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(Received 14 March 1994, accepted 10 May 1994)

Abstract. — The nonlinear dispersion relation of monochromatic optical Bloch modes in a periodic half-space is derived for both normal and oblique incidence. It is shown that strong nonlinear beam steering can occur in the vicinity of the Bragg condition. By studying the field microstructure, and making use of diagrams of wavevector versus frequency, a fresh perspective is gained on the physical mechanisms that govern the behaviour of light in nonlinear periodic media. A linearised analysis is used to identify parameter ranges of temporal and spatial instability, and to quantify the gain of the associated modulational instability.

1. Introduction.

For some years there has been considerable interest in the behaviour of light in nonlinear periodic structures. Winful [1] in an early paper pointed out that bistability, self-pulsing and chaos could occur in distributed feedback (DFB) geometries. The later realization that strong group velocity dispersion appears close to a Bragg condition (positive on the red-shifted, negative on the blue-shifted side) suggested that gratings could be used for pulse compression [2, 3] and for soliton formation in spectral regions (for optical fibres, in the visible) where the material dispersion is positive. This has led to a series of theoretical studies, driven by the prospect of solitons at any wavelength in any nonlinear medium whose refractive index can be periodically patterned [4-9]. In most of these contributions, a coupled wave formalism is used, leading to nonlinear differential equations describing the behaviour of the so-called gap-solitons that can form in the vicinity of the linear stop-band. Our pre-occupation in this paper is the alternative Bloch wave (BW) approach, including two-dimensional spatial effects (i.e., Bragg angles less than 90°). The BW approach has been used for over a decade to explain wave propagation in periodic structures such as corrugated planar waveguides [10-15]. It has, however, only recently been introduced to study nonlinear propagation in periodic structures [16-19]. In the temporal domain, nonlinear Bloch pulses can represented by a centre-band single-frequency BW which is modulated by a slowly-varying envelope function [16, 17].

What advantage is there in choosing a BW (and not a coupled-wave) approach to propagation in periodic structures? The coupled-wave approach sketches a physical picture of
power being continuously exchanged among a group of plane waves by multiple reflections at the grating planes. The BW approach, on the other hand, provides a complementary picture based on the actual eigenmodes of the periodic structure, which consist of superpositions of constant amplitude plane waves that propagate as independent (and in the linear case, always stable) groups through the grating (see Fig. 1). These BW's play the same role in uniform periodic media as infinite plane waves do in isotropic media [20]. In linear periodic media, this provides a useful alternative viewpoint from which to interpret, and make use of, the many curious aspects of propagation in periodic media (e.g., BW interference [12], focusing elements based on group velocity dispersion [11] and BW superlattice modulators [21]). One further advantage of the BW approach is that it enables, and indeed encourages, one to think in terms of field microstructure, which leads to a range of useful explanations for the behaviour of light in linear gratings [15].

![Diagram of Bloch wave](image)

Fig. 1. — Boundary condition at periodic half-space. The fringes are shown. They are identical inside and outside the periodic region; however the partial waves are «pinned together» inside the periodic region, sharing a common group velocity.

A criticism sometimes levelled at the use of Bloch waves (and eigen-modes in general) to treat nonlinear propagation is that they are not a reliable concept under these circumstances; this view is, however, inconsistent given that plane waves (the normal modes of linear isotropic media) and their rays are commonly used to treat a wide range of nonlinear phenomena. Our specific aim in this paper is to treat the excitation of nonlinear photonic Bloch waves in semi-infinite periodic structures. By restricting the discussion to a periodic half-space (i.e., not the more usual parallel-sided slab), the quantizing and cross-phase effects of multiple Bloch wave reflections [13] are avoided, permitting the discussion to concentrate on the BW's themselves. The approach is thus reminiscent of work on reflection at a nonlinear dielectric interface [22]. The Bloch waves are used as the starting point of a stability analysis; if they are stable, it is clear that they are a useful simple solution to the half-space problem; if not, their degree of instability can be assessed. The intention is not to present a global analysis of the behaviour of short pulses in nonlinear periodic media. Instead, we aim to provide a physical picture (and an analytical formulation) that may be useful in interpreting the complex results of full-blown numerical simulations.
OUTLINE OF PAPER. — The paper is organized as follows: the dispersion relation of monochromatic nonlinear BW’s is derived in section 2. The nonlinear dispersion diagram is plotted and its behaviour related to the field microstructure (and associated effective grating strength) in section 3, where reflectivity versus nonlinearity is also explored. Group velocity and nonlinear beam-steering are discussed in section 4, and the stability of the BW’s is then assessed (Sect. 5) by introducing additional weak Bloch wave side-bands, linearising the resulting equations and obtaining a modulational instability (MI) matrix equation describing the behaviour of the MI modes. This enables the MI gain spectrum of the side-bands to be plotted (Sect. 6), and hence the stability of the pump wave to be assessed. Various experimental situations in which these instabilities might be observed are explored in section 7, and conclusions are presented in section 8. The validity of the two-wave approximation is briefly touched on in an appendix.

