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Submitted on 1 Jan 1992

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Linearized unsteady aerodynamics for turbomachinery aeroelastic applications

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(Received 15 February 1991, accepted 23 September 1991)

Abstract. — The unsteady aerodynamic analyses developed for turbomachinery aeroelastic predictions must be applicable to the moderate through high frequency unsteady flows that are excited by structural motions and aerodynamic interactions between adjacent blade rows and occur over a wide range of operating conditions. In addition, because of the large number of controlling parameters involved, there is a stringent requirement on computational efficiency. The traditional practice has been to invoke a number of simplifying assumptions, e.g., small-amplitude unsteady excitations, high Reynolds number flow, etc., in the development of analyses intended for use in design applications. In particular, during the past decade researchers have developed general inviscid unsteady aerodynamic linearizations that account for the effects of the nonuniform steady-flow phenomena associated with real blade geometry and mean blade loading on the unsteady aerodynamic response of two-dimensional cascades. In this paper several recent advances to this theory are reviewed. Example results are also presented to illustrate the current status of the theoretical model and the associated numerical prediction capabilities.

1. Introduction.

The development of theoretical analyses to predict unsteady flows in axial-flow turbomachines has been motivated primarily by the need to predict the aeroelastic behavior of the blading, i.e., the onset of blade flutter and the amplitudes of forced blade vibration. For the most part, the blades of an isolated two-dimensional cascade are considered, viscous effects are usually neglected and the unsteady fluctuations are assumed to be sufficiently small so that a linearized treatment of the unsteady flow is justified. To determine the aeroelastic characteristics of the blading, the resulting analyses must be capable of predicting the unsteady loads that act on the blades due to various sources of unsteady excitation, i.e., prescribed structural (blade) motions and external aerodynamic excitations. The latter include variations in total temperature and total pressure (entropy and vorticity waves) at

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inlet and variations in static pressure (acoustic waves) at inlet and exit. In particular, for blade flutter applications it is only necessary to predict the unsteady loads due to prescribed blade motions, while for forced response applications the unsteady loads due to incident entropic, vortical and acoustic disturbances are also required.

It should also be noted that the unsteady aerodynamic analyses developed for aeroelastic applications are also useful in aeroacoustic response studies, i.e., for determining the noise generated by the aerodynamic interactions between a given blade row and its neighbors. For aeroacoustic applications, the pressure waves that are produced by the foregoing excitations and carry energy away from the blade row are the response quantities of interest.

The unsteady aerodynamic analyses currently used for turbomachinery aeroelastic and aeroacoustic design predictions are essentially based on classical linearized unsteady aerodynamic theory (see Ref. [1] for an informative review). The classical formulation admits very efficient semi-analytic solutions for entirely subsonic [2], entirely supersonic [3] and high-frequency transonic [4] flows, but it does not account for interactions between steady and unsteady disturbances. Since the latter can be crucial to the successful prediction of the unsteady aerodynamic phenomena that occur over a wide range of practical operating conditions, solutions based on the classical linearization often fail to meet the needs of turbomachinery designers.

To partially overcome the limitations of classical theory, unsteady aerodynamic linearizations have been developed during the past decade which include the effects of important design features, such as real blade geometry, mean or steady blade loading and operation at transonic Mach numbers, on unsteady aerodynamic response. Here the unsteady flow is regarded as a small-amplitude perturbation about a fully nonuniform steady flow. The steady flow is governed by a nonlinear inviscid equation set; the unsteady flow, by a system of linear equations with variable coefficients that depend upon the underlying steady flow. This type of model has received considerable attention in recent years [5-11], and numerical methods for solving the linearized unsteady equations have reached the state where it is appropriate to apply them in design studies. Detailed reviews on this general inviscid unsteady aerodynamic linearization can be found in references [12, 13].

In the present paper, we describe advances in the linearized analysis of unsteady cascade flows that have been made since the publication of references [12, 13]. For the most part, these build upon the LINFLO (Linearized INviscid FLOW) analysis developed by Verdon and Caspar [8, 10]. In particular, the linearized unsteady aerodynamic formulation has been extended so that the response of subsonic cascades to incident entropic and vortical gusts can be predicted. In addition, more complete treatments of the linearized unsteady flow behavior in the far field of an isolated blade row [14] and the aerodynamic response at a moving blade surface [15] have been determined. Finally, numerical procedures have been developed for the prediction of unsteady transonic flows through vibrating cascades [16] as well as for the prediction of the unsteady subsonic flows excited by entropic and vortical gusts [17]. These developments are outlined in the present paper. Example results are also presented to illustrate current prediction capabilities for linearized unsteady subsonic and transonic cascade flows.

2. The physical problem.

Consider time-dependent, adiabatic flow, with negligible body forces, of an inviscid non-heat-conducting, perfect gas through a two-dimensional cascade such as the one shown in figure 1. The time-dependent or unsteady fluctuations in the flow arise from one or more of the following sources (Fig. 2): blade motions, upstream entropic and vortical disturbances, and upstream and downstream acoustic disturbances which carry energy toward the blade row.
Fig. 1. — Two-dimensional transonic compressor cascade; \( M_{+\infty} < M_{-\infty} < 1 \).

\[
\tilde{s}_{-\infty}(x - V_{-\infty}t)
\]

\[
\tilde{p}_{I,-\infty}(x_1, t)
\]

\[
\tilde{\xi}_{-\infty}(x - V_{-\infty}t)
\]

Fig. 2. — Unsteady excitations: blade motion, entropic and vortical gusts from upstream, and acoustic gusts from upstream and downstream.
These excitations are assumed to be of small amplitude and periodic in time. The external aerodynamic excitations are also spatially periodic, while structural excitations are periodic in the cascade- or $\eta$-direction.

In the following discussion all variables are dimensionless. Lengths have been scaled with respect to blade chord, time with respect to the ratio of blade chord to upstream free-stream speed, density and velocity with respect to the upstream free-stream density and flow speed, respectively, pressure with respect to the product of the upstream free-stream density and the square of the upstream free-stream speed, and entropy with respect to the fluid specific heat at constant pressure.

The mean or steady-state positions of the blade chord lines coincide with the line segments $\eta = \xi \tan \Theta + mG$, $0 < \xi < \cos \Theta$, $m = 0, \pm 1, \pm 2, \ldots$, where $\xi$ and $\eta$ are coordinates in the axial-flow and cascade directions, respectively, $m$ is a blade number index, $\Theta$ is the cascade stagger angle, and $G$ is the cascade gap vector which is directed along the $\eta$-axis with magnitude equal to the blade spacing. We assume that far upstream (say $\xi < \xi_-$) and far downstream ($\xi > \xi_+$) from the blade row, the mean flow is at most a small irrotational perturbation from a uniform free stream. In addition, blade shape and orientation relative to the inlet free-stream direction, the inlet to exit mean static pressure ratio and the amplitude, frequency and mode of the unsteady excitation are such that the flow remains attached to the blade surfaces. Thus thin vortex sheets or unsteady wakes emanate from the blade trailing edges and extend downstream. Finally, any shocks that might occur are assumed to be of weak to moderate strength and to have small curvature. Thus, changes in the entropy and vorticity of a fluid particle as it passes through a shock are regarded as negligible.

The (prescribed) blade motions are of the form

$$\mathbf{r}_B(x + mG, t) = \text{Re} \left\{ r_B(x) \exp[i(\omega t + m\sigma)] \right\}, \quad x \in B. \quad (1)$$

Here, $\mathbf{r}_B$ measures the displacement of a point on a moving blade surface relative to its mean or steady-state position, $x$ is a position vector, $t$ is time, $r_B$ is a complex displacement-amplitude vector, $\omega$ is the frequency of the blade motion, $\sigma$ is the phase angle between the motions of adjacent blades, Re denotes the real part and B denotes the reference ($m = 0$) blade surface. The incident disturbances are prescribed as functions of $x$ and $t$ which satisfy the field equations that govern the unsteady flow. For example, small-amplitude harmonic perturbations of a uniform stream must be of the form

$$\mathbf{s}(x, t) = \text{Re} \left\{ s_{-\infty} \exp[i(\kappa_{-\infty} x + \omega t)] \right\}, \quad \xi < \xi_-, \quad (2)$$

$$\mathbf{s}(x, t) = \text{Re} \left\{ s_{-\infty} \exp[i(\kappa_{-\infty} x + \omega t)] \right\}, \quad \xi < \xi_-, \quad (3)$$

and

$$\mathbf{p}_l(x, t) = \text{Re} \left\{ p_{l, \infty} \exp[-\beta \infty x + i(\kappa_{\infty} x + \omega t)] \right\}, \quad \xi > \xi_+. \quad (4)$$

Here $s_{-\infty}, \xi_{-\infty}$ and $p_{l, \infty}$ are the (prescribed) complex amplitudes of the incident entropic, vortical and pressure fluctuations, $\mathbf{s}(x, t), \mathbf{s}(x, t)$ and $\mathbf{p}_l(x, t)$, respectively, far upstream ($-\infty$) and far downstream ($+\infty$) from the blade row, $\kappa$ is the wave number of an incident disturbance, $\kappa_{-\infty} = \sigma G^{-1}$, and $\beta$ is an attenuation constant. The temporal frequency and wave number of an incident entropic or vortical disturbance are related by $\omega = -\kappa_{-\infty} V_{-\infty}$, where $V_{-\infty}$ is the uniform relative inlet velocity, but a more complicated relationship exists between $\omega$ and $\kappa_{-\infty}$ for incident pressure disturbances [14].

