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A mathematical model for the shape of wave-soldered joints on printed circuit boards

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Abstract. — The problem of automatic inspection of solder joints in Printed Circuit Boards by 3D machine vision, is addressed. To provide an appropriate theoretical background to its solution, a mathematical model for the geometric shape of a wide class of wave-soldered joints is elaborated. The formation of such joints is considered approximately as a static phenomenon and studied as a moving boundary constrained variational problem, leading to a general differential equation, which offers a satisfactory model of the joint’s shape. The case of axisymmetric through-hole-soldered joints is studied in detail, by integrating numerically the corresponding equations in various cases. The use of the model for simulation of sensor data is illustrated and the possibilities of using the information provided by the model for the development of inspection algorithms, are discussed.

1. Introduction.

Today, there is an urgent need for the development of advanced automatic optical inspection systems for industrial assembly lines of Printed Circuit Boards (PCBs). This need emerges from current trends of Electronic and Information technology industries, like the miniaturization of components, increasing complexity of the PCBs and strong demand for defect levels at the range of parts per million. Today, 100% automatic inspection of components and assemblies is generally considered desirable.

Currently, the most critical and demanding problem in the area of automatic inspection of assembled PCBs, is the inspection of solder joints. Despite the increasing interest and research efforts all over the world, still no satisfactory solution exists meeting the industrial requirements. At present solder joint inspection is mainly done by humans with the aid of microscopes and is based largely on subjective criteria. Human error is unavoidable and the current industrial trends mentioned above, are expected to make soon this practice inapplicable. On the other hand, electrical tests cannot guarantee the reliability of a solder joint. Miniaturization of components poses also limitations on in-circuit electrical tests with pin-beds.
Various techniques for the automatic inspection of solder joints have been investigated and proposed [3]. Most of them employ grey level 2-dimensional (2D) machine vision, while the use of more sophisticated sensors, providing 3D (range) data, seems to be advantageous and more promising. Therefore, there is an increasing interest for research in this field [4]. The main drawback of 2D machine vision for solder joint inspection, is that the input information depends strongly on the scene illumination and the object's surface reflectivity, whereas for solder joints what matters more, is their geometrical shape. A grey level image contains information for the object's shape too, but this information is incomplete and difficult to extract. On the other hand, direct 3D sensors acquire $x$, $y$, and $z$ coordinates, for every individual nonhidden surface element of the object. In this way a complete map of the whole surface of a scene is generated. The availability of this information fascilitates the development of inspection algorithms and overcomes the bottleneck of grey level image processing.

The advantages and prospects of 3D machine vision has been the main motivation for the initiation of the EEC Esprit 2017-TRIOS research project and the present work has been conducted in this context. The project started in March 1989 and is expected to finish in 1993. It aims at the development of prototype, high-speed, high-resolution 3D inspection systems, to be applied on the automated inspection of bare and assembled PCBs. The project has so far advanced successfully and two different prototype inspection systems are currently half way to their completion. Each system includes a 3D sensor, 3D data processing hardware and software and special CAD links for the extraction of reference information on the design features of the boards under inspection. The 3D sensor of the first system (for bare board inspection), features a resolution in $x$, $y$ directions of $15 \times 15$ $\mu$m, a scan length of 300 mm and a data rate of 15 Megapixels/s. The second sensor (for assembled board inspection), has a resolution of $40 \times 40 \times 40$ $\mu$m in $x$, $y$, $z$ directions and its data rate is around 1.5 Megapixel/s. Both sensors are synchronized, triangulation-based laser scanners and make use of polygonal mirrors. The reliability targets of both systems are high. As far as the inspection system for assembled PCBs is concerned, a reliability of more than 98 % is desired, while the false alarm rate should be less than 0.005 % of the inspected objects. The most important task of the inspection system for the assembled PCBs, will be the inspection of solder joints.

The use of 3D sensors presents the solder joint inspection problem in a new perspective and makes necessary the development of appropriate 3D data processing techniques. The range data offer direct 3D scene information and thus it is possible to extract various 3D features and surface characteristics that were not available in 2D images. But even with 3D vision, the classification of a solder joint with a high degree of reliability, is a difficult task. The definition of effective 3D features and operators, is far from evident. This is especially true when certain, difficult to detect, defects come under consideration (bad wetted or cold joints). While the geometric shape of a solder joint is the key determinant of its quality [1], and classification schemes for solder joint defects exist [2], a serious difficulty remains due to the complexity of solder joint shapes. Good solder joints have very complex and widely differing shapes — no two joints being exactly alike — and the same is true for defective ones. Therefore, a thorough study of the geometry of solder joint shapes and their underlying invariant geometrical properties, seems to be the essential starting point for an effective solution of the problem.

Following this line of thought, we will attempt to provide in this paper, a firm mathematical basis and a clear and detailed understanding of the geometrical properties of the expected shapes of wave-soldered joints. For this purpose, we will work out in detail a mathematical model for the shape of a wide class of wave-soldered joints. This class, which we call the class of «normal joints», contains all joints of acceptable quality as well as certain types of defective ones (see Sect. 6). The model is based on the physics of the formation process of a solder joint and it is derived under certain reasonable simplifying assumptions that are
presented below. The behaviour of the model in various cases is also studied. The proposed model provides us with valuable a priori knowledge for the geometrical shape of a solder joint and it is expected to be useful for the interpretation of range data and to guide effectively the search for appropriate 3D features and the development of defect detection algorithms.

Seen as a whole, the process of the formation of a solder joint during wave-soldering is a rather complicated process, involving fluid dynamics and thermodynamics (see Fig. 1).

