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1. Introduction.

In a previously published work [1] a method based on the concept of the overall period was derived for the analysis of a non Fourier-transformable function in the usual sense [2, 3]. In this context, we propose the spectral analysis theory of a signal represented by a bi-valued function whose transitions are modulated by a sinusoidal waveform function. On the other hand, it is customary to use the harmonic approach [4] when simultaneous generation of small number of harmonics is required. These approaches use as building blocks a set of multipliers, dividers and mixers. The encountered problems are spurious outputs with an associated degradation in phase noise performances. An other design approach of the component frequency generator block based on the spectral properties derived from the above analysis is proposed.
2. Theory.

Let us define the following square waveform function $\alpha(t, \varepsilon, \infty)$ as follows,

$$\alpha(t, \varepsilon, \infty) = \sum_{m=0}^{\infty} \left[ \theta(t - T_m(\varepsilon)) - 2 \cdot \theta(t - T_{m+1/2}(\varepsilon)) + \theta(t - T_{m+1}(\varepsilon)) \right].$$  \hfill (1)

The transitions $T_m(\varepsilon)$, $T_{m+1/2}(\varepsilon)$, $T_{m+1}(\varepsilon)$ are given by,

$$T_i(\varepsilon) = iT - A \sin(4 \pi \varepsilon x),$$  \hfill (2)

$x$ is an integer or half-integer. $A$ and $T$ are constants.

The signal switches between two levels, the time of the $x$-th transition is varied from that of the reference square waveform by the delay $A \sin (4 \pi \varepsilon x)$. $\varepsilon$ is a parameter. $t \in [0, \infty]$, the $\infty$ symbol in the argument of the function (1) refers to the upper limit of the sum. This form of $\alpha$ assures, when the sum is truncated to $N$, to have no tail (i.e., $\alpha(t \sim NT, \varepsilon, N) = 0$). Other forms are however possible. $\theta(t)$ is the heaviside function. The constant $A$ is chosen to be at most $T/4$ in order to avoid overlapping of adjacent $T$, for finite $\varepsilon$. We note that if the frequency $2 \varepsilon = 1/p$ where $p$ is an integer, $\alpha(t, \varepsilon, \infty)$ becomes a periodic function repeating itself every $p$ switches. The frequency $2 \varepsilon$ is then of the form,

$$2 \varepsilon = \frac{q}{\ell}, \quad \ell \neq 1; \quad \ell, q \text{ are mutual prime numbers}.$$  \hfill (3)

When $\varepsilon$ satisfies the above equation, the period of the signal $\alpha(t, \varepsilon, \infty)$ is $\ell T$, where $\ell$ is the above defined integer. This periodicity is however lost when $\varepsilon$ becomes irrational, $\alpha(t, \varepsilon, \infty)$ is hence a non Fourier-transformable function. Our using of (3) is due to the fact that every irrational number is in fact a rational number in any finite precision computation. The Fourier coefficients of $\alpha(t, \varepsilon, \infty)$, calculated independently from the nature of $\varepsilon$ in the appendix, are given by,

$$C(b) = \frac{1}{i \pi b} \sum_{n=-\infty}^{\infty} J_n \left( \frac{2 \pi A}{T} b \right) \cdot \psi(b + 2 n \varepsilon) \cdot (1 - \exp i \pi (b + 2 n \varepsilon))$$  \hfill (4)

$$\psi(x) = \begin{cases} 1, & x \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}; \quad C(0) = 0$$

$J_n(x)$ is the Bessel function of the first kind of integral order $n$ and argument $x$. The coefficients $C(b)$ are different from zero for frequencies $b$ satisfying the following fundamental relation,

$$b = 2 \varepsilon p + p' \cdot \quad p \text{ is an integer, } p' \text{ in an odd integer}.$$  \hfill (5)