The equations governing the fluid motion follow from the mass, momentum and energy conservation laws and the thermodynamic relations for a perfect gas. These provide a coupled set of nonlinear differential equations (i.e., the Euler equations) in continuous regions of the
flow and jump conditions at surfaces across which the inviscid flow variables are discontinuous, i.e., at vortex-sheet wakes and shocks. In continuous regions the energy equation can be replaced by the requirement that the entropy following a fluid particle must remain constant. In addition to the foregoing field equations and jump conditions, the attached flow assumption requires that the unsteady flow must be tangential to the moving blade surfaces and information on the uniform flow conditions at inlet and exit, the entropic and vortical fluctuations at inlet and the incident static pressure disturbances at inlet and exit must be specified. The remaining steady and unsteady departures from the uniform inlet and exit conditions (i.e., the steady and unsteady pressure response at inlet and exit, and the entropic and vortical fluctuations at exit) must be determined as part of the time-dependent solution.

The foregoing unsteady aerodynamic problem is a formidable one as it involves a system of nonlinear time-dependent partial differential equations along with conditions imposed on moving blade, wake and shock surfaces in which the instantaneous positions of the wakes and shocks must, in principle, be determined as part of the solution. Because of these features and the prohibitive expense that would be involved in obtaining the unsteady aerodynamic response information required for turbomachinery aeroelastic applications, the traditional approach has been to examine limiting forms of the full governing equations with the intention of providing efficient theoretical analyses for the prediction of blade flutter and forced response. One such approach, in which the unsteady flow is regarded as a small perturbation of a fully nonuniform mean flow, is described below.


For small-amplitude unsteady excitations, i.e., \(|S_{\infty}|, |\xi_{\infty}|, \ldots\), \(O(\varepsilon) \ll 1\), the time-dependent flow can be regarded as a small perturbation about an underlying nonlinear mean or steady background flow. Thus, for example, the nonlinear time-dependent fluid velocity \(\tilde{V}(x, t)\) can be expressed in the form

\[
\tilde{V}(x, t) = V(x) + \tilde{v}(x, t) + \ldots,
\]

where \(V(x)\) is the local mean velocity and \(\tilde{v}(x, t)\) is the first-order (in \(\varepsilon\)) unsteady velocity. In addition, Taylor series expansions, e.g.,

\[
\tilde{V}_S = (\tilde{V} + \mathcal{R} \cdot \nabla \tilde{V} + \ldots)|_S,
\]

where \(\mathcal{R}(x, t)\) describes the displacement of the moving surface relative to its mean position, and surface-vector equations which relate the unit tangent, \(\mathbf{t}\), and normal, \(\mathbf{n}\), vectors, where \(\mathbf{n} \times \mathbf{t}\) points out from the page, at a point on a moving surface \(S\) to the corresponding vectors at the location of this point on the mean surface \(S\), can be applied to refer information at a moving blade, wake or shock surface to the mean position of this surface.

Equations governing the steady and first-order unsteady flow properties are determined by substituting the foregoing expansions into the full time-dependent equations, collecting terms of like power in \(\varepsilon\) and neglecting terms of \(O(\varepsilon^2)\) or higher. If the steady flow variables are assumed to stay within \(O(\varepsilon)\) of their upstream free-stream values, the equations of classical linearized theory are recovered. Here, we consider the more general case in which only limited restrictions (i.e., those on far-field behavior) are placed on the departures of the mean or steady flow variables from their upstream free-stream values.

The first-order unsteady motions arising from the various modes of excitation are not coupled and hence, can be determined separately. Indeed, it is sufficient to develop solution procedures for a single harmonic (in \(t\) and \(\eta\)) component of a given disturbance. Solutions for arbitrary disturbances and arbitrary combinations of various disturbances can then be
obtained by Fourier superposition. The first-order or linearized unsteady flow associated with an unsteady excitation at temporal frequency $\omega$ will be harmonic in time, e.g.,

$$v(x, t) = \text{Re} \{v(x) e^{i\omega t}\} . \quad (7)$$

In addition, the steady and, for an excitation at wave number $\kappa = \sigma G^{-1}$, the first-order unsteady properties will satisfy circumferential periodicity conditions, e.g.,

$$V(x + mG) = V(x) \quad \text{and} \quad v(x + mG) = v(x) e^{im\sigma}, \quad (8)$$

where $\sigma$ is the interblade phase angle of the unsteady excitation. Thus, solutions to time-independent nonlinear steady and linearized unsteady flow problems are required only over a single extended blade-passage region of the cascade. In addition, since analytic far-field solutions can be determined [14], the numerical solution domain can be restricted further to a single extended blade-passage region of finite extent in the axial-flow direction, as indicated in figure 3.

![Figure 3](image)

Fig. 3. — Extended blade-passage solution domain and computational grids used in unsteady flow calculations.

**THE STEADY BACKGROUND FLOW.** — As a consequence of our assumptions regarding shocks and the flow far upstream of the blade row, the steady background flow will be isentropic and irrotational; i.e., $V = \nabla \Phi$, where $\Phi$ is the steady velocity potential. Thus, the field equations that govern the underlying steady flow are

$$\nabla \cdot (\bar{\rho} \nabla \Phi) = 0 \quad (9)$$

and

$$(M_{-\infty} A)^2 = \bar{\rho}^{-1} = (\gamma M_{-\infty}^2 P)^{(\gamma - 1)/\gamma} = 1 - \frac{\gamma - 1}{2} M_{-\infty}^2 [(\nabla \Phi)^2 - 1], \quad (10)$$
where \( M, A, \bar{p} \) and \( P \) are the local steady Mach number, speed of sound propagation, density and pressure, respectively, and \( \gamma \) is the specific heat ratio of the fluid. Since, by assumption, the flow remains attached to the blade surfaces \( (B_m) \), a flow tangency condition applies at such surfaces. In addition, mass and tangential momentum must be conserved across shocks \( (Sh_{m,n}) \), and the steady pressure and normal velocity component must be continuous across blade wakes \( (W_m) \). Here the subscript \( n \) refers to the \( n \)-th shock associated with the \( m \)-th blade.

Numerical procedures for determining two-dimensional steady potential flows through cascades have been developed extensively, see, e.g., references [18, 19], particularly for flows with subsonic relative inlet and exit Mach numbers. In such calculations far-field boundary conditions are imposed at axial stations placed at finite distances upstream and downstream (i.e., at \( \xi = \xi_{\pm} \)) from the blade row, where linearized solutions describing the behavior of the steady potential can be matched to a nonlinear near-field solution. In addition, a Kutta condition is usually imposed at blade trailing edges in lieu of prescribing an exit free-stream property. Finally the usual practice is to solve the conservative form of the mass-balance equation (9) throughout the entire fluid domain. Thus, shock- and wake-jump conditions are usually not imposed explicitly. Instead, shock phenomena are captured through the use of special differencing techniques, and wake conditions are satisfied implicitly, because the fluid properties are continuous and differentiable across wakes.

