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The Elastohydrodynamic Lubrication Force in the Squeeze Motion of Hertzian Deformable Bodies

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Abstract. — The elastohydrodynamic force on two slightly deformable bodies approaching one another head on is calculated in the limit of small gap sizes and small deformations by combining Hertz’s theory of deformation and lubrication theory. An expression for the lubrication force analytical in the deformation and gap size is obtained. The effect of the gap size, the elasticity and the velocity of approach on the lubrication force is demonstrated. Further, the lubrication force with a pressure dependent viscosity is calculated.

1. Introduction

The hydrodynamics of deformable bodies is of great interest in many experimental and practically important systems on a mesoscopic scale, e.g. in studies with the surface forces apparatus of particles in liquids and in the rheology of suspensions and colloidal systems. Historically, elastohydrodynamic lubrication theory has been worked out in the context of understanding the workings of highly loaded gears and bearings, see e.g. [1]. Here this theory is specialized to elastic bodies approaching one another head on.

A hydrodynamic force between two elastic bodies deforms these objects changing their surface profile which in turn leads to a modification of the hydrodynamic force. If the distance between the surfaces is small enough, i.e. typically much smaller than the radii of curvature of the surfaces of the bodies, the Reynolds equation can be used to calculate the hydrodynamic pressures in the gap with respect to ambient pressure. If the deformations are not too large the Hertzian approach [2,3] can be used to calculate the counteracting elastic pressures. Christensen [4] and more recently Davis et al. [5] used the Hertzian approach to calculate the deformation and the hydrodynamic pressure numerically. The additional assumptions made here to calculate the lubrication force analytically are an explicit reference to the surface profile of the deformed body (it is assumed to consist of two parts: a flat central region and a curved outside region which is not deformed), to the elastic pressure profile (semi-elliptic) and to the velocity of approach (constant in time and space). It should also be mentioned that Sekimoto and Leibler [6] presented an expression for the tangential lubrication force on deformable bodies.

For the sake of completeness and practical importance the lubrication force is also calculated with an exponentially dependent viscosity. The results of this exercise are presented in the Appendix.

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2. Theory

In Figure 1 the principal geometrical parameters for the surfaces of the solid bodies are presented. The reduced radius for two curved surfaces with radii of curvature $R_1$ and $R_2$ in a tangentially parallel orientation is defined as \[ R = \frac{R_1 R_2}{R_1 + R_2} \] (1)

For length scales small compared to the radii of curvature the distance $H$ between the undeformed surfaces is expressed as a parabolic function in $r$ (see Fig. 1).

\[ H = H_0 + \frac{r^2}{2R} \] (2)

where $H_0$ is the distance of closest approach between the surfaces for $r = 0$. For deformed surfaces the above equation is replaced by

\[ h = H_0 + \frac{r^2}{2R} + W(r) \] (3)

where $h$ denotes the distance between deformed surfaces and $W(r) = W_1(r) + W_2(r)$ denotes the deformation of the surfaces at position $r$. In the present calculation I assume that the deformability is Hertzian, i.e. a flat central region for $r \leq a$ and a curved outside region for $r > a$ can be distinguished if both surfaces are deformed. The distance between the flat regions on both surfaces for $r = 0$ with $W(0) = \delta$ is

\[ h_0 = H_0 + \delta \] (4)

The distance between the flat regions on both surfaces for $r \leq a$ is

\[ h = h_0 \] (5)

and the distance between the deformed surfaces for $r > a$ is [1]

\[ \frac{(h - h_0)R}{a^2} = \frac{1}{\pi} \left( \frac{r^2}{a^2} - 2 \arccos(a/r) + \sqrt{\frac{r^2}{a^2} - 1} \right) \] (6)

For the present calculations this expression is replaced by the approximation

\[ \frac{(h - h_0)R}{a^2} = \frac{r^2}{2a^2} - \frac{1}{2} \] (7)

Fig. 1. — Geometry of two curved surfaces before deformation (thick line) separated by a gap of size $H$, $H_0$ for $r = 0$, and a gap of size $h$ after deformation (thin line) with the radius of the flat central region $a$. 

\[ \text{Fig. 1. Geometry of two curved surfaces before deformation (thick line) separated by a gap of size } H, H_0 \text{ for } r = 0, \text{ and a gap of size } h \text{ after deformation (thin line) with the radius of the flat central region } a. \]
The difference between the R.H.S. of equations (6) and (7) is small which can be understood from the fact that they are both continuously increasing functions of \( r/a \) and their limits for \( r/a < 1 \) and \( r/a \to \infty \) are equal.

