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Magnetic and Electric Field Induced Periodic Deformations in Nematics

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Abstract. — This work reports theoretical studies on the occurrence of a static periodic distortion (PD) in a nematic sample under the action of crossed electric (E) and magnetic (H) fields applied on a uniformly aligned initial director configuration (n_o). Linear stability analysis with rigid anchoring hypothesis is employed. For homeotropic alignment of a nematic such as M1 which has negative dielectric anisotropy (\(\epsilon_A < 0\)) and positive diamagnetic anisotropy (\(\chi_A > 0\)), PD occurs when E and H are both in the sample plane. The direction of periodicity depends on the angle between E and H. PD is unfavorable when H is tilted sufficiently away from E. In a material (such as CCH-7) with \(\epsilon_A > 0\) and \(\chi_A < 0\), the direction of periodicity changes continuously in the H, E plane as a function of the strength and tilt of the stabilizing H relative to E. Possible effects of flexoelectricity, weak anchoring and director pretilt are discussed briefly.

1. Introduction

The many interesting deformations produced by the application of E and H fields on nematics are well understood in the framework of the continuum theory [1-4]. The application of H normal to n_o of a nematic sample between flat plates leads to an aperiodic or homogeneous deformation (HD) above a second order Freedericksz threshold when \(\chi_A > 0\) and the nematic has low anisotropy of curvature elastic constant. But in nematic systems with high elastic anisotropy [5,6], a periodic distortion (PD) is observed. A satisfactory description of PD has been given [5,7-10] on the basis of the continuum theory taking into account effects of director pretilt at the boundaries, director anchoring energy at interfaces [11], oblique magnetic tilt [12], etc.

Unlike the effect of H, the varied effects of E on a nematic sample are properly understandable theoretically provided that the modifications of the field inside the sample are taken into account [3,4,12]. These modifications can be due to a spatial change of the local dielectric tensor as well as flexoelectricity [13] which becomes important especially for static fields. Static PD can occur under the action of a static field even in nematics with low elastic anisotropy [14] due to flexoelectricity, PD is also found to be caused by high frequency fields [15] mainly due
to dielectric interactions. In materials (such as 5CB) with high, positive $\epsilon_A$, even a first order Fredericksz transition has been detected [16,17].

The combined action of $\mathbf{H}$ and $\mathbf{E}$ leads to the observation and prediction of many new effects [12,18,19]. When $\mathbf{H}$ stabilizes $\mathbf{n}_0$, a destabilizing ac $\mathbf{E}$ field can cause the sample to become optically biaxial [19] and also lead to the formation of PD above a threshold. A simple mathematical model [20] interprets the occurrence of PD above a first order transition. A linear stability analysis [21] takes into account the modification of $\mathbf{E}$ inside the nematic sample and explains PD as occurring above a second order transition; some of the results of [21] are in good agreement with experimental observations [17,19] while some of the predictions are yet to be tested experimentally. Recent work [22] indicates the possibility that interest in these effects is likely to be revived and it seems appropriate to reconsider the previous theoretical studies. In addition, new materials can be studied. The earlier work [17,19,22] concerns 5CB which has $\epsilon_A, \chi_A > 0$; this makes it difficult to consider all possible geometries involving the relative directions of $\mathbf{H}$ and $\mathbf{E}$. The synthesis of nematics with different dielectric and diamagnetic properties [23,24] enables a more complete study of available geometries.

With this motivation, the governing equations for static distortions are set up in Section 2, the boundary conditions discussed and expressions for the aperiodic deformation (HD) threshold obtained. In Section 3, results on periodic deformations are presented for a material having $\epsilon_A < 0$ and $\chi_A > 0$. Section 4 deals with periodic distortions in a material with $\epsilon_A > 0$ and $\chi_A < 0$. The last section contains the main conclusions and limitations of the mathematical model used.

2. Governing Equations, Boundary Conditions

Consider an insulating nematic sample of thickness $2h$ contained between glass plates $z = \pm h$ and sandwiched between electrodes at $x = \pm g$; the electrodes lie in the $yz$ plane and are connected to a power supply. It is assumed that the electrode gap $2g$ is large compared to the sample thickness $2h$ and that the sample is studied near $x = 0$, midway between the electrodes. The nematic is initially oriented uniformly in the $yz$ plane such that the unit director field is

$$\mathbf{n}_0 = (0, S, C); \quad S = \sin \phi_0, \quad C = \cos \phi_0; \quad \phi_0 = \text{constant}. \quad (1)$$

The applied magnetic field is (depending upon the situation under investigation) either along $\mathbf{n}_0$,

$$\mathbf{H}_\parallel = (0, H_\parallel S, H_\parallel C) \quad (2)$$

or in a plane normal to $\mathbf{n}_0$,

$$\mathbf{H}_\perp = (H_\perp C, H_\perp S, H_\perp S); \quad (3)$$

where $S_\alpha = \sin \alpha$ and $C_\alpha = \cos \alpha$. When $\alpha = 0$, $\mathbf{H}_\perp$ is along $x$, while for $\alpha = \pi/2$, it is in the $yz$ plane. In the absence of perturbations, $\mathbf{E}_0 = (E_\alpha, 0, 0)$ inside the sample with $E_{xo} = V_o/2g$ where $V_o$ is the potential difference applied between the electrodes. Under perturbations, the director and the $\mathbf{E}$ field take the form

$$\mathbf{n} = [\sin \theta, \cos \theta \sin (\phi_0 + \phi), \cos \theta \cos (\phi_0 + \phi)]; \quad \mathbf{E} = \mathbf{E}_0 + \mathbf{E}'; \quad \mathbf{E}' = -\text{grad} \psi \quad (4)$$

where the perturbations $\theta, \phi, \psi$ are functions of $x, y, z$; Maxwell’s curl equation is employed to write the electric field perturbation ($\mathbf{E}'$) as the gradient of the scalar potential $\psi$. At constant
potential difference $V_0$, the total free energy density

$$
F = \frac{1}{2}[K_1(\text{div } \mathbf{n})^2 + K_2(\mathbf{n} \cdot \text{curl } \mathbf{n})^2 + K_3(\mathbf{n} \cdot \text{grad } \mathbf{n})^2 - \chi_A(\mathbf{H} \cdot \mathbf{n})^2] - \frac{1}{8\pi}(\mathbf{D} \cdot \mathbf{E});
$$

$$
D_\perp = \epsilon_{ij}E_j\text{; } \epsilon_{ij} = \epsilon_\perp \delta_{ij} + \epsilon_A n_i n_j; \text{ } \epsilon_A = \epsilon_\parallel - \epsilon_\perp,
$$

(5)

where the $K_i$ are the curvature elastic constants, $\mathbf{D}$ the electric displacement; $\epsilon_{\parallel,\perp}$ are, respectively, the dielectric constants parallel to and normal to the director; summation is assumed with respect to repeated indices. For linear perturbations, terms up to the second order in perturbations and their gradients are retained in $F$. In (5), the $\mathbf{H}$ field is taken as either $\mathbf{H}_\parallel$ or as $\mathbf{H}_\perp$. When the total free energy is minimized with respect to $\psi$ holding other quantities constant, Maxwell’s divergence equation is recovered:

$$
div \mathbf{D} = -[\epsilon_{\perp}\psi_{,xx} + (\epsilon_{\perp} + \epsilon_A S^2)\psi_{,yy} + 2\epsilon_A SC\psi_{,yz} + (\epsilon_{\perp} + \epsilon_A C^2)\psi_{,zz}] + \epsilon_A E_{zo}(S\theta_y + C\theta_z) = 0
$$