4. The linearized unsteady equations.

The field equations that govern the first-order entropy \( (\bar{s}) \), velocity \( (\bar{v}) \) and pressure \( (\bar{p}) \) in the unsteady perturbation of a nonlinear isentropic and irrotational steady background flow can be written in the form [12, 20].

\[
\frac{\partial \bar{s}}{\partial t} = 0 ,
\]

\[
\frac{\partial}{\partial t} (\bar{v} - \bar{s} \nabla \Phi / 2) + \left[ (\bar{v} - \bar{s} \nabla \Phi / 2) \cdot \nabla \right] \nabla \Phi = - \nabla (\bar{p} / \bar{\rho}) \tag{12}
\]

and

\[
\frac{\partial}{\partial t} \left[ \bar{p} / (\bar{\rho} \bar{A}^2) \right] + \bar{\rho}^{-1} \nabla \cdot (\bar{\rho} \bar{v}) = 0 ,
\]

where \( \partial / \partial t = \partial / \partial t + \nabla \Phi \cdot \nabla \) is a convective derivative based on the mean-flow velocity. In general, we require a solution to this system of equations subject to prescribed flow conditions far upstream and far down-stream from the blade row, the flow tangency condition at moving blade surfaces, and, since the field equations apply only in continuous regions, jump conditions based on the fluid-dynamic conservation laws at moving shocks and blade wakes. As a consequence of the small-amplitude unsteady-disturbance approximation, conditions on the linearized unsteady perturbation at moving blade, shock and wake surfaces can be transferred to the mean positions of these surfaces, with the mean wake \( (W_m) \), i.e., the downstream stagnation streamlines, and shock \( (Sh_{m,n}) \) locations being determined from the nonlinear steady solution.

As indicated by Goldstein [20], the foregoing system of equations can be recast into a very convenient form by decomposing the unsteady velocity into rotational \( (\bar{v}_R) \) and irrotational \( (\nabla \bar{\phi}) \) parts, i.e.,

\[
\bar{v} = \bar{v}_R + \nabla \bar{\phi} . \tag{14}
\]
Here, the rotational velocity is divergence-free far upstream of the blade row, i.e., \( \nabla \cdot \tilde{\mathbf{v}}_R = 0 \) for \( \xi < \xi_- \), and the unsteady pressure depends on the unsteady potential \( \tilde{\phi} \) through the relation \( \tilde{p} = -\tilde{p} \tilde{D} \tilde{\phi} / \partial t \). The unsteady vorticity is given by \( \tilde{\zeta} = \nabla \times \tilde{\mathbf{v}}_R \). The field equations for the unsteady flow variables \( \tilde{\mathbf{v}}_R \) and \( \tilde{\phi} \) can then be written as

\[
\frac{\tilde{D}}{\partial t} (\tilde{\mathbf{v}}_R - \tilde{s} \nabla \phi/2) + [(\tilde{\mathbf{v}}_R - \tilde{s} \nabla \phi/2) \cdot \nabla] \nabla \phi = 0
\]

and

\[
\frac{\tilde{D}}{\partial t} \left( A \tilde{s} \tilde{D} \tilde{\phi} / \partial t \right) - \tilde{p}^{-1} \nabla \cdot (\tilde{p} \nabla \tilde{\phi}) = \tilde{p}^{-1} \nabla \cdot (\tilde{p} \tilde{\mathbf{v}}_R) .
\]

Equations (11), (15) and (16) are coupled only sequentially. Moreover, since the entropy fluctuation is governed by a convection equation, it can be determined in terms of the prescribed upstream entropy field. The rotational velocity fluctuation is governed by a modified convection equation and can be determined in terms of the entropy and the prescribed upstream rotational velocity field. Thus, the unsteady equations can be solved in order to determine \( \tilde{s} \), \( \tilde{\mathbf{v}}_R \) and \( \tilde{\phi} \). For subsonic inlet and exit conditions the velocity potential, \( \phi \), is governed by an elliptic or a mixed elliptic/hyperbolic equation and depends, therefore, upon boundary condition information both far upstream and far downstream from the blade row as well as upon conditions at the blade, wake and shock surfaces.

**Far-Field Behavior.** We have assumed that the mean or steady flow is at most a small (i.e., \( O(\epsilon) \)) perturbation from a uniform stream both far upstream (\( \xi < \xi_- \)) and far downstream (\( \xi > \xi_+ \)) from the blade row. Therefore, in these regions, the first-order (in \( \epsilon \)) unsteady field equations can be reduced to constant coefficient equations for which analytic solutions can be determined. For example, it follows after replacing \( \nabla \phi \) by \( \nabla \phi_{-\infty} \) in equations (11) and (15) that the entropy and rotational velocity fluctuations far upstream (\( \xi < \xi_- \)) of the blade row must be of the form

\[
\tilde{s}(x, t) = \tilde{s}_{-\infty}(x - V_{-\infty} t) = \text{Re} \left\{ s_{-\infty} \exp \left[ i (\mathbf{k}_{-\infty} \cdot \mathbf{x} + \omega t) \right] \right\} ,
\]

and

\[
\tilde{\mathbf{v}}_R(x, t) = \mathbf{v}_{R, -\infty}(x - V_{-\infty} t) = \text{Re} \left\{ v_{R, -\infty} \exp \left[ i (\mathbf{k}_{-\infty} \cdot \mathbf{x} + \omega t) \right] \right\} ,
\]

where \( s_{-\infty} \) and \( \xi_{-\infty} = i \mathbf{k}_{-\infty} \times \mathbf{v}_{R, -\infty} \), the complex amplitudes (at \( x = 0 \)) of the incident entropic and vortical gusts, are prescribed. Since the rotational velocity is divergence free far upstream of the blade row, the vectors \( \mathbf{k}_{-\infty} \) and \( \mathbf{v}_{R, -\infty} \) are orthogonal. Therefore,

\[
\mathbf{v}_{R, -\infty} = i (\mathbf{k}_{-\infty} \times \xi_{-\infty})/|\mathbf{k}_{-\infty}|^2
\]

The velocity potential fluctuations in the far upstream and far downstream regions depend upon the acoustic excitation as well as the acoustic and vortical response of the blade row to the imposed unsteady excitations. Thus, we can write

\[
\tilde{\phi}(x, t) = \tilde{\phi}_E(x) + \tilde{\phi}_R(x) \quad \text{for} \quad \xi \equiv \xi_+ .
\]

The potential component \( \tilde{\phi}_E = \phi_E \exp(i \omega t) \) accounts for acoustic excitations, i.e., pressure disturbances that either attenuate as they approach the blade row or propagate and carry
5. Entropy and rotational velocity fluctuations.

Closed form solutions can be determined for the entropy and rotational velocity fluctuations that arise from arbitrary disturbances at inlet [17, 20], but here, we restrict our consideration to the fluctuations that arise from a single harmonic excitation at frequency $\omega$ and wave number $\kappa_n$. We introduce the vector

$$X = V_{-\infty} \Delta + \Psi e_z \times V_{-\infty},$$

(23)

where $\Delta$ and $\Psi$ are the drift and stream functions of the steady background flow, and $e_z$ is a unit vector that points out from the page. For the present application $\Delta$ and $\Psi$ are defined as follows

$$\Delta(x) = \Delta(x_-) + \int_{x_- + e_z \times V_{-\infty}}^{x} V^{-1} d\tau \psi$$

(24)

and

$$\Psi(x) = \Psi(x_-) + \int_{x_- + e_z \times V_{-\infty}}^{x} \overline{\beta}(e_z \times V) \cdot d\tau,$$

(25)

where the constants $\Delta(x_-)$ and $\Psi(x_-)$ are chosen so that $X \to x_z$ as $\xi \to -\infty$. In equations (24) and (25), $x_-$ is the position vector to the point of intersection $(\xi_-, \eta_-)$ of the reference blade stagnation streamline and the axial line $\xi = \xi_-, \ d\tau \psi$ is a differential element of arc length along a streamline, and $d\tau$ is a differential vector tangent to the path of integration in equation (25).

As the reader can easily verify, the solution to the entropy transport equation (11) which satisfies the upstream condition (17) is

$$\bar{s}(x, t) = \text{Re} \left\{ s_{-\infty} \exp[i\kappa_{-\infty} (X - V_{-\infty} t)] \right\},$$

(26)

and the solution to the rotational velocity equation (15) which satisfies the far upstream condition (18) is given by

$$\bar{\nu}_R(x, t) = \text{Re} \left\{ [\nabla(X \cdot \mathcal{A}_{-\infty}) + s_{-\infty} \nabla \Phi/2] \times \exp [i\kappa_{-\infty} (X - V_{-\infty} t)] \right\},$$

(27)
where
\[ \mathcal{A}_{-\infty} = v_{R,-\infty} - s_{-\infty} V_{-\infty}/2 \]
\[ = - (\kappa_{-\infty} \times \xi_{-\infty})/|\kappa_{-\infty}|^2 - s_{-\infty} V_{-\infty}/2. \]

Thus, the entropy and rotational velocity fluctuations can be evaluated once the drift and stream functions, and their derivatives, are determined from the solution for the underlying steady flow. The unsteady vorticity, \( \xi = \nabla \times v_R \), and the source term, \( \bar{\rho}^{-1} \nabla \cdot (\bar{\rho} v_R) \), in the unsteady potential equation (16) depend also upon the second partial derivatives of the mean flow drift and stream functions. Thus, an accurate solution for the nonlinear steady background flow is a critical prerequisite to determining the unsteady effects associated with inlet entropic and vortical excitations.

