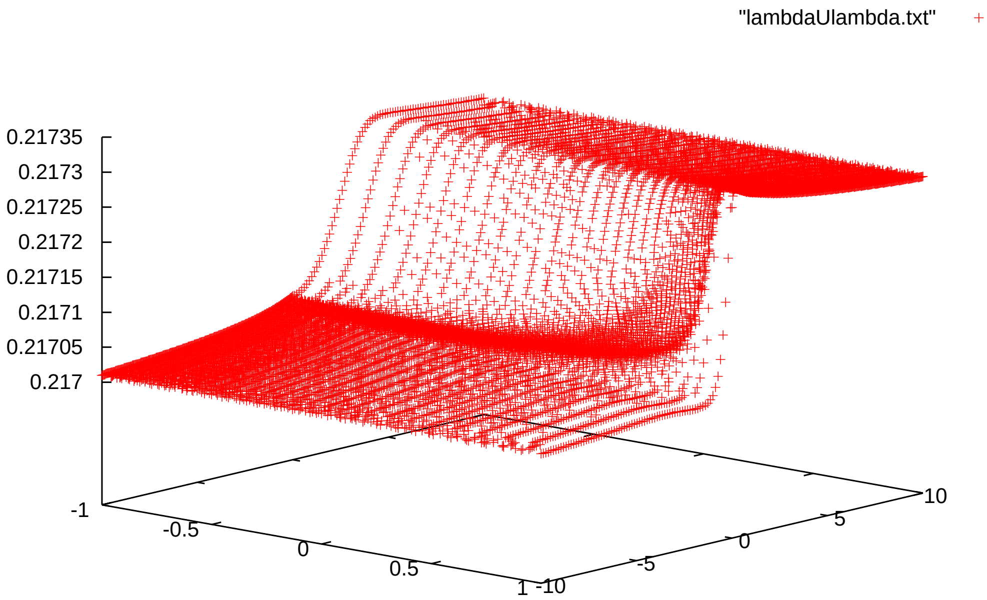


Figure 1.3: *From the top to the bottom:* λu_λ *for* $\lambda = 1.0$ *and* $\lambda = 10^{-1}$, *with* $g(y,z) = \frac{1}{2\pi\sigma_z\sigma_y} e^{-\frac{1}{2}\frac{(z-Y)^2}{\sigma_z^2}} e^{-\frac{1}{2}\frac{y^2}{\sigma_y^2}}$ *and* $\sigma_z = \sigma_y = \sqrt{1/200}$.



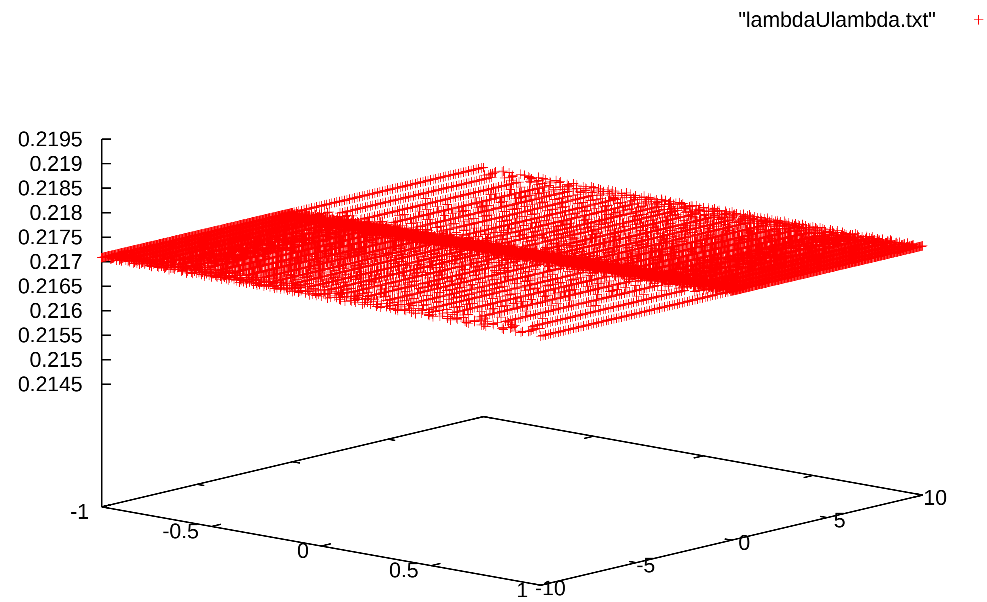


Figure 1.4: *From the top to the bottom:* λu_λ *for* $\lambda = 10^{-4}$ *and* $\lambda = 10^{-9}$*, with* $g(y,z) = \frac{1}{2\pi\sigma_z\sigma_y} e^{-\frac{1}{2}\frac{(z-Y)^2}{\sigma_z^2}} e^{-\frac{1}{2}\frac{y^2}{\sigma_y^2}}$ *and* $\sigma_z = \sigma_y = \sqrt{1/200}$.

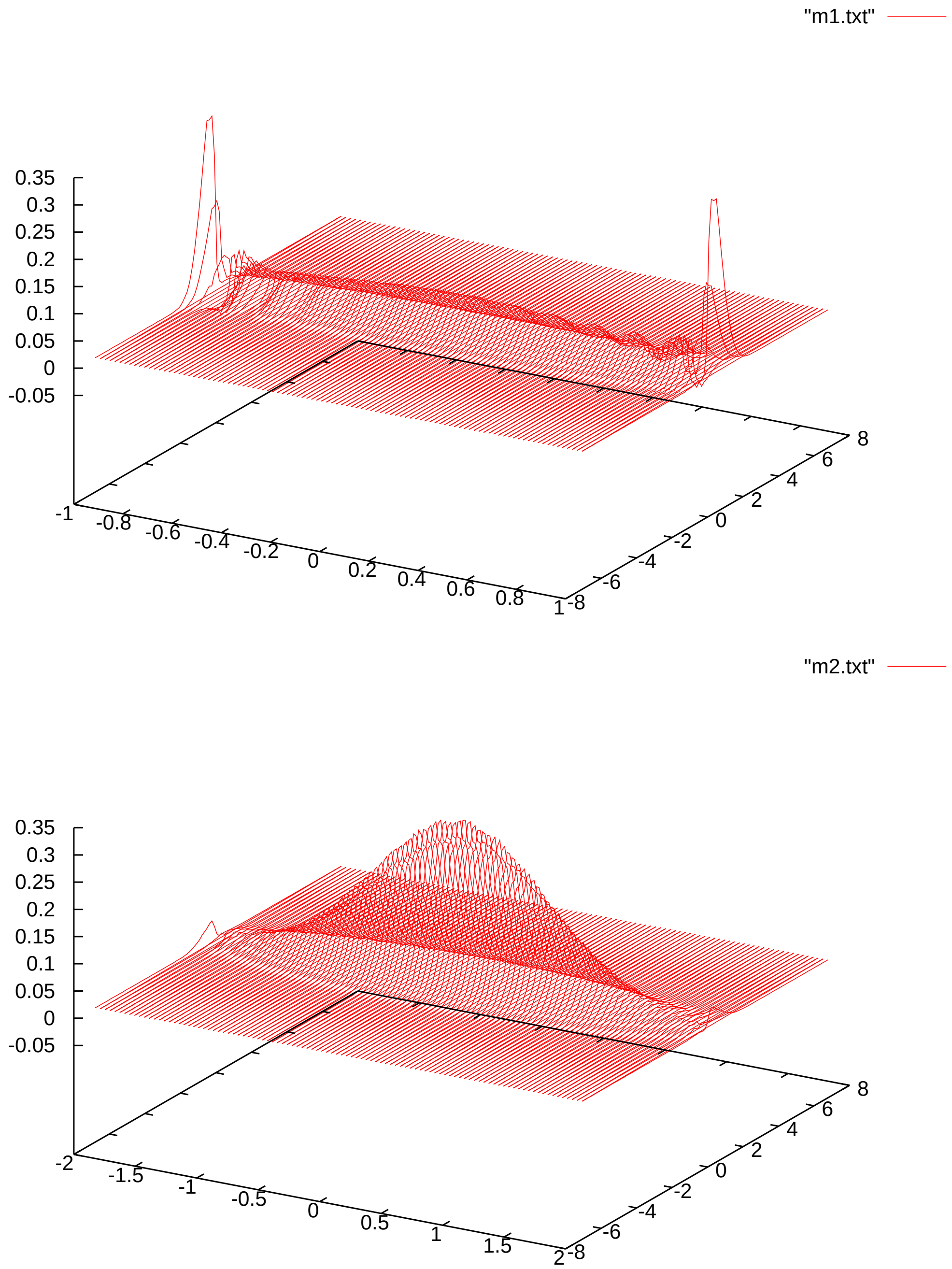
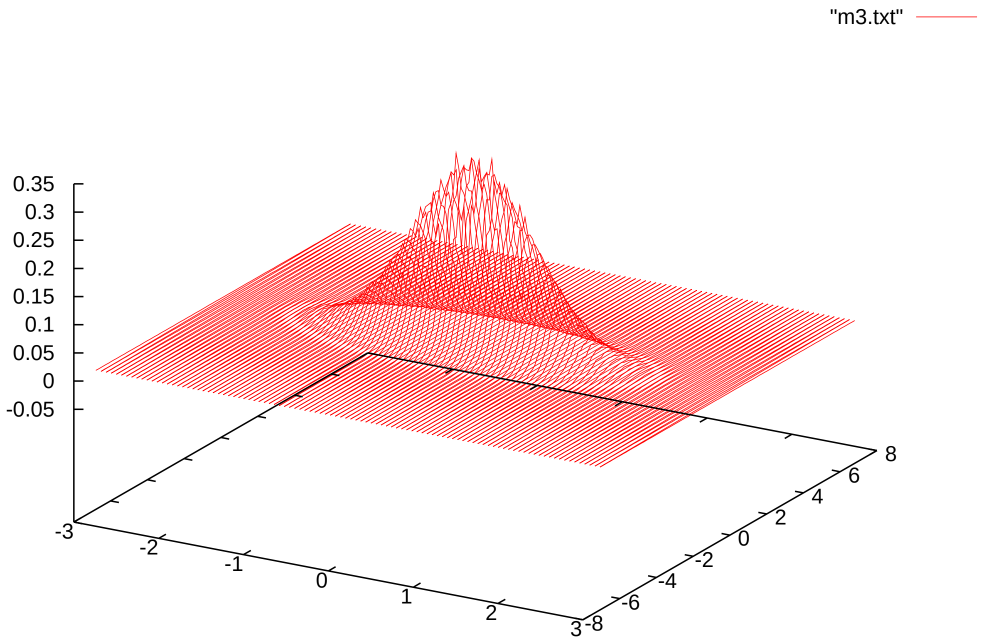


Figure 1.5: *m_N computed by the Ultra Weak Method, from the top to the bottom we consider $Y = 1, 2$. When Y increases the plastic part decreases.*



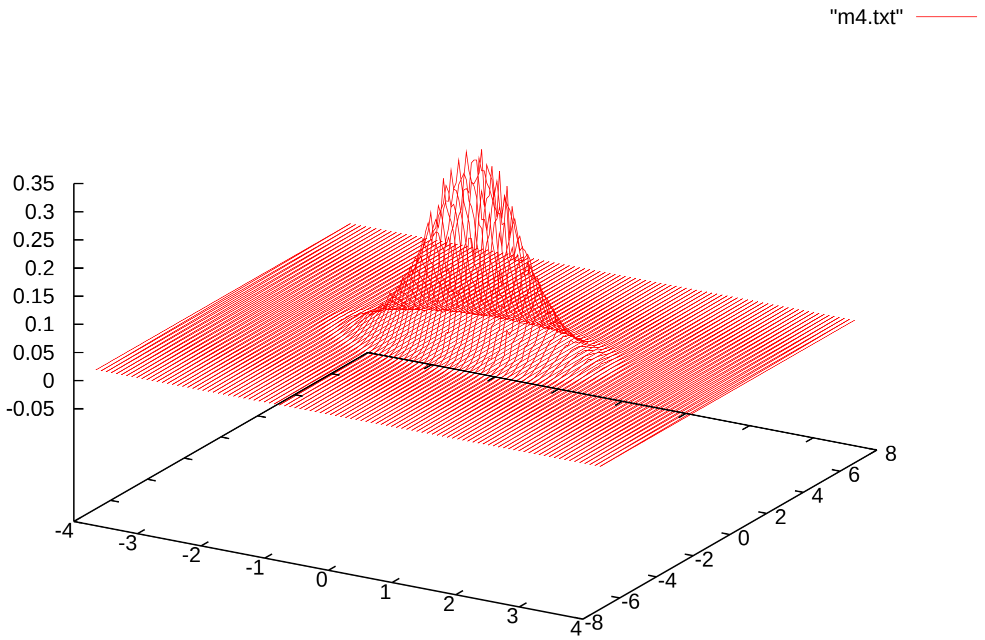


Figure 1.6: *m_N computed by the Ultra Weak Method, from the top to the bottom we consider $Y = 3, 4$. When Y increases the plastic part decreases.*

1.4 Monte-Carlo simulations

1.4.1 Reformulation with stopping times

We recall from [7] the stochastic differential equations governing the process $(y(t), z(t))$ of (1.1) corresponding to the elastic and plastic regimes, respectively.

For $n = 0, 1, ..$ and starting with $\theta_0 = \tau_0 = 0$, we define two sets of stopping times in the following way:

$$\left\{ \begin{array}{lll} \theta_{n+1} & = & \inf\{t > \tau_n, \quad |z(t)| = Y\} \\ \tau_{n+1} & = & \inf\{t > \theta_{n+1}, \quad y(t) = 0\} \end{array} \right. \tag{1.10}$$

When $t \in [\tau_n, \theta_{n+1}[$, we have $|z(t)| < Y$ and $(y(t), z(t))$ satisfies:

$$\left\{ \begin{array}{lll} \mathrm{d}y(t) & = & -(c_0 y(t) + kz(t))\mathrm{d}t + \mathrm{d}w(t) \\ \mathrm{d}z(t) & = & y(t)\mathrm{d}t \end{array} \right. \tag{1.11}$$

When $t \in [\theta_{n+1}, \tau_{n+1}[$, we have $|z(t)| = Y$ and $(y(t), z(t))$ satisfies

$$\left\{ \begin{array}{lll} \mathrm{d}y(t) & = & -(c_0 y(t) + kz(t))\mathrm{d}t + \mathrm{d}w(t) \\ \mathrm{d}z(t) & = & 0 \end{array} \right. \tag{1.12}$$

Remark 2. *$(\theta_n)_{n\geqslant 1}$ corresponds to each entry in plastic regime. During plastic regime, when sign$(y(t))$ changes the behavior of the oscillator comes back to the elastic regime. Then $(\tau_n)_{n\geqslant 1}$ corresponds to each exit of plastic regime.*

1.4.2 Analytic formulae

We will use later the fact that in each elastic and plastic time segments there is an analytic formula for the solution. Let $(y(0), z(0)) = (y, z)$ and let $\omega = \frac{\sqrt{4k - c_0^2}}{2}$. From now on, we assume $4k > c_0^2$. Note that the condition $4k > c_0^2$ is needed so that $(y(t), z(t))$ have real valued solutions. We have

$$\left\{ \begin{array}{lll} z(t) & = & e^{\frac{-c_0 t}{2}}\{z\cos(\omega t) + \frac{1}{\omega}(y + \frac{c_0}{2}z)\sin(\omega t)\} \\ & & +\frac{1}{\omega}\int_0^t e^{-\frac{c_0}{2}(t-s)}\sin(\omega(t-s))dw(s), \\ y(t) & = & -\frac{c_0}{2}z(t) + e^{-\frac{c_0 t}{2}}\{-\omega z\sin(\omega t) + (y + \frac{c_0}{2}z)\cos(\omega t)\} \\ & & +\int_0^t e^{-\frac{c_0}{2}(t-s)}\cos(\omega(t-s))dw(s). \end{array} \right. \tag{1.13}$$

$z(t)$ is a gaussian variable with mean equal to $e_z(t, z, y)$ and variance equal to $\sigma_z^2(t)$, where

$$\begin{array}{lll} e_z(t, y, z) & = & e^{\frac{-c_0 t}{2}}\{z\cos(\omega t) + \dfrac{1}{\omega}(y + \dfrac{c_0}{2}z)\sin(\omega t)\}, \\ \sigma_z^2(t) & = & \dfrac{1}{\omega^2}\displaystyle\int_0^t e^{-c_0 s}\sin^2(\omega s)ds. \end{array}$$

$y(t)$ is a gaussian variable with mean equal to $e_y(t, y, z)$ and variance equal to $\sigma_y^2(t)$, where

$$\begin{array}{lll} e_y(t, y, z) & = & -\dfrac{c_0}{2}e_z(t, z, y) + e^{-\frac{c_0 t}{2}}\{-\omega z\sin(\omega t) + (y + \dfrac{c_0}{2}z)\cos(\omega t)\}, \\ \sigma_y^2(t) & = & \displaystyle\int_0^t e^{-c_0 s}\cos^2(\omega s)ds - \dfrac{4c_0^2}{\omega^2}\int_0^t e^{-c_0 s}\sin^2(\omega s)ds - \dfrac{c_0}{2\omega^2}e^{-c_0 t}\sin^2(\omega t). \end{array}$$

The correlation between $y(t)$ and $z(t)$ is given by

$$\sigma_{yz}(t) \quad = \quad \frac{1}{2\omega}\int_0^t e^{-c_0 s}\sin(2\omega s)ds - \frac{c_0}{4\omega^2}\int_0^t e^{-c_0 s}\sin^2(\omega s)ds.$$

When the system is in a plastic state there is an analytic solution also. Let $(y(0), z(0)) = (y, \pm Y)$ then

$$\begin{cases} z(t) &= \pm Y, \\ y(t) &= ye^{-c_0 t} \mp \frac{kY}{c_0}(1 - e^{-c_0 t}) + \frac{e^{-c_0 t}}{\sqrt{2c_0}} w(e^{2c_0 t} - 1). \end{cases} \tag{1.14}$$

1.4.3 Monte-Carlo method for computing the invariant measure

Based on these analytic solutions, a C code was written to simulate $(y(t), z(t))$. Let $T > 0, N \in \mathbb{N}$, and $(t_n)_{n=0..N}$ a family of time which discretize $[0, T]$, such that $t_n = n\delta t$ where $\delta t := \frac{T}{N}$.
We set $\Sigma \in \mathcal{M}_{2,2}(\mathbb{R}^2)$ such that

$$\Sigma\Sigma^T = \begin{pmatrix} \sigma_z^2(\delta t) & \sigma_{z,y}(\delta t) \\ \sigma_{z,y}(\delta t) & \sigma_y^2(\delta t) \end{pmatrix}. \tag{1.15}$$

Let $(G_{n,m})_{n=0..N, m=1,2}$ be random independent Gaussian $\mathcal{N}(0,1)$ variables. Here, all gaussian random variables are generated by the Box-Muller formula [23] and the C function `rand()`. The finite difference scheme for (1.1) is as follows:

if $|z(t_n)| < Y$,

$$\begin{pmatrix} z(t_{n+1}) \\ y(t_{n+1}) \end{pmatrix} \quad = \quad \begin{pmatrix} e_z(\delta t, y(t_n), z(t_n)) \\ e_y(\delta t, y(t_n), z(t_n)) \end{pmatrix} + \Sigma \begin{pmatrix} G_{n,1} \\ G_{n,2} \end{pmatrix}, \tag{1.16}$$

if $|z(t_n)| = Y$,

$$\begin{pmatrix} z(t_{n+1}) \\ y(t_{n+1}) \end{pmatrix} \quad = \quad \begin{pmatrix} \pm Y \\ y(t_n)e^{-c_0\delta t} \mp \frac{kY}{c_0}(1 - e^{-c_0\delta t}) + e^{-c_0\delta t}\sqrt{\frac{e^{2c_0\delta t}-1}{2c_0}}G_{n,2} \end{pmatrix}. \tag{1.17}$$

Figure 1.7 shows a sample of trajectory of the process at T=10. Then, to compute the probability density function of $(y(T), z(T))$, we define L sufficiently large and the domains

$$D_L := (-L, L) \times (-Y, Y), \quad D_L^- := (0, L) \times \{Y\}, \quad D_L^- := (-L, 0) \times \{-Y\}$$

and the three uniform grids on D_L, D_L^+ and D_L^-

$$\mathcal{G}(N_y, N_z), \quad \mathcal{G}(N_y), \quad \mathcal{G}(N_y)$$

for given integers N_y, N_z.
The i,j-cell on D_L is

$$\mathcal{D}_{i,j} = (-L + \frac{2jL}{N_y}, -L + \frac{2(j+1)L}{N_y}) \times (-Y + \frac{2iY}{N_z}, -Y + \frac{2(i+1)Y}{N_z}])$$

and the j-cell on $D_L^\pm$ is

$$\mathcal{D}_j^+ = (j\frac{L}{N_y}, (j+1)\frac{L}{N_y}), \quad \mathcal{D}_j^- = (-L + j\frac{L}{N_y}, -L + (j+1)\frac{L}{N_y})$$

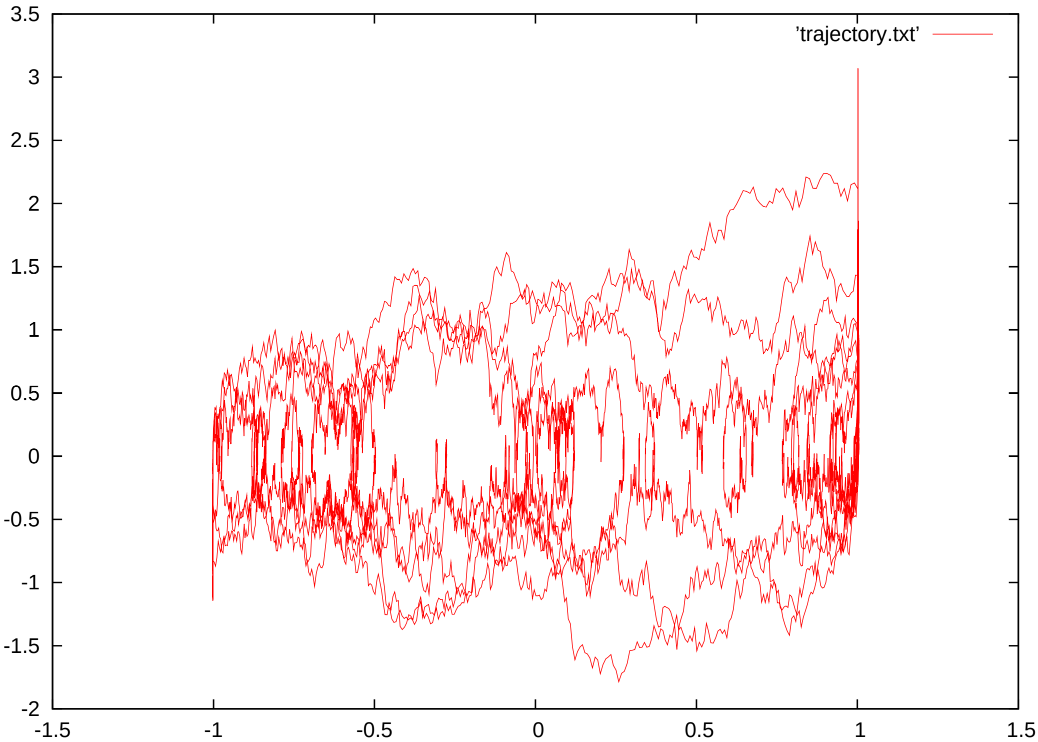


Figure 1.7: *Example of trajectory* $(y(t), z(t))_{t\in[0,T]}, T = 10$.

for given $0 \leqslant i < N_z, 0 \leqslant j < N_y$. Let us generate numerically N trajectories up to time T and count the number $n_{i,j}, n_j^+$ and n_j^- of trajectories ending in $\mathcal{D}_{i,j}$,$\mathcal{D}_j^+$ and $\mathcal{D}_j^-$. By the law of large numbers we can approximate the probability that $((y(T), z(T)) \in \mathcal{D}_{i,j})$ by $\bar{X}_{i,j}^N := \frac{n_{i,j}}{N}$ and the probability that $((y(T), z(T)) \in \mathcal{D}_{j^\pm})$ by $\bar{X}_j^{\pm,N} := \frac{n_j^\pm}{N}$. By the central limit theorem we also know the error: for instance at 5 % error

$$\mathbb{P}((y(T), z(T)) \in \mathcal{D}_{i,j}) \in \left(\bar{X}_{i,j}^N - \frac{1.96\bar{X}_{i,j}^N(1-\bar{X}_{i,j}^N)}{\sqrt{N}}, \bar{X}_{i,j}^N + \frac{1.96\bar{X}_{i,j}^N(1-\bar{X}_{i,j}^N)}{\sqrt{N}}\right)$$

and also

$$\mathbb{P}((y(T), z(T)) \in \mathcal{D}_j^\pm) \in \left(\bar{X}_j^{\pm,N} - \frac{1.96\bar{X}_j^{\pm,N}(1-\bar{X}_j^{\pm,N})}{\sqrt{N}}, \bar{X}_j^{\pm,N} + \frac{1.96\bar{X}_j^{\pm,N}(1-\bar{X}_j^{\pm,N})}{\sqrt{N}}\right)$$

The invariant measure m of the process is computed as the asymptotic limit for large T of $\bar{X}_{i,j}^N$ and $\bar{X}_j^{\pm,N}$. In figure 1.8-1.9 m is shown for different values of Y, $Y = 1, 2, 3, 4$, with $L = 7, T = 10, N = 10^7$. Moreover, for each Y, we can estimate the plastic part denoted by p_Y^{MC},

$$p_Y^{MC} := \mathbb{P}\big((y(T), z(T)) \in (0, +\infty)\big) \times \{Y\} + \mathbb{P}\big((y(T), z(T)) \in (-\infty, 0) \times \{-Y\}\big)$$

i.e.

$$p_Y^{MC} = \int_0^\infty m(y, Y)dy + \int_{-\infty}^0 m(y, -Y)dy$$

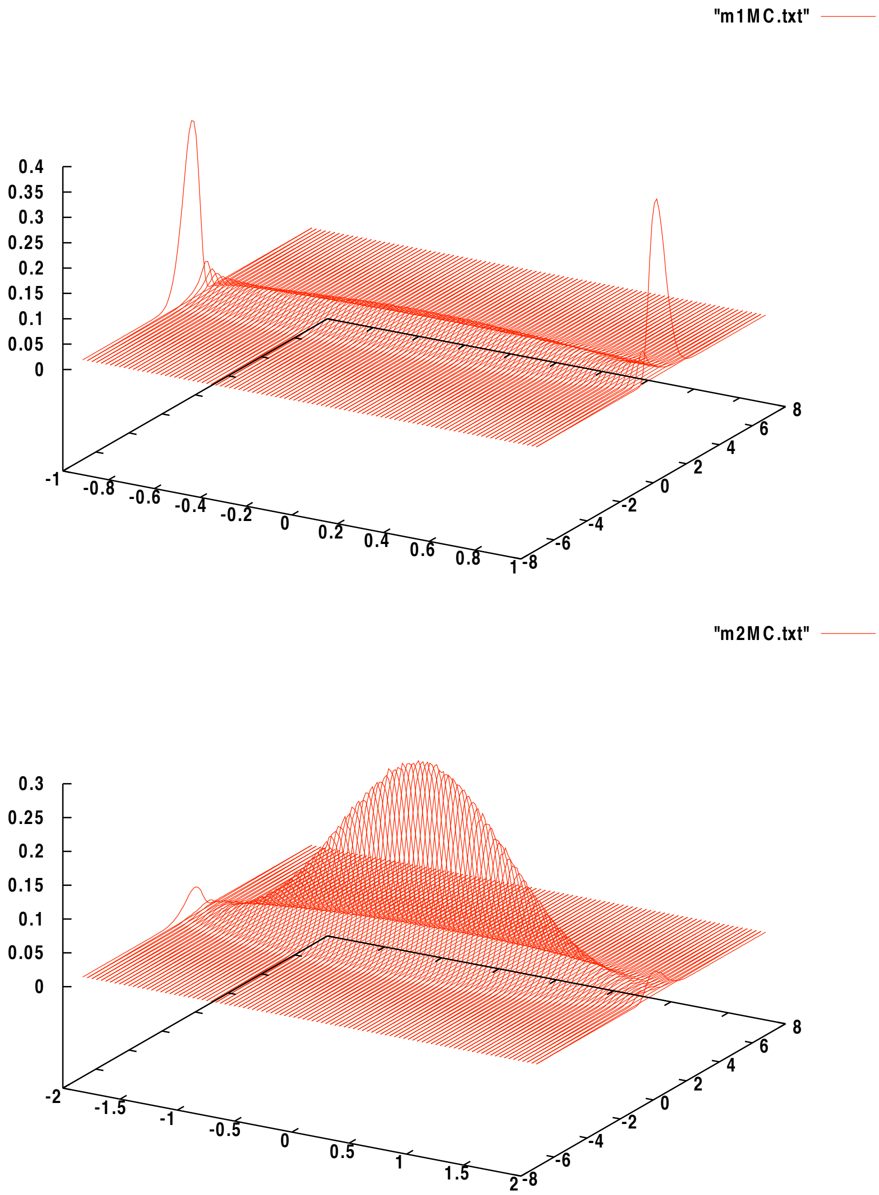


Figure 1.8: *Plots of m_{MC} computed by the Monte-Carlo Method; from the top to the bottom we consider $Y = 1, 2$. When Y increases the plastic part decreases, indeed* $p_1^{MC} = 84.9_{[84.7,85.0]}\%, p_2^{MC} = 9.62_{[9.61,9.63]}\%$

and the confidence interval for a 5 % error. We denote m_{MC} the solution given by the Monte-

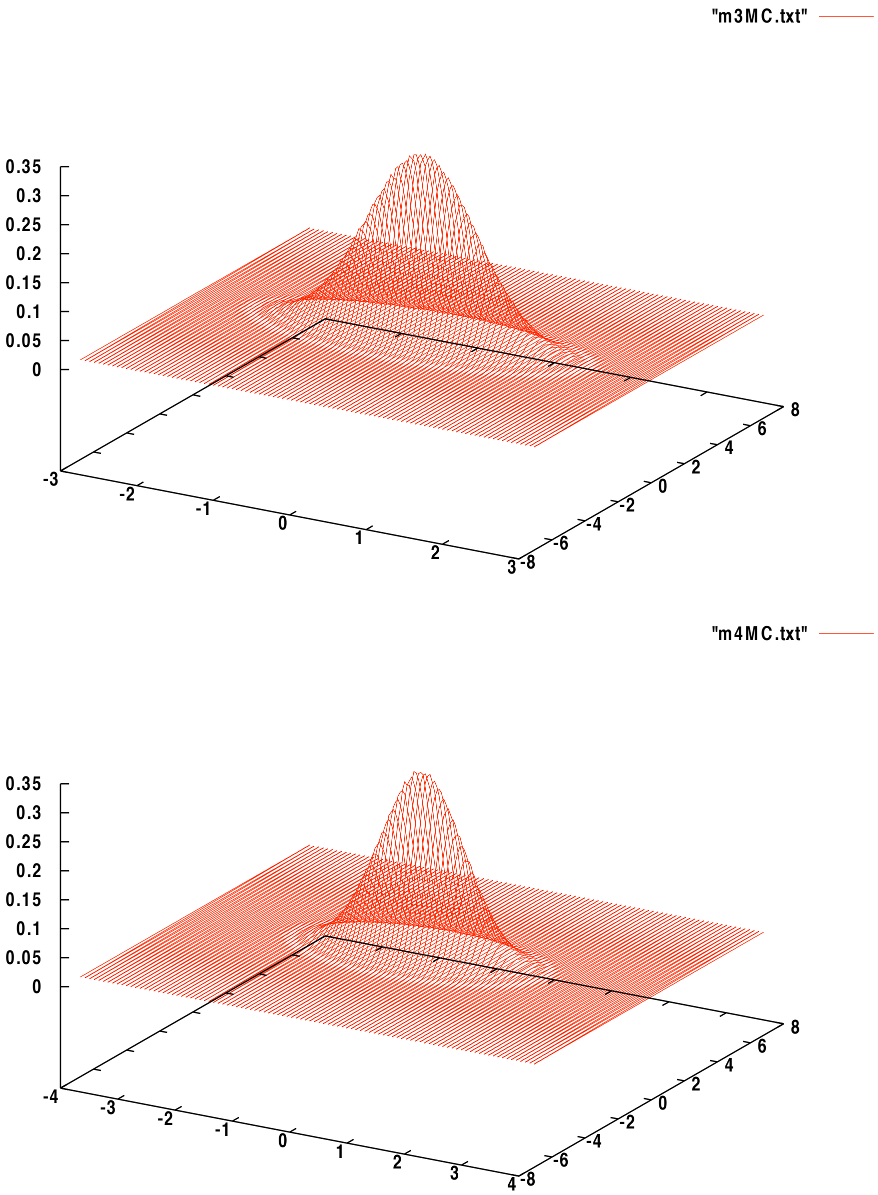


Figure 1.9: *Plots of m_{MC} computed by the Monte-Carlo Method; from the top to the bottom we consider $Y = 3, 4$. When Y increases the plastic part decreases, indeed $p_3^{MC} = 0.04764_{[0.04760,0.04768]}\%, p_4^{MC} = 0.0\ \%$.*

Carlo method.

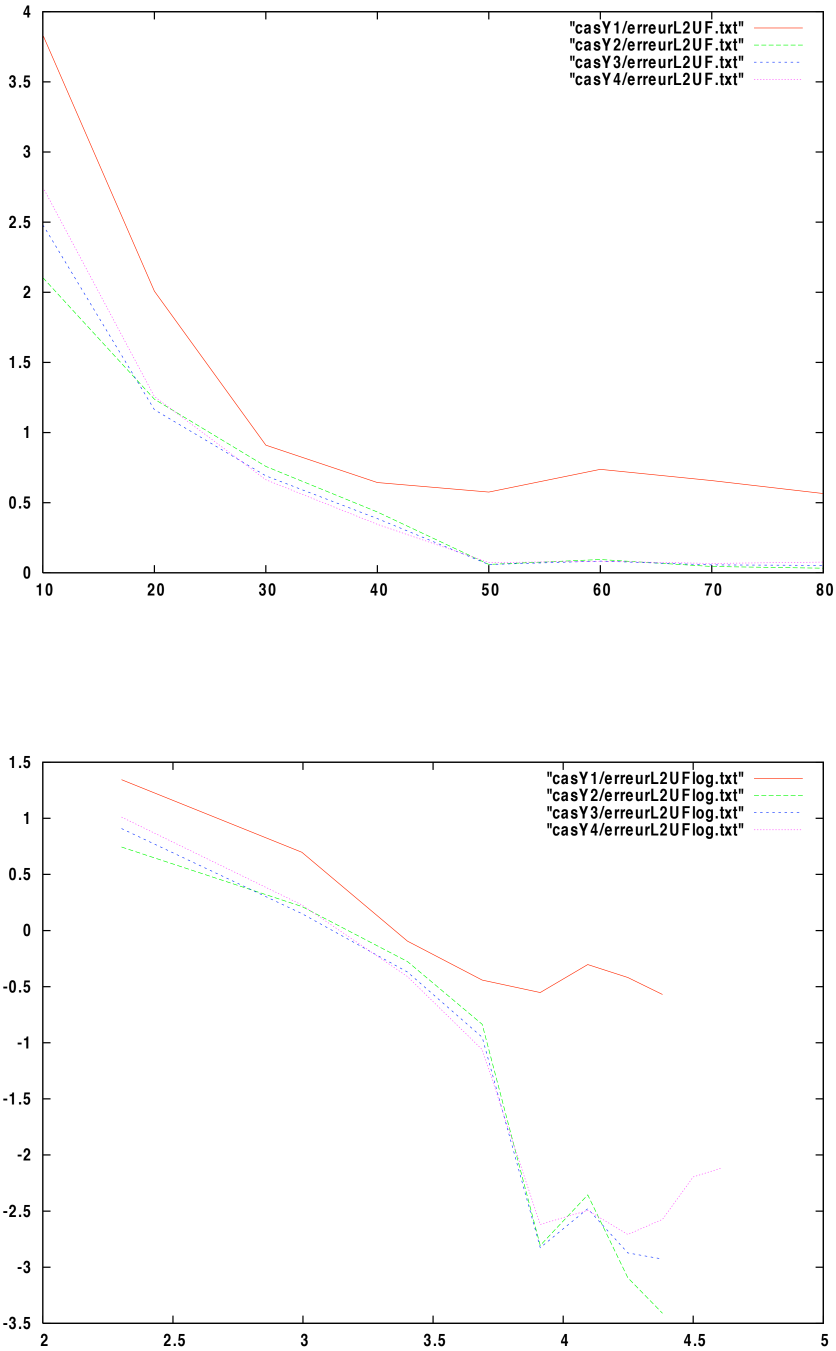


Figure 1.10: *Top: L^2-Relative error : $\frac{\|m_{MC}-m_N\|_{L^2}}{\|m_{MC}\|_{L^2}}$ between the result given by the Monte-Carlo method and the results given by the Ultra Weak method, versus $\sqrt{N}$, where N is the number of basis functions. Bottom: The same plot in log-log scale.*

1.5 Conclusion

In this work, we compared a deterministic method to a Monte-Carlo computation to compute the stationnary state of the process $(y(t), z(t))$ when $t \to \infty$. The Monte-Carlo method is straightforward but expensive. The probability density of $\lim_{t\to\infty}(y(t), z(t))$ is also characterized by a PDE. As we saw, classical methods, on this PDE do not work because m belong to L^2 but not to H^1. So, an Ultra Weak method has been proposed to compute m. The main idea is to solve the dual problem of m on each function of a basis of L^2. Comparing the result between the two methods, we found less than 3 % of L^2 relative difference. This deterministic method is also expensive but it is more precise than the Monte-Carlo method at equal computing time so it is faster at comparable precisions. In the future, we shall extend this approach to multidimensional problems, more relevant to application for earthquake engineering.

1.6 Appendix: A numerical study of elastic phasing

Let us exhibit the phenomenon of micro-elastic phases which are also small as well as numerous.

1.6.1 Direct numerical simulations

Figure 1.11 shows (a) the sojourns in the elastic and plastic phases, (b) the sojourns in micro-elastic and micro-plastic phases.

1.6.2 Approximation of the second marginal of m

In this subsection, we consider the numerical resolution of problem (P_λ). Concentration of micro-elastic phasing must appear in temporal averaging of $(y(t), z(t))$ on long time period. We recall that ergodicity implies

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(y(s), z(s))\mathrm{d}s = \int_{D^-} m(y, -Y) f(y, -Y)\mathrm{d}y + \int_{D^+} m(y, Y) f(y, Y)\mathrm{d}y + \int_D m(y, z) f(y, z)\mathrm{d}y\mathrm{d}z,$$

for all bounded continuous function $\forall f$. Consequently, m has to be affected by this phenomenon. So, we are interested in computing the second marginal of m in elastic phase that is

$$U : s \to \int_{-\infty}^{+\infty} m(y, s)\mathrm{d}y, \quad s \in (-Y, Y).$$

For this purpose, we introduce

- $U_n(s)$ to approximate $U(s)$,

$$U_n(s) := \int_{-\infty}^{+\infty} \int_{-Y}^{Y} m(y, z)\chi_n(z - s)\mathrm{d}z\mathrm{d}y.$$

- $W_n(s)$ (the derivative of $U_n(s)$) to compute sensibility to the threshold $U'(s)$.

$$W_n(s) := n \int_{-\infty}^{+\infty} \int_{-Y}^{Y} m(y, z)(z - s)\chi_n(z - s)\mathrm{d}z\mathrm{d}y,$$

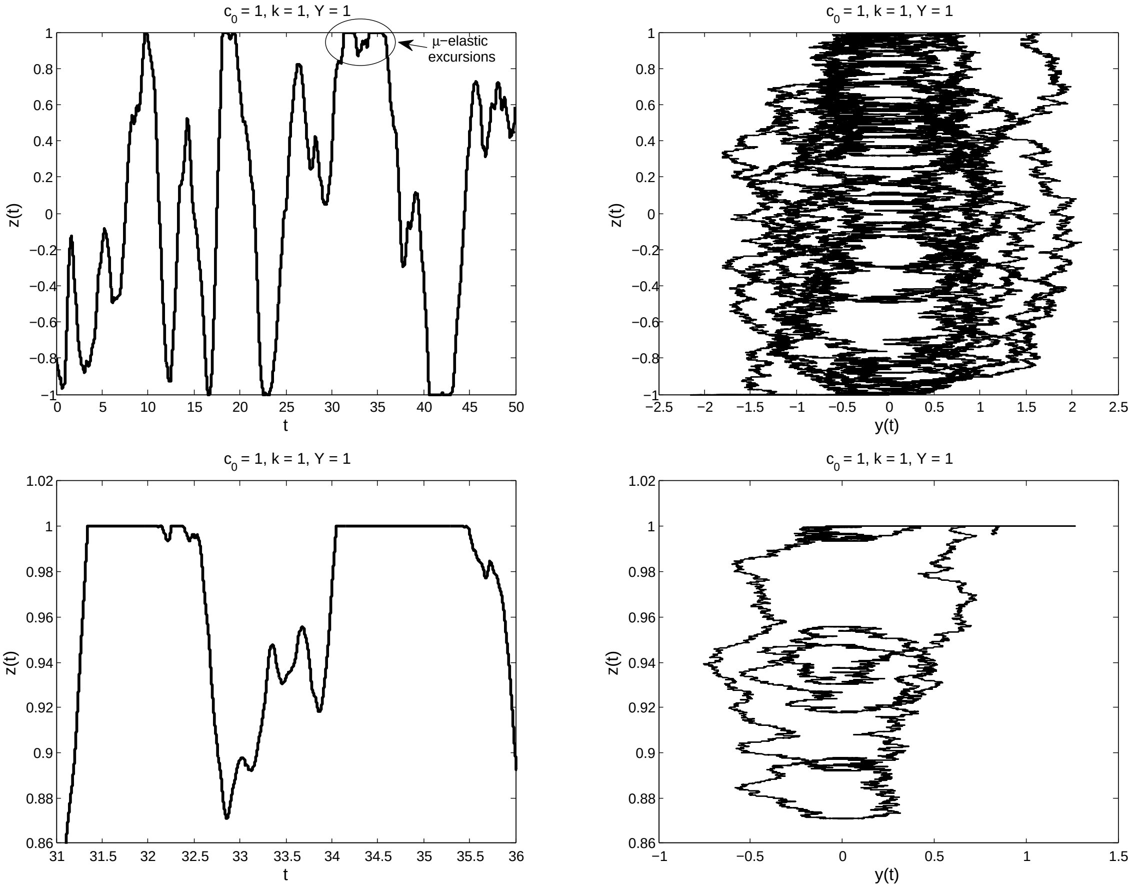


Figure 1.11: *Visualization of Micro-Elastic Phasing Phenomenon: At the top-left we have sample trajectory of $(t, z(t))$, at the top-right we have the corresponding trajectory $(y(t), z(t))$ in the phase space; we observe regular elastic and plastic phasing. At the bottom in the both cases, a zoom on plastic phasing reveals an illustrative micro-elastic phasing.*

where χ_n is an approximation of the Dirac function

$$\chi_n(x) := \sqrt{\frac{n}{2\pi}}\exp(-\frac{nx^2}{2}), \quad n \text{ is sufficiently large.} \tag{1.18}$$

Proposition 2. *$\forall s \in (-Y, Y)$, $U_n(s), W_n(s)$ are converging sequences when n goes to ∞. Furthermore, they can be expressed by the behavior at zero of the problem (P_λ) with $\chi_n(z-s)$ and $(z-s)\chi_n(z-s)n$ respectively at the right hand side. More precisely, consider $\tilde{u}$ such that*

$$\begin{cases} \lambda\tilde{u} - \frac{1}{2}\tilde{u}_{yy} + (c_0 y + kz)\tilde{u}_y - y\tilde{u}_z &= f(y,z) \quad y \in \mathbb{R}, \quad |z| < Y, \\ \lambda\tilde{u} - \frac{1}{2}\tilde{u}_{yy} + (c_0 y + kY)\tilde{u}_y &= f(y,Y) \quad y > 0, \quad z = Y, \\ \lambda\tilde{u} - \frac{1}{2}\tilde{u}_{yy} + (c_0 y - kY)\tilde{u}_y &= f(y,-Y) \quad y < 0, \quad z = -Y. \end{cases}$$

Then

- $\lim_{\lambda\to 0} \lambda\tilde{u} = U_n(s)$, *if* $f(y,z) = \chi_n(z-s)$ *or*
- $\lim_{\lambda\to 0} \lambda\tilde{u} = W_n(s)$, *if* $f(y,z) = n(z-s)\chi_n(z-s)$.

Proof. As m is assumed sufficiently regular inside D, we deduce convergence of the sequences of Dirac's approximation $U_n(s) \to U(s)$ and $W_n(s) \to U'(s)$ as $n \to \infty$. □

Computational results on the invariant distribution

In Figure 1.12, 1.13 and 1.14 (left) computations of pdf of z show up a significant concentration of $\lim_{t\to\infty} z(t)$ in the neighborhood of plastic boundaries. So, in Figures 1.12,1.13 and 1.14 (right) points of inflection of U' are located. For applications, we believe that we located thresholds $s^{\pm}$ sufficiently far away from the boundary. Indeed, note that $s^{\pm}$ are near but not on the boundary, thereby justifying existence of micro-elastic phasing. This phenomenon is also present in the probabilistic numerical simulation of U_n (not shown here) but it is quantified much more accurately with the PDEs related to the invariant distribution.

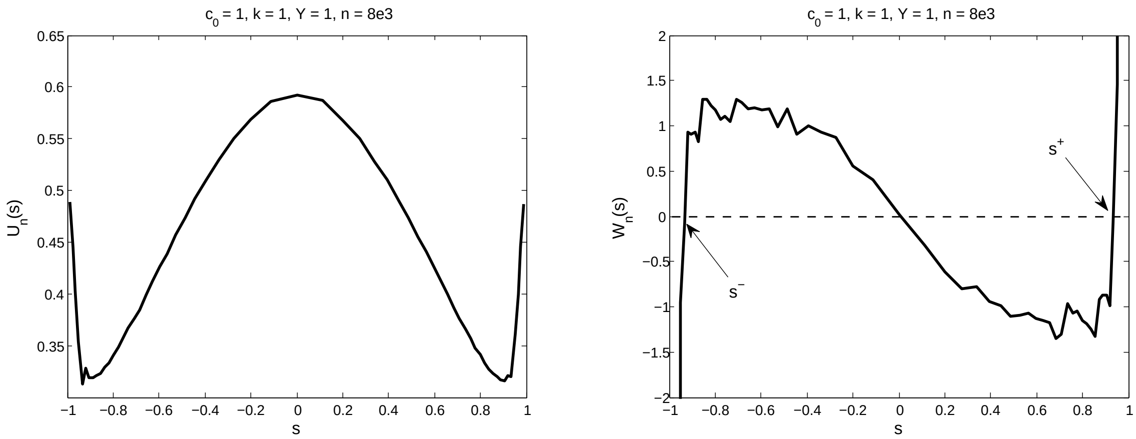


Figure 1.12: $U_n(s), W_n(s), s \in (-Y, Y)$ and location of $s^{\pm}, Y = 1$.

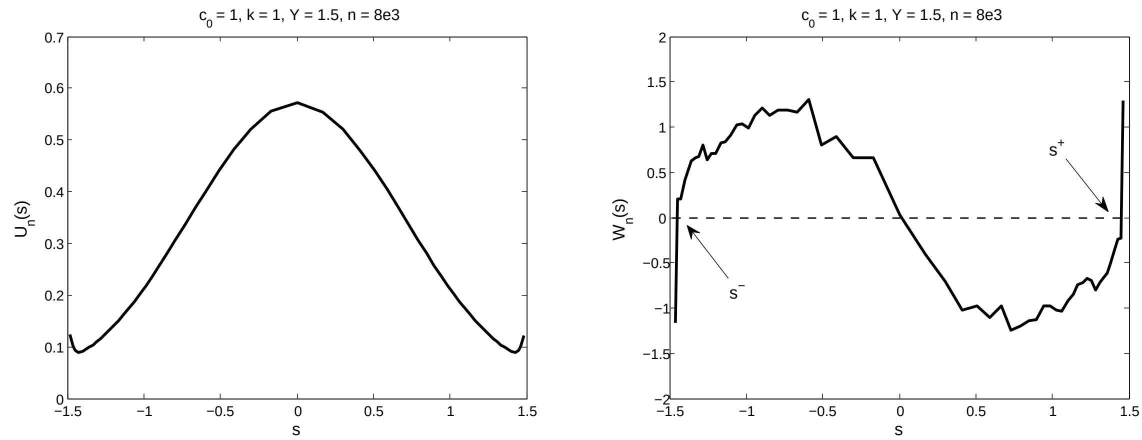


Figure 1.13: $U_n(s), W_n(s), s \in (-Y, Y)$ and location of $s^{\pm}, Y = 1.5$.

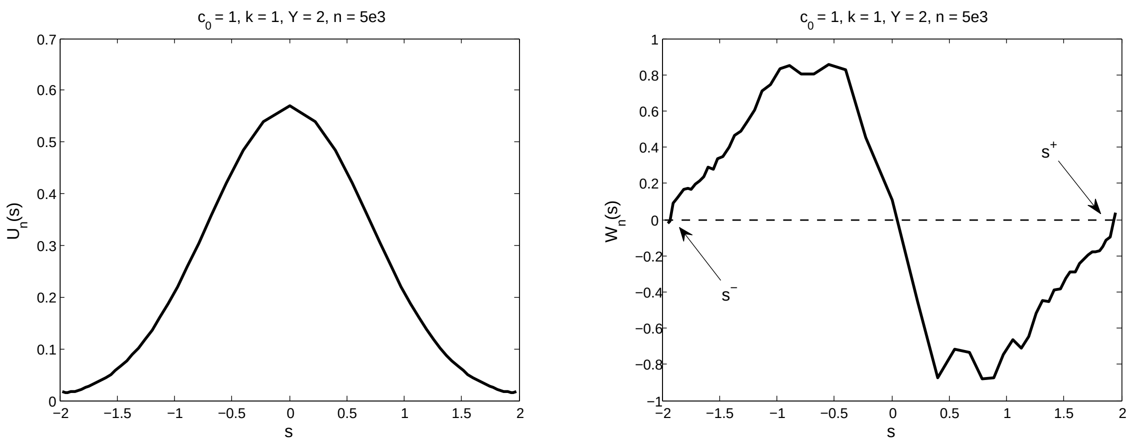


Figure 1.14: $U_n(s), W_n(s), s \in (-Y, Y)$ and location of $s^{\pm}, Y = 2$.

Chapter 2

An analytic approach to the ergodic theory of stochastic variational inequalities

Ce chapitre fait l'objet d'une note soumise au Comptes Rendus Acad. Sci. Paris [5] en collaboration avec Alain Bensoussan.

