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Adequate mathematical tools for superconductivity

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Abstract. — This paper has for essential objective to point out the existence of, at least, two original mathematical techniques, particularly well adapted to the problems arised from the superconductors ; taking the interest and the complexity of the latter into account, it is really urgent to investigate a concrete cooperation between the concerned communities (Mathematicians, Physicists, Engineers) in order to improve their respective competences.

Introduction.

The superconducting state is ruled by the existence of characteristic surfaces in the phases space : $\{\vec{B}, T, \mathbf{J}\}$, where $\vec{B}$ is the magnetic induction ; T, the temperature and $\mathbf{J}$, the current density vector ; for this reason, the modelling and solution of relevant mathematical problems have to be founded on the « free boundary » techniques ; these have enormously improved with the birth of « Variational Inequalities » between 1960 and 1965 [12, 17, 11, 15] ; they now allow to systematically take into account the inequalities which describe, either the behaviour of media, or the boundary conditions.

For some technical reasons, the superconductors have often to be used in the form of « Superconductor Multifilamentary Composites » (S.M.C.) ; consequently, « Homogenization methods », born around the seventies [10, 21, 2, 24] should play a decisive role in the approach of the macroscopic behaviour of these composites.

The two mathematical tools mentioned above have, as a common origin, the « theory of distribution » and « variational methods » ; consequently they constitute an ideal frame for the numerical computing of fields, particularly by finite elements techniques.

1. Illustrations of problems arisen from the behaviour, and the multifilamentary conditioning, of superconductors.

1.1 THE SUPERCONDUCTING BEHAVIOUR (AT THE CONTINUOUS MEDIA SCALE). — Without giving too much details, let us analyse here a classical scheme proposed to characterize the behaviour of a « type II superconductor ». At the sight of the figure 1, we note the existence of
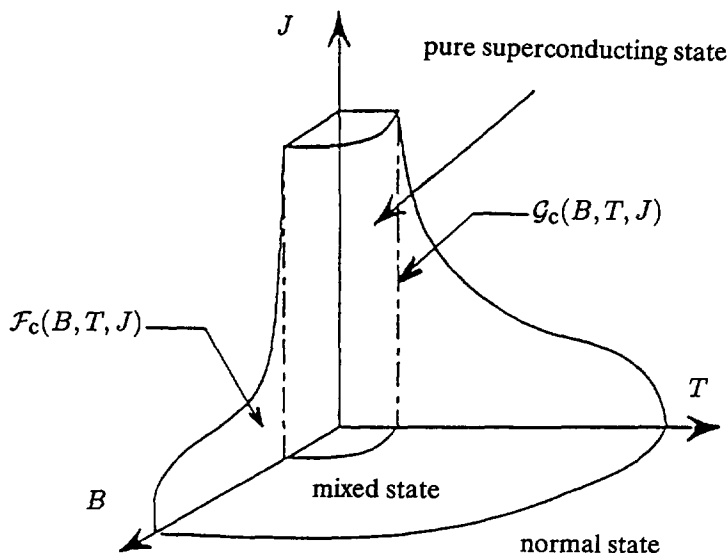
Fig. 1. — Definition of the different states of a type II superconductor in phase space. With : B and J : modulus of magnetic inductions an current density ; T : temperature.

two threshold functionals (critical ones), $\mathcal{G}_c$ and $\mathcal{F}_c$, defined in phase space, and delimiting three possible states of the medium :

Super state (non resistive one) : $\mathcal{G}_c(B, T, J) < 0$

Normal state (resistive one) : $\mathcal{F}_c(B, T, J) > 0$

Mixed state (of transition) : $\mathcal{G}_c(B, T, J) \geq 0$ and $\mathcal{F}_c(B, T, J) \leq 0$,

where B and J designate respectively the modulus of $\vec{\mathbf{B}}$ and $\mathbf{J}$, and T the temperature.