It is important to note that if the steady background flow stagnates at blade leading edges, as it will for realistic cascade configurations, the rotational velocity will be singular along the mean blade and wake surfaces. This feature stems from the singular behavior of the drift function, i.e., \( \Delta \to a_0 \ln n \), as \( n \to 0 \), where \( n \) is the normal distance from a mean blade or wake surface and \( a_0 \) is a constant [21].

**Numerical Evaluation of \( \Delta \) and \( \Psi \).** — It is convenient to use a grid in which one set of mesh lines are the streamlines of the steady background flow for evaluating the steady-flow drift and stream functions and their spatial derivatives. Thus, an \( H \)-grid which covers the solution domain, i.e., one which is bounded by the upstream and downstream axial lines \( \xi = \xi_{\pm} \) and two neighboring mean-flow stagnation streamlines, is appropriate. The locations of the latter can be determined \textit{a posteriori} from the solution for the nonlinear steady background flow by particle tracing, i.e., numerically by integrating the equation \( \partial\xi/\partial t = V \) from the leading-edge stagnation and the trailing-edge points of a blade to the far upstream \( (\xi = \xi_-) \) and far downstream \( (\xi = \xi_+) \) boundaries. Once the boundaries of the \( H \)-grid are established, the locations of the interior grid points can be determined by using an elliptic grid generation technique similar to that developed by Thompson \textit{et al.} [22]. A typical \( H \)-grid, generated for a compressor cascade, is shown in figure 3.

Because a streamline mesh is used, the drift function can be evaluated at each point in the computational domain by a straightforward numerical integration of equation (24). The procedure used in reference [17] is simply to specify the drift function along the far upstream boundary \( \xi = \xi_- \), and then to evaluate this function along each streamline using a second-order accurate difference approximation. The derivatives of the drift and stream functions at a given grid point are determined using the finite difference operators developed by Caspar and Verdon [23]. Because the drift function is singular at blade and wake surfaces, one-sided difference approximations are used to evaluate its derivatives at points on the first streamlines removed from these surfaces.

6. The unsteady potential.

The unsteady perturbation velocity \( \bar{v} = \bar{v}_R + \nabla \bar{\phi} \) must be finite at mean blade and wake surfaces. Therefore, if the rotational velocity is singular, \( \nabla \bar{\phi} \) must also be singular at such surfaces. This feature can prohibit a successful numerical solution for the unsteady potential, \( \bar{\phi} \). Therefore, Atassi and Grzedzinski [21] decomposed the unsteady potential by setting \( \bar{\phi} = \bar{\phi}' + \bar{\phi}'' \), where \( \bar{\phi}' = \Re \{ \phi' \exp(i\omega t) \} \) is a convected or pressure-less potential, i.e., \( \bar{D} \bar{\phi}'/Dt = 0 \), which renders \( \bar{v}_R = \bar{v}_R + \nabla \bar{\phi}' \), and therefore \( \nabla \bar{\phi}'' \), finite at mean blade and wake surfaces. For periodic, unsteady, two-dimensional, cascade flows they set
\[ \phi' = [-i \omega^{-1} \mathcal{A}_{-\infty} V_{-\infty} + F(\Psi)] \exp(i\kappa_{-\infty} X), \]
where
\[ F(\Psi) = -\frac{c_2 G \cos \Omega_{-\infty}}{2 \pi (1 - ia_0 \omega)} \sin \left( \frac{2 \pi [\Psi(x) - \Psi(x_-)]}{G \cos \Omega_{-\infty}} \right) \] (30)

is a complex function that depends upon, among other things, the behavior of the mean flow in the vicinity of a stagnation point, and \( c_2 = -\omega^{-1}(\kappa_{-\infty} \times \mathbf{A}_{-\infty}) \cdot \mathbf{e}_z \). This choice for \( \phi' \) also ensures that \( \mathbf{v}_h \cdot \mathbf{n} = 0 \) at blade and wake mean positions.

The unsteady potential \( \tilde{\phi}'' = \text{Re} \{ \phi'' \exp(i \omega t) \} \) must be determined numerically by solving the field equation
\[ \frac{\partial \phi''}{\partial t} \mathbf{D}_\omega \left( A^{-2} \frac{\partial \phi''}{\partial t} \right) - \tilde{\rho}^{-1} \nabla \cdot (\tilde{\rho} \nabla \phi'') = \tilde{\rho}^{-1} \nabla \cdot (\tilde{\rho} \mathbf{v}_h'), \] (31)

where \( \mathbf{D}_\omega/\partial t = i \omega + \nabla \Phi \cdot \nabla \), subject to the appropriate surface and far-field (cf. Eqs. (20-22)) conditions.

The following conditions [12, 13] apply at blade (\( B_m \)), wake (\( W_m \)) and shock (\( Sh_{m,n} \)) mean positions. The first-order flow tangency condition has the form
\[ \nabla \phi'' \cdot \mathbf{n} = [i \omega \mathbf{r}_B + (\nabla \Phi \cdot \mathbf{\tau}) (\mathbf{\tau} \cdot \nabla) \mathbf{r}_B - (\mathbf{r}_B \cdot \nabla) \nabla \Phi] \cdot \mathbf{n}, \ x \in B_m, \] (32)

where \( \mathbf{r}_B \) is the complex amplitude of the blade displacement. The wake-jump conditions require that the fluid pressure and the normal component of the fluid velocity must be continuous across blade wakes. Therefore,
\[ \left[ \frac{\mathbf{D}_\omega \phi''/\partial t}{\partial t} \right] = 0 \quad \text{and} \quad \left[ \nabla \phi'' \right] \cdot \mathbf{n} = 0, \ x \in W_m, \] (33)

respectively, where the symbol \( \left[ \right] \) indicates the jump or change in a quantity at a surface at which the flow variables are discontinuous. Finally, if we neglect changes in entropy and rotational velocity across shocks, the conservation laws for mass and tangential momentum require that
\[ \left[ \tilde{\rho} \left( \nabla \phi'' + \mathbf{v}_h' - A^{-2} \frac{\partial \phi''}{\partial t} \nabla \Phi \right) \right] \cdot \mathbf{n} = \]
\[ - \left[ \tilde{\rho} \right] [i \omega + (\nabla \Phi \cdot \mathbf{\tau}) \mathbf{\tau} \cdot \nabla] \left[ \left[ \nabla \Phi \right] \cdot \mathbf{n} \right]^{-1} \left[ \phi'' \right] \]
\[ - \left( \left[ \nabla \Phi \right] \cdot \mathbf{n} \right)^{-1} \left[ \phi'' \right] \nabla \left( \left[ \tilde{\rho} \right] \nabla \Phi \cdot \mathbf{\tau} \right), \ x \in Sh_{m,n} \] (34)

for a shock that terminates in the fluid. Equation (34) provides a relation for determining the jump in the unsteady potential, \( \left[ \phi'' \right] \), at the mean position of a shock. The shock displacement normal to the mean shock locus is then given by \( \mathbf{r}_{sh} \cdot \mathbf{n} = - \left( \left[ \nabla \Phi \right] \cdot \mathbf{n} \right)^{-1} \left[ \phi'' \right] \).

**Numerical Evaluation of \( \phi'' \).** — A numerical resolution of the foregoing linear, variable-coefficient, boundary-value problem for \( \phi'' \) is required over a single, extended, blade-passage region of finite extent. The field equation, (31) must be solved in continuous regions of the flow subject to surface conditions, equations (32-34), at the mean blade, wake and shock surfaces. In addition, the near-field numerical solution for the potential must be matched to far-field analytical solutions at finite distances (\( \xi = \xi_\infty \)) upstream and downstream from the blade row. Numerical methods for determining \( \phi'' \) for subsonic and transonic flows have been reported in references [8, 10, 16, 23]. At present, transonic solutions have been determined
only for flows in which a single normal shock occurs in each blade passage and for unsteady flows in which \( s = v_R = 0 \).

