Damage Description with Related Crack Initiation and Propagation Conditions

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Abstract. The damage accumulation condition expressed in terms of traction components on a physical plane is considered for both monotonic and cyclic loading conditions. The crack initiation is assumed to correspond to a critical value of damage parameter on a maximum damage plane. A non-local condition is formulated for singular stress or strain regimes. The model is applied to predict damage distribution within the element for cyclic loading condition; in particular for combined torsion and bending. The damage tensors are introduced to describe the predicted damage distribution.

1. INTRODUCTION

The present paper is concerned with the description of damage evolution for brittle materials such as rock, concrete, ceramics, some glassy polymers, and for metals under variable loading inducing fatigue crack initiation and propagation within the nominally elastic regime. The macroscopic plastic deformation can then be neglected and the inelastic strain is associated with the microcracking process. For high-cycle fatigue, the microplastic effects occur at the dislocation level and the macroscopic response is usually treated as elastic with negligible effect of microcracks on the effective elastic moduli.

In formulating constitutive models the damage is usually described by scalar or tensor state variables representing average crack density and orientation within the macroscopic element. The evolution rules for damage and compliance are then formulated. Starting from the early Kachanov work [1] who defined the scalar measure of damage, the subsequent investigators used different measures, cf. Lemaitre [2], Dragon and Mróz [3], Krajcinovic [4], Murakami [5], Onat and Leckie [6], Simo and Ju [7], Chaboche [8] and others. The anisotropic crack or fabric distribution by various order tensors was thoroughly discussed by Onat and Leckie [6], Kanatani [9], and recently by Lubarda and Krajcinovic [10].

In the present work, the damage distribution within the material element will be described by providing the rule of distribution of a scalar damage variable $\omega_n(\underline{n})$ on planes of varying orientation $\underline{n}$. The macrocrack initiation is assumed to occur along the plane of maximal damage value. For the specified distribution of damage the respective tensor measure of crack density distribution can be identified. The elastic compliance variation due to damage can be neglected for multiaxial fatigue problems. However, when needed, this variation can be calculated by introducing the interfacial strain components due to damage and averaging them over all plane orientations.

2. NON-LOCAL STRESS OR STRAIN CONDITION OF BRITTLE FAILURE

To provide uniform treatment of crack initiation and propagation from singular and regular stress concentrations, the non-local stress condition was proposed by Seweryn and Mróz [12] and applied to predict both critical load value for crack initiation and also orientation of crack propagation. The fatigue

Figure 1. Physical plane Δ associated with the local coordinate system (ξ_1, ξ_2, ξ_3) and the global system (x_1, x_2, x_3) .

damage accumulation and crack propagation was studied using this conditions by Seweryn and Mróz [13,14].

Consider a local plane Δ specified by a position vector $\underline{x}_0$ and a unit normal vector $\underline{n}$, Figure 1. The traction vector $\underline{t} = \underline{\sigma} \underline{n}$ acting on the physical plane can be expressed in the local coordinate system by

$$\underline{\Sigma}(\tau_{n1}, \tau_{n2}, \sigma_n) = \underline{Q} \underline{t} = \underline{Q} \underline{\sigma} \underline{n} \quad (1)$$

where $\underline{Q}$ is an orthogonal tensor of transformation from the global system (x_1, x_2, x_3) to the local system (ξ_1, ξ_2, ξ_3) with ξ_3 following the normal to the plane. The shear stresses within the plane follow coordinate axes ξ_1 and ξ_2 and the resultant shear stress equals

$$\tau_n = (\tau_{n1}^2 + \tau_{n2}^2)^{1/2} \quad (2)$$

The local stress condition of brittle failure is formulated in terms of the damage stress function on any physical plane, thus the crack initiation condition is

$$R_{f\sigma} = \max_{(\underline{n}, \underline{x}_0)} R_{\sigma}(\sigma_n / \sigma_c, \tau_n / \tau_c) = 1 \quad (3)$$

where $0 \leq R_{f\sigma} \leq 1$ is the brittle failure factor, σ_c and τ_c denote the critical values of normal and shear stresses. The damage stress function is expressed in terms of contact stress components σ_n and τ_n . Let us note that besides the crack initiation condition, we may specify also the variation of R_{σ} with respect to plane orientation and assume that damage distribution follows the variation of R_{σ} .