We have then to expect *three zones* in the domain Ω filled by the medium :

● The *pure superconducting* part, in which no resistance is opposed to the passage of the current and, out of which the magnetic induction is repelled

$$\Omega_s = \{x \in \Omega , \quad \text{such as} \quad \mathcal{G}_c[B(x), T(x), J(x)] < 0\} .$$

● *The normal part.* which has come back to the resistive state

$$\Omega_n = \{x \in \Omega , \quad \text{such as} \quad \mathcal{F}_c[B(x), T(x), J(x)] > 0\} .$$

● The *mixed part,* in which the resistance is still quasi null but where the magnetic induction penetrates partially

$$\Omega_m = \{x \in \Omega , \quad \text{such as} \quad \mathcal{G}_c[B(x), T(x), J(x)] \geq 0 \quad \text{and} \quad \mathcal{F}_c[B(x), T(x), J(x)] \leq 0\} .$$

The sets of points defined by

$$\mathcal{G}_c(B(x), T(x), J(x)) = 0 \quad \text{and} \quad \mathcal{F}_c[B(x), T(x), J(x)] = 0$$

are called « Free boundaries », they evolve with the values of the three arguments and are thus *a priori* unknown. Consequently, we are led to write conditional behaviour relations, according to the situation of the considered point in Ω_s, Ω_n or Ω_m.

It is also important to point out the existence of hysteresis phenomena which oblige to take into account, not only the actual values of the fields but still, a part of their anterior histories, which are also *a priori* unknown.

The techniques of modelling by « variational inequalities », offer a lot of tricks to write down explicitly the conditional situations. About this point, it could be really instructive for any one, to look at the piece of work [11] written in 1972 ; it contains, in particular a chapter about Maxwell equations. However, since 1972, many new works about theses techniques, have been produced, especially in mechanics, biology and thermic. It is interesting to point out in particular, two branches : plasticity and, melting or solidification of media, which present some analogies with superconductivity ; variational inequalities techniques have actually promoted considerable improvements in these two situations.

Concerning the superconductors, the mathematicians have some difficulties to understand the diverse microscopic and macroscopic approaches which are propounded ; they become now passionately fond of the « Ginzburg-Landau » equations [6, 16], which appeal sophisticated theories, lead to really difficult mathematical problems, and, however rise far above material cares of the engineers. A more popular approach, from the point of view of the applications, is Bean's critical state model ; it is also much more accessible with the mathematical tools available today [4, 14]. It will be already useful to understand the link between « Ginzburg-Landau » and « Bean » in order that, the two approaches be not completely disconnected. In a general way ; it appears as indispensable that, technicians of superconductivity, define themselves for the Mathematicians, what they consider as the fundamental basis of their art, then cooperate with them to build up concrete models the most satisfying for everyone.

1.2 THE MULTIFILAMENTARY CONDITIONING OF THE SUPERCONDUCTORS. — Assuming that the behaviour of a superconductor has been clearly defined, it will be necessary anyway to situate it in its practical conditioning, that is, the superconducting multifilamentary composite (S.M.C.). Figure 2 (borrowed from [19] *via* [4]) gives rise to several remarks.

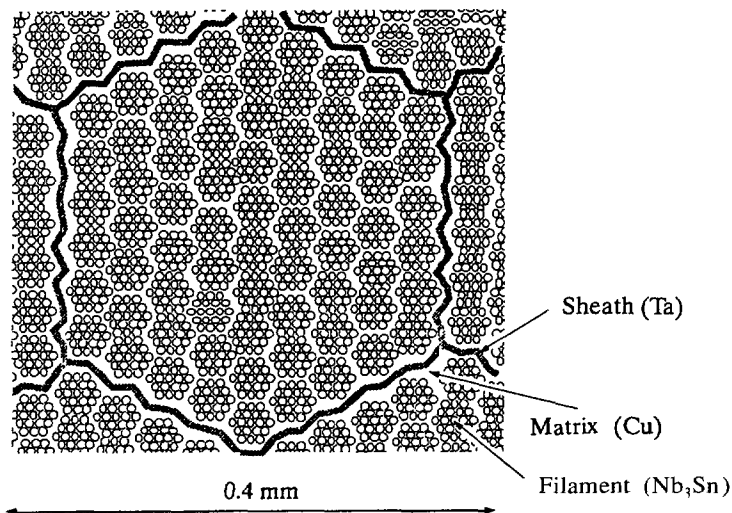i) It points out the multiperiodical feature of a S.M.C.