In an earlier work made by the first author with J. Turi (Degenerate Dirichlet Problems Related to the Invariant Measure of Elasto-Plastic Oscillators, AMO, 2008), the solution of a stochastic variational inequality modeling an elasto-perfectly-plastic oscillator has been studied. The existence and uniqueness of an invariant measure have been proven. Nonlocal problems have been introduced in this context. In this work, we present a new characterization of the invariant measure. The key finding is the connection between nonlocal PDEs and local PDEs which can be interpreted with short cycles of the Markov process solution of the stochastic variational inequality.

Résumé

Une approche analytique de la théorie ergodique des inéquations variationnelles stochastiques. Dans un travail précédent du premier auteur en collaboration avec Janos Turi (Degenerate Dirichlet Problems Related to the Invariant Measure of Elasto-Plastic Oscillators, AMO, 2008), la solution d'une inéquation variationnelle stochastique modélisant un oscillateur élastique-parfaitement-plastique a été étudiée. L'existence et l'unicité d'une mesure invariante ont été prouvées. Des problèmes nonlocaux ont été introduits dans ce contexte. La conclusion importante est la connexion entre des EDPs nonlocales et des EDPs locales qui peuvent être interprétées comme les cycles courts du processus de Markov solution de l'inéquation variationnelle stochastique.

Version française abrégée

La dynamique de l'oscillateur élastique-parfaitement-plastique s'exprime à l'aide d'une équation à mémoire (voir (2.1)-(2.2)). A. Bensoussan et J. Turi ont montré que la relation entre la vitesse et la composante élastique de l'oscillateur est un processus de Markov ergodique qui satisfait une inéquation variationnelle stochastique (voir (2.3)). La solution admet une mesure invariante caractérisée par dualité à l'aide d'une équation aux dérivées partielles avec des conditions de bord non-locales (voir (2.4)). Dans ce travail, une nouvelle preuve de la théorie ergodique est présentée ainsi qu'une nouvelle caractérisation de l'unique distribution invariante. Dans ce contexte, nous déduisons des nouvelles formules reliant des équations aux dérivées partielles avec des conditions de bord non-locales à des problèmes locaux (voir (2.10)).

2.1 Introduction

In the engineering literature, the dynamics of the elastic-perfectly-plastic (EPP) oscillator has been formulated as a process $x(t)$ which stands for the displacement of the oscillator, evolving with hysteresis. The evolution is defined by the problem

$$\ddot{x} + c_0\dot{x} + \mathbf{F}(x(s), 0 \leqslant s \leqslant t) = \dot{w} \tag{2.1}$$

with initial conditions of displacement and velocity $x(0) = x, \quad \dot{x}(0) = y$. Here $c_0 > 0$ is the viscous damping coefficient, $k > 0$ the stiffness, w is a Wiener process; $\mathbf{F}(x(s), 0 \leqslant s \leqslant t)$ is a nonlinear functional which depends on the entire trajectory $\{x(s), 0 \leqslant s \leqslant t\}$ up to time t. The plastic deformation denoted by $\Delta(t)$ at time t can be recovered from the pair $(x(t), \mathbf{F}(x(s), 0 \leqslant s \leqslant t))$ by the following relationship:

$$\mathbf{F}(x(s), 0 \leqslant s \leqslant t) = \begin{cases} kY & if \quad x(t) = Y + \Delta(t), \\ k(x(t) - \Delta(t)) & if \quad x(t) \in] - Y + \Delta(t), Y + \Delta(t)[, \\ -kY & if \quad x(t) = -Y + \Delta(t). \end{cases} \tag{2.2}$$

where Y is an elasto-plastic bound. Such elasto-plastic oscillator is simple and representative of the elasto-plastic behavior of a class of structure dominated by their first mode of vibration, they are employed to estimate prediction of failure of mechanical structures. Karnopp & Scharton [26] proposed a separation between elastic states and plastic states and introduced a fictitious variable $z(t) := x(t) - \Delta(t)$.

Recently, the right mathematical framework of stochastic variational inequalities (SVI) modeling an EPP oscillator with noise has been introduced by one of the authors in [7]. Although SVI have been already studied in [2] to represent reflection-diffusion processes in convex sets, no connection with random vibration had been made so far. The inequality governs the relationship between the velocity $y(t)$ and the variable $z(t)$:

$$\mathrm{d}y(t) = -(c_0 y(t) + kz(t))\mathrm{d}t + \mathrm{d}w(t), \quad (\mathrm{d}z(t) - y(t)\mathrm{d}t)(\phi - z(t)) \geqslant 0, \quad \forall |\phi| \leqslant Y, \quad |z(t)| \leqslant Y. \tag{2.3}$$

Let us introduce some notations.

Notation 3. *$D := \mathbb{R} \times (-Y, +Y)$, $D^+ := (0, \infty) \times \{Y\}$, $D^- := (-\infty, 0) \times \{-Y\}$, and the differential operators $A\zeta := -\frac{1}{2}\zeta_{yy} + (c_0 y + kz)\zeta_y - y\zeta_z$, $B_+\zeta := -\frac{1}{2}\zeta_{yy} + (c_0 y + kY)\zeta_y$, $B_-\zeta := -\frac{1}{2}\zeta_{yy} + (c_0 y - kY)\zeta_y$. where ζ is a regular function on D.*

In [7], it has been shown that the probability distribution of $(y(t), z(t))$ converges to an asymptotic probability measure on $D \cup D^+ \cup D^-$ namely ν. Moreover, ν is the unique invariant distribution of $(y(t), z(t))$. In addition, from [8] we know also that there exists a unique solution u_λ to the following partial differential equation (PDE)

$$\lambda u_\lambda + Au_\lambda = f \quad \text{in} \quad D, \quad \lambda u_\lambda + B_+ u_\lambda = f \quad \text{in} \quad D^+, \quad \lambda u_\lambda + B_- u_\lambda = f \quad \text{in} \quad D^- \tag{2.4}$$

with the nonlocal boundary conditions given by the fact that $u_\lambda(y, Y)$ and $u_\lambda(y, -Y)$ are continuous, where $\lambda > 0$ and f is a bounded measurable function. The function u_λ satisfies $\|u_\lambda\|_\infty \leqslant \frac{\|f\|_\infty}{\lambda}$, u_λ is continuous and for all $(\eta, \zeta) \in \bar{D}$, we have $\lim_{\lambda \to 0} \lambda u_\lambda(\eta, \zeta) = \nu(f)$. We use the notation $u_\lambda(y, z; f)$.

Now, we introduce the short cycles to provide a new proof of the ergodic theory for (2.3). In this context, we derive new formulas linking PDEs with nonlocal boundary conditions to local problems.

2.1.1 Short cycles

Let $\lambda > 0$, consider $v_\lambda(y, z)$ the solution of

$$\lambda v_\lambda + Av_\lambda = f \quad \text{in} \quad D, \quad \lambda v_\lambda + B_+ v_\lambda = f \quad \text{in} \quad D^+, \quad \lambda v_\lambda + B_- v_\lambda = f \quad \text{in} \quad D^- \tag{2.5}$$

with the local boundary conditions $v_\lambda(0^+, Y) = 0$ and $v_\lambda(0^-, -Y) = 0$. Also, if f is symmetric (resp. antisymmetric) then v_λ is symmetric (resp. antisymmetric). We use the notation $v_\lambda(y, z; f)$. As $\lambda \to 0$, $v_\lambda \to v$ with

$$Av = f \quad \text{in} \quad D, \quad B_+ v = f \quad \text{in} \quad D^+, \quad B_- v = f \quad \text{in} \quad D^- \tag{P_v}$$

with the local boundary conditions $v(0^+, Y) = 0$ and $v(0^-, -Y) = 0$. We use the notation $v(y, z; f)$. We call $v(y, z; f)$ a short cycle. We detail the solution of (P_v) in the next section. We introduce next $\pi^+(y, z)$ and $\pi^-(y, z)$ such that

$$A\pi^+ = 0 \quad \text{in} \quad D, \quad \pi^+ = 1 \quad \text{in} \quad D^+, \quad \pi^+ = 0 \quad \text{in} \quad D^- \tag{2.6}$$

and

$$A\pi^- = 0 \quad \text{in} \quad D, \quad \pi^- = 0 \quad \text{in} \quad D^+, \quad \pi^- = 1 \quad \text{in} \quad D^-. \tag{2.7}$$

We have $\pi^+ + \pi^- = 1$, so the existence and uniqueness of a bounded solution to (2.6) and (2.7) are clear. A new formulation of the invariant distribution is given by the following theorem.

Theorem 2.1.1 (New formulation of the invariant distribution ν)**.** *Let f be a bounded measurable function on $\bar{D}$, we have the following analytical characterization of the invariant distribution:*

$$\nu(f) = \frac{v(0^-, Y; f) + v(0^+, -Y; f)}{2v(0^+, Y; 1)}.$$

Denote $\nu_\lambda(f) := \frac{v_\lambda(0^-, Y; f) + v_\lambda(0^+, -Y; f)}{2v_\lambda(0^-, Y; 1)}$. As $\lambda \to 0$,

$$u_\lambda(y, z; f) - \frac{\nu_\lambda(f)}{\lambda} \to u(y, z; f), \quad \nu_\lambda(f) \to \nu(f) \tag{2.8}$$

where u satisfies

$$Au = f - \nu(f) \quad \text{in} \quad D, \quad B_+u \quad = \quad f - \nu(f) \quad \text{in} \quad D^+, \quad B_-u \quad = \quad f - \nu(f) \quad \text{in} \quad D^- \tag{2.9}$$

with the nonlocal boundary conditions given by the fact that

$$u(y,Y) \quad \textit{and} \quad u(y,-Y) \quad \textit{are continuous.}$$

Then, we obtain also the representation formula

$$u(y,z;f) = v(y,z;f) - \nu(f)v(y,z;1) + \frac{\pi^+(y,z) - \pi^-(y,z)}{4\pi^-(0^-,Y)}(v(0^-,Y;f) - v(0^+,-Y;f)) \tag{2.10}$$

2.2 Analysis of the short cycles

We describe the solution of (P_v). We can write $v(y,z;f) = v_e(y,z;f) + v^+(y,z;f) + v^-(y,z;f)$ with v_e, v^+, v^- satisfying

$$Av_e = f(y,z) \quad \text{in} \quad D, \quad v_e = 0 \quad \text{in} \quad D^+, \quad v_e = 0 \quad \text{in} \quad D^-, \tag{2.11}$$

$$Av^+ = 0 \quad \text{in} \quad D, \quad v^+(y,Y) = \varphi^+(y;f) \quad \text{in} \quad D^+, \quad v^+ = 0 \quad \text{in} \quad D^-, \tag{2.12}$$

and

$$Av^- = 0 \quad \text{in} \quad D, \quad v^- = 0 \quad \text{in} \quad D^+, \quad v^-(y,-Y) = \varphi^-(y;f) \quad \text{in} \quad D^-. \tag{2.13}$$

where $\varphi^+(y;f)$ and $\varphi^-(y;f)$ are defined by

$$-\frac{1}{2}\varphi^+_{yy} + (c_0y + kY)\varphi^+_y = f(y,Y), \quad y > 0, \quad \varphi^+(0^+;f) = 0 \tag{2.14}$$

and

$$-\frac{1}{2}\varphi^-_{yy} + (c_0y - kY)\varphi^-_y = f(y,-Y), \quad y < 0, \quad \varphi^-(0^-;f) = 0. \tag{2.15}$$

We check easily the formula $\varphi^+(y;f) = 2\int_0^\infty d\xi \exp(-(c_0\xi^2 + 2kY\xi)) \int_\xi^{\xi+y} f(\zeta;Y)\exp(-2c_0\xi(\zeta - \xi))d\zeta$, if $y \geqslant 0$ and also $\varphi_-(y;f) = 2\int_0^\infty d\xi \exp(-(c_0\xi^2 - 2kY\xi)) \int_{y-\xi}^{-\xi} f(\zeta;-Y)\exp(-2c_0\xi(\zeta - \xi))d\zeta$, if $y \leqslant 0$.

2.2.1 Solution to Problem (2.11)

The proof will be based on solving a sequence of Interior Exterior Dirichlet problems and a fixed point argument. Thus, we need to state the two following lemmas as preliminary results. It is sufficient to consider $f = 1$, with no loss of generality.

Interior Dirichlet problem

We begin with the interior problem, let $D_1 := (-\bar{y}_1, \bar{y}_1)\times(-Y,Y)$, $D_1^+ := [0,\bar{y}_1)\times\{Y\}$, $D_1^- := (-\bar{y}_1, 0]\times\{-Y\}$. Let us consider the space C_1^+ of continuous functions on $[-Y,Y]$ which are 0 on Y and the space C_1^- of continuous functions on $[-Y,Y]$ which are 0 on $-Y$. Let $\varphi^+ \in C_1^+$ and $\varphi^- \in C_1^-$. We consider the problem

$$-\frac{1}{2}\zeta_{yy} + (c_0y + kz)\zeta_y - y\zeta_z = 1 \quad \text{in} \quad D_1, \quad \zeta(y,Y) = 0 \quad \text{in} \quad D_1^+, \quad \zeta(y,-Y) = 0 \quad \text{in} \quad D_1^- \tag{2.16}$$

with $\zeta(\bar{y}_1, z) = \varphi^+(z)$ and $\zeta(-\bar{y}_1, z) = \varphi^-(z)$, if $-Y < z < Y$.

Lemma 1. *There exists a unique bounded solution to the equation* (2.16).

Proof. It is sufficient to prove an a priori bound. For that we can assume $\varphi^+, \varphi^- = 0$. Consider $\lambda > 0$ and the function $\theta(y,z) = \exp(\lambda c_0(y^2 + kz^2))$ then $-\frac{1}{2}\theta_{yy} + (c_0 y + kz)\theta_y - y\theta_z = \theta\left(-\lambda c_0 + 2\lambda c_0^2 y^2(1-\lambda)\right)$. Set next $H := -(\theta + \zeta)$ then

$$-\frac{1}{2}H_{yy} + (c_0 y + kz)H_y - yH_z = -1 + \theta\left(\lambda c_0 - 2\lambda c_0^2 y^2(1-\lambda)\right). \tag{2.17}$$

If we pick $\lambda > \max(1, \frac{1}{c_0})$ the right hand side of (2.17) is positive. Therefore the minimum of H can occur only on the boundary $y = \bar{y}_1$ and $z = Y$ with $y > 0$ or $z = -Y$ with $y < 0$. It follows that $H(y,z) \geqslant -\exp(\lambda c_0(\bar{y}_1^2 + Y^2))$ and thus also $0 \leqslant \zeta \leqslant \exp(\lambda c_0(\bar{y}_1^2 + Y^2))$. □

Exterior Dirichlet problems

Now, we proceed by considering two exterior Dirichlet problems. Let $0 < \bar{y} < \bar{y}_1$, we define $D_{\bar{y}<y} := \{y > \bar{y}, \quad -Y < z < Y\}$, $D^+_{\bar{y}<y} := \{y > \bar{y}, \quad z = Y\}$ and $D_{y<-\bar{y}} := \{y < -\bar{y}, \quad -Y < z < Y\}$, $D^-_{y<-\bar{y}} := \{y < -\bar{y}, \quad z = -Y\}$ and consider

$$-\frac{1}{2}\eta^+_{yy} + (c_0 y + kz)\eta^+_y - y\eta^+_z = 1 \quad \text{in} \quad D_{\bar{y}<y}, \quad \eta^+(y,Y) = 0 \quad \text{in} \quad D^+_{\bar{y}<y} \tag{2.18}$$

with the condition $\eta^+(\bar{y}, z) = \zeta(\bar{y}, z)$ if $-Y < z < Y$, and

$$-\frac{1}{2}\eta^-_{yy} + (c_0 y + kz)\eta^-_y - y\eta^-_z = 1 \quad \text{in} \quad D_{y<-\bar{y}}, \quad \eta^-(y,-Y) = 0 \quad \text{in} \quad D^-_{y<-\bar{y}} \tag{2.19}$$

with the condition $\eta^-(-\bar{y}, z) = \zeta(-\bar{y}, z)$, if $-Y < z < Y$. We use the same notation $\eta(y,z)$ for the two problems (2.18),(2.19) for the convenience of the reader. We have

Lemma 2. *For any $\bar{y} > 0$ there exists a unique bounded solution of (2.18),(2.19).*

Proof. It is sufficient to prove the bound, we claim that $\|\zeta\|_\infty \leqslant \eta(y,z) \leqslant \|\zeta\|_\infty + \frac{Y-z}{\bar{y}}$, for $y > \bar{y}$ and $\|\zeta\|_\infty \leqslant \eta(y,z) \leqslant \|\zeta\|_\infty + \frac{Y+z}{\bar{y}}$, for $y < -\bar{y}$. Consider for instance $\rho(z) = \|\zeta\|_\infty + \frac{Y-z}{\bar{y}}$ for $y > \bar{y}, -Y < z < Y$ then $-\frac{1}{2}\rho_{yy} + \rho_y(c_0 y + kz) - y\rho_z = \frac{y}{\bar{y}} > 1$, $\rho(\bar{y}, z) = \|\zeta\|_{L^\infty} + \frac{Y-z}{\bar{y}} > \zeta(\bar{y}, z)$, $\rho(\bar{y}, z) = \|\zeta\|_{L^\infty} > 0$. So clearly $\eta(y,z) \leqslant \rho(z)$. So in all cases we can assert that $\|\eta\|_\infty \leqslant \|\zeta\|_\infty + \frac{2Y}{\bar{y}}$. □

Solution to Problem (2.11)

Proposition 3. *There exists a unique bounded solution to Problem* (2.11).

Proof. Uniqueness comes from maximum principle. Setting $\Phi = (\varphi^+(z), \varphi^-(z))$ and using the notation $\Phi(\bar{y}_1, z) = \varphi^+(z), \quad \Phi(-\bar{y}_1, z) = \varphi^-(z)$, we can next define $\Gamma\Phi(\bar{y}_1, z) = \eta(\bar{y}_1, z)$ and $\Gamma\Phi(-\bar{y}_1, z) = \eta(-\bar{y}_1, z)$. We thus have defined a map Γ from C_1^+, C_1^- into itself. If Γ has a fixed point, then it is clear that the function

$$v_e(y,z) = \begin{cases} \zeta(y,z), & -\bar{y}_1 < y < \bar{y}_1, \\ \eta(y,z), & y > \bar{y}, \quad y < -\bar{y} \end{cases}$$

is a solution of (2.11) since $\zeta = \eta$ for $\bar{y} < y < \bar{y}_1, z \in (-Y, Y)$ and for $-\bar{y}_1 < y < -\bar{y}, z \in (-Y, Y)$ and the required regularity is available at boundary points $\bar{y}, \bar{y}_1, -\bar{y}, -\bar{y}_1$. The result will follow from the property : Γ is a contraction mapping. This property will be an easy consequence of the following result. Consider the exterior problem

$$-\frac{1}{2}\psi_{yy} + \psi_y(c_0 y + kz) - y\psi_z = 0 \quad \text{in} \quad D_{\bar{y}<y}, \quad \psi(y, Y) = 0 \quad \text{in} \quad D^+_{\bar{y}<y}, \tag{2.20}$$

where $\psi(\bar{y}, z) = 1$ if $\; -Y < z < Y$, then $\sup_{-Y<z<Y} \psi(\bar{y}_1, z) \leqslant \rho < 1$.

Indeed if $\sup_{-Y<z<Y} \psi(\bar{y}_1, z) = 1$, then the maximum is attained on the line $y = \bar{y}_1$, and this is impossible because it cannot be at $z = Y$, nor at $z = -Y$, nor at the interior, by maximum principle considerations. □

2.2.2 Solution to Problems (2.12) and (2.13)

We now consider the function φ^+ and φ^- solution of (2.14) and (2.15). Note that if $y < 0$, we have $\varphi^-(y; 1) = \varphi^+(-y; 1)$. So it is sufficient to consider (2.14) and we easily see that

$$\varphi^+(y; 1) = \int_0^\infty \exp(-(c_0\xi^2 + 2kY\xi))\frac{1 - \exp(-2c_0 y\xi)}{2c_0\xi}\mathrm{d}\xi, \quad \text{if} \quad y > 0$$

and we have $\varphi^+(y; 1) \leqslant \frac{1}{c_0}\log(\frac{c_0 y + kY}{kY})$, if $\; y > 0$. We next want to solve the problem (2.12). We proceed as follow. We extend φ^+ for $y < 0$, by a function which is C^2 on $\mathbb{R}$ and with compact support on $y < 0$. It is convenient to call $\varphi(y)$ the C^2 function on $\mathbb{R}$, with compact support for $y < 0$ and $\varphi(y) = \varphi^+(y; 1)$ for $y > 0$. We set $w^+(y, z) = v^+(y, z) - \varphi(y)$ then we obtain the problem

$$Aw^+ = g \quad \text{in} \quad D, \quad w^+(y, Y) = 0, \quad \text{in} \quad D^+, \quad w^+(y, -Y) = -\varphi(y), \quad \text{in} \quad D^- \tag{2.21}$$

with $g(y, z) = -\left(-\frac{1}{2}\varphi_{yy} + (c_0 y + kz)\varphi_y\right)$.

But, $g(y, z) = \mathbf{1}_{\{y>0\}}\left(-1 + k(Y - z)\varphi_y(y)\right) + \mathbf{1}_{\{y<0\}}\left(-(-\frac{1}{2}\varphi_{yy} + (c_0 y + kz)\varphi_y)\right)$ and thus, taking into account the definition of φ when $y < 0$, we can assert that $g(y, z)$ is a bounded function. Again, from the definition of $\varphi(y)$ when $y < 0$, we obtain that on the boundary, w^+ is bounded. It follows from what was done for Problem (2.11) that (2.21) has a unique solution. So we can state the following proposition.

Proposition 4. *There exists a unique solution to* (2.12) *of the form* $v^+(y, z) = \varphi^+(y)\mathbf{1}_{\{y>0\}} + \tilde{v}^+(y, z)$ *where* $\tilde{v}^+(y, z)$ *is bounded. Similarly, there exists a unique solution to* (2.13) *of the form* $v^-(y, z) = \varphi^-(y)\mathbf{1}_{\{y<0\}} + \tilde{v}^-(y, z)$ *where* $\tilde{v}^-(y, z)$ *is bounded.*

Proof. We just define $\varphi(y)$ extension of $\varphi^+(y)$ for $y < 0$ as explained before and consider $w^+(y, z)$ solution of (2.21). We know that $w^+(y, z)$ is bounded and we have $v^+(y, z) = \varphi(y) + w^+(y, z) = \varphi^+(y)\mathbf{1}_{\{y>0\}} + \varphi(y)\mathbf{1}_{\{y<0\}} + w^+(y, z)$ which is of the form (2.12) with $\tilde{v}^+(y, z) = \varphi(y)\mathbf{1}_{\{y<0\}} + w^+(y, z)$. □

2.2.3 The complete Problem (P_v)

Finally, we consider the complete Problem (P_v), we can state

Theorem 2.2.1. *There exists a unique solution of* (P_v) *of the form* $v(y, z; f) = \varphi^+(y; f)\mathbf{1}_{\{y>0\}} + \varphi^-(y; f)\mathbf{1}_{\{y<0\}} + \tilde{w}(y, z)$ *where* $\tilde{w}(y, z)$ *is a bounded function which can be written as* $\tilde{w} = v_e + w^+ + w^-$.

Proof. We just collect the results of Propositions 3 and 4. □

2.3 Ergodic Theorem

Proof of Theorem 2.1.1. We first prove the result when f is symmetric. In that case, we can write

$$u_\lambda(y,z;f) = v_\lambda(y,z;f) + \frac{v_\lambda(0^-,Y;f)}{v_\lambda(0^-,Y;1)}\left(\frac{1}{\lambda} - v_\lambda(y,z;1)\right) \tag{2.22}$$

Indeed, we know that $u_\lambda(y,z;f)$ and $v_\lambda(y,z;f)$ are symmetric. Setting $\tilde{u}_\lambda(y,z;f) = u_\lambda(y,z;f) - v_\lambda(y,z;f)$, we obtain

$$\lambda\tilde{u}_\lambda + A\tilde{u}_\lambda = 0 \quad \text{in} \quad D, \quad \lambda\tilde{u}_\lambda + B_+\tilde{u}_\lambda = 0 \quad \text{in} \quad D^+, \quad \lambda\tilde{u}_\lambda + B_-\tilde{u}_\lambda = 0 \quad \text{in} \quad D^- \tag{2.23}$$

with the boundary conditions $\tilde{u}_\lambda(0^+,Y;f) - \tilde{u}_\lambda(0^-,Y;f) = v_\lambda(0^-,Y;f)$ and $\tilde{u}_\lambda(0^+,-Y;f) - \tilde{u}_\lambda(0^-,-Y;f) = -v_\lambda(0^+,-Y;f)$. This last condition is automatically satisfied, thanks to the previous one and the symmetry. The function $\frac{1}{\lambda} - v_\lambda(y,z;1)$ satisfies the three partial differential equations on D, D^+ and D^-. So, $\tilde{u}_\lambda = C\left(\frac{1}{\lambda} - v_\lambda(y,z;1)\right)$ and writing the first boundary condition, we have $\tilde{u}_\lambda(0^+,Y;f) - \tilde{u}_\lambda(0^-,Y;f) = -C\left(v_\lambda(0^+,Y;1) - v_\lambda(0^-,Y;1)\right) = Cv_\lambda(0^-,Y;1)$. Hence, $C = \frac{v_\lambda(0^-,Y;f)}{v_\lambda(0^-,Y;1)}$ and formula (2.22) has been obtained. Now, we have $\nu_\lambda(f) \to \nu(f) = \frac{v(0^-,Y;f)}{v(0^-,Y;1)}$, as $\lambda \to 0$. If we define $u^\star_\lambda(y,z;f) = u_\lambda(y,z;f) - \frac{\nu_\lambda(f)}{\lambda} = v_\lambda(y,z;f) - \nu_\lambda(f)v_\lambda(y,z;1)$. The function $u^\star_\lambda(y,z;f) \to v(y,z;f) - \nu(f)v(y,z;1) = v(y,z;f-\nu(f)), \quad \lambda \to 0$. Also from its definition the function $u^\star_\lambda(y,Y;f)$ and $u^\star_\lambda(y,-Y;f)$ are continuous. From the choice of $\nu(f)$ the function $v(y,Y;f-\nu(f))$ is continuous. Now, since $f-\nu(f)$ is symmetric $v(0^+,-Y;f-\nu(f)) - v(0^-,-Y;f-\nu(f)) = v(0^+,Y;f-\nu(f)) - v(0^-,Y;f-\nu(f)) = 0$. So the result is proven when f is symmetric. We now consider the situation when f is antisymmetric. We know that $u_\lambda(y,z;f)$ is antisymmetric. Similarly $v_\lambda(y,z;f)$ is antisymmetric. Consider π^-_λ and π^+_λ defined by

$$\lambda\pi^+_\lambda + A\pi^+_\lambda = 0 \quad \text{in} \quad D, \quad \lambda\pi^+_\lambda + B_+\pi^+_\lambda = 0 \quad \text{in} \quad D^+, \quad \pi^+_\lambda = 0 \quad \text{in} \quad D^- \tag{2.24}$$

with the boundary condition $\pi^+_\lambda(0^+,Y) = 1$ and

$$\lambda\pi^-_\lambda + A\pi^-_\lambda = 0 \quad \text{in} \quad D, \quad \pi^-_\lambda = 0 \quad \text{in} \quad D^+, \quad \lambda\pi^-_\lambda + B_+\pi^-_\lambda = 0 \quad \text{in} \quad D^- \tag{2.25}$$

with the boundary condition $\pi^-_\lambda(0^-,-Y) = 1$. We have $\pi^-_\lambda(y,z) = \pi^-_\lambda(-y,-z)$, we then state the formula $u_\lambda(y,z;f) = v_\lambda(y,z;f) - \frac{\left(\pi^+_\lambda(y,z) - \pi^-_\lambda(y,z)\right)v_\lambda(0^+,-Y;f)}{1-\pi^+_\lambda(0^-,Y)+\pi^+_\lambda(0^+,-Y)}$. So we see that $u_\lambda(y,z;f)$ converges as $\lambda \to 0$,without substracting a number $\frac{\nu_\lambda(f)}{\lambda}$. The function $u_\lambda(y,z;f)$ converges pointwise to $u(y,z;f) = v(y,z;f) - \frac{\left(\pi^+(y,z) - \pi^-(y,z)\right)v(0^+,-Y;f)}{2\pi^-(0^-,Y)}$. So when f is antisymmetric, the results (2.8)-(2.9) hold with $\nu_\lambda(f) = 0$ and $\nu(f) = 0$. For the general case, we can write $f = f_{\text{sym}} + f_{\text{asym}}$ with $f_{\text{sym}}(y,z) = \frac{f(y,z)+f(-y,-z)}{2}$, $\quad f_{\text{asym}}(y,z) = \frac{f(y,z)-f(-y,-z)}{2}$. We have $\nu(f_{\text{sym}}) = \frac{v(0^-,Y;f_{\text{sym}})}{v(0^-,Y;1)}$ and thus $\nu(f_{\text{sym}}) = \frac{v(0^-,Y;f)+v(0^+,-Y;f)}{2v(0^+,-Y;1)}$ Since $\nu(f_{\text{asym}}) = 0$, we deduce $\nu(f) = \nu(f_{\text{sym}}) = \frac{v(0^-,Y;f)+v(0^+,-Y;f)}{2v(0^+,-Y;1)}$. We obtain also the representation formula $u(y,z;f) = v(y,z;f) - \nu(f)v(y,z;1) + \frac{\pi^+(y,z)-\pi^-(y,z)}{4\pi^-(0^-,Y)}\left(v(0^-,Y;f) - v(0^+,-Y;f)\right)$ and the result is obtained. $\square$

Chapter 3

Behavior of the plastic deformation

Ce chapitre fait l'objet d'une note soumise au Comptes Rendus Acad. Sci. Paris, [6] en collaboration avec Alain Bensoussan.

Earlier works in engineering, partly experimental, partly computational have revealed that asymptotically, when the excitation is a white noise, plastic deformation and total deformation for an elasto-perfectly-plastic oscillator have a variance which increases linearly with time with the same coefficient. In this work, we prove this result and we characterize the corresponding drift coefficient. Our study relies on a stochastic variational inequality governing the evolution between the velocity of the oscillator and the non-linear restoring force. We then define long cycles behavior of the Markov process solution of the stochastic variational inequality which is the key concept to obtain the result. An important question in engineering is to compute this coefficient. Also, we provide numerical simulations which show successful agreement with our theoretical prediction and previous empirical studies made by engineers.

Résumé

Le comportement de la déformation plastique pour un oscillateur élastique-parfaitement-plastique excité par un bruit blanc. Des résultats expérimentaux en sciences de l'ingénieur ont montré que, pour un oscillateur élasto-plastique-parfait excité par un bruit blanc, la déformation plastique et la déformation totale ont une variance, qui asymptotiquement, croît linéairement avec le temps avec le même coefficient. Dans ce travail, nous prouvons ce résultat et nous caractérisons le coefficient de dérive. Notre étude repose sur une inéquation variationnelle stochastique gouvernant l'évolution entre la vitesse de l'oscillateur et la force de rappel non-linéaire. Nous définissons alors le comportement en cycles longs du processus de Markov solution de l'inéquation variationnelle stochastique qui est le concept clé pour obtenir le résultat. Une question importante en sciences de l'ingénieur est de calculer ce coefficient. Les résultats numériques confirment avec succès notre prédiction théorique et les études empiriques faites par les ingénieurs.

Version française abrégée

Dans cet article, nous étudions la variance de l'oscillateur élastique-parfaitement-plastique (EPP) excité par un bruit blanc. La dynamique de l'oscillateur s'exprime à l'aide d'une équation à mémoire. (voir (3.1)-(3.2)). A.Bensoussan and J.Turi ont montré que la relation entre la vitesse et la composante élastique satisfait une inéquation variationnelle stochastique (voir ($\mathcal{SVI}$)). Dans ce cadre, nous introduisons les cycles long indépendants (définis plus loin) et nous justifions qu'ils permettent de caractériser la variance de la déformation totale et de la déformation plastique (voir (3.4)).

3.1 Introduction

In civil engineering, an elasto-perfectly-plastic (EPP) oscillator with one single degree of freedom is employed to estimate prediction of failure of mechanical structures under random vibrations. This elasto-plastic oscillator consists in a one dimensional model simple and representative of the elasto-plastic behavior of a class of structure dominated by their first mode of vibration. The main difficulty to study these systems comes from a frequent occurrence of plastic phases on small intervals of time. A plastic deformation is produced when the stress of the structure crosses over an elastic limit. The dynamics of the EPP-oscillator has memory, so it has been formulated in the engineering literature as a process with hysteresis $x(t)$, which stands for the displacement of the oscillator. We study the problem

$$\ddot{x} + c_0\dot{x} + \mathbf{F}_t = \dot{w} \tag{3.1}$$

with initial conditions of displacement and velocity $x(0) = 0, \quad \dot{x}(0) = 0$. Here $c_0 > 0$ is the viscous damping coefficient, $k > 0$ the stiffness, w is a Wiener process; $\mathbf{F}_t := \mathbf{F}(x(s), 0 \leqslant s \leqslant t)$ is a non-linear functional which depends on the entire trajectory $\{x(s), 0 \leqslant s \leqslant t\}$ up to time t. The plastic deformation denoted by $\Delta(t)$ at time t can be recovered from the pair $(x(t), \mathbf{F}_t)$ by the following relationship:

$$\mathbf{F}_t = \begin{cases} kY & if \quad x(t) = Y + \Delta(t), \\ k(x(t) - \Delta(t)) & if \quad x(t) \in]-Y + \Delta(t), Y + \Delta(t)[, \\ -kY & if \quad x(t) = -Y + \Delta(t). \end{cases} \tag{3.2}$$

where $\Delta(t) = \int_0^t y(s)\mathbf{1}_{\{|\mathbf{F}_t|=kY\}}\mathrm{d}s$ and Y is an elasto-plastic bound. Karnopp & Scharton [26] proposed a separation between elastic states and plastic states. They introduced a fictitious variable $z(t) := x(t) - \Delta(t)$ and noticed the simple fact that between two plastic phases $z(t)$ behaves like a linear oscillator. In addition, other works made by engineers [11], partly experimental, partly computational, have revealed that total deformation has a variance which increases linearly with time:

$$\lim_{t\to\infty} \frac{\sigma^2(x(t))}{t} = \sigma^2. \tag{3.3}$$

where σ^2 is a positive real number.

Recently, the right mathematical framework of stochastic variational inequalities (SVI) modeling an elasto-plastic oscillator with noise (presented below) has been introduced by the first author with J. Turi in [7]. The inequality governs the relationship between the velocity $y(t)$ and the elastic component $z(t)$:

$$\mathrm{d}y(t) = -(c_0 y(t) + kz(t))\mathrm{d}t + \mathrm{d}w(t), \quad (\mathrm{d}z(t) - y(t)\mathrm{d}t)(\phi - z(t)) \geqslant 0, \quad \forall|\phi| \leqslant Y, \quad |z(t)| \leqslant Y. \qquad (\mathcal{SVI})$$

The plastic deformation $\Delta(t)$ can be recovered by $\int_0^t y(s)\mathbf{1}_{\{|z(s)|=Y\}}\mathrm{d}s$.

Throughout the paper, the objective is to prove (3.3) and to provide a exact characterization of (3.3) based on ($\mathcal{SVI}$). Now, we introduce *long cycles* in view to the fact that we are interested in identifying independent sequences in the trajectory.

3.1.1 Long cycles

Denote $\tau_0 := \inf\{t > 0, \quad y(t) = 0 \quad \text{and} \quad |z(t)| = Y\}$ and $s := \mathrm{sign}(z(\tau_0))$ which labels the first boundary hit by the process $(y(t), z(t))$. Define $\theta_0 := \inf\{t > \tau_0, \quad y(t) = 0 \quad \text{and} \quad z(t) = -sY\}$. In a recurrent manner, for $n \geqslant 0$, knowing θ_n we can define

$$\begin{aligned}\tau_{n+1} &:= \inf\{t > \theta_n, \quad y(t) = 0 \quad \text{and} \quad z(t) = sY\},\\ \theta_{n+1} &:= \inf\{t > \tau_{n+1}, \quad y(t) = 0 \quad \text{and} \quad z(t) = -sY\}.\end{aligned}$$

Now, according to the previous setting we can define the *n-th* long cycle (resp. first part of the cycle, second part of the cycle) as the piece of trajectory delimited by the interval $[\tau_n, \tau_{n+1})$, (resp. $[\tau_n, \theta_{n+1})$ and $[\theta_{n+1}, \tau_{n+1})$). Indeed, at the instants $\{\tau_n, n \geqslant 1\}$, the process $(y(t), z(t))$ is in the same state at the instant τ_0. In addition, there are two types of cycles depending on the sign of s. The set of stopping times $\{\tau_n, n \geqslant 0\}$ represents the times of occurence of long cycles.

Let us remark that the plastic deformation and the total deformation are the same on a long cycle since $\int_{\tau_0}^{\tau_1} y(t)\mathrm{d}t = \int_{\tau_0}^{\tau_1} y(t)\mathbf{1}_{\{|z(s)|=Y\}}\mathrm{d}t + \int_{\tau_0}^{\tau_1} y(t)\mathbf{1}_{\{|z(s)|<Y\}}\mathrm{d}t$ and that $\int_{\tau_0}^{\tau_1} y(t)\mathbf{1}_{\{|z(s)|<Y\}}\mathrm{d}t = z(\tau_1) - z(\tau_0) = 0$. As main result, we have obtained the following theorem.

Theorem 3.1.1 (Characterization of the variance related to the plastic/total deformation)**.** *In the previously defined context, we have shown*

$$\lim_{t\to\infty} \frac{\sigma^2(x(t))}{t} = \frac{\mathbb{E}\left(\int_{\tau_0}^{\tau_1} y(t)\,dt\right)^2}{\mathbb{E}(\tau_1 - \tau_0)}. \tag{3.4}$$

Our proofs are based on solving nonlocal partial differential equations related to long cycles. Simpler formula will be given below at equation (3.13).

3.2 The issue of long cycles and plastic deformations

Let us introduce notations.

Notation 4. $D := \mathbb{R} \times (-Y, +Y), \quad D^+ := (0,\infty) \times \{Y\}, \quad D^- := (-\infty, 0) \times \{-Y\}$, *and the differential operators* $A\zeta := -\frac{1}{2}\zeta_{yy} + (c_0 y + kz)\zeta_y - y\zeta_z, \quad B_+\zeta := -\frac{1}{2}\zeta_{yy} + (c_0 y + kY)\zeta_y, \quad B_-\zeta := -\frac{1}{2}\zeta_{yy} + (c_0 y - kY)\zeta_y$. *where ζ is a regular function on D.*

Let f be a bounded measurable function, we want to solve

$$Av^+ = f(y,z) \quad \text{in} \quad D, \quad B_+ v^+ = f(y, Y) \quad \text{in} \quad D^+, \quad B_- v^+ = f(y, -Y) \quad \text{in} \quad D^- \qquad (P_{v^+})$$

with the nonlocal boundary conditions $v^+(y,Y)$ continuous and $v^+(0^-,-Y)=0$, and

$$Av^- = f(y,z) \quad \text{in} \quad D, \quad B_+v^- = f(y,Y) \quad \text{in} \quad D^+, \quad B_-v^- = f(y,-Y) \quad \text{in} \quad D^- \qquad (P_{v^-})$$

with the nonlocal boundary conditions $v^-(0^+,Y)=0$ and $v^-(y,-Y)$ continuous. The functions $v^+(y,z;f)$ and $v^-(y,z;f)$ are called long cycles. In addition, we define $\pi^+(y,z)$ and $\pi^-(y,z)$ by

$$A\pi^+ = 0 \quad \text{in} \quad D, \quad \pi^+(y,Y) = 1 \quad \text{in} \quad D^+, \quad \pi^+(y,-Y) = 0 \quad \text{in} \quad D^- \qquad (P_{\pi^+})$$

and

$$A\pi^- = 0 \quad \text{in} \quad D, \quad \pi^-(y,Y) = 0 \quad \text{in} \quad D^+, \quad \pi^-(y,-Y) = 1 \quad \text{in} \quad D^- \qquad (P_{\pi^-})$$

Note that $\pi^+(y,z)+\pi^-(y,z)=1$, so the existence and uniqueness of a bounded solution (P_{π^+}) and (P_{π^-}) is clear.

Proposition 5. *We have the properties $\pi^-(0^+,-Y)>0$ and $\pi^+(0^-,Y)>0$.*

Proof. We check only the first property. We consider the elastic process $(y_{yz}(t), z_{yz}(t))$:

$$z_{yz}(t) = e^{\frac{-c_0 t}{2}}\{z\cos(\omega t) + \frac{1}{\omega}(y+\frac{c_0}{2}z)\sin(\omega t)\} + \frac{1}{\omega}\int_0^t e^{-\frac{c_0}{2}(t-s)}\sin(\omega(t-s))\mathrm{d}w(s),$$

$$y_{yz}(t) = -\frac{c_0}{2}z_{yz}(t) + e^{-\frac{c_0 t}{2}}\{-\omega z\sin(\omega t) + (y+\frac{c_0}{2}z)\cos(\omega t)\} + \int_0^t e^{-\frac{c_0}{2}(t-s)}\cos(\omega(t-s))\mathrm{d}w(s)$$

where $\omega := \frac{\sqrt{4k-c_0^2}}{2}$ (we assume $4k>c_0^2$). Note that the condition $4k>c_0^2$ is needed so that $(y_{yz}(t), z_{yz}(t))$ have real valued solutions. Set $\tau_{yz} := \inf\{t>0, \quad |z_{yz}(t)| \geqslant Y\}$ then we have the probabilistic interpretation $\pi^+(y,z) = \mathbb{P}(z_{yz}(\tau_{yz})=Y), \quad \pi^-(y,z)=\mathbb{P}(z_{yz}(\tau_{yz})=-Y)$. We can state

$$\pi^-(y,z) \to 1 \quad \text{as} \quad y\to-\infty, \quad z\in[-Y,Y]. \tag{3.5}$$

Indeed, $\forall t$ with $0<t<\frac{\pi}{\omega}$ we have $z_{yz}(t)\to-\infty$, as $y\to-\infty$ a.s. Therefore $\forall t$ with $0<t<\frac{\pi}{\omega}$, a.s. $z_{yz}(t)<-Y$ for y sufficiently large. Hence, a.s. $\tau_{yz}<t$ for y sufficiently large. Therefore a.s. $\limsup_{y\to-\infty}\tau_{yz}<t$. Since t is arbitrary, necessarily a.s. $\tau_{yz}\to 0$, as $y\to-\infty$ which implies (3.5). Moreover the function $\pi^+(y,Y)$ cannot have a minimum or a maximum at any finite $y<0$. It is then monotone and since $\pi^-(-\infty,Y)=1$, it is monotone decreasing. It follows that $\pi^-(0^-,Y)<1$. It cannot be 0. Otherwise, $\pi^-(y,Y)$ is continuous at $y=0$, and $(0,Y)$ is a point of minimum of $\pi^-(y,z)$. Since for $y<0$, $\pi^-_y(y,Y)<0$ from the equation of π^- we get $\limsup_{y\to 0}\pi^-_{yy}(y,Y)\leqslant 0$ which is not possible since $(0,Y)$ is a minimum. □

We next define $\eta(y,z)$ by

$$-\frac{1}{2}\eta_{yy} + (c_0y+kz)\eta_y - y\eta_z = f(y,z) \quad \text{in} \quad D, \quad \eta(y,Y)=0 \quad \text{in} \quad D^+, \quad \eta(y,-Y)=0 \quad \text{in} \quad D^- \qquad (P_\eta)$$

with the local boundary conditions $\eta(0^+,Y)=0$ and $\eta(0^-,-Y)=0$. For f bounded (P_η) has a unique bounded solution [4]. Then define $\varphi_+(y;f)$ by solving

$$-\frac{1}{2}\varphi_{+,yy} + (c_0y+kY)\varphi_{+,y} = f(y,Y), \quad y>0, \quad \varphi_+(0;f)=0. \tag{3.6}$$

We check easily the formula

$$\varphi_+(y;f) = 2\int_0^\infty \mathrm{d}\xi \exp(-(c_0\xi^2+2kY\xi))\int_\xi^{\xi+y} f(\zeta;Y)\exp(-2c_0\xi(\zeta-\xi))\mathrm{d}\zeta, \quad y\geqslant 0. \tag{3.7}$$

We consider $\psi_+(y,z;f)$ defined by

$$\begin{cases} -\frac{1}{2}\psi_{+,yy} + (c_0 y + kz)\psi_{+,y} - y\psi_{+,z} &= 0 \quad \text{in} \quad D, \\ \psi_+(y,Y) &= \varphi_+(y;f) \quad \text{in} \quad D^+, \\ \psi_+(y,-Y) &= 0 \quad \text{in} \quad D^- \end{cases} \tag{P_{ψ_+}}$$

and similarly $\varphi_-(y;f), \psi_-(y,z;f)$ defined by

$$-\frac{1}{2}\varphi_{-,yy} + (c_0 y - kY)\varphi_{-,y} = f(y,-Y), \quad y < 0, \quad \varphi_-(0;f) = 0. \tag{3.8}$$

which leads to

$$\varphi_-(y;f) = 2\int_0^\infty \mathrm{d}\xi \exp(-(c_0\xi^2 - 2kY\xi)) \int_{y-\xi}^{-\xi} f(\zeta;-Y)\exp(-2c_0\xi(\zeta-\xi))\mathrm{d}\zeta, \quad y \leqslant 0.$$

and

$$\begin{cases} -\frac{1}{2}\psi_{-,yy} + (c_0 y + kz)\psi_{-,y} - y\psi_{-,z} &= 0 \quad \text{in} \quad D, \\ \psi_-(y,Y) &= 0 \quad \text{in} \quad D^+, \\ \psi_-(y,-Y) &= \varphi_-(y;f) \quad \text{in} \quad D^- \end{cases} \tag{P_{ψ_-}}$$

We can state

Proposition 6. *The solution of* (P_{v^+}) *is given by*

$$\begin{aligned} v^+(y,z;f) =& \eta(y,z;f) + \psi_+(y,z;f) + \psi_-(y,z;f) \\ &+ \frac{\eta(0^-,Y;f) + \psi_+(0^-,Y;f) + \psi_-(0^-,Y;f)}{\pi^-(0^-,Y)}\pi^+(y,z). \end{aligned} \tag{3.9}$$

and the solution of (P_{v^-}) *is given by*

$$\begin{aligned} v^-(y,z;f) =& \eta(y,z;f) + \psi_+(y,z;f) + \psi_-(y,z;f) \\ &+ \frac{\eta(0^+,-Y;f) + \psi_+(0^+,-Y;f) + \psi_-(0^+,-Y;f)}{\pi^+(0^-,Y)}\pi^-(y,z). \end{aligned} \tag{3.10}$$

Proof. Direct checking. □

3.3 Complete cycle

First, let us check that $\mathbb{E}[x(t)] = 0$ and then $\sigma^2(x(t)) = \mathbb{E}[x^2(t)]$. Indeed, by symmetry of the inequality $(\mathcal{SVI})$ and by the choice of initial conditions $y(0) = 0, z(0) = 0$, the processes $(y(t), z(t))$ and $-(y(t), z(t))$ have same law. Then, $\mathbb{E}[x(t)] = \mathbb{E}\left[\int_0^t y(s)\mathrm{d}s\right] = 0$. In addition, $(y(\tau_1), z(\tau_1))$ is equal to $(0,-Y)$ or $(0,Y)$ with probability 1/2 for both, therefore with no loss of generality we can suppose that $(y(0), z(0)) = (0,-Y)$ or $(0,Y)$ with probability 1/2 for both.