However as a first approximation and for an acceptable solder joint, it is reasonable to assume the following:

(i) the relaxation time-scale of transient phenomena is much smaller than the cooling and solidification time-scale. This means that during the process, transient phenomena induced by the passing solder wave, have enough time to relax before solidification starts. In this way the joint reaches static equilibrium, while it is still liquid. Let us call the moment when this equilibrium is reached, by $t_0$ and the moment when solidification starts, by $t_1$;

(ii) the difference between the above mentioned time-scales suggests that, it is reasonable to assume that after $t_0$, cooling and solidification can be considered as quasi-static processes, i.e. that static equilibrium is preserved during these processes;

(iii) the shape of the solder joint does not change substantially after time $t_1$, when solidification starts. Thermal contraction due to cooling towards the room temperature has been calculated and it has been found that there is a volume reduction less than 2.5 %. This corresponds to solder height variations of the order of 0.8 %, or less than 5 $\mu$m for typical solder joints. Such variations are much beyond the resolution capabilities of the sensors under consideration and correspond to changes of the solder shape which have no practical importance.

Therefore, under the above assumptions, defining the class of normal joints, the problem is reduced to the determination of the shape of a solder joint in static equilibrium just before solidification starts at time $t_1$ and in a steady, homogeneous temperature field.

In section 2 we consider an axisymmetric model and we minimize the energy of the joint due to gravity and surface tensions. This is a moving-boundary variational problem. The corresponding Lagrange equation determining the profile of the solder surface, is derived and it is found to be directly related to important geometrical characteristics of the surface.

In section 3 this equation is numerically integrated for realistic values of solder volume, solder alloy physical parameters and copper pin and pad diameters. The behaviour of the model is studied for various sets of parameters and the results are plotted in 2D and 3D diagrams.
In section 4 we drop the assumption of axial symmetry and by a similar, though mathematically more sophisticated method, we derive the equation determining the shape of a solder joint in this general case.

In section 5 the use of the model for simulation purposes is illustrated. Here the resolution and noise characteristics of an existing 3D scanner are taken into account and the result of the simulation is compared with real data.

Finally in section 6 the domain of applicability of the present model, possible generalizations and extensions and the possibilities of using the model for the solution of solder joint inspection problems are discussed.

2. The axisymmetric model.

We look for the shape of the surface of a liquid solder joint, under static equilibrium and constant temperature. We suppose that (a) gravity acts in the z-direction, perpendicularly to the metallic pad. (b) the copper-pin has a cylindrical shape of radius \( R_1 \), normal to the pad (c) the metallic pad is a circular disk of radius \( R_2 \), with its center on the axis of the lead (see Fig. 2). (d) the liquid solder forms a solid of revolution, so that its shape is completely determined by its profile \( z = h(r) \), where \( r \) is the polar radius on the \( xy \)-plane (the metallic pad).

The system is composed by the liquid solder (substance 1) the copper-pin (2), the air (3), and the metallic pad (4). The joint is formed under the influence of gravity, surface tension and adhesion. To keep the analysis as simple as possible we first suppose that the solder covers the whole basis area (i.e. \( h(R_2) = 0 \)), which is constant, so that the surface energy between the solder and the metallic pad is constant and plays no role in the determination of \( h(r) \) (see below). Moreover the work done by the solder against the atmospheric pressure is constant since the solder volume \( V_0 \) is constant, which is an important constraint of the problem determined by the soldering wave dynamics.
Under these assumptions the total energy $E$ is

$$E = E_g + E_{13} + E_{12} + E_{23}$$

(2.1)

where $E_g$ is the solder's gravitational potential energy and $E_{ij}$ is the surface energy due to the substances $i$ and $j$. If $a_{ij}$, $S_{ij}$ are the surface tension constant and separating area of $i$ and $j$, respectively, then

$$E_{ij} = a_{ij} S_{ij}.$$  

(2.2)

Clearly

$$V = 2 \pi \int_{R_1}^{R_2} \frac{r h(r)}{r} dr = V_0$$

(2.3)

$$E_g = 2 \pi \varepsilon \int_{R_1}^{R_2} r h'(r) dz = 2 \pi \varepsilon \int_{R_1}^{R_2} r h'(r) dr$$

(2.4)

where $\varepsilon$ is the solder specific weight. By axial symmetry

$$E_{13} = 2 \pi a_{13} \int_{R_1}^{R_2} r \sqrt{1 + (h'(r))^2} dr, \quad h' = \frac{dh}{dr}$$

(2.5)

whereas using that $h(R_2) = 0$ we obtain

$$E_{12} = -2 \pi a_{12} R_1 \int_{R_1}^{R_2} h'(r) dr$$

(2.6a)

$$E_{23} = \text{Const.} + 2 \pi a_{23} R_1 \int_{R_1}^{R_2} h'(r) dr.$$  

(2.6b)

Substituting (2.4-2.6) in (2.1) we get

$$E = 2 \pi \int_{R_1}^{R_2} \left( a_{13} r \sqrt{1 + (h'(r))^2} + \frac{\varepsilon rh'^2}{2} + (a_{23} - a_{12}) h'(r) R_1 \right) dr$$

(2.1')

which has to be minimized under the constraint (2.3). This is variational problem with a conditional extremum, and with one boundary point moving along the straight line $r = R_1$.

In general an extremum of the functional

$$J(h) = \int_{r_1}^{r_2} L(r, h(r), h'(r)) dr$$

with moving boundaries $(r_1 + \delta r_1, h_1 + \delta h_1), \ (r_2 + \delta r_2, h_2 + \delta h_2)$ subject to the constraint (2.3), is obtained by solving the unconstrained Lagrange equation

$$\frac{\partial L^*}{\partial h} - \frac{d}{dr} \left( \frac{\partial L^*}{\partial h'} \right) = 0, \quad L^* = L - 2 \pi \lambda rh$$

(2.7)

where $\lambda$ is a constant Lagrange multiplier, together with the condition

$$\left[ \left( L^* - h' \frac{\partial L^*}{\partial h'} \right) \delta r + \frac{\partial L^*}{\partial h'} \delta h \right]_{r_1}^{r_2} = 0$$