Fig. 2. — Typical structure for a Superconducting Multifilamentary Composite (SMC).

ii) It shows up the coexistence of three media, with different electromagnetic behaviours : the superconducting filaments, the matrix and the sheath.

iii) It reveals the existence of several characteristic scales. Choosing, for instance, the cable diameter as the unit length, we count at least three small parameters, that is : the respective diameters of threads, strands and filaments. Moreover, the threads and strands are twisted in helical packings whose constant pitches may constitute some intermediate scales.

Homogenization techniques offer systematic ways to reach, distinguish one from the others and, eventually optimize the different macroscopic behaviours of S.M.C., according to the respective orders of magnitude of the multiple parameters ; the periodicity of S.M.C. is a quite propitious factor of efficacy for these techniques.

It is interesting to mention a recent thesis [1] which paid particular attention to multiscale composites with applications to the modelling of bones (whose structure is practically as complicated as that of S.M.C.).

1.3 CONSEQUENCES OF THE DOUBLE COMPLEXITY OF C.M.S. — The brief presentation that we gave above of the superconductor has pointed out a double difficulty :

• that of the « threshold behaviour law », depending of three physical fields and generating two free boundaries ;

• that of the composite structure, with coexistence of several small parameters.

We have to admit that, if the conjugate use of variational inequalities and homogenization techniques, does not offer *a priori* difficulties, the complex conjoncture of C.M.S. calls for particular efforts of adaptation.

2. Examples of problems related to the superconductivity, already solved by the above mathematical tools.

The two problems described below did not need the conjugate use of inequalities and homogenization ; the first one because it has previously been « homogenized » in a way very different from ours [13] ; the second one because the applied magnetic field was supposed weak enough to allow the filaments to stay permanently in the pure superconducting state ; hence the absence of free boundaries.

2.1 STEFAN TYPE FREE BOUNDARY PROBLEMS RELATED TO THE CURRENT DISTRIBUTION IN A SUPERCONDUCTOR [13, 22, 23, 26]. — The authors of [13] suggest a model to find J, modulus of the « global current density » vector, in a cylindrical S.M.C., with circular cross section ; the latter, being submitted to a exterior magnetic field dependent of the time τ, perpendicular to its axis. J cannot exceed a critical value that depends on the magnetic induction.

In [22, 23], a few reliable approximations (suggested by physicists), as well as some variables and functions changes, allowed to obtain the simplified model :

Find u, time dependent function, defined on a circle Ω such that :

$$u(\mathbf{x}, t) \leq C \quad \forall \mathbf{x} \in \Omega \quad \text{and} \quad \forall t \tag{1}$$

$$\frac{\partial u}{\partial t} - \Delta u = 0 \quad \text{in the part of } \Omega \text{ where } u(\mathbf{x}, t) < C \tag{2}$$

$$u(\mathbf{x}, 0) = 0 \quad \forall \mathbf{x} \in \Omega \tag{3}$$

$$\int_{\Omega} u(\mathbf{x}, t) \, dx = \lambda t . \tag{4}$$

Here, u represents yet the unknown J, while $t = B(\tau)$ is the new temporal scale (this change of temporal scale is only possible if the magnetic induction modulus B is a regular and strictly increasing function).

C and λ are strictly positive given constant numbers.

This problem is an « inequation » because of the threshold condition (1) ; its originality lies in the fact that there is no boundary condition, this one being replaced by a global flux condition (4) through the conductor cross section.