Let us treat the case $y(0) = 0, z(0) = Y$. So, $\tau_0 = 0$ and $\theta_1 = \inf\{t > 0, \quad z(t) = -Y, y(t) = 0\}$. We can assert that $\mathbb{E}\theta_1 = v^+(0,Y;1)$ hence $\theta_1 < \infty$ a.s.. Next we define $\tau_1 = \inf\{t > \theta_1, \quad z(t) = Y, y(t) = 0\}$ then $\mathbb{E}\tau_1 = v^+(0,Y;1) + v^-(0,-Y;1) = 2v^+(0,Y;1)$. At time τ_1 the state of the system is again $(0,Y)$. So the sequence $\{\tau_n, n \geqslant 0\}$ is such that $\tau_n < \tau_{n+1}$ and in the interval (τ_n, τ_{n+1}) we have a cycle identical to $(0,\tau_1)$. Consider the variable $\int_0^{\tau_1} y(t)\mathrm{d}t$. We have

$\mathbb{E}\int_0^{\tau_1} y(t)\mathrm{d}t = \mathbb{E}\int_0^{\theta_1} y(t)\mathrm{d}t + \mathbb{E}\int_{\theta_1}^{\tau_1} y(t)dt = v^+(0,Y;y) + v^-(0,-Y;y)$. However if f is antisymmetric $f(-y,-z) = -f(y,z)$ we have $v^+(0,Y;f) = -v^-(0,-Y;f)$, therefore $\mathbb{E}\int_0^{\tau_1} y(t)\mathrm{d}t = 0$. Hence, $\mathbb{E}\left(\int_0^{\tau_n} y(t)\mathrm{d}t\right)^2 = \mathbb{E}\left(\sum_{j=0}^{n-1}\int_{\tau_j}^{\tau_{j+1}} y(t)\mathrm{d}t\right)^2 = n\mathbb{E}\left(\int_0^{\tau_1} y(t)\mathrm{d}t\right)^2$. Then

$$\frac{\mathbb{E}\left(\int_0^{\tau_n} y(t)dt\right)^2}{\mathbb{E}\tau_n} = \frac{\mathbb{E}\left(\int_0^{\tau_1} y(t)dt\right)^2}{\mathbb{E}\tau_1}.$$

Let $T>0$ and N_T with $\tau_{N_T} \leqslant T < \tau_{N_{T+1}}, \quad N_T = 0 \quad \text{if} \quad \tau_1 > T, \quad \tau_0 = 0$ and then calculations lead to the following

$$\frac{\mathbb{E}\left(\int_0^{\tau_{N_T+1}} y(t)\mathrm{d}t\right)^2}{\mathbb{E}\tau_{N_T+1}} = \frac{\mathbb{E}\left(\int_0^{\tau_1} y(t)\mathrm{d}t\right)^2}{\mathbb{E}\tau_1}.$$

Next, we can justify that

$$\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left(\int_0^T y(t)\mathrm{d}t\right)^2 = \lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left(\int_0^{\tau_{N_T+1}} y(t)\mathrm{d}t\right)^2 \tag{3.11}$$

and that we have a lower bound and an upper bound for (3.11), that is

$$\frac{\mathbb{E}\left(\int_0^{\tau_1} y(t)\mathrm{d}t\right)^2}{\mathbb{E}\tau_1} \leqslant \frac{1}{T}\mathbb{E}\left(\int_0^{\tau_{N_T+1}} y(t)\mathrm{d}t\right)^2 \leqslant \left(1+\frac{\mathbb{E}\tau_1}{T}\right)\frac{\mathbb{E}\left(\int_0^{\tau_1} y(t)\mathrm{d}t\right)^2}{\mathbb{E}\tau_1}.$$

Therefore

$$\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left(\int_0^T y(t)\mathrm{d}t\right)^2 = \frac{\mathbb{E}\left(\int_0^{\tau_1} y(t)\mathrm{d}t\right)^2}{\mathbb{E}\tau_1}. \tag{3.12}$$

Moreover, we can simplify (3.12), indeed we have

$$\mathbb{E}\left(\int_0^{\tau_1} y(t)\mathrm{d}t\right)^2 = \mathbb{E}\left[\left(\int_0^{\theta_1} y(t)\mathrm{d}t\right)^2 + \left(\int_{\theta_1}^{\tau_1} y(t)\mathrm{d}t\right)^2 + 2\int_0^{\theta_1} y(t)\mathrm{d}t\int_{\theta_1}^{\tau_1} y(t)\mathrm{d}t\right]$$

anf therefore, we can justify that

$$\mathbb{E}\left(\int_0^{\tau_1} y(t)\mathrm{d}t\right)^2 = 2\left[\mathbb{E}\left(\int_0^{\theta_1} y(t)\mathrm{d}t\right)^2 - \left(\int_0^{\theta_1} y(t)\mathrm{d}t\right)^2\right] = 2\left[\mathbb{E}\left(\int_0^{\theta_1} y(t)\mathrm{d}t\right)^2 - (v^+(0,Y;y))^2\right].$$

Since $\mathbb{E}\tau_1 = 2v^+(0,Y;1)$ we obtain

$$\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left(\int_0^T y(t)\mathrm{d}t\right)^2 = \frac{\mathbb{E}\left(\int_0^{\theta_1} y(t)\mathrm{d}t\right)^2 - (v^+(0,Y;y))^2}{v^+(0,Y;1)}. \tag{3.13}$$

3.4 Numerical evidence in support of our result

In this section, we provide computational results which confirm our theoretical results. A C code has been written to simulate $(y(t),z(t))$. See [3] for the numerical scheme considered to do direct simulation. Let $T>0$, $N\in\mathbb{N}$ and $\{t_n = n\delta t, 0\leqslant n\leqslant N\}$ where $\delta t = \frac{T}{N}$. Then, to compute the left hand side of (3.4), we consider $MC\in\mathbb{N}$ and we generate MC numerical solutions of $(\mathcal{SVI})$ $\{y^i(t), 0\leqslant t\leqslant T, 1\leqslant i\leqslant MC\}$ up to the time T. By the law of large numbers, we can approximate $\frac{1}{T}\mathbb{E}\left(\int_0^T y(s)ds\right)^2$ by $X_{MC} := \frac{1}{T}\frac{1}{MC}\sum_{i=1}^{MC}\left(\sum_{i=1}^{N} y^i(t_n)\delta t\right)^2$ and also

$c_0 = 1, k = 1$				
Y	$X_{MC}, T = 500$	$\frac{\delta_{MC}}{\tau_{MC}}$	τ_{MC}	Relative error %
0.1	$0.807^{\pm 0.031}$	$0.834^{\pm 0.069}$	$6.61^{\pm 0.11}$	3.2
0.2	$0.649^{\pm 0.026}$	$0.624^{\pm 0.047}$	$8.74^{\pm 0.13}$	3.8
0.3	$0.493^{\pm 0.020}$	$0.464^{\pm 0.034}$	$10.45^{\pm 0.16}$	5.8
0.4	$0.361^{\pm 0.014}$	$0.355^{\pm 0.026}$	$12.12^{\pm 0.18}$	1.7
0.5	$0.266^{\pm 0.011}$	$0.257^{\pm 0.019}$	$13.80^{\pm 0.21}$	3.3
0.6	$0.195^{\pm 0.008}$	$0.198^{\pm 0.014}$	$16.15^{\pm 0.26}$	1.5
0.7	$0.137^{\pm 0.005}$	$0.149^{\pm 0.011}$	$18.84^{\pm 0.31}$	8
0.8	$0.103^{\pm 0.004}$	$0.112^{\pm 0.008}$	$22.80^{\pm 0.39}$	8
0.9	$0.071^{\pm 0.003}$	$0.086^{\pm 0.006}$	$26.79^{\pm 0.47}$	15

Table 3.1: Monte-Carlo simulations to compare numerical solution of the left and right hand sides of (3.4), $T = 500$, $\delta t = 10^{-4}$ and $MC = 5000$.

$\frac{1}{T^2}\mathbb{E}\left(\int_0^T y(s)ds\right)^4$ by $X_{MC}^2 := \frac{1}{T^2}\frac{1}{MC}\sum_{i=1}^{MC}\left(\sum_{i=n}^{N} y^i(t_n)\delta t\right)^4$. Denote $s_X := \sqrt{X_{MC}^2 - (X_{MC})^2}$, we also know by the central limit theorem that $\frac{1}{T}\mathbb{E}\left(\int_0^T y(s)ds\right)^2 \in [X_{MC} - \frac{1.96 s_X}{\sqrt{MC}}, X_{MC} + \frac{1.96 s_X}{\sqrt{MC}}]$ with 95% of confidence. Similarly, to compute the right hand side of (3.4), we generate MC numerical long cycles. For each trajectory $\{y^i(t), t \geqslant 0\}$, we consider N_c^i the required number of time step to obtain a completed cycle. Denote $\delta_{MC} := \frac{1}{MC}\sum_{i=1}^{MC}\left(\sum_{n=0}^{N_c^i} y^i(t_n)\delta t\right)^2$, $\delta_{MC}^2 := \frac{1}{MC}\sum_{i=1}^{MC}\left(\sum_{n=0}^{N_c^i} y^i(t_n)\delta t\right)^4$, $\tau_{MC} := \frac{1}{MC}\sum_{i=1}^{MC} N_c^i\delta t$, $\tau_{MC}^2 := \frac{1}{MC}\sum_{i=1}^{MC}(N_c^i\delta t)^2$, $s_\delta := \sqrt{\delta_{MC}^2 - (\delta_{MC})^2}$ and $s_\tau := \sqrt{\tau_{MC}^2 - (\tau_{MC})^2}$. We also know that $\frac{\delta_{MC}}{\tau_{MC}} \in [\frac{\delta_{MC} - \frac{1.96 s_\delta}{\sqrt{MC}}}{\tau_{MC} + \frac{1.96 s_\tau}{\sqrt{MC}}}, \frac{\delta_{MC} + \frac{1.96 s_\delta}{\sqrt{MC}}}{\tau_{MC} - \frac{1.96 s_\tau}{\sqrt{MC}}}]$ with 95% of confidence. In table 3.1, is shown a comparison of the results obtained for X_{MC} and $\frac{\delta_{MC}}{\tau_{MC}}$ where $T = 500$, $\delta t = 10^{-4}$ and $MC = 5000$.

Remark 3. *From a numerical point of view, the behavior in long cycles is very relevant. Indeed, computing the left hand side of* (3.4) *is much more expensive in time compared to the right hand side (see the τ_{MC}-column of table 3.1).*

Chapter 4

Stochastic variational inequalities with vanishing jumps

Ce chapitre fait l'objet d'un article soumis à Asymptotic Analysis [1] en collaboration avec Alain Bensoussan, Hector Jasso-Fuentes et Stéphane Menozzi.

In a previous work by the first author with J. Turi [7], a stochastic variational inequality has been introduced to model an elasto-plastic oscillator with noise. A major advantage of the stochastic variational inequality is to overcome the need to describe the trajectory by phases (elastic or plastic). This is useful, since the sequence of phases cannot be characterized easily. In particular, when a change of regime occurs, there are numerous small elastic phases which may appear as an artefact of the Wiener process. However, it remains important to have informations on both the elastic and plastic phases. In order to reconcile these contradictory issues, we introduce an approximation of stochastic variational inequalities by imposing artificial small jumps between phases allowing a clear separation of the elastic and plastic regimes. In this work, we prove that the approximate solution converges on any finite time interval, when the size of jumps tends to 0.

4.1 Introduction

The elastic-perfectly-plastic (EPP) oscillator under standard white noise excitation is the simplest structural model exhibiting a hysteretic behavior. Moreover, the model is representative of the behavior of mechanical structures which vibrate mainly on their first deformation mode. In the context of earthquake engineering, relevant applications to piping systems under random vibrations can be accessed this way [19, 20]. The main difficulty to study these systems comes from a frequent occurrence of nonlinear phases (plastic phases) on small time intervals. A nonlinear phase corresponds to a permanent deformation, or in other words to a plastic deformation. A plastic deformation is produced when the stress of the structure exceeds an elastic limit. Denoting by $x(t)$ the elasto-plastic displacement, we consider the problem

$$\ddot{x} + c_0\dot{x} + \mathbf{F}(x(s), 0 \leqslant s \leqslant t) = \dot{w}, \tag{4.1}$$

with initial conditions of displacement and velocity

$$x(0) = x \quad , \quad \dot{x}(0) = y.$$

Here $c_0 > 0$ is the viscous damping coefficient, $k > 0$ the stiffness, w is a Wiener process and $\mathbf{F}(\{x(s), 0 \leqslant s \leqslant t\})$ is a nonlinear functional which depends on the entire trajectory $\{x(s), 0 \leqslant s \leqslant t\}$ up to time t. Denote $y(t) := \dot{x}(t)$. Equation (4.1) written as a stochastic differential equation (SDE) reads

$$\mathrm{d}y(t) = -(c_0 y(t) + F(\{x(s), 0 \leqslant s \leqslant t\}))\mathrm{d}t + \mathrm{d}w(t), \quad dx(t) = y(t)\mathrm{d}t. \tag{4.2}$$

Beyond a given threshold $|F(\{x(s), 0 \leqslant s \leqslant t\})| = kY$ for the nonlinear restoring force, the material goes through plastic deformation (see e.g. [26]). Introducing $\Delta(t)$, the total plastic yielding accumulated up to time t, we can define a new state variable $z(t)$ as $z(t) := x(t) - \Delta(t)$. It follows that in the plastic regime, $\dot{z}(t) = 0$. From now on, we choose to express the restoring force $F(\{x(s), 0 \leqslant s \leqslant t\})$ in (4.2) in terms of the new variable $z(t)$ as $F(\{x(s), 0 \leqslant s \leqslant t\}) := kz(t)$ where $|z(t)| \leqslant Y$. In other words we consider a linear restoring force of the variable $z(t)$ (whose modulus equals Y during the plastic phases). This type of force characterizes the elasto-perfectly-plastic behavior.

In [7], for the previous choice for F, it is shown that (4.2) is equivalent to a stochastic variational inequality (SVI). In addition, existence and uniqueness of an invariant measure for the solution of SVI have also been proven. For a general framework dealing with this class of inequalities we refer the reader to [2] and to [18] for specific deterministic applications to mechanics. Although SVIs have been already studied in [2] to represent reflection-diffusion processes in convex sets, no connection with random vibration problems had been made so far. From [7], the solution $(y(t), z(t)) \in \mathbb{R}^2$ of (4.2) satisfies

$$\dot{y}(t) = -(c_0 y(t) + kz(t)) + \dot{w}(t), \quad (\dot{z}(t) - y(t))(\phi - z(t)) \geqslant 0, \quad \forall |\phi| \leqslant Y, \quad |z(t)| \leqslant Y. \tag{4.3}$$

In terms of dynamics of the process $(y(t), z(t))$, a plastic deformation begins when $z(t)$ reaches and is absorbed by Y (resp. $-Y$) with positive (resp. negative) slope, $y(t) > 0$. (resp. $y(t) < 0$) i.e. when $\mathrm{sign}(y(t))z(t) = Y$. Then, the plastic behavior ends when the velocity changes sign. At that time, the elastic behavior is reactivated. However, around 0, the velocity which is subjected to white noise, changes sign an infinite number of times during any small time interval. Often, this leads to a return into plastic behavior in a short time duration. This phenomenon is called *micro-elastic phasing* and has been studied in [21] using the numerical method developed in [3] for the SVI (4.3). It plays a crucial role on frequency and statistics of plastic deformations. Because of this phenomenon, frequency of occurence, statistics (time duration or absolute plastic deformation) and the sequence of entry in plastic phase (as well as the sequence of exit) are not well defined. In this paper, we consider an EPP oscillator under standard white noise excitation subjected to jumps (presented below) to study phase transitions. It has the advantage of separating phases clearly, while being an approximation. We prove the convergence of the approximated process towards the solution of the stochastic variational inequality (4.3).

4.1.1 Model definition and convergence results

In this subsection, we introduce a stochastic variational inequality whose dynamics is "almost" similar to the one of (4.3) except that the second component is subjected to jumps of magnitude $\varepsilon > 0$ at some random times corresponding to the various exits of the plastic phases.

Precisely, we describe the evolution of the new process $(y^\epsilon(t), z^\epsilon(t))$ by the following procedure; we start by defining $\tau_0^\epsilon := 0$ and by $(y_0^\epsilon(t), z_0^\epsilon(t))$ the solution of (4.3), with initial conditions:

$$y_0^\epsilon(0) = y \quad \text{and} \quad z_0^\epsilon(0) = z, \ (y, z) \in \mathbb{R} \times (-Y, Y) := D.$$

Then, we define

$$\tau_1^\epsilon := \inf\{t > 0, \quad y_0^\epsilon(t) = 0 \quad \text{and} \quad |z_0^\epsilon(t)| = Y\}.$$

For $t \geqslant \tau_1^\epsilon$, let $(y_1^\epsilon(t), z_1^\epsilon(t))$ be the solution of (4.3) with initial conditions:

$$y_1^\epsilon(\tau_1^\epsilon) = 0 \quad \text{and} \quad z_1^\epsilon(\tau_1^\epsilon) = \text{sign}(z_0^\epsilon(\tau_1^\epsilon))\,(Y - \epsilon)\,,$$

again, we define

$$\tau_2^\epsilon := \inf\{t > \tau_1^\epsilon, \quad y_1^\epsilon(t) = 0 \quad \text{and} \quad |z_1^\epsilon(t)| = Y\}.$$

In a recurrent manner, knowing τ_n^ϵ, $y_n^\epsilon(t)$, and $z_n^\epsilon(t)$, we define

$$\tau_{n+1}^\epsilon := \inf\{t > \tau_n^\epsilon, \quad y_n^\epsilon(t) = 0 \quad \text{and} \quad |z_n^\epsilon(t)| = Y\},$$

and $(y_{n+1}^\epsilon(t), z_{n+1}^\epsilon(t))$ be the solution of (4.3) with initial conditions:

$$y_{n+1}^\epsilon(\tau_{n+1}^\epsilon) = 0 \quad \text{and} \quad z_{n+1}^\epsilon(\tau_{n+1}^\epsilon) = \text{sign}(z_n^\epsilon(\tau_{n+1}^\epsilon))\,(Y - \epsilon)\,.$$

Now, we define the process $(y^\epsilon(t), z^\epsilon(t))$ on each interval of time $[\tau_n^\epsilon, \tau_{n+1}^\epsilon)$ as follows:

$$\dot{y}^\epsilon(t) = -(c_0 y^\epsilon(t) + k z^\epsilon(t)) + \dot{w}(t), \quad (\dot{z}^\epsilon(t) - y^\epsilon(t))(\phi - z^\epsilon(t)) \geqslant 0, \quad \forall |\phi| \leqslant Y, \quad |z^\epsilon(t)| \leqslant Y \tag{4.4}$$

with the following jump-conditions:

$$y^\epsilon(\tau_n^\epsilon -) = 0, \qquad z^\epsilon(\tau_n^\epsilon -) = z_{n-1}^\epsilon(\tau_n^\epsilon),$$

and

$$y^\epsilon(\tau_n^\epsilon) = 0, \qquad z^\epsilon(\tau_n^\epsilon) = \text{sign}(z_{n-1}^\epsilon(\tau_n^\epsilon))(Y - \epsilon).$$

Remark 4. *By construction, the process $(y^\epsilon(t), z^\epsilon(t))$ is càdlàg; hence it is regular. In particular, for each fixed time $T > 0$, the number of jumps arise in $(0, T]$, is finite a.s.*

We will prove that the solution $(y^\epsilon(t), z^\epsilon(t))$ converges to $(y(t), z(t))$ on any finite time interval, when ϵ goes to 0 in the sense described below.

4.2 Main results

Our main result is the following theorem.

Theorem 4.2.1. *Fix $T > 0$, and consider the processes $(y(t), z(t))$ and $(y^\epsilon(t), z^\epsilon(t))$ satisfying (4.3) and (4.4) respectively. Suppose that $k > X_+(c_0) := \frac{1}{2}\left(-\frac{c_0}{3} + c_0\sqrt{\frac{1}{9} + 4\frac{c_0}{6}}\right)$. Then, the following convergence property holds:*

$$\frac{1}{\epsilon}\mathbb{E}\left[\sup_{0 \leqslant t \leqslant T}\left\{|y(t) - y^\epsilon(t)|^2 + k\,|z(t) - z^\epsilon(t)|^2\right\}\right] \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Remark 5. *Observe that the above condition relating k and c_0 is purely technical. It will appear clearly in the proof of Lemma 4 below.*

4.2.1 Preliminary results

For $(y, z) \in D := \mathbb{R} \times (-Y, Y)$, we consider the "elastic" process $(y_{yz}(t), z_{yz}(t))$:

$$z_{yz}(t) = e^{\frac{-c_0 t}{2}}\{z\cos(\omega t) + \frac{1}{\omega}(y + \frac{c_0}{2}z)\sin(\omega t)\} + \frac{1}{\omega}\int_0^t e^{-\frac{c_0}{2}(t-s)}\sin(\omega(t-s))\mathrm{d}w(s),$$
$$y_{yz}(t) = -\frac{c_0}{2}z_{yz}(t) + e^{-\frac{c_0 t}{2}}\{-\omega z\sin(\omega t) + (y + \frac{c_0}{2}z)\cos(\omega t)\} + \int_0^t e^{-\frac{c_0}{2}(t-s)}\cos(\omega(t-s))\mathrm{d}w(s).$$

where, assuming $4k > c_0^2$,

$$\omega := \frac{\sqrt{4k - c_0^2}}{2}.$$

Remark 6. *The terminology "elastic" is justified from the observation that $(y_{yz}(t), z_{yz}(t))$ is actually the solution of*

$$\dot{y}(t) = -(c_0 y(t) + kz(t)) + \dot{w}(t), \quad \dot{z}(t) = y(t), \quad (y_{yz}(0), z_{yz}(0)) = (y, z),$$

that is the explicit solution of (4.3) when the threshold $Y = \infty$ (purely elastic case). Note that the condition $4k > c_0^2$ is needed so that $(y(t), z(t))$ have real valued solutions.

Define

$$\theta(y, z) := \inf\{t > 0, \quad |z_{yz}(t)| = Y\}, \tag{4.5}$$

where $(y_{yz}(0), z_{yz}(0)) = (y, z)$. For $t \in [0, T]$, we set $u(y, z, t) := \mathbb{P}[\theta(y, z) > T - t]$. This function is regular and satisfies the mixed Cauchy-Dirichlet parabolic PDE

$$-u_t + Au = 0, \text{ in } D; \quad u(y, Y, t) = 0, \quad y > 0; \quad u(y, -Y, t) = 0, \quad y < 0; \quad u(y, z, T) = 1, \tag{4.6}$$

with

$$Au = -\frac{1}{2}u_{yy} + (c_0 y + kz)u_y - yu_z.$$

For $t < T$, the function $u(y, z, t)$ is locally smooth. On the other hand, in the particular case when $(y_{yz}(0), z_{yz}(0)) := (0, Y - \epsilon)$, we consider the probability density function p^ϵ of $(y_{0,Y-\epsilon}(t), z_{0,Y-\epsilon}(t))$. It is also known that p^ϵ satisfies Chapman-Kolmogorov's equation

$$p_t^\epsilon + A^* p^\epsilon = 0, \quad p^\epsilon(y, z, 0) = \delta_{0,Y-\epsilon}(y, z), \tag{4.7}$$

where A^* represents the adjoint operator of A; that is

$$A^* p^\epsilon = -\frac{1}{2}p_{yy}^\epsilon - ((c_0 y + kz)p^\epsilon)_y + yp_z^\epsilon.$$

Next, observe that the processes $z_{0,Y-\epsilon}(t)$ and $y_{0,Y-\epsilon}(t)$ are gaussian processes. The key point is to express the solution of (4.6) through its variational formulation with p^ϵ as test function (see proof of Lemma 3).

The mean, variance and covariance of $z_{0,Y-\epsilon}(t)$ and $y_{0,Y-\epsilon}(t)$ write:

$$m^\epsilon(t) := (Y - \epsilon)e^{-\frac{c_0 t}{2}}(\cos\omega t + \frac{c_0}{2\omega}\sin\omega t), \quad \sigma_z^2(t) := \frac{1}{\omega^2}\int_0^t e^{-c_0 s}\sin^2(\omega s)\mathrm{d}s, \tag{4.8}$$

$$q^\epsilon(t) := -(Y - \epsilon)\frac{k}{\omega}e^{-\frac{c_0 t}{2}}\sin\omega t, \quad \sigma_y^2(t) := \int_0^t e^{-c_0 s}(\cos\omega s - \frac{c_0}{2\omega}\sin\omega s)^2\mathrm{d}s, \tag{4.9}$$

and

$$\sigma_{yz}(t) := \frac{1}{2\omega^2} e^{-c_0 t} \sin^2 \omega t. \tag{4.10}$$

The density p^ϵ then explicitly writes

$$\begin{aligned} p^\epsilon(y,z,t) =& \frac{1}{2\pi\sigma_z(t)\sigma_y(t)(1-\rho^2(t))^{1/2}} \exp\left\{ -\frac{1}{2(1-\rho^2(t))} \left[\frac{(y-q^\epsilon(t))^2}{\sigma_y^2(t)} + \frac{(z-m^\epsilon(t))^2}{\sigma_z^2(t)} \right.\right. \\ & \left.\left. - \frac{2\rho(t)(y-q^\epsilon(t))(z-m^\epsilon(t))}{\sigma_y(t)\sigma_z(t)} \right] \right\}, \end{aligned} \tag{4.11}$$

where the correlation coefficient $\rho(t)$ is defined by $\sigma_{yz}(t)/\sigma_y(t)\sigma_z(t)$. Observe that for $\epsilon = 0$, (4.11) reduces to

$$\begin{aligned} p^0(y,z,t) =& \frac{1}{2\pi\sigma_z(t)\sigma_y(t)(1-\rho^2(t))^{1/2}} \exp\left\{ -\frac{1}{2(1-\rho^2(t))} \left[\frac{(y-q^0(t))^2}{\sigma_y^2(t)} + \frac{(z-m^0(t))^2}{\sigma_z^2(t)} \right.\right. \\ & \left.\left. - \frac{2\rho(t)(y-q^0(t))(z-m^0(t))}{\sigma_y(t)\sigma_z(t)} \right] \right\}, \end{aligned} \tag{4.12}$$

with

$$m^0(t) := Y e^{-\frac{c_0 t}{2}} (\cos\omega t + \frac{c_0}{2\omega} \sin\omega t), \quad q^0(t) := -Y \frac{k}{\omega} e^{-\frac{c_0 t}{2}} \sin\omega t. \tag{4.13}$$

From (4.8)-(4.13) we can easily see that

$$m^\epsilon(t) = m^0(t) - \epsilon f(t), \quad q^\epsilon(t) = q^0(t) + \epsilon g(t), \tag{4.14}$$

with

$$f(t) := e^{-\frac{c_0 t}{2}} (\cos\omega t + \frac{c_0}{2\omega} \sin\omega t) \quad \text{and} \quad g(t) := \frac{k}{\omega} e^{-\frac{c_0 t}{2}} \sin\omega t.$$

Plugging (4.14) into (4.11), we obtain

$$\begin{aligned} p^\epsilon(y,z,t) =& \frac{1}{2\pi\sigma_z(t)\sigma_y(t)(1-\rho^2(t))^{1/2}} \exp\left\{ -\frac{1}{2(1-\rho^2(t))} \left[\frac{(y-\left[q^0(t)+\epsilon g(t)\right])^2}{\sigma_y^2(t)} + \right.\right. \\ & \left.\left. \frac{(z-\left[m^0(t)-\epsilon f(t)\right])^2}{\sigma_z^2(t)} - \frac{2\rho(t)(y-\left[q^0(t)+\epsilon g(t)\right])(z-\left[m^0(t)-\epsilon f(t)\right])}{\sigma_y(t)\sigma_z(t)} \right] \right\}. \end{aligned} \tag{4.15}$$

Now, notice that

$$\begin{aligned} & \frac{(y-\left[q^0(t)+\epsilon g(t)\right])^2}{\sigma_y^2(t)} + \frac{(z-\left[m^0(t)-\epsilon f(t)\right])^2}{\sigma_z^2(t)} - \frac{2\rho(t)(y-\left[q^0(t)+\epsilon g(t)\right])(z-\left[m^0(t)-\epsilon f(t)\right])}{\sigma_y(t)\sigma_z(t)} \\ =& \frac{(y-q^0(t))^2}{\sigma_y^2(t)} + \frac{(z-m^0(t))^2}{\sigma_z^2(t)} - \frac{2\rho(t)(y-q^0(t))(z-m^0(t))}{\sigma_y(t)\sigma_z(t)} + \epsilon^2 \left[\frac{g^2(t)}{\sigma_y^2(t)} + \frac{f^2(t)}{\sigma_z^2(t)} + \frac{2\rho(t)g(t)f(t)}{\sigma_y(t)\sigma_z(t)} \right] \\ & - 2\epsilon \left[\frac{(y-q^0(t))g(t)}{\sigma_y^2(t)} - \frac{(z-m^0(t))f(t)}{\sigma_z^2(t)} + \frac{\rho(t)}{\sigma_y(t)\sigma_z(t)} \left((y-q^0(t))f(t) - (z-m^0(t))g(t) \right) \right]. \end{aligned} \tag{4.16}$$

Then considering (4.12), we have

$$p^\epsilon(y,z,t) = p^0(y,z,t) \exp\left\{ -\frac{\epsilon^2}{2} A(t) + \epsilon \frac{(y-q_0(t))r(t) - (z-m_0(t))s(t)}{(1-\rho^2(t))\sigma_y(t)\sigma_z(t)} \right\}. \tag{4.17}$$

where

$$
\begin{aligned}
A(t) &:= \frac{1}{1-\rho^2(t)}\left(\frac{g^2(t)}{\sigma_y^2(t)}+\frac{f^2(t)}{\sigma_z^2(t)}+\frac{2\rho(t)g(t)f(t)}{\sigma_y(t)\sigma_z(t)}\right),\\
r(t) &:= \frac{g(t)\sigma_z(t)}{\sigma_y(t)}+\rho(t)f(t),\\
s(t) &:= \frac{f(t)\sigma_y(t)}{\sigma_z(t)}+\rho(t)g(t).
\end{aligned}
$$

We now give a representation of $u(0,Y-\epsilon,0)$ in terms of the densities p^ϵ and p^0 of the Gaussian processes $(z_{0,Y-\epsilon}(t),y_{0,Y-\epsilon}(t))$ and $(z_{0,Y}(t),y_{0,Y}(t))$ respectively. The proof is postponed to Section 4.3.

Lemma 3. *Let u be a solution of* (4.6)*. Then, it satisfies*

$$
\begin{aligned}
u(0,Y-\epsilon,0) &= \int_D \left[p^\epsilon(y,z,T)-p^0(y,z,T)\right] dy\,dz\\
&+\int_{D_T^-} yu(y,Y,t)p^0(y,Y,t)\left[\exp\left\{-\frac{1}{2}\epsilon^2A(t)+\frac{\epsilon(yr(t)-Yh(t))}{(1-\rho^2(t))\sigma_y(t)\sigma_z(t)}\right\}-1\right]dy\,dt\\
&-\int_{D_T^+} yu(y,-Y,t)p^0(y,-Y,t)\left[\exp\left\{-\frac{1}{2}\epsilon^2A(t)+\frac{\epsilon(yr(t)-Yl(t))}{(1-\rho^2(t))\sigma_y(t)\sigma_z(t)}\right\}-1\right]dy\,dt,
\end{aligned}
\tag{4.18}
$$

with

$$
\begin{aligned}
Yh(t) &:= q_0(t)r(t)+(Y-m_0(t))s(t),\; h(t):=-g(t)r(t)+(1-f(t))s(t),\\
Yl(t) &:= q_0(t)r(t)-(Y+m_0(t))s(t),\; l(t):=-g(t)r(t)-(1+f(t))s(t),\\
D_T^+ &:= (0,T)\times(0,\infty)\text{ and } D_T^-:=(0,T)\times(-\infty,0).
\end{aligned}
$$

Now consider the terms

$$
\begin{aligned}
H^\epsilon &= \int_D \left[p^\epsilon(y,z,T)-p^0(y,z,T)\right]\mathrm{d}y\mathrm{d}z,\\
I^\epsilon &= \int_{D_T^-} yu(y,Y,t)p^0(y,Y,t)\left[\exp\left\{-\frac{1}{2}\epsilon^2A(t)+\frac{\epsilon(yr(t)-Yh(t))}{(1-\rho^2(t))\sigma_y(t)\sigma_z(t)}\right\}-1\right]\mathrm{d}y\mathrm{d}t,\\
J^\epsilon &= -\int_{D_T^+} yu(y,-Y,t)p^0(y,-Y,t)\left[\exp\left\{-\frac{1}{2}\epsilon^2A(t)+\frac{\epsilon(yr(t)-Yl(t))}{(1-\rho^2(t))\sigma_y(t)\sigma_z(t)}\right\}-1\right]\mathrm{d}y\mathrm{d}t.
\end{aligned}
\tag{4.19}
$$

Next, we study the behavior of these last integrals, with u satisfying (4.6) so that the previous lemma holds, when ϵ is sufficiently small. The proof is also postponed to Section 4.3.

Lemma 4. *Let J^ϵ, I^ϵ, and H^ϵ be the integrals of above. Suppose that*

$$
k > X_+(c_0) := \frac{1}{2}\left(-\frac{c_0}{3}+c_0\sqrt{\frac{1}{9}+4\frac{c_0}{6}}\right).
$$

Then,

- $\liminf_{\epsilon\to 0} \frac{I^\epsilon}{\epsilon} = +\infty$,
- $\lim_{\epsilon\to 0} \frac{J^\epsilon}{\epsilon}$ *is finite,*
- $\lim_{\epsilon\to 0} \frac{H^\epsilon}{\epsilon}$ *is finite.*

Therefore,

$$\lim_{\epsilon\to 0} \frac{u(0, Y-\epsilon, 0)}{\epsilon} = +\infty \quad \textit{and} \quad \lim_{\epsilon\to 0} \frac{u(0, -Y+\epsilon, 0)}{\epsilon} = +\infty.$$

4.2.2 Proof of Theorem 4.2.1

We shall use the notation $\sigma_n^\epsilon = \mathrm{sign}(z^\epsilon(\tau_n^\epsilon -))$. Recall that, for each $n \geqslant 1$, the stopping time τ_n^ϵ represents the instant of the $n-$th jump of the process $(y^\epsilon(t), z^\epsilon(t))$. Hence, for all $\tau_n^\epsilon \leqslant t < \tau_{n+1}^\epsilon$ and $n \geqslant 1$, we deduce from (4.3) and (4.4) that

$$\begin{aligned} &\dot{y}(t) - \dot{y}^\epsilon(t) = -\left[c_0(y(t) - y^\epsilon(t)) + k(z(t) - z^\epsilon(t))\right], \quad \text{and} \\ &(\dot{z}^\epsilon(t) - y^\epsilon(t))(z(t) - z^\epsilon(t)) \geqslant 0, \\ &(\dot{z}(t) - y(t))(z^\epsilon(t) - z(t)) \geqslant 0. \end{aligned}$$

By using the notation d/dt of derivatives, we obtain

$$\frac{d}{dt}(y(t) - y^\epsilon(t)) = -c_0(y(t) - y^\epsilon(t)) - k(z(t) - z^\epsilon(t)), \tag{4.20}$$

$$\left(\frac{d}{dt}(z(t) - z^\epsilon(t)) - (y(t) - y^\epsilon(t))\right)(z(t) - z^\epsilon(t)) \leqslant 0, \tag{4.21}$$

Multiplying by $(y(t) - y^\epsilon(t))$ in (4.20) and using the product rule for derivatives, we get from (4.20) and (4.21)

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|y(t) - y^\epsilon(t)|^2 + c_0|y(t) - y^\epsilon(t)|^2 &\leqslant -k(z(t) - z^\epsilon(t))(y(t) - y^\epsilon(t)) \\ &\leqslant -\frac{k}{2}\frac{d}{dt}|(z(t) - z^\epsilon(t))|^2 \end{aligned} \tag{4.22}$$

for all $\tau_n^\epsilon \leqslant t < \tau_{n+1}^\epsilon$ and $n \geqslant 1$. Now, integrating (4.22) on $[\tau_n^\epsilon, \tau_{n+1}^\epsilon)$ and noting that $y(\tau_n^\epsilon -) = y(\tau_n^\epsilon)$, $y^\epsilon(\tau_n^\epsilon -) = y^\epsilon(\tau_n^\epsilon) = 0$, $z(\tau_n^\epsilon -) = z(\tau_n^\epsilon)$, for all $n \geqslant 1$, we obtain

$$\left|y(\tau_{n+1}^\epsilon)\right|^2 - |y(\tau_n^\epsilon)|^2 + 2c_0\int_{\tau_n^\epsilon}^{\tau_{n+1}^\epsilon} |y(t) - y^\epsilon(t)|^2 \; \mathrm{d}t + k\left|z(\tau_{n+1}^\epsilon) - z^\epsilon(\tau_{n+1}^\epsilon -)\right|^2 - k\left|z(\tau_n^\epsilon) - z^\epsilon(\tau_n^\epsilon)\right|^2 \leqslant 0. \tag{4.23}$$

But

$$\begin{aligned} k\left|z(\tau_n^\epsilon) - z^\epsilon(\tau_n^\epsilon)\right|^2 &= k\left|(z(\tau_n^\epsilon) - z^\epsilon(\tau_n^\epsilon -)) + (z^\epsilon(\tau_n^\epsilon -) - z^\epsilon(\tau_n^\epsilon))\right|^2 \\ &= k\left|z(\tau_n^\epsilon) - z^\epsilon(\tau_n^\epsilon -)\right|^2 + k\,\epsilon^2 \\ &\quad +2k\epsilon\,\sigma_n^\epsilon\,(z(\tau_n^\epsilon) - z^\epsilon(\tau_n^\epsilon -)). \end{aligned} \tag{4.24}$$

Plugging (4.24) into (4.23), and rearranging terms, we obtain

$$\begin{aligned} &\left|y(\tau_{n+1}^\epsilon)\right|^2 - |y(\tau_n^\epsilon)|^2 + k\left|z(\tau_{n+1}^\epsilon) - z^\epsilon(\tau_{n+1}^\epsilon -)\right|^2 - k\left|z(\tau_n^\epsilon) - z^\epsilon(\tau_n^\epsilon -)\right|^2 \\ &+2c_0\int_{\tau_n^\epsilon}^{\tau_{n+1}^\epsilon} |y(t) - y^\epsilon(t)|^2 \; \mathrm{d}t \leqslant k\epsilon^2 + 2k\epsilon(\sigma_n^\epsilon z(\tau_n^\epsilon) - Y). \end{aligned}$$

We can drop the term $2k\epsilon(\sigma_n^\epsilon z(\tau_n^\epsilon) - Y) \leqslant 0$ and get

$$\begin{aligned}&\left|y(\tau_{n+1}^\epsilon)\right|^2 - |y(\tau_n^\epsilon)|^2 + k\left|z(\tau_{n+1}^\epsilon) - z^\epsilon(\tau_{n+1}^\epsilon-)\right|^2 - k\left|z(\tau_n^\epsilon) - z^\epsilon(\tau_n^\epsilon-)\right|^2\\ &+2c_0\int_{\tau_n^\epsilon}^{\tau_{n+1}^\epsilon} |y(t) - y^\epsilon(t)|^2 \, \mathrm{d}t \leqslant k\epsilon^2.\end{aligned} \tag{4.25}$$

Observe that, for $N \in \mathbb{N}^\star$, we can iterate (4.25) for $1 \leqslant n \leqslant N$ to obtain

$$\begin{aligned}&\left|y(\tau_{N+1}^\epsilon)\right|^2 - |y(\tau_1^\epsilon)|^2 + k\left|z(\tau_{N+1}^\epsilon) - z^\epsilon(\tau_{N+1}^\epsilon-)\right|^2 - k\left|z(\tau_1^\epsilon) - z^\epsilon(\tau_1^\epsilon-)\right|^2\\ &+2c_0\int_{\tau_1^\epsilon}^{\tau_{N+1}^\epsilon} |y(t) - y^\epsilon(t)|^2 \, \mathrm{d}t \leqslant kN\epsilon^2.\end{aligned}$$

Also, recalling that $y(\tau_1^\epsilon) = 0$, $|z(\tau_1^\epsilon) - z^\epsilon(\tau_1^\epsilon-)|^2 = 0$, and that $\int_0^{\tau_1^\epsilon} |y(t) - y^\epsilon(t)|^2 \, \mathrm{d}t = 0$, we derive:

$$\left|y(\tau_{N+1}^\epsilon)\right|^2 + k\left|z(\tau_{N+1}^\epsilon) - z^\epsilon(\tau_{N+1}^\epsilon-)\right|^2 + 2c_0\int_0^{\tau_{N+1}^\epsilon} |y(t) - y^\epsilon(t)|^2 \, \mathrm{d}t \leqslant k\epsilon^2 N. \tag{4.26}$$

Denote the total number of jumps of the process $(y^\epsilon(t), z^\epsilon(t))$ arising in the time interval $(0,T)$ by $N_T^\epsilon := \max_N \{\tau_N^\epsilon \leqslant T\}$. Note that $T < \tau_{N_T^\epsilon+1}$. Hence, from (4.26), we deduce

$$\sup_{1\leqslant n\leqslant N_T^\epsilon+1} |y(\tau_n^\epsilon)|^2 + k \sup_{1\leqslant n\leqslant N_T^\epsilon+1} |z(\tau_n^\epsilon) - z^\epsilon(\tau_n^\epsilon-)|^2 + 2c_0\int_0^T |y(t) - y^\epsilon(t)|^2 \, \mathrm{d}t \leqslant k\epsilon^2 N_T^\epsilon. \tag{4.27}$$

Assume first $z(0) = Y - \epsilon$. According to the definition of (4.5) set $\theta^\epsilon := \theta(0, Y-\epsilon) = \inf\{t > 0, \quad |z^\epsilon(t)| = Y\} = \inf\{t > 0, \quad |z_{0,Y-\varepsilon}(t)| = Y\}$. It is clear that $\tau_1^\epsilon > \theta^\epsilon$ a.s. and then $\mathbb{P}(\tau_1^\epsilon > T) > \mathbb{P}(\theta^\epsilon > T)$. Now, let us assume $z(0) = -Y + \epsilon$. It is easy to verify that $u(-y, -z, t) = u(y, z, t)$, which gives

$$\mathbb{P}(\theta^\epsilon > T) = u(0, Y - \epsilon, 0) = u(0, -Y + \epsilon, 0).$$

Thus, by Lemma 4 we have $\frac{\mathbb{P}(\theta^\epsilon > T)}{\epsilon} \to +\infty$. Therefore, if the initial condition $z(0)$ associated to (4.4), is a random variable $\Gamma \overset{(\text{law})}{=} p_1\delta_{Y-\epsilon} + (1-p_1)\delta_{-Y+\epsilon}$ independent of the Wiener process $w(t)$, then again setting $\theta_\Gamma^\epsilon := \inf\{t > 0, \quad |z_{0,\Gamma}(t)| = Y\}$,

$$\frac{\mathbb{P}(\theta_\Gamma^\epsilon > T)}{\epsilon} \to +\infty \quad \text{as} \quad \epsilon \to 0.$$

Coming back to equation (4.26) and noting that $N_T^\epsilon = \sum_n \chi_{\{\tau_n^\epsilon \leqslant T\}}$, we get

$$\mathbb{E}N_T^\epsilon = \sum_{n=1}^\infty \mathbb{E}\chi_{\{\tau_n^\epsilon \leqslant T\}} = \mathbb{E}\chi_{\{\tau_1^\epsilon \leqslant T\}} + \sum_{n=2}^\infty \mathbb{E}\chi_{\{\tau_n^\epsilon \leqslant T\}}. \tag{4.28}$$

Observe that for all $n \geqslant 2$ and that $\tau_n^\epsilon - \tau_{n-1}^\epsilon$ is independent of τ_{n-1}^ϵ.

$$\mathbb{E}\chi_{\{\tau_n^\epsilon \leqslant T\}} = \mathbb{E}\left[\chi_{\{\tau_{n-1}^\epsilon \leqslant T\}}\chi_{\{\tau_n^\epsilon - \tau_{n-1}^\epsilon \leqslant T - \tau_{n-1}^\epsilon\}}\right] \leqslant \mathbb{E}\chi_{\{\tau_{n-1}^\epsilon \leqslant T\}}\mathbb{E}\chi_{\{\tau_n^\epsilon - \tau_{n-1}^\epsilon \leqslant T\}}. \tag{4.29}$$

But note that

$$\mathbb{E}\chi_{\{\tau_n^\epsilon - \tau_{n-1}^\epsilon \leqslant T\}} \leqslant \mathbb{P}(\theta^\epsilon \leqslant T).$$

From the last inequality and using (4.29), we deduce

$$\mathbb{E}\chi_{\{\tau_n^\epsilon \leqslant T\}} \leqslant \mathbb{E}\chi_{\{\tau_1^\epsilon \leqslant T\}}(1-u(0,Y-\epsilon,0))^{n-1}.$$

This yields

$$\mathbb{E}N_T^\epsilon \leqslant \mathbb{E}\chi_{\{\tau_1^\epsilon \leqslant T\}}\frac{(1-u(0,Y-\epsilon,0))}{u(0,Y-\epsilon,0)} \leqslant \frac{\epsilon\mathbb{E}\chi_{\{\tau_1^\epsilon \leqslant T\}}}{\epsilon u(0,Y-\epsilon,0)}. \tag{4.30}$$

Hence, from Lemmas 3 and 4

$$\epsilon\ \mathbb{E}N_T^\epsilon \to 0 \quad \text{as } \epsilon \to 0. \tag{4.31}$$

Thus, as ϵ goes to 0, (4.27) and (4.31) yield

$$\frac{1}{\epsilon}\left\{\mathbb{E}\left[\sup_{1\leqslant n\leqslant N_T^\epsilon+1}|y(\tau_n^\epsilon)|^2\right] + 2c_0\mathbb{E}\int_0^T |y(t)-y^\epsilon(t)|^2\ \mathrm{d}t + k\mathbb{E}\left[\sup_{1\leqslant n\leqslant N_T^\epsilon+1}|z(\tau_n^\epsilon)-z^\epsilon(\tau_n^\epsilon-)|^2\right]\right\} \to 0. \tag{4.32}$$

Since the forced jumps have magnitude ϵ, this implies:

$$\frac{1}{\epsilon}\left\{\mathbb{E}\left[\sup_{1\leqslant n\leqslant N_T^\epsilon+1}|y(\tau_n^\epsilon)|^2\right] + 2c_0\mathbb{E}\int_0^T |y(t)-y^\epsilon(t)|^2\ \mathrm{d}t + k\mathbb{E}\left[\sup_{1\leqslant n\leqslant N_T^\epsilon+1}|z(\tau_n^\epsilon)-z^\epsilon(\tau_n^\epsilon)|^2\right]\right\} \to 0. \tag{4.33}$$

Also, by (4.22), we can see that any $\tau_n^\epsilon \leqslant t < \tau_{n+1}^\epsilon$ satisfies

$$|y(t)-y^\epsilon(t)|^2 - |y(\tau_n^\epsilon)|^2 + k|z(t)-z^\epsilon(t)|^2 - k\,|z(\tau_n^\epsilon)-z^\epsilon(\tau_n^\epsilon)|^2 \leqslant 0.$$

This gives

$$\sup_{\tau_n^\epsilon\leqslant t<\tau_{n+1}^\epsilon}\left\{|y(t)-y^\epsilon(t)|^2 + k|z(t)-z^\epsilon(t)|^2\right\} \leqslant |y(\tau_n^\epsilon)|^2 + k\,|z(\tau_n^\epsilon)-z^\epsilon(\tau_n^\epsilon)|^2.$$

Hence,

$$\begin{aligned}\sup_{\tau_1^\epsilon\leqslant t<T}\left\{|y(t)-y^\epsilon(t)|^2 + k|z(t)-z^\epsilon(t)|^2\right\} &\leqslant \sup_{1\leqslant n\leqslant N_T^\epsilon+1}\left\{\sup_{\tau_n^\epsilon\leqslant t<\tau_{n+1}^\epsilon}\left\{|y(t)-y^\epsilon(t)|^2 + k\,|z(t)-z^\epsilon(t)|^2\right\}\right\}\\ &\leqslant \sup_{1\leqslant n\leqslant N_T^\epsilon+1}\left\{|y(\tau_n^\epsilon)|^2 + k\,|z(\tau_n^\epsilon)-z^\epsilon(\tau_n^\epsilon)|^2\right\}.\end{aligned}$$

Also,

$$\sup_{0\leqslant t\leqslant T}\left\{|y(t)-y^\epsilon(t)|^2 + k\,|z(t)-z^\epsilon(t)|^2\right\} \leqslant \sup_{1\leqslant n\leqslant N_T^\epsilon+1}\left\{|y(\tau_n^\epsilon)|^2 + k\,|z(\tau_n^\epsilon)-z^\epsilon(\tau_n^\epsilon)|^2\right\}.$$

Therefore, (4.33) gives

$$\frac{1}{\epsilon}\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\left\{|y(t)-y^\epsilon(t)|^2 + k\,|z(t)-z^\epsilon(t)|^2\right\}\right] \to 0 \quad \text{as } \epsilon \to 0.$$

4.3 Proof of the technical lemmas

This section is devoted to the proofs of Lemmas 3 and 4.

Proof of Lemma 3. From (4.6), we have

$$\begin{aligned} 0 &= \int_0^T \int_D (-u_t + Au)p^\epsilon \,\mathrm{d}y\mathrm{d}z\mathrm{d}t \\ &= \int_0^T \int_D (-u_t - \frac{1}{2}u_{yy} + (c_0 y + kz)u_y - yu_z)p^\epsilon \,\mathrm{d}y\mathrm{d}z\mathrm{d}t \\ &= -\int_D p^\epsilon(y,z,T)dydz + u(0, Y-\epsilon, 0) + \int_0^T \int_D up_t^\epsilon \mathrm{d}y\mathrm{d}z\mathrm{d}t \\ &\quad - \int_0^T \int_D \frac{1}{2}up_{yy}^\epsilon dydzdt - \int_0^T \int_D u((c_0 y + kz)p^\epsilon)_y \mathrm{d}y\mathrm{d}z\mathrm{d}t \\ &\quad - \int_{D_T^-} yu(y,Y,t)p^\epsilon(y,Y,t)dydt + \int_{D_T^+} yu(y,-Y,t)p^\epsilon(y,-Y,t)\mathrm{d}y\mathrm{d}t \\ &\quad + \int_0^T \int_D yup_z^\epsilon \mathrm{d}y\mathrm{d}z\mathrm{d}t. \end{aligned} \tag{4.34}$$

By using (4.7) and rearranging terms, (4.34) becomes

$$u(0,Y-\epsilon,0) = \int_D p^\epsilon(y,z,T)\mathrm{d}y\mathrm{d}z + \int_{D_T^-} yu(y,Y,t)p^\epsilon(y,Y,t)\mathrm{d}y\mathrm{d}t - \int_{D_T^+} yu(y,-Y,t)p^\epsilon(y,-Y,t)\mathrm{d}y\mathrm{d}t. \tag{4.35}$$

In addition, $p^0(y,z,0) := \delta_{0,Y}(y,z)$, and

$$0 = \int_D p^0(y,z,T)\mathrm{d}y\mathrm{d}z + \int_{D_T^-} yu(y,Y,t)p^0(y,Y,t)\mathrm{d}y\mathrm{d}t - \int_{D_T^+} yu(y,-Y,t)p^0(y,-Y,t)\mathrm{d}y\mathrm{d}t. \tag{4.36}$$

Using (4.17) and substracting (4.36) to (4.35), we can deduce the result (4.18).

Proof of Lemma 4. First note that, on a neighborhood of $t = 0$, we have the following expansions:

- $f(t) = e^{-\frac{c_0 t}{2}}(\cos\omega t + \frac{c_0}{2\omega}\sin\omega t) = 1 - k\frac{t^2}{2} + \frac{c_0}{12}(c_0^2 + 2\omega^2)t^3 + o(t^3)$,
- $g(t) = \frac{k}{\omega}e^{-\frac{c_0 t}{2}}\sin\omega t = kt(1 - \frac{c_0}{2}t) + o(t^2)$.

From (4.8)-(4.10), we also have

- $\sigma_y^2(t) = t - c_0 t^2 + o(t^2), \quad \sigma_y(t) = \sqrt{t}\left((1 - \frac{c_0}{2}t) + o(t)\right)$,
- $\sigma_z^2(t) = \frac{t^3}{3} - c_0\frac{t^4}{4} + o(t^4), \quad \sigma_z(t) = \frac{t^{\frac{3}{2}}}{\sqrt{3}}\left((1 - \frac{c_0 3}{8}t) + o(t)\right)$,
- $\frac{\sigma_z(t)}{\sigma_y(t)} = \frac{t}{\sqrt{3}}\left((1 + \frac{c_0}{8}t) + o(t)\right)$,
- $\rho(t) = \frac{\sqrt{3}}{2}\left(1 - \frac{c_0}{8}t + o(t)\right)$, recalling that $\rho(t) = \frac{\sigma_{yz}(t)}{\sigma_y(t)\sigma_z(t)}$.

Equation (4.13) yields

- $q^0(t) = -Ykt(1 - \frac{c_0}{2}t) + o(t^2)$,
- $m^0(t) = Y(1 - k\frac{t^2}{2}) + o(t^2)$.