(2.8)
(see e.g. [5] chap. 7, [6] Sect. 14). In our case only the first boundary is moving and in fact
\( \delta r_1 = 0 \) \( (r_1 = R_1) \) so that (2.8) gives

\[ \frac{\partial L^*}{\partial h'} \bigg|_{-R_1} = 0 . \]  

(2.8')

Equations (2.1'), (2.7), (2.8') readily imply

\[ \frac{h''}{(1 + (h')^2)^{3/2}} + \frac{h'}{r(1 + (h')^2)^{1/2}} - \frac{\varepsilon h - \lambda}{a_{13}} = 0 \]  

(2.9a)

\[ h'(R_1) = - \cot \theta \]  

(2.9b)

\[ h(R_2) = 0 \]  

(2.9c)

where

\[ \cos \theta = (a_{23} - a_{12})/a_{13} . \]  

(2.9'b)

Equations (2.9), (2.3) form a boundary value problem for the determination of \( h(r) \) and \( \lambda \). Notice that the surface energies \( E_{12}, E_{23} \) enter the problem only through (2.9b), determining the angle of contact (wetting angle) \( \theta \).

Remarks: (i) For a reversed solder joint \( h(r) \leq 0 \) and (2.9a, b) hold with \( \varepsilon, \theta, \lambda \) having the opposite sign.

(ii) If the solder wets completely the pin, the latter is covered by a solder thin film for \( z > h(R_1) \), so that \( \theta = 0 \) or \( h'(R_1) = \infty \). Notice that in this case (2.9'b) is no longer valid since \( E_{23} = 0, E_{12} = \text{Const.} \), and the variation method leading to (2.8') has no meaning at all.

(iii) For a « big » metallic basis, where the solder cannot cover the whole disk, instead of (2.9c) we have \( h(R) = 0 \), for some \( R < R_2 \), and now there are two moving boundaries, the second one moving along the \( r \)-axis. In addition, the surface energies \( E_{34}, E_{14} \) between air and metallic basis and solder and metallic basis respectively, must be taken into account:

\[ E_{14} = \pi a_{14} (R^2 - R_1^2) = 2 \pi a_{14} \int_{R_1}^{R_2} dr \]  

(2.10a)

\[ E_{34} = \pi a_{34} (R_1^2 - R^2) = \text{Const.} - 2 \pi a_{34} \int_{R_1}^{R} r dr \]  

(2.10b)

so that the Lagrangian is

\[ L = 2 \pi \left[ a_{13} r (1 + (h')^2)^{3/2} + r \left( \frac{\varepsilon h^2}{2} - \lambda h \right) + (a_{23} - a_{12}) h' R_1 + (a_{14} - a_{34}) r \right] . \]  

(2.11)

Since \( h(R) = 0 \), so that \( \delta h_2 = 0 \) in (2.8), equations (2.11, 2.7, 2.8), lead to (2.9a, b) and

\[ \tan \theta = - h'(R) , \quad h(R) = 0 \]  

(2.9'c)

where

\[ \cos \theta = (a_{34} - a_{14})/a_{13} . \]

(iv) The first two terms in (2.9a) are just twice the mean curvature \( H \) of the solder surface given in the parametric form

\[ x = \cos \varphi , \quad y = \sin \varphi , \quad z = h(r) \]

\[ H = \frac{1}{2} \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right) \]
where \( \rho_1, \rho_2 \) are the principal radii of curvature of this surface (see e.g. [7] Sect. V.6). Therefore (2.9a) can be rewritten as

\[
\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{\varepsilon h - \lambda}{a_{13}} \tag{2.9'a}
\]

In section 4 it will be shown that (2.9'a) is independent of the assumption of axial symmetry and implies that \( H \) is an important feature, to the extent that it can be used in the classification of joints (see Sect. 6).

(v) Equation (2.9'a) and equations (2.9b), (2.9c) could be obtained by using different types of argument (see e.g. [8, 9]), but it is interesting that a unified treatment can be given, based on the fact that the problem can be formulated as a constrained variational problem with moving boundaries.

(vi) It must be emphasized that the boundary value problem defined by (2.9), (2.3) may not have a solution for arbitrary values of \( V_0, \theta \) and \( a_{13} \). In fact, numerical integration is not possible beyond a maximum value of \( V_0 \), which is expected upon physical considerations, since for given values of \( \theta, a_{13}, \) and \( \varepsilon \), there can be no static equilibrium of the model for arbitrarily large values of \( V_0 \).

3. Integration of the Lagrange equation.

Here we consider in detail a reversed axisymmetric solder joint and we integrate (2.9) with \( \varepsilon, \lambda, \theta \) having the opposite sign, subject to the constraint (2.3) (see remark (i) in Sect. 2). This is not an initial value problem (cf. Eqs. (2.3, 2.9b, c)), but it can be reduced to a two-point boundary value problem ([10] chap. 16) for the following system

\[
h' = g, \quad \frac{g'}{(1 + g^2)^{3/2}} + \frac{g}{r(1 + g^2)^{1/2}} + \frac{\varepsilon h - \lambda}{a_{13}} = 0 \tag{3.1a}
\]

\[
\lambda' = 0, \quad V' = 2 \pi h(r) \tag{3.1b}
\]

with

\[
g(R_1) = -\cot \theta, \quad h(R_2) = 0 \tag{3.1c}
\]

\[
V(R_1) = 0, \quad V(R_2) = V_0 \tag{3.1d}
\]

where an accent denotes differentiation with respect to \( r \). Equations (3.1) have been integrated numerically using the «shooting method» ([10] Sect. 16.1). Thus a combination of a 4th order Runge-Kutta (RK) method for the integration of (3.1) and a two-dimensional Newton-Raphson (NR) method for the determination of \( \lambda, h(R_1) \) has been used: for each step of the NR, (3.1) had to be integrated 3 times by RK and the NR iterations were continued until

\[ h(R_2) < 10^{-7} \text{mm}, \quad V - V_0 < 10^{-5} \text{mm}^3 \]

Moreover the number of steps for each RK was 5701, corresponding to a step of \( r \) equal to 0.12 \( \mu \text{m} \). The initial values of \( \lambda, h(R_1) \) had to be chosen carefully to avoid computational overflow.