2. General nonlinear dispersion relation.

Under circumstances when the Bragg angle is large and the grating strength small it is valid (see appendix) to adopt the two-wave approximation, i.e., to model the Bloch waves by a pair of superimposed plane partial waves

\[ E(z, t) = \frac{1}{2} E_0 \left[ V_1 \exp(-j k_i \cdot r) + V_b \exp(-j k_b \cdot r) \right] \exp(j \omega t) + c.c. \]  

where \( E_0 \) is the electric field amplitude, \( \omega = 2 \pi c/\lambda \) is the optical frequency in radians per second, \( c \) and \( \lambda \) the optical velocity and wavelength in vacuo. \( V_1, V_b \) are the (constant) complex partial wave amplitudes, \( f \) and \( b \) are forward and backward labels and \( r \) and \( t \) are the position vector and time coordinates respectively. The wavevectors in (1) are related by Floquet’s theorem:

\[ k_i = k_b + K \]  

(2)

where \( A = 2 \pi /|K| \) is the grating period and \( K \) the grating vector. This pair of waves propagates through the grating at a fixed group velocity or decay rate. Its direction of power flow is determined by the relative strengths of each wave. We define the dominant wavevector as the one whose associated partial wave amplitude is strongest. For BW’s with positive group velocities, the dominant wavevector is the forward one. The wavevectors \( k_b \) and \( k_i \) are written as follows (3):

\[ k_i = k_b + K = k_i \hat{y} + \beta \hat{z}, \]
\[ k_i = (k_b + K) \]  

(3)

where \( \beta \) is the wavevector component along the boundary, parallel to the grating planes (see Fig. 2). For purely distributed feed-back (DFB) structures, \( \beta = 0 \) and \( k_b \) and \( k_i \) are anti-parallel. Putting (1) into Maxwell’s equations, assuming a grating with first order susceptibility

\[ \chi^{(1)} = \chi_0^{(1)} + \chi_m^{(1)} \cos(Ky), \quad \chi_m^{(1)} > 0 \]  

(4)

\( \chi^{(1)} \) is the susceptibility per unit area. The electric field in the grating plane is

\[ E_\parallel(z, t) = \frac{1}{2} E_0 \left[ V_1 \exp(-j k_i \cdot r) + V_b \exp(-j k_b \cdot r) \right] \exp(j \omega t) + c.c. \]

\[ E_\parallel(z, t) = \frac{1}{2} E_0 \left[ V_1 \exp(-j k_i \cdot r) + V_b \exp(-j k_b \cdot r) \right] \exp(j \omega t) + c.c. \]

where \( E_\parallel \) is the parallel field component, and \( E_\perp \) the perpendicular field component.
Fig. 2. — Wavevectors and geometry used to describe an arbitrary Bloch wave.

A third order susceptibility $\chi^{(3)}$, and making standard approximations, it is straightforward to show that the field amplitudes obey:

$$
\begin{pmatrix}
-k + \Delta (|V_1|^2 + |V_b|^2) & \kappa + \Delta V_b^* V_l \\
\kappa + \Delta V_b V_l^* & 2 \gamma + 2 \vartheta + \Delta (|V_1|^2 + |V_b|^2)
\end{pmatrix}
\begin{pmatrix}
V_1 \\
V_b
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}
$$

(5)

where the coupling constant

$$
\kappa = \frac{\chi_m^{(1)} k_0}{4 n_0^2} \frac{1}{\sqrt{1 - (\beta/k_0)^2}} = \kappa_N \frac{1}{\sqrt{1 - (\beta/k_0)^2}}
$$

(6)

$n_0$ is the average index and $\kappa_N$ the coupling constant at normal incidence ($\beta = 0$). The mean wavevector $k_0$ is given by:

$$
k_0 = 2 \pi n_0/\lambda = \omega n_0/c
$$

(7)

and the nonlinear dephasing parameter $\Delta$ is given by:

$$
\Delta = \frac{\Delta_N}{\sqrt{1 - (\beta/k_0)^2}}, \\
\Delta_N = \frac{3 \chi^{(3)} \omega \mu S_0}{4 n_0^2} = \frac{3 \chi^{(3)} E_0^2 k_0}{8 n_0^2}
$$

(8)

where $\Delta_N$ is its value at normal incidence and $S_0$ is the incident Poynting vector. The parameter $\gamma$ is the perturbation to $k_{10}$ (the wavevector component normal to the boundary in the absence of a grating, i.e., based on the average index) that appears in the vicinity of a Bragg condition, i.e.,

$$
k_1 = \sqrt{k_0^2 - \beta^2} + \gamma = k_{10} + \gamma.
$$

(9)

The parameter $\vartheta$, which signifies dephasing from the Bragg condition, is given by (2)

$$
\vartheta = k_{10} - K/2 = k_0 \sin \theta - K/2
$$

$$
= (\theta - \vartheta_B) [k_{0B} \cos \theta_B] + (\omega - \omega_B) [(n_0/c) \sin \theta_B]
$$

$$
= - (\beta - \beta_B) \cos \theta_B + (\omega - \omega_B) [(n_0/c) \sin \theta_B]
$$

(10)

(2) This definition of $\theta$ differs by a factor of two from the one used in previous publications [15], and results in simpler algebra.
where the subscript B indicates a Bragg condition, which occurs when the angle of incidence $\theta$ (see Fig. 2) and the wavevector $k_0$ are related by $\sin \theta = K/2 k_0$.

Analytical solutions of (5) are easily found for a given boundary condition, yielding up to three different sets of values $(\gamma_1, V, \gamma_2)$, each describing a different nonlinear Bloch wave. Expressed in their most symmetrical form, the solutions are as follows:

$$\dot{\gamma} = -\hat{\theta} \frac{2 V^2_b}{V^2_b + V^2_f} + \hat{\Delta} (V^2_f - V^2_b)$$  \hspace{1cm} (11)$$

and

$$\frac{2 V_b V_f}{V^2_b + V^2_f} = -\hat{\Delta} + \frac{1}{2\hat{\Delta} + [3 \hat{\Delta} (V^2_b + V^2_f)/2]}.$$  \hspace{1cm} (12)$$

where only travelling Bloch waves are considered (for which $V_f$ and $V_b$ are real-valued). This restriction is needed to avoid evanescent waves, whose field amplitudes suffer exponential growth or decay, altering the level of nonlinearity and preventing the formation of nonlinear eigenmodes. In (11) and (12) the normalized parameters (with hats) are defined as follows:

$$\hat{\theta} = \theta/\kappa, \quad \hat{\Delta} = \Delta/\kappa, \quad \dot{\gamma} = \gamma/\kappa.$$  \hspace{1cm} (13)$$