To solve this problem one had to call on a maximum principle [20], show that the flux integral condition could be interpreted as a free boundary condition at $r = s(t)$ and then, pass on to the following free boundary formulation of the problem (where r is the cylindrical radius and $\theta(r, t) = u(\mathbf{x}, t)$) :

Find $\theta(r, t) \leqslant C$ and $s(t) \leqslant R$ defined for $r \in\,]0, R[$ and $t \in\,]0, T[$ such as :

$$\frac{\partial \theta}{\partial t} - \Delta \theta = 0 \quad \text{as far as} \quad r < s(t)$$

$$\theta[s(t), t] = C$$

$$\frac{\partial \theta}{\partial r}[s(t), t] = \frac{\lambda}{2\pi s(t)}$$

$$\theta(r, 0) = \theta_0(r) \quad \text{with} \quad \theta_0(R) = C$$

$$s(0) = R .$$

The finite time T characterizes the state : $s(T) = 0$ and $\theta(r, T) = C$ everywhere ; it is *a priori* computed from the flux integral relation.

This new model describes only the second phase of evolution for the unknown function u ; that is from time t_1 (taken as the new origine) where the saturation $u(\mathbf{x}, t) = C$ appears at $|x| = R$.

The actual results for this problem were :

i) two completely different proofs of the existence and uniqueness of the solution, giving in addition some interesting properties of the latter ;

ii) an original numerical method ;

iii) some numerical solutions for varying sets of parameters ;

iv) a generalization when C is a decreasing function of B (which is the realistic situation).

2.2 ESTIMATION OF THE GLOBAL CONDUCTIVITY OF A S.M.C. SUBMITTED TO A WEAK MAGNETIC FIELD [18]. — This work allowed, among others things, to test the homogenization techniques ; it compares them first with experiments [9] and, secondly with other computations ([5] in particular).

The steps and results have been (under the control of specialists) the following ones.

i) Modelling and solution of a general problem of induction calculus, for a composite with superconducting inclusions ; this composite being embedded in an exterior uniform magnetic field $\vec{\mathbf{H}}(t)$, weak enough to let the inclusions in a pure superconducting state. Under these conditions, the relations to write in $\mathbf{R}^3$ are :

— Maxwell's equations everywhere ;
— the material laws :

$$\vec{\mathbf{B}} = \mu_0 \vec{\mathbf{H}} \quad \text{and} \quad \mathbf{J} = 0 \quad \text{outside the composite} ;$$

$$\vec{\mathbf{B}} = \mu_0 \vec{\mathbf{H}} \quad \text{and} \quad \mathbf{J} = \sigma_m \mathbf{E} \text{ in the matrix} ;$$

$$\vec{\mathbf{B}} = \vec{\mathbf{0}} \quad \text{and} \quad \mathbf{E} = 0 \quad \text{in the inclusions} ;$$

— the transmission conditions at the various interfaces ;

— the initial conditions.

The proof of the existence and uniqueness of the solution $\vec{B}$ for this general problem allowed to check the validity of the chosen model.

ii) *A priori* properties of the solution in the particular case of a cylindrical S.M.C. situated, either perpendicularly, or paralelly, to an exterior given magnetic field of fixed direction.

iii) Taking the large number of fibers and their periodicity into account :

letting ε be the ratio « diameter of the period/diameter of the cylinder » it was then obvious that the solution $\vec{B}_\varepsilon$ would depend on this small parameter as well as the basic cell. The exact mathematical study of $\vec{B}_\varepsilon$ behaviour when ε tends to 0, led to the limit $\vec{B}_0$; moreover it was showed :

— on the one hand, that $\vec{B}_0$ is an approximation of $\vec{B}_\varepsilon$ all the better as ε is smaller ;

— on the other hand, that $\vec{B}_0$ is the solution of the same boundary value problem, for a homogeneous conductor of same shape ; an anisotropy being introduced by the direction of fibers (parallel to the axis).

iv) Theoretical computation of the « equivalent homogeneous conductor ».