(6)

where a subscripted comma denotes partial differentiation. When $\psi$ is held constant and the total free energy is minimized with respect to $\theta, \phi$, the following torque equations result:

$$
K_1\theta_{,xx} + (K_2C^2 + K_3S^2)\theta_{,yy} + (K_2S^2 + K_3C^2)\theta_{,zz} + 2(K_3 - K_2)SC\theta_{,yz} + \theta(\frac{\epsilon_A E_{zo}}{4\pi} + a_\theta) + a_\phi \phi + (K_1 - K_2)(\phi_{,xy}C - \phi_{,yz}S)
$$

$$
- \frac{\epsilon_A E_{zo}}{4\pi}(\psi_{,yy}S + \psi_{,z}C) = 0
$$

(7)

$$
K_2\phi_{,xx} + (K_1C^2 + K_3S^2)\phi_{,yy} + (K_3C^2 + K_1S^2)\phi_{,zz} + 2(K_3 - K_1)SC\phi_{,yz} + b_\theta \theta + b_\phi \phi + (K_1 - K_2)(\phi_{,xy}C - \phi_{,yz}S) = 0
$$

(8)

$$
a_\theta = \chi_A H_1^2 C_\perp \text{ and } a_\phi = \chi_A H_1^2 S_\perp C_\parallel; \text{ or } a_\theta = -\chi_A H_\parallel^2 \text{ and } a_\phi = 0;
$$

$$
b_\phi = \chi_A H_1^2 S_\perp C_\parallel; \text{ or } b_\phi = -\chi_A H_\parallel^2 \text{ and } b_\theta = 0;
$$

(9)

While the director anchoring energy is finite [11], the rigid anchoring hypothesis is adopted for the sake of simplicity; then the director perturbations must vanish at the sample planes. As per the requirements of the electromagnetic theory [25], the normal component of the electric displacement has to be continuous at the sample planes. As the sample walls are generally made of an isotropic dielectric, the $\mathbf{E}$ field as well as the induced polarization will be along $z$ axis inside the glass plates, hence, the $z$ component of $\mathbf{D}$ must vanish at the sample planes. The boundary conditions become [26]

$$
\theta(z = \pm h) = 0; \text{ } \phi(z = \pm h) = 0;
$$

$$
\epsilon_A CE_{zo}\theta - \epsilon_A SC\psi_{,y} - (\epsilon_{\perp} + \epsilon_A C^2)\psi_{,z} = 0 \text{ at } z = \pm h
$$

(10)

The next sections describe different solutions of (6)-(10) which reduce to an eigenvalue problem. In every case, a subset of terms from (6)-(10) is chosen depending on the assumptions made. The neglect of flexoelectricity in (5)-(10) may be a reasonable assumption if $E_0$ is an ac field with sufficiently high frequency; a qualitative discussion on flexoelectric effects is given in the concluding section.

Consider, for instance, the threshold for HD obtainable from the assumption that the perturbations are functions of $z$ alone. In general, all three perturbations are present. Then
the vanishing of \( \theta, \phi \) and \( d\psi/dz \) at the boundaries is the required set of conditions (10). Noting the modal structure of the equations and seeking solutions of the form \((\theta, \phi, \psi) = (\theta_p \cos qz, \phi_p \cos qz, \psi_p \sin qz)\), the compatibility of (6)-(8) requires that \( q = \pi/2h \) and the threshold condition is given by

\[
\left( \frac{\pi}{2} \right)^4 - \omega_1 \left( \frac{\pi}{2} \right)^2 + \omega_2 = 0;
\]

\[
\omega_1 = \left( \frac{b_0}{\omega_5} + \frac{\omega_3}{\omega_4} \right) h^2; \quad \omega_2 = \left( \frac{b_0 \omega_3 - a_0^2}{\omega_1 \omega_5} \right) h^4; \quad \omega_3 = a_0 + \frac{\varepsilon_1 \varepsilon_A E_{zo}^2}{4\pi(\varepsilon_\parallel + \varepsilon_A C^2)};
\]

\[
\omega_4 = K_2 S^2 + K_3 C^2; \quad \omega_5 = K_1 S^2 + K_3 C^2;
\]

(11)

this can be regarded as the definition of the lowest eigenvalue. Clearly, only the ratios of the perturbation amplitudes are known at threshold.

From (11), different threshold fields can be worked out for a homeotropic sample \((\phi_o = 0)\). With a stabilizing \(E_o\) acting along \(x\) (relevant to a material with \(\varepsilon_A < 0\) and \(\chi_A > 0\)), the magnetic threshold \(H_F\) for the \(H_l\) field becomes:

\[
(H_F)^2 = \frac{(H_B)^2(q^2 - \sigma_E)}{(q^2 - \sigma_E S_0^2)}; \quad q = \frac{\pi}{2};
\]

\[
H_B = \left( \frac{q}{h} \right) \left( \frac{K_3}{\chi_A} \right)^{1/2}; \quad \sigma_E = \frac{\varepsilon_1 \varepsilon_A E_{zo}^2 h^2}{4\pi \varepsilon_\parallel K_3}
\]

(12)

In a material with \(\chi_A < 0\) and \(\varepsilon_A > 0\), and a stabilizing \(H_l\) field, the electric threshold necessary to produce a HD in a homeotropic sample is.

\[
(E_F)^2 = \frac{(E_B)^2(q^2 - \sigma_H)}{(q^2 - \sigma_H S_0^2)}; \quad q = \frac{\pi}{2};
\]

\[
\sigma_H = \frac{\chi_A h^2 H_{zo}^2}{K_3}; \quad E_B = \left( \frac{q}{h} \right) \left( \frac{4\pi K_3 \varepsilon_\parallel}{\varepsilon_A \varepsilon_\parallel} \right)^{1/2}
\]

(13)

3. Material with \(\varepsilon_A < 0\) and \(\chi_A > 0\)

As the pretilt angle \(\phi_o\) is a parameter in the theory, its effect should also be studied. Unfortunately, results for PD cannot be analytically derived as in the case of HD (11). A reason for this is the presence of mixed derivatives in (6)-(10) involving \(z\) which makes it impossible to seek solutions in closed form involving pure spatial modes as was done for deriving (11). In order to present analytical expressions in a preliminary effort, results are presented in detail only for the homeotropic alignment \((\phi_o = 0)\). This can be achieved by the simple substitution, \(C = 1\) and \(S = 0\) in (6)-(10).

In this case, \(E_o\) stabilizes \(n_o\) which is along \(z\) and the field \(H_l\) tends to destabilize. The action of \(E_o\) is to turn the director towards the \(z\)-axis if there are any perturbations which take the director towards \(E_o\). Hence, the case \(\alpha = \pi/2\) is of interest, with \(H_l\) acting along \(y\). With dependence on \(z\) alone for aperiodic perturbations, only \(\phi\) gets coupled to \(H_l\); \(\theta\) and \(\psi\) damp out and one recovers the bend Freedericksz threshold \(H_B\) above which HD should develop with the \(\phi\) deformation occurring in the \(yz\) plane. On the other hand, when \(\alpha = 0\) and \(H_l\) is along \(x\), the destabilizing influence of \(H_l\) is fully opposed by the stabilizing action of \(E_o\). Apart from the HD which can develop above the threshold \(H_F\) (12), more general distortions (PD) seem possible owing to the interaction between the director and electric fields.
The three varieties of PD that can be studied correspond to perturbations which depend on y, z (yz Mode), x, z (xz Mode) and x, y, z (xyz Mode), respectively. In the absence of fields, there exists cylindrical symmetry about n_o which is along z. Under the action of E_0 along x, x and y axes become symmetry directions. When H_\perp is imposed in the xy plane, one cannot easily arrive at symmetry directions for general magnetic tilts relative to E_0. On grounds of symmetry, therefore, PD should have a general direction of periodicity in the xy plane and its inclination with x should be determinable from the equations of equilibrium. Yet, energetics seems to suggest differently; this becomes clear presently. Hence, the xz and yz Modes are studied initially and a brief description of the xyz Mode is presented towards the end of this section.