The force which arises due to small compressions, \( \delta \ll R \), is modelled as the Hertzian force \([2,3]\).

\[
F_{\text{Hertz}} = \delta^{3/2} \frac{R^{1/2}}{D}
\]  

where \( D \) is the Hertzian elastic constant. The relation between \( a \) and the compression \( \delta \) is \([3]\)

\[
a = \sqrt{\delta R}
\]

To calculate the lubrication force on the two bodies I used Reynolds equation expressed in cylinder coordinates and restricted to the squeeze motion

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r h(r)^2 \frac{\partial P}{\partial r} \right) = 12 \eta V
\]

where \( V \) is the velocity along the line of centres of the surfaces, \( \eta \) the viscosity of the medium between the surfaces and \( P \) the pressure between the surfaces. The viscosity, pressure and velocity are considered to be constant through the thickness of fluid film between the two bodies. In the derivation of equation (10) inertia terms are assumed to be small. This is justified for \( \text{Re} h_0/R < 1 \), where the Reynolds number is defined by \( \text{Re} \equiv \rho V/\eta \) and \( \rho \) is the medium density \([5]\). Equation (10) can be integrated once for both the flat and the parabolic region

\[
\frac{\partial P}{\partial r} = 6 \eta V \frac{r}{h^3}
\]

The pressure profile between the deformed surfaces in the present approximation consist of two parts, for \( r \leq a \) the pressure with respect to the ambient pressure at \( r = \infty \) is

\[
- [P(r) - P(\infty)] = \int_r^a (\partial P/\partial r)_{\text{flat}} dr + \int_a^\infty (\partial P/\partial r)_{\text{parabolic}} dr
\]

Integrating this, using \( \partial h/\partial r = r/R \) for the parabolic region (see Eq. (7)) gives

\[
P(r) - P(\infty) = \frac{-3 \eta V}{h_0} \left( \left( \frac{a}{h_0} \right)^2 - \left( \frac{r}{h_0} \right)^2 + \frac{R}{h_0} \right)
\]

For \( r > a \) the integration

\[
- [P(r) - P(\infty)] = \int_r^\infty (\partial P/\partial r)_{\text{parabolic}} dr
\]

leads to

\[
P(r) - P(\infty) = \frac{-3 \eta V R}{h^2}
\]

The contribution to the lubrication force for \( r \leq a \) is obtained by integrating equation (13) over the area of the flat region of the two surfaces

\[
F_{\text{lubr}}(r \leq a) = 2 \pi \int_0^a [P(r) - P(\infty)] r dr
\]
Working this out gives

\[ F_{\text{lubr}}(r \leq a) = -3\pi\eta V h_0 \left( \frac{a^4}{h_0^4} + \frac{Ra^2}{h_0^3} - \frac{2a^3}{3h_0^2} \right) \quad (17) \]

The contribution to the lubrication force for \( r > a \) is obtained by integrating equation (15) over the area outside the flat region of the two surfaces

\[ F_{\text{lubr}}(r > a) = 2\pi \int_a^{\infty} [P(r) - P(\infty)]r dr \quad (18) \]

Working this out results in

\[ F_{\text{lubr}}(r > a) = -6\pi\eta V \frac{R^2}{h_0} \quad (19) \]

The lubrication force \( F_{\text{lubr}} = F_{\text{lubr}}(r \leq a) + F_{\text{lubr}}(r > a) \) is

\[ F_{\text{lubr}} = -3\pi\eta V h_0 \left( \frac{a^4}{h_0^4} + \frac{Ra^2}{h_0^3} - \frac{2a^3}{3h_0^2} + 2 \left( \frac{R}{h_0} \right)^2 \right) \quad (20) \]

The radius \( a \) of the flat region is determined by balancing the contribution to the lubrication force for \( r < a \), equation (17) with the Hertzian force, equation (8). That this contribution and not the total lubrication force is equated with the Hertzian force is in line with the approximation that the surfaces outside the flat region are not deformed. Further, it should be noted that the parabolic approximation for the tangential distances used in both the calculation of the Hertzian force and the lubrication force is only valid for \( \delta \ll R \) and \( H_0 \ll R \).

