Comparison of high- and low-frequency electromagnetic field analysis
A. Konrad, I. Tsukerman

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Abstract. — Research in the area of electromagnetic field computation has evolved within two distinct groups which are distinguished by their areas of applications. High frequency field computation has developed mostly out of the need to analyze antennas and scatterers in open space and also microwave devices. Low frequency field analysis, on the other hand, has developed around electrostatic, magnetostatic and eddy current applications. The paper reviews the two favourite methods of these groups: the method of moments and the finite element method. The objective is to show the common origin of the two methods. With the common grounds clearly established, it is hoped that there will be cross-fertilization of ideas between the low- and high-frequency groups.

1. Introduction.

When one talks about the numerical solution of electromagnetic field problems by this method or that method, one must think of the following two items:

(i) the kind of formulation used, and

(ii) the mathematical tool employed.

The combination of these two ingredients determines what method one is using.

There exist two basic types of formulations of electromagnetic field problems:

(i) integral equation formulations, and

(ii) differential equation formulations.

However, mixed integral/differential equation formulations are also possible and could be thought of as the third type.

In integral equation formulations usually volume or surface charges or currents are taken as the unknown functions. Since the electromagnetic field can be expressed as an integral of these unknown functions, one obtains either a Fredholm integral equation of the first kind or the second kind. Linear problems with piecewise-homogeneous media can be formulated in terms
of surface charges or currents. An advantage of integral equation formulations is that threedimensional problems can be reduced to two-dimensional ones and two-dimensional problems to one-dimensional ones.

In differential equation formulations usually fields or frequently their potentials are the unknowns. Since Maxwell’s first-order differential equations are coupled, the differential equations in terms of a single field quantity or potential function are often second-order differential equations.

By applying a mathematical tool to a given formulation, one obtains a method. From the mathematical tool box one may choose either an analytical or a numerical approach. Analytical approaches are very useful but more often than not they are restricted to simple problem geometries. From among the available numerical tools, one may choose a numerical quadrature or the general method of moments (GMM, Sect. 2). The famous Galerkin’s method is a particular case of the GMM.

By applying the GMM to differential, integral or mixed integrodifferential formulations, one arrives at some of the well known basic solution methods. For example, the finite element method (FEM) is a special case of the GMM (frequently Galerkin’s method) combined with a differential or integrodifferential equation formulation. The FEM has been a favourite of low-frequency electromagnetic field computation groups since the late Sixties.

Similarly, the GMM combined with a formulation based on a Fredholm integral equation of the first kind yields Harrington’s well known method of moments (MOM) which is often referred to simply as the moment method without further specification. This is somewhat confusing because the FEM is also a particular case of the GMM. The MOM has been around since the early Sixties and had and still has numerous advocates among high-frequency electromagnetic field computation groups.

Now that the preferences of the low- and high-frequency electromagnetic field computation groups in the past have been mentioned, the two objectives of this paper can be clearly stated:

The first objective, already partially achieved, is to show that both groups use methods that can be traced back to a common mathematical tool, namely the GMM.

The second objective is to show that inherent difficulties may arise in either group’s preferred methods. In particular, one may single out the nasty problem of nonphysical solutions obtained with the FEM. Also, one should mention spurious resonances in solutions that gave headaches to MOM advocates for many years.

2. The general method of moments.

Let us first recall that a linear functional $f$ in a linear space $X$ is a linear mapping of $X$ to the space of real numbers $R^1$. In other words, for each $x \in X$ there is a real number denoted by $f(x)$ or $\langle x, f \rangle$ such that for any $y \in X$ and any real number pair $a, b$ the following equality holds:

$$\langle ax + by, f \rangle = a \langle x, f \rangle + b \langle y, f \rangle.$$  \hspace{1cm} (1)

Consider now the solution of the linear operator equation

$$Lx = g$$  \hspace{1cm} (2)

where $x$ is the unknown and the known right-hand side $g$ belongs to the range $R$ of the linear operator $L$. If $x$ is the exact solution then for any linear functional $f$ (whose domain contains $g$) we have

$$\langle Lx, f \rangle = \langle g, f \rangle.$$  \hspace{1cm} (3)
Seeking \( x \) as a linear combination of a finite number of basis elements \( \{N_j\} \) \((j = 1, 2, \ldots, n)\), i.e.