3.1. Dependence on y and z; yz Mode, \( \alpha = 0; \phi_0 = 0 \). — H_\perp is along x with strength \( H_z \). As (8) gets decoupled from (6) and (7), \( \phi \) damps out. The equations being linear, the variable \( y \) can be separated with \( \theta \) and \( \psi \) having the same \( y \) dependence, say, \((\theta, \psi) = \{T(z), U(z)\} \cos(q_y y)\). This substitution reduces (6), (7) to a set of coupled, linear, ordinary differential equations in \( T(z) \) and \( U(z) \) with the \( y \) dependence being replaced by the wavevector \( q_y \). For a mode with \( T(z) \) symmetric and \( U(z) \) antisymmetric about \( z = 0 \), we seek solutions of the form \( \{T(z), U(z)\} = (T_o \cos p z, U_o \sin p z) \) with constant coefficients \( T_o \) and \( U_o \). The compatibility of (6), (7) along with (10) leads to \( p = \pi/2h \) and the threshold condition

\[
\frac{\chi_A h^2 H_z^2}{K_3} = \frac{(\beta_3 Q_y^4 + \beta_4 Q_y^2 + \beta_5)}{(q^2 + \beta_1 Q_y^2)}; \quad q = \frac{\pi}{2}; \quad Q_y \approx q_y h;
\]

\[
\beta_1 = \frac{\epsilon_\perp}{\epsilon_\parallel}; \quad \beta_2 = \frac{K_2}{K_3}; \quad \beta_3 = \beta_1 \beta_2; \quad \beta_4 = q^2(\beta_1 + \beta_2) - \sigma_E; \quad \beta_5 = q^2(q^2 - \sigma_E)
\]

(14)

\( H_x \) is a function of the dimensionless wavevector \( Q_y \). At \( Q_y = 0 \), (14) reduces to (12), the HD threshold. As \( \beta_1, \beta_2 > 0 \), the extremum \( H_P \) of \( H_x(\theta_y) \) occurs at \( \theta_y = \theta_P \):

\[
Q_P^2 = \frac{-\beta_3 q^2 + \sqrt{\beta_3^2 q^4 - \beta_1 \beta_2 \beta_5}}{\beta_1 \beta_3};
\]

\[
\beta_6 = \beta_4 q^2 - \beta_1 \beta_5
\]

(15)

A real solution for \( Q_P \) exists only if \( \beta_6 < 0 \), implying that

\[
E_{2x0}^2 > E_C^2; \quad E_C^2 = \frac{\pi^2 K_2 \epsilon_\parallel^2}{\epsilon_\perp \epsilon_A^2 h^2}
\]

(16)

Calculations are performed for the parameters of M1 [23] at a reduced temperature of 0.9:

\[
(K_1, K_2, K_3) = (16.1, 5.2, 18.3) \times 10^{-7} \text{ dyne}; \quad \chi_A = 1.39 \times 10^{-7} \text{ emu};
\]

\[
\epsilon_\parallel = 8.9; \quad \epsilon_\perp = 22.8; \quad \epsilon_A = -13.9
\]

(17)

For an electrode gap of 2g = 4 mm and a semisample thickness \( h = 250 \mu m \), 2gEC corresponds to about 6.46 V. Computation is done by fixing \( E_{x0} \) at a sufficiently high value so that the reduced electric field \( r_E = E_{x0}/E_C \) is high enough (say, 10 or 20); \( E_C \) is defined in (16). A direct computation of (14) shows that when \( Q_y \) is increased from zero, \( H_x \) diminishes from \( H_P \) (12), the HD threshold and attains a minimum, \( H_P = H_x(\theta_P) \) when \( Q_y = Q_P \) given by (15); further increase of \( Q_y \) causes \( H_x \) to increase above \( H_P \). We can use the HD threshold \( H_P \) of
(12) to define the *reduced magnetic threshold* for the $yz$ Mode \( r_Y = H_P/H_F \). As \( r_Y < 1 \) at high \( r_E \), \( H_P \) and \( Q_P \) can be regarded as the $yz$ Mode threshold and wavevector at threshold, respectively.

Having ascertained that \( H_x \) of (14) has a minimum at \( Q_P \), it is straightforward to compute \( Q_P \) from (15), determine \( H_P \) from (14) and calculate \( r_Y \) as a function of \( r_E \). When \( E_{zo} \) is diminished from a high value, \( Q_P \) decreases and \( r_Y \) increases; when \( r_E \to 1 \), \( Q_P \to 0 \) and \( H_P \to H_F \) of (12) proving that (16) defines the critical point separating the HD and the $yz$ Mode. For \( E_{zo} < E_C \), PD with modulation along $y$ cannot appear; HD alone is possible. Thus, the *stabilizing* action of \( E_o \) has to be strong enough for the $yz$ Mode to occur under the *destabilizing* action of \( H_{\perp} \). This result is analogous to the case of 5CB [17,21] where PD can be caused by a *destabilizing* \( E_o \) field only in the presence of a sufficiently strong *stabilizing* \( H_{\parallel} \). The curve depicting the variation of $r_Y$ and \( Q_P \) with \( r_E \) is similar in shape to curves shown elsewhere (see, for instance, Fig 1 of [21]). For \( E_{zo} > E_C \), \( Q_P \) increases continuously with \( r_E \). In other words, \( E_C \) (16) is the only critical point separating the $yz$ Mode and HD. Substitution of (15) into (14) shows that in the limit of very high stabilizing \( E_{zo} \), \( r_Y \to (\epsilon_{\perp}/\epsilon_{\parallel})^{1/2} \) and \( Q_P^2 \approx (\pi/2)\sqrt{\sigma_{E}(1-\beta_{1})/\beta_{2}B_{z}^{2}} \); this is found to be correct from numerical computation of (14), (15) Extremization of \( E_C \) (16) with respect to different parameters shows that for a material with negative \( \epsilon_A \), \( E_C \) increases (the range of existence of the $yz$ Mode shrinks) with increase of \( K_2 \) and \( \epsilon_{\parallel} \); \( E_C \) diminishes when \( \epsilon_{\perp} \) is enhanced.

A qualitative explanation for some of the above results can be given as follows. If we put \( E_{zo} = 0 \) in (14), \( H_x \) continuously increases with \( Q_y \) without exhibiting a minimum, hence, PD cannot occur in the absence of \( E_o \). With an electric field, both HD and PD are possible as solutions For HD, the perturbations \( E_z' \) and \( \theta \) exist and are functions of \( z \) with \( E_z' \) being coupled to \( \theta \) by (6). The only elastic stabilizing effect comes from \( K_3 \), the bend constant. As \( \epsilon_A < 0 \), \( E_z' \) exerts a destabilizing torque \((\sim \psi_{z}\text{ in (7)})\) which partially annuls the stabilizing influence of \( E_o \), this is responsible for the reducing factor \( \epsilon_{\perp}/\epsilon_{\parallel} \) in \( \sigma_{E} \) (12).

For PD, the perturbations vary with \( z \) and also have the same dependence on \( y \). \( E' \) has both \( z \) and \( y \) components. The variation with \( y \) brings in additional twist distortion energy (elastic constant \( K_2 \)) which should actually increase the threshold of PD with respect to that of HD. But with PD, the destabilizing torque corresponding to \( E_z' \) itself becomes a function of \( y \) with the same \( y \) dependence as for \( \theta \). When the periodicity wavevector is high enough, this destabilizing influence can offset the total stabilizing torques (elastic as well as electric) to make the PD threshold (at the given, stabilizing \( E_{zo} \)) less than the HD threshold; this is the main message conveyed by (14) - (16). The fact that the additional twist elastic energy has to be overcome to produce the PD threshold implies that \( E_o \) of a minimum strength be present; this explains the presence of the term \( K_2/h^2 \) in \( E_C \) (16). From the purely mathematical viewpoint, the terms in (14) containing the ratio of dielectric constants are the ones which enter through the modification of the dielectric tensor; terms containing \( \sigma_{E} \) (which is negative) represent the purely stabilizing action of the electric field.

Along with \( E'_y \), we have \( E'_y' \) also for the $yz$ Mode. As \( \epsilon_A < 0 \), \( E'_y \) has a stabilizing influence. But by Maxwell’s equations, \( E'_{y,y} = E'_{y,z} \). This has two indications. Firstly, \( E'_{y} \) is out of phase along \( y \) with respect to \( E'_{z} \) and \( \theta \). Secondly, \( E'_{y} \) is antisymmetric with respect to \( z = 0 \); hence, it vanishes at \( z = 0 \) where the deformation \( \theta \) has a maximum. This could be the reason for the destabilizing effect of \( E'_{z} \) becoming prominent enough to cause PD to set in.

