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Roll and hexagonal patterns in a phase-contrast oscillator

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Résumé. — Cet article présente une nouvelle approche au problème des instabilités optiques dans un milieu Kerr face à un miroir de réaction. Il montre comment les instabilités apparaissant peuvent être comprises en termes de modes transverses d'une cavité consistant en un miroir normal et un milieu possédant à la fois les propriétés d'un miroir à conjugaison de phase et celles d'un miroir normal appelé miroir à contraste de phase. La condition de seuil pour l'instabilité est établie ainsi que le comportement de diverses formes (rouleaux, hexagones) au-dessus du seuil d'oscillation.

Abstract. — This paper presents a new approach to the problem of the optical instabilities for a thin slice of Kerr medium with a feedback mirror. We show how to interpret this instability in terms of the transverse modes of a cavity consisting of a normal mirror and a medium having the properties of both a phase-conjugate mirror and a normal mirror that we called phase-contrast mirror. The threshold condition for this instability and the properties of a few patterns (rolls and hexagons) above the oscillation threshold are discussed.

Pattern formation is currently a subject of much interest in optics [1]. In particular, hexagonal patterns appear to be encountered very often in the study of light propagation in a nonlinear medium. The first report of the observation of spontaneous formation of hexagons in nonlinear optical systems came in 1988 [2]. In the experimental study of the instability of a standing wave in a nonlinear medium, it was shown that above some intensity threshold, new coherent beams are generated that display an hexagonal pattern when projected on a screen. In a related theoretical analysis [3], the origin of the instability was related to four-wave mixing interactions between the incident and generated beams and the origin of hexagons was associated with nonlinear interactions that only exist for the hexagon geometry. This point was verified later by studying theoretically the bifurcation diagrams close to threshold for several patterns including hexagons [4]. It was also shown by a very interesting numerical computation that hexagons are the only stable pattern close to threshold [5]. Several other experimental observations of the appearance of hexagons for this instability have been reported in the litterature [6, 7]. In particular, experimental recordings of bifurcation diagrams for hexagons and other patterns (rings, rolls) are shown in reference [7]. As expected, the existence of hexagonal patterns is not restricted to the particular geometry discussed above. Hexagonal
patterns were also predicted [8] and observed [9] for the case of a thin slice of Kerr medium with a feedback mirror. They have also been predicted for a Kerr medium in a cavity [10].

Patterns in optics are very interesting not only by themselves, but also because of possible connections with hydrodynamics. In fact, it is well known for almost one century now that hexagonal patterns often appear in hydrodynamical instabilities, the most famous example being the Bénard cells of the Bénard-Marangoni instability [11]. As pointed out by several authors, this similarity is not fortuitous but corresponds to a deep link between some nonlinear terms of the partial differential equations leading to pattern formation in optics and hydrodynamics. Because the study of pattern formation and the route to turbulence is currently of major interest in hydrodynamics, one may hope that nonlinear optics will provide interesting clues for a quantitative theory of these processes.

The aim of the paper is to revisit the problem of a Kerr slice with a feedback mirror considered by Firth and d’Alessandro [8] and to show how this problem can be related to the study of the transverse modes of a cavity consisting of a real mirror and a phase-contrast mirror [12]. Such a mirror transforms phase-variations of the incident field \( E \) into amplitude variations in the reflected field [12] because the reflected field is the sum of two terms: one proportional to \( E \) and the second to \( E^* \). Because a phase-contrast mirror can have reflection coefficients much larger than 100% [13] such a cavity may spontaneously oscillate leading to a new type of oscillator that we call « phase-contrast oscillator ». We show that this new description of the instability described by Firth and d’Alessandro [8] gives a new insight in the origin of the instability and permits the easy recovery of some earlier results.

The paper is organized as follows: after recalling and extending several results about phase-contrast mirrors (Sect. 1), we show how to relate the problem of a Kerr slice with a feedback mirror to the study of the transverse modes of a phase-contrast oscillator (Sect. 2). We then investigate the behaviour of rolls and hexagons (Sect. 3) above the oscillation threshold. Finally, we discuss in the last section the quantum properties of these hexagons and show that some combination of the photon numbers in the different directions should exhibit subpoissonian fluctuations (Sect. 4), this property being an extension to hexagons of the quantum noise reduction that was predicted earlier for roll structures [14].