Equation (12) represents the visibility of the fringes in the periodic field microstructure of the Bloch wave; in both linear and nonlinear gratings, this is 100% at the stop-band edges ($\hat{\theta} = \pm 1$ in the linear case). In the presence of nonlinearity, however, the stop-band edges shift away from these points to the positions:

$$\hat{\theta} = -3 \hat{\Delta} (V^2_b + V^2_f)/2 \pm 1;$$  \hspace{1cm} (14)$$

notice that the stop-band width is always equal to 2 in normalized units (2 $\kappa$ in wavevector units), regardless of the level of nonlinearity. The expression for $\dot{\gamma}$ in (11) contains a linear and a nonlinear term. At the stop-band edges, $V_f = \pm V_b$ and the nonlinear second term is zero; at these points $\dot{\gamma}$ is given by $-\hat{\theta}$ at any level of nonlinearity. In the linear case, (11) and (12) can be solved to yield to well known solution:

$$\dot{\gamma} = -\hat{\theta} \pm \sqrt{\hat{\theta}^2 - 1}$$  \hspace{1cm} (15)$$

as recorded for example in reference [15]. A number of different ways of expressing the solutions of (11) and (12) exist. For the purposes of this paper, we select two examples: $V^2_b + V^2_f = 1$, and $V^2_f = 1$.

2.1 Solutions for $V^2_f + V^2_b = 1$. — In this case (see (11) and (12)), the effects of nonlinearity are controlled by $\hat{\Delta}$ and the difference term $V^2_f - V^2_b$. The resulting solutions constitute a general purpose description of isolated nonlinear Bloch waves, inside an extended periodic medium far from any boundary. The dispersion relation takes the form:

$$\dot{\gamma} + \hat{\theta} = \pm (\hat{\Delta} + \hat{\theta}) \sqrt{1 - (\hat{\theta} + 3 \hat{\Delta}/2)^{-2}}$$  \hspace{1cm} (16)$$

at a reflectivity $\eta = V^2_b(0 = \eta < 1)$ given by:

$$\eta = \left[1 \mp \sqrt{1 - (\hat{\theta} + 3 \hat{\Delta}/2)^{-2}}\right]/2.$$  \hspace{1cm} (17)$$
These expressions are real-valued outside the range:

\[-3 \Delta/2 - 1 < \delta < -3 \Delta/2 + 1\]  \hspace{1cm} (18)

which defines the stop-band. The expression in (16) is zero, i.e. \(k = K/2\), at three points: the stop-band edges (the two limits in (18)), and \(\delta = -\Delta\). If the third point occurs outside the stop-band, then a loop appears in the wavevector diagram. The onset of this loop occurs when the upper stop-band edge (the right-hand limit in (18)) coincides with the third condition, i.e., when \(\Delta = 2\). Some examples of the resulting stop-band shapes are given in figure 3.

2.2 Solutions for \(V_1^2 = 1\). — When a particular boundary condition is to be treated, however, these general-purpose solutions are not appropriate. As an example we now analyze

![Diagram](image-url)

Fig. 3. — Stop-band for case when \(V_0^2 + V_1^2 = 1\), for increasing levels of nonlinearity. The dashed portions of the curves represent BW's with backward group velocities.
the case of a semi-infinite periodic space with a boundary at the plane \( y = 0 \), excited by an infinite plane wave from the isotropic side \( y < 0 \). Under these circumstances, at least for linear structures [15], a single Bloch wave is excited in the periodic half-space. It is logical, given the way the equations are cast, to set \( V^2_1(y = 0) = 1 \), i.e., to normalize the field amplitudes to that of the incident wave; changes in the incident power level are then handled by the parameter \( \hat{A} \). Under these conditions (5) yields:

\[
\hat{y} + \hat{\theta} = \hat{\theta} + V_b + \hat{A}(1 + 2 \ V^2_b) = - \hat{\theta} - (1/V_b) - \hat{A}(2 + V^2_b) \quad (19)
\]

where, since \( V^2_1 = 1 \), \( V^2_b \) is equivalent to the reflectivity \( \eta \) from the grating. It is useful to define a parameter \( s \) to represent the sign of the backward partial wave amplitude, i.e., \( V_b = s \sqrt{\eta} \); this parameter determines the phase of the interference fringes relative to the grating (see Sect. 3 for a discussion). From (12), \( s \) is given by:

\[
|\psi| = 1 \quad \text{sign} (s) = \text{sign} [ - \hat{\theta} - 3 \hat{\Delta}(1 + \eta) / 2 ], \quad (20)
\]

and (19) can be further simplified to:

\[
\hat{y} + \hat{\theta} = \hat{\theta} + s \sqrt{\eta} + \hat{A}(1 + 2 \eta) = - \hat{\theta} - (s / \sqrt{\eta}) - \hat{A}(2 + \eta) . \quad (21)
\]

The nonlinear dispersion diagrams in section 3 are plots of the wavevector \( (k_l - K/2)/\kappa \) (= \( \hat{y} + \hat{\theta} \)) against frequency deviation \( \hat{\theta} \). They are obtained by taking a range of values of \( \eta \) and solving (21) straightforwardly for \( \hat{y} \) and \( \hat{\theta} \). As in the previous case (Sect. 2.1), there are three possible points where \( k_l = K/2 (\hat{y} + \hat{\theta} = 0) \): the first two are at the stop-band edges:

\[
\hat{\theta} = - 3 \hat{\Delta} + 1 \quad (22)
\]

where \( V_b = \mp 1 \), and the third occurs at:

\[
\hat{\theta} = - \hat{\Delta} - 1 / \hat{\Delta} . \quad (23)
\]

when it turns out that \( V^2_b = 1 / \hat{\Delta} \). The right-hand stop-band edge coincides with this last point at \( \hat{\Delta} = 1 \); at \( \hat{\Delta} > 1 \), three distinct points exist where \( \hat{y} + \hat{\theta} = 0 \).