\[
x = \sum_{j=1}^{n} \alpha_j N_j
\]

(4)

and choosing a system of \( n \) linear functionals \( \{f_i\} \) \((i = 1, 2, \ldots, n)\), one may attempt to satisfy (3) at least for these functionals, i.e.

\[
\left\langle L \sum_{j=1}^{n} \alpha_j N_j, f_i \right\rangle = \left\langle g, f_i \right\rangle \quad \text{where} \quad i = 1, 2, \ldots, n.
\]

(5)

The elements of the coefficient matrix of the above system are given by \( \left\langle LN_j, f_i \right\rangle \); this system of \( n \) simultaneous algebraic equations in the unknowns \( \alpha_j \) constitutes the mathematical tool known as the Method of Moments. To distinguish it from its particular applications in electromagnetics, which are also known as « the moment method », we shall call the mathematical tool described above the « General Method of Moments » (GMM).

Notice that only the concepts of linear space, linear operator and linear functional were used to arrive at the GMM. Even a norm and a scalar product were not needed; however, convergence and accuracy of the approximate solution will make sense only if a norm is specified. The GMM can be applied to both integral and differential equation formulations of electromagnetic field problems.

By solving the system (5) for \( \alpha_j \) one may try to obtain an approximate solution to (2) as a linear combination (4). However, there is no guarantee that (5) has a solution and even if it does it may or may not bear resemblance to the exact solution. More specific information about the linear space \( X \) and the linear operator \( L \) has to be provided to ensure that (5) is a useful mathematical tool. Detailed treatment of this subject can be found in the mathematical literature.

When the functional space \( X \) has a scalar product denoted by \( (,\) \) and the norm induced by this scalar product, then the functional \( f_i \) in (5) may be defined as

\[
\left\langle x, f_i \right\rangle = (x, \eta_i)
\]

(6)

where \( \{\eta_i\} \) is a chosen set of elements belonging to \( X \) called a « projection system » or a system of « trial functions ». When the trial functions are the same as the basis elements (i.e. \( N_i = \eta_i \) for \( i = 1, 2, \ldots, n \)), then the GMM is called Galerkin’s method.

One specific choice of functionals \( f_i \) in (5), the Dirac \( \delta \)-function, is well worth noting. It is a distribution [1, 20, 21], not a regular function, defined as a linear functional (we could call it the \( \delta \)-functional),

\[
\left\langle x(t), \delta \right\rangle = x(0)
\]

(7)

where \( x(t) \) is a function belonging to an appropriate functional space \( X \). The particular case of the method of moments with Dirac \( \delta \)-functions used as trial functions is called collocation or point-matching.

Remark. — It is stated in [11, 12] that the \( \delta \)-functional is discontinuous. However, the continuity of \( \delta \) depends on the nature of the space \( X \) and the norm or topology on it. In the mathematical literature distributions (with the Dirac functional among them) are by definition continuous functionals on a special topological vector space of infinitely differentiable functions [1, 20, 21]. Discussion of this matter, though, is far beyond the scope of the present paper.
3. An integral equation formulation.

Consider a very simple integral equation formulation in electrostatics: electrostatic field $E$ in a homogeneous dielectric in the presence of a conducting body or surface. It will be assumed that there are no volume charges so Poisson's equation simplifies to Laplace's equation

$$\text{div} \text{ grad} \ (V) = 0 \quad (8)$$

where $V$ is the electric scalar potential. The solution of Laplace's equation may be sought as the potential of a single layer of charges. The potential $V$ need not be created by real charges; one may look for a fictitious distribution of surface charges whose potential is the same as that of the actual field. Since the potential created by surface charges satisfies Laplace's equation (8), it is sufficient to satisfy the boundary conditions which may have two forms: (i) given potential at the conductor surface, or (ii) given the total charge on the conductor surface.