In all the paper, we only consider the static properties of the patterns. However, one should note that this physical problem also has interesting extensions in the temporal domain as shown by the experiments of Giusfredi et al. [15] and the theoretical analysis of Firth [8] and Le Berre et al. [16]. Furthermore, one can note that the linear stability analysis performed by Le Berre et al. [16] leads to a description of the nonlinear medium in terms analogous to those used in the present paper.

1. Phase-contrast mirror.

1.1 General. — Most nonlinear mirrors using four-wave mixing behave as simple phase-conjugate mirrors only for very large angles \( \theta \) between probe and pump. For small angles, it was shown [12] that the reflected beam originates from two nonlinear contributions: the phase-conjugate term equal to \( r_c E^* \) and the distributed feedback term equal to \( r_d E \) (\( r_c \) and \( r_d \) are the amplitude reflection coefficients associated with these nonlinear effects). These two terms have similar values for angles \( \theta \) small compared to \( \sqrt{\lambda / \ell} \) where \( \lambda \) is the wavelength of light and \( \ell \) is the length of the nonlinear medium. For larger values of \( \theta \), \( r_d \) decreases because of the phase-matching condition. In the present paper, we always assume that \( \theta \) is sufficiently small so that we can neglect the variation of \( r_d \) with \( \theta \). In these conditions, the reflected beam is the sum of the two contributions mentioned above and can exhibit interferences [12] because of the difference in phase between \( r_d E \) and \( r_c E^* \) (hence the name: phase-contrast mirror). Such a mirror can be amplifying because both
and \( r_d \) can be larger than 1. This point was checked experimentally by Vallet et al. [13] who have observed reflectivities much larger than 100\% with this type of mirror. This property implies that an oscillation can build up in a cavity made with an amplifying phase-contrast mirror and a usual mirror. Such an oscillation was indeed observed in reference [13] but, because of the particular polarization properties of this experiment, different reflecting mirrors were used for the pump and oscillating beams. For this situation, a longitudinal oscillation was obtained for particular values of the distance between the two mirrors. In the case where the same mirror is used, such a flexibility does not exist and one may expect the oscillation to occur for some angle between the pump and the oscillation beam. This is in fact how we will reinterpret the instability of d’Alessandro and Firth [8].

Before studying the modes of oscillation, we first derive the value of \( r_c \) and \( r_d \) for the case of a simple Kerr medium assuming total wash-out of the spatial grating having the shortest spatial period (1). We call \( E_f \) and \( E_b \) the two beams that counterpropagate in the nonlinear medium (Fig. 1). If the medium is described by a nonlinear refractive index \( n_2 \), the incident field \( E_i \) at the exit of the nonlinear medium is proportional to:

\[
E_i^{(\text{out})} = E_i^{(\text{in})} \exp \left( i \left( \frac{\ell n_i}{E_i} \right) \left( E_f^* E_f + E_b^* E_b^* \right) \right). \tag{1}
\]

We now assume that the backward field is actually the sum of two fields: \( E_b \) and \( E' \). We replace \( E_b \) by \( E_b + E' \) in equation (1) and assume \( \ell n_i E_b E' \ll 1 \) and \( \ell n_i E' E' \ll 1 \) to expand the exponential of equation (1) in power series. To describe the saturation mechanism that limit the intensity of the phase-contrast oscillator, we need to make the expansion up to third order in \( E' \). We find:

\[
E_i^{(\text{out})} = E_i^{(\text{in})} \left[ \exp i \left( \frac{\ell n_i}{2} \right) \left( E_f^* E_f + E_b^* E_b^* \right) \right] \times \left[ 1 + i \frac{\ell n_i}{6} \left( E_b^* E' + E_b E' E' \right) \right. \\
\left. - \frac{\ell n_i^2}{2} \left( E_b^* E' + E_b E' E' \right) \right] \left( E_b^* E' + E_b E' E' \right)^3 + \cdots \right]. \tag{2}
\]

Fig. 1. — Scheme of a phase-contrast oscillator. A thin nonlinear medium (longitudinal dimension \( \ell \)) is set at a distance \( d \) from the mirror M. A pump wave \( E_p \) is incident on the nonlinear medium. Above a certain intensity of \( E_i \), an oscillation builds up inside the cavity consisting of the mirror and the nonlinear medium.