If instead the reflectivity for given values of \( \hat{\theta} \) and \( \hat{\Delta} \) is sought, it may be shown to be given by solutions of the cubic:

\[
\{9 \hat{\Delta}^2\} \eta + \{6 \hat{\Delta}(2 \hat{\theta} + 3 \hat{\Delta}) - 1\} \eta^2 + \{(2 \hat{\theta} + 3 \hat{\Delta})^2 - 2\} \eta - 1 = 0 . \quad (24)
\]

Finally, the level of nonlinearity \( \hat{\Delta} \) turns out to be related to \( \eta \) and \( \hat{\theta} \) by the simple expression:

\[
\hat{\Delta} = - 1 / 3 \left[ \frac{2 \hat{\theta} + s}{\eta + 1} \right] . \quad (25)
\]

3. Field microstructure and dispersion diagrams.

The behaviour of the eigenvalues of matrix equation (5) is best understood by reference to the field microstructure of the Bloch waves. The discussion in this section is restricted to the case when \( V^2_1 = 1 \) (the solution of Sect. 2.2).
In the absence of a grating, the solution is the straight « light » line \((k_i - K/2)/\kappa = \delta + \Delta\) expected of a monochromatic plane wave travelling into an isotropic nonlinear half-space. It is the sloping (−−−−−−−−−−) line on three left-hand diagrams in figure 4, and shifts to the left as the level of nonlinearity increases.

In the absence of any nonlinearity (\(\Delta = 0\)) but in the presence of a linear grating, the usual stop-band opens up at the Bragg condition (Fig. 4, top left). The branches with negative group velocities (\(\partial k_i/\partial \delta < 0\)) are plotted with thinner dashed lines — the assumption of no sources at infinity means that they play no role in the grating half-space. As explained previously [15], the main consequence of Floquet's theorem is that the field microstructure (created by interference of backward and forward partial waves) mimics the grating structure. Fast and

\[\text{index profiles}\]

\[\text{position } K y/\pi\]

\[\text{index profiles}\]

\[\text{Fig. 4. — Stop-bands and refractive index profiles for case when } V_1^2 = 1 \text{ and } V_b^2 = \eta \text{ for three levels of nonlinearity. The slanting (dash-dot-dot) lines on the stop-band diagrams are the solutions for a plane in an isotropic nonlinear medium. Except in the linear case, only the loci of forward travelling Bloch waves are included (the backward travelling waves do not appear in a half-space). The refractive index profiles show the original grating (dash-dot), the nonlinear index change (dash-dot-dot) and the net index profile (full line). Note that cancellation of the linear grating occurs at points 1 and 3. All the nonlinear distortions in the stop-band shape can be understood from these index modulation patterns. Although the branches (terminating at 4 and 5, and 1 and 2) may be extended above the horizontal axis (dashed lines), the solutions fail to conserve power (\(\eta > 1\)). see text for discussion.}\]
slow BW's exist, for which the optical power is partially or fully redistributed by interference into, respectively, the low and high refractive index regions of the grating. In the linear case, the slow (high index) BW's appear in the range $\hat{\delta} < -1$ and the fast BW's in the range $\hat{\delta} > +1$. Inside the stop-band the waves are evanescent, and the fringe microstructure is neither exactly in-phase nor exactly out-of-phase with the gratings.

When a nonlinearity is introduced ($\hat{\delta} > 0$), the mean wavevectors in the grating are increased by $\hat{\delta}(V_1^2 + V_2^2) = \hat{\delta}(1 + \eta)$ (on-diagonal matrix elements in (5)), and the effective grating strength by $\hat{\delta} V_0 V_1 = \hat{\delta} s \sqrt{\eta}$ (off-diagonal matrix elements in (5)). The fringe microstructure of the BW, having the same periodicity as the grating, acts either to enhance or to reduce its strength. A fast BW in a grating with a positive $\hat{\delta}$ will experience a gradual diminution in net grating strength as the intensity rises, leading to cancellation of the linear grating at

$$\kappa = - \Delta V_0 V_1 = - \Delta s \sqrt{\eta}.$$ (26)

The slow BW's, on the other hand, will experience a nonlinear enhancement in grating strength as the power is raised.

These effects may be understood by means of an effective first-order dielectric susceptibility (i.e., the superposition of linear and nonlinear components). This is evaluated by comparing the matrix in (5) with the results of a linear analysis, and (relative to $\chi_0^{(1)}$) and normalized to $\chi_m^{(1)}$ takes the form:

$$\chi_{eff}^{(1)} = \hat{\delta}(1 + \eta)/2 + (1 + s \sqrt{\eta}) \cos K y.$$ (27)

The sign of $s$ thus determines whether the associated BW enhances or depletes the linear grating strength. We already know that $s = +1$ yields slow BW's and $s = -1$ yields fast BW's. Thus, increasing the optical nonlinearity or the input power level, two effects are seen (Fig. 4, lower two left-hand plots): i) the Bragg condition shifts to lower frequencies owing to the nonlinear increase in the average propagation constant; ii) the stop-band branches develop distortions caused by the nonlinear grating. These distortions are most pronounced where the nonlinear grating is strongest, i.e., close to the band edges where $\eta$ and hence the fringe visibility is greatest. Notice also that on the slow stop-band branch there is the potential for bistability since two or more travelling-wave solutions can exist, with high and low reflection states. The issue of whether these states are stable is addressed in section 5. To illustrate these phenomena, the linear, nonlinear and net grating susceptibility profiles across the grating planes, from (25), are plotted in figure 4 for ten selected BW's. In cases 1 and 3, the net grating strength is zero; in case 5 the nonlinear grating strength actually exceeds the linear, reversing the sign of the net grating ripple: a fast Bloch wave turns into a slow.