Recalling that $r(t) = \frac{g(t)\sigma_z(t)}{\sigma_y(t)} + \rho(t)f(t)$ and $s(t) := \frac{f(t)\sigma_y(t)}{\sigma_z(t)} + \rho(t)g(t)$ and using the previous estimations, we can check that $-g(t)r(t) = -\frac{k\sqrt{3}}{2}t + \frac{5\sqrt{3}}{16}c_0kt^2 + o(t^2)$ and $(1-f(t))s(t) = \frac{k\sqrt{3}}{2}t + \frac{\sqrt{3}}{4}\left(k^2 - \frac{3}{4}c_0k - \frac{c_0^3}{6}\right)t^2 + o(t^2)$. Therefore, $h(t) \sim \frac{\sqrt{3}}{4}P(c_0,k)t^2$ where

$$P(c_0,k) := k^2 + \frac{c_0}{3}k - \frac{c_0^3}{6}.$$

Denote $X_+(c_0) := \frac{1}{2}\left(-\frac{c_0}{3} + c_0\sqrt{\frac{1}{9} + 4\frac{c_0}{6}}\right)$. Since we have assumed that $k > X_+(c_0)$, it then follows that $h'(0) = 0$ and $h''(0) = \frac{\sqrt{3}P(c_0,k)}{2} > 0$. We can thus consider a fixed interval $(0,\tilde{t})$ such that $h''(t) > 0$ on $[0,\tilde{t}]$, hence $h(t) > 0$ on $(0,\tilde{t})$. Also, we have $r(t) = \frac{\sqrt{3}}{2}(1 - \frac{c_0}{8}t + o(t))$. Hence, there exists a positive constant $\bar{t}$ such that $r(t) > 0$ on $[0,\bar{t}]$. Let $t_0 := \min\{\tilde{t},\bar{t}\}$. This implies that $h(t) > 0$ and $r(t) > 0$ on $(0,t_0)$. Recall that $h(0) = 0$. Now write, from (4.19)

$$I^\epsilon = I_1^\epsilon + I_2^\epsilon,$$

with

$$\begin{aligned}
I_1^\epsilon &:= \int_0^{t_0\wedge T}\int_{-\infty}^0 yu(y,Y,t)p^0(y,Y,t)\left[\exp\left\{-\frac{1}{2}\epsilon^2 A(t) + \frac{\epsilon(yr(t) - Yh(t))}{(1-\rho^2(t))\sigma_y(t)\sigma_z(t)}\right\} - 1\right]\mathrm{d}y\mathrm{d}t,\\
I_2^\epsilon &:= \int_{t_0\wedge T}^{T}\int_{-\infty}^0 yu(y,Y,t)p^0(y,Y,t)\left[\exp\left\{-\frac{1}{2}\epsilon^2 A(t) + \frac{\epsilon(yr(t) - Yh(t))}{(1-\rho^2(t))\sigma_y(t)\sigma_z(t)}\right\} - 1\right]\mathrm{d}y\mathrm{d}t.
\end{aligned}$$

From the definition of t_0, $h(t) \geqslant 0$ and $r(t) > 0$ for $0 < t < t_0 \wedge T$. Moreover, $y < 0$ in I_1^ϵ, so we have

$$-\frac{1}{2}\epsilon^2 A(t) + \frac{\epsilon(yr(t) - Yh(t))}{(1-\rho^2(t))\sigma_y(t)\sigma_z(t)} \leqslant 0.$$

Therefore, the integrand in I_1^ϵ is a positive function. Now, using the basic inequality $\exp\{-x\} - 1 \leqslant -x\exp\{-x\}$, for $x \geqslant 0$, we can write

$$\begin{aligned}
\frac{I_1^\epsilon}{\epsilon} \geqslant\ & -\int_0^{t_0\wedge T}\int_{-\infty}^0 \left[yu(y,Y,t)p^0(y,Y,t)\left[\frac{1}{2}\epsilon A(t) - \frac{(yr(t) - Yh(t))}{(1-\rho^2(t))\sigma_y(t)\sigma_z(t)}\right]\right.\\
&\left.\times \exp\left\{-\frac{1}{2}\epsilon^2 A(t) + \frac{\epsilon(yr(t) - Yh(t))}{(1-\rho^2(t))\sigma_y(t)\sigma_z(t)}\right\}\right]\mathrm{d}y\mathrm{d}t.
\end{aligned}$$

As $A(t) \geqslant 0$, we get

$$\begin{aligned}
\frac{I_1^\epsilon}{\epsilon} \geqslant\ & \int_0^{t_0\wedge T}\int_{-\infty}^0 \left[yu(y,Y,t)p^0(y,Y,t)\left[\frac{(yr(t) - Yh(t))}{(1-\rho^2(t))\sigma_y(t)\sigma_z(t)}\right]\right.\\
&\left.\times \exp\left\{-\frac{1}{2}\epsilon^2 A(t) + \frac{\epsilon(yr(t) - Yh(t))}{(1-\rho^2(t))\sigma_y(t)\sigma_z(t)}\right\}\right]\mathrm{d}y\mathrm{d}t. \qquad (4.37)
\end{aligned}$$

As the integrand in the right hand side of (4.37) is a positive function, Fatou's lemma yields the following inequality,

$$\liminf_{\epsilon\to 0}\frac{I_1^\epsilon}{\epsilon} \geqslant \int_0^{t_0\wedge T}\int_{-\infty}^0 yu(y,Y,t)p^0(y,Y,t)\left[\frac{(yr(t) - Yh(t))}{(1-\rho^2(t))\sigma_y(t)\sigma_z(t)}\right]\mathrm{d}y\mathrm{d}t. \qquad (4.38)$$

Note that in (4.38) the right hand side may be $+\infty$. For I_2^ϵ, since $t \geqslant t_0 \wedge T$, there is no singularity at $t = 0$. Therefore, taking the limit of I_2^ϵ/ϵ, we obtain

$$\liminf_{\epsilon\to 0} \frac{I_2^\epsilon}{\epsilon} = \int_{t_0\wedge T}^{T} \int_{-\infty}^{0} \frac{yu(y,Y,t)p^0(y,Y,t)(yr(t) - Yh(t))}{(1-\rho^2(t))\sigma_y(t)\sigma_z(t)} \mathrm{d}y\mathrm{d}t \tag{4.39}$$

which is finite. Note that

$$J = -\int_0^T \frac{|h(t)|}{(1-\rho^2(t))\sigma_y(t)\sigma_z(t)} \left[\int_{-\infty}^{0} yu(y,Y,t)p^0(y,Y,t)\mathrm{d}y\right] \mathrm{d}t$$

is finite. Indeed, from the expansion of $h(t)$ we have that locally in time

$$\frac{h(t)}{\sigma_y(t)\sigma_z(t)}$$

is bounded. Moreover from (4.36) above

$$-\int_0^T \int_{-\infty}^{0} yu(y,Y,t)p^0(y,Y,t)\mathrm{d}y\mathrm{d}t < \infty.$$

Collecting results we can assert that

$$\begin{aligned}\liminf_{\epsilon\to 0} \frac{I_\epsilon}{\epsilon} \geqslant & \int_0^T \int_{-\infty}^{0} \frac{y^2 r(t)u(y,Y,t)p^0(y,Y,t)\mathrm{d}y\mathrm{d}t}{(1-\rho^2(t))\sigma_y(t)\sigma_z(t)} \\ & - Y\int_0^T \frac{h(t)}{(1-\rho^2(t))\sigma_y(t)\sigma_z(t)} \left(\int_{-\infty}^{0} yu(y,Y,t)p^0(y,Y,t)\mathrm{d}y\right) \mathrm{d}t.\end{aligned} \tag{4.40}$$

The second integral is finite. Now, let us show that the first integral is $+\infty$. We check that

$$\lim_{t\to 0} \int_{-\infty}^{0} y^2 u(y,Y,t)p^0(y,Y,t)\mathrm{d}y > 0.$$

The function $u(y,z,t)$ is increasing in t. Indeed, from the probabilistic representation we have

$$u(y,z,t_1) = \mathbb{P}[\theta(y,z) > T - t_1] \leqslant \mathbb{P}[\theta(y,z) > T - t_2] = u(y,z,t_2), \quad \forall t_1 \leqslant t_2.$$

Therefore, we have

$$\lim_{t\to 0} \int_{-\infty}^{0} y^2 u(y,Y,0)p^0(y,Y,t)\mathrm{d}y \leqslant \lim_{t\to 0} \int_{-\infty}^{0} y^2 u(y,Y,t)p^0(y,Y,t)\mathrm{d}y \tag{4.41}$$

Now,

$$u_y(0-,Y,0) < 0.$$

Indeed, $\forall c > 0, \quad u(-c,Y,0) > 0$ and $u(0,Y,0) = 0$. So,

$$u_y(0-,Y,0) \leqslant 0.$$

It cannot be equal to 0, otherwise the derivative exists and $u_y(0,Y,0) = 0$. But then by minimum properties we have $u_{yy}(0,Y,0) > 0$, that contradicts

$$-u_t(0,Y,0) - \frac{1}{2}u_{yy}(0,Y,0) = 0.$$

Therefore for $y < 0$ close to 0 we have

$$u(y, Y, 0) \sim ay, \quad a < 0.$$

On the interval $(-\eta, 0)$, we can assume $u(y, Y, 0) > \frac{a}{2}y$. So,

$$\lim_{t \to 0} \int_{-\eta}^{0} y^2 u(y, Y, 0) p^0(y, Y, t) \mathrm{d}y \geqslant \lim_{t \to 0} \int_{-\eta}^{0} \frac{a}{2} y^3 p^0(y, Y, t) \mathrm{d}y.$$

From (4.40), since $\int_0^T \frac{r(t)}{(1-\rho^2(t))\sigma_y(t)\sigma_z(t)} \mathrm{d}t = +\infty$, it is sufficient to check the property

$$\lim_{t \to 0} \int_{-\eta}^{0} y^3 p^0(y, Y, t) \mathrm{d}y < 0.$$

Set

$$\tilde{q}_0(t) := q_0(t) + \frac{\rho(t) Y (1 - f(t)) \sigma_y(t)}{\sigma_z(t)},$$

then

$$p^0(y, Y, t) = \frac{1}{2\pi \sigma_y(t) \sigma_z(t) (1 - \rho^2(t))^{\frac{1}{2}}} \exp\left(-\frac{1}{2} \frac{Y^2 (1 - f(t))^2}{\sigma_z^2(t)}\right) \exp\left(-\frac{(y - \tilde{q}_0(t))^2}{2(1 - \rho^2(t)) \sigma_y^2(t)}\right).$$

Hence, denoting

$$L_\eta := \int_{-\eta}^{0} \frac{y^3}{\sqrt{2\pi} \sigma_y(t) (1 - \rho^2(t))^{1/2}} \exp\left(-\frac{(y - \tilde{q}_0(t))^2}{2(1 - \rho^2(t)) \sigma_y^2(t)}\right) \mathrm{d}y,$$

we get

$$\int_{-\eta}^{0} y^3 p^0(y, Y, t) \mathrm{d}y = \frac{1}{\sqrt{2\pi} \sigma_z(t)} \exp\left(-\frac{1}{2} \frac{Y^2 (1 - f(t))^2}{\sigma_z^2(t)}\right) L_\eta. \tag{4.42}$$

In addition, by change of variables, we have

$$L_\eta = \int_{\frac{-\eta - \tilde{q}_0(t)}{(1-\rho^2(t))^{1/2} \sigma_y(t)}}^{\frac{-\tilde{q}_0(t)}{(1-\rho^2(t))^{1/2} \sigma_y(t)}} (\tilde{q}_0(t) + (1 - \rho^2(t))^{1/2} \sigma_y(t) u)^3 \exp(-\frac{1}{2} u^2) \frac{\mathrm{d}u}{\sqrt{2\pi}}. \tag{4.43}$$

Therefore, for t close to 0 we have $\tilde{q}_0(t) \sim -\frac{Ykt}{4}$ and we can check using formula (4.42) and (4.43) that

$$\lim_{t \to 0} \int_{-\eta}^{0} y^3 p^0(y, Y, t) \mathrm{d}y = \frac{1}{2\pi} \frac{\sqrt{3}}{8} \int_{-\infty}^{0} u^3 \exp(-\frac{1}{2} u^2) \mathrm{d}u.$$

Finally, since $\frac{r(t)}{\sigma_y(t)\sigma_z(t)} = \frac{3}{2t^2}(1 + \frac{3c_0}{4} t + o(t))$, the first integral in the right hand side in (4.40) is $+\infty$. We thus have proven

$$\lim_{\epsilon \to 0} \frac{I^\epsilon}{\epsilon} = +\infty.$$

Next, consider from (4.19) the term

$$\frac{J^\epsilon}{\epsilon} = -\frac{1}{\epsilon} \int_0^T \int_0^\infty y u(y, -Y, t) p^0(y, -Y, t) \left[\exp\left\{ -\frac{1}{2} \epsilon^2 A(t) + \frac{\epsilon (y r(t) - Y l(t))}{(1 - \rho^2(t)) \sigma_y(t) \sigma_z(t)} \right\} - 1 \right] \mathrm{d}y \mathrm{d}t.$$

From (4.12) we have

$$p^0(y,-Y,t) = \frac{1}{2\pi\sigma_z(t)\sigma_y(t)(1-\rho^2(t))^{1/2}} \exp\left\{-\frac{1}{2(1-\rho^2(t))}\left[\frac{(y-q^0(t))^2}{\sigma_y^2(t)} + \frac{(Y+m^0(t))^2}{\sigma_z^2(t)} + \frac{2\rho(t)(y-q^0(t))(Y+m^0(t))}{\sigma_y(t)\sigma_z(t)}\right]\right\}.$$

Due to the term $\exp\{-\frac{(Y+m^0(t))^2}{2(1-\rho^2(t))\sigma_z^2(t)}\}$, we do not have a singularity because of $\sigma_z(t)$. Indeed, $Y + m^0(t) \geqslant Y(1-\exp(-\frac{c_0\pi}{2\omega}))$, and for t close to 0 we have

$$\begin{aligned}
&\frac{1}{\sigma_z(t)}\exp\left(-\frac{(Y+m^0(t))^2}{2(1-\rho^2(t))\sigma_z^2(t)}\right)\\
&\leqslant \frac{\sqrt{2(1-\rho^2(t))}}{Y(1-\exp(-\frac{c_0\pi}{2\omega}))}\left(\frac{Y+m^0(t)}{\sqrt{2(1-\rho^2(t))}\sigma_z(t)}\right)\exp\left(-\frac{(Y+m^0(t))^2}{2(1-\rho^2(t))\sigma_z^2(t)}\right)\\
&\leqslant C\exp\left(-\frac{(Y+m^0(t))^2}{4(1-\rho^2(t))\sigma_z^2(t)}\right)
\end{aligned}$$

where $C > 0$ is a constant depending on T. From the above equation and recalling the asymptotics $\sigma_y(t) \sim \sqrt{t}$ for t close to 0, we deduce that the quantity

$$-\int_0^T\int_0^\infty yu(y,-Y,t)p^0(y,-Y,t)[\frac{(yr(t)-Yl(t))}{(1-\rho^2(t))\sigma_y(t)\sigma_z(t)}]dydt$$

is well defined. In the same way

$$\frac{H^\epsilon}{\epsilon} = \frac{1}{\epsilon}\int_D \left(p^\epsilon(y,z,T) - p^0(y,z,T)\right)\mathrm{d}z\mathrm{d}y,$$

has a well defined limit. Therefore, from (4.18), we deduce

$$\frac{u(0,Y-\epsilon,0)}{\epsilon} \to +\infty \quad \text{as} \quad \epsilon\to 0. \tag{4.44}$$

As before, let us assume that $z(0) = -Y+\epsilon$. It is easy to see that $u(-y,-z,t) = u(y,z,t)$. This yields

$$\mathbb{P}(\theta^\epsilon > T) = u(0,Y-\epsilon,t) = u(0,-Y+\epsilon,0),$$

so $\frac{u(0,-Y+\epsilon,0)}{\epsilon} \to +\infty$ as well. This completes the proof.

Chapter 5

Degenerate Dirichlet problems related to the ergodic theory for an elasto-plastic oscillator excited by a filtered white noise

Ce chapitre fait l'objet d'un article soumis à IMA Journal of Applied Mathematics [5] en collaboration avec Alain Bensoussan.

A stochastic variational inequality is proposed to model an elasto-plastic oscillator excited by a filtered white noise. We prove the ergodic properties of the process and characterize the corresponding invariant measure. This extends Bensoussan-Turi's method (Degenerate Dirichlet Problems Related to the Invariant Measure of Elasto-Plastic Oscillators, AMO, 2008) with a significant additional difficulty of increasing the dimension. Two points boundary value problem in dimension 1 is replaced by elliptic equations in dimension 2. In the present context, Khasminskii's method (Stochastic Stability of Differential Equations, Sijthoff and Noordhof,1980) leads to the study of degenerate Dirichlet problems with partial differential equations and nonlocal boundary conditions.

5.1 Introduction

Nonlinear oscillators subjected to vibrations represent useful models for predicting the response of mechanical structures when stressed beyond the elastic limit. When the excitation is a white noise, it has received considerable interest over the past decades. In a previous work [7], the authors have considered the response of a white noise excited elasto-plastic oscillator using a stochastic variational inequality formulation. The results in [7] provide a framework to assess the accuracy of calculations made in the literature (see e.g., [19, 20, 26], and the references therein). In this paper, instead of considering white noise input signal whose power spectral density (PSD) is constant, we consider an excitation with a non-constant PSD which could be a more realistic framework. We consider the excitation as the velocity of a "reflected" Ornstein-Uhlenbeck process. Therefore, comparing with the elasto plastic oscillator excited by white noise, a third process occurs in the variational inequality. Consider $w(t)$ and $\tilde{w}(t)$, two independent Wiener processes and $x(t)$ an Ornstein-Ulhenbeck reflected process

$$\mathrm{d}x(t) = -\alpha x(t)\mathrm{d}t + \mathrm{d}w(t) + 1_{\{x(t)=-L\}}\mathrm{d}\xi_t^1 - 1_{\{x(t)=L\}}\mathrm{d}\xi_t^2. \tag{5.1}$$

We have $-L \leqslant x(t) \leqslant L$. In this model, the stochastic excitation is given by

$$-\beta x(t)\mathrm{d}t + \mathrm{d}\tilde{w}(t).$$

The stochastic variational inequality model is given by:

$$\begin{cases} \mathrm{d}x(t) = -\alpha x(t)\mathrm{d}t + \mathrm{d}w(t) + 1_{\{x(t)=-L\}}\mathrm{d}\xi_t^1 - 1_{\{x(t)=L\}}\mathrm{d}\xi_t^2, \\ \mathrm{d}y(t) = -(\beta x(t) + c_0 y(t) + kz(t))\mathrm{d}t + \mathrm{d}\tilde{w}(t), \\ (\mathrm{d}z(t) - y(t)\mathrm{d}t)(\zeta - z(t)) \geqslant 0, \\ |\zeta| \leqslant Y, \\ |z(t)| \leqslant Y. \end{cases} \tag{5.2}$$

When $\beta \neq 0$, $x(t)$ is involved in the dynamic of $y(t)$ and then this model will be referred as <u>the 2d case</u>. Whereas if $\beta = 0$, $x(t)$ is not involved in the dynamic of $y(t)$ and then $(y(t), z(t))$ satisfy the elasto-plastic oscillator problem of [7] which will be referred as <u>the 1d case</u>.

Notation 5. *Introduce the operators*

$$Au := \frac{1}{2}u_{yy} + \frac{1}{2}u_{xx} - \alpha x u_x - (\beta x + c_0 y + kz)u_y + y u_z,$$

$$B_+ u := \frac{1}{2}u_{yy} + \frac{1}{2}u_{xx} - \alpha x u_x - (\beta x + c_0 y + kY)u_y,$$

$$B_- u := \frac{1}{2}u_{yy} + \frac{1}{2}u_{xx} - \alpha x u_x - (\beta x + c_0 y - kY)u_y.$$

The infinitesimal generator of the process $(x(t), y(t), z(t))$, denoted by Λ is given by:

$$\Lambda : \ \phi \ \mapsto \ \begin{cases} A\phi & \text{if } z \in]-Y, Y[, \\ B_\pm \phi & \text{if } z = \pm Y, \pm y > 0. \end{cases}$$

Notation 6.

$\mathcal{O} := (-L, L)\times\mathbb{R}\times(-Y, Y)$; $\quad \mathcal{O}^+ := (-L, L)\times(0, +\infty)\times\{Y\}$; $\quad \mathcal{O}^- := (-L, L)\times(-\infty, 0)\times\{-Y\}$.

As the main result of the paper we prove the following:

Theorem 5.1.1. *There exists one and only one probability measure ν on $\mathcal{O}\cup\mathcal{O}^-\cup\mathcal{O}^+$ satisfying*

$$\int_{\mathcal{O}} A\phi d\nu + \int_{\mathcal{O}^-} B_-\phi d\nu + \int_{\mathcal{O}^+} B_+\phi d\nu = 0, \quad \forall\phi \quad \text{smooth}.$$

Moreover, ν has a probability density function m such that

$$\int_{\mathcal{O}} m(x, y, z)\, dx\, dy dz + \int_{\mathcal{O}^+} m(x, y, Y)\, dx\, dy + \int_{\mathcal{O}^-} m(x, y, -Y)\, dx\, dy = 1,$$

where

- *$\{m(x, y, z),\quad (x, y, z) \in \mathcal{O}\}$ is the elastic component,*
- *$\{m(x, y, Y),\quad (x, y) \in \mathcal{O}^+\}$ is the positive plastic component,*
- *and $\{m(x, y, -Y),\quad (x, y) \in \mathcal{O}^-\}$ is the negative plastic component.*

In addition, m satisfies in the sense of distributions the following equation in $\mathcal{O}$,

$$\alpha\frac{\partial}{\partial x}[xm] + \frac{\partial}{\partial y}[(\beta x + c_0 y + kz)m] - y\frac{\partial m}{\partial z} + \frac{1}{2}\frac{\partial^2 m}{\partial x^2} + \frac{1}{2}\frac{\partial^2 m}{\partial y^2} = 0 \qquad \text{in } \mathcal{O},$$

and on the boundary

$$ym + \frac{\partial}{\partial x}[xm] + \frac{\partial}{\partial y}[(\beta x + c_0 y + kY)m] + \frac{1}{2}\frac{\partial^2 m}{\partial x^2} + \frac{1}{2}\frac{\partial^2 m}{\partial y^2} = 0, \qquad \text{in } \mathcal{O}^+$$

$$-ym + \frac{\partial}{\partial x}[xm] + \frac{\partial}{\partial y}[(\beta x + c_0 y - kY)m] + \frac{1}{2}\frac{\partial^2 m}{\partial x^2} + \frac{1}{2}\frac{\partial^2 m}{\partial y^2} = 0, \qquad \text{in } \mathcal{O}^-$$

$$m = 0, \qquad \text{in } (-L, L) \times (-\infty, 0) \times \{Y\} \cup (-L, L) \times (0, \infty) \times \{-Y\}.$$

The proof will be based on solving a sequence of interior and exterior Dirichlet problems, which are interesting in themselves. We will put in parallel the 1d and 2d cases, in order to facilitate the reader's work. In the 1d case, the variable x disappears ($\beta = 0$), we will still use the notation A, B_+, B_- for the operators defined above without x.

Let us mention that our study presents a mathematical interest because it generalizes the method proposed by the first author and J. Turi [7] in the case of higher dimension. Non-local boundary conditions expressed in the form of differential equations in dimension 1 are replaced by elliptic partial differential equations (PDEs) in dimension 2. In the first case, there are two semi-explicit solutions. Thus, the non-local boundary conditions are reduced to two unknown numbers. In the second case, we do not know explicit formulas for solutions of the both elliptic on the boundary. In this context, these two solutions depend on two unknown functions respectively defined on the set $(-L, L)$.

In addition, the choice of the excitation (5.1) is also motivated by two technical considerations:

- the first is to force $x(t)$ (through the processes ξ^1, ξ^2) to evolve in the compact set $[-L, L]$. Thus, as part of our proof, a compactness argument allows to show the ergodic property of the triple $(x(t), y(t), z(t))$. Note that this is not a problem in terms of applications, because if we choose L large enough, then the process $x(t)$ is similar to an Ornstein-Ulhenbeck process.
- the second is the uncorrelation of $w(t)$ and $\tilde{w}(t)$. In our approach, based on PDEs associated to the triple $(x(t), y(t), z(t))$, we avoid the appearance of cross-derivative terms in the infinitesimal generator Λ. In the case where $w(t)$ and $\tilde{w}(t)$ are correlated, these cross-derivative terms yield more technical difficulties.

5.2 The interior Dirichlet problem

In this section, we prove existence and uniqueness to the homogeneous interior Dirichlet problem.

5.2.1 Some background on the interior Dirichlet problem in the 1d case

Let us recall the interior Dirichlet problem from [7]. Let $\bar{y}_1 > 0$,

Notation 7.

$$D_1 := (-\bar{y}_1, \bar{y}_1) \times (-Y, Y); \quad D_1^+ := (0, \bar{y}_1) \times \{Y\}; \quad D_1^- := (-\bar{y}_1, 0) \times \{-Y\}$$

and

$$D_1^\epsilon := (-\bar{y}_1 + \epsilon, \bar{y}_1 - \epsilon) \times (-Y, Y), \epsilon > 0.$$

Denote $\bar{\tau}_1 := \inf\{t > 0, |y(t)| = \bar{y}_1\}$ and consider $\phi \in L^\infty(-Y, Y)$. We will use the following notation $\mathbb{E}_p(\cdot) := \mathbb{E}\{\cdot \ |(y(0), z(0)) = p\}$. It is shown that $\mathbb{E}_{(y,z)}(\phi(z(\bar{\tau}_1)))$ solves a nonlocal Dirichlet problem: Find $\eta \in L^\infty(D_1) \cap \mathcal{C}^0(D_1^\epsilon), \forall \epsilon > 0$ such that

$$A\eta = 0 \text{ in } D_1, \quad B_+\eta = 0 \text{ in } D_1^+, \quad B_-\eta = 0 \text{ in } D_1^- \tag{5.3}$$

with

$$\eta(\bar{y}_1, z) = \phi(z), \quad \eta(-\bar{y}_1, z) = 0, \quad \text{if } z \in (-Y, Y).$$

Since $\eta(\bar{y}_1, Y) = \phi(Y)$ and $\eta(-\bar{y}_1, -Y) = 0$, there are semi-explicit solutions by solving ordinary differential equations for η on the boundary at $z = Y$, and $z = -Y$ respectively,

$$\eta(y, Y) = \eta_Y I(y, \bar{y}_1) + \phi(Y) I(0, y), \quad 0 < y \leqslant \bar{y}_1; \quad \eta(y, -Y) = \eta_{-Y} I(-\bar{y}_1, y), \quad -\bar{y}_1 \leqslant y < 0,$$

where η_Y and η_{-Y} are constants, and

$$I(a, b) := \frac{\int_a^b \exp(c_0\lambda^2 + 2kY\lambda) \mathrm{d}\lambda}{\int_0^{\bar{y}_1} \exp(c_0\lambda^2 + 2kY\lambda) \mathrm{d}\lambda}.$$

The nonlocal condition is restricted to the value of these two constants. Based on these semi-explicit expressions a subset K of $H^1(D_1)$ is defined for proving existence of the solution to (5.3) , by

$$K := \begin{cases} v \in H^1(D_1), \\ v(-\bar{y}_1, z) = \phi(z), \quad v(-\bar{y}_1, z) = 0, \\ v(y, Y) = v_Y I(y, \bar{y}_1) + \phi(Y) I(0, y), \quad 0 \leqslant y \leqslant \bar{y}_1 \\ v(y, -Y) = v_{-Y} I(-\bar{y}_1, y), \quad -\bar{y}_1 \leqslant y \leqslant 0 \\ v_Y, v_{-Y} \quad \text{are constant with} \quad |v_{\pm Y}| \leqslant \|\phi\|_{L^\infty}. \end{cases}$$

The set K is convex and not empty if $\phi(z) \in H^1(-Y, Y)$. We take $v(y, z) = \phi(z) I(0, y) \mathbf{1}_{\{y > 0\}}$.

5.2.2 The interior Dirichlet problem in the 2d case

Notation 8.

$$\Delta_1 := (-L, L) \times (-\bar{y}_1, \bar{y}_1) \times (-Y, Y),$$

$$\Delta_1^+ := (-L, L) \times (0, \bar{y}_1) \times \{Y\}, \quad \Delta_1^- := (-L, L) \times (-\bar{y}_1, 0) \times \{-Y\}$$

and

$$\Delta_1^\epsilon := (-L, L) \times (-\bar{y}_1 + \epsilon, \bar{y}_1 - \epsilon) \times (-Y, Y), \forall \epsilon > 0.$$

Denote $\bar{\tau}_1 := \inf\{t > 0, |y(t)| = \bar{y}_1\}$ and consider $\phi \in L^\infty((-L,L) \times (-Y,Y))$. Similarly as before we use the notation $\mathbb{E}_p(\cdot) := \mathbb{E}\{\cdot\ |(x(0),y(0),z(0)) = p\}$. We want to define $\mathbb{E}_{(x,y,z)}(\phi(x(\bar{\tau}_1), z(\bar{\tau}_1)))$ as the solution of the interior Dirichlet problem stated below.

Statement of the problem 1. *Find $\eta \in L^\infty(\Delta_1) \cap \mathcal{C}^0(\Delta_1^\epsilon), \forall \epsilon > 0$ such that*

$$A\eta = 0 \ \text{in}\ \Delta_1, \quad B_+\eta = 0 \ \text{in}\ \Delta_1^+, \quad B_-\eta = 0 \ \text{in}\ \Delta_1^-$$

and

$$\begin{array}{rcll} \eta_x(\pm L, y, z) & = & 0 & \text{in} \quad (y,z) \in (-\bar{y}_1, \bar{y}_1) \times (-Y,Y), \\ \eta(x, \bar{y}_1, z) & = & \phi(x,z) & \text{in} \quad (x,z) \in (-L,L) \times (-Y,Y), \\ \eta(x, -\bar{y}_1, z) & = & 0 & \text{in} \quad (x,z) \in (-L,L) \times (-Y,Y). \end{array}$$

This is formal. We should consider the case of ϕ smooth first and precise the functional space, then proceed with the regularization.

As in the 1d-case, this problem is a nonlocal problem but the boundary condition are in two dimensions. Thus we need to solve partial differential equations for η on the boundary at $z = Y$, and $z = -Y$ respectively. Here we do not have semi-explicit solution, indeed $\eta(x,y,Y)$ solves

$$B_+\eta = 0 \text{ on } (-L,L) \times (0,\bar{y}_1) \text{ with } \eta_x(\pm L, y, Y) = 0, \quad \eta(x,\bar{y}_1,Y) = \phi(x,z) \tag{5.4}$$

with $\eta(x,0,Y) = \eta_Y(x)$, and $\eta(x,y,-Y)$ solves

$$B_-\eta = 0 \text{ on } (-L,L) \times (-\bar{y}_1, 0) \text{ with } \eta_x(\pm L, y, -Y) = 0, \quad \eta(x,-\bar{y}_1,-Y) = 0 \tag{5.5}$$

with $\eta(x,0,-Y) = \eta_{-Y}(x)$ where $\eta_Y(x)$ and $\eta_Y(x)$ are unknown function with $\|\eta_{\pm Y}\|_{L^\infty} \leqslant \|\phi\|_{L^\infty}$. Next, we give a convenient formulation of the boundary condition (5.4)-(5.5).

Boundary conditions

In order to reformulate Problem 1, we consider first the equation (5.4) on the boundary Δ_1^+. Define $\beta^+(x,y)$ the solution of the mixed Dirichlet-Neuman problem

$$\left\{ \begin{array}{rcll} B_+\beta^+ & = & 0, & \text{in } (-L,L) \times (0,\bar{y}_1), \\ \beta_x^+(\pm L, y) & = & 0, & \text{in } (0,\bar{y}_1), \\ \beta^+(x,\bar{y}_1) & = & \phi(x,Y), & \text{in } (-L,L), \\ \beta^+(x,0) & = & 0, & \text{in } (-L,L). \end{array} \right. \tag{5.6}$$

Proposition 7. *If $\phi(x,Y) \in H^1(-L,L)$ then there exists a unique solution to the equation* (5.6)

$$\beta^+ \in H^1((-L,L) \times (0,\bar{y}_1)) \quad \textit{satisfying} \quad \|\beta^+\|_{L^\infty} \leqslant \|\phi\|_{L^\infty}.$$

Proof. Define $D_Y^+ := (-L,L) \times (0,\bar{y}_1)$ and consider on $H^1(D_Y^+)$ the bilinear form

$$b_Y(\xi,\chi) = \frac{1}{2}\int_{D_Y^+} \big(\xi_x\chi_x + \xi_y\chi_y\big)\mathrm{d}x\mathrm{d}y + \int_{D_Y^+} \big(\alpha x\xi_x + (c_0 y + kY + \beta x)\xi_y\big)\chi\mathrm{d}x\mathrm{d}y.$$

For λ sufficiently large $b_Y(\xi,\chi) + \lambda(\xi,\chi)$ is coercive. Now, define the convex set

$$K_Y = \{\xi \in H^1(D_Y^+), \xi(x,\bar{y}_1) = \phi(x,Y) \text{ and } \xi(x,0) = 0\}$$

which is not empty since $\phi(x,Y) \in H^{\frac{1}{2}}(-L,L)$. Take $z \in K_Y$ with $\|z\|_{L^\infty} \leqslant \|\phi\|_{L^\infty}$, we define ξ_λ to be the unique solution of

$$b_Y(\xi_\lambda, \chi - \xi_\lambda) + \lambda(\xi_\lambda, \chi - \xi_\lambda) \geqslant \lambda(z, \chi - \xi_\lambda), \qquad \xi_\lambda \in K_Y, \forall \chi \in K_Y \tag{5.7}$$

We have defined a map $T_\lambda(z) = \xi_\lambda$ from $K_Y \to K_Y$. Let us check that

$$\|\xi_\lambda\|_{L^\infty} \leqslant \|\phi\|_{L^\infty} = C. \tag{5.8}$$

Indeed, in (5.7) we take $\chi = \xi_\lambda - (\xi_\lambda - C)^+ \in K_Y$. Hence

$$-b_Y(\xi_\lambda, (\xi_\lambda - C)^+) - \lambda(\xi_\lambda, (\xi_\lambda - C)^+) \geqslant \lambda(z, (\xi_\lambda - C)^+)$$

and

$$b_Y((\xi_\lambda - C)^+, (\xi_\lambda - C)^+) + \lambda|(\xi_\lambda - C)^+|_{L^2} \leqslant -\lambda(z + C, (\xi_\lambda - C)^+).$$

Since $z + C \geqslant 0$ it follows that $(\xi_\lambda - C)^+ = 0$, hence $\xi_\lambda \leqslant C$. Similarly, we check that $(-\xi_\lambda - C)^+ = 0$, hence we have (5.8). Consider then the sequence ξ_λ^n defined by

$$b_Y(\xi_\lambda^{n+1}, \chi - \xi_\lambda^{n+1}) + \lambda(\xi_\lambda^{n+1}, \chi - \xi_\lambda^{n+1}) \geqslant \lambda(\xi_\lambda^n, \chi - \xi_\lambda^{n+1}) \tag{5.9}$$

with

$$\xi_\lambda^0 \in K_Y, \|\xi_\lambda^0\|_{L^\infty} \leqslant \|\phi\|_{L^\infty}.$$

We can take $\xi_\lambda^0(x,y) = \frac{y}{\bar{y}_1}\phi(x,Y)$. From (5.8), we have $\|\xi_\lambda^n\|_{L^\infty(D_Y^+)} \leqslant C$ and from (5.9), we have $\|\xi_\lambda^n\|_{H^1(D_Y^+)} \leqslant C'$. Then, we can consider a subsequence, also denoted by ξ_λ^n such that

$$\xi_\lambda^n \to \xi \qquad \text{in } H^1(D_Y^+) \text{ weakly and in } L^\infty(D_Y^+) \text{ weakly} \star$$

also,

$$\xi_\lambda^n \to \xi \qquad \text{in } L^2(D_Y^+) \text{ strongly.}$$

From (5.9) we obtain

$$b_Y(\xi, \chi - \xi) \geqslant 0, \qquad \xi \in K_Y, \forall \chi \in K_Y.$$

We conclude easily that ξ is a solution of (5.6) also $\|\xi\|_{L^\infty} \leqslant \|\phi\|_{L^\infty}$. Now, in order to prove uniqueness, we must prove that a solution $\xi \in H^1(D_Y^+)$ satisfying

$$\begin{aligned} B_+\xi &= 0 \text{ in } (-L,L) \times (0,\bar{y}_1) \\ \xi_x(-L,y) &= \xi_x(L,y) = 0 \\ \xi(x,\bar{y}_1) &= \xi(x,0) = 0 \end{aligned} \tag{5.10}$$

is identically zero.
Consider $\chi := \xi_x$ then we have

$$\begin{aligned} B_+\chi + \alpha\chi + \beta\xi_y &= 0 \text{ in } D_Y^+, \\ \chi(-L,y) &= \chi(L,y) = 0, \\ \chi(x,\bar{y}_1) &= \chi(x,0) = 0, \end{aligned}$$

hence $\chi \in H_0^1(D_Y^+)$. This implies

$$\xi_{xx} \in L^2(D_Y^+), \qquad \xi_{yx} \in L^2(D_Y^+).$$

From equation (5.10) we deduce $\xi_{yy} \in L^2(D_Y^+)$. Hence $\xi \in H^2(D_Y^+)$. In particular, ξ is continuous on $\bar{D}_Y^+$. We have $\|\xi\|_{L^\infty} = 0$, so $\xi = 0$. $\square$

Now, consider the following convex sets,

$$K_{Y,M} := \{\chi^+ \in H^1((-L,L)\times(0,\bar{y}_1)) \text{ satisfying (5.12)}; \quad \chi^+(x,\bar{y}_1)=0; \quad \|\chi^+\|_{L^\infty} \leqslant M\} \quad (5.11)$$

where

$$\begin{array}{l}\int_{-L}^{L}\int_0^{\bar{y}_1}\{\frac{1}{2}(\chi_x^+\psi_x+\chi_y^+\psi_y)+(\alpha x\chi_x^+ +(\beta x+c_0 y+kY)\chi_y^+)\psi\}\mathrm{d}x\mathrm{d}y=0,\\ \forall\psi\in H^1((-L,L)\times(0,\bar{y}_1)) \text{ with } \psi(x,0)=\psi(x,\bar{y}_1)=0.\end{array} \quad (5.12)$$

and

$$K_{-Y,M} := \{\chi^- \in H^1((-L,L)\times(-\bar{y}_1,0)) \text{ satisfying (5.14)}; \quad \chi^-(x,-\bar{y}_1)=0; \quad \|\chi^-\|_{L^\infty} \leqslant M\} \quad (5.13)$$

where

$$\begin{array}{l}\int_{-L}^{L}\int_{-\bar{y}_1}^{0}\{\frac{1}{2}(\chi_x^-\psi_x+\chi_y^-\psi_y)+(\alpha x\chi_x^- +(\beta x+c_0 y-kY)\chi_y^-)\psi\}\mathrm{d}x\mathrm{d}y=0,\\ \forall\psi\in H^1((-L,L)\times(-\bar{y}_1,0)) \text{ with } \psi(x,0)=\psi(x,-\bar{y}_1)=0.\end{array} \quad (5.14)$$

These sets are not empty since they contain 0.

Remark 7. • *$\forall\pi(y)$ function of y such that $\pi(0)=0$,*

$$\{\chi\pi;\quad \chi\in K_{Y,M}\} \quad \textit{and} \quad \{\chi\pi;\quad \chi\in K_{-Y,M}\} \quad \textit{are } H^1-\textit{bounded.}$$

- *If $\chi \in K_{Y,\|\phi\|_{L^\infty}}$, denoting $\omega := \beta^+ + \chi$, we have*

$$B_+\omega = 0 \qquad \textit{and} \qquad \max(|\omega(x,0)|,|\omega(x,\bar{y}_1)|) \leqslant \|\phi\|_{L^\infty},$$

so a maximum principle implies $\|\omega\|_{L^\infty} \leqslant \|\phi\|_{L^\infty}$.

Using the sets $K_{Y,\|\phi\|}$ and $K_{-Y,\|\phi\|}$, the following result gives a convenient formulation of the boundary conditions in Problem 1.

Proposition 8. *The Problem 1 can be reformulated in the following way: find $\eta \in L^\infty(\Delta_1)\cap \mathcal{C}^0(\Delta_1^\epsilon), \forall\epsilon>0$ such that*

$$A\eta = 0 \textit{ in } \Delta_1, \quad \eta(x,y,Y)-\beta^+(x,y) \in K_{Y,\|\phi\|}, \quad \eta(x,y,-Y)\in K_{-Y,\|\phi\|}$$

and

$$\begin{array}{rcll}\eta_x(\pm L,y,z) &=& 0, & \textit{in} \quad (y,z)\in(-\bar{y}_1,\bar{y}_1)\times(-Y,Y),\\ \eta(x,\bar{y}_1,z) &=& \phi(x,z), & \textit{in} \quad (x,z)\in(-L,L)\times(-Y,Y),\\ \eta(x,-\bar{y}_1,z) &=& 0, & \textit{in} \quad (x,z)\in(-L,L)\times(-Y,Y).\end{array}$$

Proof. First, we can obtain the generic solution of (5.4) by considering any function χ^+ which satisfies

$$\chi^+ \in H^1((-L,L)\times(0,\bar{y}_1))$$

and

$$B_+\chi^+ = 0, \quad \chi^+_x(\pm L, y) = 0, \quad \chi^+(x, \bar{y}_1) = 0. \tag{5.15}$$

Note that we have not defined the value of χ^+ for $y = 0$, hence χ^+ is certainly not unique. We add the condition that χ^+ is bounded by $\|\phi\|_{L^\infty}$. We define in a similar way the function χ^- such that

$$\chi^- \in H^1((-L, L) \times (-\bar{y}_1, 0))$$

and

$$B_-\chi^- = 0, \quad \chi^-_x(\pm L, y) = 0, \quad \chi^-(x, -\bar{y}_1) = 0. \tag{5.16}$$

Then, interpreting (5.15) as (5.12) and (5.16) as (5.14) repectively, we obtain $\chi^+ \in K_{Y,\|\phi\|}$ and $\chi^- \in K_{-Y,\|\phi\|}$. Hence, the set of solutions of (5.4) and (5.5) can be written as follows

$$\eta(x, y, Y) - \beta^+(x, y) \in K_{Y,\|\phi\|}, y > 0; \quad \eta(x, y, -Y) \in K_{-Y,\|\phi\|}, \quad y < 0.$$

□

Approximation (part 1)

We study Problem 1 by a regularization method in the next proposition. Define $A^\epsilon := A + \frac{\epsilon}{2}\frac{\partial^2}{\partial z^2}$.

Proposition 9. *The following problem: find $\eta^\epsilon \in L^\infty(\Delta_1) \cap H^1(\Delta_1)$ such that $\|\eta^\epsilon\|_{L^\infty} \leqslant \|\phi\|_{L^\infty}$,*

$$A^\epsilon\eta^\epsilon = 0 \text{ in } \Delta_1, \quad \eta^\epsilon(x, y, Y) - \beta_+(x, y) \in K_{Y,\|\phi\|}, \quad \eta^\epsilon(x, y, -Y) \in K_{-Y,\|\phi\|} \tag{5.17}$$

and

$$\begin{array}{lcll}
\eta^\epsilon_x(\pm L, y, z) & = & 0, & \text{in} \quad (y, z) \in (-\bar{y}_1, \bar{y}_1) \times (-Y, Y), \\
\eta^\epsilon_z(x, y, \pm Y) & = & 0, & \text{in} \quad (x, \mp y) \in (-L, L) \times (0, \bar{y}_1), \\
\eta^\epsilon(x, \bar{y}_1, z) & = & \phi(x, z), & \text{in} \quad (x, z) \in (-L, L) \times (-Y, Y), \\
\eta^\epsilon(x, -\bar{y}_1, z) & = & 0, & \text{in} \quad (x, z) \in (-L, L) \times (-Y, Y),
\end{array}$$

has a unique solution.

As in the 1d case, we formulate a variational inequality to prove existence of solutions. In the present context, we consider a convex subset of $H^1(\Delta_1)$ which is adapted to the two dimensional boundary condition by

$$K = \begin{cases}
\psi \in H^1(\Delta_1), \quad \|\psi\|_\infty \leqslant \|\phi\|_\infty, \\
\psi(x, \bar{y}_1, z) = \phi(x, z), \quad \text{for } (x, z) \in (-L, L) \times (-Y, Y), \\
\psi(x, -\bar{y}_1, z) = 0, \quad \text{for } (x, z) \in (-L, L) \times (-Y, Y), \\
\psi(., ., Y) - \beta^+(x, y) \in K_{Y,\|\phi\|_{L^\infty}}, \\
\psi(., ., -Y) \in K_{-Y,\|\phi\|_{L^\infty}}.
\end{cases}$$

Proposition 10. *The set K is a closed non-empty subset of $H^1(\Delta_1)$.*

Proof. The fact that K is closed follows from the continuity of the trace operator. Now, pick the function,

$$\psi(x, y, z) = \frac{y}{\bar{y}_1}\big[\phi(x, z) - \frac{z}{2Y}\phi(x, Y) - \frac{1}{2}\phi(x, Y)\big]1_{\{y \geqslant 0\}} + (\frac{z}{2Y} + \frac{1}{2})\beta_+(x, y)1_{\{y \geqslant 0\}}. \tag{5.18}$$

We have

$$\begin{aligned}
\psi(x,\bar{y}_1,z) &= \phi(x,z),\\
\psi(x,-\bar{y}_1,z) &= 0,\\
\psi(x,y,Y) &= \beta_+(x,y), \quad y>0,\\
\psi(x,y,-Y) &= 0, \quad y<0.
\end{aligned}$$

So, if $\phi(x,z) \in H^1(\partial\Delta_1)$, $\phi(x,Y) \in H^1(-L,L)$, then the function ψ defined by (5.16) belongs to K. □

Remark 8. *If $u \in K$ and $w \in H^1(\Delta_1)$ with $w(x,\pm\bar{y}_1,z)=0$ and $w(x,y,\pm Y)=0$, for $0<\pm y<\bar{y}_1$ then $u+w \in K$.*

Consider the bilinear form

$$\begin{aligned}
a(u,v) &= \frac{1}{2}\int_{\Delta_1}\{\epsilon u_z v_z + u_y v_y + u_x v_x\}\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&+\int_{\Delta_1}(\beta x + c_0 y + kz)u_y v\mathrm{d}x\mathrm{d}y\mathrm{d}z - \int_{\Delta_1} y u_z v\mathrm{d}x\mathrm{d}y\mathrm{d}z + \int_{\Delta_1}\alpha x u_x v\mathrm{d}x\mathrm{d}y\mathrm{d}z.
\end{aligned}$$

Equation (5.17) is formulated as follows $a(u,v-u) \geqslant 0, \quad \forall v \in K, u \in K$.

Proof of Proposition 9. First, existence is proved by variational argument. For λ sufficiently large $a(u,v)+\lambda(u,v)$ is coercive on $H^1(\Delta_1)$ and for $f \in L^2(\Delta_1)$ we can solve the variational inequality

$$a(u,v-u)+\lambda(u,v-u) \geqslant (f,v-u), \qquad \forall v \in K, u \in K.$$

We define the map $u = T_\lambda w$ where u is the unique solution of

$$a(u,v-u)+\lambda(u,v-u) \geqslant \lambda(w,v-u), \qquad \forall v \in K, u \in K.$$

The following lemma shows that T_λ is a contracting map. (see proof in Appendix)

Lemma 5. *If $\|w\|_{L^\infty} \leqslant \|\phi\|_{L^\infty}$ then $\|u\|_{L^\infty} \leqslant \|\phi\|_{L^\infty}$.*

Moreover, taking $v = u_0 \in K$ we deduce

$$a(u,u)+\lambda|u|^2_{L^2} \leqslant a(u,u_0)+\lambda(u,u_0)+\lambda\gamma|u_0-u|_{L^1}.$$

That implies

$$\|u\|_{H^1(\Delta_1)} \leqslant M,$$

for a constant M which depends only of λ, ϵ, H^1 norm of u_0 and γ.
Now, we define

$$\bar{K} = \{w \in K : \|w\|_{L^\infty} \leqslant \|\phi\|_{L^\infty}, \|w\|_{H^1} \leqslant M\}.$$

We have $T_\lambda : L^2(\Delta_1) \to L^2(\Delta_1)$ is continuous, maps $\bar{K}$ into itself and $\bar{K}$ is a compact subset of L^2. Then Schauder's theorem implies T_λ has a fixed point $u \in \bar{K}$ which satisfies

$$a(u,v-u) \geqslant 0, \qquad \forall v \in K.$$

Let us check that u is solution of (5.17). As $u \in K$, we have

$$\begin{array}{rcll} B_+u & = & 0 & \text{in} \quad \Delta^+ \\ B_-u & = & 0 & \text{in} \quad \Delta^- \\ u(x,\bar{y}_1,z) & = & \phi(x,z) & \text{in} \quad (-L,L)\times(-Y,Y) \\ u(x,-\bar{y}_1,z) & = & 0 & \text{in} \quad (-L,L)\times(-Y,Y) \end{array}$$

Moreover,

$$\forall v \in \mathcal{H} := \left\{ v \in H^1(\Delta_1) \text{ such that } \left\{ \begin{array}{l} v(x,\pm\bar{y}_1,z) = 0 \\ v(x,y,\pm Y) = 0, \quad 0 < \pm y < \bar{y}_1 \end{array} \right. \right\},$$

we have $u + v \in K$ and $a(u,v) \geqslant 0$. Then,

$$A^\epsilon u = 0 \text{ in the sense } \mathcal{H}'$$

and integration by parts gives

$$\frac{\epsilon}{2}\int_{-L}^{L}\int_{-\bar{y}_1}^{0} u_z(x,y,Y)v(x,y,Y)\mathrm{d}x\mathrm{d}y - \frac{\epsilon}{2}\int_{-L}^{L}\int_{0}^{\bar{y}_1} u_z(x,y,-Y)v(x,y,-Y)\mathrm{d}x\mathrm{d}y$$
$$+\int_{-\bar{y}_1}^{\bar{y}_1}\int_{-Y}^{Y} u_x(L,y,z)v(L,y,z)\mathrm{d}y\mathrm{d}z - \int_{-\bar{y}_1}^{\bar{y}_1}\int_{-Y}^{Y} u_x(-L,y,z)v(-L,y,z)\mathrm{d}y\mathrm{d}z = 0.$$

Uniqueness of the solution to problem (5.17) comes from $\|u\|_{L^\infty} \leqslant \|\phi\|_{L^\infty}$. □

Approximation (part 2)

When ϕ is smooth, we can exhibit a solution to the problem (5.19) by extracting a converging subsequence of η^ϵ. Let θ a smooth fonction such that $\theta(\pm\bar{y}_1) = 0$. Denote $\pi(y) := y^p\theta(y)^q$ for some p, q.

Proposition 11. *The following problem: find $\eta \in L^\infty(\Delta_1)$ such that $\eta\pi \in H^1(\Delta_1)$,*

$$A\eta = 0 \textit{ in } \Delta_1, \quad \eta(x,y,Y) - \beta_+(x,y) \in K_{Y,\|\phi\|}, \quad \eta(x,y,-Y) \in K_{-Y,\|\phi\|} \tag{5.19}$$

and

$$\begin{array}{rcll} \eta_x(\pm L,y,z) & = & 0 & \textit{in} \quad (y,z) \in (-\bar{y}_1,\bar{y}_1)\times(-Y,Y), \\ \eta(x,\bar{y}_1,z) & = & \phi(x,z) & \textit{in} \quad (x,z) \in (-L,L)\times(-Y,Y), \\ \eta(x,-\bar{y}_1,z) & = & 0 & \textit{in} \quad (x,z) \in (-L,L)\times(-Y,Y), \end{array}$$

has a unique solution.

As in the 1d case, the key ingredient of the proof is to bound uniformly the norm of first derivative w.r.t. z using the auxiliary function π.