Relation (5) shows that the produced spectrum exhibits a particular structure. The produced frequencies are not given by the classical rule stating that harmonic frequencies are integral multiples of the fundamental frequency [2, 3] and used in the process of harmonic generation from a sinusoidal waveform reference function [4]. For $q/\ell = 0.9$ and $q/\ell = 0.93$ (e.g.) only harmonics of frequencies containing respectively one figure and two figures are created with harmonics of integral frequencies (see the Appendix). With $2 \varepsilon$ an integer, only integral frequencies are available but when $2 \varepsilon$ possesses one figure (0.9), the empty segment between every successive harmonic frequencies of the former amplitude spectrum will be occupied by new harmonics possessing one figure according to relation (5). This bifurcation scheme will repeat itself for the two figures (0.93) and for every new figure that $2 \varepsilon$ may possess (see Fig. 1). This spreading is tenfold because we are using a decimal base. For another base number $B$, it is $B$-fold. Figure 2 displays the spectra for $2 \varepsilon = 0.9$ and $2 \varepsilon = 0.91$; the longer
The structure of the produced spectrum for different figures (a, b, c, ...) of the modulation frequency $2\,\varepsilon$. When $2\,\varepsilon$ possesses one figure ($2\,\varepsilon = a$), the empty segment between every successive integer harmonic frequencies of the former spectrum with $2\,\varepsilon = E$ ($E$ is an integral value), will be occupied by new harmonics possessing one figure according to the relation $b = 2\,p\,\varepsilon + p'$, $p$ is an integer, $p'$ is an odd integer. We have a bifurcation process dependent on the number of figures that $2\,\varepsilon$ may possess.

The fractional part of $2\,\varepsilon$ is the more figures the value of the produced harmonic $b$ can have. The effect of the figures of the frequency $2\,\varepsilon$ independently form the nominal value of $2\,\varepsilon$ is shown in figure 1. Due to the nominal value of $2\,\varepsilon$, some harmonics having the required number of figures as $2\,\varepsilon$ may not appear; this happens when the last figure of $2\,\varepsilon$ is 5 or even. The missing peaks in figure 2 are due to their small magnitude.

3. The harmonic approach in frequency synthesis.

In frequency synthesis operation, the output frequency $f_o$ is intended to be a rational multiple of a single standard frequency $f_s$ so that $f_o/f_s = N/M$ where $N$ and $M$ are integers. When the
Fig. 2. — Typical calculated spectra in a logarithmic scale for different modulation frequencies $2 \varepsilon$ with their different figures (Fig. 2a and Fig 2b). Frequency is in arbitrary units.

Simultaneous generation of small number of frequencies is required, it is customary to use the brute-force or the harmonic approach [4]. These approaches use as building blocks a reference frequency source with a set of multipliers, dividers and mixers. The synthesizer depicted in figure 3 generates each output frequency as a programmed rational multiple of a reference frequency $f_1$. We emphasize its component frequency generator block. The latter consists of one divider and 4 multipliers producing the following set $R$ of the 4 frequencies, $R = [45.5984, 44.2368, 40.9600, 39.3216]$ for an input frequency $f_1 = 8$ (frequencies are expressed in arbitrary units). The basic problems of the component frequency generator block are the spurious outputs generated in frequency multiplication and division processes with an important degradation in terms of phase noise performances. The phase noise is amplified by the multiplication factor of each frequency multiplier that feeds the « Mix & Divide » stages of the synthesizer. The global performances of the instrument are then reduced. We propose in the sequel a component frequency generator block based on the developed theory in section 2.
4. Possible realisation of the method.

4.1 Component frequency generator block. — \( \alpha(t, \varphi, \infty) \) given by equation (1) appears as a bi-valued function with transitions \( T_i(\varphi) \) modulated by a sinusoidal waveform function (Eq. (2)). Figure 4 shows the block diagram of the proposed component frequency generator block. The square wave signal represented by the function \( \alpha(t, p, \infty) \) where \( p \) is an integer may be obtained from sinusoidal reference frequency source by means of a limiter and an amplifier. The sinusoidal modulation signal whose frequency has the appropriate chosen figures may be derived from the sinusoidal reference source by means of a divider. The modulation process as defined by equations (1) and (2) may be obtained by various means.

4.1.1 Source and processing chain noise effects. — Let us assume the transitions \( T_i \) of the bi-valued function \( \alpha(t, \varphi, \infty) \) when the noise is present to be given by,

\[
T_i(\varphi) = \pm T - A \sin(4 \pi \varphi \pm \lambda_i)
\]

or

\[
T_i(\varphi) = \pm T - A \sin(4 \pi \varphi \pm \lambda_i + \tau_i)
\]

depending on the type of noise we are considering. \( \chi \) stands for an integer or half integer. The random variables \( \tau_i \) and \( \lambda_i \) measure respectively the total periodicity fluctuations and the total phase fluctuations introduced by the contribution of the source and the processing chain: limiter, amplifier, divider and the modulator. We call the periodicity noise the \( \tau \)-type noise and the phase noise the \( \lambda \)-type noise. \( \tau_i \) are assumed uniformly distributed in the interval \( [-\Delta T/2, \Delta T/2] \) around nominal times \( \mp T \) of frequency switch. \( \lambda_i \) are also assumed uniformly distributed in the interval \( [-\Delta \phi/2, \Delta \phi/2] \) around nominal times of the frequency 2 \( \varphi \) switch. In these conditions the periodicity of the function \( \alpha(t, \varphi, \infty) \) is lost. Its spectral analysis follows the same treatment as for the free noise signal (see the Appendix). The Fourier
Fig. 4. — The proposed design approach of the block diagram of the component frequency generator block. The limiter and the amplifier generates the square wave signal $\alpha (t, p, \infty)$ where $p$ is an integer. The modulation signal of frequency $2 \pi$ is available at the output of the frequency divider.