The solutions are analytical into regions where $\eta > 1$. Power conservation prohibits these solutions in the steady-state case with unslanted boundaries (unless counter-propagating pump plane waves with the correct phases and amplitudes are launched into the periodic region). If such solutions are nevertheless retained, two « unphysical » sections of curve (each originating from opposite sides of the stop-band) appear — the dashed curves in figure 4 (lower two left-hand plots). At intermediate levels of nonlinearity, curious regimes can appear where the only solution is one of these unphysical BW's. An example occurs in figure 4 for $\hat{\delta} = 1$ at $\hat{\delta} = -2.5$. The reflection efficiency in this case is 134%, which cannot be a stable steady-state value; although to test this requires a full dynamic analysis, the half-space grating response appears to have no choice but to oscillate if the time averaged reflection efficiency is
to be less than 100%. These « unphysical » states acquire genuine physical meaning when the
input boundary slants at a positive angle $\delta$ to the horizontal axis. Under these circumstances the
maximum ratio of backward to forward field intensities is limited by power conservation to:

$$\frac{V_b^2}{V_i^2} \leq \frac{\sin (\theta_B - \delta)}{\sin (\theta_B + \delta)}, \quad |\theta_B| > |\delta|.$$  \hspace{1cm} (28)

This shows that, for $\delta < 0$, $V_b^2$ is permitted to exceed $V_i^2$ without violation of power
conservation.

Finally, in figure 5 the reflectivity is plotted against $\Delta$ for a number of different values of
$\dot{\delta}$ (using (25)), and the field microstructure given in a number of cases. Once again, parameter
regions exist where two states are possible. For $\dot{\delta} > 0$ only one positive value of $\Delta$
exists, implying (for media with $\chi^{(3)} > 0$) that only one solution is physically meaningful; this
is because the Bragg condition moves further away (instead of closer) as the power is
raised.

---

**Fig. 5.** — Reflectivity against $\Delta$ for three values of dephasing $\dot{\delta}$. 
4. Group velocity and nonlinear beam-steering.

The standard definition of group velocity (as $\partial \omega / \partial k$) breaks down in nonlinear systems, owing to the invalidity of linear superposition. A suitable alternative definition follows from the standard expression for the Poynting vector: $\langle S \rangle = (\text{stored energy per unit volume}) \times (\text{group velocity})$. $\langle S \rangle$ is expressible either a) as the superposition of the Poynting vectors of two partial waves (with group velocities $c/n_0$ and amplitudes $V_i$ and $V_b$) or b) as the product of the stored energy in the BW and its group velocity. Equating these two alternatives (factoring out $E_0^2$):

$$\frac{c_0}{2} \left( (V_i^2 + V_b^2) \hat{z} \cos \theta_B + (V_i^2 - V_b^2) \hat{y} \sin \theta_B \right) = \frac{c_0}{2} (V_i^2 + V_b^2) v_g$$

leads to the convenient definition

$$v_g = \frac{c_0}{n_0} \left[ \hat{z} \cos \theta_B + \hat{y} \left( \frac{V_i^2 - V_b^2}{V_i^2 + V_b^2} \right) \sin \theta_B \right].$$

The parameter $\hat{\theta}$ in section 2 allows for both temporal and spatial frequency deviations from the Bragg condition. Changing the frequency or direction of incident light causes variations in BW group velocity that result in strong beam steering within the periodic structure. Altering the incident power level has a similar effect, producing strong nonlinear beam steering. The solutions of section 2 show that regions of bistability may exist, and that under some conditions the system seems inescapably unstable or oscillatory. This means that the direction of propagation of the light inside the periodic region will under the right conditions oscillate or exhibit bistability. To illustrate this, the beam steering angle $\alpha$, defined by:

$$\alpha = \arctan \left[ \frac{V_i^2 - V_b^2}{V_i^2 + V_b^2} \tan \theta_B \right],$$

is plotted alongside the reflectivity in figure 6 for $\hat{\theta} = -8$ and $\theta_B = 45^\circ$.

Fig. 6. — The beam steering angle $\alpha$ (and reflectivity) assuming a Bragg angle of $45^\circ$, for the third case in figure 5.
5. Linearized stability analysis.

We now address the important issue of whether the nonlinear Bloch waves discussed in the previous sections are stable against small perturbations. The approach we adopt is the conventional one of adding small-amplitude disturbances at upper and lower sidebands of the pump frequency (see Fig. 7), linearising the equations describing their dynamics, and quantifying the gain, if any, of their modulational instability. The physical situation is that of degenerate four-wave (i.e., four Bloch wave or eight partial wave) mixing. We start with a group of three BW's consisting of one strong pump at $\omega$ and two weak sidebands at temporal frequencies $\omega \pm \Omega$ and spatial frequencies $\beta \pm \sigma$ (in the $z$ direction, see Fig. 7):

$$E(z, t)E_0 = \frac{1}{2} \left( V_1 \exp \{- j \phi_0\} + V_b \exp \{- j \psi_0\} \right) +$$

$$+ \frac{1}{2} \left( f_1 \exp \{- j (\phi_0 + \phi_1)\} + b_1 \exp \{- j (\psi_0 + \phi_1)\} \right) +$$

$$+ \frac{1}{2} \left( f_2^* \exp \{- j (\phi_0 - \phi_1^*)\} + b_2^* \exp \{- j (\psi_0 - \phi_1^*)\} \right) + \text{c.c.} \quad (32)$$

where $f_1$, $f_2$, $b_1$ and $b_2$ are small constant amplitudes (subscript 1 for the upper sideband and 2 for the lower) and

$$\phi_0 = \beta z + k_1 y - \omega t = \psi_0 + K y, \quad \phi_1 = q y + \sigma z - \Omega t. \quad (33)$$