In the previous study [12], the elliptic condition combined with the shear condition was applied, thus

$$R_{\sigma} = \left[\left(\frac{\langle \sigma_n \rangle}{\sigma_c} \right)^2 + \left(\frac{\tau_n}{\tau_c} \right)^2 \right]^{1/2} \quad (4)$$

where $\langle \sigma_n \rangle = \sigma_n$ for $\sigma_n > 0$ and $\langle \sigma_n \rangle = 0$ for $\sigma_n < 0$. An alternative condition is provided by the Coulomb condition combined with the normal stress condition, thus

$$R_{\sigma} = \min \left\{ \frac{1}{\tau_c} (|\tau_n| + \sigma_n \tan \varphi), \frac{\langle \sigma_n \rangle}{\sigma_c} \right\} \quad (5)$$

Figures 2a,b present these two conditions in the stress plane.

Figure 2. Stress failure functions: a) elliptical condition combined with the maximum shear condition, b) Coulomb condition combined with limit tensile strength condition.

For singular or quasi-singular stress distributions with very large stress gradients, the failure condition is averaged over the area $d_0 \times d_0$, thus

$$\bar{R}_{f\sigma} = \max_{(\underline{n}, \underline{x}_0)} \bar{R}_\sigma \left(\frac{\sigma_n}{\sigma_c}, \frac{\tau_n}{\tau_c} \right) = \max_{(\underline{n}, \underline{x}_0)} \left[\frac{1}{d_0^2} \int_0^{d_0} \int_0^{d_0} R_\sigma \left(\frac{\sigma_n}{\sigma_c}, \frac{\tau_n}{\tau_c} \right) d\xi_1 d\xi_2 \right] = 1 \quad (6)$$

where d_0 specifies the non-locality domain. The value of d_0 can be related to microstructural parameters, for instance, grain size. To make the non-local model equivalent to Griffith-Irwin crack propagation condition $K_I = K_{Ic}$ in Mode I, the value of d_0 can be identified as [11]

$$d_0 = \frac{1}{2\pi} \left(\frac{2K_{Ic}}{\sigma_c} \right)^2 \quad (7)$$

where K_{Ic} is the critical value of the stress intensity factor.

The damage accumulation and temperature effect can be accounted for by assuming that

$$\begin{aligned} \sigma_c &= \sigma_c(T, \omega_{n\sigma}) = \sigma_{c0}(T)(1 - \omega_{n\sigma})^p \\ \tau_c &= \tau_c(T, \omega_{n\tau}) = \tau_{c0}(T)(1 - \omega_{n\tau})^p \end{aligned} \quad (8)$$

where σ_{c0} and τ_{c0} are the failure stresses for the undamaged material.

An alternative condition for failure can be expressed in terms of contact strain components γ_{n1} , γ_{n2} , ε_n on the plane Δ . Introducing the resultant shear stress $\gamma_n = [\gamma_{n1}^2 + \gamma_{n2}^2]^{1/2}$, the local failure condition can be expressed as follows

$$R_{f\varepsilon} = \max_{(\underline{n}, \underline{x}_0)} R_\varepsilon \left(\frac{\varepsilon_n}{\varepsilon_c}, \frac{\gamma_n}{\gamma_c} \right) = 1 \quad (9)$$

where ε_c , γ_c are the critical failure strain values and R_ε is the strain failure function. Similarly to (4), we can postulate

$$R_\varepsilon = \left[\left(\frac{\langle \varepsilon_n \rangle}{\varepsilon_c} \right)^2 + \left(\frac{\gamma_n}{\gamma_c} \right)^2 \right]^{1/2} \quad (10)$$

Figure 3. Stress failure function (4) in the strain plane for $\tau_c/\sigma_c = 1/\sqrt{3}$, $\nu = 0.3$: a) plane stress, b) plane strain state.