3.2. Dependence on \( x \) and \( z \); \( xz \) Mode; \( \alpha = 0; \phi = 0 \). — Calculation of the \( xz \) Mode threshold is analogous to that of the $yz$ Mode (Sect. 3.1; Eqs. (14)-(16)) with the splay elastic constant \( K_1 \) replacing \( K_2 \) and the dimensionless wavevector \( Q_x \) substituting for \( Q_y \).
The condition for the occurrence of the $xz$ Mode is

$$E_{zo}^2 > E_M^2; \quad E_M = \frac{\pi^3 K_1 \epsilon_0^2}{\epsilon_x^2 \hbar^2}$$  \hspace{1cm} (18)$$

For the same geometry as earlier, the voltage $2gE_M$ for material parameters (17) becomes 11.36 V. This means, the $yz$ Mode is more favorable than the $xz$ Mode when $H_L$ and $E_o$ are mutually parallel and act normal to the homeotropic $n_0$. This can be intuitively expected as $K_1 > K_2$ and the $xz$ Mode will involve higher elastic energy than the $yz$ Mode. An alternate statement is that the stabilizing elastic torque associated with the periodic perturbation (see Eq. (7)) is weaker for the $yz$ Mode ($K_2 \theta_{yy}$) than for the $xz$ Mode ($K_1 \theta_{xz}$). This also implies that the wavevector at threshold for the $xz$ Mode should be less than that for the $yz$ Mode. Direct computation with (17) shows this to be true for any $E_{zo} > E_M$. The discussion given at the end of Section 3.1 can be extended to the $xz$ Mode with $r_X$ being the reduced magnetic threshold.

3.3. Dependence on $y$ and $z$: $yz$ Mode; $\alpha \neq 0; \phi_o = 0$. — Now $\theta, \phi, \psi$ become coupled and the elastic free energy of the deformation gets enhanced compared to the case of $\alpha = 0$. $\phi$, which is a distortion in the $yz$ plane arises due to the magnetic torques $a_\phi \phi$ (7) and $b_\theta \theta$ (8) but does not disturb the electric field. The tilting of $H_L$ away from $x$ in the $xy$ plane diminishes its destabilizing influence. These factors combine to augment the $yz$ Mode threshold (at a given $E_{zo}$) when the magnetic tilt $\alpha$ is increased from zero. A solution is possible with $\theta$ and $\phi$ symmetric and $\psi$ antisymmetric about $z = 0$ such that $(\theta, \phi, \psi) = (T_o \cos pz \cos qy, P_o \cos pz \cos qy, U_o \sin pz \cos qy)$ with constant coefficients. The compatibility condition for (6)-(8) reduces to a cubic in $p^2$ with roots $p_1^2, p_2^2, p_3^2$. The boundary conditions (10) again demand the vanishing of $\theta, \phi$ and $\psi$ at $z = \pm h$ which implies that a quantity of the type $\cos(p_1 h) \cos(p_2 h) \cos(p_3 h)$ should vanish. As a cubic has at least one real root, we assume that $p_1 = \pi/2h$. Substituting for $p_1$ in the cubic, $H_L$ is found to be a function of $Q_y$:

$$H_L^2 = \frac{t_1 Q_y^6 + t_2 Q_y^4 + t_3 Q_y^2 + t_4}{u_1 Q_y^4 + u_2 Q_y^2 + u_3}; \quad \gamma_1 = \frac{K_1}{K_3}; \quad q = \frac{\pi}{2};$$

$$\gamma_2 = \frac{\chi A h^2}{K_3}; \quad f(\alpha) = \gamma_1 C_2 + \beta_2 S_2; \quad t_1 = \gamma_1 \beta_1 \beta_2;$$

$$t_2 = q^2 [\gamma_1 \beta_2 + \beta_1 (\gamma_1 + \beta_2)] - \sigma E \gamma_1;$$

$$t_3 = q^2 (\beta_1 + \gamma_1 + \beta_2) - \sigma E (\gamma_1 + 1)];$$

$$t_4 = q^4 (q^2 - \sigma E); \quad u_1 = \beta_1 \gamma_2 f(\alpha);$$

$$u_2 = \gamma_2 [q^2 (f(\alpha) + \beta_1) - \sigma E S_2];$$

$$u_3 = q^2 \gamma_2 (q^2 - \sigma E S_2^2);$$

$$\hspace{1cm} (19)$$

Obviously, (19) reduces identically to (14) when $\alpha = 0$ (provided that the nonzero common factor $q^2 + \gamma_1 Q_y^2$ is canceled between the numerator and denominator). An extremization of $H_L$ with respect to $Q_y$ leads to a fourth degree equation to be solved to obtain $Q_y^2$ at threshold; hence, direct computation of (19) with (17) is more convenient. At a given, strong stabilizing $E_o$, (19) is used to get $H_L$ as a function of $Q_y$; this is the neutral stability curve. If the $yz$ Mode is more favorable than HD, $H_L$ exhibits a minimum at $Q_y = Q_{PY}$. Then, $H_L(Q_{PY})$ and $Q_{PY}$ are taken as the $yz$ Mode threshold and wavevector at threshold, respectively.
The critical point for the \( yz \) Mode can be found by expanding \( H_\perp^2 \) (19) as a function of \( Q_y \) for small \( Q_y \). Then up to second order in \( Q_y \),

\[
H_\perp^2 = H_F^2 \left( 1 + \frac{\gamma_2 \pi^4 \Omega_Y Q_y^2}{16 t_4 w_3} \right),
\]

\[
\Omega_Y = \gamma_1 S_y^2 \sigma_E^2 - 2(e(\alpha) q^2 \sigma_E + q^4 j(\alpha));
\]

\[
e(\alpha) = \gamma_1 S_y^2 + \frac{1}{2} (1 - \beta_1) C_\alpha^2;
\]

\[
j(\alpha) = \beta_2 C_{\alpha}^2 + \gamma_1 S_y^2, \quad \alpha \neq 0.
\] (20)

If \( H_\perp < H_F \) in the limit of small wavevectors, \( \Omega_Y < 0 \). The critical equation for the \( yz \) Mode is \( \Omega_Y = 0 \). If \( \alpha = 0 \) is assumed \textit{ab initio}, then \( \Omega_Y < 0 \) leads to the definition of \( E_C \) (16). As \( \Omega_Y \) is a quadratic in \( \sigma_E \), a real solution for the critical point exists if the discriminant is positive definite; i.e.,

\[
\tan^2 \alpha < \tan^2 \alpha_Y; \quad \tan^2 \alpha_Y = \frac{K_3^2 \varepsilon_A^2}{4 K_1 \varepsilon_\parallel (K_2 \varepsilon_\parallel - K_3 \varepsilon_A)}.
\] (21)

Hence, the \( yz \) Mode cannot exist when \( \alpha > \alpha_Y \) regardless of the stabilizing \( \mathbf{H}_\perp \) or the destabilizing \( \mathbf{H}_x \) applied. For (17), \( \alpha_Y = 0.55 \) radian. We shall see below that for \( \alpha > \alpha_Y \), the \( xz \) Mode may exist for M1. An interesting feature of (20) is that there are two roots for the critical equation in the permitted range of \( \alpha \) yielding two critical points for the \( yz \) Mode as compared to only one for the case \( \alpha = 0 \). Thus for a given \( \alpha \), these two values of the stabilizing \( \mathbf{H}_x \) field (say, \( E_{y1} \) and \( E_{y2} \) with \( 0 < E_{y1} < E_{y2} \)) determine the range of existence of the \( yz \) Mode. As the wavevector tends to zero at two limiting values of \( E_x \), the nature of variation of the threshold and wavevector will be different from that described by (14) and (15) for zero magnetic tilt. It should be stated that (21) can be arrived at using the approach of [27].