Permitted values of the wavevector $q$ for specified temporal and spatial frequencies $\Omega$ and $\sigma$ are now sought. If complex values of $q$ exist, then instability is likely, although whether it is seen in practice will depend on the level of gain and the path lengths in the periodic region. Putting (32) into the nonlinear wave equation, collecting terms with identical frequencies and phase velocities, equating their coefficients to zero and linearising the equations (i.e., neglecting products and powers of order $f_1^2$) yields the following eigenvalue matrix for the wavevectors $\hat{q} = q/\kappa$:

$$\begin{pmatrix}
\hat{\xi}_1 - \hat{q} & 1 + 2 \hat{\Delta}V_1 V_b & \hat{\Delta}V_b^2 & 2 \hat{\Delta}V_1 V_b \\
-1 - 2 \hat{\Delta}V_1 V_b & -\hat{\xi}_b - \hat{q} & -2 \hat{\Delta}V_1 V_b & -\hat{\Delta}V_b^2 \\
-\hat{\Delta}V_1^2 & -2 \hat{\Delta}V_1 V_b & \hat{\xi}_b - \hat{q} & -1 - 2 \hat{\Delta}V_1 V_b \\
2 \hat{\Delta}V_1 V_b & \hat{\Delta}V_b^2 & 1 + 2 \hat{\Delta}V_1 V_b & \hat{\xi}_b - \hat{q}
\end{pmatrix} \times \begin{pmatrix}
f_1 \\
h_1 \\
f_2 \\
h_2
\end{pmatrix} = \begin{pmatrix}0 \\
0 \\
0 \\
0
\end{pmatrix} \quad (34)$$

Fig. 7. — Diagram of wavevectors for modulational instability calculation. Both spatial ($\sigma \neq 0$) and temporal ($\Omega \neq 0$) frequencies are allowed for. Phase matching is automatically achieved by assigning the lower and upper sidebands wavevector corrections of $-q^*$ and $+q$ respectively.
with

$$\xi_t = \hat{\Omega} \pm \left( \Delta V_1^2 - \frac{V_b}{V_t} \right)$$

$$\xi_b = \hat{\Omega} \pm \left( \Delta V_2^2 - \frac{V_t}{V_b} \right).$$

(35)

The parameter

$$\hat{\Omega} = \frac{(\Omega \Delta n/c) - (\sigma \beta/k_0)}{\kappa_N}$$

(36)

is a « unified » MI frequency, consisting of a normalized amalgamation of both temporal and spatial components. Solution of the eigenvalue problem yields four values of \( \hat{\Omega} = \hat{\omega}_R + j\hat{\omega}_I \) and four modal shapes. Equation (34) is valid for a general pump Bloch wave, with the proviso that it is restricted to propagating pump Bloch waves, i.e., those with real-valued wavevectors. In the specific example treated in section 2.2, the pump amplitudes are replaced with \( V_t = \) and \( V_b = s \sqrt{\eta} \), yielding

$$\begin{pmatrix}
\hat{\xi}_t - \hat{\omega} & 1 + 2\hat{\Delta}s \sqrt{\eta} & \hat{\Delta} & 2\hat{\Delta}s \sqrt{\eta} \\
-1 - 2\hat{\Delta}s \sqrt{\eta} & -\hat{\xi}_b - \hat{\omega} & -2\hat{\Delta}s \sqrt{\eta} & -\hat{\Delta} \eta \\
-\hat{\Delta} & -2\hat{\Delta}s \sqrt{\eta} & \hat{\xi}_t - \hat{\omega} & 1 - 2\hat{\Delta}s \sqrt{\eta} \\
2\hat{\Delta}s \sqrt{\eta} & \hat{\Delta} \eta & 1 + 2\hat{\Delta}s \sqrt{\eta} & -\hat{\xi}_b - \hat{\omega}
\end{pmatrix} \begin{pmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}$$

(37)

with

$$\xi_t = \hat{\Omega} \pm (\hat{\omega} - s \sqrt{\eta})$$

$$\xi_b = \hat{\Omega} \pm \eta \left( \hat{\omega} - \frac{s}{\sqrt{\eta}} \right).$$

(38)

The matrix in (34) and (37) is not Hermitian, so does not produce orthogonal eigenmodes. This means that the solutions of the linearized model do not conserve power even when the eigenvalues \( \hat{\omega} \) are real-valued. This is not surprising since the pump BW can act as a source of gain. It is straightforward to assess whether the modal solutions of section 2 are stable or not. In regions where there is MI gain or loss (a mode with gain is always accompanied by one with loss, since the eigenvalues \( \hat{\omega} \) come in conjugate pairs), interactions of type

$$\{ (\omega \pm \Omega), \omega, \omega \} \rightarrow \{ (\omega \pm \Omega), (\omega \pm \Omega), (\omega \mp \Omega) \}$$

(39)

in time, and

$$\{ (\beta \pm \sigma), \beta, \beta \} \rightarrow \{ (\beta \pm \sigma), (\beta \pm \sigma), (\beta \mp \sigma) \}$$

(40)

in space will occur. When stimulated by a single photon at \( (\omega \pm \Omega), (\beta \pm \sigma) \), a pair of jump photons at \( (\omega, \beta) \) is converted into a pair of photons at \( (\omega \pm \Omega, \beta \pm \sigma) \), which leads to the growth of the side-band amplitudes and instability. A consequence is that the photon flux in each side-band (which grows or decays at a rate \( 2q_i \)) must be equal. Within the approximations of the model (i.e., \( \Omega \ll \omega \) and \( \sigma \ll \beta \)), a conservation law is therefore expected :

$$|f_1|^2 - |b_1|^2 = |f_2|^2 - |b_2|^2$$

(41)
This can be proven by manipulation of the matrix equation (34). Where there is no parametric gain or loss (i.e., the \( \dot{q} \) parameters are real-valued), this conservation law no longer holds, for under these conditions the power in the two side-bands is independent of the pump power and each other, the two sideband BW's merely experiencing self and cross-phase modulation.