The equivalent homogeneous medium does not depend on the applied exterior field ; however, each particular boundary condition allows only to compute a part of the homogenized coefficients. So, in the case of an exterior field parallel to the composite axis, it has been found :

$$\mu_{33} = (1 - \theta^s)\, \mu_0\,, \quad \sigma_{\alpha\beta} = \frac{1}{\mu_{33}} \frac{q_{\alpha\beta}}{q_{11}\, q_{22} - q_{12}^2} \quad \text{for} \quad \alpha,\ \beta = 1 \text{ or } 2$$

(longitudinal permeability and transversal conductivity).

Here θ_s is the volume proportion of superconducting medium in the S.M.C. and

$$q_{\alpha\beta} = \frac{1}{\sigma_m\, \mu_0} \left[\delta_{\alpha\beta} - \frac{1}{|Y^*|} \int_{Y^*} \frac{\partial \chi^\alpha}{\partial y_\beta}\, dy \right]\,; \quad \delta_{\alpha\beta} = \begin{cases} 1 & \text{if} \quad \alpha = \beta \\ 0 & \text{if} \quad \alpha \neq \beta\,. \end{cases}$$

In this last relation, Y^* is the domain filled by the matrix (of conductivity σ_m) in the basic cell (Fig. 3) ; its area is $|Y^*|$; χ^α is the solution of an intermediate problem to compute over Y^*.

Thanks to the symmetries of the basic cell, it was possible to show that $q_{11} = q_{22} = q$ and $q_{12} = 0$; from which the calculus of the global transverse conductivity

$$\sigma_T = \frac{1}{(1 - \theta^s)\, q}$$

v) Numerical computation by finite elements methods and comparison with experimental results and other theoretical predictions (Fig. 4).

3. Basic notions about the methods : variational inequalities and homogenization.

In this section, we first mention the necessity to introduce the distribution spaces that we briefly describe. We recall then the variational formulation of a problem and we show how to gather in an admissible subset, all the *a priori* properties of its solution : if these properties contains some inequalities coming from either the constitutive law (like in superconductivity) or the boundary conditions (like in contact problems), the admissible subset is not a vectorial subspace ; in these conditions, we obtain a variational inequality. The last subparagraph is devoted to the description of the homogenization techniques.
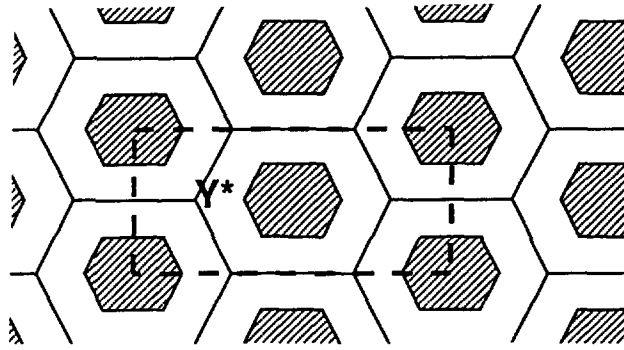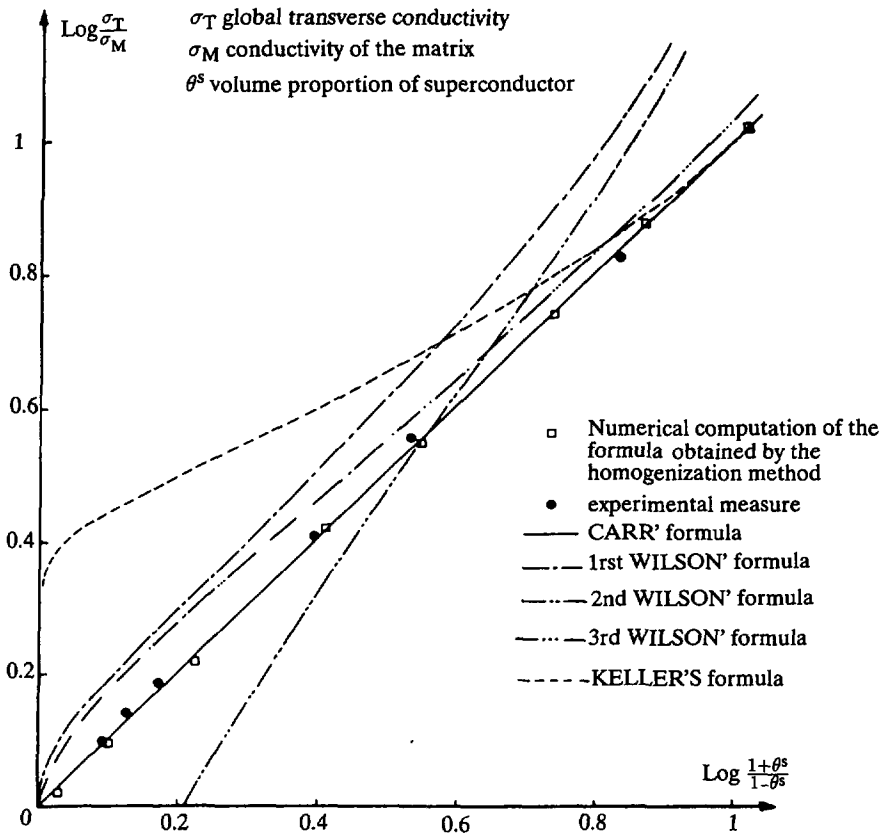
Fig. 3. — Basic cell for a hexagonal structure.