For our calculations a solder alloy of 60% Sn-40% Pb with \( a_{13} = 0.485 \text{ J/m}^2 \) at 350 °C and specific weight \( \varepsilon = 87,878 \text{ Nt/m}^3 \) was used, according to data provided by the Siemens Production Department. The pin and pad radius were taken \( R_1 = 3 \text{ mm}, R_2 = 1 \text{ mm}, \) on the basis of measurements of a certain reference joint on a sample PCB. Values of \( a_{23}, a_{12} \) were not available, so a direct knowledge of the contact angle \( \theta \) was not possible. However
for a good solder joint, the solder is expected to wet the pin completely, so that \( \theta = 0 \) (see Remark (ii) in Sect. 2). Of course for numerical calculations we must use an angle \( \theta > 0 \) or \( h'(R_j) < \infty \), but it was found that for \( h'(R_j) > 4 \) no substantial change of the profile appear (see Fig. 7).

In order to investigate the behaviour of our model when \( V, \theta, a_{13} \), are varied, 18 solder joints have been calculated and plotted. Their general characteristics are given in table I whereas some of them are given in figures 3-9.

Joint 1 (Fig. 3) is an acceptable solder joint, close to our reference joint. Joint 2 (Fig. 4), for which \( \theta = 166^\circ \), shows the effect of bad wetting. This situation can arise when the pin has not been cleaned before soldering. What is unrealistic in figure 4 is the fact that axial symmetry is preserved, given that usually dirtiness of the pin is not uniform. In joint 7 (Fig. 5) bad wetting is also shown, though not so bad as in figure 4. Moreover the solder volume is twice that of

![Fig. 3. Joint 1: \( V_0 = 0.60 \text{ mm}^3, \theta = 14.03 \text{ deg.} \)](image)

![Fig. 4. Joint 2: \( V_0 = 0.60 \text{ mm}^3, \theta = 165.96 \text{ deg.} \)](image)

![Fig. 5. Joint 7: \( V_0 = 1.20 \text{ mm}^3, \theta = 59.03 \text{ deg.} \)](image)

![Fig. 6. Joint 8: \( V_0 = 1.80 \text{ mm}^3, \theta = 14.03 \text{ deg.} \)](image)
Table I. — General characteristics of calculated joints.

<table>
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<tr>
<th>Joint No</th>
<th>R₁ (µm)</th>
<th>R₂ (µm)</th>
<th>r (Nt/m³)</th>
<th>a₁₁ (J/m²)</th>
<th>V₀ (mm³)</th>
<th>h'(R₁) (µm)</th>
<th>h(R₁) (µm)</th>
<th>λ (Nt/m²)</th>
<th>E₁ (µJ)</th>
<th>E₁₂ + E₁₃ (µJ)</th>
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<td>4.0</td>
<td>-712</td>
<td>2.55</td>
<td>-0.0991</td>
<td>0.1940</td>
<td>0.0650</td>
</tr>
<tr>
<td>16</td>
<td>300</td>
<td>1000</td>
<td>87 878</td>
<td>1 × 10⁴</td>
<td>0.60</td>
<td>4.0</td>
<td>-660</td>
<td>5 × 10⁶</td>
<td>-0.0085</td>
<td>3.78 × 10⁶</td>
<td>1.21 × 10⁶</td>
</tr>
<tr>
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<td>1000</td>
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<td>0.485</td>
<td>0.60</td>
<td>4.0</td>
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<td>-0.0000</td>
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<td>0.60</td>
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<td>1.8302</td>
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</table>
joints 1 and 2. In joint 8 (Fig. 6) there is good wetting but an excessive solder volume equal three times that of joint 1.

Figure 7 illustrates the effect of contact angle on the solder joint shape. Five solder joint profiles with the same $V_0$, $a_{13}$, but different $\theta$, are plotted. The influence of $V_0$ on the shape of the joint is illustrated in figure 8, where 5 joints with the same $a_{13}$, $\theta$ but different $V_0$ are plotted. Finally in figure 9, $a_{13}$ is varied while $V_0$ and $\theta$ are kept constant. Here we see that an increase of $a_{13}$ in joint 1 ($0.485 \text{ J/m}^2$) does not affect the shape of the joint appreciably.

This reflects the fact that gravity is not important for large values of $a_{13}$. On the other hand for smaller values of $a_{13}$ gravity becomes important (cf. $\varepsilon h/\lambda$ may be greater than 1 in (3.1a) — see joints 13, 14, 15 in Tab. 1).

---

**Fig. 7.** — The influence of the contact angle on the solder joint shape.

**Fig. 8.** — The influence of the solder volume on the solder joint shape.
It is important to notice here that although our model is intended to describe all acceptable solder joints, it obviously describes certain types of defective ones, like joints with an excess or less than acceptable solder volume, or with a bad contact angle (see also Sect. 6).

Another fact revealed by our calculations is, that gravity plays a secondary role in the determination of the joint shape. By changing or discarding $\varepsilon$, the modification of the joint was found to be insignificant for values of $a_{13}$ greater than 0.485 J/m$^2$ (cf. joints 1, 16, 17, 18 in Tab. I). We may notice here that if $\varepsilon = 0$, then (3.1a) is easily integrated, leading to the following expression for $h'$

$$h'(1 + h'^2)^{-1/2} = \lambda r/2 a_{13} + c/r \tag{3.2}$$

where $c$ is a constant of integration. In view of remarks (iv), (i) in section 2, equation (3.2) is equivalent to

$$h'(1 + h'^2)^{-1/2} = \cos \theta (r) = Hr + c/r \tag{3.2}'$$

where $\theta (r)$ is the angle between the tangent to the profile $z = h (r)$ and the $z$-axis, measured in the counterclockwise direction (cf. Fig. 2). Given the boundary condition

$$\cos \theta (r)|_{r = R} = \cos \theta = w$$

$\theta$ being the wetting angle — cf. (2.9'b) — , we can determine $c$, so that from (3.2') we finally obtain

$$w = (r \cos \theta (r) - H(r^2 - R_1^2)) R_1^{-1} \tag{3.3}$$

Consequently, if $\cos \theta (r)$ is known at two points $r_1$, $r_2$ say, preferably far from the pin (cf. Sect. 6), then $H$ can be eliminated from (3.3), thus offering an expression for the wetting angle $\theta$. Therefore, (3.3) can be used as an indirect means for obtaining the wetting angle, which otherwise cannot be measured directly (cf. Sect. 6). Finally we may notice that, (3.2) yields $h(r)$ as an elliptic integral which can be calculated with the aid of Jacobian elliptic functions. However, in the context of the present paper, numerical integration of the original equation is preferable.
4. The non-axisymmetric model.