Whether or not a MI wave with a complex value of \( \dot{q} \) is a growing or a decaying disturbance may be established by reference to the group velocity \( v_g \), for the \( i \)-th sideband (\( i = 1, 2 \)), given by

\[
v_g - \frac{c}{n_0} \hat{z} \cos \theta_B = \frac{c}{n_0} \dot{q} \left( \frac{|f_i|^2 - |b_i|^2}{|f_i|^2 + |b_i|^2} \right) \sin \theta_B.
\] (42)

Although the magnitudes of the group velocities in each sideband are in general unequal. from (41) and (42) it is clear that their \( y \)-components point in the same direction. If this direction coincides with the direction of exponential field decay, then power will flow from the MI wave into the pump wave and the system is stable; if it points against the direction of field decay, then power flows into the MI wave and the system is unstable.

An interesting case arises when one of the sideband BW’s is evanescent at \( \hat{q} = 0 \). This will occur when the pump is incident at \( \hat{\theta} \neq 0 \) and \( \hat{q} \) is chosen to place one of the sidebands inside the linear stop-band; under these circumstances, the linear BW associated with this side-band has zero group velocity. As the pump wave power is increased from zero, however, the group velocity of the “evanescent” side-band BW gradually deviates from zero and is coupled by the nonlinearity to the other side-band, permitting MI gain to appear and forcing the system to obey the conservation law (41).

6. Modulational instability gain spectra.

In might be thought that the stop-band width “perceived” by a weak signal at a frequency different from the pump would depend on the strength of the effective grating (Eq. (27)). If this is zero, then the MI stop-bands would have zero width. If the pump Bloch wave is fast, then the MI stop-band would be narrower than \( 2 \kappa \) owing to cancellation of the linear by the nonlinear grating; and if it is slow, then the MI stop-band would conversely be wider than \( 2 \kappa \). In practice, however, four wave mixing between the sideband waves and the pump wave causes additional gain or loss to appear, significantly complicating the picture.

In this section, a few cases are illustrated from the rich range of possibilities. In each, the real and imaginary parts of \( \dot{q} \) are plotted, together with the beam steering angles of the upper and lower side-band waves in ranges of MI gain/loss. The interested reader may wish to study the plots, referring to their captions for parameter values and brief explanations (Figs. 8-12). Some general observations may be made. For example, the group velocities of the upper and lower side-bands are not normally equal; indeed, they are only equal when they are both zero, or when (not illustrated) the pump wave sits on the edge of the left-hand stop-band branch \((\eta = 1, s = +1, \hat{\theta} = 1 - 3 \hat{\Delta})\). It is interesting to note that one range exists where stability is guaranteed; this is on the fast stop-band branch \((s = -1)\), under the conditions:

\[
\hat{\Delta} > 1, \quad \hat{\theta} \leq - \hat{\Delta} - 1/\hat{\Delta}
\] (43)

which hold for example to the left of points 1 and 3 in figure 4. Outside this range, MI gain is always present; whether this results in appreciable instability will depend on the path lengths, the noise level at the side-band frequencies, and the group velocities of the three participating waves. A useful result is that the asymptotic level of gain at high values of \( \hat{\Omega} \) (beyond the MI
Fig. 8. — Real part of $\hat{\eta}$, MI gain (imaginary part of $\hat{\eta}$) and beam steering angle $\alpha$ ($\theta_0 = 45^\circ$) at $\hat{\eta} = -6.36$ and $\hat{\Delta} = 2.5$ for $\eta = 0.046$ and $s = +1$ (conditions similar to point 7 in Fig. 4). The dashed curves represent solutions with real-valued $\hat{\eta}$ and thus zero gain. Growth and decay exists in the range $\hat{\Delta} > 0.54$, the direction of growth being into the periodic region ($\alpha > 0$). The horizontal dashed line on the third figure is $\alpha$ for the pump wave: the upper side-band experiences stronger beam steering as $\hat{\Delta}$ increases. The region where the instability will be most effective is around $\hat{\Delta} = 6$, where all three waves (pump and two side-bands) share approximately the same group velocity.

The plots provide a powerful means of establishing the frequencies at which instability is most likely to appear. When the group velocities (or beam steering angles $\alpha$) of the pump and

\[
\hat{\eta}_1 \to \pm \sqrt{2 \hat{\Delta} s / \sqrt{\eta - \eta}}, \quad \text{and} \quad \pm \sqrt{2 \hat{\Delta} s / \sqrt{\eta - 1/\eta}}.
\]

Note that these expressions are useful only when real-valued.
side-band waves are similar, then long phase-matched interaction lengths are possible, and strong instabilities will occur if the periodic region is extended enough.

7. Discussion: experimental parameters.

Although the effects discussed in this paper are in themselves intriguing, one would like to know if the parameter ranges are accessible in any existing material systems. We take two
Fig. 10. — Real part of \( \dot{q} \) at \( \dot{\omega} = -6.36 \) and \( \dot{\Delta} = 2.5 \) for \( \eta = 0.962 \) and \( s = -1 \) (conditions similar to point 5 in Fig. 4). No regions of side-band gain or decay exist — the pump BW is stable.

For example: optical fibre gratings (in which temporal instability might occur) and corrugated two-dimensional thin film waveguides of tantalum on silica (in which both temporal and spatial instabilities might be observed).