Consider the boundary condition corresponding to the known potential $V_0$ of the conductor. If the solution is sought as the potential of a single layer of charge, the following integral equation is obtained from the principle of superposition of the potentials of point charges:

$$\frac{1}{4 \pi \varepsilon} \int_S \frac{\sigma(y)}{|x - y|} \, dS_y = V_0, \quad x \in S. \quad (9)$$

This is a Fredholm integral equation of the first kind which is an ill-posed problem [2, 14, 22]. Although the solution of (9) does exist, numerical instability can generally be expected when this type of equation is solved. As a rule of thumb, the smoother the kernel, the more ill-conditioned is the Fredholm integral equation of the first kind [2]. From this point of view, the kernels with singularities appearing in electromagnetic problems (e.g., $1/|x - y|$) may be expected to exhibit «reasonable» behaviour.


The delicate matter of instability regarding Fredholm integral equations of the first kind is not considered in Harrington's work [11, 12]. He knew that despite theoretical instability, straightforward solution of integral equations of the first kind often yields good results in practice.

Harrington introduced the simplest variant of the GMM in the Sixties. This variant consists of the following steps:

(a) subdivide the conducting surfaces $S_j$ into subsections $\Delta s_i$;
(b) approximate the charge density $\sigma$ by a linear combination of pulse functions $f_i$ with unknown coefficients $\sigma_i$:

$$\sigma = \sum_{i=1}^{n} \sigma_i f_i \quad (10)$$

where

$$f_i = \begin{cases} 
1 & \text{on } \Delta s_i \\
0 & \text{on all other } \Delta s_k (k \neq i),
\end{cases}$$

(c) substitute the approximation into the integral equation and, using point-matching, obtain the algebraic equations

$$\sum_{j=1}^{n} a_{ij} \sigma_j = v_i, \quad i = 1, 2, \ldots, n \quad (11)$$
where \( \phi_i \) is the given potential of the conducting surface on which the subsection \( \Delta s_i \) is located, and the coefficients \( a_{ij} \) are given by

\[
a_{ij} = \int_{\Delta s_j} \frac{dS}{4 \pi \epsilon r_{ij}} \tag{12}
\]

\[
r_{ij} = ((x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2)^{1/2} \tag{13}
\]

(d) solve the system of algebraic equations to determine the unknown coefficients \( \sigma_i \).

With the coefficients \( \sigma_i \) known, the approximate charge density is expressed according to (10); then, with the known distribution of charge density \( \sigma \), the electric scalar potential can be computed as a single layer potential given by (9).

Note that with Harrington’s MOM the capacitance of a single conductor is computed numerically from the following simple expression

\[
C = \frac{Q}{V} = \sum_{i=1}^{n} \sigma_i \Delta s_i \tag{14}
\]

For the implementation of the MOM one needs the coefficients \( a_{ij} \). A procedure to calculate these coefficients is given in [11, 12].

One can now estimate the computer memory and the number of arithmetic operations required to implement the moment method. The number of subsections of the size \( O(h) \) on a two-dimensional surface is obviously \( O(h^{-2}) \). The dimension of the matrix of the moment method is equal to the number of subsections; therefore this is an \( O(h^{-2}) \times O(h^{-2}) \) square matrix. Being a full matrix, it contains \( O(h^{-4}) \) nonzero elements which have to be kept in the computer memory. Thus the moment method requires \( O(h^{-4}) \) units of memory. Numerical solution of an \( n \times n \) system by Gaussian elimination requires \( O(n^3) \) arithmetic operations [10]. Since \( n = O(h^{-2}) \) for the MOM system, the required number of operations is \( O(h^{-6}) \).

After the system is solved, one needs \( O(h^{-2}) \) arithmetic operations to compute the capacitance using (14). This number is negligible as compared to \( O(h^{-6}) \). However, if, besides the capacitance, the potential or field values are needed at \( m \) points, this will require \( O(mh^{-2}) \) additional operations.

The advantages of the MOM are as follows:

1. simplicity;
2. handles non-closed conducting surfaces;
3. solves problems in unbounded domains.