Fig. 4. — Variation of the transverse conductivity for a SMC according to the ratio of superconductor.

3.1 DISTRIBUTION SPACES [8]. — The right context to solve partial differential equations is no longer the one of « regular spaces » (continuous differentiability up to a certain order m). The frame of « distribution spaces » (often of Sobolev type) is much more comfortable : here the functions need only to be defined « almost everywhere » and however, are differentiable to all orders under the « distribution » interpretation ; it is only required that the partial derivatives,

up to the order m, be of p-th power integrable (p to be defined, $p = 2$ in linear problems). These spaces, whose properties are well-known to mathematicians, are of easy use for the non specialists ; they have the following advantages.

i) They can be equipped with a norm which gives them a topology of complete space (real advantages in the case of approximation of solutions by sequences ; we are then sure, when this last one converges, that the limit is still in the working space).

ii) They allow to write terms like

$$\sum \frac{\partial}{\partial x_i} \left[a_{ij} \frac{\partial u}{\partial x_j} \right]$$

even when the coefficients a_{ij} are only piecewise constant numbers (ideal context to study heterogeneous media).

iii) They are ordered (in the « inclusion relation » meaning) relatively to the regular spaces ; the regular spaces being generally dense in the distribution spaces for a given m.

3.2 ADMISSIBLE SETS FOR A GIVEN PROBLEM AND ASSOCIATED VARIATIONAL INEQUALITIES [25, 8, 11]. — For any problem and each unknown quantity of this problem, it is necessary to define an « admissible set $\mathcal{U}_{ad}$ », in which the solution has to be looked for ; in this set, one have to insert the different constraints that this one is supposed to satisfy, for instance, some of the boundary conditions. When $\mathcal{U}_{ad}$ is defined, we have to choose a largest distribution space V, containing $\mathcal{U}_{ad}$, which will act as the resolution space for the problem.