(1) Such a situation can occur in the case of optical pumping nonlinearities as shown by Pinard, Boyd and Grynberg (Phys. Rev. A to be published).
The exit of the nonlinear medium is taken as the reference plane $z = 0$. In particular, we choose the phase origin to have $E_f$ real in this plane. To third order in $E'$, we obtain by retaining only the terms useful for future developments:

$$E_f(\text{out}) = E_f[1 + i \alpha (E_b^* E' + E_b E'*)] +$$
$$+ E_f \left[ i \alpha E' E'^* - \frac{(\alpha)^2}{2} (E_b^* E' + E_b E'^*)^2 \right] +$$
$$+ E_f \left[ - \alpha^2 E' E'^* (E_b^* E' + E_b E'^*) - i \frac{\alpha^3}{6} (E_b^* E' + E_b E'^*)^3 \right]$$

with

$$\alpha = k n_2 \ell.$$  

1.2 REFLECTION COEFFICIENTS. — Consider first the term of equation (3) which is linear in $E'^*$. This term $i \alpha E_f E_b E'^*$ describes the phase conjugate reflection of beam $E'$. The term of equation (3) linear in $E'$ i.e. $i \alpha E_f E_b^* E'$ describes the distributed feedback reflection of beam $E'$. For a weak incident field $E'$, one finds the reflection coefficients $r_d$ and $r_c$ for distributed feedback and phase-conjugation to be:

$$r_d = i \alpha E_f E_b^* = i r_0 \frac{E_b^*}{E_b}$$  

$$r_c = i \alpha E_f E_b = i r_0 \frac{E_b}{|E_b|}$$

where we have introduced the dimensionless quantity

$$r_0 = kn_2 \ell |E_f E_b|.$$  

It can be noticed that, contrary to the case of a pure phase-conjugate mirror, the phase-conjugate reflectivity given by equation (5) does not diverge when $\ell$ increases. This behaviour (which was observed previously for the case of cross-polarized beams [13]) is particular to the case of small angles $\theta$.

1.3 TRANSMISSION COEFFICIENTS. — We can derive the transmission coefficients of the phase-contrast mirror by a similar approach. The field $E_b$ in the plane $z = -\ell$ (i.e. after transmission through the nonlinear medium) is equal to

$$E_b(\text{out}) = E_b(\text{in}) \exp i [(kn_2 \ell) (E_f E_f^* + E_b E_b^*)].$$

Replacing $E_b$ by $E_b + E'$, one obtains to first order in $E'$

$$E_b(\text{out}) = [E_b + E' + i kn_2 \ell (E_b^* E' + E_b E_f^*)] e^{i\phi_n}$$

with

$$\phi_n = kn_2 \ell (E_f E_f^* + E_b E_b^*).$$

One can thus define a transmission coefficient $t$ for beam $E'$

$$t = e^{i\phi_n} \left[ 1 + i r_0 |E_b| E_f \right]^{-1}.$$
Besides, four-wave mixing generates a beam in a direction symmetric with respect to Oz of the propagation direction of \( E' \). The amplitude \( E'_m \) of this beam is

\[
E'_m = t_c E'^* ,
\]

with

\[
t_c = i r_0 \frac{E_b^2}{|E_f E_b|} e^{i \phi_n} \] (11b)

2. Phase-contrast oscillator.

2.1 Threshold condition. — We now consider that the preceding phase-contrast mirror is placed in front of a real mirror \( M \) having an amplitude reflection coefficient \( r = \sqrt{R} \) and located at a distance \( d \) (see Fig. 1). Such an ensemble of two mirrors constitutes a cavity and we look for the onset of self-oscillation in this cavity. We call \( E_+ \) and \( E_- \) the fields propagating in the + \( z \) direction which make respectively an angle + \( \theta \) and − \( \theta \) with Oz (Fig. 1). We call \( E'_+ \) and \( E'_- \) the fields propagating in the − \( z \) direction. All the fields are evaluated in the plane \( z = 0 \) corresponding to the exit of the nonlinear medium. The relation between these fields are given by the propagation between the two mirrors and by the reflection relations on the surfaces. We obtain:

\[
E_b = r E_f e^{2 i k d} \] (12a)

\[
E_+ (x) = r (E_+ (x - 2 d \theta )) \exp 2 i k d \left( 1 + \frac{\theta^2}{2} \right) \] (12b)

\[
E_- (x) = r (E_- (x + 2 d \theta )) \exp 2 i k d \left( 1 + \frac{\theta^2}{2} \right) . \] (12c)