The disadvantages of the MOM are as listed below:

1. it is difficult to solve problems with given charges as opposed to given potentials at the conductor surfaces;
2. the method is not well suited for inhomogeneous problems or bounded domains;
3. numerical instability is possible;
4. computer memory and the number of arithmetic operations grow very quickly with any attempt to increase accuracy by increasing the number of subsections.

5. The Finite Element Method (FEM).

To apply the General Method of Moments described in section 2 to a boundary value problem, one, roughly speaking, needs to do the following:
(i) formulate this problem in the operator form in a suitable functional space;
(ii) choose an appropriate system of basis functions and the projection system of trial functionals;
(iii) solve the GMM algebraic system similar to (5) to obtain the approximate solution.

**FEM is a method of moments** with special basis functions. The basis functions in FEM are chosen to be nonzero only on a small part of the whole computational domain (on a few adjacent elements). Traditionally the basis functions were associated with the nodes of the elements; however, edge elements (see below) are more flexible and powerful in 3D electromagnetic problems.

FEM is very often viewed also as a Ritz method (minimization of a functional). However, Ritz formulation is less general and applicable only to self-adjoint positive definite operators, whereas the moment method can be, in principle, applied to an arbitrary problem.

Finite elements in which the basis functions are associated with the edges rather than nodes are called **edge elements**. They were first proposed in 1980 by Nedelec [17]. In 1982-83 Bossavit and Vérité [6] used these elements in a new formulation of the eddy current problem. Edge elements can be formed on tetrahedral, brick or hexahedral elements [17, 4, 7, 24, 16] and may be of the first or higher orders [13]. These elements are now becoming increasingly popular.

Edge elements are connected with some basic concepts of differential geometry which treats fields as differential forms (loosely speaking, elementary circulations and fluxes) rather than vector quantities. The differential form approach is more general than the conventional vector field analysis, and this explains the greater flexibility of edge elements. For a detailed discussion of this and other related issues please see [3-8]. Nedelec's elements are part of « Whitney's complex » of elements [4, 7] which is related to de Rham's complex of differential geometry. A « Whitney element of degree 0 » is just a conventional node element; the Nedelec's edge element is a « Whitney element of degree 1 »; and a « Whitney element of degree 2 » is a « facet element ».

Let \( W^k \) be the finite dimensional space spanned by Whitney’s elements of degree \( k \ (k = 0, 1, 2) \) in a domain \( G \) (boundary conditions should be specified). The Whitney’s complex \( \{ W^1, W^2, W^3 \} \) has the following « exactness property » [7]:

\[
\begin{align*}
\nabla W^0 & \text{ is the kernel of curl in } W^1; \\
\text{curl } W^1 & \text{ is the kernel of div in } W^2
\end{align*}
\]

(15)

This implies, in particular, that any curl-free field \( \tilde{\mathbf{J}} \) in \( W^1 \) can be expressed as \( \nabla \Omega \), where \( \Omega \) belongs to \( W^0 \) (i.e. \( \Omega \) is continuous in \( G \) and linear on each tetrahedron of a given mesh). In a similar way, any solenoidal field that belongs to \( W^2 \) is a curl of some vector field of \( W^1 \). These properties mean that the finite dimensional Whitney’s spaces resemble the properties of spaces of continuous scalar/vector fields with respect to the operators \( \text{grad} \), \( \text{curl} \) and \( \text{div} \).

Although this assertion seems rather abstract, it has very important practical implications. For example, it is due to the resemblance between discrete and continuous spaces that nonphysical modes in 3D waveguide and cavity computations are eliminated (Sect. 6). It can be predicted that in the near future most of 3D numerical modeling in high and low frequency electromagnetics will be based on edge elements.

To estimate the **computational complexity** of the FEM, consider a finite element mesh in a bounded domain. For one-dimensional problems, the mesh consists of segments; for two-dimensional problems, the mesh may consist of triangles, rectangles etc., for three-dimensional problems — of tetrahedra, « brick elements » (parallelepipeds), hexahedra etc.
For definiteness, let us discuss the 3D case. If the elements have the size $O(h)$ and the domain is bounded, then there are $O(h^{-3})$ basis functions; therefore one deals with an $O(h^{-3}) \times O(h^{-3})$-matrix. However, this matrix is very sparse and contains only a few nonzero elements per row. The typical total number of nonzero elements in the matrix is $O(h^{-3})$.