Fibre gratings 10 cm long, with index modulation depths as high as 0.01, are feasible in the present state-of-the-art [23]. For a grating with \( \chi_m^{(1)} = 10^{-5} \) and \( n_0 = 1.458 \), at \( \lambda = 1.06 \mu\text{m} \) and \( \dot{\omega} = -8 \) (i.e., 17 GHz on the red-shifted side of the Bragg condition), the coupling constant \( \kappa = 10.2 \) per metre. Under these conditions, \( \dot{\omega} = 1 \) represents a MI frequency of 330 MHz (at normal incidence); at this frequency normalized MI gains of about 5 are possible, representing an actual linear growth rate of 0.51 per cm. For a nonlinear index of \( n_2 = 3 \times 10^{-8} \mu\text{m}^2/\text{W} \) and a core area 1 \( \mu\text{m}^2 \), the power needed to observe this is of the order of 500 W per \( \mu\text{m}^2 \). The normalized group velocity \( v_g n_0/c \) of this disturbance is about 0.08, which implies that the group velocity will be about a tenth that of light. A possible experimental scenario is of a weak injected pulse of FWHM bandwidth 80 MHz (say) and centre frequency 330 MHz from the primary optical frequency (which is itself 17 GHz from the Bragg condition), growing as \( \exp(51 \xi) \) with a physical pulse length a tenth that of a similar pulse in a fibre, i.e., 8.8 cm instead of 1.1 m.

Corrugated waveguides in \( \text{Ta}_2\text{O}_5 \) (index 2.12) have been successfully realized [16] with \( \kappa \) values as high as 120 per mm. Could spatially unstable Bloch waves be seen in these waveguides? Path lengths of 1 cm are feasible, and a single-mode guide is ~ 100 nm thick at \( \lambda = 1 \mu\text{m} \). For a guided beam 10 \( \mu\text{m} \) wide the cross-sectional area is ~ 1 \( \mu\text{m}^2 \), and the nonlinearity in \( \text{Ta}_2\text{O}_5 \) might be 10 \( \times \) that of silica. Under these circumstances, 500 W would be sufficient to attain the same parameter regime as for the fibre mentioned above.

8. Conclusions.

In summary, the two waves bound together by the grating form (in the linear case at least) an entity insensitive against weak perturbations. Slight changes in refractive index (\( \dot{\Delta} \ll 1 \)) will slow down or speed up the Bloch wave without disturbing its field microstructure or group velocity dispersion (GVD). This occurs for example on the \( \dot{\omega} > 1 \) side of the stop-band far
from the band edge, where the negative GVD of the linear Bloch waves is relatively undisturbed by the nonlinear index changes needed for soliton formation. Dispersion and nonlinearity then operate independently, and the conventional explanation for soliton formation (as a balance between self-phase modulation and negative GVD) can be used. A quite different situation occurs when $\hat{\gamma} \sim 1$, for then the nonlinear index perturbation is comparable to the grating index modulation and the Bloch wave entity is susceptible to gross deviations from its usual shape. This causes the stop-band distortions seen in figure 4, and results in an inextricable intertwining of dispersion and nonlinearity. These distortions arise
Fig. 12. — The gain spectrum at two levels of nonlinearity ($\tilde{\xi} = 0.05$ and 1) in two cases: (a) $\tilde{\xi} = -3.8$, $\kappa = 1$ and (b) $\tilde{\xi} = +2$, $\kappa = -1$. In (a) the reflectivities at $\tilde{\xi} = 0.05$ and 1 are 0.019 and 0.057 respectively, and in (b) they are 0.066 and 0.021 respectively. Notice that a region of gain opens up at higher frequencies in case (a) and at lower frequencies in (b), and that the stop-band moves in opposite directions in each case. In (b), the edges of the stop-bands are delineated with vertical dashed lines (notice the very narrow bridge close to $\tilde{\xi} = 1.1$).
through the interference of the incident and reflected partial waves of the Bloch wave, which acts to enhance or deplete the grating strength. Close to the Bragg condition, this effect pushes $k_0$ to values higher than suggested by a simplistic physical picture. The full picture is of a complex interplay of Bragg condition dephasing (caused by changes in the average refractive index) and grating enhancement/depletion. To observe the more dramatic of the effects described, the nonlinear index parameter $\Delta$ must be comparable to the index modulation depth of the linear grating. A linearised instability analysis identifies regions of instability, and quantifies the level of modulational instability gain. Side-bands whose Poynting vectors point in the same direction as the pump BW experience the maximum overall gain, free from walk-off.

In conclusion, the nonlinear BW approach leads to an easily solvable algebraic dispersion relation and a useful field-microstructural explanation for BW behaviour. It may be used to delineate regions of stability and instability for incidence on a grating half-space. An interesting future project would be to relate the solutions of the Bloch wave approach to the results of full-blown numerical simulations, for example in the field of gap solitons.

Appendix.

Validity of two-wave approximation.

The two-wave approximation essentially rests on the validity of assuming that the field microstructure is sinusoidal. In a rigorous analysis, the full infinite set of higher order partial waves required by Floquet’s theorem must be included. This gives rise to field periodicities whose spatial frequencies are integral multiples of $K$; and the result is a non-sinusoidal but periodic field microstructure described by a complex Fourier series, with a fundamental period equal to the grating pitch. The range of validity of the two-wave approximation is easily identified: it holds provided the grating momentum $\hbar K$ is much greater than the fractional photon momentum across the stop-band, $\hbar \kappa$, i.e., it breaks down if $\hbar K$ is of the same order or less than $\hbar \kappa$. Two well-known regimes of experimental interest when the two-wave approximation breaks down are a) the Raman-Nath regime when the Bragg angle is very small (i.e., $K \sim \kappa$) and many higher order partial waves become important, and b) the photonic band-gap regime where the grating modulation depth is sufficiently large to yield a $\hbar \kappa$ that is comparable with the average photon momentum $\hbar k_p$. Throughout this paper we assume that the two-wave approximation holds. It is worth noting here that if the periodic structure is a multilayer stack of alternating refractive indices, versatile plane-wave transfer-matrix formalisms may be used [10, 24], thus avoiding the need to use a Floquet expansion. For general smoothly varying periodic grating profiles, however, the Floquet expansion is indispensable.

References