In this equation, we have specified the point in the transverse direction \( O_x \) where the field is evaluated because the phases of the fields are not constant and evolve according to \( E_+(x) = E_+ e^{i k \phi_1} \) and \( E_- (x) = E_- e^{-i k \phi_1} \). The relations on the phase-contrast mirror are:

\[
E_+ = r_d E'_+ + r_c (E'_-)^* \] (13a)

\[
E_- = r_d E'_- + r_c (E'_+)^* \] (13b)

Using equations (5) and (12), one can rewrite equations (13) as

\[
E_+ = (i r_0 r e^{i k \phi_1} E_+ e^{-2 i k \phi_1 d} + (i r_0 r e^{-i k \phi_1} d) E^*_+ e^{2 i k \phi_1 d} \] (14a)

\[
E_- = (i r_0 r e^{i k \phi_1} E_- e^{-2 i k \phi_1 d} + (i r_0 r e^{-i k \phi_1} d) E^*_+ e^{2 i k \phi_1 d} \] (14b)

These equations admit a nonzero solution provided that

\[
[(1 - i r_0 r e^{-i k \phi_1} d) (1 + i r_0 r e^{i k \phi_1} d) - (r_0 r)^2] = 0 \] (15)

i.e.

\[
1 - 2 r r_0 \sin k \phi_1 d = 0 . \] (16)

The oscillation occurs when

\[
|r_0| \geq \frac{1}{2 r} \] (17)
Because $|E_0| = rE_0$, we obtain using equation (6) the threshold condition:

$$kn_2 I_f \ell > \frac{1}{2 R}$$

(18)

where $I_f = |E_f|^2$.

The angle of oscillation is different for a self-focusing medium ($r_0 > 0$) and a self-defocusing mirror ($r_0 < 0$). In the first case, one should have $k\theta^2d = \pi/2$

$$n_2 > 0 \Rightarrow \theta = \sqrt{\frac{\lambda}{4d}}$$

(19)

while for a self-defocusing medium, equation (16) is satisfied provided that $k\theta^2d = 3 \pi/2$

$$n_2 < 0 \Rightarrow \theta = \sqrt{\frac{3 \lambda}{4d}} .$$

(20)

The preceding results are identical to those presented originally by Firth and d’Alessandro [8]. By comparison with the analysis of Vallet et al. [13], the main difference (apart from the polarization dependence) is that the two propagation lengths appearing in equation (12a) on the one hand and in equations (12b, c) on the other hand were different in [13].

Using equations (14), (16), (19) and (20), one finds the following relation between $E_+$ and $E_-$ at threshold:

$$E^* = - E_+ .$$

(21)

The phase relation between the oscillating beams is

$$\phi_+ = \sigma - \phi_+ + 2 p \pi \text{ (p integer)} .$$

(22)

The beams may have the same phase provided that $\phi = \pm \pi/2$.

2.2 **FIELDS OUTSIDE THE CAVITY.** — It may be interesting to know the values of the fields outside the cavity. After the ordinary mirror, the output fields have amplitudes obtained by multiplying the amplitude of $E_+$ or $E_-$ by the transmission coefficient of the mirror. Outside

the nonlinear medium, $A_+$ and $A_-$ can be obtained from equations (10) and (11):

$$A_+ = tE'_+ + t_c E'_+^*$$

(23a)

$$A_- = tE'_- + t_c E'_-^*$$

(23b)

which yields when equations (12) are used:

$$A_+ = e^{i(2kd + \phi_{ad})} [r e^{-i \theta^2 d} (1 + ir_0 r) E_+ + ir_0 r^2 e^{i \theta^2 d} E_+^* ]$$