In this section we drop the assumption of axial symmetry and we suppose that the surface $S$ of the liquid solder has the following parametric form in cylindrical polar coordinates $(r, \varphi, z)$

$$
x = r \cos \varphi, \quad y = r \sin \varphi, \quad z = h(r, \varphi).
$$

Clearly

$$
V = \int_0^{2\pi} \int_{R_1}^R rh \, dr \, d\varphi \tag{4.1a}
$$

$$
E_g = \varepsilon \int_0^{2\pi} \int_{R_1}^R \frac{rh^2}{2} \, dr \, d\varphi \tag{4.1b}
$$

$$
E_{13} = a_{13} \int_0^{2\pi} \int_{R_1}^R \sqrt{r^2 + h^2 + h^2_\phi} \, dr \, d\varphi \tag{4.1c}
$$

(cf. Appendix B). Here $h_r = \frac{\partial h}{\partial r}, \ h_\varphi = \frac{\partial h}{\partial \varphi}$ and we suppose that

$$
h(R_1, \varphi) = z_1 \tag{4.2a}
$$

$$
h(R, \varphi) = 0 \tag{4.2b}
$$

where as in section 2, $z_1$ and $R$ are moving boundaries to be specified by the solution of the problem. Here we assume that the pin is an arbitrary cylindrical surface, taken for simplicity to be vertical, whose (constant) cross-section is bounded by a given but otherwise, arbitrary curve $c_1(s) = (R_1(s), \varphi(s))$, $s$ being the arc-length. Similarly (4.2b) holds on a curve $c(s) = (R(s), \varphi(s))$. Therefore

$$
E_{14} = a_{14}(A - A_1)
$$

$$
E_{34} = a_{34}(A_2 - A) = a_{34}(A_2 - A_1 + A_1 - A) = \text{Const.} + a_{34}(A_1 - A)
$$

where $A_1$, $A$, $A_2$ are the areas of the closed regions of the xy-plane bounded by $c_1$, $c$ and the boundary of the pad. Thus

$$
E_{14} + E_{34} = \text{Const.} + (a_{14} - a_{34}) \int_0^{2\pi} \int_{R_1(\phi)}^R r \, dr \, d\varphi \tag{4.3}
$$

On the other hand (cf. Eq. (2.6))

$$
E_{12} + E_{23} = \text{Const.} + (a_{12} - a_{23}) S_{12}
$$

where $S_{12}$ is the area of the cylindrical surface of the pin, between $z = 0$ and $z = z_1$. This surface has the parametrization, in cylindrical coordinates $(\varphi, z)$

$$
r(\varphi, z) = (R_1(\varphi) \cos \varphi, R_1(\varphi) \sin \varphi, z)
$$

so that its normal is

$$
\frac{\partial r}{\partial \varphi} \times \frac{\partial r}{\partial z} = (R_1 \sin \phi + R_1 \cos \phi, -R_1 \cos \phi + R_1 \sin \phi, 0), \tag{4.4}
$$
\[ R_1 = \frac{dR_1}{d\phi} \]

hence

\[
S_{12} = \int_0^{2\pi} \int_0^{R_1} \sqrt{(R_1')^2 + R_1^2} \, dz \, d\phi = \int_0^{2\pi} h(R_1, \phi) \sqrt{(R_1')^2 + R_1^2} \, d\phi
\]

\[
= - \int_0^{2\pi} \sqrt{(R_1')^2 + R_1^2} \left( \int_{R_1}^R \frac{\partial h}{\partial \phi} \, dr \right) \, d\phi
\]

where (4.2b) has been used. Consequently

\[
E_{12} + E_{23} = \text{Const.} + (a_{23} - a_{12}) \int_0^{2\pi} \int_{R_1}^R h, \sqrt{(R_1')^2 + R_1^2} \, dr \, d\phi.
\]

The total energy \( E \) (cf. Eq. (2.1)) has to be extremized subject to the constraint

\( V = V_0 \) (cf. (4.1a)), or as explained in Appendix A, we have to extremize the functional

\[
E^*(h) = \int_0^{2\pi} \int_{R_1}^R L^* \, dr \, d\phi
\]

\[
L^* = a_{13} \sqrt{r^2(1 + h_\phi^2) + h_\phi^2} + r \left( \frac{\varepsilon k^2}{2} - \lambda h \right) + (a_{14} - a_{34}) r + (a_{23} - a_{12}) h_r \sqrt{(R_1')^2 + R_1^2}
\]

(4.6)

with moving boundaries described by (4.2).