The non-local strain failure condition can be formulated analogously to (6), thus

$$\bar{R}_{f\varepsilon} = \max_{(\underline{n}, \underline{x}_0)} \bar{R}_\varepsilon = \max_{(\underline{n}, \underline{x}_0)} \left[\frac{1}{d_0^2} \int_0^{d_0} \int_0^{d_0} R_\varepsilon \left(\frac{\varepsilon_n}{\varepsilon_c}, \frac{\gamma_n}{\gamma_c} \right) d\xi_1 d\xi_2 \right] = 1 \quad (11)$$

It should be noted that the stress and strain conditions are not equivalent since ε_n depends on σ_n and σ_{t1} , σ_{t2} acting within the contact plane and σ_n depends on strain components ε_n , ε_{t1} , ε_{t2} . Figures 3a,b present the stress failure condition (4) in the strain plane for plane stress and plane strain state. It is seen that the elliptical condition is affected by the ratio $\varepsilon_t/\varepsilon_n$.

3. MULTIAXIAL FATIGUE LOADING

Let us now discuss the damage evolution rule in the case of multiaxial fatigue loading within the elastic domain so the macroplastic strains do not occur. The growth of brittle damage is now governed by the relation

$$d\omega_{n\sigma} = d\omega_{n\sigma}(\underline{\Sigma}, d\underline{\Sigma}, \omega_{n\sigma}) \quad (12)$$

The domain of no damage accumulation is specified by the inequality

$$R_{\sigma\sigma} \left(\frac{\sigma_n}{\sigma_o}, \frac{\tau_n}{\tau_o} \right) < 1 \quad (13)$$

and the damage initiation locus corresponds to the value $R_{f\sigma\sigma} = R_{\sigma\sigma} = 1$. Here σ_o and τ_o are the damage initiation stresses in tension and shear. For large values of stress gradient the non-local condition is expressed as follows

$$\bar{R}_{f\sigma\sigma} = \max_{(\underline{n}, \underline{x}_o)} \bar{R}_{\sigma\sigma} = \max_{(\underline{n}, \underline{x}_o)} \left[\frac{1}{d_o^2} \int_0^{d_o} \int_0^{d_o} R_{\sigma\sigma} \left(\frac{\sigma_n}{\sigma_o}, \frac{\tau_n}{\tau_o} \right) d\xi_1 d\xi_2 \right] = 1 \quad (14)$$

Consider the domain Ω^* bounded by damage initiation and stress failure curves, Figure 4. Assume, for simplicity that $\sigma_o = f\sigma_c$, $\tau_o = f\tau_c$. Consider a family of curves

$$R_{\sigma} = \left[\left(\frac{\langle \sigma_n \rangle}{\sigma_c} \right)^2 + \left(\frac{\tau_n}{\tau_c} \right)^2 \right]^{1/2} = \text{const.} \quad (15)$$

For $R_{\sigma} = 1$, the curve coincides with the stress failure locus, for $R_{\sigma} = f$, with the damage initiation locus $R_{\sigma\sigma} = 1$. The damage growth is specified by the following relation

$$d\omega_{n\sigma} = \Psi_{\sigma}(R_{\sigma}) d\hat{R}_{\sigma} \quad (16)$$

where the damage accumulation function Ψ_{σ} can be assumed in the form

$$\Psi_{\sigma}(R_{\sigma}) = A_{\sigma} \left(\frac{R_{\sigma} - R_{\sigma oc}}{1 - R_{\sigma oc}} \right)^{n_{\sigma}} \frac{1}{1 - R_{\sigma oc}} \quad (17)$$

$$R_{\sigma oc} = R_{\sigma} / R_{\sigma o}$$

The material parameters n_{σ} and A_{σ} are specified from experimental data. The increment $d\hat{R}_{\sigma}$ is specified by the formula

$$d\hat{R}_{\sigma} = \begin{cases} dR_{\sigma} & \text{for } dR_{\sigma} \geq 0 \text{ and } R_{\sigma} \geq f \\ 0 & \text{for } dR_{\sigma} < 0 \text{ or } R_{\sigma} < f \end{cases} \quad (18)$$

and

$$dR_{\sigma} = \frac{\partial R_{\sigma}}{\partial \sigma_n} d\sigma_n + \frac{\partial R_{\sigma}}{\partial \tau_n} d\tau_n + \frac{\partial R_{\sigma}}{\partial \omega_{n\sigma}} d\omega_{n\sigma} \quad (19)$$