3.4. Dependence on \( x \) and \( z \); \( xz \) Mode: \( \alpha \neq 0; \phi_0 = 0 \); Comparison of Results for \( xz \) and \( yz \) Modes. — In this case, the results can be written down by simply interchanging \( K_1 \) and \( K_2 \) (or, \( \gamma_1 \) and \( \beta_2 \)) in (19)-(21) and writing the reduced wavevector \( Q_x \) for \( Q_y \). In particular, the critical point is determined from

\[
\Omega_X = \beta_2 S_{\alpha}^2 \sigma_E^2 - 2e'(\alpha) q^2 \sigma_E + q^4 f(\alpha) = 0;
\]

\[
e'(\alpha) = \beta_2 S_{\alpha}^2 + \frac{1}{2} (1 - \beta_1) C_{\alpha}^2; \quad \alpha \neq 0
\] (22)

When \( \alpha = 0 \) is assumed \textit{ab initio}, (22) reduces to the definition of \( E_M \) (18); the condition for the existence of the \( xz \) Mode is

\[
\tan^2 \alpha < \tan^2 \alpha_X; \quad \tan^2 \alpha_X = \frac{K_3^2 \varepsilon_A^2}{4 K_2 \varepsilon_\parallel (K_1 \varepsilon_\parallel - K_3 \varepsilon_A)}
\] (23)

For (17), \( \alpha_X = 0.75 \) radian which is greater than \( \alpha_Y \) (21). When \( \alpha_X < \alpha < \pi/2 \), only the HD is possible above a threshold. For M1, the \( \alpha \) range of existence of the \( yz \) Mode is overlapped by that of the \( xz \) Mode; this is discussed below in connection with Figure 1. Again, (22) yields two roots for \( \sigma_E \) corresponding to two critical points for the \( xz \) Mode at \( E_{x0} = E_{X1} \) and \( E_{x0} = E_{X2} \) and these define the existence of the \( xz \) Mode over the range, say, \( 0 < E_{X1} < E_{X2} \) for a given \( \alpha \). Results for the two PD Modes can now be compared. As \( E_M \) (18) is higher than \( E_C \) (16), the critical value of \( E_{x0} \) for a given PD Mode is measured in terms of the dimensionless
Fig. 1. — Homeotropic nematic initially aligned along z; \( \epsilon_A < 0, \chi_A > 0, \) \( \mathbf{E}_o \) along \( \alpha \) and \( \mathbf{H}_L \) in the \( xy \) plane making angle \( \alpha \) with \( x \) axis. Periodic distortions with \( x \) and \( y \) modulation (\( zz \) Mode and \( yz \) Mode) are possible at different thresholds above second order transitions. \( E_X(\alpha) \) and \( E_Y(\alpha) \) correspond to the critical points separating the \( xx \) and \( yy \) Modes, respectively, from the HD (see Eqs. (20) to (23)). Plots of \( R_Y = E_Y/E_M \) (curve 1) and \( R_X = E_X/E_M \) (curve 2) as functions of \( \alpha \) where \( E_M \) is given by (18). The material parameters of M1 (17) are used. It is seen that for \( \alpha \neq 0 \), the regions of existence of both \( xx \) and \( yz \) Modes are delimited by two values of \( R \). The \( yz \) Mode appears to be favorable when \( \alpha \) and \( R \) are small, the \( xx \) Mode dominates when \( \mathbf{H}_L \) is sufficiently tilted away from \( \mathbf{E}_o \). For \( \alpha > \alpha_X \), only the HD exists. In the overlap region where both \( yz \) and \( xx \) Mode exist, the Mode having lower magnetic threshold is taken to be of real interest; this has to be done by explicit computation. For the parameters of M1 the more general \( xyz \) Mode does not appear to exist (see Sects. 3.3, 3 4).

quantity \( E_{zo}/E_M \). For a given \( \alpha \), we define \( R_X = E_X/E_M \) where \( E_X \) takes the critical values \( E_{X1}, E_{X2} \) for the \( xx \) Mode computed from (22). Similarly, \( R_Y = E_Y/E_M \) is defined for the \( yz \) Mode where \( E_Y \) takes the critical values \( E_{Y1}, E_{Y2} \) for the \( yz \) Mode found from (20) at the same \( \alpha \). In general, \( R_X \) and \( R_Y \) have two values at a given \( \alpha \). Clearly, \( R_X, R_Y \) correspond only to the critical points and not to any arbitrary value of \( E_{zo} \); they should be distinguished from \( r_E \) used in Section 3.1 which is relevant only to the case \( \alpha = 0 \).

The plots of \( R_X \) and \( R_Y \) are shown in Figure 1 for M1 (17). The diagram has four regions

i) For \( \alpha > \alpha_X \) (23) and also for very weak stabilizing \( \mathbf{E}_o \), only the HD exists (this is marked HD). Here, both \( \Omega_X \) and \( \Omega_Y \) are positive.

ii) The region marked \( XZ \) corresponds to the existence of only the \( xx \) Mode; only \( \Omega_X \) is negative but \( \Omega_Y \) is positive.

iii) Similarly, when \( \alpha \) and \( E_{zo} \) are both small enough, only the \( yz \) Mode seems favorable (region marked \( YZ \)); here, \( \Omega_Y \) is negative but \( \Omega_X \) is positive.

iv) The region labeled by \( XZ \) and \( YZ \) is the overlap region where both \( yz \) and \( xx \) Modes can exist at thresholds less than \( H_F \) (12). Here, either the \( yz \) Mode or the \( xx \) Mode will be favorable depending upon which Mode has the lower threshold. This comparison has to be done by direct computation as shown in Figures 2 and 3.

v) Of some interest is the point of intersection between the curves 1 and 2 in Figure 1. At this point, \( E_{zo} = E_B \) and \( \alpha = \alpha_B \) such that both (20) and (22) are satisfied. It is found
that (see Eqs. (14-19))

\[ E_B^2 = E_C^2 + E_M^2; \quad \sin^2 \alpha_B = \frac{(\beta_1 - 1)^2}{(\beta_1 - 1)^2 + (\beta_1 - 1 + \gamma_1 + \gamma_2)^2} \]

For the M1 parameters (17), \( E_B = 1.15E_M \) and \( \alpha_B = 0.52 \) radian, in good agreement with Figure 1.

Figure 1 illustrates the existence of two critical points for both \( yz \) and \( xz \) Modes when the magnetic tilt is sufficiently different from zero; the critical points merge into one when \( \alpha \) tends to the limits \( \alpha_Y \) (21) and \( \alpha_X \) (23), respectively, for the two Modes. Using (20) it is found that the difference between the critical points for the \( yz \) Mode has an inverse dependence on \( S_\alpha \), from (22), a similar conclusion can be reached for the \( xz \) Mode. This explains why the two branches of the curve for a given Mode diverge as \( \alpha \to 0 \); in this limit, the higher critical point moves off to infinity. As long as \( \epsilon_\alpha < 0 \), both \( \alpha_X \) and \( \alpha_Y \) will remain \( < \pi/2 \) due to the curvature elastic constants being positive. By comparing (21) and (23) it is seen that \( \alpha_X = \alpha_Y \) if \( K_1 = K_2 \).

It is convenient to measure \( \mathbf{H}_\perp \) in terms of the reduced quantity \( r_Y = H_F/H_F \) where \( H_F \) is given by (12) and \( H_P \) is the \( yz \) Mode threshold. Similarly, for the \( xz \) Mode, \( r_X \) is the ratio between the \( xz \) Mode threshold and \( H_F \). The dimensionless wavevectors at threshold shall be denoted by \( Q_{PX} \) and \( Q_{PY} \), respectively, for the two Modes. The strength of the stabilizing \( \mathbf{E}_o \) field is measured in terms of the dimensionless quantity \( R_E = E_{zo}/E_M \) where \( E_M \) is the critical field for the \( xz \) Mode defined in (18) when \( \mathbf{H}_\perp \) acts along \( x \) axis. It will be realized that \( R_E \) is employed for measuring arbitrary values of \( E_{zo} \) while the earlier defined \( R_X, R_Y \) correspond only to the critical points.