Proof. From the previous section, we have $\|\eta^\epsilon\|_\infty \leqslant \|\phi\|_\infty$, hence $\eta^\epsilon \to \eta$ in $L^\infty\star$. We also have $a(\eta^\epsilon, u_0 - \eta^\epsilon) \geqslant 0$, for some $u_0 \in K$. So, we deduce estimates in the following lemma: (see proof in Appendix)

Lemma 6. *We have*

$$\epsilon\int_{\Delta_1}(\eta_z^\epsilon)^2\,dx\,dy\,dz \leqslant C; \quad \int_{\Delta_1}(\eta_x^\epsilon)^2\,dx\,dy\,dz \leqslant C; \quad \int_{\Delta_1}(\eta_y^\epsilon)^2\,dx\,dy\,dz \leqslant C.$$

It is licit to test (5.17) with $\eta_z^\epsilon y^{2p-1}\theta^{2q}$. We have

$$\int_{\Delta_1}\big(\frac{\epsilon}{2}\eta_{zz}^\epsilon+\frac{1}{2}\eta_{yy}^\epsilon+\frac{1}{2}\eta_{xx}^\epsilon+y\eta_z^\epsilon-\alpha x\eta_x^\epsilon-(\beta x+c_0y+kz)\eta_y^\epsilon\big)\eta_z^\epsilon y^{2p-1}\theta^{2q}=0.$$

So, we obtain

$$\begin{array}{rl} & \int_{\Delta_1}(\eta_z^\epsilon\pi)^2\leqslant \\ & \frac{1}{4}\int_0^L\int_0^{\bar y_1}\{(\eta_y^\epsilon(x,y,Y))^2+(\eta_x^\epsilon(x,y,Y))^2\}y^{2p-1}\theta^{2q}\mathrm{d}x\mathrm{d}y \\ + & \frac{1}{4}\int_0^L\int_{-\bar y_1}^0\{(\eta_y^\epsilon(x,y,-Y))^2+(\eta_x^\epsilon(x,y,-Y))^2\}|y|^{2p-1}\theta^{2q}\mathrm{d}x\mathrm{d}y \\ + & \frac{1}{2}\int_{\Delta_1}\eta_y^\epsilon\eta_z^\epsilon\big(((2p-1)y^{2p-2}\theta^{2q}+y^{2p-1}2q\theta'\theta^{2q-1})+2(\beta x+c_0y+kz)y^{2p-1}\theta^{2q}\big)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ + & \int_{\Delta_1}\eta_x^\epsilon\eta_z^\epsilon(\alpha xy^{2p-1}\theta^{2q})\mathrm{d}x\mathrm{d}y\mathrm{d}z. \end{array} \tag{5.20}$$

Moreover, Remark 7 allows to bound the two integrals on the boundary that yields the following estimate:

$$\|\eta_z^\epsilon\pi\|_{L^2}\leqslant C_\pi.$$

Denoting $v^\epsilon:=\eta^\epsilon\pi$, we have $\|v^\epsilon\|_{H^1(\Delta_1)}\leqslant\tilde C_\pi$ so we can extract a weakly converging subsequence $v^\epsilon\to v$ in $H^1(\Delta_1)$ and $v=\eta\pi\in H^1(\Delta_1)$. We can check that η satisfies the boundary condition of Problem 1. First, let us check that

$$\pi(y)A\eta=0\text{ in }H^{-1}(\Delta_1),\quad \pi(y)\eta_x(\pm L,y,z)=0\text{ in }(H^{\frac{1}{2}}((-L,L)\times(-\bar y_1,\bar y_1)))'.$$

As $v^\epsilon\in H^2(\Delta_1)$ and $\pi A^\epsilon\eta^\epsilon=0$, we have

$$-A^\epsilon v^\epsilon=f(\eta,\eta_y)\quad\text{ in strong sense}$$

with $f(\eta,\eta_y):=-\frac{1}{2}\{\pi''\eta^\epsilon+2\pi'\eta_y^\epsilon\}+(c_0y+kz+\alpha x)\pi'\eta^\epsilon$.

We obtain that $\forall\phi\in H^1(\Delta_1),\phi(x,y,\pm Y)=0$ and $\phi(x,\pm\bar y_1,z)=0$,

$$\frac{\epsilon}{2}\int_{\Delta_1}v_z^\epsilon\phi_z+\frac{1}{2}\int_{\Delta_1}v_y^\epsilon\phi_y+\frac{1}{2}\int_{\Delta_1}v_x^\epsilon\phi_x+\int_{\Delta_1}\big(\alpha xv_x^\epsilon+(\beta x+c_0y+kz)v_y^\epsilon-yv_z^\epsilon\big)\phi=\int_{\Delta_1}f(\eta^\epsilon,\eta_y^\epsilon)\phi.$$

Now, when ϵ goes to 0, we have

$$\frac{1}{2}\int_{\Delta_1}v_y\phi_y+\frac{1}{2}\int_{\Delta_1}v_x\phi_x+\int_{\Delta_1}\big(\alpha xv_x+(\beta x+c_0y+kz)v_y-yv_z\big)\phi=\int_{\Delta_1}f(\eta,\eta_y)\phi.$$

We deduce we have in $H^{-1}(\Delta_1)$, firstly $-Av=f(\eta,\eta_y)$ which is equivalent to $\pi A\eta=0$ and secondly that choice of test function implies $\pi\eta_x(\pm L,y,z)=0$ in $(H^{\frac{1}{2}}((-L,L)\times(-\bar y_1,\bar y_1))'$. Then, we check that

$$\eta(x,y,Y)-\beta^+(x,y)\in K_{Y,\|\phi\|};\quad \eta(x,y,-Y)\in K_{-Y,\|\phi\|}.$$

We know that $\eta^\epsilon\in H^1(\Delta_1)$, its trace is well defined and satisfies

$$\gamma(\eta^\epsilon)(x,y,Y)=\chi^{+,\epsilon}+\beta^+;\quad \chi^{+,\epsilon}\in K_{Y,\|\phi\|};\quad y>0$$
$$\gamma(\eta^\epsilon)(x,y,-Y)=\chi^{-,\epsilon};\quad \chi^{-,\epsilon}\in K_{-Y,\|\phi\|};\quad y<0$$

with

$$\|\chi^{\pm,\epsilon}\|_{L^\infty} \leqslant \|\phi\|_{L^\infty}. \tag{5.21}$$

We also have $\chi^{-,\epsilon}, \chi^{+,\epsilon}$ satisfy respectively (5.22) and (5.23)

$$\begin{aligned}&\textstyle\int_{-L}^{L}\int_0^{\bar{y}_1}\{\frac{1}{2}(\chi_x^{+,\epsilon}\psi_x + \chi_y^{+,\epsilon}\psi_y) + (\alpha x\chi_x^{+,\epsilon} + (\beta x + c_0 y + kY)\chi_y^{+,\epsilon})\psi\}\mathrm{d}x\mathrm{d}y = 0,\\ &\forall\psi \in H^1((-L,L)\times(0,\bar{y}_1)) \text{ with } \psi(x,0) = \psi(x,\bar{y}_1) = 0.\end{aligned} \tag{5.22}$$

and

$$\begin{aligned}&\textstyle\int_{-L}^{L}\int_{-\bar{y}_1}^{0}\{\frac{1}{2}(\chi_x^{-,\epsilon}\psi_x + \chi_y^{-,\epsilon}\psi_y) + (\alpha x\chi_x^{-,\epsilon} + (\beta x + c_0 y - kY)\chi_y^{-,\epsilon})\psi\}\mathrm{d}x\mathrm{d}y = 0,\\ &\forall\psi \in H^1((-L,L)\times(-\bar{y}_1,0)) \text{ with } \psi(x,0) = \psi(x,-\bar{y}_1) = 0.\end{aligned} \tag{5.23}$$

First, we study convergence of the sequence $\chi^{\pm,\epsilon}$ and we deduce the PDEs satisfied by $\lim_{\epsilon\to 0}\chi^{\pm,\epsilon}$. In particular (5.21) implies

$$\begin{aligned}\chi^{+,\epsilon} &\to \chi^+ \text{ in } L^2((-L,L)\times(0,\bar{y}_1)) \text{ weakly},\\ \chi^{-,\epsilon} &\to \chi^- \text{ in } L^2((-L,L)\times(-\bar{y}_1,0)) \text{ weakly}.\end{aligned}$$

And (5.22) and (5.23) imply

$$\begin{aligned}\|\chi^{+,\epsilon}\pi\|_{H^1} \leqslant C \quad &\text{and} \quad \chi^{+,\epsilon}\pi \to \chi^+\pi \text{ in } H^1 \text{ weakly},\\ \|\chi^{-,\epsilon}\pi\|_{H^1} \leqslant C \quad &\text{and} \quad \chi^{-,\epsilon}\pi \to \chi^-\pi \text{ in } H^1 \text{ weakly}.\end{aligned}$$

Denote $\xi^{\pm,\epsilon} := \chi^{\pm,\epsilon}y^2$. From (5.22), we obtain $B_+\xi^{+,\epsilon} = 0$ in $(-L,L)\times(-\bar{y}_1,\bar{y}_1)$, in $H^{-1}((-L,L)\times(0,\bar{y}_1))$. Since the operator B_+ is strictly elliptic then $\xi^{+,\epsilon} \in H^2((-L,L)\times(0,\bar{y}_1))$. We also have $\xi_x^{+,\epsilon}(\pm L, y) = 0$ in $(H^{\frac{1}{2}}(0,\bar{y}_1))'$. As $\xi^{+,\epsilon} \in H^2(\Delta_1)$ and $\pi B_+\chi^{+,\epsilon} = 0$, we obtain

$$-B_+\xi^{+,\epsilon} = g(\chi^{+,\epsilon}, \chi_y^{+,\epsilon}) \quad \text{ in a strong sense}$$

with $g(\chi^{+,\epsilon}, \chi_y^{+,\epsilon}) := -\chi^{+,\epsilon}\{1 - 2y(\alpha x + c_0 y + kY)\} - 2y\chi_y^{+,\epsilon}$. We obtain that $\forall\psi \in H^1((-L,L)\times(0,\bar{y}_1)), \psi(x,0) = \psi(x,\bar{y}_1) = 0$,

$$\int_{-L}^{L}\int_0^{\bar{y}_1}\frac{1}{2}(\xi_x^{+,\epsilon}\psi_x + \xi_y^{+,\epsilon}\psi_y) + \big(\alpha x\xi_x^{+,\epsilon} + (\beta x + c_0 y + kY)\xi_y^{+,\epsilon} - y\xi_z^{+,\epsilon}\big)\psi = \int_{-L}^{L}\int_0^{\bar{y}_1} g(\chi^\epsilon, \chi_y^\epsilon)\psi.$$

Now, when ϵ goes to 0, we have

$$\int_{-L}^{L}\int_0^{\bar{y}_1}\frac{1}{2}(\xi_x^{+}\psi_x + \xi_y^{+}\psi_y) + \big(\alpha x\xi_x^{+} + (\beta x + c_0 y + kY)\xi_y^{+} - y\xi_z^{+}\big)\psi = \int_{-L}^{L}\int_0^{\bar{y}_1} g(\chi, \chi_y)\psi.$$

We deduce that in $H^{-1}((-L,L)\times(0,\bar{y}_1))$, we firstly have $-B_+\xi^+ = g(\chi^+, \chi_y^+)$, which is equivalent to $y^2B_+\chi^+ = 0$ and secondly that choice of test functions implies $y^2\chi_x^+(\pm L, y) = 0$ in $(H^{\frac{1}{2}}((0,\bar{y}_1)))'$. To summarize, we have

$$\xi^+ \in H^1((-L,L)\times(0,\bar{y}_1))$$

and

$$\begin{cases} -B_+\xi^+ &= g(\chi^+,\chi^+_y) & \text{in} & (-L,L)\times(0,\bar{y}_1),\\ \xi^+_x(\pm L,y) &= 0 & \text{in} & (0,\bar{y}_1),\\ \xi^+(x,\bar{y}_1) &= 0 & \text{in} & (-L,L). \end{cases}$$

Hence

$$\pi\chi^+ \in H^1((-L,L)\times(0,\bar{y}_1)), \quad \|\chi^+\|_{L^\infty} \leqslant \|\phi\|_{L^\infty}$$

and

$$\begin{cases} \pi B_+\chi^+ &= 0 & \text{in} & (-L,L)\times(0,\bar{y}_1),\\ \pi\chi^+_x(\pm L,y) &= 0 & \text{in} & (0,\bar{y}_1),\\ \chi^+(x,\bar{y}_1) &= 0 & \text{in} & (-L,L). \end{cases} \tag{5.24}$$

Similarly, we have

$$\pi\chi^- \in H^1((-L,L)\times(-\bar{y}_1,0)), \quad \|\chi^-\|_{L^\infty} \leqslant \|\phi\|_{L^\infty}$$

and

$$\begin{cases} \pi B_+\chi^- &= 0 & \text{in} & (-L,L)\times(-\bar{y}_1,0),\\ \pi\chi^-_x(\pm L,y) &= 0 & \text{in} & (-\bar{y}_1,0),\\ \pi\chi^-(x,-\bar{y}_1) &= 0 & \text{in} & (-L,L). \end{cases} \tag{5.25}$$

First, $\gamma(\pi\eta^\epsilon) \to \pi(\chi^+ + \beta^+)$ in $H^1((-L,L)\times(0,\bar{y}_1))$ weakly. Secondly, the weak convergence of $\pi\eta^\epsilon \to \pi\eta$ in $H^1(\Delta_1)$ implies the weak convergence of $\gamma(\pi\eta^\epsilon) \to \gamma(\pi\eta)$ in $H^{\frac{1}{2}}(\partial\Delta_1)$. By uniqueness of the limit, we deduce $\gamma(\pi\eta) = \pi(\chi^+ + \beta^+)$. Finally, we verify that

$$\eta(x,\bar{y}_1,z) = \phi(x,z); \quad \eta(x,\bar{y}_1,z) = 0.$$

Using Green formula, we obtain

$$\forall\psi \in H^1(\Delta_1), \quad \int_{\Delta_1} \eta^\epsilon_y\psi + \int_{\Delta_1} \eta^\epsilon\psi_y = \int_{\partial\Delta_1} \phi\psi\vec{n}(y)\mathrm{d}\sigma.$$

Now, we can let ϵ tend to 0, we obtain

$$\forall\psi \in H^1(\Delta_1), \quad \int_{\Delta_1} \eta_y\psi + \int_{\Delta_1} \eta\psi_y = \int_{\partial\Delta_1} \phi\psi\vec{n}(y)\mathrm{d}\sigma.$$

□

Approximation (part 3)

Now, $\phi \in L^\infty(\partial\Delta_1)$, we introduce a sequence of function $\{\phi^k, k \geqslant 0\} \subset H^1(\partial\Delta_1)$ such that $\phi^k \to \phi$ in $L^2(\partial\Delta_1)$. We denote η^k the solution of the Problem 5.19 with ϕ^k as boundary condition. From the previous section we have $\eta^k \in L^\infty(\Delta_1)$ satisfies $\pi(y)\eta^k \in H^1(\Delta_1)$,

$$A\eta^k = 0 \text{ in } \Delta_1, \quad \eta^k(x,y,Y) - \beta^{k,+}(x,y) \in K_{Y,\|\phi\|}, \quad \eta^k(x,y,-Y) \in K_{-Y,\|\phi\|}$$

and

$$\begin{aligned} \eta^k_x(\pm L,y,z) &= 0 & \text{in} \quad & (y,z) \in (-\bar{y}_1,\bar{y}_1)\times(-Y,Y),\\ \eta^k(x,\bar{y}_1,z) &= \phi^k(x,z) & \text{in} \quad & (x,z) \in (-L,L)\times(-Y,Y),\\ \eta^k(x,-\bar{y}_1,z) &= 0 & \text{in} \quad & (x,z) \in (-L,L)\times(-Y,Y). \end{aligned}$$

where $\beta^{k,+}(x,y) \in H^1((-L,L) \times (0,\bar{y}_1))$ solves the problem (5.6) with $\phi^k(x,Y)$ as boundary condition. Moreover $\|\eta^k\|_\infty \leqslant \|\phi\|_\infty$ and $\|\beta^k\|_\infty \leqslant \|\phi\|_\infty$. Let us check that the sequence β^k has a limit.

Proposition 12. *We have*

$$\beta^{k,+}(x,y) \to \beta^+ \text{ in } L^2((-L,L) \times (0,\bar{y}_1)) \text{ weakly,}$$
$$\pi\beta^{k,+}(x,y) \to \pi\beta^+ \text{ in } H^1((-L,L) \times (0,\bar{y}_1)) \text{weakly}$$

and the limit β^+ solves the problem (5.6) with $\phi(x,Y)$ as boundary condition.

Proof. We have $\forall\psi \in H^1((-L,L) \times (0,\bar{y}_1))$ with $\psi(x,0) = \psi(x,\bar{y}_1) = 0$,

$$\frac{1}{2}\iint \left(\beta_x^{k,+}\psi_x + \beta_y^{k,+}\psi_y\right) + \iint \left(\alpha x\beta_x^{k,+} + (c_0 y + kY + \beta x)\beta_y^{k,+}\right)\psi = 0. \tag{5.26}$$

In particular, the choice of $\psi = \pi(y)\beta^{k,+}$ with $\pi(0) = \pi(\bar{y}_1) = 0$ gives

$$\frac{1}{2}\iint \left((\pi\beta_x^{k,+})^2 + (\pi\beta_y^{k,+})^2\right) + \iint \pi(y)\pi'(y)\beta_y^{k,+}\beta^{k,+} + \iint \left(\alpha x\beta_x^{k,+} + (c_0 y + kY + \beta x)\beta_y^{k,+}\right)\pi^2\beta^{k,+} = 0.$$

This implies

$$\iint (\pi\beta_x^{k,+})^2 \leqslant C_\pi \quad \text{and} \quad \iint (\pi\beta_y^{k,+})^2 \leqslant C_\pi.$$

We deduce that we can extract a subsequence such that we have

$$\beta^{k,+} \to \beta^+ \text{ in } L^2 \text{ weakly} \qquad \text{and} \qquad \pi\beta^{k,+} \to \pi\beta^+ \text{ in } H^1 \text{ weakly.}$$

Now, denote $\gamma^k := \pi(y)\beta^{k,+}$. We have $B_+\gamma^k = -\beta^{k,+}(\frac{\pi''}{2} - (\beta x + c_0 y + kY)\pi') - \beta_y^{k,+}\pi'$. Also $\forall\psi \in H^1((-L,L) \times (0,\bar{y}_1))$ with $\psi(x,0) = \psi(x,\bar{y}_1) = 0$,

$$\begin{aligned}\frac{1}{2}\iint \left(\gamma_x^k\psi_x + \gamma_y^k\psi_y\right) + \iint \left(\alpha x\gamma_x^k + (c_0 y + kY + \beta x)\gamma_y^k\right)\psi \\ = \iint \{-\beta^{k,+}(\frac{\pi''}{2} - (\beta x + c_0 y + kY)\pi') - \beta_y^{k,+}\pi'\}\psi.\end{aligned} \tag{5.27}$$

When k goes to $+\infty$ in (5.27), we obtain

$$B_+\beta^+ = 0; \quad \beta_x^+(\pm L, y) = 0.$$

Then, from (5.28),

$$\forall\psi \in H^2 \cap H_0^1, \quad \iint \beta^{k,+}\psi_{yy} = -\iint \beta_y^{k,+}\psi_y + \int \phi^k\psi_y\vec{n}(x)\mathrm{d}\sigma, \tag{5.28}$$

we deduce taking limit when k goes to $+\infty$

$$\forall\psi \in H^2 \cap H_0^1, \quad \iint \beta^+\psi_{yy} = -\iint \beta_y^+\psi_y + \int \phi\psi_y\vec{n}(x)\mathrm{d}\sigma,$$

and we obtain the Dirichlet boundary condition

$$\beta^+(x,\bar{y}_1) = \phi(x,Y), \quad \beta^+(x,0) = 0.$$

□

Theorem 5.2.1. *The Problem 1 has a unique solution.*

Proof. Testing $A\eta^k$ with $\eta\theta^2$, we obtain

$$\frac{1}{2}\iint(\eta^k_y\theta)^2+\frac{1}{2}\iint(\eta^k_x\theta)^2 \leqslant \frac{1}{2}\iint(\eta^k)^2(\theta\theta')'+\frac{1}{2}\iint(c_0y+kz+\beta x)(\eta^k)^2(\theta^2)'+\frac{1}{2}\iint\alpha\theta^2(\eta^k)^2$$
$$-\frac{1}{2}\int\alpha x(\eta^k\theta)^2\vec{n}(x)d\sigma+\frac{1}{2}\int y\theta^2(\eta^k)^2\vec{n}(z)\mathrm{d}\sigma.$$

So, we have

$$\iint(\eta^k_x\theta)^2\leqslant C;\quad \iint(\eta^k_y\theta)^2\leqslant C.$$

Testing $(A\eta^k)$ with $\eta^k_z y^{2p-1}\theta^{2q}$, we deduce the following lemma

Lemma 7. *We have*

$$\begin{aligned}
\textstyle\int_{\Delta_1}(\eta^k_z\pi)^2 \leqslant\quad & \tfrac{1}{4}\textstyle\int_0^L\int_0^{\bar{y}_1}\{(\eta^k_y(x,y,Y))^2+(\eta^k_x(x,y,Y))^2\}y^{2p-1}\theta^{2q}dxdy\\
+\quad & \tfrac{1}{4}\textstyle\int_0^L\int_{-\bar{y}_1}^0\{(\eta^k_y(x,y,-Y))^2+(\eta^k_x(x,y,-Y))^2\}|y|^{2p-1}\theta^{2q}dxdy\\
+\quad & (p-\tfrac{1}{2})\textstyle\int_{\Delta_1}(\eta^k_y\theta)(\eta^k_z\pi)y^{p-2}\theta^{q-1}+q\int_{\Delta_1}(\eta^k_y\theta)(\eta^k_z\pi)y^{p-1}\theta^{q-2}\theta' \qquad (5.29)\\
+\quad & \textstyle\int_{\Delta_1}(c_0y+kz+\beta x)(\eta^k_y\theta)(\eta^k_z\pi)y^{p-1}\theta^{q-1}\\
+\quad & \textstyle\int_{\Delta_1}\alpha x(\eta^k_x\theta)(\eta^k_z\pi)y^{p-1}\theta^q.
\end{aligned}$$

Moreover, $\|\eta^k_z\pi\|_{L^2}\leqslant C_\pi$.

Denoting $v^k:=\eta^k\pi$, we have $\|v^k\|_{H^1(\Delta_1)}\leqslant\tilde{C}_\pi$ so we can extract a weakly converging subsequence $v^k\to v$ in $H^1(\Delta_1)$ and $v=\eta\pi\in H^1(\Delta_1)$. Similarly as before, we can check that η satisfies the boundary condition of Problem 1 which is summarized in the following lemma. (see proof in Appendix).

Lemma 8. *We have* $\pi(y)A\eta=0$ *in* $H^{-1}(\Delta_1)$, $\quad\pi(y)\eta_x(\pm L,y,z)=0$ *in* $(H^{\frac{1}{2}}((-L,L)\times(-\bar{y}_1,\bar{y}_1)))'$, $\eta(x,y,Y)-\beta^+(x,y)\in K_{Y,\|\phi\|}$, $\quad\eta(x,y,-Y)\in K_{-Y,\|\phi\|}$, $\quad$ *and* $\quad\eta(x,\bar{y}_1,z)=\phi(x,z)$; $\quad\eta(x,\bar{y}_1,z)=0$.

□

Local regularity in the interior Dirichlet problem

In this section, we derive local regularity properties of the function v related to the interior problem. We recall that $v:=\pi\eta\in H^1(\Delta_1)$ and we have

$$-\frac{1}{2}v_{xx}-\frac{1}{2}v_{yy}+\rho_0(x,y,z)v_y+\alpha xv_x-yv_z=\eta\rho_1(x,y,z)+\eta_y\rho_2(y) \qquad (5.30)$$

where we denote

$$\rho_0(x,y,z):=\beta x+c_0y+kz,\quad \rho_1(x,y,z):=\frac{\pi''}{2}-\rho_0(x,y,z)\pi',\quad \rho_2(y)=\pi'(y).$$

Let $\bar{y}<\bar{y}_1$,

Notation 9.

$$\begin{aligned} \Delta_1(\delta) &:= \{(x,y,z) \in \Delta_1, \quad |y-\bar{y}_1| > \delta, \quad |y+\bar{y}_1| > \delta\} \\ \Delta_1(\delta,\gamma) &:= \{(x,y,z) \in \Delta_1(\delta), \quad |z-Y| > \gamma\} \\ H_1(\delta) &:= H^1(-Y,Y;H^1(-L,L;H^1(-\bar{y}+\delta,\bar{y}-\delta))) \end{aligned}$$

Recall that $\eta \in H_1(\delta)$ means $\eta, \eta_x, \eta_y, \eta_z, \eta_{xy}, \eta_{xz}, \eta_{zy}, \eta_{xyz} \in L^2(\Delta_1(\delta))$.

Proposition 13. *We have*

$$\forall \delta, \gamma > 0, \quad \eta \in H_1(\delta); \quad \eta \in \mathcal{C}^0(\Delta_1(\delta)); \quad \eta \in H^2(\Delta_1(\delta,\gamma)), \tag{5.31}$$

and

$$\|\eta\|_{H_1(\delta)} \leqslant M_1; \quad \|\eta\|_{H^2(\Delta_1(\delta,\gamma))} \leqslant M_2. \tag{5.32}$$

Moreover, the trace of η at $y=\bar{y}$, denoted by $h(x,z) := \eta(x,\bar{y},z)$ satisfies

$$h \in H^1((-L,L)\times(-Y,Y)) \cap \mathcal{C}^0((-L,L)\times(-Y,Y)). \tag{5.33}$$

Proof. The proof relies on the following estimates (see proof in Appendix). We have

$$\eta_{xx}\pi^2 \in L^2(\Delta_1); \quad \eta_{xy}\pi^2 \in L^2(\Delta_1); \quad \eta_{yy}\pi^2 \in L^2(\Delta_1),$$
$$\eta_{xz}y^2\pi^4 \in L^2(\Delta_1); \quad \eta_{yz}y^2\pi^3 \in L^2(\Delta_1).$$

For p and q large enough, we have

$$\eta_{xyz}y^p\pi^q \in L^2(\Delta_1); \quad \eta_{yyz}y^p\pi^q \in L^2(\Delta_1); \quad \rho(z)\eta_{xxz}y^p\pi^q \in L^2(\Delta_1); \quad \rho(z)\eta_{zz}y^p\pi^q \in L^2(\Delta_1).$$

□

5.3 The exterior Dirichlet problem

In this section, we prove existence and uniqueness to the homogeneous exterior Dirichlet problem.

5.3.1 Some background on the exterior Dirichlet problem in the 1d case

Notation 10.

$$D_d := (-\infty,-\bar{y})\times(-Y,Y), \quad D_u := (\bar{y},+\infty)\times(-Y,Y),$$
$$D := D_d \cup D_u, \quad D^+ := (\bar{y},+\infty)\times\{Y\}, \quad D^- := (-\infty,-\bar{y})\times\{-Y\}$$

and

$$D^\epsilon := D \cap \{|y| > \bar{y}+\epsilon\}, \epsilon > 0.$$

Let us recall the exterior Dirichlet problem from [7]. Denote $\bar{\tau} := \inf\{t > 0, |y(t)| = \bar{y}\}$ and consider $h_\pm \in L^\infty(-Y,Y)$. It is shown that $\mathbb{E}_{(y,z)}(h_\pm(z(\bar{\tau})))$ solves a Dirichlet problem: Find $\zeta \in L^\infty(D) \cap \mathcal{C}^0(D^\epsilon), \forall \epsilon > 0$ such that

$$A\zeta = 0 \text{ in } D, \quad B_\pm\zeta = 0, \text{ in } D^\pm \tag{5.1}$$

with

$$\zeta(\pm\bar{y}, z) = h_{\pm}(z), \quad -Y < z < Y.$$

By solving the ordinary differential equation on the boundary, there is

$$\zeta_y(y, \pm Y) = K \exp(c_0 y^2 \pm 2kYy).$$

As a bounded solution is sought, K must be 0. Hence, $\zeta(y, \pm Y) = h_{\pm}(\pm Y)$ and then problem 5.1 was recast in [7] by

$$A\zeta = 0 \text{ in } D, \quad \zeta(y, \pm Y) = h_{\pm}(\pm Y), \text{ in } D^{\pm} \tag{5.2}$$

with

$$\zeta(\pm\bar{y}, z) = h_{\pm}(z), \quad -Y < z < Y.$$

5.3.2 The exterior Dirichlet problem in the 2d case

Notation 11.

$$\Delta_d := (-L, L) \times (-\infty, -\bar{y}) \times (-Y, Y), \quad \Delta_u := (-L, L) \times (\bar{y}, +\infty) \times (-Y, Y)$$

$$\Delta := \Delta_d \cup \Delta_u, \quad \Delta^+ := (-L, L) \times (\bar{y}, +\infty) \times \{Y\}, \quad \Delta^- := (-L, L) \times (-\infty, -\bar{y}) \times \{-Y\}$$

and

$$\Delta_u^{\epsilon} := \Delta_u \cap \{|y| > \bar{y} + \epsilon\}, \epsilon > 0.$$

Due to the symmetry in the exterior Dirichlet problem, we can consider positive values of y only. Denote $\bar{\tau} := \inf\{t > 0, y(t) = \bar{y}\}$ and consider $h \in L^{\infty}((-L, L) \times (-Y, Y))$, we define $\mathbb{E}_{(x,y,z)}(h(x(\bar{\tau}), z(\bar{\tau})))$ as the solution of the exterior Dirichlet Problem 2:

Statement of the problem 2. *Find $\zeta \in L^{\infty}(\Delta_u) \cap \mathcal{C}^0(\Delta_u^{\epsilon}), \forall \epsilon > 0$ such that*

$$A\zeta = 0 \text{ in } \Delta_u, \quad B_+\zeta = 0 \text{ in } \Delta^+$$

and

$$\begin{aligned}
\zeta_x(\pm L, y, z) &= 0, && \text{in} \quad (y, z) \in (\bar{y}, +\infty) \times (-Y, Y), \\
\zeta(x, \bar{y}, z) &= h(x, z), && \text{in} \quad (x, z) \in (-L, L) \times (-Y, Y).
\end{aligned}$$

Boundary condition

Find $\zeta^+ \in L^{\infty}(\Delta^+)$ such that

$$B_+\zeta^+ = 0 \text{ in } \Delta^+, \tag{5.3}$$

and

$$\begin{aligned}
\zeta_x^+(\pm L, y, z) &= 0 && \text{in} \quad (y, z) \in (\bar{y}, +\infty) \times (-Y, Y), \\
\zeta^+(x, \bar{y}, z) &= h(x, z) && \text{in} \quad (x, z) \in (-L, L) \times (-Y, Y).
\end{aligned}$$

Define

$$H_2^1(\Delta^+) := \left\{ u : \Delta^+ \to \mathbb{R}, \quad \int_{\Delta^+} \frac{u^2 + u_x^2 + u_y^2}{1 + y^2} dxdy < \infty \right\}.$$

Proposition 14. *Assume that there exists $H \in H_2^1(\Delta^+)$ such that $H(x,\bar{y}) = h(x,Y)$. Then there exists one and only one solution to the problem (5.3) with*

$$\zeta^+ \in H_2^1(\Delta^+), \quad \|\zeta^+\|_{L^\infty} \leqslant \|h(.,Y)\|_{L^\infty}.$$

Proof. First, we prove uniqueness. It is sufficient to prove that if we have $B_+\zeta = 0$, $\zeta_x(\pm L, y) = 0$ and $\zeta(x,\bar{y}) = 0$ with ζ bounded then we obtain $\zeta = 0$. Set $u(x,y) := \zeta(x,y)\exp(-\frac{c_0}{2}y^2)$ then

$$\begin{aligned}
\frac{1}{2}u_{xx} + \frac{1}{2}u_{yy} - \alpha x u_x - u_y(\beta x + kY) + uc_0(-\frac{c_0}{2}y^2 - y(\beta x + kY) + \frac{1}{2}) &= 0, \quad (x,y) \in \Delta^+, \\
u_x(\pm L, y) &= 0, \quad y > \bar{y}, \\
u(x,\bar{y}) &= 0, \quad x \in (-L, L).
\end{aligned}$$

We can assume that $-\frac{c_0}{2}y^2 - y(\beta x + kY) + \frac{1}{2} < 0, \quad y \geqslant \bar{y}$ ($\bar{y}$ sufficiently large). Let us prove that $u = 0$. Indeed if u has a positive maximum if cannot be at $y = \infty$. But then this contradicts maximum principle from (5.3). Similarily, we cannot have a negative minimum.

Now, we adress existence. Let

$$W_2^1(\Delta^+) := \left\{ u \in H_2^1(\Delta^+), \quad \|u\|_{W_2^1(\Delta^+)} := \|u\|_{L^\infty} + \left(\int_{\Delta^+} \frac{u_x^2 + u_y^2}{(1+y^2)^2} dxdy \right)^{\frac{1}{2}} < \infty \right\}.$$

We define a bilinear form on $H_2^1(\Delta^+) \times W_2^1(\Delta^+)$ by

$$\begin{aligned}
a(u,v) \quad := \quad & \frac{1}{2}\int_{\Delta^+} \frac{u_x v_x}{(1+y^2)^2} dxdy + \frac{1}{2}\int_{\Delta^+} \frac{u_y v_y}{(1+y^2)^2} dxdy - 2\int_{\Delta^+} \frac{u_y v y}{(1+y^2)^3} dxdy \\
& + \alpha \int_{\Delta^+} \frac{x u_x v}{(1+y^2)^2} dxdy + \int_{\Delta^+} \frac{(\beta x + c_0 y + kY) u_y v}{(1+y^2)^2} dxdy.
\end{aligned}$$

We next define

$$a_\gamma(u,v) := a(u,v) + \gamma \int_{\Delta^+} \frac{uv}{(1+y^2)^2} dxdy.$$

We finally define a bilinear form on $H_2^1(\Delta^+) \times H_2^1(\Delta^+)$ by

$$\begin{aligned}
a_{\gamma,\delta}(u,v) \quad := \quad & \frac{1}{2}\int_{\Delta^+} \frac{u_x v_x}{(1+y^2)^2} dxdy + \frac{1}{2}\int_{\Delta^+} \frac{u_y v_y}{(1+y^2)^2} dxdy - 2\int_{\Delta^+} \frac{u_y v y}{(1+y^2)^3} dxdy \\
& + \alpha \int_{\Delta^+} \frac{x u_x v}{(1+y^2)^2} dxdy + \int_{\Delta^+} (\beta x + c_0 \frac{y - \bar{y}}{1 + \delta y} + c_0 \bar{y} + kY) \frac{u_y v}{(1+y^2)^2} dxdy \\
& + \gamma \int_{\Delta^+} \frac{uv}{(1+y^2)^2} dxdy.
\end{aligned}$$

If $v \in W_2^1(\Delta^+)$,

$$a_{\gamma,\delta}(u,v) = a_\gamma(u,v) - \int_{\Delta^+} \frac{c_0(y - \bar{y})\delta y}{1 + \delta y} \frac{u_y v}{(1+y^2)^2} dxdy.$$

If $u \in W_2^1(\Delta^+)$, we can compute

$$\begin{aligned} a_{\gamma,\delta}(u,u) &= \frac{1}{2}\int_{\Delta^+}\frac{u_x^2}{(1+y^2)^2}dxdy + \frac{1}{2}\int_{\Delta^+}\frac{u_y^2}{(1+y^2)^2}dxdy - 2\int_{\Delta^+}\frac{u_y u y}{(1+y^2)^3}dxdy \\ &+\alpha\int_{\Delta^+}\frac{x u_x u}{(1+y^2)^2}dxdy + \int_{\Delta^+}\frac{(\beta x + c_0\bar{y} + kY)}{(1+y^2)^2}u_y u dxdy \\ &-\frac{c_0}{2}\int_{\Delta^+}\frac{u^2}{(1+y^2)^2}\left[\frac{1}{1+\delta y} - \frac{\delta(y-\bar{y})}{(1+\delta y)^2} - \frac{4(y-\bar{y})y}{(1+\delta y)(1+y^2)}\right]dxdy \\ &+\gamma\int_{\Delta^+}\frac{u^2}{(1+y^2)^2}dxdy. \end{aligned}$$

And see that

$$\begin{aligned} a_{\gamma,\delta}(u,u) &\geqslant \frac{1}{2}\int_{\Delta^+}\frac{u_x^2}{(1+y^2)^2}dxdy + \frac{1}{2}\int_{\Delta^+}\frac{u_y^2}{(1+y^2)^2}dxdy - 2\int_{\Delta^+}\frac{u_y u y}{(1+y^2)^3}dxdy \\ &+\alpha\int_{\Delta^+}\frac{x u_x u}{(1+y^2)^2}dxdy + \int_{\Delta^+}\frac{(\beta x + c_0\bar{y} + kY)}{(1+y^2)^2}u_y u dxdy \\ &+(\gamma - \frac{c_0}{2})\int_{\Delta^+}\frac{u^2}{(1+y^2)^2}dxdy \end{aligned}$$

and the right hand does not depend on δ. Moreover, we can define a constant $\bar{\gamma}$ depending only on the constants $\alpha, \beta, c_0, \bar{y}, k, Y$ but not on δ such that

$$a_{\gamma,\delta}(v,v) \geqslant a_0\|v\|^2_{H_2^1(\Delta^+)}, \quad a_0 > 0. \tag{5.4}$$

The constant a_0 depends only on $\alpha, \beta, c_0, \bar{y}, k, Y$. If $f(x,y)$ is bounded, we consider the problem

$$a_{\gamma,\delta}(u, v-u) \geqslant \gamma\int_{\Delta^+}\frac{f(v-u)}{(1+y^2)^2}dxdy, \quad \forall v \in K, \quad u \in K \tag{5.5}$$

where

$$K := \{v \in H_2^1(\Delta^+), \quad v(x,\bar{y}) = h(x,Y)\}$$

which is not empty from the assumption. Then from the coercivity (5.4) and results of the theory of Variational Inequalities (5.5) has one and only one solution $u_{\gamma\delta}(f)$ (writing u for $u_{\gamma\delta}(f)$ to simplify notation). If $w \in H_2^1(\Delta^+)$ satisfies $w(x,\bar{y}) = 0$ then $u + w \in K$, hence

$$a_{\gamma,\delta}(u,w) = \gamma\int_{\Delta^+}\frac{fw}{(1+y^2)^2}dxdy$$

and thus also

$$\begin{aligned} -\frac{1}{2}u_{xx} - \frac{1}{2}u_{yy} + \alpha x u_x + (\beta x + c_0\frac{y-\bar{y}}{1+\delta y} + c_0\bar{y} + kY)u_y + \gamma u &= \gamma f, \quad (x,y) \in \Delta^+, \\ u_x(\pm L, y) &= 0, \quad y \in (\bar{y}, +\infty), \\ u(x,\bar{y}) &= h(x,Y), \quad x \in (-L,+L). \end{aligned}$$

Also if

$$M_f := \max\{\|h(.,Y)\|_\infty, \|f\|_{L^\infty}\}$$

then

$$\|u_{\gamma\delta}(f)\|_{L^\infty} \leqslant M_f.$$

Moreover,

$$a_{\gamma\delta}(u, H-u) \geqslant \gamma \int_{\Delta^+} \frac{f(H-u)}{(1+y^2)^2} dxdy$$

hence

$$a_{\gamma\delta}(u, u) \leqslant a_{\gamma\delta}(u, H) - \gamma \int_{\Delta^+} \frac{f(H-u)}{(1+y^2)^2} dxdy$$

and

$$\begin{aligned} a_{\gamma\delta}(u, H) &= a_\gamma(u, H) - \int_{\Delta^+} \frac{c_0(y-\bar{y})\delta y}{(1+\delta y)} \frac{u_y H}{(1+y^2)^2} dxdy \\ &\leqslant C\|u\|_{H_2^1(\Delta^+)} \|H\|_{W_2^1(\Delta^+)} + \gamma C M_f \|h(.,Y)\|_\infty \end{aligned}$$

where C depends only on constants $\alpha, \beta, c_0\bar{y}, k, Y$. From (5.4), we deduce easily that

$$\|u_{\gamma\delta}(f)\|_{H_2^1(\Delta^+)} \leqslant C_\gamma(f). \tag{5.6}$$

Letting $\delta \to 0$, we obtain

$$u_{\gamma\delta} \to u_\gamma(f) \quad \text{in} \quad H_2^1(\Delta^+) \quad \text{weakly and in} \quad L^\infty(\Delta^+) \quad \text{weakly-}\star, \quad u_\gamma(f) \in K$$

We deduce easily from (5.5) that $u_\gamma(f)$ satisfies

$$a_\gamma(u, v-u) \geqslant \gamma \int_{\Delta^+} \frac{f(v-u)}{(1+y^2)^2} dxdy \tag{5.7}$$

$$\forall v \in W_2^1(\Delta^+), \quad v \in K, \quad u \in K, \quad \|u\|_\infty \leqslant M_f, \quad \|u\|_{H_2^1(\Delta^+)} \leqslant C_\gamma(f)$$

where again we write u for $u_\gamma(f)$. The solution of (5.7) is unique. Indeed, we first claim that

$$a(u, w) = \gamma \int_{\Delta^+} \frac{fw}{(1+y^2)^2} dxdy, \quad \forall w \in W_2^1(\Delta^+), \quad w(x, \bar{y}) = 0.$$

But then if u^1, u^2 are two solutions

$$a_\gamma(u^1, u^1-u^2) = \gamma \int_{\Delta^+} f \frac{u^1-u^2}{(1+y^2)^2} dxdy, \quad a_\gamma(u^2, u^1-u^2) = \gamma \int_{\Delta^+} f \frac{u^1-u^2}{(1+y^2)^2} dxdy,$$

hence $a_\gamma(u^1-u^2, u^1-u^2) = 0$. However, from (5.4) we also have

$$a_\gamma(v, v) \geqslant a_0 \|v\|^2_{H_2^1(\Delta^+)}, \quad \forall v \in H_2^1(\Delta^+)$$

Therefore $u^1 - u^2 = 0$. We next consider a sequence ζ^n with $\zeta^0 = \|h(.,Y)\|_{L^\infty}$ given by the solution of

$$a_\gamma(\zeta^{n+1}, v-\zeta^{n+1}) \geqslant \gamma \int_{\Delta^+} \frac{\zeta^n(v-\zeta^{n+1})}{(1+y^2)^2} dxdy, \tag{5.8}$$

where

$$\zeta^{n+1} \in K, \quad \forall v \in W_2^1(\Delta^+), \quad v \in K, \quad \|\zeta^{n+1}\|_{L^\infty} \leqslant \max\left(\|h(.,Y)\|_{L^\infty}, \|\zeta^n\|_{L^\infty}\right).$$

Considering ζ^1, we have

$$a_\gamma(\zeta^1, v-\zeta^1) \leqslant \gamma \int_{\Delta^+} \frac{\zeta^0(v-\zeta^1)}{(1+y^2)^2} dxdy.$$

Take $v = \zeta^1 - (\zeta^1 - \zeta^0)^+$, which is admissible, then

$$a_\gamma(\zeta^1, (\zeta^1-\zeta^0)^+) \leqslant \gamma \int_{\Delta^+} \frac{\zeta^0(\zeta^1-\zeta^0)^+}{(1+y^2)^2} dxdy$$

and

$$a_\gamma(\zeta^1-\zeta^0, (\zeta^1-\zeta^0)^+) = a_\gamma(\zeta^1, (\zeta^1-\zeta^0)^+) - \gamma \int_{\Delta^+} \frac{\zeta^0(\zeta^1-H)^+}{(1+y^2)^2} dxdy \leqslant 0$$

from the assumptions. Therefore $(\zeta^1-\zeta^0)^+ = 0$ which implies $\zeta^1 \leqslant \zeta^0$ and also

$$\|\zeta^1\|_{L^\infty} \leqslant \|h(.,Y)\|_{L^\infty}.$$

Then by induction if $\zeta^n \leqslant \zeta^{n-1}$, $\|\zeta^n\|_{L^\infty} \leqslant \|h(.,Y)\|_{L^\infty}$. We obtain $\zeta^{n+1} \leqslant \zeta^n$, $\|\zeta^{n+1}\|_{L^\infty} \leqslant \|h(.,Y)\|_{L^\infty}$. Also

$$a_\gamma(\zeta^{n+1}, \zeta^1-\zeta^{n+1}) \geqslant \gamma \int_{\Delta^+} \frac{\zeta^n(\zeta^1-\zeta^{n+1})}{(1+y^2)^2} dxdy$$

$$\begin{aligned} a_\gamma(\zeta^{n+1}, \zeta^{n+1}) &\leqslant a_\gamma(\zeta^{n+1}, \zeta^1) - \gamma \int_{\Delta^+} \frac{\zeta^n(\zeta^1-\zeta^{n+1})}{(1+y^2)^2} dxdy \\ &\leqslant a(\zeta^{n+1}, \zeta^1) + \gamma \int_{\Delta^+} \frac{\zeta^1\zeta^{n+1}}{(1+y^2)^2} dxdy - \gamma \int_{\Delta^+} \frac{\zeta^n(\zeta^1-\zeta^{n+1})}{(1+y^2)^2} dxdy \end{aligned}$$

Hence

$$\|\zeta^{n+1}\|_{H_2^1(\Delta^+)} \leqslant C_\gamma \left(\|\zeta^1\|_{H_2^1(\Delta^+)} + \|h(.,Y)\|_\infty \right).$$

The sequence $\zeta^n \to \zeta$ is monotone decreasing and $\zeta^n \to \zeta$ in $H_2^1(\Delta^+)$ weakly, $\zeta \in K$, $\|\zeta\|_{L^\infty} \leqslant \|h(.,Y)\|_{L^\infty}$.
Hence in the limit,

$$a(\zeta, v-\zeta) \geqslant 0, \quad \forall v \in W_2^1(\Delta^+), v \in K, \quad \zeta \in K, \quad \|\zeta\|_{L^\infty} \leqslant \|h(.,Y)\|_{L^\infty} \tag{5.9}$$

and ζ is solution of equation (5.3). □

The Cauchy problem

The exterior Dirichlet Problem 2 is equivalent to

$$\begin{aligned} A\zeta &= 0, && (x,y,z) \in (-L,L) \times (\bar{y},\infty) \times (-Y,Y), \\ \zeta(x,\bar{y},z) &= h(x,z), && (x,z) \in (-L,L) \times (-Y,Y), \\ \zeta(x,y,Y) &= \zeta^+(x,y), && (x,y) \in (-L,L) \times (\bar{y},\infty), \\ \zeta_x(\pm L,y,z) &= 0, && (y,z) \in (\bar{y},\infty) \times (-Y,Y) \end{aligned} \tag{5.10}$$

with

$$\begin{aligned} B_+\zeta^+ &= 0, && (x,y) \in (-L,L) \times (\bar{y},\infty), \\ \zeta^+(x,\bar{y}) &= h(x,Y), && x \in (-L,L), \\ \zeta_x^+(\pm L,y) &= 0, && y \in (\bar{y},\infty) \end{aligned}$$

and

$$\|\zeta^+\|_{L^\infty} \leqslant \|h(.,Y)\|_{L^\infty}.$$

This is a Cauchy problem, with z taking place of time and (x,y) as the space variable. We write it as

$$\zeta_z + \frac{1}{y}\left(\frac{1}{2}\zeta_{yy} + \frac{1}{2}\zeta_{xx} - \alpha x\zeta_x - (\beta x + c_0 y + kz)\zeta_y\right) = 0 \tag{5.11}$$

and using the notation

$$\mathcal{A}(z)u(x,y) := -\frac{1}{y}\left(\frac{1}{2}u_{yy} + \frac{1}{2}u_{xx} - \alpha x u_x - (\beta x + c_0 y + kz)u_y\right)$$

we obtain

$$\begin{array}{rcll} -\frac{\partial\zeta}{\partial z} + \mathcal{A}(z)\zeta & = & 0, & (x,y) \in \Delta^+, z < Y, \\ \zeta(x,y,Y) & = & \zeta^+(x,y), & (x,y) \in (-L,L) \times (\bar{y},\infty), \\ \zeta_x(\pm L,y,z) & = & 0, & (y,z) \in (\bar{y},\infty) \times (-Y,Y), \\ \zeta(x,\bar{y},z) & = & h(x,z), & (x,z) \in (-L,L) \times (-Y,Y). \end{array} \tag{5.12}$$

So it is a Cauchy problem with mixed Dirichlet-Neuman boundary conditions. We consider the space $H^1_2(\Delta^+)$, with a new norm

$$\|u\|^2_\star = \int_{\Delta^+} \frac{1}{y}\frac{u_x^2 + u_y^2}{(1+y^2)^2}\mathrm{d}x\mathrm{d}y + \int_{\Delta^+} \frac{u^2}{(1+y^2)^2}\mathrm{d}x\mathrm{d}y$$

which is not equivalent to

$$\|u\|^2 = \int_{\Delta^+} \frac{u_x^2 + u_y^2}{(1+y^2)^2}\mathrm{d}x\mathrm{d}y + \int_{\Delta^+} \frac{u^2}{(1+y^2)^2}\mathrm{d}x\mathrm{d}y.$$

The norm in $L^2_2(\Delta^+)$ is defined by

$$|u|^2_\star := \int_{\Delta^+} \frac{u^2}{(1+y^2)^2}\mathrm{d}x\mathrm{d}y \leqslant \|u\|^2_\star.$$

We define on $H^1_2(\Delta^+)$ the bilinear continuous form

$$\begin{aligned} \mathcal{A}(z)(u,v) \;:=\; & \frac{1}{2}\int_{\Delta^+} \frac{u_x v_x + u_y v_y}{y(1+y^2)^2}\mathrm{d}x\mathrm{d}y - \frac{1}{2}\int_{\Delta^+} \frac{u_y v(1+3y^2)}{y^2(1+y^2)^3}\mathrm{d}x\mathrm{d}y \\ & + \int_{\Delta^+} \frac{(\alpha x u_x + (\beta x + c_0 y + kz)u_y)\,v}{y(1+y^2)^2}\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Moreover,

$$\begin{aligned} \mathcal{A}(z)(u,u) \;=\; & \frac{1}{2}\int_{\Delta^+} \frac{u_x^2 + u_y^2}{y(1+y^2)^2}\mathrm{d}x\mathrm{d}y - \frac{1}{2}\int_{\Delta^+} \frac{u_y u(1+3y^2)}{y^2(1+y^2)^3}\mathrm{d}x\mathrm{d}y \\ & + \int_{\Delta^+} \frac{(\alpha x u_x + (\beta x + c_0 \bar{y} + kz)u_y)\,u}{y(1+y^2)^2}\mathrm{d}x\mathrm{d}y \\ & - \frac{1}{2}\int_{\Delta^+} u^2 c_0 \frac{d}{dy}\left(\frac{y-\bar{y}}{y(1+y^2)^2}\right)\mathrm{d}x\mathrm{d}y \end{aligned}$$

and for $|z| < Y$,

$$\leqslant a_0 \int_{\Delta^+} \frac{u_x^2 + u_y^2}{y(1+y^2)^2} \mathrm{d}x\mathrm{d}y - b_0 \int_{\Delta^+} \frac{u^2}{(1+y^2)^2} \mathrm{d}x\mathrm{d}y$$

where the constants a_0, b_0 depend only on $\alpha, \beta, c_0, k, \bar{y}, Y$. Therefore also,

$$< \mathcal{A}(z)u, u > \geqslant a_0 \|u\|^2_{H^1_2(\Delta^+)} - b \int_{\Delta^+} \frac{u^2}{(1+y^2)^2} \mathrm{d}x\mathrm{d}y.$$