Neglecting the effect of damage accumulation on function R_{σ} , for the damage condition (4) we have

Figure 4. The brittle failure function and the microplastic damage initiation function. The damage accumulation occurs due to: a) variation of the failure function, b) variation of the stress state

$$dR_{\sigma} = \frac{1}{R_{\sigma}} \left[\frac{\langle \sigma_n \rangle d\sigma_n}{\sigma_c^2} + \frac{\tau_n d\tau_n}{\tau_c^2} \right] \quad (20)$$

An alternative specification of loading-unloading domains can be proposed by introducing two planes moving with stress point [13]

$$\begin{aligned} \phi_1 &= \hat{\sigma}_n - \sigma_n^+(t) = 0 \\ \phi_2 &= \hat{\tau}_n^2 - \tau_n^2(t) = 0 \end{aligned} \quad (21)$$

where $\sigma_n^+(t)$ and $\tau_n(t)$ are the actual values of tensile and shear stresses and damage growth is assumed to occur when

$$\begin{aligned} \phi_1 &= 0, & d\phi_1 &> 0, & \hat{\sigma}_n &= \sigma_n^+(t), & d\hat{\sigma}_n &= d\sigma_n > 0 \\ \phi_2 &= 0, & d\phi_2 &> 0, & \hat{\tau}_n^2 &= \tau_n^2(t), & \hat{\tau}_n d\hat{\tau}_n &= \tau_n d\tau_n > 0 \end{aligned} \quad \text{or} \quad (22)$$

Figure 4b presents four domains of loading-unloading specified by the corner regime at P. The non-local measure of increment of damage function is specified as follows

$$d\bar{R}_{\sigma} = \frac{1}{d^2} \int_0^{\hat{R}_d} \int_0^{\hat{R}_d} d\hat{R}_{\sigma} d\xi_1 d\xi_2 \quad (23)$$

Let us apply the damage accumulation model to study crack initiation and damage distribution in the case of cyclic loading of a cylinder under combined bending and torsion. Assuming that $\sigma_o/\sigma_c = \tau_o/\tau_c = f$ and neglecting the effect of damage on values of σ_c and τ_c , the crack initiation condition is assumed in the form

$$R_d = \max_{(n, x_o)} \omega_{n\sigma} = \max_{(n, x_o)} \left[\int_0^{\hat{R}_d} A_{\sigma}^* \left(\frac{R_{\sigma} - f}{1 - f} \right)^{n_{\sigma}^*} \frac{d\hat{R}_{\sigma}}{1 - f} \right] = 1 \quad (24)$$

Geometrically, for any stress point P within the domain Ω^* , the radial distance of this point from the curve $R_{\sigma o} = 1$ equals PP_o and the radial distance between the curves $R_{\sigma} = 1$ and $R_{\sigma o} = 1$ equals $P_c P_o$. We have then

$$\left(\frac{R_{\sigma} - f}{1 - f} \right)^{n_{\sigma}^*} = \left(\frac{PP_o}{P_c P_o} \right)^{n_{\sigma}^*} \quad (25)$$

Consider a cylindrical element under cyclic loading by the bending and torsional moments, Figure 5. The resulting stress components are the axial stress σ_z and the shear stress $\tau_{r\theta}$. The maximal stress values occur on the cylindrical surface and can be presented as follows

$$\begin{aligned} \sigma_z(t) &= \sigma_a \sin \omega t + \sigma_m \\ \tau_{r\theta}(t) &= \tau_a \sin(\omega t - \delta) + \tau_m \end{aligned} \quad (26)$$

where σ_a , τ_a are the normal and shear stress amplitudes, σ_m , τ_m denote their mean values, and δ is the phase-angle of two stress components. The stress components acting on any plane normal to the cylindrical surface are