To evoke quickly what is the variational formulation of the studied problem, we may express the followings steps.

i) Multiplication of the equation by the difference « $v - u$ » (where u, solution, has to belong to $\mathcal{U}_{ad}$, and v is an arbitrary element of $\mathcal{U}_{ad}$).

ii) Integration by parts, on Ω (domain filled by the physical media) of the relation obtained. We then use all the properties of the elements of $\mathcal{U}_{ad}$.

iii) Showing up (if the differential operator is linear) a bilinear form $a(,)$ and a linear one $L( )$, defined on $\mathcal{U}_{ad}$ and such as, if u is the solution of the primitive problem, then it also has to be a solution of the following variational one :

$$\boxed{\text{Find} \quad u \in \mathcal{U}_{ad} \quad \text{such as} \quad a(u, v - u) \geq L(v - u) \quad \forall v \in \mathcal{U}_{ad}} .$$

The theoretical solution of such a problem is founded on some theorems which give necessary and sufficient conditions for existence and uniqueness of u. One has then to come back to the original problem, showing in fact that u is more regular than the distributions space V did require.

The numerical solution of the variational problem consists on a computation of the same one, replacing $\mathcal{U}_{ad}$ by its intersection with a finite dimension subset of V (let n be this dimension) ; then we show that the obtained solution u_n converges toward u when n increases. The frame of finite elements allows to find such adequate subsets.

An important result to know is that, when the bilinear form $a(,)$ is symmetric, the above variational problem is equivalent to the following one :

$$\boxed{\text{Find} \quad u \in \mathcal{U}_{ad} \quad \text{such as if} \quad J(v) = 1/2 \, a(v, v) - L(v) \quad \text{then} \quad J(u) \leq J(v) \quad \forall v \in \mathcal{U}_{ad}} .$$

We rejoin there the « concept of energy minimization » more vivid for a physicist.

If $\mathcal{U}_{ad}$ is a subvector space of V, then the variational problem can be reduced to the variational equation

$$\boxed{\text{Find} \quad u \in \mathcal{U}_{ad} \quad \text{such as} \quad a(u, v) = L(v) \quad \forall v \in \mathcal{U}_{ad}}.$$

If $\mathcal{U}_{ad}$ is only a convex subset of V, then we have to keep the inequality and the problem is called : variational inequality.

Let us mention that analogous processes are also developed for nonlinear situations.

3.3 NOTIONS ABOUT THE TECHNIQUES OF HOMOGENIZATION (PERIODICAL CASE) [7, 21, 18, 1, 2].

— Four important steps have to be mentioned to briefly introduce « Homogenization ».

i) First it is essential that the mathematical model about the heterogeneous medium be previously clear. It is not yet necessary to take the periodicity and the small size of the representatif element of this medium into account. The existence and number of solutions of an associated general boundary value problem have to be *a priori* studied to check the validity of the model. In fact, the mathematician thinks about homogenization only for numerical computations ; he discovers that the necessary discretisation will not be fine enough to take the details of the structure into account ; hence the idea to look for an *a priori* theoretical approximation.

ii) One has to characterize the basic period of the structure, by its internal organization, but also by the small parameter ε which represents its size : for instance :

$$\varepsilon = \max \varepsilon_i \quad \text{if} \quad \varepsilon_i \text{ is the reduced period for the direction } x_i .$$

The homogenization process consists on the « *a priori* study of the problem when ε goes to zero » ; for this reason, it is necessary to associate a macroscopic fixed cell P to the microscopic basic period ; we obtain that by a similitude of ratio $1/\varepsilon$ which maintains

$$p_i = \frac{\varepsilon_i}{\varepsilon} = \text{Cte} , \quad [\text{Fig. 5}]$$

in the limit process $\varepsilon \to 0$. This operation allows in particular to attach to each particule of the medium, two kinds of coordinates : the macroscopic ones, x_i in Ω (domain filled by the medium) and the microscopic ones : $y_i = \dfrac{x_i}{\varepsilon}$ in P.