Coefficients are modified as follows,

$$E(C^A(b)) = \frac{1}{\pi b} \sum_{n=-\infty}^{\infty} J_n \left( \frac{2 \pi A}{T} b \cdot \Omega_\lambda(\Delta \phi) \right) \cdot \psi(b + 2n\pi) \cdot (1 - \exp i \pi (b + 2n\pi))$$  \hspace{1cm} (7a)$$

$$E(C^\tau(b)) = \Omega_\tau(\Delta T, b) \cdot C(b)$$  \hspace{1cm} (7b)$$

where,

$$\Omega_\lambda(\Delta \phi) = \frac{\sin (\Delta \phi/2)}{\Delta \phi/2}$$  \hspace{1cm} (8a)$$

$$\Omega_\tau(\Delta T, b) = \frac{\sin (b \pi \Delta T/T)}{b \pi \Delta T/T}$$  \hspace{1cm} (8b)$$

$C(b)$ is given by equation (4). $E$ is the expectation operator. $\Omega_\tau(\Delta T, b)$ is the Fourier transform of the distribution function of the random variables $\tau$. The $\tau$-type noise induce a reduction of spectral lines of the free-noise square waveform function $\alpha(t, \pi, \infty)$ directly by the amount $\Omega_\tau(\Delta T, b)$, contrary to the $\lambda$-type noise. This allows to distinguish between both
types of noise just by considering the envelope function of the amplitude spectrum. The distinction is achieved by defining the following ratio-function,

\[ Q(b) = \frac{E(C^*(b))}{C(b)} \quad \text{or} \quad \tau. \quad (9) \]

If \( Q(b) \) matches the form (8a), the \( \tau \)-type noise is dominant. \( C(b) \) is given then by the theoretical model of \( \alpha (t, \varepsilon, \infty) \). The factor \( Q(b) \) when measured at the output of the modulator, expresses the quality factor of the processing chain: limiter, amplifier, divider and the modulator.

4.1.2 Frequency generating source. — Section 2 shows that a large range of harmonics with the desired frequency figures are made available by an adequate choice of the modulating frequency \( 2 \varepsilon \). The amplitude spectrum of \( \alpha (t, \varepsilon, \infty) \) will contain the spectral lines whose frequencies are in the following set \( S \) close to the set \( R \) of section 3. \( S = [44.2426, 42.5978, 40.9094, 39.3116] \) when we choose \( 2 \varepsilon = 0.4221 \) with an input frequency source \( F_i = 1 = f_i / 8 \) (frequencies are expressed in arbitrary units). \( F_i \) is much lower than \( f_i \) present at the input of the component frequency generator block of section 3 (Fig. 3). These lines can be extracted by various filtering techniques. The proposed block diagram combines a small number of electronic functions and does not include any frequency multiplication process. Better noise performances can therefore be expected.

5. Conclusion.

The developed theory of a novel mode of fractional frequency harmonic generation is presented. Results show that the produced spectrum possesses a bifurcation structure created by the number of figures of the transition modulating wave frequency. The longer the fractional part of the frequency of the modulating wave is, the more figures the value of the produced harmonic frequency has. We show that this property can open the way to a new generation of frequency synthesizers. It can provide significant advantages in terms of noise performances of component frequency generator blocks of frequency synthesizers.

Appendix.