Figure 5. Cylindrical element and coordinate system (θ, r, z) .

$$\begin{aligned}\sigma_n &= \sigma_{n0} + k_0 \cos(2\vartheta + 2\beta) \\ \tau_n &= k_0 \sin(2\vartheta + 2\beta)\end{aligned}\quad (27)$$

where

$$\sigma_{n0} = \frac{1}{2} \sigma_z, \quad k_0 = \left[\left(\frac{\sigma_z}{2} \right)^2 + \tau_{z\vartheta}^2 \right]^{1/2}$$

and ϑ denotes the angle of orientation of the plane with respect to circumferential direction, β denotes the orientation of the major principal stress plane. When the reference axes coincide with the principal axes, we set $\beta = 0$ and then $\sigma_n = \sigma_{n0} + k_0 \cos 2\vartheta$, $\tau_n = k_0 \sin 2\vartheta$.

Figure 6. Damage accumulation on the extremal plane during one cycle of bending loading for: a) $\tau_c/\sigma_c = 1/\sqrt{3}$, b) $\tau_c/\sigma_c = \sqrt{3}$,

Figures 6a,b present the damage accumulation on the extremal plane during one cycle of in-phase loading ($\delta = 0$) and for the stress amplitude corresponding to $R_{\sigma} = 0.5$. It is assumed that $n_{\sigma} = 1$, $f = 0.2$, and the bending loading only, $\tau_a = \tau_m = 0$, $\sigma_m = 0$. Figure 6a corresponds to $\sigma_c/\tau_c = 1/\sqrt{3}$ and Figure 6b to $\sigma_c/\tau_c = \sqrt{3}$. The portions of stress cycle for which damage accumulation occurs are marked as thickened segments. It is seen that damage occurs on different portions of the stress cycle and $\omega_{n\sigma}$ attains different values.

Figures 7a,b present the damage accumulation diagrams for the out-of phase combined loading, $\tau_m = \sigma_m = 0$, $\tau_a = \sigma_a$, $\delta = \pi/2$ and for two values of σ_c/τ_c as in previous diagrams.

Figure 8 (referring to the same case as Figure 6) presents rosette diagrams of damage parameter distribution on physical planes inclined at the angle ϑ to the plane normal to cylinder axis, for the bending cycles and for three values of mean and amplitude stress, namely $m = -1$, $m = -0.5$ and $m = 0$. It is seen that the maximal damage plane orientation is essentially affected by the values of m . Figure 9 presents similar diagrams for non-proportional loading, and refers the case of loading illustrated in Figure 7.

Figure 7. Damage accumulation on the extremal plane during one cycle of out-of-phase bending and torsional loading for: a) $\tau_c/\sigma_c = 1/\sqrt{3}$, b) $\tau_c/\sigma_c = \sqrt{3}$,

Figure 8. Rosette diagrams of damage distribution for the bending loading: a) $\tau_c/\sigma_c = 1/\sqrt{3}$, b) $\tau_c/\sigma_c = \sqrt{3}$,

Figure 9. Rosette diagrams of damage distribution for the out-of-phase bending and torsional loading: a) $\tau_c/\sigma_c = 1/\sqrt{3}$, b) $\tau_c/\sigma_c = \sqrt{3}$,

4. DAMAGE TENSOR REPRESENTATION

The scalar damage distribution on physical planes predicted by the present model can be used in order to construct the tensor state variable of damage. In fact, the function $\omega_n(\underline{n})$ can be generated for any loading program. The mean value of damage within the element equals

$$\omega_o = \frac{1}{4\pi} \int_{4\pi} \omega_n(\underline{n}) d\Omega \quad (28)$$

and the maximal value

$$\omega_d = \max_{(\underline{n})} \omega_n(\underline{n}) \quad (29)$$

is used to specify the crack initiation condition. Following the previous work of Kanatani [9] and Lubarda and Krajcinovic [10], the use of second and fourth-order damage tensors will be discussed.