With the dimensionless quantities \( \delta^* \equiv H_0/R, \delta^* \equiv \delta/R, \) and \( D^* \equiv D/(3\pi\eta V) \) the dimensionless force is defined as \( F^* \equiv F/(3\pi\eta VR) \). Inserting this in equation (20) leads to the dimensionless lubrication force

\[ F_{\text{lubr}}^* = - \left( \frac{2}{H^*} + \frac{5\delta^*}{H^*} - \frac{2}{3} \frac{\delta^*}{H^*} + \frac{4\delta^*}{H^*} - \frac{2}{3} \frac{\delta^*}{H^*} \right) / \left( 1 + \frac{\delta^*}{H^*} \right)^3 \quad (21) \]

The above equation reduces to the expression for the squeeze lubrication force on an undeformable body in the limit \( \delta^* = 0 \)

\[ F_{\text{lubr}}^* = - \frac{2}{H^*} \quad (22) \]

The dimensionless Hertzian force is

\[ F_{\text{Hertz}}^* = \frac{\delta^*^{3/2}}{D^*} \quad (23) \]

For small values of \( \delta^* \) an approximative expression of \( \delta^* \) is obtained by equating an expansion to third order in \( \sqrt{\delta^*} \) of \( -F_{\text{lubr}}^*(r \leq a) \) with \( F_{\text{Hertz}}^* \).

\[ \delta^* \approx \frac{25}{H^*} \frac{1}{\left( 1/D^* + (2/3)1/H^* \right)^2} \quad (24) \]

Since this approximation is better for smaller \( \delta^* \) its accuracy improves for smaller \( D^* \) and increasing values of \( H^* \). To leading order in both \( D^* \) and \( H^* \) the deformation is \( \delta^* \approx 25D^*/H^{*4} \). To leading order in \( \delta^* \) the lubrication force is \( F_{\text{lubr}}^* \approx -(2/H^* - \delta^*/H^*). \) The resulting approximative equation for the lubrication force is

\[ F_{\text{lubr}}^* \approx - \left( \frac{2}{H^*} - 25 \frac{D^*}{H^{*6}} \right) \quad (25) \]
3. Results and Discussion

The deformation $\delta^*$ is larger for larger values of $D^*$ and decreases for increasing values of $H^*$ as shown in Figures 2 and 3. Higher values of $\delta^*$ correspond to smaller values of the lubrication force, as reflected by the curves in Figure 4. For the limiting value of $D^* = 0$ the hard body limit is reached.

Fig. 2. — Lubrication force, $-F_{\text{lub}}^*(r < a)$, and Hertzian force, $F_{\text{Hertz}}^*$, as a function of the deformation $\delta^*$ (gap size $H^* = 0.2$). Monotonically increasing curves are for $F_{\text{Hertz}}^*$ with $D^* = 0.01, 0.05$ and $0.1$ from left to right.

Fig. 3. — Lubrication force, $-F_{\text{lub}}^*(r \leq a)$, and Hertzian force, $F_{\text{Hertz}}^*$, as a function of the deformation $\delta^*$ with the elastic constant $D^* = 0.1$. The curves which decrease for larger $\delta^*$, are for $-F_{\text{lub}}^*(r \leq a)$ with $H^* = 0.2, 0.4$ and $0.6$ from top to bottom.
Fig. 4. — Lubrication force, $-F_{\text{lubr}}^*$, as a function of the gap size $H^*$. Curves are from left to right for values of the elastic constant $D^* = 0.1, 0.05, 0.01$ and 0.

Fig. 5. — Lubrication force, $-F_{\text{lubr}}^*$, as a function of the gap size $H^*$. Upper curves are numerically calculated, lower curves are obtained by use of the equation (25) valid for small deformations. Solid curves are for the elastic constant $D^* = 0.01$; dashed curves are for $D^* = 0.05$.