In this appendix, we show the steps leading to relation (3). Let us consider an overall period \( NT \) where \( N \) is an integer. The function \( \alpha (t, \varepsilon, N) \) in the restricted interval \([0, NT]\) is rewritten as follows,

\[ \alpha (t, \varepsilon, N) = \sum_{\hat{m} = 0}^{N-1} \left[ \theta (t - \hat{m}T - A \sin 4 \pi \varepsilon m) - 2 \theta (t - (\hat{m} + 1/2)T - A \sin 2 \pi \varepsilon (2 \hat{m} + 1)) + \theta (t - (\hat{m} + 1)T - A \sin 2 \pi \varepsilon (2 \hat{m} + 2)) \right] \quad (A.1) \]

\[ \alpha (t, \varepsilon, N) = \alpha_1 (t, \varepsilon, N) + \alpha_2 (t, \varepsilon, N) + \alpha_3 (t, \varepsilon, N) \quad (A.2) \]

\( \alpha_1 (t, \varepsilon, N) \) are the three sums in (A.1). The Fourier coefficients of \( \alpha_1 (t, \varepsilon, N) \) are given by,

\[ R_k = \frac{1}{NT} \int_0^{NT} \exp (-i 2 \pi k t / NT) \times \]

\[ \times \sum_{\hat{m} = 0}^{N-1} \sum_{\rho = 0}^\infty \frac{(-1)^\rho}{\rho!} A^\rho \sin^\rho (4 \pi \varepsilon m) \cdot \delta (\hat{m} - 1)(t - \hat{m}T) \, dt. \quad (A.3) \]
In the above expression, \( \alpha_j(t, \varepsilon, N) \) is Taylor series expanded around \( t - mT \). \( \delta^{(n)}(t) \) is the \( n \)-th derivative of the Dirac distribution function. The Fourier coefficients become,

\[
R_k = \frac{1}{i \pi k} \left( \frac{2 \pi k}{NT} \right)^{\rho-1} \sum_{\rho=0}^{\infty} \frac{(-1)^\rho}{\rho!} A^\rho \left( \frac{i 2 \pi k}{NT} \right)^{\rho-1} \times \\
\times \sum_{m=0}^{N-1} \sin(4 \pi \varepsilon m) \cdot \exp(-i 2 \pi km/N). \tag{A.4}
\]

Let us compute the coefficients \( R_k \) for the orders \( p = 0 \) and \( p = 1 \), this gives,

\[
R_{k}^{(0, 1)} = \frac{1}{4 i \pi k} + \frac{1}{2 i \pi k} \left[ \frac{N}{k} \text{ integer} \right] - \frac{A}{2 i NT} \left[ \frac{N}{2 \varepsilon - k} \text{ integer} \right] - \\
- \frac{A}{2 i NT} \left[ \frac{N}{2 \varepsilon - k} \text{ integer} \right] \tag{A.5}
\]

where,

\[
G(\lambda) = \frac{\exp(i 2 \pi (\lambda - k/N)) - \exp(i 2 \pi \lambda N)}{1 - \exp(i 2 \pi (\lambda - k/N))} \tag{A.6}
\]

and the superscripts 0 and 1 refers to the considered orders being summed.

The result in (A.5) is obtained for \( \alpha(t, \varepsilon, N) \) defined in the restricted interval \([0, NT]\). The dummy index \( k \) has to change when \( N \) varies in order to keep the ratio \( k/N \) finite \([1]\). We assume that the mean spectrum of (1) is given by (A.5) when \( N \) extends to infinity. This infinity limit means: the more figures \( 2 \varepsilon \) has, the larger \( N \) must be. For \( 2 \varepsilon \) irrational, \( N \) is infinite. Let \( b = k/N \), we perform this limit on the coefficient \( R_k^{(0, 1)} \), the latter is relabelled \( R_{k}^{(0, 1)}(b) \). We get,

\[
R_{k}^{(0, 1)}(b) = \frac{1}{i \pi b} \psi(b) + \frac{A}{2 i T} \left[ \psi(b + 2 \varepsilon) - \psi(b - 2 \varepsilon) \right]. \tag{A.7}
\]

The \( n \)-th order \( (p = n) \) is given by,

\[
R_{k}^{(n)}(b) = \frac{A^n \pi^{n-1} b^{n-1}}{T^n 2 i \pi n!} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \cdot \psi(b + 2(n - k) \varepsilon) \tag{A.8}
\]

where,

\[
\left( \begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!} \tag{A.9}
\]

Relation (A.7) shows the spectral lines created by the successive orders. We note that the different even orders contribute to the creation of harmonics of integral frequency. Every higher order contribute to harmonics created by lower orders and create new ones with their appropriate figures. \( R(b) \) is obtained by summing all orders,

\[
R(b) = \frac{1}{i \pi b} \sum_{n=-\infty}^{\infty} J_n(2 \pi Ab/T) \cdot \psi(b + 2 n \varepsilon). \tag{A.10}
\]

where the series representation of the Bessel function is used. The same calculations are performed for \( \alpha_z(t, \varepsilon, N) \) and \( \alpha_3(t, \varepsilon, N) \). The collection of results leads to the coefficient \( C(b) \) given by (3).
References