For a second order damage tensor ω_{ij} , the damage measure on the physical plane equals

$$\omega_n(\underline{n}) = \omega_{ij} n_i n_j \quad (30)$$

Consider a tensor ω_{ij}^* obtained from the known distribution $\omega_n(\underline{n})$, namely

$$\omega_{ij}^* = \frac{3}{4\pi} \int_{4\pi} \omega_n(\underline{n}) n_i n_j d\Omega \quad (31)$$

It is easy to show that the tensors ω_{ij} and ω_{ij}^* are interrelated by the equation

$$\omega_{ij} = \frac{5}{2} \omega_{ij}^* - \frac{3}{2} \delta_{ij} \omega_o \quad (32)$$

Thus, the damage tensor ω_{ij} can be identified in terms of ω_{ij}^* and ω_o specified from the known distribution of scalar damage on physical planes.

Figure 10. Rosette diagrams of damage distribution $\omega_n(\vartheta)$ and approximations by the second and fourth order tensors

When the fourth order damage tensor ω_{ijkl} is applied, the damage measure on the plane equals

$$\omega_n(\underline{n}) = \omega_{ijkl} n_i n_j n_k n_l \quad (33)$$

On the other hand, knowing $\omega_n(\underline{n})$, the auxiliary damage tensor ω_{ijkl}^* can be obtained, namely

$$\omega_{ijkl}^* = \frac{5}{4\pi} \int_{4\pi} \omega_n(\underline{n}) n_i n_j n_k n_l d\Omega \quad (34)$$

The tensors ω_{ijkl} and ω_{ijkl}^* are now interrelated by the following equation

$$\omega_{ijkl} = \frac{63}{8} \omega_{ijkl}^* - \frac{35}{4} A_{ijkl}^* + \frac{15}{8} I_{ijkl} \omega_o \quad (35)$$

where

$$\begin{aligned} A_{ijkl}^* &= \frac{1}{6} (\delta_{ij} \omega_{kl}^* + \delta_{kl} \omega_{ij}^* + \delta_{ik} \omega_{jl}^* + \delta_{il} \omega_{jk}^* + \delta_{jk} \omega_{il}^* + \delta_{jl} \omega_{ik}^*) \\ I_{ijkl} &= \frac{1}{3} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \end{aligned} \quad (36)$$

The relation (35) provides the damage tensor ω_{ijkl} in terms of the specified distribution $\omega_n(\underline{n})$. For the plane case, the respective formulae are

$$\begin{aligned} \omega_o &= \frac{1}{2\pi} \int_{2\pi} \omega_n(\vartheta) d\vartheta \\ \omega_{ij}^* &= \frac{1}{\pi} \int_{2\pi} \omega_n(\vartheta) n_i n_j d\vartheta \\ \omega_{ijkl}^* &= \frac{4}{3\pi} \int_{2\pi} \omega_n(\vartheta) n_i n_j n_k n_l d\vartheta \end{aligned} \quad (37)$$

and the damage tensors ω_{ij} and ω_{ijkl} are expressed as follows

$$\begin{aligned}\omega_{ij} &= 2\omega_{ij}^* - \delta_{ij}\omega_o \\ \omega_{ijkl} &= 6\omega_{ijkl}^* - 6A_{ijkl}^* + I_{ijkl}\omega_o\end{aligned}\quad (38)$$

Figures 10a,b present the description of damage distribution by the second and fourth order damage tensors for the cases loading discussed in the previous section, cf. Figures 6-9. It is seen that the second order damage tensor cannot describe adequately the damage distribution on physical planes. Much better description is provided by the fourth-order tensor.

The elastic compliance variation due to damage can be derived similarly assuming normal and tangential compliance on each material plane to be specified by the local damage $\omega_n(\underline{n})$.

5. CONCLUDING REMARKS

The damage and fracture conditions were expressed in terms of interface stresses or strains. The damage distribution within the element can then be determined and the crack initiation is assumed to occur within the plane of maximal damage. The singular and regular stress regimes can be uniformly treated by applying the non-local condition. The damage distribution function can be used to specify the damage tensor and compliance variation.

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