Because of the stringent and conflicting requirements placed on computational meshes for cascade flows, a composite-mesh, which is constructed by overlaying a polar-type local mesh and an \( H \)-type cascade mesh, has been adopted for determining the unsteady potential. The \( H \) mesh is used to resolve unsteady phenomena over the entire solution domain; the local surface-fitted mesh, to resolve phenomena in the vicinities of a rounded blade leading edge and/or a shock. The cascade mesh facilitates the imposition of the cascade periodicity conditions \( (8) \) and the matching of the analytic and numerical unsteady solutions at the far upstream \( (\xi = \xi_-) \) and far downstream \( (\xi = \xi_+) \) boundaries. Use of this mesh alone is often sufficient for resolving unsteady subsonic flows. The local mesh allows an accurate modeling of unsteady leading-edge and shock phenomena. It is constructed so that two radial lines coincide with the predicted mean shock locus to provide upstream and downstream shock-mesh lines for the accurate imposition of shock-jump conditions.

Since the cascade and local body-fitted meshes differ topologically, a zonal solution procedure for overlapping meshes has been adopted for determining \( \phi'' \) \cite{16}. In the region of intersection between the two meshes, i.e., the region covered by the local mesh, certain cascade mesh points are eliminated depending upon their location within the local mesh domain. The discrete equations are written separately for the cascade and local meshes and coupled implicitly through special interface conditions resulting in a single composite system of finite-difference equations that describes the unsteady flow over the entire solution domain.

The finite-difference model used to approximate the unsteady equations on the cascade and local meshes has been described in detail in reference \cite{23}. Algebraic approximations to the various linear operators, which make up the unsteady boundary-value problem, are obtained using an implicit, least-squares, interpolation procedure that is applicable on arbitrary grids. This procedure employs a nine point centered difference star at subsonic field points. At supersonic points the field equation is expressed in local canonical coordinates that are aligned with and normal to the local steady flow direction. The discrete approximation to the streamwise second-derivative of the potential is then retarded or upwind along the cascade tangential and the local circumferential mesh lines using a nine point one-sided difference star. The remaining terms in the field equation are approximated using the standard nine point centered star, resulting in a twelve point difference star at supersonic field points.

At a blade boundary point a nine point one-sided difference star is used on the cascade mesh, while nine- or six-point one-sided stars are used on the local mesh. Shocks are fitted in the local-mesh calculation by approximating the shock-jump condition \( (34) \) using one-sided difference expressions to evaluate the normal derivatives of the unsteady potential on the upstream (supersonic) and downstream (subsonic) sides of the shock. At those points along the shock mesh lines at which the steady flow is continuous (i.e., at points lying beyond the end of the shock), the condition \( [\phi] = 0 \) is imposed. It is important to note that the unsteady shock-jump condition involves jumps in the steady or mean-flow derivatives. Since current steady potential-flow codes use shock capturing methods, the steady shock-jump information needed in the unsteady calculation must be determined by extrapolating mean-flow information from upstream and downstream of a smeared shock to the upstream and downstream sides, respectively, of an estimated mean shock position. The latter can be defined as the locus of all points in the smeared shock at which \( M = 1 \). The errors associated with this approximation can be minimized by using a very dense calculation mesh in the vicinity of the shock, thereby reducing the distance over which the shock is smeared.

The resulting systems of linear algebraic equations that approximate the unsteady boundary-value problem on the cascade and local meshes are block-tridiagonal for subsonic
flow and block-pentadiagonal for transonic flow. A subsonic solution on the \(H\)-mesh alone is determined using a direct block inversion scheme. Composite (cascade/local) mesh subsonic or transonic solutions are determined using a different scheme. Because of the cascade/local mesh coupling conditions, the composite system of discrete equations contains a sparse coefficient matrix of large band width. Consequently, special storage and inversion techniques must be applied to achieve an efficient solution. Once the composite system of unsteady equations is cast into an appropriate format, it can be solved using Gaussian elimination [16].

7. Aerodynamic response at a blade surface.

At this point we have presented a linearized unsteady aerodynamic formulation that describes the general first-order fluid-dynamic perturbation of an isentropic and irrotational steady background flow, and we have outlined numerical procedures that have been used to evaluate the unsteady entropy, rotational velocity and velocity potential. Solutions to the linearized unsteady flow problem are required to determine the aerodynamic response information needed for aeroelastic and aeroacoustic design predictions, i.e., the unsteady pressures acting on the blades and the unsteady pressure field far from the blade row. Parameters that describe the unsteady aerodynamic response at a blade surface provide the information needed turbomachinery aeroelastic applications. Such information permits one to determine whether an airstream tends to support or suppress a prescribed blade motion, i.e., whether or not the blades will flutter, and to evaluate the structural response of the blading to external aerodynamic excitations.

7.1 Surface Pressures. — The pressure acting at the instantaneous position of the reference \((m = 0)\) blade surface is given by

\[
\tilde{P}(\tau_B, t) = P(\tau_B) + \tilde{p}_B(\tau_B, t) + \sum_n \tilde{p}_{sh_n}(\tau_B, t) + \cdots,
\]

where

\[
\tilde{p}_B(\tau_B) = \Re \{p_B(\tau_B) \exp(i \omega t)\}, \tag{35}
\]

\[
p_B = \left[ -\tilde{\phi} \frac{\partial^2 \phi}{\partial \tau^2} + (r_B \cdot \nabla) P \right]_B, \tag{36}
\]

and \(\tau\) measures distance in the counterclockwise direction along the blade surface. The first two terms on the right-hand-side of equation (35) account for the steady and the first-harmonic unsteady contributions to the time-dependent fluid pressure acting at the moving blade surface, \(B\). The third term accounts for the anharmonic contribution caused by the motions of shocks and is determined by analytically continuing the solutions to the steady and the linearized unsteady boundary-value problems from the mean to the instantaneous shock locations [24]. Thus, for example,

\[
\tilde{p}_{sh_n}(\tau, t) = -\frac{R_{sh_n}}{|R_{sh_n}|} U[(\tau - \tau_{sh_n})(\tau_{sh} - \tau)] \times
\]

\[
\times \left\{ \left[ P_{sh_n} \right]_{sh_n} + (\tau - \tau_{sh_n}) \left[ \frac{\partial P_{sh_n}}{\partial \tau} \right]_{sh_n} + \left[ \tilde{p}_B \right]_{sh_n} + \cdots \right\}, \tag{37}
\]

where \(R_{B \cdot B} = \Re \left\{ (r_{sh_n} - r_B) \cdot \tau_B \exp(i \omega t) \right\}\) is the relative displacement of the shock foot in the counterclockwise or \(\tau\)-direction along the moving blade surface, \(U(\tau)\) is the unit-step
function, and the subscripts $\delta h$ and $\delta h$ refer to the instantaneous and mean shock locations, respectively.

Although the unsteady pressure disturbance is not everywhere harmonic, its regions of anharmonicity are small. Consequently, the first-order global unsteady airloads are harmonic in time [25]. For example, if $F(X)$ is an integrable vector function which is real and single-valued at the intersection of a shock and a blade surface, then

$$
\oint_B \tilde{P}_{\delta h, \gamma}(\tau, t) F(\tau) d\tau = -R_{\delta h, \gamma} B F(\tau_{\delta h, \gamma}) \left[ P_B \right]_{\delta h} + . \tag{38}
$$

7.2 Unsteady Airloads. — For aeroelastic calculations, it is convenient to introduce a set of generalized forces corresponding to a set of independent, but otherwise arbitrary, structural deformation modes [26]. Thus, the reference blade displacement can be expressed in the form

$$
R_B(x, t) = \sum_{i=1}^l \tilde{\delta}_i(t) R_j(x) = \sum_{i=1}^l \text{Re} \left\{ \delta_i e^{i\omega t} \right\} R_j(x) = \text{Re} \left\{ r_B(x) e^{i\omega t} \right\}, \tag{39}
$$

where the generalized coordinate $\tilde{\delta}_i(t)$ describes the amplitude of the motion in the $i$-th mode and $R_j(x)$ describes the shape and direction of this displacement. The $i$-th generalized force is then defined as

$$
\tilde{Q}_i(t) = -\oint_B \tilde{P} n \cdot R_j \, d\tau = Q_i + \tilde{q}_i + . \tag{40}
$$

where $n$ is a unit normal vector pointing out from the blade, and $Q_i$ and $\tilde{q}_i = \text{Re} \left\{ q_i \exp(i\omega t) \right\}$ are the steady and the first-harmonic unsteady components, respectively, of the time-dependent generalized force $\tilde{Q}_i(t)$. After replacing the surface integration in equation (40) by an integration over the mean blade surface and performing some algebra [15], we find that

$$
Q_i = -\oint_B P n \cdot R_j \, d\tau, \tag{41}
$$

and

$$
q_i = -\oint_B \left[ P \frac{\partial R_B}{\partial \tau} \times e_J + P_B n \right] \cdot R_j \, d\tau + \sum_n r_{\delta h, \gamma} B \left( \left[ P_B \right] n \cdot R_j \right)_{\delta h} . \tag{42}
$$