(24)

and a similar formula for $A_-$. In particular, near threshold, when equation (21) is true, equation (24) gives:

$$A_+ = r e^{i(2kd + \phi_{ad})} [e^{-i \theta^2 d} + 2 r_0 r \sin k\theta^2 d] E_+ .$$

(25)

Using equation (16), one finds the value of the output beam intensity

$$|A_+|^2 = 2R |E_+|^2$$

(26)
2.3 CONDITION OF OSCILLATION. — Experiences are always done with beams having a finite transverse dimension $w$. The instability can occur only if the lateral shift of the oscillating beam after one round trip (which is of the order of $d \theta$) is much smaller than $w$. Using equation (19) or (20), the condition can be written

$$d \ll \frac{w^2}{\lambda},$$

and expresses that the length of the cavity is much smaller than the Rayleigh range of the incident beam.

We note finally that the threshold conditions has other roots than those given by equations (19) and (20) which correspond to larger values of $\theta$ for which $\sin k \theta^2 d = \pm 1$. We do not consider these roots in the following. In fact, experimentally, $r_0$ is generally a decreasing function of $\theta$ (because of grating wash-out effects) so that it is probably reasonable to consider only the root given by equation (19) or (20) close to threshold. There is still a better way to select the smallest angle for the oscillating beam. It is to introduce a lens and a pinhole inside the cavity and adjust the diameter of the pinhole so that oscillation for angles larger than those given by equation (19) or (20) is impossible. In the following, we assume that this pinhole is present and prevents also the spatial harmonics of the oscillating field to propagate between the phase-contrast mirror and mirror $M$.

3. Pattern formation.

3.1 GENERAL. — We now discuss the basis of pattern formation. Because the system is invariant under any rotation around the pump axis, one might think that the only possible pattern is a ring whose angle is determined by equation (19) or (20). However, this is not always the case and experiments done with counterpropagating waves [2, 7] showed several patterns in the far field (two dots (rolls), hexagons, rings, squares). There are fundamental reasons for this symmetry breaking. For instance, it was shown in [3] that a peculiar high-order coupling between the oscillating beams favours hexagon formations. Indeed, it turns out that in many circumstances, the oscillation threshold is smaller for hexagons than for other patterns [4, 8, 10]. Even for patterns that have the same threshold, there might be different emitted intensities for the same distance to threshold. In the problem considered here for instance, rolls are predicted to give an output higher than squares while squares give an output larger than rings. Of course, even if the symmetry breaking occurs because of fundamental reasons, the particular orientation of the privileged pattern is generally due to experimental imperfections which do not respect the cylindrical symmetry. It is possible from the preceding equations to determine the intensity of the instability for each of these patterns. Of course, there is generally a competition between these patterns and all of them are not stable [8].

If we consider a far-field pattern consisting of $2N$ off-axis dots regularly spaced, the field can be described in the plane orthogonal to $Oz$ as

$$E = \sum_{j=1}^{2N} E_j e^{iK_j \cdot r} = \sum_{j=1}^{N} [E_j e^{iK_{j-1} \cdot r} + E_{j+N} e^{-iK_{j-1} \cdot r}]$$

(28a)

with

$$K_j = \left[ e, \cos j \frac{\pi}{N} + e, \sin j \frac{\pi}{N} \right] k \theta.$$

(28b)

The second expression of $E$ in equation (28a) shows that the pattern can also be considered as the algebraic superposition of $N$ rolls. In fact, at lowest order, the four-wave mixing
processes couple the fields $E_j$ and $E_j + N$ which propagate in symmetric directions (they correspond to the fields $E_+$ and $E_-$ of Sect. 2). Higher-order terms can also couple all these fields. Two generic cases can however be distinguished. If for $j \neq 1$ $e^{i K_j \cdot r}$ $e^{i K_f \cdot r}$ can be equal to $e^{i K_0 \cdot r}$, one may observe a transcritical bifurcation because terms like $i \alpha E_i E^* E^*$ of equation (3) redistributes energy between the oscillating beams. In the opposite case, a supercritical bifurcation is generally obtained.