The problem (5.12) is equivalent to

$$-(\frac{\partial \zeta}{\partial z}, w) + \mathcal{A}(z)(\zeta, w) = 0, \quad \forall w \in H^1_2(\Delta^+), \quad w(x, \bar{y}) = 0 \tag{5.13}$$

and

$$\zeta(x, y, Y) = \zeta^+(x, y) \quad , \quad \zeta(x, \bar{y}, z) = h(x, z).$$

We assume that there exists

$$H(z) \in H^1(-Y, Y; H^1_2(\Delta^+)) \quad \text{with} \quad H(z)(x, \bar{y}) = h(x, z), \quad \forall x, z. \tag{5.14}$$

Writing $\tilde{\zeta}(x, y, z) = \zeta(x, y, z) - H(z)(x, y)$, we deduce

$$\begin{aligned}
-(\tfrac{\partial \tilde{\zeta}}{\partial z}, w) + \mathcal{A}(z)(\tilde{\zeta}, w) &= (\tfrac{\partial H}{\partial z}(z)(x, y), w) - \mathcal{A}(z)(H(z), w), \quad \forall w \in H^1_2(\Delta^+), \quad w(x, \bar{y}) = 0. \\
\tilde{\zeta}(x, y, Y) &= \zeta^+(x, y) - H(Y)(x, y) \\
\tilde{\zeta}(x, \bar{y}, z) &= 0
\end{aligned}$$

Under this form, we obtain one and only one solution

$$\tilde{\zeta}(x, y, z) \in L^2\left(-Y, Y; H^1_{2,0}(\Delta^+)\right), \quad \frac{\partial \tilde{\zeta}}{\partial z}(x, y, z) \in L^2\left(-Y, Y; H^1_{2,0}(\Delta^+)'\right)$$

where $H^1_{2,0}(\Delta^+)$ denotes the subspace of $H^1_2(\Delta^+)$ of function which vanish at $\bar{y}$. We now prove that

$$\|\zeta\|_{L^\infty} \leqslant \|h(., Y)\|_{L^\infty} \tag{5.15}$$

We will consider

$$\Delta^+_R := \{-L < x < L, \quad \bar{y} < y < R\}$$

for R large. We begin with an approximation of the boundary condition ζ^+ with ζ^+_R solution of (we delete $+$)

$$\begin{aligned}
\tfrac{1}{2}\zeta_{R,xx} + \tfrac{1}{2}\zeta_{R,yy} - \alpha x \zeta_{R,x} - (\beta x + c_0 y + kY)\zeta_{R,y} &= 0, & (x, y) \in (-L, L) \times (\bar{y}, Y), \\
\zeta_{R,x}(\pm L, y) &= 0, & y \in (\bar{y}, \infty), \\
\zeta_R(x, \bar{y}) &= h(x, Y), & x \in (-L, L), \\
\zeta_R(x, R) &= 0, & x \in (-L, L).
\end{aligned} \tag{5.16}$$

We can assume $h \geqslant 0$, otherwise we decompose $h = h^+ - h^-$. We extend $\zeta_R(x, y)$ by 0 for $y > R$. The sequence of functions $\zeta_R(x, y)$ is increasing and $\|\zeta_R\|_{L^\infty} \leqslant \|h(., Y)\|_{L^\infty}$. Let $\theta(y) = 1$ if $0 < y < \frac{1}{2}$ and 0 if $y > 1$ be a smooth function. We may assume $\bar{y} < \frac{R}{2}$. Let $v \in W^1_2(\Delta^+), v \in K$

and test (5.16) with $\frac{v\theta(\frac{y}{R})-\zeta_R}{(1+y^2)^2}$ which vanishes at $y=\bar{y}$ and $y=R$. Setting $\theta_R(y)=\theta(\frac{y}{R})$, we obtain,

$$\begin{aligned}&\frac{1}{2}\int_{\Delta_R^+}\frac{\zeta_{R,x}(v_x\theta_R-\zeta_{R,x})}{(1+y^2)^2}\mathrm{d}x\mathrm{d}y+\frac{1}{2}\int_{\Delta_R^+}\frac{\zeta_{R,y}(v_y\theta_R-\zeta_{R,y})}{(1+y^2)^2}\mathrm{d}x\mathrm{d}y\\&+\frac{1}{2R}\int_{\Delta_R^+}\frac{\zeta_{R,y}v\theta_R'}{(1+y^2)^2}\mathrm{d}x\mathrm{d}y-\int_{\Delta_R^+}\frac{2\zeta_{R,y}(v\theta_R-\zeta_R)y}{(1+y^2)^3y}\mathrm{d}x\mathrm{d}y\\&+\int_{\Delta_R^+}\frac{\alpha x\zeta_R(v\theta_R-\zeta_R)}{(1+y^2)^2}\mathrm{d}x\mathrm{d}y+\int_{\Delta_R^+}\frac{(\beta x+c_0y+kY)}{(1+y^2)^2}\zeta_{R,y}(v\theta_R-\zeta_R)\mathrm{d}x\mathrm{d}y=0.\end{aligned}\tag{5.17}$$

and thus also $a(\zeta_R, v\theta_R-\zeta_R)=0$. Recalling that

$$a(u,u)+\gamma\int_{\Delta^+}\frac{u^2}{(1+y^2)^2}dxdy\geqslant a_0\|u\|^2_{H^1_2(\Delta^+)},$$

we get

$$a_0\|\zeta_R\|^2_{H^1_2(\Delta^+)}\leqslant a(\zeta_R,v\theta_R)+C\|h(.,Y)\|^2_{L^\infty},$$

from which we deduce easily,

$$\|\zeta_R\|_{H^1_2(\Delta^+)}\leqslant C.$$

We then consider the limit

$$\zeta_R\to\zeta \text{ monotone increasing },\quad \zeta_R\to\zeta\quad\text{in}\quad H^1_2(\Delta^+)\quad\text{weakly}$$

Hence,

$$a(\zeta,v)\leqslant a(\zeta,\zeta)$$

which implies that ζ is the solution ζ^+ of (5.3). We next consider the approximation of (5.10) for $\bar{y}<y<R$. We write

$$\begin{aligned}A\zeta_R&=0, && (x,y)\in\Delta^R\times(-Y,Y),\\ \zeta_R(x,y,Y)&=\zeta_R^+(x,y), && (x,y)\in\Delta^R,\\ \zeta_R(x,\bar{y},z)&=h(x,z), && (x,z)\in(-L,L)\times(-Y,Y),\\ \zeta_R(x,R,z)&=0, && (x,z)\in(-L,L)\times(-Y,Y),\\ \zeta_{R,x}(\pm L,y,z)&=0, && (y,z)\in(\bar{y},R)\times(-Y,Y).\end{aligned}\tag{5.18}$$

We write (5.18) as a Cauchy problem

$$\begin{aligned}\tfrac{\partial\zeta_R}{\partial z}+\mathcal{A}(z)\zeta_R&=0, && \text{in}\quad\Delta_R,\\ \zeta_R(x,y,R)&=\zeta_R^+(x,y), && (x,y)\in\Delta_R^+,\\ \zeta_{R,x}(\pm L,y,z)&=0, && (y,z)\in(\bar{y},R)\times(-Y,Y),\\ \zeta_R(x,\bar{y},z)&=h(x,z), && (x,z)\in(-L,L)\times(-Y,Y),\\ \zeta_R(x,R,z)&=0, && (x,z)\in(-L,L)\times(-Y,Y)\end{aligned}\tag{5.19}$$

and extend ζ_R by 0 for $y>R$. If $w\in H^1_2(\Delta^+), w(x,\bar{y})=0$, we can write

$$\begin{aligned}-(\tfrac{\partial\zeta_R}{\partial z},w\theta_R)+\mathcal{A}(z)(\zeta_R,\theta_Rw)&=0,\\ \zeta_R(x,y,Y)&=\zeta_R^+(x,y),\quad(x,y)\in\Delta_R^+,\\ \zeta_R(x,\bar{y},z)&=h(x,z),\qquad(x,z)\in(-L,L)\times(-Y,Y).\end{aligned}\tag{5.20}$$

However, from (5.19), we deduce

$$\|\zeta_R\|_{L^\infty} \leqslant \|h(.,Y)\|_{L^\infty}. \tag{5.21}$$

We can pass to the limit in (5.21) and check that $\zeta_R \to \zeta$ solution of (5.13). This proves (5.15).

5.3.3 Local regularity in the exterior Dirichlet problem

In this section, we derive local regularity properties of the solution ζ to the exterior Dirichlet problem.
Recall that $\bar{y} < \bar{y}_1$,

Notation 12.

$$\Delta_d(\delta) := \{(x,y,z) \in \Delta_d, \quad y < -\bar{y} - \delta\}, \quad \Delta_u(\delta) := \{(x,y,z) \in \Delta_u, \quad y > \bar{y} + \delta\},$$
$$\Delta^-(\delta) := \Delta_d(\delta) \cap \Delta^-, \quad \Delta^+(\delta) := \Delta_u(\delta) \cap \Delta^+,$$
$$\Delta_d(\delta,\gamma) := \{(x,y,z) \in \Delta_d(\delta), \quad -Y + \gamma < z\}, \quad \Delta_u(\delta,\gamma) := \{(x,y,z) \in \Delta_u(\delta), \quad z < Y - \gamma\},$$

$$H_d(\delta) := H^1(-Y,Y; H^1(-L,L; H^1(-\bar{y}_1 - \delta, -\bar{y}_1 + \delta))),$$
$$H_u(\delta) := H^1(-Y,Y; H^1(-L,L; H^1(\bar{y}_1 - \delta, \bar{y}_1 + \delta)))$$

and

$$\Delta(\delta) := \Delta_d(\delta) \cup \Delta_u(\delta), \quad \Delta(\delta,\gamma) := \Delta_d(\delta,\gamma) \cup \Delta_u(\delta,\gamma), \quad H(\delta) := H_d(\delta) \cap H_u(\delta).$$

Again, thanks to the symmetry in the exterior Dirichlet problem, we consider only positive values of y.

Proposition 15. *[Local regularity of the exterior Dirichlet problem] We have*

$$\forall \delta, \gamma > 0, \quad \zeta \in H_u(\delta); \quad \zeta \in \mathcal{C}^0(\Delta_u(\delta)); \quad \zeta \in H^2(\Delta_u(\delta,\gamma)), \tag{5.22}$$

and

$$\|\zeta\|_{H_u(\delta)} \leqslant M_{1,\|\phi\|}; \quad \|\zeta\|_{H^2(\Delta_u(\delta,\gamma))} \leqslant M_{2,\|\phi\|}. \tag{5.23}$$

Moreover, the trace of ζ at $y = \bar{y}_1$, denoted $g(x,z) := \zeta(x,\bar{y},z)$ satisfies

$$g \in H^1((-L,L) \times (-Y,Y)) \cap \mathcal{C}^0((-L,L) \times (-Y,Y)).$$

Proposition 16. *[Local regularity of the exterior Dirichlet problem on the boundary $z = Y$] We have*

$$\forall \delta > 0, \quad \zeta \in H^2(\Delta^+(\delta)); \quad \|\zeta\|_{H^2(\Delta^+(\delta))} \leqslant M_{3,\|\phi\|}. \tag{5.24}$$

Similar results hold for negative values of y. Proofs of Proposition 15 and Proposition 16 rely on similar estimates related to the local regularity of the interior Dirichlet problem.

5.4 The ergodic operator P

Notation 13.

$$\bar{\Gamma}_1^\pm := (-L,L) \times \{\pm \bar{y}_1\} \times (-Y,Y); \quad \bar{\Gamma}^\pm := (-L,L) \times \{\pm \bar{y}\} \times (-Y,Y)$$

and

$$\bar{\Gamma}_1 := \bar{\Gamma}_1^- \cup \bar{\Gamma}_1^+; \quad \bar{\Gamma} := \bar{\Gamma}^- \cup \bar{\Gamma}^+$$

5.4.1 Construction of the operator P

Consider $\phi := (\phi_-, \phi_+) \in L^\infty(\bar{\Gamma}_1)$. Following the same procedure of the 1d case, we first solve the interior Dirichlet problem for η with the boundary condition

$$\begin{cases} \eta &= \phi_+ \quad \text{in } \bar{\Gamma}_1^+, \\ \eta &= \phi_- \quad \text{in } \bar{\Gamma}_1^-. \end{cases}$$

Then

$$\|\eta\|_{L^\infty} \leqslant \max(\|\phi_+\|, \|\phi_-\|).$$

Then, we solve the exterior Dirichlet problem for ζ with the boundary condition

$$\begin{cases} \zeta &= \eta \quad \text{in } \bar{\Gamma}^+, \\ \zeta &= \eta \quad \text{in } \bar{\Gamma}^-. \end{cases}$$

Then

$$\|\zeta\|_{L^\infty} \leqslant \|\eta\|_{L^\infty} \leqslant \|\phi\|_{L^\infty}.$$

For $\bar{p}_1 \in \bar{\Gamma}_1$, let us define

$$\mathbf{P}\phi(\bar{p}_1) := \zeta(\bar{p}_1).$$

5.4.2 Probabilistic interpretation of the operator P

Let us recall from Khasminskii [27] the probabilistic interpretation of the operator $\mathbf{P}$. For $\bar{p} \in \bar{\Gamma}$ and for $\bar{p}_1 \in \bar{\Gamma}_1$, we denote by $\bar{\gamma}_1(\bar{p};.)$ the distribution of $z(\bar{\tau}_1)$ starting from $\bar{p}$ and by $\bar{\gamma}(\bar{p}_1;.)$ the distribution of $z(\bar{\tau})$ starting from $\bar{p}_1$. We showed

$$\eta(\bar{p}) = \int_{\bar{\Gamma}_1} \phi(u)\bar{\gamma}_1(\bar{p}; du) \quad \text{ and } \quad \zeta(\bar{p}_1) = \int_{\bar{\Gamma}} \eta(v)\bar{\gamma}(\bar{p}_1; dv).$$

Also, using Fubini theorem we can write

$$\mathbf{P}\phi(\bar{p}_1) = \int_{\bar{\Gamma}_1} \phi(u)\bar{\gamma}\bar{\gamma}_1(\bar{p}_1, du),$$

where

$$\bar{\gamma}\bar{\gamma}_1(\bar{p}_1, du) := \int_{\bar{\Gamma}} \bar{\gamma}(\bar{p}_1; dv)\bar{\gamma}_1(v; du).$$

By construction, the operator $\mathbf{P}$ is the transition probability associated to the Markov chain

$$\{(x(\bar{\tau}_{1,k}), y(\bar{\tau}_{1,k}), z(\bar{\tau}_{1,k}))\}_{k \geqslant 0}$$

where $\bar{\tau}_{1,0} = 0$ and

$$\bar{\tau}_{k+1} := \inf\{t > \bar{\tau}_{1,k}, \quad |y(t)| = \bar{y}\}; \quad \bar{\tau}_{1,k+1} := \inf\{t > \bar{\tau}_{k+1}, \quad |y(t)| = \bar{y}_1\}.$$

Note that $\bar{\tau}_{k+1} = \bar{\tau} \circ \theta_{\bar{\tau}_{1,k}}$ and $\bar{\tau}_{1,k+1} = \bar{\tau}_1 \circ \theta_{\bar{\tau}_{k+1}}$ where θ_t is the shift operator.

5.4.3 Ergodic property for P

Theorem 5.4.1. *The operator* **P** *is ergodic.*

Proof. A Borel subset of $\bar{\Gamma}_1$ can be written as $B := B_- \times \{y_1\} \cup B_+ \times \{-y_1\}$ with B_+, B_- are Borel subsets of $(-L, L) \times (-Y, Y)$. We have

$$B_+ = \{(x,z) : (x, \bar{y}_1, z) \in B\} \text{ and } B_- = \{(x,z) : (x, -\bar{y}_1, z) \in B\}$$

also

$$\mathbf{1}_{B_+}(x,z) = \mathbf{1}_B(x, \bar{y}_1, z); \quad \mathbf{1}_{B_-}(x,z) = \mathbf{1}_B(x, -\bar{y}_1, z).$$

Consider $\phi_+ = \mathbf{1}_{B_+}$ and $\phi_- = \mathbf{1}_{B_-}$ in the interior Dirichlet problem, then

$$\mathbf{P}(\mathbf{1}_B)(x,y,z) = \begin{cases} \zeta(x, \bar{y}_1, z) & \text{if } y = \bar{y}_1 \\ \zeta(x, -\bar{y}_1, z) & \text{if } y = -\bar{y}_1. \end{cases}$$

Let $p, \tilde{p} \in \bar{\Gamma}_1$, define

$$\lambda_{p,\tilde{p}}(B) := \mathbf{P}(\mathbf{1}_B)(p) - \mathbf{P}(\mathbf{1}_B)(\tilde{p}).$$

We will prove the ergodic property of the operator **P** in the four following steps.

1. **Doob's criterion from [17].**
   The operateur **P** is ergodic if we prove the following Doob's criterion

   $$\sup \lambda_{p,\tilde{p}}(B) < 1, \quad \forall p, \tilde{p} \in \bar{\Gamma}_1 \text{ and } \forall B.$$

2. **Negation of Doob's criterion.**
   Suppose Doob's criterion is not verified. Then, there exists two sequences $p_k := (x_k, y_k, z_k)$, $\tilde{p}_k := (\tilde{x}_k, \tilde{y}_k, \tilde{z}_k)$ in $\bar{\Gamma}_1$ and a sequence of Borel subset B_k such that

   $$\lambda_{p_k,\tilde{p}_k}(B_k) \to 1.$$

   Denote η^k and ζ^k the solution of the interior and exterior Dirichlet problems with $\phi_+ = \mathbf{1}_{B_{+k}}$ and $\phi_- = \mathbf{1}_{B_{-k}}$, where B_{+k} and B_{-k} are deduced from B_k as previously. We have

   $$\lambda_{p_k,\tilde{p}_k}(B_k) = \zeta^k(x_k, y_k, z_k) - \zeta^k(\tilde{x}_k, \tilde{y}_k, \tilde{z}_k).$$

   Now, extracting a subsequence of p_k and $\tilde{p}_k$, we deduce

   $$p_k \to p^\star \quad \text{and} \quad \tilde{p}_k \to \tilde{p}^\star \text{ in } \bar{\Gamma}_1.$$

   Also,

   $$p^\star = (x^\star, \bar{y}_1, z^\star) \quad \text{or} \quad (x^\star, -\bar{y}_1, z^\star)$$

   and

   $$\tilde{p}^\star = (\tilde{x}^\star, \bar{y}_1, \tilde{z}^\star) \quad \text{or} \quad (\tilde{x}^\star, -\bar{y}_1, \tilde{z}^\star).$$

   From (5.22), we know that

   $$\|\zeta^k\|_{H(\delta)} \leqslant M_{1,1} \text{ and } \|\zeta^k\|_{H^2(\Delta(\delta,\gamma))} \leqslant M_{2,1}$$

then

$$\begin{aligned}&\zeta^k \in H(\delta) \subset \mathcal{C}^0(\Delta(\delta)),\\&\zeta^k \to \zeta^\star \quad \text{in} \quad H(\delta) \quad \text{weakly},\\&\zeta^\star \in \mathcal{C}^0(\Delta(\delta)),\end{aligned} \tag{5.1}$$

and

$$\zeta^k \to \zeta^\star \quad \text{in} \quad \mathcal{C}^0(\Delta(\delta,\gamma)).$$

From (5.23), we know that

$$\|\zeta^k\|_{H^2(\Delta^\pm(\delta))} \leqslant M_{3,1},$$

so, we obtain

$$\zeta^k \to \zeta^\star \quad \text{in} \quad \mathcal{C}^0(\Delta^\pm(\delta)). \tag{5.2}$$

Proposition 17. *Under the hypothesis that P is not ergodic, we have $\zeta^\star(p^\star) = 1$.*

Proof. We must have $\lim_{k\to\infty} \zeta^k(p_k) = 1$.

(a) If $|z_\star| < Y$ or $z_\star y_\star = -Y\bar{y}_1$ then (5.1) implies $\lim_{k\to\infty} \zeta^k(p_k) = \zeta^\star(p_\star)$.

(b) If $z_\star y_\star = Y\bar{y}_1$ then (5.2) implies $\lim_{k\to\infty} \zeta^k(x_k, \pm\bar{y}_1, \pm Y) = \zeta^\star(p_\star)$. Indeed, as for all $k \geqslant 0, \zeta^k$ is continuous, there exists $\sigma_k \geqslant k$ such that $|\zeta^k(p_{\sigma_k}) - \zeta^k(x_{\sigma_k}, \pm\bar{y}_1, \pm Y)| \leqslant 2^{-k}$. Also, $\lim_{k\to\infty} \zeta^k(p_{\sigma_k}) = \lim_{k\to\infty} \zeta^k(x_{\sigma_k}, \pm\bar{y}_1, \pm Y) = \zeta^\star(p^\star)$.

□

3. **Contradiction.**
Suppose $p^\star = (x^\star, \bar{y}_1, z^\star)$ and set

$$\Xi_1 := \bar{\Gamma}_1 \cap \{x = \pm L\}; \quad \Xi_2 := \bar{\Gamma}_1 \cap \{z = Y\}.$$

Maximum principle for parabolic operator applied to ζ^* implies $p^\star \in \Xi_1 \cup \Xi_2$. Then $p^\star$ cannot be in Ξ_1 because of the Neuman condition and $p^\star$ cannot be in Ξ_2 because of the boundary condition $z = Y$. A similar argument yields a contradiction with $p^\star = (x^\star, -\bar{y}_1, z^\star)$.

4. **Conclusion.**
From ergodic theory, there exists a unique probability mesure $\gamma_\star = (\gamma_\star^-, \gamma_\star^+)$ on $\bar{\Gamma}_1$ and $K, \rho > 0$ such that

$$\forall n \geqslant 0, \quad \left| \mathbf{P}^n\phi - \iint_{\bar{\Gamma}_1^-} \phi(r,s)\mathrm{d}\gamma_\star^-(r,s) - \iint_{\bar{\Gamma}_1^+} \phi(r,s)\mathrm{d}\gamma_\star^+(r,s) \right| \leqslant K\|\phi\| \exp(-\rho n). \tag{5.3}$$

□

5.5 The nonhomogeneous interior Dirichlet problem

Consider f a bounded function on Δ_1, we want to define $\mathbb{E}_{(x,y,z)}\big(\int_0^{\bar{\tau}_1} f(x(s),y(s),z(s))\mathrm{d}s\big)$ as the solution of the non-homogeneous interior Dirichlet Problem 3:

Statement of the problem 3. *Find* $\chi \in L^\infty(\Delta_1) \cap \mathcal{C}^0(\Delta_1^\epsilon), \forall \epsilon > 0$ *such that*

$$\begin{aligned} A\chi + f &= 0 \quad \text{in} \quad \Delta_1, \\ B_\pm\chi + f &= 0 \quad \text{in} \quad \Delta_1^\pm, \\ \chi(x, \pm\bar{y}_1, z) &= 0 \quad \text{in} \quad (-L, L) \times (-Y, Y), \\ \chi_x(\pm L, y, z) &= 0 \quad \text{in} \quad (-\bar{y}_1, \bar{y}_1) \times (-Y, Y). \end{aligned}$$

Consider

$$\Phi(x,y,z) := \exp\lambda(c_0kz^2 + c_0y^2); \quad \lambda \geqslant 1.$$

Theorem 5.5.1. *The Problem 3 has a unique solution. Moreover, we have*

$$\|\chi\|_{L^\infty} \leqslant \exp\lambda(c_0kY^2 + c_0\bar{y}_1^2),$$

where λ *depends on* f.

Proof. The uniqueness of a solution of Problem 3 is argued as the uniqueness for the homogeneous Dirichlet interior problem. The existence can be proven by the regularization technique used previously. Now, we give a L^∞ bound for χ. We have

$$\begin{aligned} &\phi_x = 0; \quad \phi_{xx} = 0 \\ &\phi_y = 2yc_0\lambda\phi; \quad \phi_{yy} = 2c_0\lambda\phi + (2c_0\lambda y)^2\phi \\ &\phi_z = 2zc_0k\lambda\phi \end{aligned}$$

and

$$\begin{aligned} y\phi_z - \alpha x\phi_x - \phi_y(\beta x + c_0y + kz) + \frac{1}{2}\phi_{yy} + \frac{1}{2}\phi_{xx} &= -2\beta c_0\lambda xy\phi - 2c_0^2\lambda y^2\phi + c_0\lambda\phi + 2(c_0\lambda y)^2 \\ &= c_0\lambda\phi - c_0\lambda 2\beta xy\phi + 2(c_0y)^2\lambda(\lambda-1)\phi \\ &\geqslant c_0\lambda(1 - 2\beta L\bar{y}_1)\phi + 2(c_0y)^2\lambda(\lambda-1)\phi \\ &\geqslant \|f\| \end{aligned}$$

where $\bar{y}_1$ can be chosen as $2\bar{y}_1 < \frac{1}{2\beta L}$, and $\lambda \geqslant \max(1, \frac{\|f\|}{c_0(1-2\beta L\bar{y}_1)})$.
Using that $\phi(x,y,\pm Y) = \exp\lambda(c_0kY^2)\exp\lambda(c_0^2y^2)$ we obtain for $z = Y, 0 < y < \bar{y}_1$.

$$\begin{aligned} -\alpha x\phi_x - \phi_y(\beta x + c_0y + kY) + \frac{1}{2}\phi_{yy} + \frac{1}{2}\phi_{xx} &= -\phi_y(x,y,Y)(c_0y + kY + \beta x) + \frac{1}{2}\phi_{yy}(x,y,Y) \\ &= \lambda c_0\phi(x,y,Y)\big(1 - 2(kY + \beta x)y + 2c_0(\lambda-1)y^2\big) \\ &\geqslant \lambda c_0\phi(x,y,Y)\big(1 - 2(kY + \beta L)y + 2c_0(\lambda-1)y^2\big) \\ &\geqslant \lambda c_0\big(1 - \frac{(kY + \beta L)^2}{2c_0(\lambda-1)}\big) \\ &\geqslant \|f\|, \end{aligned}$$

if $\lambda \geqslant \max(2\frac{\|f\|}{c_0}, 1+\frac{(kY+\beta L)^2}{c_0})$. Similar estimates hold for $z=-Y, -\bar{y}_1 < y < 0$. Consider $u := \phi - \chi$, then u satisfies the following inequalities

$$Au \geqslant 0 \text{ in } \Delta_1; \quad B_\pm u \geqslant 0 \text{ in } \Delta_1^\pm \tag{5.1}$$

and the following boundary conditions

$$\begin{array}{rcll} u(x, \pm\bar{y}_1, z) & = & \exp(\lambda(c_0 k z^2 + c_0 \bar{y}_1^2)) & \text{in } (-L, L) \times (-Y, Y), \\ u_x(\pm L, y, z) & = & 0 & \text{in } (-\bar{y}_1, \bar{y}_1) \times (-Y, Y). \end{array} \tag{5.2}$$

The maximum of u cannot be attained in Δ_1 or on the boundaries $z = \pm Y, x = \pm L$. It can only be attained for $y = \bar{y}_1$ or $y = \bar{y}_1$. Hence,

$$u(x, y, z) \leqslant \exp(\lambda(c_0 k Y^2 + c_0 \bar{y}_1^2))$$

which implies

$$\chi(x, y, z) \geqslant -\exp(\lambda(c_0 k Y^2 + c_0 \bar{y}_1^2)).$$

Now, consider $v = -(\phi + \chi)$, then v satisfies inequalities (5.3)

$$Av \leqslant 0 \text{ in } \Delta_1; \quad B_\pm v \leqslant 0 \text{ in } \Delta_1^\pm \tag{5.3}$$

and the boundary condition (5.4)

$$\begin{array}{rcll} v(x, \pm\bar{y}_1, z) & = & -\exp(\lambda(c_0 k z^2 + c_0 \bar{y}_1^2)) & \text{in } (-L, L) \times (-Y, Y), \\ v_x(\pm L, y, z) & = & 0 & \text{in } (-\bar{y}_1, \bar{y}_1) \times (-Y, Y). \end{array} \tag{5.4}$$

Again, the minimum of v cannot be attained in Δ_1 or on the boundaries $z = \pm Y, x = \pm L$. Hence

$$v(x, y, z) \geqslant -\exp(\lambda(c_0 k Y^2 + c_0 \bar{y}_1^2))$$

which implies

$$\chi(x, y, z) \leqslant \exp(\lambda(c_0 k Y^2 + c_0 \bar{y}_1^2)).$$

Now, let us derive some estimates on derivatives. Let $M := \exp(\lambda(c_0 k Y^2 + c_0 \bar{y_1}^2))$. Using the test function $m_0(x, y, z) = \exp(-c_0(y^2 + kz^2)) \exp(-\alpha x^2)$ we obtain now a priori estimates on the partial derivatives of χ. We test (3) with $m_0 \chi$

$$\begin{aligned} \int_{\Delta_1} \chi_y^2 m_0 + \int_{\Delta_1} \chi_x^2 m_0 &= \int_{\partial\Delta_1, z=Y} y\chi^2 m_0 - \int_{\partial\Delta_1, z=-Y} y\chi^2 m_0 + 2\int_{\Delta_1} f\chi m_0 - \int_{\Delta_1} c_0 \beta x \chi^2 m_0 \\ &\leqslant \int_{\partial\Delta_1, z=Y, y>0} y\chi^2 m_0 - \int_{\partial\Delta_1, z=-Y, y<0} y\chi^2 m_0 + 2\int_{\Delta_1} f\chi \\ &\leqslant C(M). \end{aligned}$$

Hence,

$$\int_{\Delta_1} \chi_y^2 m_0 \leqslant C(M) \quad ; \quad \int_{\Delta_1} \chi_x^2 m_0 \leqslant C(M).$$

We have then,

$$\int_{\Delta_1} (y^2 \chi_z)^2 m_0 \leqslant C(M) \quad \text{and} \quad \int_{\Delta_1} (y^2 \chi_{yy})^2 m_0 \leqslant C(M).$$

The function χ is smooth outside a neighborhood of $y = 0$. □

5.6 The nonhomogeneous exterior Dirichlet problem

Consider the function

$$\Psi(y) := \log(y) + K,\ K \text{ such that } \log(\bar{y}_1) + K > 0$$

and the space of functions

$$X_\Psi^\infty := \{u\psi, \quad u \in L^\infty(\Delta_u)\}.$$

We want to define $\mathbb{E}_{(x,y,z)}\big(\int_0^{\bar{\tau}} f(x(s), y(s), z(s))ds\big)$ as the solution of the nonhomogeneous exterior Dirichlet Problem 4:

Statement of the problem 4. *Find* $\xi \in X_\Psi^\infty \cap \mathcal{C}^0(\Delta_u^\epsilon), \forall \epsilon > 0$ *such as*

$$\begin{array}{rcll} A\xi + f &=& 0 & in \quad \Delta_u, \\ B_+\xi + f &=& 0 & in \quad \Delta^+, \\ \xi(x, \bar{y}, z) &=& 0 & in \quad (-L, L) \times (-Y, Y), \\ \xi_x(\pm L, y, z) &=& 0 & in \quad (\bar{y}, +\infty) \times (-Y, Y). \end{array}$$

Theorem 5.6.1. *The Problem 4 has a unique solution. Moreover, we have* $-\psi(y) \leqslant \xi(x,y,z) \leqslant \psi(y)$.

Proof. We justify uniqueness of the solution taking $f = 0$ and setting ξ

$$\xi = w\psi^\alpha, \quad \alpha > 1.$$

We show $w = 0$. In particular, w satisfies

$$\begin{array}{rcl} \xi &=& w\psi^\alpha \\ \xi_x &=& w_x\psi^\alpha; \quad \xi_y = w_y\psi^\alpha + w\alpha\psi_y\psi^{\alpha-1}; \quad \xi_z = w_z\psi^\alpha \\ \xi_{xx} &=& w_{xx}\psi^\alpha; \quad \xi_{yy} = w_{yy}\psi^\alpha + 2w_y\alpha\psi_y\psi^{\alpha-1} + w\alpha\big(\psi_{yy}\psi^{\alpha-1} + (\psi_y)^2\psi^{\alpha-2}(\alpha-1)\big) \end{array}$$

then

$$\begin{array}{l} \frac{1}{2}w_{xx}\psi^\alpha + \frac{1}{2}\{w_{yy}\psi^\alpha + 2w_y\alpha\psi_y\psi^{\alpha-1}w\alpha\big(\psi_{yy}\psi^{\alpha-1} + (\psi_y)^2\psi^{\alpha-2}(\alpha-1)\} \\ -(c_0y + kz + \beta x)\{w_y\psi^\alpha + w\alpha\psi_y\psi^{\alpha-1}\} - \alpha x w_x\psi^\alpha + y w_z\psi^\alpha = 0. \end{array}$$

Collecting terms related to ψ^α, we obtain

$$\begin{array}{l} \frac{1}{2}w_{xx} + \frac{1}{2}\{w_{yy} + 2w_y\alpha\frac{\psi_y}{\psi} + w\alpha\big(\frac{\psi_{yy}}{\psi} + \frac{(\psi_y)^2}{\psi^2}(\alpha-1)\} \\ +(-c_0y - kz - \beta x)\{w_y + w\alpha\frac{\psi_y}{\psi}\} - \alpha x w_x + y w_z = 0, \end{array}$$

which implies for $z \in (-Y, Y)$ and $y > \bar{y}$,

$$\begin{array}{l} \frac{1}{2}w_{xx} + \frac{1}{2}w_{yy} + w_y\{\alpha\frac{\psi_y}{\psi} - c_0 - kz - \beta x\} + \frac{w\alpha}{\psi}\{\frac{1}{2}\psi_{yy} + \frac{(\alpha-1)}{2}\frac{(\psi_y)^2}{\psi} \\ -(c_0y + kz + \beta x)\alpha\psi_y\} - \alpha x w_x + y w_z = 0 \end{array}$$

and for $z = Y; \quad y > \bar{y}$,

$$\begin{aligned}&\frac{1}{2}w_{xx} + \frac{1}{2}w_{yy} + w_y\{\alpha\frac{\psi_y}{\psi} - c_0 - kY - \beta x\} + \frac{w\alpha}{\psi}\{\frac{1}{2}\psi_{yy} + \frac{(\alpha-1)}{2}\frac{(\psi_y)^2}{\psi}\\ &- (c_0 y + kY + \beta x)\alpha\psi_y\} - \alpha x w_x + y w_z = 0\end{aligned}$$

and

$$w(x, \bar{y}, z) = 0.$$

We can find $\alpha > 1$ with $\alpha - 1$ small so that

$$\frac{1}{2}\psi_{yy} - (c_0 y + kz + \beta x) + (\alpha - 1)\frac{(\psi_y)^2}{\psi} \leqslant -\gamma\frac{c_0}{2} + \frac{(\alpha-1)\gamma}{y^2(\log(y) + K)} < 0.$$

Since $w \to 0$ as $y \to \infty$, a positive maximum can be attained only for a finite $y^\star$. But this is impossible from the equation. So w is negative,$w \leqslant 0$. But, w is also positive, $w \geqslant 0$. Hence $\xi = 0$ and uniqueness follows.
Now, existence is demonstrated by the following approximation

$$\begin{aligned}&A\xi^R + f = 0, \text{ in } \Delta_R,\\ &B_+\xi^R + f = 0, \text{ in } \Delta_R^+,\\ &\xi(x, \bar{y}, z) = 0, \quad \xi(x, R, z) = 0,\\ &\xi_x(\pm L, y, z) = 0.\end{aligned}$$

Using $u^R = \xi^R - \psi$, we obtain

$$\begin{aligned}&Au^R \geqslant 0, \text{ in } \Delta_R,\\ &B_+u^R \geqslant 0, \text{ in } \Delta_R^+,\\ &u^R(x, \bar{y}, z) = -\psi(\bar{y}),\\ &u_x^R(\pm L, y, z) = 0.\end{aligned}$$

Necessarily, $u^R \leqslant 0$. The sequence ξ^R is monotone increasing and converges towards a solution. Let us show estimates on u. Suppose $f \geqslant 0$, we will show that $0 \leqslant \xi \leqslant \psi$. Then,

$$\begin{aligned}\psi_y(-c_0 y - kz - \beta x) + \frac{1}{2}\psi_{yy} &= \gamma[\frac{1}{y}(-c_0 y - kz - \beta x) - \frac{1}{2y^2}]\\ &\leqslant \gamma[\frac{1}{y}(-c_0 y - kz) - \frac{1}{2y^2}]\\ &\leqslant \gamma[\frac{1}{y}(-c_0 y - kz) - \frac{1}{2y^2}]\\ &\leqslant -\gamma\frac{c_0}{2}, \quad \text{if } \bar{y} > \frac{2kY}{c_0}.\end{aligned}$$

Define γ such that $\frac{\gamma c_0}{2} \geqslant \|f\|$. We then have, setting $u = \xi - \psi$,

$$\begin{aligned}&Au \geqslant 0, \text{ in } \Delta\\ &B_+u \geqslant 0, \text{ in } \Delta^+\\ &u(x, \bar{y}, z) = -\psi(\bar{y})\\ &u_x(\pm L, y, z) = 0\end{aligned}$$

It follows that $u \leqslant 0$.
We can show similarly that $\xi + \phi \geqslant 0$. Hence, we have

$$-\psi \leqslant \xi \leqslant \psi.$$

□

5.7 The operator T and the invariant measure ν

5.7.1 Construction of the operator T

Consider $f \in L^\infty(\mathcal{O})$, following the same procedure of the 1d case, we first solve the interior non-homogeneous Dirichlet problem for χ with f at the right hand side, then we solve the exterior non-homogeneous Dirichlet problem for ξ with f at the right hand side and χ as a boundary condition. Also, for any $\bar{p}_1 \in \bar{\Gamma}_1$, we define the operator

$$\mathbf{T}f(\bar{p}_1) := \xi(\bar{p}_1).$$

This defines a linear operator from $L^\infty\big(\bar{\Gamma}_1\big)$ in $L^\infty(\bar{\Gamma}_1)$.

5.7.2 Probabilistic interpretation of the operator T

For any A a Borel subset of $\mathcal{O}$, consider the two following measures of occupation of A by the process $(x(t), y(t), z(t))$ starting at $p \in \mathcal{O}$, namely

$$\begin{cases} \nu_\chi(p;A) & := & \mathbb{E}_p\big(\int_0^{\bar{\tau}_1} \mathbf{1}_A(x(s),y(s),z(s))\mathrm{d}s\big) \\ \nu_\xi(p;A) & := & \mathbb{E}_p\big(\int_0^{\bar{\tau}} \mathbf{1}_A(x(s),y(s),z(s))\mathrm{d}s\big). \end{cases}$$

We have shown

$$\chi(p) = \int_{\mathcal{O}} f(q)\mathrm{d}\nu_\chi(p;\mathrm{d}q) \quad \text{ and } \quad \xi(\tilde{p}) = \int_{\mathcal{O}} f(\tilde{q})\mathrm{d}\nu_\xi(\tilde{p};\mathrm{d}\tilde{q}).$$

For any $\bar{p}_1 \in \bar{\Gamma}_1$, we have

$$\mathbf{T}(f)(\bar{p}_1) := \mathbb{E}_{\bar{p}_1}\Big(\int_0^{\bar{\tau}_1} f(x(s),y(s),z(s))\mathrm{d}s\Big) + \int_{\bar{\Gamma}} \mathbb{E}_q\Big(\int_0^{\bar{\tau}} f(x(s),y(s),z(s))\mathrm{d}s\Big)\mathrm{d}\gamma(\bar{p}_1;\mathrm{d}q)$$

$\mathbf{T}(f)$ integrates f over a time cycle of the Markov chain stopped on $\bar{\Gamma}_1$, that is $(x(\bar{\tau}_{1,k}), y(\bar{\tau}_{1,k}), z(\bar{\tau}_{1,k}))$.

5.7.3 Construction of the invariant measure ν

Now, define

$$\nu(f) := \frac{\int_{-L}^{L}\int_{-Y}^{Y} Tf(r,\bar{y}_1,s)\gamma_\star^+(\mathrm{d}r,\mathrm{d}s) + \int_{-L}^{L}\int_{-Y}^{Y} Tf(r,-\bar{y}_1,s)\gamma_\star^-(\mathrm{d}r,\mathrm{d}s)}{\int_{-L}^{L}\int_{-Y}^{Y} T\mathbf{1}(r,\bar{y}_1,s)\gamma_\star^+(\mathrm{d}r,\mathrm{d}s) + \int_{-L}^{L}\int_{-Y}^{Y} T\mathbf{1}(r,-\bar{y}_1,s)\gamma_\star^-(\mathrm{d}r,\mathrm{d}s)}.$$

The denominator is well defined and is stricly positive. Now, we want to solve the complete problem.

Statement of the problem 5. *Find u such that $u\psi^{-1}$ is bounded for $|y| > |\bar{y}|$ and*

$$\begin{aligned} Au + f &= 0, \quad \textit{in} \quad \mathcal{O} \\ B_+u + f &= 0, \quad \textit{in} \quad \mathcal{O}^+ \\ B_-u + f &= 0, \quad \textit{in} \quad \mathcal{O}^- \\ u_x(\pm L, y, z) &= 0, \quad \textit{in} \quad \{x = \pm L\} \cap \mathcal{O} \end{aligned}$$

Considering Problems 1, 2, 3 and 4, in the following result we use a functional analysis method to prove that ν is the unique invariant distribution associated to the solution $(x(t), y(t), z(t))$ of the stochastic variational inequality (5.2).

Theorem 5.7.1. *The Problem 5 has a unique solution if and only if $\nu(f) = 0$.*

Proof. Similar arguments as the 1d case are considered. Uniqueness is guaranteed by the ergodic property of the operator $\mathbf{P}$. Suppose $\nu(f) = 0$. We define χ the solution of the Interior non-homogeneous Dirichlet Problem 3 and ξ the solution of the Exterior non-homogeneous Dirichlet Problem 4. We set $\chi^0 := \chi$ and $\xi^0 := \xi$, and for $k \geqslant 0$, we define χ^{k+1} by

$$\begin{aligned} A\chi^{k+1} &= 0, \quad \text{in} \quad \Delta_1, \\ B_+\chi^{k+1} &= 0, \quad \text{in} \quad \Delta_1^+, \\ B_-\chi^{k+1} &= 0, \quad \text{in} \quad \Delta_1^-, \\ \chi_x^{k+1}(\pm L, y, z) &= 0, \quad \text{in} \quad (-\bar{y}_1, \bar{y}_1) \times (-Y, Y) \end{aligned}$$

with

$$\chi^{k+1}(x, \bar{y}_1, z) = \xi^k(x, \bar{y}_1, z); \quad \chi^{k+1}(x, -\bar{y}_1, z) = \xi^k(x, -\bar{y}_1, z).$$

Afterwards, we define ξ_{k+1} by

$$\begin{aligned} A\xi^{k+1} &= 0, \quad \text{in} \quad \Delta_u, \\ B_+\xi^{k+1} &= 0, \quad \text{in} \quad \Delta_u^+, \\ \xi_x^{k+1}(\pm L, y, z) &= 0, \quad \text{in} \quad (\bar{y}, +\infty) \times (-Y, Y) \end{aligned}$$

with

$$\xi^{k+1}(x, \bar{y}, z) = \chi^{k+1}(x, \bar{y}, z).$$

Similarly, we define ξ^{k+1} by

$$\begin{aligned} A\xi^{k+1} &= 0, \quad \text{in} \quad \Delta_d, \\ B_-\xi^{k+1} &= 0, \quad \text{in} \quad \Delta_d^-, \\ \xi_x^{k+1}(\pm L, y, z) &= 0, \quad \text{in} \quad (-\infty, -\bar{y}) \times (-Y, Y) \end{aligned}$$

with

$$\xi^{k+1}(x, -\bar{y}, z) = \chi^{k+1}(x, -\bar{y}, z).$$

That means

$$\xi^0|_{\bar{\Gamma}_1} = \mathbf{T}f; \quad \xi^{k+1}|_{\bar{\Gamma}_1} = \mathbf{P}(\xi^k|_{\bar{\Gamma}_1}).$$

Next, we set

$$\tilde{\xi}^k = \xi^0 + \xi^1 + ... + \xi^k; \quad \tilde{\chi}^k = \chi^0 + \chi^1 + ... + \chi^k$$

then

$$\tilde{\xi}^k|_{\bar{\Gamma}_1} = \mathbf{T}f + \mathbf{P}(\mathbf{T}f) + ... + \mathbf{P}^k(\mathbf{T}f).$$

Now, let us remark that $\tilde{\chi}^k$ satisfies the following equation:

$$\begin{array}{rcll} A\tilde{\chi}^k + f &=& 0, & \text{in} \quad \Delta_1, \\ B_+\tilde{\chi}^k + f &=& 0, & \text{in} \quad \Delta_1^+, \\ B_-\tilde{\chi}^k + f &=& 0, & \text{in} \quad \Delta_1^-, \\ \tilde{\chi}_x^k(\pm L, y, z) &=& 0, & \text{in} \quad (-\infty, -\bar{y}) \times (-Y, Y) \end{array} \tag{5.1}$$

with

$$\tilde{\chi}^k(x, \bar{y}_1, z) = \tilde{\xi}^{k-1}(x, \bar{y}_1, z); \quad \tilde{\chi}^k(x, -\bar{y}_1, z) = \tilde{\xi}^{k-1}(x, -\bar{y}_1, z)$$

$\tilde{\xi}^k$ satisfies the following equations:

$$\begin{array}{rcll} A\tilde{\xi}^k + f &=& 0, & \text{in} \quad \Delta_u, \\ B_+\tilde{\xi}^k + f &=& 0, & \text{in} \quad \Delta_u^+, \\ \tilde{\xi}_x^k(\pm L, y, z) &=& 0, & \text{in} \quad (\bar{y}, +\infty) \times (-Y, Y) \end{array} \tag{5.2}$$

$$\tilde{\xi}^k(x, \bar{y}, z) = \tilde{\chi}^k(x, \bar{y}, z)$$

and

$$\begin{array}{rcll} A\tilde{\xi}^k + f &=& 0, & \text{in} \quad \Delta_d, \\ B_+\tilde{\xi}^k + f &=& 0, & \text{in} \quad \Delta_d^-, \\ \tilde{\xi}_x^k(\pm L, y, z) &=& 0, & \text{in} \quad (-\infty, -\bar{y}) \times (-Y, Y) \end{array} \tag{5.3}$$

$$\tilde{\xi}^k(x, -\bar{y}, z) = \tilde{\chi}^k(x, -\bar{y}, z).$$

The condition $\nu(f) = 0$ means

$$\int_{\bar{\Gamma}_1} \mathbf{T}f(x)\mathrm{d}\pi(x) = \int_{\bar{\Gamma}_1} \mathbf{T}f(r, -\bar{y}_1, s)\mathrm{d}\pi_1(r, s) + \int_{\bar{\Gamma}_1} \mathbf{T}f(r, \bar{y}_1, s)\mathrm{d}\pi_2(r, s) = 0.$$

From the estimate (5.3), we obtained

$$\tilde{\xi}^k \text{ converges in } L^\infty(\bar{\Gamma}_1).$$

Now, we notice that $\tilde{\chi}^k - \chi$ is a solution of the interior homogeneous Dirichlet problem with $(\tilde{\chi}^k - \chi)|_{\bar{\Gamma}_1} = \tilde{\xi}^{k-1}|_{\bar{\Gamma}_1}$ and $\tilde{\xi}^k - \xi$ is a solution of the Exterior homogeneous Dirichlet problem with $(\tilde{\xi}^k - \xi)|_{\bar{\Gamma}} = (\tilde{\chi}^k - \chi)|_{\bar{\Gamma}}$. Then, we obtain

$$\|\tilde{\xi}^k - \xi\|_{L^\infty} \leqslant \|\tilde{\chi}^k - \chi\|_{L^\infty} \leqslant \|\tilde{\xi}^{k-1}\|_{L^\infty} \leqslant C.$$

We can extract a subsequence such that

$$\tilde{\xi}^k \to \tilde{\xi}; \quad \tilde{\chi}^k \to \tilde{\chi}.$$

Moreover, $\tilde{\chi}$ satisfies equation:

$$\begin{array}{rcll} A\tilde{\chi}+f &=& 0, & \text{in} \quad \Delta_1, \\ B_+\tilde{\chi}+f &=& 0, & \text{in} \quad \Delta_1^+, \\ B_-\tilde{\chi}+f &=& 0, & \text{in} \quad \Delta_1^-, \\ \tilde{\chi}_x(\pm L,y,z) &=& 0, & \text{in} \quad (-\bar{y}_1,\bar{y}_1)\times(-Y,Y) \end{array} \tag{5.4}$$

$$\tilde{\chi}(x,\bar{y}_1,z)=\tilde{\xi}(x,\bar{y}_1,z); \quad \tilde{\chi}(x,-\bar{y}_1,z)=\tilde{\xi}(x,-\bar{y}_1,z)$$

and $\tilde{\xi}$ satisfies equations:

$$\begin{array}{rcll} A\tilde{\xi}+f &=& 0, & \text{in} \quad \Delta_u, \\ B_+\tilde{\xi}+f &=& 0, & \text{in} \quad \Delta_u^+, \\ \tilde{\xi}_x(\pm L,y,z)+f &=& 0, & \text{in} \quad (\bar{y},+\infty)\times(-Y,Y) \end{array} \tag{5.5}$$

$$\tilde{\xi}(x,\bar{y},z)=\tilde{\chi}(x,\bar{y},z)$$

$$\begin{array}{rcll} A\tilde{\xi}+f &=& 0, & \text{in} \quad \Delta_d, \\ B_-\tilde{\xi}+f &=& 0, & \text{in} \quad \Delta_d^-, \\ \tilde{\xi}_x(\pm L,y,z) &=& 0, & \text{in} \quad (-\infty,-\bar{y})\times(-Y,Y) \end{array} \tag{5.6}$$

$$\tilde{\xi}(x,-\bar{y},z)=\tilde{\chi}(x,-\bar{y},z).$$

Then, we must have

$$\tilde{\chi}=\tilde{\xi} \quad \text{in} \quad \bar{y}<\pm y<\bar{y}_1$$

Thus, the function

$$u=\begin{cases}\tilde{\chi} & \text{in } \Delta_1,\\ \tilde{\xi} & \text{in } \Delta_1^c\end{cases}$$

is the solution of Problem 5.