As an example, consider bending, torsional and chordwise bending motions of an actual three-dimensional blade. To describe such motions a translation in the $\gamma$-direction (i.e., $R_\gamma = e_\gamma$), a counterclockwise rotation about an axis at $x = x_p$ ($R_2 = e_x \times R_p$) and an elastic deflection normal to the blade chord (e.g., $R_3 = e_\gamma \sin \pi x$) are usually chosen as the independent modes of motion for each two-dimensional blade section; i.e.,

$$
r_B = \delta_1 e_\gamma + \delta_2 (e_x \times R_p) + \delta_3 e_\gamma \sin \pi x, \tag{43}
$$

where $R_p(x) = x_B - x_p$ is a vector that extends from the mean position of the reference blade axis of rotation to a point on the mean position of the reference blade surface. The complex amplitudes of the first-harmonic generalized forces that correspond to these modes of motion are

$$
q_1 = f \cdot e_\gamma, \quad q_2 = m_\pi - \oint_B P \left[ R_B - R_B(x_p) \right] \cdot \tau \, d\tau
$$
and
\[ q_3 = -\oint_{\Omega} \left( P \frac{\partial \mathbf{r}_B}{\partial \tau} \times \mathbf{e}_z + p_\mathbf{B} \mathbf{n} \right) \cdot \mathbf{e}_y \sin \pi x \, d\tau + \sum_n r_{\mathbf{Sh}_nB} \left( \mathbf{n} \cdot \mathbf{e}_y \sin \pi x \right)_{\mathbf{Sh}_n}, \tag{44} \]

where \( f \) and \( m_\tau \) are the complex amplitudes of the first-harmonic unsteady aerodynamic force and moment about the moving pitching axis at \( x = x_p = x_p + \mathbf{R}_B(x_p) \). The equations of motion for the three-dimensional blade structure can then be written using the generalized coordinates, \( \delta_1, \delta_2 \) and \( \delta_3 \), and the generalized forces, \( q_1, q_2 \) and \( q_3 \), as the dependent variables that describe the structural motion and the aerodynamic forcing functions, respectively, for each two-dimensional blade section. An important special case for turbomachinery aeroelastic calculations is one in which each incremental two-dimensional blade section vibrates only in a rigid-body mode (i.e., \( \delta_3 = 0 \)). In this case \( q_3 = 0 \) and \( q_2 = m_\tau + \delta_2 M_p \), where \( M_p \) is the zeroth-order or steady moment about the mean pitching axis at \( x = x_p \).

7.3 AERODYNAMIC WORK PER CYCLE AND PRESSURE-DISPLACEMENT FUNCTION. — The aerodynamic work per cycle and the pressure-displacement function are particularly convenient global and local unsteady aerodynamic response parameters. They are time-independent, real quantities that can be used to describe the aerodynamic response to any mode of blade vibration. The work per cycle is the work done by an airstream on a given blade over one cycle of its motion. Therefore, a prescribed blade motion is stable, neutrally stable or unstable according to whether the work per cycle is less than, equal to, or greater than zero, respectively. The pressure-displacement function describes the distribution of the work per cycle over a blade surface. It contains both harmonic and anharmonic local response information, and its integral over a blade surface, i.e., the aerodynamic work per cycle, is a physically meaningful quantity.

By definition
\[ W_C = \oint_{\Omega} \frac{d\tilde{W}}{dt} \, dt = -\omega^{-1} \int_{\phi}^{\phi+2\pi} \oint_{\mathbf{B}} \tilde{P} \frac{\partial \mathbf{R}_B}{\partial t} \cdot \mathbf{n} \, d\tau \, d(\omega t), \tag{45} \]

where \( d\tilde{W}/dt \) is the rate at which work is done by the airstream on a given blade. After transferring the surface integration in equation (45) from the moving to the mean blade surface, interchanging the order of integration, and carrying out the integration with respect to \( \omega t \), we find that [15]
\[ W_C = -\pi \oint_{\mathbf{B}} \text{Im} \left\{ \left[ P \frac{\partial \mathbf{r}_B}{\partial \tau} \times \mathbf{e}_z + p_\mathbf{B} \mathbf{n} \right] \cdot \mathbf{r}^* \right\} \, d\tau + \pi \text{Im} \left\{ \sum_n r_{\mathbf{Sh}_nB} \left( \mathbf{n} \cdot \mathbf{r}^*_B \right)_{\mathbf{Sh}_n} \right\}, \tag{46} \]

where the superscript * denotes the complex conjugate. The first term on the right-hand-side of equation (46) represents the contribution to the work per cycle from the harmonic component of the unsteady aerodynamic response; the second term (i.e., \( W_{C,\mathbf{Sh}} \)) the contribution from the anharmonic component associated with shock motion. If the blade motion is expressed in terms of generalized coordinates, as in equation (39), then
\[ W_C = \pi \text{Im} \left\{ \sum_{i=1}^{1} \delta_i^* q_i \right\}, \tag{47} \]

where \( q_i \) is the complex amplitude of the first-harmonic generalized force resulting from the motion in the \( i \)-th mode. If only rigid blade motions occur, e.g., if \( \mathbf{r}_B = \delta_1 \mathbf{e}_y + \delta_2 \mathbf{e}_z \times \mathbf{R}_p \), then \( W_C = \pi \text{Im} \{ \delta_1^* f_y + \delta_2^* m_\tau \} \).
The work per cycle can also be expressed as the integral of the function \( w(\tau) \) over the mean blade surface, i.e.,

\[
W_C = \oint_B w(\tau) \, d\tau ,
\]

where

\[
w(\tau) = -\pi \text{Im} \left\{ \left[ P \frac{\partial \mathbf{r}_B}{\partial \tau} \times \mathbf{e}_z + p_\beta \mathbf{n} \right] \cdot \mathbf{r}_B^* - \sum_n r_{Sh_n B} \left( [P_B] \mathbf{n} \cdot \mathbf{r}_B^* \right)_{Sh_n} \delta (\tau - \tau_{Sh_n}) \right\} .
\]

We call \( w \) the pressure-displacement function. It describes the distribution of the work per cycle over a blade surface. The pressure-displacement function is continuous in \( \tau \) except at shock roots where, because of the concentrated shock load, it exhibits a delta-function behavior. This function reveals those intervals on the blade surface at which the local unsteady loads act to resist or enhance a prescribed blade motion.

8. Example response predictions.

Theoretical results will be presented to indicate the status of the numerical procedures used to solve the foregoing nonlinear steady and linear variable-coefficient unsteady boundary-value problems, and to demonstrate several important features, associated with nonuniform steady flow, of the aerodynamic response to prescribed blade vibrations and incident vortical disturbances. We will consider two different compressor-type cascades operating at subsonic inlet conditions. The first, a high-speed compressor cascade, consists of cambered NACA 0006 airfoils and operates at high subsonic inlet Mach numbers; the second, a compressor exit guide vane (EGV), consists of highly cambered NACA 0012 blades and operates at low inlet Mach number. For purposes of comparison, results for flat-plate cascades operating in uniform mean flows with \( M = M_{\infty} \) and \( \Omega = \Omega_{\infty} = \Theta \) will also be presented.

The steady background flows are assumed to satisfy a Kutta condition at blade trailing edges. Therefore, only information on the inlet free-stream conditions, e.g., inlet Mach number and flow angle, is required to calculate the steady flow fields. The unsteady flows through the NACA 0006 cascade are excited by prescribed single-degree-of-freedom (SDOF) blade motions at unit amplitude, \( (|\delta| = 1) \); those through the EGV, by prescribed vortical gusts with \( v_{R,\infty} \mathbf{e}_N = (1, 0) \), where \( \mathbf{e}_N = \mathbf{e}_x \times \mathbf{V}_\infty \) is a unit vector normal to the inlet free-stream flow direction. These unsteady excitations are termed subresonant if all acoustic response waves attenuate with increasing axial distance from the blade row and superresonant \((m, n)\) if \( m \) and \( n \) such waves persist far upstream and/or far downstream, respectively, and carry energy away from the blade row. An acoustic resonance occurs if at least one wave persists in the far field and carries energy along the blade row.