Figure 2 illustrates the variations of \( r_X, r_Y \) and \( Q_{PX}, Q_{PY} \) as functions of \( R_E \) for two different angles of magnetic tilt. At a given \( \alpha \neq 0 \) when \( R_E \) is sufficiently small, the \( yz \) Mode is more favorable than the \( xz \) Mode; in this limit, \( Q_{PY} > Q_{PX} \). When \( \mathbf{E}_o \) becomes stronger, the \( xz \) Mode should become observable. This also implies that the direction of periodicity should change by \( \pi/2 \) radian and the wavelength of the pattern should decrease discontinuously (provided that the \( xz \) Mode does not intervene). On further increase of \( E_{zo} \), \( Q_{PX} \) decreases (see Figs. 2d, 2e drawn for \( \alpha = 0.2 \) radian), attaining zero at some limit; in the same limit, \( r_X \to 1 \). The \( yz \) Mode gets quenched at much lower \( E_{zo} \). Values of \( R_E \) at the two critical points for the \( xz \) and \( yz \) Modes agree with those calculated from (22) and (20), respectively (they can also be read off at the given \( \alpha \) by drawing a line normal to the \( \alpha \) axis in Fig 1). Beyond the upper critical point (\( E_{X2} \)) for the \( xz \) Mode, only HD is possible. The extrema exhibited by \( r \) and \( Q_P \) for the two Modes are a direct consequence of the presence of two critical points when \( \alpha \neq 0 \). An increase in the stabilizing effect of \( \mathbf{E}_o \) initially favors the onset of PD and subsequently quenches it; these results are reminiscent of calculations reported earlier [8,28] on the effect of crossed fields on the occurrence of PD in the splay geometry of nematics with high elastic anisotropy. It is found (diagrams are not presented) that the \( R_E \) ranges of existence for both \( yz \) and \( xz \) Modes shrink as \( \alpha \) is increased; the magnetic thresholds move closer to \( H_F \) and the peak values of \( Q_P \) diminish.

The variations of \( r \) and \( Q_P \) depicted as functions of the magnetic tilt angle \( \alpha \) at fixed values of \( R_E \) (Fig. 3) complement the results of Figures 1 and 2. The increase of \( r_X, r_Y \) to 1 and the diminution of \( Q_{PX}, Q_{PY} \) to zero is monotonous for the two PD Modes, but the rates of change are different. At a given \( R_E \), therefore, the \( yz \) Mode is more favorable than the \( xz \) Mode in the low \( \alpha \) range while the reverse holds in the high \( \alpha \) range. The quenching of the \( yz \) Mode at lower \( \alpha \) than for the \( xz \) Mode at a given \( R_E \) as well as the shrinking of the \( \alpha \) ranges of existence of both Modes with increasing \( R_E \) are in qualitative agreement with Figure 1.
Fig 2. — Geometry and material parameters as in Figure 1. Plots of reduced thresholds and periodicity wavevectors as functions of reduced electric field at fixed magnetic tilt $\alpha$ (Figs 2a, 2b, 2d, 2e). Plots of the direction of the wavevector for the $xyz$ Modes are contained in Figures 2c, 2f. $r_Y$, $r_X$ and $r_{XY}$ are, respectively, the ratios of the $yz$, $xz$ and $xyz$ Mode thresholds to the HD threshold $H_F$ (12), see (19) as well as Sections 3.4, 3.5 for definitions of thresholds. $Q_{PX}$, $Q_{PY}$ and $Q_{PXY}$ are, respectively, the dimensionless wavevectors at threshold for the $xz$, $yz$ and $xyz$ Modes. $\mu_P$ is the angle between the wavevector and $x$ axis for the $xyz$ Mode $R_E = E_{zo}/E_M$ where $E_M$ of (18) is the critical point for the $xz$ Mode when $H_\perp$ acts along $x$ axis $\alpha = (a, b, c)$. 0.1 (d, e, f) 0.2 radian. Curves 1, 2, 3 correspond to the $yz$, $xz$ and $xyz$ Modes, respectively. Dashed lines indicate regions of academic interest. Under increase of the stabilizing $E_\alpha$, the distortion above the $H_\perp$ threshold will be periodic along $y$, periodic along $x$ and finally, aperiodic (HD) for a given magnetic tilt, $\alpha$. The $xyz$ Mode is of no real interest as it has higher threshold than the other PD Modes. The crossover from the $yz$ to the $xz$ Mode should be marked by a change in the direction of periodicity through $\pi/2$ radian and also a change in the wavevector (or stripe width). The total $R_E$ range of existence of PD diminishes with increase of $\alpha$. The critical points can be calculated from (20)-(23) (Sects. 3-3.5).

In the absence of the $xyz$ Mode, the cross over from the $yz$ to the $xz$ Mode should again be observable through discontinuous changes in the direction and wavevector of periodicity. A point worth noting is that the actual values of the PD thresholds decrease when $\alpha$ increases from zero; but the ratio of a given PD threshold to the HD threshold increases with $\alpha$. This can be established through tedious algebra using (12), (14) and (19).

Some of the above results can be interpreted as follows. It is true of both the $xz$ and the $yz$ Modes that when $\alpha = 0$, $\phi$ damps out due to decoupling from $\theta$ and $\psi$ and also absence of any destabilizing torque proportional to $\phi$. When $\alpha = \pi/2$, on the other hand, the decoupling occurs again but now $\theta$ and $\psi$ get extinguished (no destabilizing torque proportional to $\theta$), leaving $\phi$ to develop as HD in the $yz$ plane. Hence, neither PD Mode can exist when $H_\perp$ acts along $y$; this intuitively indicates cut off for the two PD Modes.

It now remains to explain why the $\alpha$ range of existence of the $xz$ Mode is broader than that of the $yz$ Mode at a given $E_{zo}$. At $\alpha = 0$, both PD Modes are associated with $\theta$ and $\psi$; $\theta$, the director deformation in the $xz$ plane, causes modification of $E$. As the $yz$ Mode is associated with twist and bend, it has lower threshold than the $xz$ Mode. When $\alpha$ is increased, $\phi$ develops
Fig. 3 — Geometry, material parameters and definitions as in Figure 2. Variation of $r_X, r_Y, r_{XY}$ and $Q_{PX}, Q_{PY}, Q_{PXY}$ with $\alpha$ for different $R_E$ (Figs 3a, 3b, 3d, 3e) Plots of $\mu_P$ for the $xyz$ Mode are shown in Figures 3c, 3f. Curves 1, 2 and 3 are drawn for the $yz$, $xz$ and $xyz$ Modes, respectively. The $xyz$ Mode is of no real interest. The reduced electric field $R_E = (a, b, c) 2; (d, e, f) 10$. In general, for a given periodic mode, the reduced magnetic threshold $r \to 1$ and the dimensionless wavevector $Q_P \to 0$ as $\alpha$ approaches an upper $\alpha$ limit which can be calculated from (21) and (23). However, at a given $R_E$, the $yz$ Mode damps out first, the appearance of the $xz$ Mode should be observable through changes in direction of periodicity and stripe width. At sufficiently high magnetic tilt with $E_0$, only HD is possible. The decrease in the $\alpha$ range of existence of the periodic deformations is in agreement with Figure 1 (Sect. 3.3-3.5).

out of the $xz$ plane due to the magnetic torque and increases the elastic free energy of both PD Modes. As $\phi$ does not modify the electric field, both $xz$ and $yz$ Mode thresholds increase with $\alpha$. If the increase in the $yz$ Mode free energy outstrips that of the $xz$ Mode, then the threshold of the $yz$ Mode can become higher than that of the $xz$ Mode when $\alpha$ crosses some limit. The elastic free energy density for the $xz$ and $yz$ Modes can be written as,

$$F_{xz} = \frac{1}{2}(K_1 \theta_x^2 + K_3 \phi_{xz}^2) + \frac{1}{2}(K_2 \phi_x^2 + K_3 \phi_{xz}^2);$$

$$F_{yz} = \frac{1}{2}(K_2 \phi_y^2 + K_3 \phi_{yz}^2) + \frac{1}{2}(K_1 \theta_y^2 + K_3 \phi_{yz}^2).$$

When $\alpha$ is increased from zero, the additional contribution from $\phi$ has the part $K_3 \phi_{xz}^2$ common to both the Modes. However, as $K_1 > K_2$, the positive contribution $K_1 \phi_y^2$ made by $\phi$ to the $yz$ Mode may outweigh the contribution $K_2 \phi_{xy}^2$ for the $xz$ Mode. A more quantitative picture would involve using the actual spatial dependence of $\theta$ and $\phi$. Then $F_{xz}$ and $F_{yz}$ can be calculated as the sum of two terms with coefficients $T_o^2$ and $P_o^2$ (see Sect. 3.3). By using (8), $P_o$ can be expressed in terms of $T_o$ for any magnetic tilt. With $q = \pi/2$,

$$P_o^2 = \frac{T_o^2(\gamma_2 H_{1s}^2 S_s C_s)^2}{q^2 - (\gamma_2 H_{1s}^2 S_s + \beta_2 Q_x^2)}; \quad (xz \text{ Mode})$$
\[ P_0^2 = \frac{T_0^4(\gamma_2^2H_0^2S_0^2C_0)^2}{q^2 - (\gamma_2^2H_0^2S_0^2 + \gamma_1Q_y^2)}; \] (yz Mode)

As \( \gamma_1 > \beta_2 \), the contribution from \( P_o^2 \) for the yz Mode can become higher than that for the xz Mode.