Solution of the problem preserving the sparsity of the matrix is a complicated matter. For elliptic equations efficient iterative methods with the operation count of $O(h^{-4})$ or even of $O(h^{-3} \log h)$ [9] are available. The computer memory required is only $O(h^{-3})$. Comparison with the estimates for the integral equation methods (Sect. 4) shows that FEM is potentially much superior in terms of numerical complexity.


Unfortunately, almost all numerical models of waveguides and cavities yield, along with physical correct solutions, a large number of nonphysical modes often referred to as «spurious» or «extraneous» (see [15] and Ref. there). For a long time, since the beginning of 70’s, this has been an annoying problem for many researchers.

Mathematically, the existence of «spurious modes» means that at least some of the numerical solutions do not converge to exact solutions. This also raises doubts about the other, «physical», modes: if part of the solutions is incorrect, is there any guarantee that the others are accurate?

Although a complete mathematical study has not appeared yet, now there is an understanding of the reasons causing nonphysical modes in FEM models and, even more importantly, there is a method to avoid these modes. It has been shown by Bossavit [8] that edge elements do not yield spurious modes. Practical results reported in one of the recent publications [18] confirm this conclusion.

«Spurious modes» are characterized by not being divergence-free. Therefore it is very important to investigate whether or not a given numerical method ensures, at least in weak form, that the divergence of the numerical solution is zero.

As an example, one may consider a waveguide problem

$$\text{curl } \epsilon^{-1} \text{curl } H - \omega^2 \mu H = 0$$  \hspace{1cm} (16)

and obtain the weak form by multiplying by a trial field $H'$ and integrating by parts:

$$\int_G \epsilon^{-1} \text{curl } H \text{ curl } H' \, dV - \omega^2 \int_G \mu H H' \, dV = 0 \, .$$  \hspace{1cm} (17)

Suppose one can choose a trial function $H'$ as a gradient, $H' = \nabla \phi'$. Then (17) becomes

$$\int_G \mu H \nabla \phi' \, dV = 0 \quad (\omega \neq 0) \, .$$

It can easily be checked (integrating by parts) that this equation is precisely the weak form of $\text{div } \mu H = 0$. In fact, taking $H' = \nabla \phi'$ is the weak form equivalent of applying the $\text{div}$ operator to the original equation (16). Thus, to ensure that the numerical solution is divergence-free, one needs to have gradients in the set of trial functions. The conventional set of nodal basis functions for $H_x, H_y, H_z$ does not generally contain gradients. As regards edge elements, due to the exactness property (15) of «Whitney’s spaces» $W^m, m = 0, 1, 2, 3$ (Sect. 5), the space $W^1$ of edge elements contains $\nabla W^0$, which is the property desired.

Thus the problem of «spurious modes» in three-dimensional waveguide and cavity field computations seems to be solved by using edge elements.
Harrington’s moment method may also be prone to nonphysical solutions. Examples are given in [25] where it is shown that most of the spurious resonances can be avoided by switching from a Fredholm equation of the first kind to the equation of the second kind. Furthermore, low frequency computations may also be affected by spurious solutions. This possibility is mentioned in [19] but, to the best of the authors’ knowledge, no detailed analysis of this subject has been reported yet. It appears, however, that for eddy current problems with real parameters (conductivity, permeability) spurious modes cannot manifest themselves, as FEM-Galerkin solutions of these problems are proven to be numerically stable [23].

7. Conclusions.

This paper was intended to encourage more interaction between the ideas and methods that are well established in low- and high-frequency electromagnetic field computations. While the finite element method is being used in high frequency computations, the high frequency group may benefit from utilizing it more extensively, especially with edge elements. On the other hand, the low frequency group should be aware of the problem of spurious modes and of the fact that edge elements yield weakly divergence-free numerical solutions.

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References