Theoretical results for steady Mach number distributions, and global and local unsteady aerodynamic response parameters are presented in figures 4 through 14 of this paper. The steady-flow predictions have been determined using the cascade full potential analysis of Caspar [18]; the unsteady results using the LINFLO (Linearized INviscid FLOw) analysis of references [8, 10, 16, 17, 23]. Unsteady solutions for the vibrating NACA 0006 cascade have been determined on a composite (cascade/local) mesh. Solutions for the EGV and the flat-plate cascades have been determined using a cascade or \( H \)-mesh alone.

8.1 HIGH-SPEED COMPRESSOR CASCADE. — This cascade has a stagger angle, \( \Theta \), of 45° and a gap/chord ratio, \( G \), of unity. The blades are constructed by superposing the thickness distribution of a NACA 0006 airfoil on a circular-arc camber line having a height at blade
midchord of $y = 0.05$. We consider two different uniform inlet operating conditions. For $M_{\infty} = 0.7$ and $\Omega_{\infty} = 55^\circ$, the mean or steady flow through the cascade is entirely subsonic; for $M_{\infty} = 0.8$ and $\Omega_{\infty} = 58^\circ$, it is transonic with a single normal shock occurring in each blade passage. The predicted blade-surface Mach number distributions are shown in figure 4. The subsonic mean flow stagnates on the pressure surface of the blade at $x = 0.0005$ and the Mach number reaches a maximum value of 0.916 on the suction surface at $x = 0.109$. The exit Mach number, $M_{+\infty}$, and flow angle, $\Omega_{+\infty}$, are 0.446 and 40.2°, respectively, and the mean lift force is 0.35. The transonic flow stagnates on the pressure surface at $x = 0.002$ and the normal shock impinges on the suction surface at $x = 0.258$. The Mach numbers at the base of the shock are 1.282 on the upstream side and 0.816 on the downstream side, and the exit Mach number and flow angle are 0.432 and 40.3°, respectively. The mean lift force acting on each blade is 0.42, and the jump in the steady pressure at the base of the shock, $[P_B]$, is $-0.477$.

![Figure 4](image)

Fig. 4. — Surface Mach number distributions for the example NACA 0006 cascade: (a) subsonic flow; $M_{\infty} = 0.7$, $\Omega_{\infty} = 55^\circ$; (b) transonic flow; $M_{\infty} = 0.8$, $\Omega_{\infty} = 58^\circ$ [16].

Work per cycle ($W_C$) responses for the NACA 0006 and the corresponding flat-plate cascades are shown in figures 5 and 6, respectively. Here the blade are undergoing unit-amplitude, i.e., $|\delta_2| = 1$ in equation (43), torsional vibrations about midchord at temporal frequencies $\omega$ of 0.5, 1.0, 1.5 and 2.0. For such motions $W_C = \pi \Im (\delta_2^* \Omega_2) = \pi \Im (\delta_2^* m_\delta)$. The flat-plate cascade has the same stagger angle ($\Theta = 45^\circ$) and blade spacing ($G = 1$) and operates at the same inlet Mach numbers as the NACA 0006 cascade, but the free-stream flow direction is aligned with the blade mean positions (i.e., $\Omega = \Theta$). The curves in figures 5 and 6 show reasonable trends with increasing vibration frequency. But, they also reveal the rather complicated nature of the global unsteady...
Fig. 5. — Work per cycle vs. interblade phase angle for pure torsional vibrations of the example NACA 0006 cascade: (a) and (b) as in figure 4.

Fig. 6. — Work per cycle vs. interblade phase angle for pure torsional vibrations of a flat-plate cascade with $\Omega = \Theta = 45^\circ$ and $G = 1$: (a) $M = 0.7$; (b) $M = 0.8$.

The work per cycle responses to the pure torsional motions of the NACA 0006 (Fig. 5) and flat-plate (Fig. 6) blades suggest the importance of nonuniform steady flow phenomena on blade response, and indicate that the torsional stability margin, $-W_C$, generally increases with increasing vibration frequency. The responses to the torsional blade motions of the subsonic and transonic NACA 0006 cascades are generally closer to the stability boundary ($W_C = 0$) than are those to the motions of the flat-plate cascades. The torsional motions of the transonic NACA 0006 cascade at $\omega = 0.5$ and interblade phase angles, $\sigma$, in the vicinity of $100^\circ$ produce large moments that oppose the blade motion. This somewhat surprising result is due to the high shock loads associated with such motions.

Work per cycle predictions are given in figure 7 for the subsonic and transonic NACA 0006 cascades undergoing pure bending (B), pure torsional (T) and pure chordwise bending or
elastic (E) blade vibrations at unit amplitude and unit frequency. The vertical lines above the curves indicate the resonant interblade phase angles for a unit-frequency excitation. The results in figure 7 indicate that the bending, torsional and chordwise bending motions at unit frequency are stable and that the torsional vibrations have the lowest stability margin.

As we have already noted, there are two contributions to the global unsteady airloads for a discontinuous transonic flow — one arising from the first harmonic unsteady aerodynamic response; the other, from the anharmonic response associated with shock motion. Shock-induced work per cycle responses, i.e., $W_{C,sh} = \text{Im} \{ r_{sh,b} (\| P_{b} \| n_{b} \cdot r_{sh}) \}$, for the transonic NACA 0006 cascade are shown in figure 8. The results in figure 8a for unit-amplitude torsional vibrations at $\omega = 0.5, 1.0, 1.5$ and 2.0 reveal that the shock loads can be quite severe, particularly at low vibration frequencies, and that these loads can have a destabilizing effect on the blade motion over a broad range of frequencies and interblade phase angles. The results in figure 8b for bending, torsional and chordwise bending vibrations at unit amplitude and frequency indicate that, except for the superresonant $(1,1)$ blade motions [i.e., those occurring in the range $\sigma \epsilon (\sigma_{-\infty}, \sigma_{+\infty})$], the shock loads tend to support the unit-frequency bending and chordwise bending motions. Also, these loads become quite severe for bending and chordwise bending vibrations at interblade phase angles near the upstream resonance condition, $\sigma = \sigma_{+\infty}$. The pure torsional motions at $\omega = 1$ give rise to relatively smaller shock loads and these usually act to resist or stabilize the blade motions.

Fig. 7. — Work per cycle vs. interblade phase angle for unit frequency, SDOF motions of the example NACA 0006 cascade: B, E and T denote bending, chordwise bending (elastic) and torsional blade motions, respectively: (a) and (b) as in figure 4 [15].

Fig. 8. — Work per cycle due to shock motion vs. interblade phase angle for the transonic NACA 0006 cascade: (a) torsional blade motions; (b) SDOF motions at unit frequency, B, E, and T as in figure 7.
One benefit of using the work per cycle to describe global unsteady aerodynamic response behavior is that the responses to different modes of blade motion, including coupled modes, can be represented via this single global response parameter. The pressure displacement function serves the same purpose in describing the local unsteady aerodynamic response. Pressure displacement function distributions for the example subsonic and transonic NACA 0006 cascades are shown in figure 9 for blades undergoing unit-amplitude, single-degree-of-freedom motions at $\omega = 1.5$. In particular, the blades of the subsonic cascade (Fig. 9a) are undergoing in-phase ($\sigma = 0^\circ$) chordwise bending vibrations with $|\delta_3| = 1$ (Eq. (43)); those of the transonic cascade, an out-of-phase ($\sigma = 180^\circ$) torsional vibration about midchord (i.e., $x_p = 0.5, 0$) with $|\delta_2| = 1$. These blade motions are stable with work per cycle ($W_c$) responses of $-5.23$ for the chordwise bending motion and $-1.24$ for the torsional motion. In the latter case the work per cycle due to the motion of the shock ($W_{C,S_b}$) is 0.16.

The curves in figure 9a indicate that, over most of the blade surface, the continuous local unsteady loads tend to suppress the blade motion, particularly those acting on the suction surface near quarter chord. However, the local response just aft of the leading edge and over the last 30 % of blade chord support the blade motion. In the transonic case the concentrated shock load, indicated by the arrow at $x = 0.256$ in figure 9b, supports the torsional blade motion.