The existence of two cut off points for each PD Mode at nonzero \( \alpha \) becomes clear from the above statements. When \( \alpha = 0 \), we know that \( r_V \) and \( r_X \) decrease monotonically with increase of \( E_{x_0} \) attaining limiting values for high \( E_{x_0} \) (Sects. 3.1, 3.2); the wavevectors at threshold increase monotonically as \( \sqrt{\sigma_F} \). When \( H_{\perp} \) is tilted away from the \( x \) axis, \( \phi \) develops adding to the elastic free energy of both PD Modes. If the threshold wavevector increases continuously, the elastic free energy of distortion becomes too expensive. Hence, the wavevector should diminish when \( E_{x_0} \) increases beyond a certain limit, this makes the existence of a second, higher cut off necessary. Mathematically, this is represented by the \( \sigma_F^2 \) terms in \( \Omega_Y \) (20) and \( \Omega_X \) (22).

3.5. Dependence on \( x, y \) and \( z, xyz \) Mode; \( \alpha \neq 0; \phi_o = 0. \) — Solutions for (6)-(8) are sought in the form

\[ (\theta, \phi, \psi) = (T_0 \cos p_2, P_0 \cos p_2, U_0 \sin p_2) \sin \{Q(xC_\mu + yS_\mu)/h\} \]

where \( \mu \) is the angle between the \( x \) axis and the dimensionless wavevector of magnitude \( Q \). The compatibility of (6) - (8) with (10) demands that \( p = \pi/2h \) and leads to an expression for \( H_\perp^2 \) of the same form as in (19) except that for \( Q_y \) we write \( Q \) and use starred coefficients \( t \) and \( u \) defined as

\[ t_1^* = t_1; \quad t_2^* = t_2 + \sigma_F(\gamma_1 - \beta_2)C^92; \]
\[ t_3^* = t_3 + q^2\sigma_F(\gamma_1 - \beta_2)C^92; \quad u_1^* = \beta_1\gamma_2(j(\alpha - \mu)); \]
\[ u_2^* = \gamma_2(q^2\{\beta_1 + j(\alpha - \mu)\} - \sigma_F^2S^2) \]

where the function \( j \) is defined in (20). Clearly, \( H_{\perp}(Q, \mu) \) is a function of both \( Q \) and \( \mu \) and its value for different \( Q \) and \( \mu \) defines a neutral stability surface. Obviously, the neutral stability surface reduces to the neutral stability curve for the \( xz \) Mode at \( \mu = 0 \) and that for the \( yz \) Mode at \( \mu = \pi/2 \). For appropriate values of \( E_{x_0} \), the surface exhibits an extremum \( H_D = H_{\perp}(Q_{PXY}, \mu_P) \) which is taken as the \( xyz \) Mode threshold if \( r_{XY} = H_D/H_F < 1 \). Then, \( Q_{PXY} \) is regarded as the wavevector at threshold and \( \mu_P \) the inclination of the wavevector with \( x \) axis. Plots of \( r_{XY}, Q_{PXY} \) and \( \mu_P \) are shown in Figures 2 and 3. The following conclusions can be reached:

i) The range of existence of the \( xyz \) Mode is never wider than that of the \( yz \) Mode

ii) The threshold of the \( xyz \) Mode is always greater than that of the \( yz \) and \( xz \) Modes; hence, the \( xyz \) Mode is of academic interest from the viewpoint of linear perturbation theory for the parameters used in Figures 2 and 3.

iii) At a given magnetic tilt, the \( xyz \) Mode threshold approaches that of the \( xz \) Mode in the low \( R_F \) limit and that of the \( yz \) Mode in the high \( R_F \) limit (Fig. 2).

A detailed interpretation of the \( xyz \) Mode is difficult to achieve analytically. Intuitively it is clear that this Mode has higher elastic free energy than the other two especially because \( K_1 \) for M1 is much higher than \( K_2 \) (17). In addition, \( E' \) will have all three components in general. The
destabilizing effect of $E'_x$ is opposed by the stabilizing influence of $E'_z$ and $E'_y$. The critical point calculation, however, is relatively simple with the critical equation $t_2^2 u_2^3 - t_4^2 u_2^4 = 0$ reducing to

$$C_\mu^2 \Omega_X + S_\mu^2 \Omega_Y + 2S_\mu \mathit{C}_\alpha s_\mu \mathit{C}_\beta \mathit{q}^2 (\gamma_1 - \beta_2) (\mathit{q}^2 - \sigma_E) = 0$$

where $\Omega_Y$ and $\Omega_X$ are defined in (20) and (22). The $xyz$ Mode has two critical values of $E_{x0}$ and these are functions of $\alpha$ and $\mu$. At a given $\alpha$, the existence of real values for the critical points (along with the condition that $S_\mu^2 \leq 1$) leads to (21). This explains why the range of existence of the $xyz$ Mode is not wider than that of the $yz$ Mode.

A calculation for hypothetical strengths determines the importance of the elastic strengths. For $\alpha = 0.1$ radian (Fig. 2a), the minima of the threshold curves occur around $R_E = 5.7$ corresponding to $V_o = 28 \text{ V}$ with a gap width $2g = 4 \text{ mm}$. At these $V_o$ and $2g$ values, the thresholds of the three PD Modes are calculated as functions of $K_1$ keeping all other parameters constant in (17). All three thresholds diminish with $K_1$. When $K_1 < K_2$, the $xyz$ Mode threshold becomes lower than the other two; this situation may occur in a material exhibiting nematic-smectic A transition. Hence, as long as $K_1 > K_2$, the $xyz$ Mode may never be more favorable than the $xz$ and $yz$ Modes. A detailed calculation involving variation of material parameters is not undertaken here.

4. Materials with $\epsilon_A > 0$ and $\chi_A < 0$. $\phi_o = 0$. $xyz$ Mode

PD can occur in materials with positive $\epsilon_A$ (the $\mathit{E}_o$ field acts as a destabilizing influence) and negative $\chi_A$ (the $\mathit{H}_\perp$ field, acting in the $xy$ plane tends to stabilize $\mathit{n}_o$). Such materials do exist; for instance, CCH-7, at reduced temperature of 0.932 has the following material parameters [24]:

$$(K_1, K_2, K_3) = (7.25, 3.48, 11.46) \times 10^{-7} \text{ dyne}; \quad \chi_A = -3.22 \times 10^{-8} \text{ emu}; \quad \epsilon_\parallel = 8.01; \quad \epsilon_\perp = 3.72; \quad \epsilon_A = +4.29.$$ (24)