The calculations are given in the appendices and the resulting Lagrange equation together with the boundary conditions on \( c_1, c \) are the following

\[
\left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right) - \frac{\varepsilon h - \lambda}{a_{13}} = 0
\]

(4.7)

\[
h(R, \phi) = 0, \quad \frac{\partial h(R, \phi)}{\partial r} = -\tan \theta
\]

(4.8a)

\[
\cos \theta = \frac{a_{14} - a_{34}}{a_{13}}
\]

(4.8b)

\[
\frac{h_\phi}{d\phi} - R_1^2 h_\phi^2 \frac{d\phi}{ds} = \frac{a_{23} - a_{12}}{a_{13}}
\]

(4.9)

\[
\int_0^{2\pi} \int_{R_1}^R h_r \, dr \, d\phi = V_0.
\]

(4.10)

Remarks: (i) here \( \rho_1, \rho_2 \) are the principal radii of curvature of \( S \), so that by (2.12) the first bracket in (4.7) is twice the mean curvature \( H \) of \( S \), which, being independent of the parametrization of \( S \) ([7] Sect. V.3), is found by standard methods to be in Cartesian coordinates \( (x, y, z = h(x, y)) \) — see [7] section V4.
\[ 2 \hat{H} = \frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{h_{x\alpha}(1 + h_{x\beta}^2) + h_{y\alpha}(1 + h_{y\beta}^2) - 2 h_{x\alpha} h_{y\alpha}}{(1 + h_{x\beta}^2 + h_{y\beta}^2)^{3/2}} \]  

(4.11)

\[ h_{x\alpha} = \frac{\delta h}{\delta x} , \quad h_{y\alpha} = \frac{\delta h}{\delta y} , \quad h_{x\alpha} = \frac{\delta^2 h}{\delta x^2} , \quad \text{etc.} \]

(ii) Equation (4.8) is evaluated along the unknown curve \( c(s) = (R(s), \varphi(s)) \). Equation (4.9) is evaluated along the pin cross section \( c_i(s) \) at the moving boundary \( z = z_1 \) (cf. (4.2b)).

(iii) Equations (4.7, 4.8) are identical with (2.9'a, 2.9'c) respectively, with \( h(r, \varphi) \) replacing \( h(r) \).

(iv) If the pin is oblique to the \( xy \)-plane, (4.7) and (4.8) do not change except that \( S_{12} \), hence \( H \) in (4.11), has a more complicated expression. Moreover, (4.9) still holds in the coordinate independent form given by (A.6).

5. Simulation of 3D scanner data.

Range data for solder joints, were provided by an available experimental 3D scanner. The specifications of the scanner are lower than those of the sensors currently under development in TRIOS project and the quality of the data is also lower. The scanner is nonsynchronized and has an addressable resolution of 70 \( \mu \text{m} \times 70 \mu \text{m} \times 39 \mu \text{m} \) in \( x, y, z \) directions respectively. In this section the use of the model to simulate the effect of the resolution and the influence of the noise characteristics of the 3D scanner on the resulting range data will be illustrated.

In figure 10 the effect of a limited resolution is presented. The data used, come from the calculated Joint 10 profile, and the grid corresponds to the scanners addressable resolution in the \( x, y, z \) coordinates. The fact that (contrary to what happens with metallic surfaces) board materials are partially transparent to the light used by the laser scanner, is also simulated here. In this way values of \( z \) obtained by the scanner and corresponding to the board level are lower than the values of metallic surfaces, whether they have the same actual level, or not.

In figure 11 the effect of noise is studied. Here, noise obtained from the scanner was added on the data of figure 10. This noise actually consists of range data obtained when scanning a completely flat metallic surface. Subsequently, a median noise filter, defined on a \( 3 \times 3 \) window, has been applied on the data of figure 11. The result is given in figure 12.

Figure 13 represents real data obtained from the scanner, on which a similar median filter has been applied. They correspond to a joint similar to joint 10, simulated in figure 10. When real and simulated data are compared we notice two main differences: first, in real data the pin is slightly inclined and not coaxial with the pad. Second, real data have false values in some regions around the pin, showing disturbances of the joint shape that do not exist in reality. These false values are due to secondary reflections, mixed up with primary ones and sensed by the scanner. In this way measurements are affected. The fact that the scanner under consideration, is a nonsynchronized one, makes such kind of errors to occur frequently. Nevertheless, it is expected that the improved design and characteristics of the scanners mentioned in the Introduction, will help to overcome problems of this kind.

Simulations like those presented so far, can help to define specifications of 3D vision machines, to evaluate the performance of noise filters and to develop inspection algorithms. The advantage of testing filters and algorithms on simulated, rather than real data, is that, in the first case, the original shape, unaffected by limited resolution and noise, is accurately known.

An extension of the simulation techniques presented so far, is possible by integrating numerically the general equation (4.7). Since this is a partial differential equation, a method of finite differences or finite elements should be employed. In this way, it would be possible to
Fig. 10. — Joint 10: Simulation of the limited resolution of the 3D scanner.

Fig. 11. — Joint 11: Scanner noise added in figure 10.

Fig. 12. — Joint 10: Scanner noise added in figure 10 and passed through a median filter.

Fig. 13. — Real 3D scanner data for a through-hole wave soldered joint.
simulate not only axisymmetric joints, but also eccentric joints, joints with inclined pins, joints with square pads etc.

6. Discussion.

As explained in section 1, the present model is valid as long as transient phenomena induced by the solder wave, have a relaxation time shorter than the cooling and solidification time-scales of the solder. Therefore normal joints (cf. Sect. 1), include all acceptable joints as well as certain types of defective joints, like, joints having excessive or less than acceptable solder volume, exhibiting poor wetting, having an exposed or missing lead (pin), dull and/or grainy joints etc. On the other hand, cold or fragmented joints, joints exhibiting icicle formation or blow holes, joints without solder etc., do not fall in the domain of applicability of the present model [12]. We may also notice here that under the assumptions made in section 1, the method of section 4 can be used to study the shape of non-THT joints (SMD wave-soldered or reflow-soldered joints) leading to the same equation (4.7). Of course the boundary conditions (4.8, 4.9) are different and depend on the type of the joint.

The model has been also applied in a way similar to that presented for axisymmetric joints, in a special case of SMD wave-soldered joints. This special case refers to SMD components having a width of the order of 4 to 5 mm. In this case, the profile of the joint in its middle area, can be calculated numerically by considering this joint as a joint of infinite length. The general equation (4.7) is simplified again, to an ordinary differential equation and can be easily integrated numerically. If gravity is neglected, this equation is further simplified and an exact solution can be found. The joint profiles in this case, are simply circular arcs.