iii) The multiple scales method (formal but intuitive and constructive one) consists on the following expectation : the solution u_ε of the real problem should be the superposition of a slowly varying function of $\mathbf{x}$ (on which the boundary conditions would play a leading part) and of a periodical fluctuation (depending of $\mathbf{y}$), taking the heterogeneity of the medium into account. That idea is expressed by looking for the solution $u_\varepsilon(\mathbf{x})$ as an asymptotic expansion

$$u_\varepsilon(\mathbf{x}) = \omega_0(\mathbf{x}, \mathbf{y}) + \varepsilon \omega_1(\mathbf{x}, \mathbf{y}) + \varepsilon^2 \omega_2(\mathbf{x}, \mathbf{y}) + o(\varepsilon^2)$$

where each term ω_k is supposed to be periodic in $\mathbf{y}$.

When this expansion is carried back to equations and boundary conditions of the studied problem, each partial derivation of the terms

$$\omega_k^*(\mathbf{x}) = \omega_k\left(\mathbf{x}, \frac{x}{\varepsilon}\right)$$

brings a factor ε^{-1}

Composition medium with two components

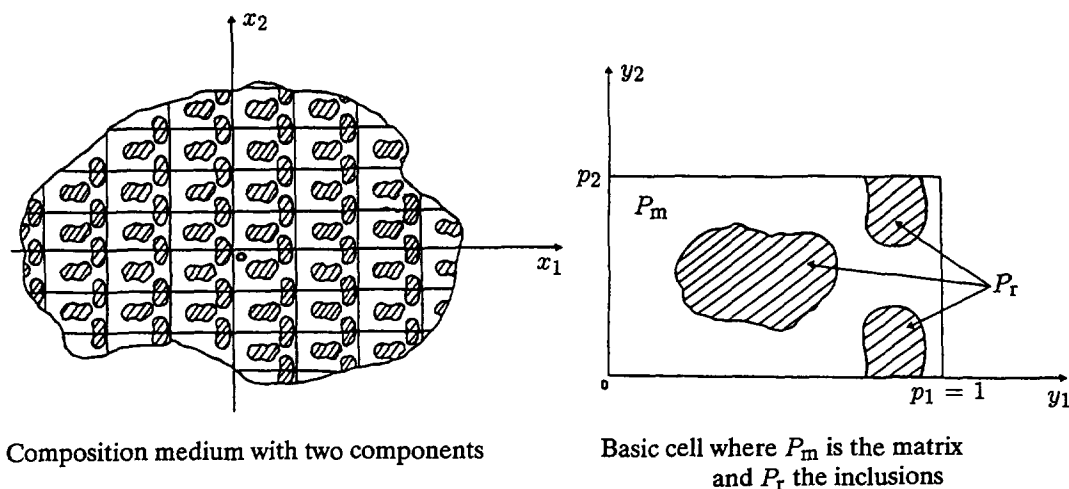Basic cell where P_m is the matrix
and P_r the inclusions

Fig. 5. — Definition of the basic cells (macroscopic and microscopic ones).

The determination of the ω_k terms is then obtained by successive identification to 0 of the coefficients A_p corresponding to ε^p, beginning by the more dangerous in the limit process ($\varepsilon \to 0$), that is $p < 0$.

One generally shows in the first identification that ω_0 does not depend on the fluctuation represented by **y**; the following identifications allow then, to characterize this term ω_0 as the solution of an analogous problem, for an equivalent homogeneous medium ; the determination of the corresponding homogenized coefficients is obtained by a two steps computation on the simple cell P.

iv) The energy method (more sophisticated but also more rigorous) consists of a direct limit process ($\varepsilon \to 0$) on the boundary values problem under study ; it allows to give the exact nature of the convergence of u_ε toward ω_0 (the same as above) and to get *a priori* estimates of the difference between the exact solution u_ε and its approximation ω_0.

4. Conclusion.

We could, with some more pages, work out other pertinent arguments in order to emphasis on the interest of the mathematical tools described above ; however the below reference list (although non exhaustive) will allow anyone, according to his or her own motivations, to thoroughly study the few ideas evoked above.

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