Now, suppose that Problem 5 has a unique solution u. Considering the same sequences $\tilde{\chi}^k$ and $\tilde{\xi}^k$, we have that $u-\tilde{\chi}^k$ is a solution of the interior homogeneous Dirichlet problem with $(u-\tilde{\chi}^k)|_{\bar{\Gamma}_1}=(u-\tilde{\xi}^{k-1})|_{\bar{\Gamma}_1}$ and $u-\tilde{\xi}^k$ is a solution of the exterior homogeneous Dirichlet problem $(u-\tilde{\xi}^k)|_{\bar{\Gamma}}=(u-\tilde{\chi}^k)|_{\bar{\Gamma}}$. Hence,

$$\|u-\tilde{\xi}^k\|_{L^\infty(\bar{\Gamma}_1)}\leqslant\|u-\tilde{\chi}^k\|_{L^\infty(\bar{\Gamma}_1)}\leqslant\|u-\tilde{\xi}^{k-1}\|_{L^\infty(\bar{\Gamma}_1)}\leqslant\|u\|_{L^\infty(\bar{\Gamma}_1)}$$

and so,

$$\|\tilde{\xi}^k\|_{L^\infty(\bar{\Gamma}_1)}\leqslant C.$$

We have

$$\tilde{\xi}^k=(k+1)\int_{\bar{\Gamma}_1}\mathbf{T}f(x)\mathrm{d}\pi(x)+\sum_{j=0}^{k}\mathbf{P}^j(\mathbf{T}(f-\nu(f)))$$

and since the sum is bounded, we obtain

$$(k+1)\int_{\bar{\Gamma}_1}\mathbf{T}f(x)\mathrm{d}\pi(x) \text{ is bounded.}$$

That leads to

$$\int_{\bar{\Gamma}_1} \mathbf{T} f(x) \mathrm{d}\pi(x) = 0$$

That implies $\nu(f) = 0$. $\square$

Consider now φ a smooth function in $[-L, L] \times \mathbb{R} \times [-Y, Y]$ with compact support. If we take

$$\begin{aligned} f &= -A\varphi, \\ f(x, y, Y) &= -B_{+}\varphi, \\ f(x, y, -Y) &= -B_{-}\varphi, \end{aligned}$$

then φ is solution of (5) for this f. From Theorem 5.7.1, we have

$$\begin{aligned} & \textstyle\int_{-L}^{L} \int_{-\infty}^{\infty} \int_{-Y}^{Y} \{\frac{1}{2}\varphi_{yy} + \frac{1}{2}\varphi_{xx} - \alpha x \varphi_x - (\beta x + c_0 y + kz)\varphi_y + y\varphi_z\} \mathrm{d}\nu(x, y, z) \\ + & \textstyle\int_{-L}^{L} \int_{0}^{\infty} \{\frac{1}{2}\varphi_{yy} + \frac{1}{2}\varphi_{xx} - \alpha x \varphi_x - (\beta x + c_0 y + kY)\varphi_y\} \mathrm{d}\nu(x, y, Y) \\ + & \textstyle\int_{-L}^{L} \int_{-\infty}^{0} \{\frac{1}{2}\varphi_{yy} + \frac{1}{2}\varphi_{xx} - \alpha x \varphi_x - (\beta x + c_0 y - kY)\varphi_y\} \mathrm{d}\nu(x, y, -Y) = 0. \end{aligned}$$

If ν has a density m, then we deduce that

$$\alpha \frac{\partial}{\partial x}[xm] + \frac{\partial}{\partial y}[(\beta x + c_0 y + kz)m] - y\frac{\partial m}{\partial z} + \frac{1}{2}\frac{\partial^2 m}{\partial x^2} + \frac{1}{2}\frac{\partial^2 m}{\partial y^2} = 0 \quad \text{in } (0, L) \times \mathbb{R} \times (-Y, Y) \tag{5.7}$$

in the sense of distributions. If we test (5.7) with φ and integrate by parts, we obtain

$$\begin{aligned} & -\int_{-L}^{L} \int_{-\infty}^{+\infty} y m(x, y, Y)\varphi(x, y, Y) dx dy + \int_{0}^{L} \int_{-\infty}^{+\infty} y m(x, y, -Y)\varphi(x, y, -Y) \mathrm{d}x\mathrm{d}y \\ & + \int_{-L}^{L} \int_{-\infty}^{\infty} \int_{-Y}^{Y} m(x, y, z)\{\frac{1}{2}\varphi_{yy} + \frac{1}{2}\varphi_{xx} - \alpha x \varphi_x - (\beta x + c_0 y + kz)\varphi_y + y\varphi_z\} \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0 \end{aligned}$$

and comparing with (5.7)

$$\begin{aligned} & -\int_{-L}^{L} \int_{-\infty}^{0} y m(x, y, Y)\varphi(x, y, Y) dx dy + \int_{0}^{L} \int_{0}^{+\infty} y m(x, y, -Y)\varphi(x, y, -Y) \mathrm{d}x\mathrm{d}y \\ & - \int_{-L}^{L} \int_{0}^{\infty} m(x, y, Y)\{\frac{1}{2}\varphi_{yy} + \frac{1}{2}\varphi_{xx} - \alpha x \varphi_x - (\beta x + c_0 y + kY)\varphi_y + y\varphi\} \mathrm{d}x\mathrm{d}y \\ & - \int_{-L}^{L} \int_{-\infty}^{0} m(x, y, -Y)\{\frac{1}{2}\varphi_{yy} + \frac{1}{2}\varphi_{xx} - \alpha x \varphi_x - (\beta x + c_0 y - kY)\varphi_y - y\varphi\} \mathrm{d}x\mathrm{d}y = 0 \end{aligned}$$

we finally deduce

$$\begin{aligned} & ym + \frac{\partial}{\partial x}[xm] + \frac{\partial}{\partial y}[(\beta x + c_0 y + kY)m] + \frac{1}{2}\frac{\partial^2 m}{\partial x^2} + \frac{1}{2}\frac{\partial^2 m}{\partial y^2} = 0, \quad \text{on } \mathcal{O}^{+} \\ & -ym + \frac{\partial}{\partial x}[xm] + \frac{\partial}{\partial y}[(\beta x + c_0 y - kY)m] + \frac{1}{2}\frac{\partial^2 m}{\partial x^2} + \frac{1}{2}\frac{\partial^2 m}{\partial y^2} = 0, \quad \text{on } \mathcal{O}^{-} \\ & m = 0, \quad \text{on } (-L, L) \times (-\infty, 0) \times \{Y\} \cup (-L, L) \times (0, \infty) \times \{-Y\} \end{aligned}$$

The proof of the main result is complete.

5.8 Appendix: Proofs of lemmas 5,6,7,8

Proof of Lemma 5. Denote $\gamma = \|\phi\|_{L^\infty}$. Notice we have

$$\begin{aligned}(u-\gamma)^+(x,\pm\bar{y}_1,z) &= 0 \quad \text{if } x \in (-L,L), \qquad z \in (-Y,Y)\\ (u-\gamma)^+(x,y,\pm Y) &= 0 \quad \text{if } x \in (-L,L), \qquad 0 < \pm y < \bar{y}_1\end{aligned}$$

and

$$\begin{aligned}(u+\gamma)^-(x,\pm\bar{y}_1,z) &= 0 \quad \text{if } x \in (-L,L), \qquad z \in (-Y,Y)\\ (u+\gamma)^-(x,y,\pm Y) &= 0 \quad \text{if } x \in (-L,L), \qquad 0 < \pm y < \bar{y}_1.\end{aligned}$$

Choose $v = u - (u-\gamma)^+ \in K$ and we obtain

$$-a(u,(u-\gamma)^+) - \lambda(u,(u-\gamma)^+) \geqslant -\lambda(w,(u-\gamma)^+)$$

$$a(u,(u-\gamma)^+) + \lambda(u,(u-\gamma)^+) \leqslant \lambda(w,(u-\gamma)^+).$$

Switching the first argument by $(u-\gamma)$ in the previous inequality, we obtain

$$a((u-\gamma)^+,(u-\gamma)^+) + \lambda|(u-\gamma)^+|^2_{L^2} \leqslant \lambda(w-\gamma,(u-\gamma)^+)$$

and $w \leqslant \gamma$ implies $(u-\gamma)^+ = 0$. Taking $v = u + (u+\gamma)^-$ similar arguments yields $(u+\gamma)^- = 0$. □

Proof of Lemma 6. We have the following expressions,

$$a(\eta^\epsilon,\eta^\epsilon) = \int_{\Delta_1} \frac{\epsilon}{2}(\eta^\epsilon_z)^2 + \frac{1}{2}(\eta^\epsilon_y)^2 + \frac{1}{2}(\eta^\epsilon_x)^2 + (\beta x + kz + c_0 y)\eta^\epsilon\eta^\epsilon_y - y\eta^\epsilon\eta^\epsilon_z + \alpha x\eta^\epsilon\eta^\epsilon_x$$

and

$$a(\eta^\epsilon,u_0) = \int_{\Delta_1} \frac{\epsilon}{2}\eta^\epsilon_z u_{0z} + \frac{1}{2}\eta^\epsilon_y u_{0y} + \frac{1}{2}\eta^\epsilon_x u_{0x} + (\beta x + kz + c_0 y)u_0\eta^\epsilon_y - yu_0\eta^\epsilon_z + \alpha x u_0\eta^\epsilon_x.$$

Inequality $a(\eta^\epsilon,\eta^\epsilon) \leqslant a(\eta^\epsilon,u_0)$ means

$$\begin{aligned}\frac{\epsilon}{2}\int_{\Delta_1}(\eta^\epsilon_z)^2 + \frac{1}{2}\int_{\Delta_1}(\eta^\epsilon_x)^2 + \frac{1}{2}\int_{\Delta_1}(\eta^\epsilon_y)^2 \leqslant{}& \frac{\epsilon}{2}\int_{\Delta_1}\eta^\epsilon_z u_{0z} + \frac{1}{2}\int_{\Delta_1}\eta^\epsilon_y u_{0y}\\ &- \int_{\Delta_1}(\beta x + c_0 y + kz)\eta^\epsilon\eta^\epsilon_y + \frac{1}{2}\int_{\Delta_1}(\beta x + c_0 y + kz)u_0\eta^\epsilon_y\\ &+ \frac{1}{2}\int_{\Delta_1}\eta^\epsilon_x u_{0x} + \int_{\Delta_1}\alpha x\eta^\epsilon\eta^\epsilon_x - \int_{\Delta_1}\alpha x u_0\eta^\epsilon_x\\ &+ \int_{\Delta_1} y\eta^\epsilon\eta^\epsilon_z - \int_{\Delta_1} yu_0\eta^\epsilon_z\end{aligned}$$

We apply Cauchy-Schwartz inequality to the first two terms, then we apply Green formula to the last one and we get

$$
\begin{aligned}
\frac{\epsilon}{2}\int_{\Delta_1}(\eta_z^\epsilon)^2+\frac{1}{2}\int_{\Delta_1}(\eta_x^\epsilon)^2+\frac{1}{2}\int_{\Delta_1}(\eta_y^\epsilon)^2 \leqslant{}& \epsilon\sqrt{\int_{\Delta_1}(\eta_z^\epsilon)^2}\frac{1}{2}(\int_{\Delta_1}(u_{0z}^\epsilon)^2)^{\frac{1}{2}} \\
&+\sqrt{\int_{\Delta_1}(\eta_y^\epsilon)^2}\{\frac{1}{2}(\int_{\Delta_1}(u_{0y}^\epsilon)^2)^{\frac{1}{2}}+\|\phi\|(\int_{\Delta_1}(\beta x+c_0y+kz)^2)^{\frac{1}{2}}\} \\
&+\sqrt{\int_{\Delta_1}(\eta_x^\epsilon)^2}\{\frac{1}{2}(\int_{\Delta_1}(u_{0x}^\epsilon)^2)^{\frac{1}{2}}+\alpha\|\phi\|(\int_{\Delta_1}x^2)^{\frac{1}{2}}+\alpha(\int_{\Delta_1}x^2u_0)^{\frac{1}{2}}\} \\
&+\|\phi\|^2\int_{-L}^{L}\int_{-\bar{y}_1}^{\bar{y}_1}|y|dxdy+\|\phi\|(\int_{\Delta_1}|yu_{0z}|)+\int_{-L}^{L}\int_{-\bar{y}_1}^{\bar{y}_1}|y||u_0(x,y,Y)| \\
&+\int_{-L}^{L}\int_{-\bar{y}_1}^{\bar{y}_1}|y||u_0(x,y,-Y)|
\end{aligned}
$$

with c_1,c_2,c_3 and c_4 are positive constant, we get

$$\|\sqrt{\epsilon}\eta_z^\epsilon\|_{L^2}^2+\|\eta_y^\epsilon\|_{L^2}^2+\|\eta_x^\epsilon\|_{L^2}^2\leqslant\sqrt{\epsilon}c_1\|\sqrt{\epsilon}\eta_z^\epsilon\|_{L^2}+c_2\|\eta_y^\epsilon\|_{L^2}+c_3\|\eta_x^\epsilon\|_{L^2}+c_4.$$

□

Proof of Lemma 7. We have

$$
\begin{aligned}
\int_{\Delta_1}\eta_{yy}^k\eta_z^ky^{2p-1}\theta^{2q} &= -\int_{\Delta_1}\eta_y^k\eta_{zy}^ky^{2p-1}\theta^{2q}-\int_{\Delta_1}\eta_y^k\eta_z^k(2p-1)y^{2p-2}\theta^{2q}-\int_{\Delta_1}\eta_y^k\eta_z^ky^{2p-1}2q\theta'\theta^{2q-1} \\
&= -\frac{1}{2}\int_{\Delta_1}((\eta_y^k)^2)_zy^{2p-1}\theta^{2q}-(2p-1)\int_{\Delta_1}(\eta_y^k\theta)(\eta_z^ky^{2p-2}\theta^{2q-1}) \\
&\quad -2q\int_{\Delta_1}(\eta_y^k\theta)(\eta_z^ky^{2p-1}\theta^{2q-2})\theta' \\
&= -\frac{1}{2}\int_{-L}^{L}\int_{-\bar{y}_1}^{\bar{y}_1}(\eta_y^k)^2(x,y,Y)y^{2p-1}\theta^{2q}+\frac{1}{2}\int_{-L}^{L}\int_{-\bar{y}_1}^{\bar{y}_1}(\eta_y^k)^2(x,y,-Y)y^{2p-1}\theta^{2q} \\
&\quad -(2p-1)\int_{\Delta_1}(\eta_y^k\theta)(\eta_z^k\pi)y^{p-2}\theta^{q-1}-2q\int_{\Delta_1}(\eta_y^k\theta)(\eta_z^k\pi)y^{p-1}\theta^{q-2}\theta',
\end{aligned}
$$

$$
\begin{aligned}
\int_{\Delta_1}\eta_{xx}^k\eta_z^ky^{2p-1}\theta^{2q} &= -\frac{1}{2}\int_{\Delta_1}((\eta_x^k)^2)_zy^{2p-1}\theta^{2q} \\
&= -\frac{1}{2}\int_{-L}^{L}\int_{-\bar{y}_1}^{\bar{y}_1}(\eta_x^k)^2(x,y,Y)y^{2p-1}\theta^{2q}+\frac{1}{2}\int_{-L}^{L}\int_{-\bar{y}_1}^{\bar{y}_1}(\eta_x^k)^2(x,y,-Y)y^{2p-1}\theta^{2q}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\Delta_1}(c_0y+kz+\beta x)\eta_y^k\eta_z^ky^{2p-1}\theta^{2q} &= \int_{\Delta_1}(c_0y+kz+\beta x)(\eta_y^k\theta)(\eta_z^k\pi)y^{p-1}\theta^{q-1} \\
\int_{\Delta_1}\alpha x\eta_x^k\eta_z^ky^{2p-1}\theta^{2q} &= \int_{\Delta_1}\alpha x(\eta_x^k\theta)(\eta_z^k\pi)y^{p-1}\theta^{q}.
\end{aligned}
$$

Testing $(A\eta^k)$ with $\eta_z^k y^{2p-1}\theta^{2q}$, we deduce

$$\begin{aligned}\int(\eta_z^k\pi)^2 &= \frac{1}{4}\int_{-L}^{L}\int_{-\bar{y}_1}^{\bar{y}_1}(\eta_y^k)^2(x,y,Y)y^{2p-1}\theta^{2q} - \frac{1}{4}\int_{-L}^{L}\int_{-\bar{y}_1}^{\bar{y}_1}(\eta_y^k)^2(x,y,-Y)y^{2p-1}\theta^{2q}\\ &+\frac{1}{4}\int_{-L}^{L}\int_{-\bar{y}_1}^{\bar{y}_1}(\eta_x^k)^2(x,y,Y)y^{2p-1}\theta^{2q} - \frac{1}{4}\int_{-L}^{L}\int_{-\bar{y}_1}^{\bar{y}_1}(\eta_x^k)^2(x,y,-Y)y^{2p-1}\theta^{2q}\\ &+(p-\frac{1}{2})\int_{\Delta_1}(\eta_y^k\theta)(\eta_z^k\pi)y^{p-2}\theta^{q-1} + q\int_{\Delta_1}(\eta_y^k\theta)(\eta_z^k\pi)y^{p-1}\theta^{q-2}\theta'\\ &+\int_{\Delta_1}(c_0y+kz+\beta x)(\eta_y^k\theta)(\eta_z^k\pi)y^{p-1}\theta^{q-1}\\ &+\int_{\Delta_1}\alpha x(\eta_x^k\theta)(\eta_z^k\pi)y^{p-1}\theta^q.\end{aligned}$$

□

Proof of Lemma 8. 1. With

$$f(\eta^k,\eta_y^k) := -\eta^k(\frac{\pi''}{2} - (c_0y+kz+\alpha x)\pi') - \eta_y^k\pi',$$

we have $\forall\psi \in H^1(\Delta_1), \psi(x,y,\pm Y) = 0$ and $\psi(x,\pm\bar{y}_1,z) = 0$,

$$\frac{1}{2}\int_{\Delta_1}v_y^k\psi_y + \frac{1}{2}\int_{\Delta_1}v_x^k\psi_x + \int_{\Delta_1}\big(\alpha x v_x^k + (\beta x + c_0y + kz)v_y^k - yv_z^k\big)\psi = \int_{\Delta_1}f(\eta^k,\eta_y^k)\psi.$$

Now, when k goes to $+\infty$, we get

$$\frac{1}{2}\int_{\Delta_1}v_y\psi_y + \frac{1}{2}\int_{\Delta_1}v_x\psi_x + \int_{\Delta_1}\big(\alpha x v_x + (\beta x + c_0y + kz)v_y - yv_z\big)\psi = \int_{\Delta_1}f(\eta,\eta_y)\psi.$$

We deduce that in $H^{-1}(\Delta_1)$ we firstly have $-Av = f(\eta,\eta_y)$ which is equivalent to $\pi A\eta = 0$ and secondly that choice of test function implies $\pi\eta_x(\pm L,y,z) = 0$ in $(H^{\frac{1}{2}}((-L,L)\times(-\bar{y}_1,\bar{y}_1))'$.

2. (a) We know that $\pi\eta^k \in H^1(\Delta_1)$, its trace is well defined and satisfies,

$$\begin{aligned}\gamma(\pi\eta^k)(x,y,Y) &= \pi(\chi^{+,k}+\beta^{k,+}); \quad y>0\\ \gamma(\pi\eta^k)(x,y,-Y) &= \pi\chi^{-,k}; \quad y<0\end{aligned}$$

with

$$\|\chi^{\pm,k}\|_{L^\infty} \leqslant \|\phi\|_{L^\infty} \tag{5.8}$$

and $\chi^{-,k}, \chi^{+,k}$ satisfy respectively (5.9) and (5.10)

$$\begin{aligned}&\textstyle\int_{-L}^{L}\int_0^{\bar{y}_1}\{\frac{1}{2}(\chi_x^{+,k}\psi_x + \chi_y^{+,k}\psi_y) + (\alpha x\chi_x^{+,k} + (\beta x + c_0y + kY)\chi_y^{+,k})\psi\}\mathrm{d}x\mathrm{d}y = 0,\\ &\forall\psi \in H^1((-L,L)\times(0,\bar{y}_1)) \text{ with } \psi(x,0) = \psi(x,\bar{y}_1) = 0\end{aligned} \tag{5.9}$$

and

$$\int_{-L}^{L}\int_{-\bar{y}_1}^{0}\{\tfrac{1}{2}(\chi_x^{-,k}\psi_x + \chi_y^{-,k}\psi_y) + (\alpha x\chi_x^{-,k} + (\beta x + c_0 y - kY)\chi_y^{-,k})\psi\}\mathrm{d}x\mathrm{d}y = 0,$$
$$\forall\psi \in H^1((-L,L)\times(-\bar{y}_1,0)) \text{ with } \psi(x,0) = \psi(x,-\bar{y}_1) = 0. \tag{5.10}$$

First, we study convergence of the sequence $\chi^{\pm,k}$ and we deduce PDEs satisfied by $\lim_{k\to 0}\chi^{\pm,k}$.
In particular (5.8) implies

$$\chi^{+,k} \to \chi^+ \text{ in } L^2((-L,L)\times(0,\bar{y}_1)) \text{ weakly,}$$
$$\chi^{-,k} \to \chi^- \text{ in } L^2((-L,L)\times(-\bar{y}_1,0)) \text{ weakly.}$$

and equalities (5.9), (5.10) imply

$$\|\chi^{+,k}\pi\|_{H^1} \leqslant C \quad \text{and} \quad \chi^{+,k}\pi \to \chi^+\pi \text{ in } H^1((-L,L)\times(0,\bar{y}_1)) \text{ weakly,}$$
$$\|\chi^{-,k}\pi\|_{H^1} \leqslant C \quad \text{and} \quad \chi^{-,k}\pi \to \chi^-\pi \text{ in } H^1((-L,L)\times(-\bar{y}_1,0)) \text{ weakly.}$$

Denote $\xi^{\pm,k} := \chi^{\pm,k}y^2$ and $g(\chi^{+,k},\chi_y^{+,k}) := -\chi^{+,k}\{1-2y(\alpha x + c_0 y + kY)\} - 2y\chi_y^{+,k}$. From (5.9), we have $B_Y\xi^{+,k} = g(\chi^{+,k},\chi_y^{+,k})$ in $(-L,L)\times(-\bar{y}_1,\bar{y}_1)$, in $H^{-1}((-L,L)\times(0,\bar{y}_1))$. The operator B_+ is strictly elliptic then $\xi^{+,k} \in H^2((-L,L)\times(0,\bar{y}_1))$. We have $\xi_x^{+,k}(\pm L,y) = 0$ in $(H^{\frac{1}{2}}(0,\bar{y}_1))'$. As $\xi^{+,k} \in H^2(\Delta_1)$ and $\pi B_+\chi^{+,k} = 0$, we have

$$-B_+\xi^{+,k} = g(\chi^{+,k},\chi_y^{+,k}) \quad \text{in a strong sense}$$

We obtain that $\forall\psi \in H^1((-L,L)\times(0,\bar{y}_1)), \psi(x,0) = \psi(x,\bar{y}_1) = 0$,

$$\int_{-L}^{L}\int_{0}^{\bar{y}_1}\frac{1}{2}(\xi_x^{+,k}\psi_x + \xi_y^{+,k}\psi_y) + \big(\alpha x\xi_x^{+,k} + (\beta x + c_0 y + kY)\xi_y^{+,k} - y\xi_z^{+,k}\big)\psi$$
$$= \int_{-L}^{L}\int_{0}^{\bar{y}_1} g(\chi^k,\chi_y^k)\psi.$$

Now, when k goes to 0, we have

$$\int_{-L}^{L}\int_{0}^{\bar{y}_1}\frac{1}{2}(\xi_x^{+}\psi_x + \xi_y^{+}\psi_y) + \big(\alpha x\xi_x^{+} + (\beta x + c_0 y + kY)\xi_y^{+} - y\xi_z^{+}\big)\psi = \int_{-L}^{L}\int_{0}^{\bar{y}_1} g(\chi,\chi_y)\psi.$$

We deduce that in $H^{-1}((-L,L)\times(0,\bar{y}_1))$ we have $-B_+\xi^+ = g(\chi^+,\chi_y^+)$ which is equivalent to $y^2B_+\chi^+ = 0$ and that choice of test function implies $y^2\chi_x^+(\pm L,y) = 0$ in $(H^{\frac{1}{2}}((0,\bar{y}_1)))'$.

(b) Firstly, $\gamma(\pi\eta^k) \to \pi(\chi^+ + \beta^+)$ in $H^1((-L,L)\times(0,\bar{y}_1))$ weakly. Secondly, weak convergence of $\pi\eta^k \to \pi\eta$ in $H^1(\Delta_1)$ implies weak convergence of $\gamma(\pi\eta^k) \to \gamma(\pi\eta)$ in $H^{\frac{1}{2}}(\partial\Delta_1)$. By uniqueness of the limit, we have $\gamma(\pi\eta) = \pi(\chi^+ + \beta^+)$.

3. Using Green formula

$$\forall\psi \in H_0^1(\Delta_1)\cap H^2(\Delta_1), \quad \int_{\Delta_1}\eta^k\psi_{yy} + \int_{\Delta_1}\eta_y^k\psi_y = \int_{\partial\Delta_1}\phi^k\psi_y\mathrm{d}\sigma.$$

Now, when k goes to ∞, we obtain

$$\forall\psi \in H_0^1(\Delta_1) \cap H^2(\Delta_1), \quad \int_{\Delta_1} \eta\psi_{yy} + \int_{\Delta_1} \eta_y\psi_y = \int_{\partial\Delta_1} \phi\psi_y \mathrm{d}\sigma.$$

□

Lemma 9. *We have*

$$\eta_{xx}\pi^2 \in L^2(\Delta_1); \quad \eta_{xy}\pi^2 \in L^2(\Delta_1); \quad \eta_{yy}\pi^2 \in L^2(\Delta_1)$$

Proof. Denote $w := v_x$. Deriving equation (5.30) with respect to x, we obtain equality (5.11),

$$-\frac{1}{2}w_{xx} - \frac{1}{2}w_{yy} + \Lambda(x,y,z)w_y + \alpha x w_x - y w_z = -\beta v_y - \alpha v_x + \rho_1\eta_x + (\rho_1)_x\eta + \rho_2(y)\eta_{xy} \quad (5.11)$$

Testing equality (5.11) with π^2, we obtain

$$\begin{aligned}\frac{1}{2}\int_{\Delta_1}(w_x\pi)^2 + \frac{1}{2}\int_{\Delta_1}(w_y\pi)^2 = &-\int_{\Delta_1} w_y\pi\pi' w - \int_{\Delta_1}\Lambda(x,y,z)w_y w\pi^2 - \int_{\Delta_1}\alpha x w_x w\pi^2\\ &+\int_{\Delta_1} y w_z w\pi^2 - \beta\int_{\Delta_1} v_y w\pi^2 - \alpha\int_{\Delta_1} v_x w\pi^2 + \int_{\Delta_1}\rho_1\eta_x w\pi^2\\ &+\int_{\Delta_1}(\rho_1)_x\eta w\pi^2 + \int_{\Delta_1}\eta_{xy}\rho_2(y)w\pi^2\end{aligned}$$

which means

$$\begin{aligned}\frac{1}{2}\int_{\Delta_1}(w_x\pi)^2 + \frac{1}{2}\int_{\Delta_1}(w_y\pi)^2 \quad = \quad &-\int_{\Delta_1} w_y\pi\pi' w - \int_{\Delta_1}\Lambda(x,y,z)w_y w\pi^2 - \int_{\Delta_1}\alpha x w_x w\pi^2\\ &+\int_{\partial\Delta_1} y\pi^2 w^2\vec{n}(z)d\sigma - \beta\int_{\Delta_1} v_y w\pi^2 - \alpha\int_{\Delta_1} v_x w\pi^2\\ &+\int_{\Delta_1}\rho_1 w^2\pi + \int_{\Delta_1}(\rho_1)_x\eta w\pi^2 + \int_{\Delta_1}\eta_{xy}\rho_2(y)w\pi^2\\ &+\int_{\Delta_1}(w_y\pi)\rho_2(y)w - \int_{\Delta_1} v_x\pi'\rho_2(y)w.\end{aligned}$$

It is easy to verify that $v_{xx}\pi, v_{xy}\pi \in L^2(\Delta_1)$.

Denote $\tilde{w} := v_y$. Deriving equation (5.30), with respect to x, we obtain equality (5.12),

$$-\frac{1}{2}\tilde{w}_{xx} - \frac{1}{2}\tilde{w}_{yy} + c_0\tilde{w} + \Lambda(x,y,z)\tilde{w}_y + \alpha x\tilde{w}_x - y\tilde{w}_z = v_z + \rho_1\eta_y + (\rho_1)_y\eta + \eta_{yy}\rho_2(y) + \eta_y\rho_2(y)'. \quad (5.12)$$

Then, we test equality (5.12) with $\pi^2\tilde{w}$, we obtain

$$\begin{aligned}\frac{1}{2}\int_{\Delta_1}(\tilde{w}_x\pi)^2 + \frac{1}{2}\int_{\Delta_1}(\tilde{w}_y\pi)^2 \quad = \quad &-\int_{\Delta_1}\tilde{w}_y\pi\pi'\tilde{w} - \int_{\Delta_1} c_0(w\pi)^2 - \int_{\Delta_1}\Lambda(x,y,z)\tilde{w}_y\tilde{w}\pi^2 - \int_{\Delta_1}\alpha x\tilde{w}_x\tilde{w}\pi^2\\ &+\int_{\Delta_1} v_z\tilde{w}\pi^2 + \int_{\Delta_1} y\tilde{w}_z\tilde{w}\pi^2 + \int_{\Delta_1}\eta_y\rho_1\tilde{w}\pi^2 + \int_{\Delta_1}\eta(\rho_1)_y\tilde{w}\pi^2\\ &+\int_{\Delta_1}\eta_{yy}\rho_2\tilde{w}\pi^2 + \int_{\Delta_1}\eta_y(\rho_2)'\pi^2\tilde{w}\end{aligned}$$

which means

$$\frac{1}{2}\int_{\Delta_1}(\tilde{w}_x\pi)^2+\frac{1}{2}\int_{\Delta_1}(\tilde{w}_y\pi)^2=-\int_{\Delta_1}\tilde{w}_y\pi\pi'\tilde{w}-\int_{\Delta_1}c_0(w\pi)^2-\int_{\Delta_1}\Lambda(x,y,z)\tilde{w}_y\tilde{w}\pi^2$$
$$-\int_{\Delta_1}\alpha x\tilde{w}_x\tilde{w}\pi^2+\int_{\Delta_1}v_z\tilde{w}\pi^2+\int_{\partial\Delta_1}y(\tilde{w}\pi)^2\vec{n}(z)d\sigma+\int_{\Delta_1}\eta_y\rho_1\tilde{w}\pi^2$$
$$+\int_{\Delta_1}\eta(\rho_1)_y\tilde{w}\pi^2+\int_{\Delta_1}(\tilde{w}_y-2\eta_y\pi'+\eta\pi'')\tilde{w}\pi+\int_{\Delta_1}\eta_y(\rho_2)'\pi^2\tilde{w}.$$

Again, it is easy to verify that $v_{yx}\pi, v_{yy}\pi \in L^2(\Delta_1)$. □

Lemma 10. *We have*

$$\eta_{xz}y^2\pi^4 \in L^2(\Delta_1); \quad \eta_{yz}y^2\pi^3 \in L^2(\Delta_1).$$

Proof. We test equality (5.11) with $y^{2p-1}\pi^{2q}w_z$, we obtain

$$-\frac{1}{2}\int w_{xx}(y^{2p-1}\pi^{2q}w_z)=\frac{1}{2}\int_{\Delta_1}w_xw_{zx}y^{2p-1}\pi^{2q}$$
$$=\frac{1}{4}\iint w_x^2(x,y,Y)y^{2p-1}\pi^{2q}dxdy-\frac{1}{4}\iint w_x^2(x,y,-Y)y^{2p-1}\pi^{2q}\mathrm{d}x\mathrm{d}y$$

$$-\frac{1}{2}\int_{\Delta_1}w_{yy}(y^{2p-1}\pi^{2q}w_z)=\frac{1}{2}\int_{\Delta_1}w_yw_{zy}y^{2p-1}\pi^{2q}+\int_{\Delta_1}w_yw_z\big((2p-1)y^{2p-2}\pi^{2q}+y^{2p-1}2q\pi^{2q-1}\pi'\big)$$
$$=\frac{1}{4}\iint w_y^2(x,y,Y)y^{2p-1}\pi^{2q}dxdy-\frac{1}{4}\iint w_y^2(x,y,-Y)y^{2p-1}\pi^{2q}\mathrm{d}x\mathrm{d}y$$
$$+(p-\frac{1}{2})\int_{\Delta_1}(w_y\pi)(w_zy^p\pi^q)(y^{p-2}\pi^{q-1})+q\int_{\Delta_1}(w_y\pi)(w_zy^p\pi^q)(y^{p-1}\pi^{q-1}\pi').$$

We have

$$\int_{\Delta_1}(y^p\pi^qw_z)^2\leqslant\frac{1}{4}\iint\{w_x^2(x,y,Y)+w_y^2(x,y,Y)\}y^{2p-1}\pi^{2q}\mathrm{d}x\mathrm{d}y$$
$$-\frac{1}{4}\iint\{w_x^2(x,y,-Y)+w_y^2(x,y,-Y)\}y^{2p-1}\pi^{2q}\mathrm{d}x\mathrm{d}y$$
$$+(p-\frac{1}{2})\int_{\Delta_1}(w_y\pi)(w_zy^p\pi^q)(y^{p-2}\pi^{q-1})+q\int_{\Delta_1}(w_y\pi)(w_zy^p\pi^q)(y^{p-1}\pi^{q-2}\pi')$$
$$+\int_{\Delta_1}\Lambda(x,y,z)(w_y\pi)(w_zy^p\pi^q)y^{p-1}\pi^{q-1}+\int_{\Delta_1}\alpha x(w_x\pi)(w_zy^p\pi^q)y^{p-1}\pi^{q-1}$$
$$+\int_{\Delta_1}\beta v_y(y^p\pi^qw_z)y^{p-1}\pi^q+\int_{\Delta_1}\alpha v_x(y^p\pi^qw_z)y^{p-1}\pi^q$$
$$-\int_{\Delta_1}\rho_1v_x(y^p\pi^qw_z)y^{p-1}\pi^{q-1}+\int_{\Delta_1}(\rho_1)_x\eta(y^p\pi^qw_z)y^{p-1}\pi^q$$
$$-\int_{\Delta_1}\rho_2(v_{xy}\pi-v_x\pi')(y^p\pi^qw_z)y^{p-1}\pi^{q-3}.$$

For $p \geqslant 2$ and $q \geqslant 3$, some calculations give $v_{xz}y^2\pi^3 \in L^2(\Delta_1)$.

Now, we test equality (5.12) with $y^{2p-1}\pi^{2q}\tilde{w}_z$, we obtain

$$\begin{aligned}
\int_{\Delta_1} (y^p\pi^q\tilde{w}_z)^2 \leqslant\ & \frac{1}{4}\iint \{\tilde{w}_x^2(x,y,Y) + \tilde{w}_y^2(x,y,Y)\} y^{2p-1}\pi^{2q}\mathrm{d}x\mathrm{d}y \\
& - \frac{1}{4}\iint \{\tilde{w}_x^2(x,y,-Y) + \tilde{w}_y^2(x,y,-Y)\} y^{2p-1}\pi^{2q}\mathrm{d}x\mathrm{d}y \\
& + (p-\frac{1}{2})\int_{\Delta_1} (\tilde{w}_y\pi)(\tilde{w}_z y^p\pi^q)(y^{p-2}\pi^{q-1}) + q\int_{\Delta_1} (\tilde{w}_y\pi)(\tilde{w}_z y^p\pi^q)(y^{p-1}\pi^{q-2}\pi') \\
& + \int_{\Delta_1} c_0\tilde{w}(y^p\pi^q\tilde{w}_z)y^{p-1}\pi^q \\
& + \int_{\Delta_1} \Lambda(x,y,z)(\tilde{w}_y\pi)(\tilde{w}_z y^p\pi^q)y^{p-1}\pi^{q-1} + \int_{\Delta_1} \alpha x(\tilde{w}_x\pi)(\tilde{w}_z y^p\pi^q)y^{p-1}\pi^{q-1} \\
& - \int_{\Delta_1} v_z(y^p\pi^q\tilde{w}_z)y^{p-1}\pi^q - \int_{\Delta_1} \rho_1(v_y-\eta\pi')(y^p\pi^q\tilde{w}_z)y^{p-1}\pi^{q-1} \\
& - \int_{\Delta_1} (\rho_1)_y\eta(y^p\pi^q\tilde{w}_z)y^{p-1}\pi^q - \int_{\Delta_1} (\rho_1)_x\eta(y^p\pi^q\tilde{w}_z)y^{p-1}\pi^q. \\
& - \int_{\Delta_1} \rho_2(y)\eta_{yy}\pi^2(y^p\pi^q\tilde{w}_z)y^{p-1}\pi^{q-2}
\end{aligned}$$

For $p \geqslant 2$ and $q \geqslant 2$, some calculations gives $v_{yz}y^2\pi^2 \in L^2(\Delta_1)$. $\square$

Lemma 11. *For p and q large enough, we have*

$$\eta_{xyz}y^p\pi^q \in L^2(\Delta_1); \quad \eta_{yyz}y^p\pi^q \in L^2(\Delta_1); \quad \rho(z)\eta_{xxz}y^p\pi^q \in L^2(\Delta_1); \quad \rho(z)\eta_{zz}y^p\pi^q \in L^2(\Delta_1).$$

Appendix A

An empirical study on plastic deformations of an elasto-plastic problem with noise

Ce chapitre fait l'objet d'un article soumis à Probabilistic Engineering Mechanics, [21] en collaboration avec Cyril Feau.

Statistical properties of the plastic deformation related to an elastic perfectly plastic oscillator under standard white noise excitation are studied in this paper. Our approach relies on a stochastic variational inequality governing the evolution between the velocity and the non-linear restoring force. Bensoussan-Turi have shown that the solution is an ergodic Markov process. In this work, we investigate elastic phasing by means of the invariant measure. First, we exhibit by probabilistic simulations the phenomenon of micro-elastic phases (small as well as numerous). The main difficulty related to micro-elastic phasing is that they interfere on quantities of interest such that frequency of plastic deformations which tends to be overestimated. Therefore, we present approximations of the probability density function limit of the elastic component and a similar expression to Rice's formula related to frequency of threshold crossings. Then, these quantities are solution of partial differential equations. Numerical experiments on these equations show that the non-linear restoring force tends to be highly distributed in the neighborhood of plastic thresholds. Finally, an interesting criterion is provided which could be useful in engineering problems to discard micro-elastic phases and to evaluate statistics of plastic deformations.

Introduction

In this paper statistics of a deformation problem are investigated in the case of a non-linear structure excited by a random process. The behavior of an elastic-perfectly-plastic (EPP) oscillator subjected to zero mean Gaussian white noise excitation is considered. This is the simplest structural model exhibiting a behavior with hysteresis.

Moreover, the model is simple and representative of the behavior of mechanical structures which vibrate mainly on their first mode of deformation. In the context of earthquake engineering, relevant applications to piping systems under random vibrations can be accessed in this way [19, 20].
The main difficulty to study these systems comes from a frequent occurrence of non-linear phases (plastic phases) on small intervals of time. One non-linear phase corresponds to a permanent deformation, or in other words a plastic deformation. A plastic deformation is produced when the stress of the structure crosses over an elastic limit. The dynamics of the EPP-oscillator has memory, so it has been formulated in the engineering literature as a process with hysteresis. Denoting $x(t)$ the displacement and $y(t) := \dot{x}(t)$ the velocity of the oscillator, we study the problem

$$\dot{y} + c_0 y + \mathbf{F}(x(s), 0 \leqslant s \leqslant t) = \dot{w}, \tag{A.1}$$

with initial conditions of displacement and velocity

$$x(0) = x \quad , \quad y(0) = y.$$

Here $c_0 > 0$ is the viscous damping coefficient, $k > 0$ the stiffness, w is a Wiener process and $\mathbf{F}(\{x(s), 0 \leqslant s \leqslant t\})$ is a non-linear functional which depends on the entire trajectory $\{x(s), 0 \leqslant s \leqslant t\}$ up to time t. The process of plastic deformation is denoted by $\Delta(t)$ at time t and can be deduced from the pair $(x(t), \mathbf{F}(x(s), 0 \leqslant s \leqslant t))$. Throughout this paper, we shall consider the following EPP-restoring force

$$\mathbf{F}(\{x(s), 0 \leqslant s \leqslant t\}) = \begin{cases} kY, & if \quad x(t) = Y + \Delta(t), \\ k(x(t) - \Delta(t)), & if \quad x(t) \in] -Y + \Delta(t), Y + \Delta(t)[, \\ -kY, & if \quad x(t) = -Y + \Delta(t), \end{cases} \tag{A.2}$$

where Y is the elastic-plastic limit. Karnopp & Scharton [26] proposed to separate elastic and plastic states by introducing a fictitious variable $z(t) := x(t) - \Delta(t)$. Indeed, they noticed the simple fact, that between two plastic phases, $z(t)$ behaves like a linear oscillator. Therefore, $x(t)$ is splitted into $x(t) = z(t) + \Delta(t)$ where $z(t)$ *(respectively* $\Delta(t)$*) is the elastic component (respectively plastic)* of $x(t)$. So, the process $(y(t), z(t))$ is relevant for engineers because estimation of statistics related to the deformations can be obtained.

Basically, one plastic deformation begins when $z(t)$ reaches and is absorbed by Y (resp. $-Y$) with positive (resp. negative) slope, $y(t) > 0$. (resp. $y(t) < 0$) i.e. when $\text{sign}(y(t))z(t) = Y$. Then, the plastic behavior ends when the velocity changes sign. At this moment, the elastic behavior is reactivated. However, the velocity which is subjected to white noise, changes sign an infinite number of times during any small time interval. Often, this leads to a return into plastic behavior in a short time duration. We refer this phenomenon as *micro-elastic phasing* (unknown so far) which plays a crucial role on frequency and statistics of plastic deformations. Because of this phenomenon, frequency of occurrence, statistics (time duration or absolute plastic deformation) and the sequence of entry in plastic phase (as well as the sequence of exit) are not well defined.

In this paper, we introduce *enlarged plastic phases* because we are interested in properties of the plastic behavior during intervals of time delimited by entries in plastic phases and by crossings by $z(t)$ of the threshold $Y - \epsilon$ (resp. $-Y + \epsilon$) with a negative (resp. positive) velocity for a small $\epsilon > 0$.

This approach allows to rigorously link the number of enlarged plastic phases to the number of times $z(t)$ crosses the threshold $\pm Y \mp \epsilon$ with negative or positive slope.

To explain the procedure, consider $T > 0$, denote $\tau_0^\epsilon := 0$ and

$$\begin{aligned} \theta_{n+1}^\epsilon &:= \inf\{t > \tau_n^\epsilon, \quad |z(t)| = Y\}, \\ \tau_{n+1}^\epsilon &:= \inf\{t > \theta_{n+1}^\epsilon, \quad |z(t)| = Y - \epsilon\}, \quad \forall n \geqslant 1. \end{aligned} \tag{A.3}$$

Also, we set $N_T^\epsilon = \sum_{n \geqslant 0} \mathbf{1}_{\{\tau_n^\epsilon \leqslant T\}}$ the number of enlarged plastic phases up to the time T. Then, for any measurable function f such that $f(y,z) = 0$ if $\mathrm{sign}(y)z \neq Y$ and satisfying $\int_{D^+} |f(y,Y)| m(y,Y) \mathrm{d}y + \int_{D^-} |f(y,-Y)| m(y,-Y) \mathrm{d}y < \infty$, we have

$$\frac{1}{T} \int_0^T f(y(s), z(s)) \mathrm{d}s = \frac{N_T^\epsilon}{T} \times \frac{1}{N_T^\epsilon} \sum_{n=1}^{N_T^\epsilon} \int_{\theta_n^\epsilon}^{\tau_n^\epsilon} f(y(s), z(s)) \mathrm{d}s. \tag{A.4}$$

So we can define separately, the frequency of crossings by $z(t)$ of both thresholds $\pm Y \mp \epsilon$ with negative or positive slope by

$$\nu(Y, \epsilon) := \lim_{T \to \infty} \frac{N_T^\epsilon}{T}$$

and the "empirical statistics" related to the enlarged plastic phase by

$$\Delta_f(Y, \epsilon) := \lim_{T \to \infty} \frac{1}{N_T^\epsilon} \sum_{n=1}^{N_T^\epsilon} \int_{\theta_n^\epsilon}^{\tau_n^\epsilon} f(y(s), z(s)) \mathrm{d}s.$$

The asymptotic behavior of the EPP-oscillator has been studied by Bensoussan and Turi in [7]. They have shown that $(y(t), z(t))$ satisfies a stochastic variational inequality (SVI) and is an ergodic Markov process. Thus, there exists a unique invariant measure also denoted by $m(y,z)$ which is composed of

1. an elastic part: $\{m(y,z), \quad |z| < Y\}$,
2. a positive plastic part: $\{m(y,Y), \quad y > 0\}$,
3. a negative plastic part: $\{m(y,-Y), \quad y < 0\}$.

Therefore, as T goes to infinity, (A.4) becomes

$$\int_0^\infty f(y,Y) m(y,Y) \mathrm{d}y + \int_{-\infty}^0 f(y,-Y) m(y,-Y) \mathrm{d}y = \nu(Y,\epsilon) \Delta_f(Y,\epsilon). \tag{A.5}$$

The above relation is essential since that states that the product of $\nu(Y,\epsilon)$ and $\Delta_f(Y,\epsilon)$ remains constant for all values of ϵ. However, in our probabilistic simulations, we observe that $\nu(Y,\epsilon)$ tends to ∞ and $\Delta_f(Y,\epsilon)$ tends to 0 as ϵ tends to 0. In this work, we provide an empirical criterion which could be useful for engineers to calibrate ϵ in order to compute a frequency and statistics of plastic deformations which does not take into account the micro-elastic phasing. Indeed, we discovered empirically that a high concentration at the neighborhood of $\{z = \pm Y\}$ can be observed in the distribution of $\lim_{t \to \infty} z(t)$. Thus, the latter admits points of minima which are identified, namely $\pm(Y - \epsilon^\star)$. Our approach relies on partial differential equations (PDEs) related to the second marginal of $\{m(y,z), \quad |z| < Y\}$, that is $s \to \int_{-\infty}^\infty m(y,s) \mathrm{d}y$.

In addition, mean frequencies of crossing thresholds $\pm(Y-\epsilon^\star)$ with negative or positive velocity are successfully computed by an expression similar to Rice's formula [31]. This is a technique widely used among engineers and is rigorously established for purely elastic systems. More precisely, if $Y = +\infty$ there is no plastic deformation, $z(t) = x(t)$ and (A.1) reduces to the dynamics response of a linear oscillator excited by a white noise excitation. It is described by the pair $(x(t), y(t))$ which is solution of the Stochastic Differential Equation (SDE):

$$\mathrm{d}y(t) = -(c_0 y(t) + kx(t))\mathrm{d}t + \mathrm{d}w(t); \quad \mathrm{d}x(t) = y(t)\mathrm{d}t. \tag{A.6}$$

The couple $(x(t), y(t))$ is an ergodic Markov process whose invariant measure $\bar{m}$ is explicitly given by [30]:

$$\bar{m}(x,y) = \frac{c_0\sqrt{k}}{\pi}\exp(-c_0kx^2)\exp(-c_0y^2). \tag{A.7}$$

Consider s, mean frequencies $\nu_s^\pm$ of threshold crossings with positive slope $y(t) > 0$ or negative slope $y(t) < 0$ are given by Rice's formula [31]:

$$\nu_s^+ = \int_0^{+\infty} y\bar{m}(y,s)\mathrm{d}y, \quad \nu_s^- = -\int_{-\infty}^0 y\bar{m}(y,s)\mathrm{d}y. \tag{A.8}$$

In the context of the EPP-oscillator ($Y < \infty$), the analogy of Rice's formula for the calculation of $\nu(Y,\epsilon)$ leads to the following approximation

$$\nu(Y,\epsilon) \approx -\int_{-\infty}^0 ym(y, Y-\epsilon)\mathrm{d}y + \int_0^{+\infty} ym(y, -Y+\epsilon)\mathrm{d}y. \tag{A.9}$$

The proof of "Rice's formula" for m is still an open problem and unfortunately we do not answer to this question. But, numerical experiments on $\{m(y,z), \quad |z| < Y\}$ to compute $\nu(Y,\epsilon)$ have provided satisfactory results in good agreement with the probabilistic algorithm.

Organization of the paper

Section 1 is devoted to setting the governing SVI of the pair $(y(t), z(t))$. As we are interested in the long time behavior, we seek the asymptotic law of $(y(t), z(t))$ which corresponds to its invariant measure. The latter is solution of a PDE. The numerical method developed in [3] to solve this PDE is also briefly summarized.
In Section 2, we investigate elastic phasing. We exhibit the phenomenon of *micro-elastic phases* which are also small as well as numerous. Micro-elastic phases are observable on numerical solution of SVI and the frequency of switching regime in (A.3) increases significantly as ϵ tends to 0. On the other hand, a high concentration at the neighborhood of $\{z = \pm Y\}$ can be observed in the probability density function (pdf) of $\lim_{t\to\infty} z(t)$. Thus, the latter admits points of minima which are identified. Therefore, points of inflection of the derivatives of the pdf of $\lim_{t\to\infty} z(t)$ are located using approximations of PDEs. To the best of our knowledge, there is no known numerical probabilistic algorithm that does the same. These points constitute appropriate thresholds $s^+ = Y - \epsilon^\star$ and $s^- = -Y + \epsilon^\star$ related to our procedure. Indeed, $\epsilon^\star$ appears to be an interesting choice to calibrate ϵ in (A.5).
In Section 3, we study applications of "Rice's formula" to $m(y,z)$. For some values of Y, we observe that a similar expression to Rice's formula is empirically valid for thresholds $s = s^-, s^+$ by using approximations of formula (A.9). Hence, in both cases, the mean frequencies of threshold crossings are deduced from solutions of PDEs. Of interest to engineering problems, an expression of this frequency which does not take into account the elastic excursions is given. Moreover in this context, statistics of plastic displacements are provided such as time duration and expected absolute deformation.

A.1 Some Background

In [7], A. Bensoussan and J. Turi showed the dynamics of EPP-oscillator (A.1),(A.2) follows a stochastic variational inequality:

$$\begin{cases} \mathrm{d}y(t) = -(c_0 y(t) + kz(t))\mathrm{d}t + \mathrm{d}w(t), \\ (\mathrm{d}z(t) - y(t)\mathrm{d}t)(\zeta - z(t)) \geqslant 0, \\ \forall\, |\zeta| \leqslant Y, \quad |z(t)| \leqslant Y. \end{cases} \tag{A.10}$$

A general framework dealing with this class of inequalities can be found in [2]. For an initial condition $(y(0), z(0))$ with a given law of probability, the system is well established. In [7] existence, uniqueness and ergodicity are proven. Hence, there exists a unique invariant measure related to the process $(y(t), z(t))$.

A.1.1 Direct numerical simulations

In Appendix, we recall how a semi analytical solution of (A.10) has been obtained in [3]. Figure A.1 shows (a) the sojourns in the elastic and plastic phases, (b) the sojourns in micro-elastic and micro-plastic phases.