The principle of the calculation is to express the reflected oscillating field by a formula identical to equation (12) but with $E$ given by equation (28a). The backward pumping beam $E_b$ should be determined in a self-consistent way to take into account pump depletion. One replaces $E_b$ and $E'$ in equation (3) by their expressions versus the forward pump after the nonlinear medium (using Eq. (12a)) and $E$ (Eqs. (12b and c)). The phase-matched contributions are kept to obtain expressions for the fields that include nonlinear terms up to third order. The steady-state equations then give the value of the intensity of the oscillating fields.

3.2 Rolls: An Example of Supercritical Bifurcation. — In the case of rolls, the oscillating field $E$ consists of two fields propagating in symmetric directions. It can be written as $E = E_+ e^{i K \cdot r} + E_- e^{-i K \cdot r}$. If we assume that the two fields have the same amplitude $E/\sqrt{2}$, and if we call $I = E^2$ the intensity of the instability propagating from the phase-contrast mirror to the mirror M, we find from equation (3):

$$E_b = r E_t [1 + (i \alpha - 2 \alpha^2 I_b) R I] e^{2 i K f}$$

$$r_d = [i \alpha (1 - \alpha^2 I_b R I) - 2 \alpha^2 R I] E_t E_b^*$$

$$r_c = [i \alpha (1 - \alpha^2 I_b R I) - 2 \alpha^2 R I] E_t E_b.$$ (29a)

(29b)

(29c)

Compatibility conditions analogous to equation (14) then yields the following condition on $I$

$$3 \alpha^2 I_b I_f R I = I_f - I_{th}$$ (30)

where $I_{th} = 1/2 \alpha R$ is the threshold intensity given in equation (18). Using the relation $\alpha R I_f \approx 1/2$ valid close to threshold, one finds that rolls intensity should vary as

$$I = \frac{4}{3} (I_f - I_{th}).$$ (31)

Another consequence of the compatibility relation is that the angle $\theta$ above threshold is generally a linear function of $I$ starting for $I = 0$ from the value given in equation (19) or (20).

The coefficient 4/3 results from the combination of two effects. The first one is pump depletion and is common to all patterns. Actually from equation (29a), one finds

$$|E_b| \approx \sqrt{R} E_t \left[ 1 - \frac{I}{2 I_{th}} \right]$$ (32)

which simply describes energy conservation. The fact that $|E_b|$ decrease above threshold stabilizes the oscillation by reducing the gain associated with phase conjugation and distributed feedback. The second effect is nonlinear mixing involving both pump and oscillating beams. The contribution of this second effect varies with the pattern. As a result, the coefficient 4/3 obtained for rolls ($N = 1$) should be replaced by 8/7 in the case of squares ($N = 2$).

3.3 Hexagon: Example of Transcritical Bifurcation. — In the case where $N = 3 p$ ($p$ integer), four-wave mixing terms such as $E_t E^* E^*$ (see Eq. (3)) can redistribute energy
between the oscillating beams and lead to an equation of lower order because it involves terms of order \( \sqrt{I} \) rather than \( I \) in equation (30). Such four-wave mixing processes which couples \( E_{j+1} E_{j+2}^* \) to \( E_j \) in the hexagonal case are very similar to the couplings considered in [2]. By including these terms, one finds the following steady-state equation of the amplitude in the hexagonal case \( (N = 3) \):

\[
E_j = 2 \alpha R I_f \left(1 - \frac{I}{2I_{th}} \right) E_j - \frac{5}{6} \alpha R I E_j + 2 \alpha R \sqrt{\frac{II_f}{6}} E_j. \tag{33}
\]

To find this equation, we have assumed that all the fields have the same amplitude \( |E_j| = \sqrt{I}/6 \) and the same phase equal to \( \pi/2 \). The first term of the right-hand side of equation (33) corresponds to the sum of the distributed feedback and phase-conjugate reflection when the effect of pump beam depletion (Eq. (32)) is taken into account. The second term arises from nonlinear coupling between the beams that appear in the cubic term of equation (3). Note that we have replaced \( \alpha R I_f \) by \( 1/2 \) to evaluate this contribution. Finally, the last term is the coupling term which is peculiar to hexagonal patterns and which arises from the quadratic term of equation (3). At second order in \( \sqrt{I} \), equation (33) becomes.