These observations can be clarified for small values of $\delta^*$ by the use of equation (25). It shows that the lubrication force for a deformable body is smaller than for an undeformable body. Further for small values of $D^*/H^*^4$ the lubrication forces will all go to the same hard body limit, as is observed in Figure 4.

For not too large values of $\delta^*$ (small $D^*$ and/or larger values of $H^*$) the approximation of $\delta^*$ with equation (24) and the lubrication force with equation (25) for the force leads to a good approximation of the numerically calculated lubrication force as illustrated in Figure 5.

Further it should be noted that the lubrication force is not linear in the velocity as for hard bodies. For small deformations this can be demonstrated by putting the dimensions back into equation (25) which for the highest order in the velocity leads to $F_{\text{lubr}} \sim V^3$. 
The effect of a pressure dependent viscosity of the type \( \eta = \eta_0 \exp(\alpha p) \) with \( \alpha \) a positive coefficient, is a decrease of the hydrodynamic pressure c.q. lubrication force between the two bodies which results in a smaller deformation than with a constant viscosity as demonstrated by Christensen [4] and as can be inferred from the equations presented in the Appendix.

4. Concluding Remarks

The present treatment of elastohydrodynamic forces is only valid for \( \delta^* \ll 1 \) due to the use of Hertz's theory for the deformation of the bodies and for \( H^* \ll 1 \) due to the use of lubrication theory. Therefore the precise validity range of the present results will have to be established by other, e.g. experimental means.

The Hertzian based formulae presented here, are expected to hold for \( \delta^* \ll 1 \) and \( H^* \ll 1 \). In particular Christensen [4] suggests that the gap size below which the Hertzian approach is valid is proportional to the squared Hertzian elastic constant \( D^2 \) and the inverse pressure coefficient \( \alpha^{-1} \).

Christensen [4] and Davis et al. [5] calculated the deformation and the hydrodynamic pressure by using a numerical iterative scheme in which the initial deformation is calculated from the hydrodynamic pressure between two undeformed spheres. This approach has the advantage of being self-consistent in the calculation of the deformation and the hydrodynamic pressure. Here the elastohydrodynamic lubrication force and pressure are analytical in the deformation and gap size, facilitating their physical interpretation and wider use.

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Appendix

Here the lubrication force with a pressure dependent viscosity according to \( \eta = \eta_0 \exp(\alpha p) \) is calculated. The procedure to do so is well known (see e.g. [1]) and can be based on the equations derived in the theory section for the constant viscosity case.

Replacing \( P(r) - P(\infty) \) on the L.H.S. of equations (13) and (15) by \( (1 - \exp(-\alpha[P(r) - P(\infty)]))/\alpha \) and calculating \( P(r) - P(\infty) \) from these equations results, after integration, in the equations for the contributions to the lubrication force with \( r \leq a \) and \( r > a \) analogous to the equations (17) and (19) for the constant viscosity case.

\[
F_{\text{lubr}}(r \leq a) = -\frac{A}{\alpha^* B} \left( -Ba^*2 - (C - Ba^*2) \log(C - Ba^*2) + C \log(C) \right)
\]

\[
\alpha^* \equiv \alpha \frac{3\eta_0 V}{R} \quad a^* \equiv \frac{a}{R}
\]

\[
A \equiv 3\pi \eta_0 VR \quad B \equiv \frac{\alpha^*}{H^*3} \quad C \equiv 1 + \frac{\alpha^*}{H^*} \left( \frac{a^*}{H^*} \right)^2 + \frac{1}{H^*}
\]
\[ F_{\text{lubr}}(r > a) = -\frac{2A}{\alpha^*} \left( \pi \sqrt{\alpha^*} - 2 \sqrt{\alpha^*} \arctan \left( \frac{H^*}{\sqrt{\alpha^*}} \right) - H^* \log \left( 1 + \frac{\alpha^*}{H^*} \right) \right) \] (27)

In order to determine the radius \( a \) of the flat region the contribution to the lubrication force for \( r \leq a \) should be balanced with the Hertzian force, as for the constant viscosity case.

References