![Diagram](https://via.placeholder.com/150)

Fig. 9. — Pressure-displacement function distributions for SDOF vibrations at $\omega = 1.5$ of the NACA 0006 cascade: (a) subsonic flow; in-phase chordwise bending vibrations; (b) transonic flow; out-of-phase torsional vibrations about midchord [15].
motion \( W_{C,sh} > 0 \), but, over most of the blade, the harmonic unsteady loads, particularly those just downstream of the shock, suppress the torsional blade motion.

8.2 **Compressor Exit Guide Vane.** — The compressor exit guide vane (EGV) has a stagger angle, \( \Theta \), of 15°, a blade spacing, \( G \), of 0.6 and operates at a prescribed inlet Mach number, \( M_{-\infty} \), and inlet flow angle, \( \Omega_{-\infty} \), of 0.3 and 40°, respectively. The blades are constructed by superposing the thickness distribution of a NACA 0012 airfoil on a circular arc camber line having a height at midchord of \( y = 0.13 \). The calculated steady Mach number contours in the field and Mach number distributions along a blade surface for this configuration are shown in figure 10. The (calculated) exit Mach number and exit flow angle are 0.226 and \(-7.4^\circ\), respectively, and the mean lift force acting on each blade is 0.36.

![Mach number contours and blade-surface Mach number distributions for steady flow at \( M_{-\infty} = 0.3 \) and \( \Omega_{-\infty} = 40^\circ \) through the EGV cascade [17].](image)

We consider unsteady flows through the EGV cascade and a corresponding flat-plate cascade (i.e., \( \Theta = \Omega = 40^\circ \), \( G = 0.6 \) and \( M = 0.3 \)) that are excited by incident vortical disturbances. Note that for this example the mean positions of the flat-plate blades are aligned with the inlet free-stream flow direction. We will examine the behavior of the unsteady vorticity and pressure fields, and the unsteady pressure-difference responses along the reference blade surfaces that result from the interaction between these cascades and a standard vortical gust defined by \( v_g = v_{R,-\infty} \), \( e_N = (1,0) \), \( \omega = 5 \) and \( \sigma = -2 \pi \), where \( v_g \) is the amplitude at \( x = 0 \) of the prescribed gust velocity component normal to the inlet free-stream flow direction. In addition, we will examine the unsteady lift responses to vortical excitations at \( v_g = (1,0) \), \( \omega = 5 \) and \(-3 \pi \leq \sigma \leq -\pi \).
Contours of the real parts of the complex amplitudes of the unsteady vorticity, \( \text{Re}\{\zeta\} \), and the unsteady pressure, \( \text{Re}\{p\} \) for the EGV and flat-plate cascades are shown in figures 11 and 12, respectively. As indicated in figure 11, the incident vortical gust is distorted as it is convected by the nonuniform mean flow through the EGV blade row. Here the mean flow stagnates in the vicinity of the blade leading edges, and consequently, the vortex lines are severely stretched through thin regions adjacent to the mean blade and wake surfaces. In addition, since the vorticity is convected at different mean velocities along the upper and lower surfaces of the EGV blades, an observer moving across the wake of an EGV blade would encounter significant and abrupt changes in vorticity. In contrast, vorticity is convected without distortion by the uniform mean flow through the flat-plate blade row. Here, the vortex lines remain straight as they are convected downstream, and the vorticity is continuous across the blades and their wakes.

\[
v_b = (1, 0), \quad \omega = 5 \quad \text{and} \quad \sigma = -2 \pi \quad [17].
\]

The pressure contours depicted in figure 12 indicate that the unsteady pressure fields associated with the EGV and flat-plate cascades are similar far upstream, but, because of the different vortical flow patterns associated with the two blade rows, the pressure fields differ substantially within the blade passages and downstream of the blade rows.

The pressure-difference responses along the reference blade of the EGV and flat-plate cascades to the standard vortical gust are shown in figure 13. The unsteady lift forces acting on
Fig. 12. — Contours of the in-phase component (real part) of the unsteady pressure for the EGV and the corresponding flat-plate cascades subjected to an incident vortical gust with $v_\theta = (1, 0)$, $\omega = 5$ and $\sigma = -2\pi$ [17].

the reference blades of the two cascades are plotted versus interblade phase angle for vortical excitations at $v_\theta = (1, 0)$, $\omega = 5$ and $-3\pi \leq \sigma \leq -\pi$ in figure 14. These results indicate the effects of nonuniform mean flow phenomena on the local and global unsteady aerodynamic responses at a blade surface for cascades subjected to incident vortical gusts. It should be noted that the unsteady lift acts in the direction of the positive $y$-axis (see Fig. 1), and this is inclined at different angles relative to the axial flow direction for the EGV ($\theta = 15^\circ$) and flat-plate ($\theta = 40^\circ$) cascades.


The linearized unsteady aerodynamic theory outlined in this paper accounts for the effects of real blade geometry, mean blade loading and operation at transonic Mach numbers on the unsteady aerodynamic response of the blades in an isolated two-dimensional cascade. This theory has been developed to meet the requirements of turbomachinery aeroelastic designers, and it should also be useful in aeroacoustic design applications. The unsteady flow is regarded
Fig. 13. — Unsteady pressure-difference response for the EGV and corresponding flat-plate cascades subjected to an incident vortical gust with $v_g = (1, 0)$, $\omega = 5$ and $\sigma = -2\pi$: (a) in-phase component (real part); (b) out-of-phase component (imaginary part); (-----) flat-plate cascade, (-----) EGV cascade [17].

Fig. 14. — Unsteady lift versus interblade phase angle for the EGV and corresponding flat-plate cascades subjected to incident vortical gusts with $v_g = (1, 0)$ and $\omega = 5$: (a), (b), (-----) and (-----) as in figure 13 [17].

as a small perturbation, which is produced by prescribed small-amplitude structural (blade) motions and external aerodynamic disturbances (incident entropic, vortical and acoustic gusts), of a fully nonuniform isentropic and irrotational mean or steady flow. Thus, the steady flow is determined as the solution of a full-potential boundary-value problem and the linearized unsteady flow as the solution of a time-independent linear, variable-coefficient problem in which the variable coefficients depend on the underlying mean flow.

The linearized unsteady problem consists of a system of three field equations along with conditions at blade, wake and shock surfaces and in the far field. These equations can be solved sequentially to determine the entropic, vortical and velocity potential fluctuations and, hence, the fluctuations in all fluid dynamic properties, throughout the required solution domain — an extended blade passage region of finite extent. Moreover, closed form solutions can be determined for the entropic and vortical fluctuations, and therefore, numerical field methods are only employed to determine the velocity potential. This feature allows a very efficient computation of the unsteady aerodynamic response information needed for aeroelastic and aeroacoustic design applications.

Because of the computational requirements associated with time-accurate resolutions of nonlinear unsteady flows and the demands for economical aeroelastic prediction methods,
linearizations relative to nonuniform steady flows offer great potential for meeting the needs of designers for efficient aerodynamic analyses that contain the essential physics of complex turbomachinery flows. However, before this potential can be fully realized, improvements in the associated numerical solution methods must be achieved, so that reliable unsteady response information can be provided over the wide range of geometric configurations and flow conditions at which blade vibrations are of practical concern (e.g., see Ref. [27]). Major advances in our ability to predict turbomachinery aeroelastic and aeroacoustic behavior should result if future research is directed toward including the effects of viscid/invviscid interactions and possibly large-scale flow-separations within a linearized invviscid unsteady aerodynamic framework. Ultimately, analyses that account for nonuniform steady flow and viscid/invviscid interaction phenomena must be extended to treat three-dimensional unsteady flows.

Acknowledgements.
This article was presented as Paper 90-2355 at the AIAA/SAE/ASME/ASEE 26th Joint Propulsion Conference, Orlando, Florida, July 16-18, 1990. Its preparation was supported under the United Technologies Corporate Research Program. The theoretical formulation and numerical solutions reported herein were achieved largely under research contracts with NASA Lewis Research Center, the Naval Air Systems Command and the Air Force Systems Command. The continuing interest and assistance provided by J. Gauntner, G. Stefko and J. Adamczyk (NASA Lewis) are gratefully acknowledged. The author would also like to thank Ms. R. Rudewicz for her help in obtaining the numerical results and in the preparation of this manuscript.

References