A.1.2 Characterization of the invariant measure

Notation 14. *Introduce*

$$D := \mathbb{R} \times (-Y, +Y), \quad D^+ := (0, \infty) \times \{Y\}, \quad D^- := (-\infty, 0) \times \{-Y\}, \tag{A.11}$$

and the differential operators

$$\begin{aligned} Au &:= -\frac{1}{2}u_{yy} + (c_0 y + kz)u_y - yu_z, \qquad (A.12)\\ B_+u &:= -\frac{1}{2}u_{yy} + (c_0 y + kY)u_y, \\ B_-u &:= -\frac{1}{2}u_{yy} + (c_0 y - kY)u_y, \end{aligned}$$

where u is a regular function on D.

By definition, the invariant measure of the process $(y(t), z(t))$ denoted by ν satisfies

$$\int_D A\varphi \mathrm{d}\nu(y,z) + \int_0^\infty B_+\varphi \mathrm{d}\nu(y,Y) + \int_{-\infty}^0 B_-\varphi \mathrm{d}\nu(y,-Y) = 0 \quad , \quad \forall \varphi \text{ regular}.$$

The measure ν has a pdf denoted by m. Then, m satisfies by nature a PDE in an ultra weak variational sense:

$$\begin{aligned} \int_{-\infty}^{+\infty}\int_{-Y}^{Y} m(y,z)\{y\partial_z\varphi - (c_0 y + kz)\partial_y\varphi + \frac{1}{2}\partial_{yy}\varphi\}\mathrm{d}y\mathrm{d}z \quad &+ \\ \int_0^{+\infty} m(y,Y)\{-(c_0 y + kY)\partial_y\varphi(y,Y) + \frac{1}{2}\partial_{yy}\varphi(y,Y)\}\mathrm{d}y \quad &+ \\ \int_{-\infty}^0 m(y,-Y)\{-(c_0 y - kY)\partial_y\varphi(y,-Y) + \frac{1}{2}\partial_{yy}\varphi(y,-Y)\}\mathrm{d}y \quad &= \quad 0. \end{aligned} \tag{A.13}$$

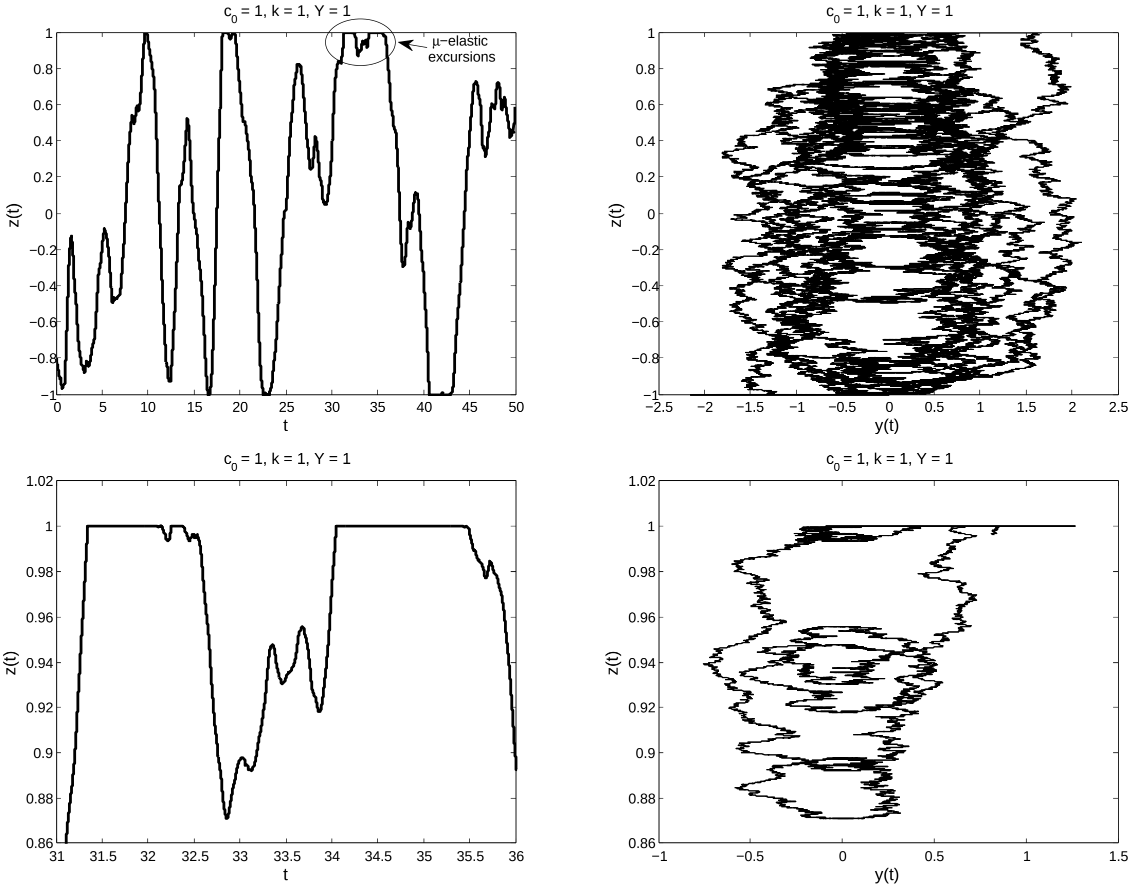


Figure A.1: *Visualization of Micro-Elastic Phasing Phenomenon: At the top-left we have sample trajectory of* $(t, z(t))$*, at the top-right we have the corresponding trajectory* $(y(t), z(t))$ *in the phase space; we observe regular elastic and plastic phasing. At the bottom in the both cases, a zoom on plastic phasing reveals an illustrative micro-elastic phasing.*

In [3], a deterministic method alternative to the Monte Carlo method has been developed for solving numerically (A.13). The method is compatible with the ultra weak variational formulation. The key point is to solve a stationary PDE with a measurable function f, satisfying

$$\int_D |f(y,z)|m(y,z)\mathrm{d}y\mathrm{d}z + \int_{D^+} |f(y,Y)|m(y,Y)\mathrm{d}y + \int_{D^-} |f(y,-Y)|m(y,-Y)\mathrm{d}y < \infty, \quad \text{(A.14)}$$

as right hand side:

$$\left\{ \begin{array}{rcll} \lambda u + Au &=& f(y,z) & \text{in } D, \\ \lambda u + B_+ u &=& f(y,Y) & \text{in } D^+, \\ \lambda u + B_- u &=& f(y,-Y) & \text{in } D^-. \end{array} \right. \quad (P_\lambda)$$

Recall from [3] that this formulation *is very significant from a numerical point of view*, since it allows to obtain $\lim_{t\to\infty} \mathbb{E}[f(y(t), z(t))]$ in a way which does not require to solve a time dependent problem. Indeed, it can be shown that $\forall (y(0), z(0)) \in \bar{D}$,

$$\lim_{\lambda\to 0} \lambda u_\lambda(y(0), z(0)) = \lim_{t\to\infty} \mathbb{E}[f(y(t), z(t))]$$

and then

$$\lim_{\lambda\to 0} \lambda u_\lambda(y(0),z(0)) = \int_{-Y}^{Y}\int_{-\infty}^{+\infty} m(y,z)f(y,z)\mathrm{d}y\mathrm{d}z$$
$$+ \int_0^{+\infty} m(y,Y)f(y,Y)\mathrm{d}y + \int_0^{+\infty} m(y,-Y)f(y,-Y)\mathrm{d}y.$$

This limit does not depend on $(y(0),z(0))$. We shall now show how to use m to compute micro-elastic phasing.

A.2 Phenomenology of micro-elastic phasing

In this section, we provide a description of elastic phasing. Recall that, the process $(y(t),z(t))$ is in elastic phase when $|z(t)| < Y$ or in plastic phase when $|z(t)| = Y$.

A.2.1 Frequency of occurence and statistics of the plastic phases

The main difficulty related to micro-elastic phasing is that it interferes on quantities of interest such as frequency of plastic deformations, which makes sense for engineers. Plastic phases consecutive to micro-elastic phases yield negligible plastic deformations. In addition, they are also small as well as numerous (see Figure A.1), therefore it becomes delicate to characterize the frequency of plastic deformations. This is a mathematical artifact due to the white noise that has to be discarded. So, we analyze this phenomenon with a numerical solution $\{(y(t),z(t)),\quad t \geqslant 0\}$ (see Appendix) and enlarged plastic phases $\{\theta_k^\epsilon, \tau_k^\epsilon, \quad k \geqslant 1\}$ for several values of $\epsilon > 0$.

We are interested in $\nu(Y,\epsilon), \Delta_1(Y,\epsilon), \Delta_{|y|}(Y,\epsilon)$: frequency of occurrence, mean duration and mean absolute value of the deformation related to enlarged plastic phases for several values of ϵ and Y. Then, these quantities are computed by numerical experiments and shown in Table A.1.

Let us explain our procedure. For any $k \geqslant 1$, duration and absolute value of the k^{th} plastic deformation are denoted by

$$\Delta_{1,k}(Y,\epsilon) := \theta_k^\epsilon - \tau_k^\epsilon \quad \text{and} \quad \Delta_{|y|,k}(Y,\epsilon) := \int_{\theta_k^\epsilon}^{\tau_k^\epsilon} |y(s)|\mathrm{d}s.$$

We can assume that $\{\Delta_{1,k}(Y,\epsilon), k \geqslant 0\}$ and $\{\Delta_{|y|,k}(Y,\epsilon), k \geqslant 0\}$ are sets of independent identically distributed (iid) random variables. Thus, we approach $\Delta_1(Y,\epsilon)$ and $\Delta_{|y|}(Y,\epsilon)$ using one trajectory. Let $M \in \mathbb{N}^\star$, the numerical simulation of $(y(t),z(t))$ is considered up to the time denoted by T_M which is required to obtain M plastic phases. For the convenience of the reader, let us drop the notation (Y,ϵ) in the context of these settings. We set

$$E_M\Delta_1 := \frac{1}{M}\sum_{k=1}^{M} \Delta_{1,k} \quad , \quad E_M\Delta_1^2 := \frac{1}{M}\sum_{k=1}^{M} \Delta_{1,k}^2,$$
$$\sigma_M^2(\Delta_1) := E_M\Delta_1^2 - (E_M\Delta_1)^2,$$

and also

$$E_M\Delta_{|y|} := \frac{1}{M}\sum_{k=1}^{M} \Delta_{|y|,k} \quad , \quad E_M\Delta_{|y|}^2 := \frac{1}{M}\sum_{k=1}^{M} \Delta_{|y|,k}^2,$$
$$\sigma_M^2(\Delta_{|y|}) := E_M\Delta_{|y|}^2 - (E_M\Delta_{|y|})^2.$$

$Y = 1$				
ϵ	$\nu(Y,\epsilon)$	$\Delta_1(Y,\epsilon)$	$\Delta_{\lvert y\rvert}(Y,\epsilon)$	$T_{M=10000}(CPU)$
1e-01	0.144591	$0.5764^{\pm 0.0056}$	$0.3567^{\pm 0.0033}$	69160
1e-02	0.150465	$0.5542^{\pm 0.0043}$	$0.3427^{\pm 0.0028}$	66460
1e-03	0.188327	$0.4440^{\pm 0.0037}$	$0.2747^{\pm 0.0025}$	53099
1e-04	0.258672	$0.3224^{\pm 0.0031}$	$0.1995^{\pm 0.0021}$	38659
1e-05	0.372639	$0.2218^{\pm 0.0023}$	$0.1372^{\pm 0.0015}$	26835
1e-06	0.542886	$0.1553^{\pm 0.0018}$	$0.0960^{\pm 0.0011}$	18420

Table A.1: Frequency of occurence, mean duration and mean absolute value of the deformation and time required to obtain $M = 10000$ enlarged plastic phases versus ϵ.The dependency on ϵ is clear.

Therefore,

$$\Delta_1 \in \left(E_M\Delta_1 - 1.96\frac{\sigma_M^2(\Delta_1)}{\sqrt{M}}, E_M\Delta_1 + 1.96\frac{\sigma_M^2(\Delta_1)}{\sqrt{M}} \right) \tag{A.15}$$

and

$$\Delta_{|y|} \in \left(E_M\Delta_{|y|} - 1.96\frac{\sigma_M^2(\Delta_{|y|})}{\sqrt{M}}, E_M\Delta_{|y|} + 1.96\frac{\sigma_M^2(\Delta_{|y|})}{\sqrt{M}} \right) \tag{A.16}$$

with 95% of confidence.

A.2.2 Comments on duration of elastic phases

In this subsection, we argue that elastic phases can have very small duration.

At the beginning of an elastic phase the velocity of the oscillator is equal to zero. Hence, the exact solution of (A.10) $y(t)$ has locally the dynamics of the Brownian motion with non zero mean. Consequently, due to the properties of the Brownian motion, elastic phases may have very small duration.

Let us prove that the velocity is positive as well as negative on any interval of time for an elastic phase starting in $(0, Y)$ (resp. $(0, -Y)$) .

Proposition 18. *We have*

$$\forall h > 0, \quad \mathbb{P}\left(\max_{t\in[0,h]} y(t) > 0 | (y(0), z(0)) = (0, Y) \right) = 1,$$

and similarly,

$$\forall h > 0, \quad \mathbb{P}\left(\max_{t\in[0,h]} y(t) < 0 | (y(0), z(0)) = (0, -Y) \right) = 1.$$

Proof. Suppose $y(0) = 0, z(0) = Y$, if the process goes into elastic regime, then $(y(t), z(t))$ satisfies (A.24) otherwise $(y(t), z(t))$ goes into plastic regime and satisfies (A.25). In both cases, we can write

$$y(t) = -k\int_0^t e^{-c_0(t-s)} z(s)\mathrm{d}s + \int_0^t e^{-c_0(t-s)}\mathrm{d}w(s). \tag{A.17}$$

Let us introduce an auxiliary process $\bar{y}(t)$ statisfying the following (SDE):

$$\mathrm{d}\bar{y}(t) = -\big(c_0\bar{y}(t) + kY\big)\mathrm{d}t + \mathrm{d}w(t); \quad \bar{y}(0) = y(0).$$

It easily follows

$$\bar{y}(t) = -k\int_0^t e^{-c_0(t-s)}Y\mathrm{d}s + \int_0^t e^{-c_0(t-s)}\mathrm{d}w(s). \tag{A.18}$$

From (A.17) and (A.18), we obtain on the interval $[0,t[$,

$$\bar{y}(t) - y(t) = -k\int_0^t (Y - z(s))\exp(-c_0(t-s))\mathrm{d}s \leqslant 0.$$

Hence,

$$\{\bar{y}(t) > 0\} \subset \{y(t) > 0\}$$

and

$$\mathbb{P}(\bar{y}(t) > 0) \leqslant \mathbb{P}(y(t) > 0).$$

An explicit formula of $\bar{y}(t)$ is given by

$$\bar{y}(t) = m(t) + s(t)G \quad \text{in law}$$

with

$$m(t) := -\frac{kY}{c_0}(1 - e^{-c_0t}); \quad s^2(t) := \frac{1}{2c_0}(1 - e^{-2c_0t}); \quad G \sim \mathcal{N}(0,1).$$

We also obtain

$$\mathbb{P}(\bar{y}(t) > 0) = \mathbb{P}\left(G > -\frac{m(t)}{s(t)}\right). \tag{A.19}$$

As for any $\epsilon > 0$, we have

$$0 \leqslant -\frac{m(t)}{s(t)} = kY\sqrt{\frac{2}{c_0}}\sqrt{\tanh(\frac{c_0t}{2})} \leqslant C\sqrt{\tanh(\frac{c_0\epsilon}{2})} \quad \text{if} \quad 0 \leqslant t \leqslant \epsilon$$

we obtain

$$\mathbb{P}\left(G > -\frac{m(t)}{s(t)}\right) \geqslant \mathbb{P}\left(G > C\sqrt{\tanh(\frac{c_0\epsilon}{2})}\right), \quad \text{if} \quad 0 < t < \epsilon. \tag{A.20}$$

Finally, denoting $c_\epsilon := \frac{1}{\sqrt{2\pi}}\int_{C\sqrt{\tanh(\frac{c_0\epsilon}{2})}}^{+\infty} e^{-\frac{x^2}{2}}\mathrm{d}x > 0$, we have shown that

$$\mathbb{P}(\bar{y}(t) > 0) \geqslant c_\epsilon$$

so that, we have

$$\mathbb{P}(y(t) > 0) \geqslant c_\epsilon$$

and Kolmogorov's zero-one law [25] implies the result. □

A.2.3 Approximation of the second marginal of m

In this subsection, we consider the numerical resolution of problem (P_λ). The main advantage in considering PDEs is to treat the case of numerically unfiltered white noise. Thus, this method is very competitive in comparison to numerical probabilistic algorithms (see remark 10 in Appendix). Concentration of micro-elastic phasing must appear in temporal averaging of $(y(t), z(t))$ on long time period. Let f be a measurable function satisfying (A.14), we recall that ergodicity implies

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(y(s), z(s))\mathrm{d}s = \int_{D^-} m(y,-Y)f(y,-Y)\mathrm{d}y + \int_{D^+} m(y,Y)f(y,Y)\mathrm{d}y.$$
$$+ \int_D m(y,z)f(y,z)\mathrm{d}y\mathrm{d}z,$$

Consequently, m has to be affected by this phenomenon. So, we are interested in computing the second marginal of m in elastic phase that is

$$U : s \to \int_{-\infty}^{+\infty} m(y,s)\mathrm{d}y, \quad s \in (-Y, Y).$$

For this purpose, we introduce

- $U_n(s)$ to approximate $U(s)$,

$$U_n(s) := \int_{-\infty}^{+\infty}\int_{-Y}^{Y} m(y,z)\chi_n(z-s)\mathrm{d}z\mathrm{d}y.$$

- $W_n(s)$ (the derivative of $U_n(s)$) to approximate $U'(s)$ which is useful to compute sensibility to the threshold

$$W_n(s) := n\int_{-\infty}^{+\infty}\int_{-Y}^{Y} m(y,z)(z-s)\chi_n(z-s)\mathrm{d}z\mathrm{d}y,$$

where χ_n is an approximation of the Dirac function

$$\chi_n(x) := \sqrt{\frac{n}{2\pi}}\exp(-\frac{nx^2}{2}), \quad n \text{ is sufficiently large.} \tag{A.21}$$

Proposition 19. *$\forall s \in (-Y,Y)$, $U_n(s), W_n(s)$ are converging sequences when n goes to ∞. Furthermore, they can be expressed by the behavior at zero of the problem (P_λ) with $\chi_n(z-s)$ and $(z-s)\chi_n(z-s)n$ respectively at the right hand side. More precisely, consider $\tilde{u}$ such that*

$$\begin{cases} \lambda\tilde{u} - \frac{1}{2}\tilde{u}_{yy} + (c_0 y + kz)\tilde{u}_y - y\tilde{u}_z &= f(y,z) & y\in\mathbb{R}, \quad |z| < Y, \\ \lambda\tilde{u} - \frac{1}{2}\tilde{u}_{yy} + (c_0 y + kY)\tilde{u}_y &= f(y,Y) & y > 0, \quad z = Y, \\ \lambda\tilde{u} - \frac{1}{2}\tilde{u}_{yy} + (c_0 y - kY)\tilde{u}_y &= f(y,-Y) & y < 0, \quad z = -Y. \end{cases}$$

Then

- $\lim_{\lambda\to 0}\lambda\tilde{u} = U_n(s)$, *if* $f(y,z) = \chi_n(z-s)$ *or*
- $\lim_{\lambda\to 0}\lambda\tilde{u} = W_n(s)$, *if* $f(y,z) = n(z-s)\chi_n(z-s)$.

Proof. As m is assumed sufficiently regular inside D, we deduce convergence of the sequences of Dirac's approximation $U_n(s) \to U(s)$ and $W_n(s) \to U'(s)$ as $n \to \infty$. □

Computational results on the invariant measure

As we have in mind to calibrate formula (A.5), minima of U are of interest. In Figure A.2, A.3 and A.4 (left) computations of pdf of z show up a significant concentration of $\lim_{t\to\infty} z(t)$ in the neighborhood of plastic boundaries. So, in Figures A.2,A.3,A.4 (right) and A.5, points of inflection of U' are located. For applications, we believe that we located thresholds $s^{\pm}$ sufficiently far away from the boundary. Indeed, note that $s^{\pm}$ are near but not on the boundary, thereby justifying existence of micro-elastic phasing. This phenomenon is also present in the probabilistic numerical simulation of U_n (not shown here) but it is quantified much more accurately with the PDEs related to the invariant measure.

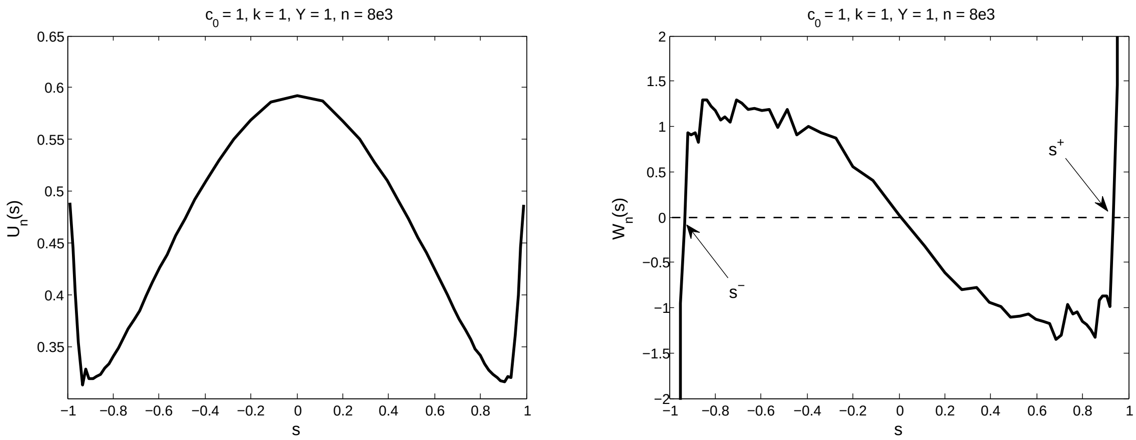


Figure A.2: $U_n(s), W_n(s), s \in (-Y, Y)$ and location of $s^{\pm}, Y = 1$.

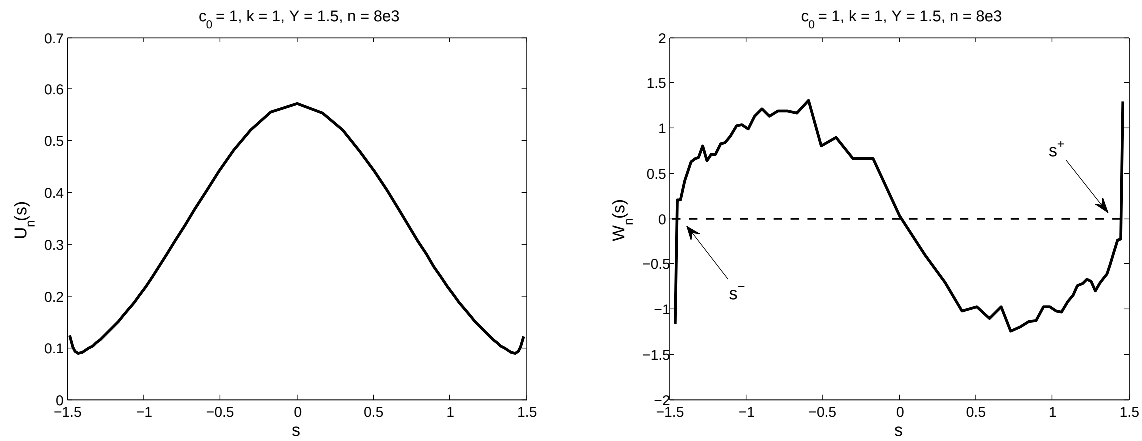


Figure A.3: $U_n(s), W_n(s), s \in (-Y, Y)$ and location of $s^{\pm}, Y = 1.5$.

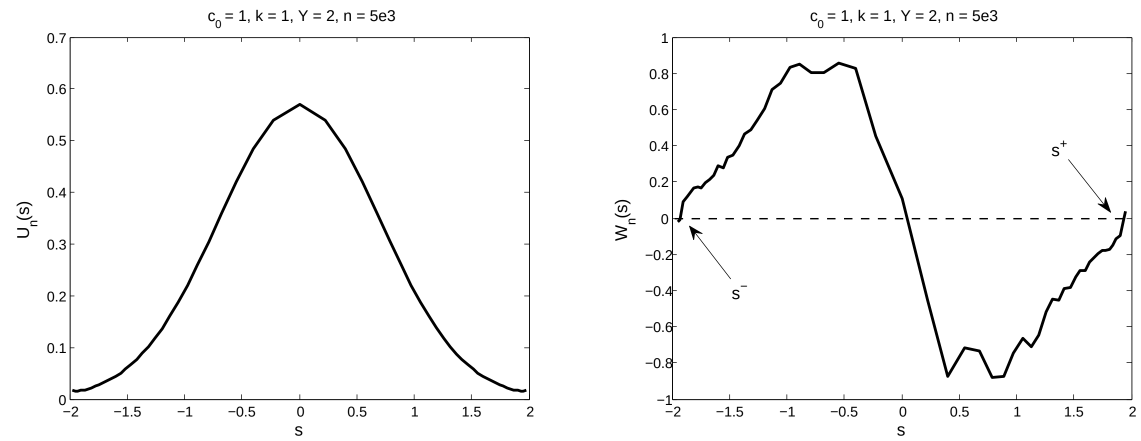


Figure A.4: $U_n(s), W_n(s), s \in (-Y, Y)$ and location of $s^{\pm}, Y = 2$.

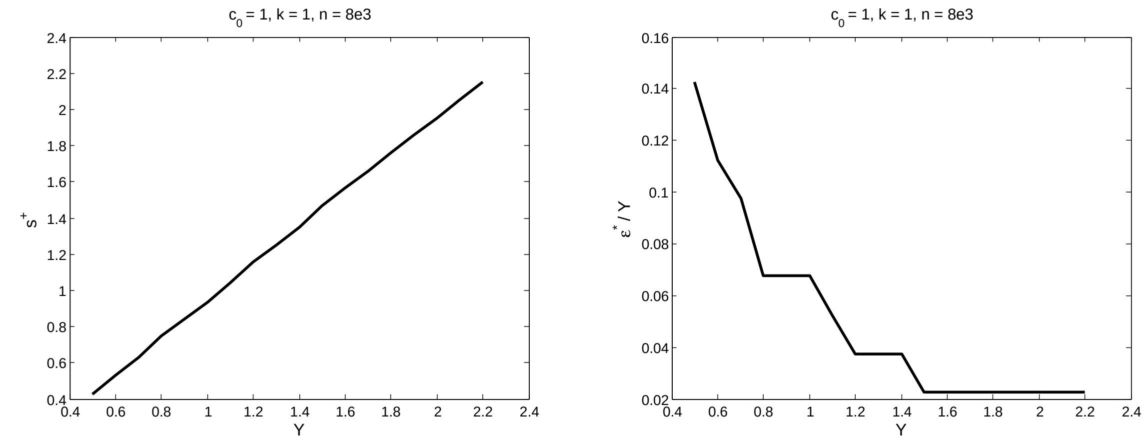


Figure A.5: Y vs s^+ at left; Y vs $\epsilon^\star/Y$ at right.

A.3 Engineering application of the invariant measure to plastic deformations

In this section, we present an empirical approach to compute frequency and statistics of plastic deformations.

A.3.1 Approximations of the frequency of threshold crossings

In this subsection, we are interested in computing "Rice's formula" on $m(y, z)$, so we consider the numerical resolution of (A.13) in a similar manner as we did for the second marginal of m. So, we are interested in

$$V^+ : s \to \int_0^{+\infty} y m(y, s) \mathrm{d}y, \quad V^- : s \to -\int_{-\infty}^0 y m(y, s) \mathrm{d}y, \quad s \in (-Y, Y).$$

Then, we introduce $V_n^{\pm}(s)$ to approximate $V^{\pm}(s)$,

$$\begin{aligned} V_n^+(s) &:= \int_0^{+\infty}\int_{-Y}^{Y} ym(y,z)\chi_n(z-s)\mathrm{d}y\mathrm{d}z \\ V_n^-(s) &:= -\int_{-\infty}^{0}\int_{-Y}^{Y} ym(y,z)\chi_n(z-s)\mathrm{d}y\mathrm{d}z \end{aligned} \tag{A.22}$$

where χ_n has been defined in (A.21).

Proposition 20. *$\forall s \in (-Y,Y)$, $V_n^+(s)$ and $V_n^-(s)$ are converging sequences when n goes to ∞. Furthermore, they can be expressed by the behavior at zero of the problem (P_λ) with $|y|\mathbf{1}_{\{y>0\}}\chi_n(z-s)$ and $|y|\mathbf{1}_{\{y<0\}}\chi_n(z-s)$ respectively at the right hand side. More precisely, consider $\tilde{u}$ such that*

$$\begin{cases} \lambda\tilde{u} - \frac{1}{2}\tilde{u}_{yy} + (c_0 y + kz)\tilde{u}_y - y\tilde{u}_z = f(y,z) & y\in\mathbb{R},\quad |z|<Y, \\ \lambda\tilde{u} - \frac{1}{2}\tilde{u}_{yy} + (c_0 y + kY)\tilde{u}_y = f(y,Y) & y>0,\quad z=Y, \\ \lambda\tilde{u} - \frac{1}{2}\tilde{u}_{yy} + (c_0 y - kY)\tilde{u}_y = f(y,-Y) & y<0,\quad z=-Y. \end{cases}$$

Then

- $\lim_{\lambda\to 0}\lambda\tilde{u} = V_n^+(s)$, *if* $f(y,z) = |y|\mathbf{1}_{\{y>0\}}\chi_n(z-s)$ *or*
- $\lim_{\lambda\to 0}\lambda\tilde{u} = V_n^-(s)$, *if* $f(y,z) = |y|\mathbf{1}_{\{y<0\}}\chi_n(z-s)$.

Proof. Similar to proof of proposition 19: As $(y,z)\to ym(y,z)$ is assumed sufficiently regular inside D, we deduce convergence of the sequences of Dirac's approximation $V_n^{\pm}(s)\to V^{\pm}(s)$ as $n\to\infty$. □

Computational results related to the frequency of deformation

Numerical results based on formulas (A.22) are shown in Figure A.6; they are in good agreement with the probabilistic simulation. Thus, "Rice's formula" is empirically valid for threshold $s = s^-, s^+$.

As mentioned in the introduction, the mean frequency of plastic deformations can be estimated using an appropriate threshold close to the elastic limit $+Y$ (resp. $-Y$), a threshold crossed by $z(t)$ with positive (resp. negative) slope corresponds to only one plastic deformation. The appropriate thresholds are $s^{\pm} = \pm(Y-\epsilon^\star)$ such that the frequency of threshold crossings does exclude the micro-elastic phasing (cf. figure A.6). The mean frequency of enlarged plastic deformations $\nu(Y,\epsilon^\star)$ is given by formula (A.9) which is equal to

$$2\int_0^{\infty} ym(y,-Y+\epsilon^\star)\mathrm{d}y, \tag{A.23}$$

because of the symmetry of the problem.

Remark 9. *For the non-linear case, it is not correct to apply directly a similar expression to Rice's formula (A.8) to $s=\pm Y$ for the frequency of plastic deformations since $m(y,\pm Y)$ is the velocity probability distribution related to the plastic component. Therefore, no informations concerning threshold crossing of the plastic boundaries by $z(t)$ can be accessed in this way.*

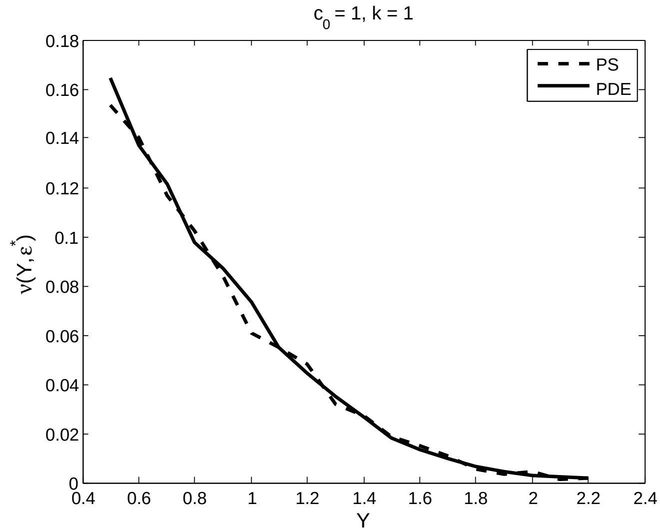


Figure A.6: The frequency of plastic deformations $\nu(Y, \epsilon^\star)$ versus Y the elasto-plastic bound. Comparison between the result given by the probabilistic simulation (PS) and the result given by solving PDEs.

A.3.2 Empirical approach to statistics of plastic deformation

Relying on formula (A.5) and $\epsilon^\star$, we consider the following estimator of mean statistic of plastic deformation, for any function $f(y, z)$ satisfying (A.14)

$$\Delta_f(Y, \epsilon^\star) = \frac{\int_{-\infty}^{0} f(y, -Y)m(y, -Y)\mathrm{d}y + \int_0^\infty f(y, Y)m(y, Y)\mathrm{d}y}{\int_0^\infty ym(y, -Y + \epsilon^\star)\mathrm{d}y}.$$

Finally, we consider the following quantities

$$\Delta_1(Y, \epsilon^\star) = \frac{\int_0^\infty m(y, Y)\mathrm{d}y}{\int_0^\infty ym(y, -Y + \epsilon^\star)\mathrm{d}y} \quad \text{and} \quad \Delta_{|y|}(Y, \epsilon^\star) = \frac{\int_0^\infty ym(y, Y)\mathrm{d}y}{\int_0^\infty ym(y, -Y + \epsilon^\star)dy}$$

to approximate the mean duration and the mean absolute quantity of plastic deformation.

Computational results related to statistic of plastic deformations

Shown in Figures A.7 and A.8 the curves which could be useful for engineering purpose.

A.4 Conclusion

The invariant measure has been used in this paper to study the statistical properties of the plastic deformation of an elastic perfectly plastic oscillator under standard white noise excitation, by means of approximation of the second marginal and "Rice's formula". From our study on micro-elastic phasing, we have deduced a useful criterion for engineers to discard this phenomenon and to compute statistics of plastic deformation. This work has a theoretical interest

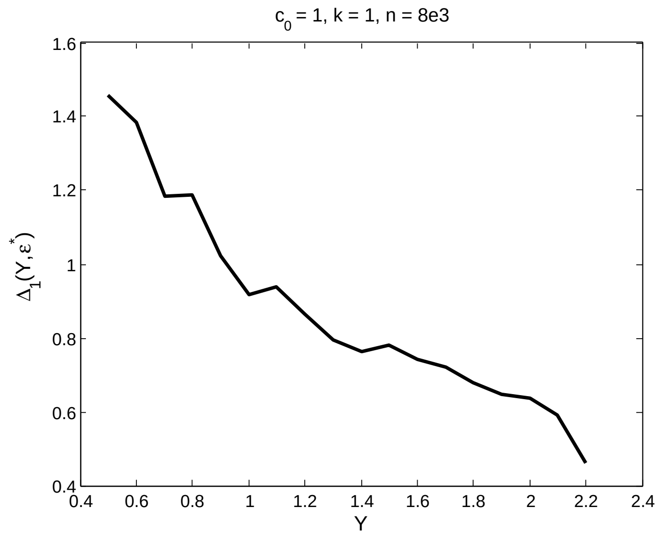


Figure A.7: $\Delta_1(Y, \epsilon^{\star})$ versus Y the elasto-plastic bound.

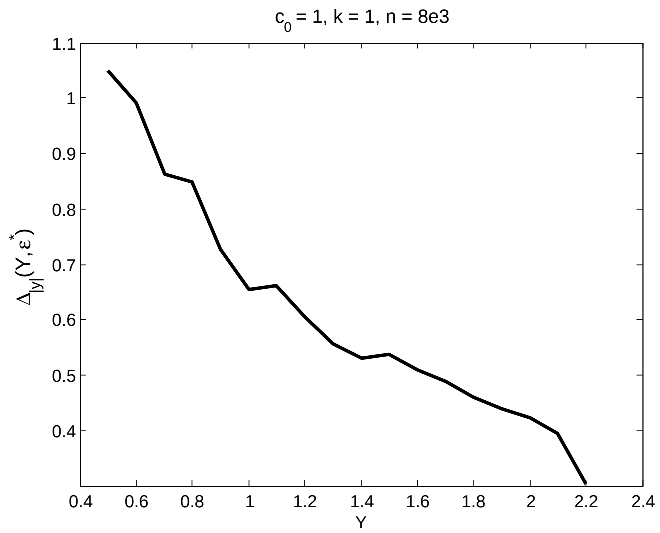


Figure A.8: $\Delta_{|y|}(Y, \epsilon^{\star})$ vs Y the elasto-plastic bound.

and contributes to a further insight into the physics of the problem.

In a future work, we aim to introduce jumps (at the phase transition from plastic state to elastic state) in the dynamics of $(y(t), z(t))$, in order to overcome difficulties related to micro-elastic phasing. For such a model, $\epsilon^{\star}$ appears to be a good candidate for the size of the jump.

A.5 Appendix

A.5.1 Appendix: Probabilistic algorithm of resolution of the SVI

We recall from [3] a discrete approximation of the solution to the SVI. We consider explicit known formula related to elastic and plastic states. They are applied on random time intervals which are determined following the trajectory. In elastic state, the process $(y(t), z(t))$ satisfies:

$$\left\{\begin{array}{rcl} \mathrm{d}y(t) & = & -(c_0 y(t) + kz(t))\mathrm{d}t + \mathrm{d}w(t), \\ \mathrm{d}z(t) & = & y(t)\mathrm{d}t, \end{array}\right. \tag{A.24}$$

whereas in plastic state we have $z(t) = \pm Y$ and $y(t)$ satisfies

$$\left\{\begin{array}{rcl} \mathrm{d}y(t) & = & -(c_0 y(t) + kz(t))\mathrm{d}t + \mathrm{d}w(t), \\ \mathrm{d}z(t) & = & 0. \end{array}\right. \tag{A.25}$$

Then, there is an explicit formula to the elastic state. Let $(y(0), z(0)) = (y, z)$ and $\omega := \frac{\sqrt{4k-c_0^2}}{2}$. Note that the condition $4k > c_0^2$ is needed so that $(y(t), z(t))$ have real valued solutions. We have

$$\left\{\begin{array}{rcl} y(t) & = & -\frac{c_0}{2}z(t) + e^{-\frac{c_0 t}{2}}\{-\omega z \sin(\omega t) + (y + \frac{c_0}{2}z)\cos(\omega t)\} \\ & & + \int_0^t e^{-\frac{c_0}{2}(t-s)} \cos(\omega(t-s))\mathrm{d}w(s), \\ z(t) & = & e^{\frac{-c_0 t}{2}}\{z\cos(\omega t) + \frac{1}{\omega}(y + \frac{c_0}{2}z)\sin(\omega t)\} \\ & & + \frac{1}{\omega}\int_0^t e^{-\frac{c_0}{2}(t-s)} \sin(\omega(t-s))\mathrm{d}w(s). \end{array}\right. \tag{A.26}$$

$y(t)$ is a gaussian variable of mean $e_y(t, z, y)$ and variance $\sigma_y^2(t)$, where

$$\begin{array}{rcl} e_y(t, y, z) & = & -\frac{c_0}{2}e_z(t, y, z) + e^{-\frac{c_0 t}{2}}\{-\omega z \sin(\omega t) + (y + \frac{c_0}{2}z)\cos(\omega t)\}, \\ \sigma_y^2(t) & = & \int_0^t e^{-c_0 s}\cos^2(\omega s)\mathrm{d}s - \frac{c_0^2}{4\omega^2}\int_0^t e^{-c_0 s}\sin^2(\omega s)\mathrm{d}s - \frac{c_0}{2\omega^2}e^{-c_0 t}\sin^2(\omega t). \end{array}$$

$z(t)$ is a gaussian variable of mean $e_z(t, z, y)$ and variance $\sigma_z^2(t)$, where

$$\begin{array}{rcl} e_z(t, y, z) & = & e^{\frac{-c_0 t}{2}}\{z\cos(\omega t) + \frac{1}{\omega}(y + \frac{c_0}{2}z)\sin(\omega t)\}, \\ \sigma_z^2(t) & = & \frac{1}{\omega^2}\int_0^t e^{-c_0 s}\sin^2(\omega s)\mathrm{d}s. \end{array}$$

The correlation between $y(t)$ and $z(t)$ are given by

$$\sigma_{yz}(t) = \frac{1}{2\omega}\int_0^t e^{-c_0 s}\sin(2\omega s)\mathrm{d}s - \frac{c_0}{2\omega^2}\int_0^t e^{-c_0 s}\sin^2(\omega s)\mathrm{d}s.$$

In plastic state, we also have an explicit formula. Let $(y(0), z(0)) = (y, \pm Y)$ then

$$\left\{\begin{array}{rcl} y(t) & = & ye^{-c_0 t} \mp \frac{kY}{c_0}(1 - e^{-c_0 t}) + \frac{e^{-c_0 t}}{\sqrt{2c_0}}w(e^{2c_0 t} - 1), \\ z(t) & = & \pm Y. \end{array}\right. \tag{A.27}$$

Considering explicit solutions, a C code has been written to simulate $(y(t), z(t))$. Let $T > 0, N \in \mathbb{N}$, and $(t_n)_{n=0..N}$ be a family of time which disretize $[0, T]$, such that $t_n = n\delta t$ where $\delta t := \frac{T}{N}$.

We set $\Sigma \in \mathcal{M}_{2,2}(\mathbb{R}^2)$ such that

$$\Sigma\Sigma^T = \begin{pmatrix} \sigma_y^2(\delta t) & \sigma_{yz}(\delta t) \\ \sigma_{yz}(\delta t) & \sigma_z^2(\delta t) \end{pmatrix}. \tag{A.28}$$

Let $(G_{n,m})_{n=0..N,m=1,2}$ a family of independent gaussian variables $\mathcal{N}(0,1)$. Gaussian variables are generated using Box-Muller formula [23] and the C function `random()`. Initialize $(y_0^{\delta t}, z_0^{\delta t}) = (y, z)$, the finite difference scheme for (A.10) is written in the following manner:

For $n = 0, 1, ..$ with $\theta_0^{\delta t} = \tau_0^{\delta t} = 0$, we define two sequences of δt-stopping time:

$$\begin{cases} \theta_{n+1}^{\delta t} & := & \inf\{t_k > \tau_n^{\delta t} \mid |z_{t_k}^{\delta t}| = Y\} \\ \tau_{n+1}^{\delta t} & := & \inf\{t_k > \theta_{n+1}^{\delta t} \mid |z_{t_k}^{\delta t}| < Y\} \end{cases} \tag{A.29}$$

- When $t_k \in [\tau_n^{\delta t}, \theta_{n+1}^{\delta t}[$, we have $|z_{t_k}^{\delta t}| < Y$, we set

$$\begin{pmatrix} \tilde{y}_{k+1} \\ \tilde{z}_{k+1} \end{pmatrix} := \begin{pmatrix} e_y(\delta t, y(t_k), z(t_k)) \\ e_z(\delta t, y(t_k), z(t_k)) \end{pmatrix} + \Sigma \begin{pmatrix} G_{k,1} \\ G_{k,2} \end{pmatrix} \tag{A.30}$$

  - If $|\tilde{z}_{k+1}| < Y$ then

$$\begin{pmatrix} y_{t_{k+1}}^{\delta t} \\ z_{t_{k+1}}^{\delta t} \end{pmatrix} := \begin{pmatrix} \tilde{y}_{k+1} \\ \tilde{z}_{k+1} \end{pmatrix} \tag{A.31}$$

  - If $|\tilde{z}_{k+1}| \geqslant Y$ then

$$\begin{pmatrix} y_{t_{k+1}}^{\delta t} \\ z_{t_{k+1}}^{\delta t} \end{pmatrix} := \begin{pmatrix} \tilde{y}_{k+1} \\ \sigma_{k+1} Y \end{pmatrix} \tag{A.32}$$

  where $\sigma_{k+1} := sign(\tilde{z}_{k+1})$.

- When $t_k \in [\theta_{n+1}^{\delta t}, \tau_{n+1}^{\delta t}[$, we have $\sigma_k z_{t_k}^{\delta t} = Y$,

  - If $\sigma_k y_{t_k}^{\delta t} > 0$, then we set

$$\begin{pmatrix} \tilde{y}_{k+1} \\ \tilde{z}_{k+1} \end{pmatrix} := \begin{pmatrix} y_{t_k}^{\delta t} e^{-c_0 \delta t} \mp \frac{kY}{c_0}(1 - e^{-c_0 \delta t}) + e^{-c_0 \delta t} \sqrt{\frac{e^{2c_0 \delta t} - 1}{2c_0}} G_{k,1} \\ \sigma_k Y \end{pmatrix} \tag{A.33}$$

    * If $\sigma_k \tilde{y}_{k+1} > 0$, then

$$\begin{pmatrix} y_{t_{k+1}}^{\delta t} \\ z_{t_{k+1}}^{\delta t} \end{pmatrix} := \begin{pmatrix} \tilde{y}_{k+1} \\ \sigma_k Y \end{pmatrix} \tag{A.34}$$

    * If $\sigma_k \tilde{y}_{k+1} < 0$, then

$$\begin{pmatrix} y_{t_{k+1}}^{\delta t} \\ z_{t_{k+1}}^{\delta t} \end{pmatrix} := \begin{pmatrix} 0 \\ \sigma_k Y \end{pmatrix} \tag{A.35}$$

  - Else, we set

$$\begin{pmatrix} \tilde{y}_{k+1} \\ \tilde{z}_{k+1} \end{pmatrix} := \begin{pmatrix} e_y(\delta t, 0, \sigma_k Y) \\ e_z(\delta t, 0, \sigma_k Y) \end{pmatrix} + \Sigma \begin{pmatrix} G_{k,1} \\ G_{k,2} \end{pmatrix} \tag{A.36}$$

    * If $|\tilde{z}_{k+1}| < Y$, then

$$\begin{pmatrix} y_{t_{k+1}}^{\delta t} \\ z_{t_{k+1}}^{\delta t} \end{pmatrix} := \begin{pmatrix} \tilde{y}_{k+1} \\ \tilde{z}_{k+1} \end{pmatrix} \tag{A.37}$$

* If $|\tilde{z}_{k+1}| \geqslant Y$, then

$$\begin{pmatrix} y^{\delta t}_{t_{k+1}} \\ z^{\delta t}_{t_{k+1}} \end{pmatrix} := \begin{pmatrix} \tilde{y}_{k+1} \\ \sigma_k Y \end{pmatrix} \tag{A.38}$$

Remark 10. *Simulations do not deal with a true white noise. Indeed, because of the step of discretization, the noise is filtered. The ideal case would be to take δt as small as possible, but our probabilistic algorithm is too slow to get satisfactory results in this way. Thus we cannot consider δt below than 10^{-6}.*

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List of Figures

List of Tables

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Résumé

Cette thèse traite des inéquations variationnelles stochastiques et de leurs applications aux vibrations de structures mécaniques. On considère d'abord un algorithme numérique déterministe pour obtenir le régime stationnaire d'une inéquation variationnelle stochastique modélisant un oscillateur élasto-plastique excité par un bruit blanc. Une famille de solutions d'équations aux dérivées partielles définissant la mesure invariante par dualité est étudiée comme alternative à la simulation probabiliste. Puis, nous présentons une nouvelle caractérisation de l'unique mesure invariante. Dans ce contexte, nous montrons une relation liant des problèmes non-locaux et des problèmes locaux en introduisant la définition des cycles courts. Dans un cadre orienté vers les applications, nous démontrons que la variance de la déformation plastique croît linéairement avec le temps et nous caractérisons rigoureusement le coefficient de dérive en introduisant la définition des cycles longs. Dans la suite, nous étudions un processus approché de la solution de l'inéquation comportant des sauts aux instants de transition de l'état plastique vers l'état élastique. Nous prouvons que la solution approchée converge sur tout intervalle de temps fini vers la solution de l'inéquation, lorsque la taille du saut tend vers 0. Ensuite, nous définissons une inéquation variationnelle stochastique pour modéliser un oscillateur élasto-plastique excité par un bruit blanc filtré. Nous prouvons la propriété ergodique du processus sous-jacent et nous caractérisons sa mesure invariante. Nous étendons la méthode de A.Bensoussan et J.Turi avec une difficulté supplémentaire due à l'accroissement de la dimension. Finalement, dans un chapitre orienté vers l'expérimentation numérique, nous mettons en évidence par les simulations probabilistes le phénomène de phases micro-élastiques. Leur impact concerne des grandeurs utiles à l'ingénieur comme la fréquence des déformations plastiques. Un critère empirique qui peut être utile à l'ingénieur est fourni afin ne pas prendre en compte les phases micro-élastiques et ainsi évaluer d'une façon réaliste, à partir de la mesure invariante, les statistiques de la déformation plastique d'un oscillateur elasto-plastique excité par un bruit blanc.

Mots-clés : inéquations variationnelles stochastiques, équations aux dérivées partielles avec des conditions non-locales, vibrations aléatoires, diffusion ergodique.

Abstract

This work is devoted to stochastic variational inequalities and their applications to random vibrations of mechanical structures. First, an efficient method for obtaining numerical solutions of a stochastic variational inequality modeling an elasto-plastic oscillator with noise is considered. Since Monte Carlo simulations for the underlying stochastic process are too slow, as an alternative, approximate solutions of the partial differential equation defining the invariant measure of the process are studied. Next, we present a new characterization of the invariant measure. The key finding is the connection between nonlocal partial differential equations and local partial differential equations which can be interpreted with short cycles of the Markov process solution of the stochastic variational inequality. For engineering applications, we prove that plastic deformation for an elasto-perfectly-plastic oscillator has a variance which increases linearly with time and we characterize the corresponding drift coefficient by defining long cycles behavior of the Markov process solution of the stochastic variational inequality. A major advantage of stochastic variational inequality is to overcome the need to describe the trajectory by phases (elastic or plastic). This is useful, since the sequence of phases cannot be characterized easily. However, it remains important to have informations on these phases. In order to reconcile these contradictory issues, we introduce an approximation of stochastic variational inequalities by imposing artificial small jumps between phases allowing a clear separation of the phases. We prove that the approximate solution converges on any finite time interval, when the size of jump tends to 0. Then to study a more general case, a stochastic variational inequality is proposed to model an elasto-plastic oscillator excited by a filtered white noise. We prove the ergodic property of the process and characterize the corresponding invariant measure. This extends Bensoussan-Turi's method with a significant additional difficulty of increasing the dimension. Finally, in a last chapter oriented to numerical experiments, we exhibit by probabilistic simulations the phenomenon of micro-elastic phases. The main difficulty related to micro-elastic phasing is that they interfere on quantities of interest such that frequency of plastic deformations. An interesting criterion is provided which could be useful in engineering problems to discard micro-elastic phases and to evaluate statistics of plastic deformations of an elasto-plastic oscillator white noise excited.

Keywords: stochastic variational inequalities, partial differential equations with nonlocal conditions, random vibration, ergodic diffusions.