Direct comparison of the joint shapes predicted by our model, and the shapes obtained by range data, provided by the existing 3D scanner mentioned previously, is not possible yet, because of the inaccuracies of the scanner data. These are due partly to the excessive noise and partly to systematic errors occurring in the solder area near the pin because of secondary reflection phenomena. In addition, the scanner's resolution is not sufficient. However, sample data that have been produced by a new scanner, under development in TRIOS project, are of high quality, suggesting that an exact experimental check of our model will be possible in the near future. Nevertheless, a first very rough check has been done by comparing the model-predicted joint 1 (see Fig. 3) with our reference (sample) joint. The measured height of the reference joint, \( h(R_1) \), agrees with the predicted one within the accuracy of our measurement (measured \( h(R_1) = 0.65 \pm 0.05 \) mm, predicted \( h(R_1) = 0.66 \)) and the predicted shape seems to fit quite well the real one.

The possibilities of using the general model of section 4 for inspection purposes arise from the fact that (4.7) expresses common geometrical properties for a wide class of solder joint shapes, and can be used in various ways in solder joint inspection problems, especially for defects that are difficult to detect otherwise. Although such a kind of work is beyond the scope of the present paper, we would like to comment briefly on some of these possibilities. For example:

(i) an important conclusion that can be drawn from (4.7) (see also (2.9'a) together with (2.1b)) is that all normal joints have the following common geometrical property: the sum of the mean curvature \( H \) with the gravity depended term \( A = \epsilon h^2 a_{13} \), is constant for all points of their surface. If the gravity is neglected (and this can be done in most cases, see Sect. 3), then these solder joints are surfaces of constant mean curvature. On the basis of this, an operator has been defined that calculates \( H \) or \( H + A \) from the range data and for every point of the solder surface. In principle, this operator would transform all normal joints to flat, horizontal surfaces, whereas most defective joints would be transformed to non-flat surfaces. Given the presence of noise in \( h(x, y) \), measures of the low frequency variations of \( H \) or \( H + A \) would be
here an appropriate family of 3D features. On the other hand, due to the nonlinear dependence of \( H \) on \( h(x, y) \), noise and inaccuracy effects of input data, may appear magnified in the output of the operator. Thus, this method depends heavily on the quality of the range data. An effective implementation, insensitive to noise as far as possible, is very important here.

(b) According to (a), a scanner recording \( H \) directly could be very advantageous for solder joint inspection.

(c) The « bad wetting angle » defect, is a very subtle one and its detection is a very difficult problem. The wetting angle of a solder joint cannot be directly measured, because the solder surface around the pin is very steep, partially hidden and measurements in this area suffer from second order reflection problems. This problem can only be solved by studying the solder’s geometry in areas quite far from the pin and through the definition of appropriate global 3D features. From the study of the present model, we have concluded that the mean curvature \( H \) of a solder joint and the wetting angle of the joint, are related monotonically, all other factors (solder volume, pin and pad geometry etc.) being kept constant. Therefore, the average value of \( H \), is a 3D feature, strongly correlated to the value of the wetting angle and together with other features (volume, pad area etc.) could form a suitable feature group for the detection of this defect.

d) As already mentioned in section 3, for axisymmetric joints for which gravity is negligible, an explicit expression can be obtained for the slope of the joint profile. This allows an exact calculation of the wetting angle in terms of the slope of the joint profile at points far from the pin, where data are accurate enough. A similar exact expression for the general case of non axisymmetric solder joints cannot be obtained. Nevertheless, such an expression could be valid to a certain degree of approximation, for a suitably defined average profile of a general through-hole solder joint.

e) Equation (4.7) could help to define effective nonlinear, smoothing filters, which will be specific to the solder surface geometry and will take into account the a priori knowledge provided by this equation. These filters could also provide output data with subpixel accuracy and permit a more reliable and accurate measurement of certain joint characteristics, like volume.

Research along these lines is under progress and the results will be given in a subsequent paper.

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Appendix A.

The solution of the variational problem of section 4 (cf. Eqs. (4.6, 4.2)) is obtained with the aid of the following theorem ([6] Sect. 37).

**Theorem**: If the functional

\[
J(h) = \int_U L(x_1, x_2, \ldots, x_r, h, h_{1r}, \ldots, h_{1r}) \, dx_1 \ldots dx_r
\]
where \( h = h(x_1, \ldots, x_n) \), \( h_i = \frac{\partial h}{\partial x_i} \) has an extremum for \( h \), then

\[
\frac{\partial L}{\partial h} - \sum_{i} \frac{\partial}{\partial x_i} \left( \frac{\partial L}{\partial h_i} \right) = 0 \tag{A1}
\]

and

\[
\int_U \left( \sum_{i} \frac{\partial}{\partial x_i} \left( \frac{\partial L}{\partial h_i} \right) \delta h + \left( L - \sum_{i} h_i \frac{\partial L}{\partial h_i} \right) \delta x_i \right) \ dx_1 \cdot dx_n = 0 \tag{A2}
\]

for all variations \( h \rightarrow h + \delta h \), \( x_i \rightarrow x_i + \delta x_i \).

Given the constraint (4.10), then as in section 2, the theorem above is valid with \( L \) replaced by \( L^* = L - \lambda r \), where \( \lambda \) is a Lagrange multiplier. In our case \( x_1 = r \), \( x_2 = \phi \) and (A1) gives

\[
\frac{\partial L^*}{\partial h} - \frac{\partial}{\partial r} \left( \frac{\partial L^*}{\partial h_i} \right) - \frac{\partial}{\partial \phi} \left( \frac{\partial L^*}{\partial h_\phi} \right) = 0 . \tag{A1'}
\]

By (4.6b), only the first two terms of \( L^* \) contribute to (A1'), the contribution of the second being \( r(eh - \lambda) \). The contribution of the first term is found in appendix B to be \(-a_{13}(\rho_1^{-1} + \rho_2^{-1}) r \), \( \rho_1, \rho_2 \) being the principal radii of curvature of \( S \). Therefore (A1') reduces to (4.7).