\[
\frac{I}{I_{th}} - \frac{1}{\sqrt{6}} \sqrt{\frac{I}{I_{th}}} = I_f - 1 \tag{34}
\]

which shows that the bifurcation is transcritical and appears with a sudden jump of intensity for \( I_f = (23/24) I_{th} \). This result is slightly different from the value found by d’Alessandro and Firth \( (I_f = (27/28) I_{th}) \). One difference between the two approaches is that we have neglected the spatial harmonics that propagate in directions making an angle 2 \( \theta \) or 3 \( \theta \) with \( O_2 \) and which are generated by the nonlinear terms in equation (3). Such an approximation may be justified if a pinhole is introduced in the cavity as mentioned before. If there is no filtering, the results of the more complete calculation of d’Alessandro and Firth [8] should be used.

4. Quantum properties.

It was shown by Lugiato and Castelli [14] that the photon numbers in the different spots of the patterns emitted by a cavity containing a nonlinear medium obey certain relations. In [14], the authors considered the case of rolls which correspond to double spots in the far field pattern and showed that the intensities should be exactly the same in each spot. More recently, we showed that a more complex relation between the photon numbers exist in the hexagonal case [17]. Similar relations should exist for the patterns emitted by a phase-contrast mirror and we justify this assertion by a simple remark on momentum conservation. To simplify the discussion we consider here the case where \( R = 100 \% \). In the case of a partially transmitting mirror, we should add the photons coming through the nonlinear medium and the photons coming through the mirror.

In the case of double spots, there are two beams \( A_+ \) and \( A_- \). The equality between \( |A_+|^2 \) and \( |A_-|^2 \) can be deduced from equations (29) and (31). However this equality remains true at the quantum level. The origin of this property is the conservation of field momentum. The incoming field propagates along the \( z \) direction so that its momentum along \( O_x \) and \( O_y \) is equal to 0. Because all the four-wave mixing processes considered here are non dissipative, the momentum of the field is conserved. If the fields \( A_+ \) and \( A_- \) propagate in the plane \( xOz \), the momentum conservation along \( O_x \) implies that

\[
N_+ \hbar K_z + N_- (-\hbar K_z) = 0 \tag{35}
\]
where \( N_+ \) and \( N_- \) are the photon numbers in fields \( A_+ \) and \( A_- \). Equation (35) shows that
\[
N_+ - N_- = 0
\]
which show that here also \( (N_+ - N_-) \) should exhibit a subpoissonian statistics.

If we have a square pattern and if we call \( A_1 \) and \( A_2 \) the fields propagating in the \( xOz \) plane and \( A_3 \) and \( A_4 \) the fields propagating in the \( yOz \) plane, conservation of momentum implies that
\[
N_1 - N_3 = 0 \quad (37a)
\]
\[
N_2 - N_4 = 0 \quad (37b)
\]

In the case of hexagons, if we label the fields \( A_i (i = 1, 6) \), conservation of momentum leads to:
\[
(N_1 - N_4) \hbar K_\lambda + (N_2 + N_6 - N_3 - N_5) \frac{\hbar K_\lambda}{2} = 0 \quad (38a)
\]
\[
(N_2 + N_3 - N_5 - N_6) \frac{\sqrt{3}}{2} \hbar K_\lambda = 0 \quad (38b)
\]

The combination of these two equations shows that the photon numbers in symmetric beams are not generally equal but there is still two independent quantities that should exhibit subpoissonian statistics because
\[
N_i + N_{i+1} - N_{i+3} - N_{i+4} = 0 \quad (i = 1, 2, 3) \quad (39)
\]

Conclusion.

We have presented here a new method to describe the instability occurring between a nonlinear medium and a mirror. This method allows the recovery of essential results with simple mathematics and is thus probably suitable for further extension. For instance, it can probably be applied to the case of the tensorial nonlinear susceptibilities that appear in many experiments made with atomic vapors. It may give access in this case to the dynamical instabilities that have been observed in the case of counterpropagating waves but which are difficult to derive theoretically because of the complexity of the problem.

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References

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