On the other hand, \( U \) in (A2) is the region between the closed curves \( c_1, c \) (see the paragraph following Eq. (4.2)), so that, by Green's theorem (A2) is

\[
\oint_{c_1} \left( -F_1 \frac{dr}{ds} + F_2 \frac{d\phi}{ds} \right) \ ds - \oint_{c} \left( -F_1 \frac{dr}{ds} + F_2 \frac{d\phi}{ds} \right) \ ds = 0 \tag{A3}
\]

where

\[
F_1 = \frac{\partial L^*}{\partial h_\phi} \delta h + \left( L^* - h_i \frac{\partial L^*}{\partial h_i} - h_\phi \frac{\partial L^*}{\partial h_\phi} \right) \delta \phi \\
F_2 = \frac{\partial L^*}{\partial h_i} \delta h + \left( L^* - h_i \frac{\partial L^*}{\partial h_i} - h_\phi \frac{\partial L^*}{\partial h_\phi} \right) \delta r .
\]

However, by (4.2) we have

\[
h_\phi(R, \varphi) = \delta h(R, \varphi) = 0 \tag{A4a}
\]

\[
\delta r(R_1, \varphi) = \delta \varphi(R_1, \varphi) = 0 \tag{A4b}
\]

where (A4a) follows from (4.2b) and (A4b) expresses the fact that (unlike \( c \) \( c_1 \) is a given curve. With the aid of (A4), equation (A3) becomes

\[
\oint_{c_1} \left( L^* - h_i \frac{\partial L^*}{\partial h_i} \right) \left( -\frac{dr}{ds} \delta \phi + \frac{d\phi}{ds} \delta r \right) \ ds + \oint_{c} \left( \frac{\partial L^*}{\partial h_\phi} \frac{dr}{ds} - \frac{\partial L^*}{\partial h_r} \frac{d\phi}{ds} \right) \delta h \ ds = 0 .
\]
Since \( \delta h, \delta r, \delta \varphi \) are arbitrary, we finally get

\[
\left( L^* - h_i \frac{\partial L^*}{\partial h_i} \right) \bigg|_{c} = 0 \quad (A5a)
\]

\[
\left( \frac{\partial L^*}{\partial h} \frac{dr}{ds} - \frac{\partial L^*}{\partial h} \frac{d\varphi}{ds} \right) \bigg|_{c} = 0 . \quad (A5b)
\]

By (4.6b), equation (A5a) is simply (4.8), whereas (A5b) reduces to (4.9) if we notice that

\[
\left( \frac{dR_i}{ds} \right)^2 + R_i^2 \left( \frac{d\varphi}{ds} \right)^2 = 1
\]

by the definition of the arc-length \( s \).

Remarks: (i) By (4.4), (B2) we readily find that, as expected on physical grounds, (A5b) is just

\[
n_i \cdot n_p = \cos \theta = (a_{23} - a_{12})/a_{13} \quad (A6)
\]

where \( n_i, n_p \) are the unit normals to the solder and pin surfaces respectively and \( \theta \) is their angle.

(ii) The Lagrange equation (A1) is the same with that obtained by varying \( J(h) \), keeping the boundaries fixed, and (A2) takes account of the moving boundaries.

Appendix B.

Here we calculate the variation of \( E_{13} \), equation (4.1c), which is proportional to the area of the surface \( S \), with the aid of the following theorem [11].

Theorem: Let \( S \) be a surface of \( \mathbb{R}^3 \) with unit normal \( n_i \) and mean curvature \( H \). Let \((x_1, x_2, x_3), (u_1, u_2)\) be local coordinates in \( \mathbb{R}^3 \) and \( S \) respectively, so that \( S \) is given in parametric form

\[
x_i = x_i(u_1, u_2), \quad i = 1, 2, 3 .
\]

If \( \sigma = \left\| \frac{\partial r}{\partial u_1} \times \frac{\partial r}{\partial u_2} \right\| \) is the norm of the normal to \( S \), \( r(u_1, u_2) \) being the position vector of any point of \( S \), then for any variation of \( S \)

\[
x_i(u_1, u_2) \rightarrow x_i(u_1, u_2) + \delta x_i(u_1, u_2)
\]

the variation of the area of \( S \) is

\[
\delta \int_S \delta \, du_1 \, du_2 = - 2 \int_S H n_i \cdot \delta r \, \sigma \, du_1 \, du_2 . \quad (B1)
\]

For the model of section 4 (see also Appendix A) we have that \((x_1, x_2, x_3)\) correspond to \((r, \varphi, z)\) and \((u_1, u_2)\) correspond to \((r, \varphi)\) and

\[
n_i = \frac{1}{\sigma} \left( \frac{\partial r}{\partial r} \times \frac{\partial r}{\partial \varphi} \right) = \frac{1}{\sqrt{r^2(1 + h^2) + h^2_\varphi}} (h_\delta \sin \varphi - rh, \cos \varphi, - h_\delta \cos \varphi - rh \sin \varphi, r) \quad (B2)
\]
where \( r = (r \cos \varphi, r \sin \varphi, h(r, \varphi)) \). Since only \( z = h(r, \varphi) \) is varied (see remark (ii) in appendix A), we have \( \delta r = \delta \varphi = 0 \) so that by (B2)

\[
\mathbf{n} \cdot \delta \mathbf{r} = r \, \delta h
\]

and therefore by (B1) the contribution of (4.1c) to the variation of (4.6), with fixed boundaries is

\[
\delta E_{13} = - \int_0^\pi \int_{R_1}^R 2 \, a_{13} \, H r \, \delta h \, dr \, d\phi
\]

which is the result used to derive (4.7) from (A1) in appendix A.

References


Proof not corrected by the authors.