When $\mathit{H}_\perp$ is along $x$, its stabilizing influence with respect to $\mathit{E}_o$ is maximum; this influence gets diminished as $\mathit{H}_\perp$ is tilted away from $x$ in the $xy$ plane. Proceeding as in Section 3.5, the compatibility condition for determining the electric threshold for the $xyz$ Mode becomes,

$$\sigma_E = \frac{t_1 Q^6 + t_2 Q^4 + t_3 Q^2 + t_4}{u_1 Q^4 + u_2 Q^2 + u_3}; \quad q = \frac{\pi}{2}; \quad t_1 = \beta_1 \beta_2 \gamma_1;$$

$$t_2' = \mathit{q}^2 [\gamma_1 \beta_2 + \beta_1 (\gamma_1 + \beta_2)] - \sigma_H \beta_1 \mathit{j}(\alpha - \mu);$$

$$t_3' = \mathit{q}^2 [\gamma_1 \beta_1 + \beta_2] - \sigma_H \{\beta_1 + \mathit{j}(\alpha - \mu)\};$$

$$t_4' = \mathit{q}^4 (\mathit{q}^2 - \sigma_H); \quad u_1' = \mathit{j}(\mu);$$

$$u_2' = \mathit{q}^2 (\mathit{j}(\mu) + 1) - \sigma_H S_{\alpha}^2; \quad u_3' = \mathit{q}^2 (\mathit{q}^2 - \sigma_H S_{\alpha}^2)$$ (25)

(see 20) for definition of function $\mathit{j}$). Obviously, $\sigma_E$ is unaffected if $\mu \rightarrow \mu \pm \pi$. Expanding $\sigma_E$ up to second order in $Q$, it is found that the $xyz$ Mode should be more favorable than HD if $\Omega_{XY} = t_3' u_3 - t_4' u_2 < 0$ with

$$\Omega_{XY} = S_\alpha^2 k(\alpha, \mu) \sigma_H^2 + \mathit{q}^2 [\mathit{j}(\mu) - k(\alpha, \mu) - \mathit{q}^2 \delta S_{\alpha}^2] \sigma_H + \mathit{q}^4 [\delta - \mathit{j}(\mu)];$$

$$\delta = \beta_1 + \beta_2 + \gamma_1 - 1; \quad k(\alpha, \mu) = \beta_1 - 1 + \mathit{j}(\alpha - \mu)$$ (26)

Even for zero stabilizing field ($\sigma_H = 0$), it is found that $\Omega_{XY} < 0$ for (24) showing that PD should occur even in the absence of $\mathit{H}_\perp$. In this case, $\Omega_{XY}$ takes minimum (most negative)
value for $\mu = \pi/2$ and therefore, the PD wavevector will be along $y$, one of the symmetry directions; in other words, for $H_\perp = 0$, the $xyz$ Mode degenerates into the $yz$ Mode. The thresholds for the $yz$ and $xz$ Modes can be found from (25) as minima of neutral stability curves by putting, ab initio, $\mu = \pi/2$ and $\mu = 0$, respectively; this can be done wherever necessary.

For non-zero values of $\sigma_H$, the $xyz$ Mode electric threshold can be computed by minimizing (25) with respect to both $Q$ and $\mu$ for (24). The neutral stability surface shows an absolute minimum corresponding to the $xyz$ threshold at $Q = Q_\text{P}$ and $\mu = \mu_\text{P}$. As the $xz$ and $yz$ Mode thresholds are generally higher, these are not considered here. In general, $0 < \mu_\text{P} < \pi/2$. Hence, the $xyz$ Mode appears to be more favorable than either of the remaining PD Modes when a stabilizing $H_\perp$ acts on the sample. This can also be seen from (26). For $\sigma_H \neq 0$, the value of $\Omega_{XY}$ (26) at general $\mu$ is lower than its values at $\mu = 0$ or $\mu = \pi/2$. Hence, only the $xyz$ Mode is studied in this section.

The $xyz$ threshold can be expressed in reduced form through division by the HD threshold $E_\text{P}$ (13) : $R_E = E_{x0}(Q_\text{P}, \mu_\text{P})/E_\text{P}$. As this Mode occurs even in the absence of $H_\perp$, there exists no critical point. We can generate a critical field by noting that at $\mu = 0$ (i.e., for the $xz$ Mode), $\Omega_{XY}$ does show zeros over certain ranges of $\sigma_H$ and $\alpha$. In particular, for $\alpha = 0$, the $xz$ Mode has the (hypothetical) critical field $H_\perp = H_C$ defined by

$$H_C^0 = -\frac{\epsilon_0 K_2 \pi^2}{4 \epsilon_A h x_A} \left[ \frac{K_1}{K_3} - \frac{\epsilon_0}{\epsilon_1} \right]$$

(27)

We can use this value to define the reduced (stabilizing) magnetic strength $r_H = H_\perp/H_C$ and plot $R_E$, $Q_\text{P}$ and $\mu_\text{P}$ as functions of $r_H$. It is not convenient to obtain analytical expressions for the different quantities. Hence, numerical results are presented first and some of these are analytically interpreted.

Figure 4 shows plots of relevant quantities as functions of stabilizing magnetic strength (Figs. 4a-4c) and magnetic tilt (Figs. 4d-4f). Figures 4a - 4c yield the following facts. When $r_H = 0$, $\mu_\text{P} = \pi/2$ (stripes have periodicity along $y$ axis and the $xyz$ Mode degenerates into the $yz$ Mode); $R_E$ and $Q_\text{P}$ also tend to their respective limits. When $H_\perp$ is strong, $R_E$ and $Q_\text{P}$ asymptotically approach their respective limits; in particular, $\mu_\text{P} \to \alpha$ (the wavevector tends to align parallel to $H_\perp$). Figures 4d-4f show that the periodicity is along $y$ when $\alpha = 0, \pi/2, \pi$; at these points again, the $xyz$ Mode degenerates into the $yz$ Mode. At $\alpha = \pi/2$ in particular ($H_\perp$ along $y$), the electric threshold and wavevector appear to be independent of the stabilizing magnetic strength. For the parameters chosen, the $xz$ and $yz$ Modes have thresholds lower than $E_\text{P}$ (13), but these are higher than the $xyz$ Mode threshold for general values of $H_\perp$; hence, results for the $xz$ and $yz$ Mode have not been included.

Consider first the case $\sigma_H = 0$ or $H_\perp = 0$. When we find the threshold, we determine the lowest possible value of $\sigma_E$ (25) by extremizing with respect to $Q$ and $\mu$. The order in which we differentiate is not important as $\sigma_E$ (25) is a single valued function of its arguments. Setting $d\sigma_E/d\mu = 0$, we find that either $\mu = 0$ ($xz$ Mode) or $\mu = \pi/2$ ($yz$ Mode). Putting $\mu = 0$ in (26), it can be shown that the $xz$ Mode cannot exist in the absence of the stabilizing $H_\perp$ (see (27), the critical point $H_C$ for the $xz$ Mode). Hence, the $xyz$ Mode should degenerate into the $yz$ Mode at $\sigma_H = 0$. With $\phi$ decoupled, the $yz$ Mode with $\theta$ and $\psi$ occurs at the reduced threshold $R_\circ$ with the threshold wavevector $Q_\circ$:

$$R_\circ = \zeta_1 + \zeta_2; \quad Q_\circ = \frac{\pi}{2} \left[ \sqrt{(\zeta_2 \zeta_1 / \beta_2 \beta_1)} - 1 \right],$$

$$\zeta_1 = \sqrt{\beta_1 (1 - \beta_2)}; \quad \zeta_2 = \sqrt{\beta_2 (1 - \beta_1)}$$

(28)
Fig. 4. — Results for CCH-7 (24) which has $\varepsilon_A > 0$ and $\chi_A < 0$. $E_o$ and $H_\perp$ act, respectively, as the destabilizing and stabilizing influences. The reduced electric threshold, $R_E$, dimensionless wavevector, $Q_P$ and $\mu_P$, the inclination of the wavevector with $x$ axis are plotted as functions of the reduced magnetic strength $r_H$ for the $xyz$ Mode in Figures 4a-4c $r_H = H_\perp/H_C$ (27) while $R_E$ is the ratio of the $xyz$ Mode threshold and the HD threshold $E_F$ (13) Curves are drawn for magnetic tilt $\alpha = (1)$ 0 4 (2) 0.8 radian. The PD exists for all strengths of the stabilizing field $H_\perp$. When $H_\perp$ is strong, $R_E$ and $Q_P$ tend to limiting values and $\mu_P \rightarrow \alpha$ When $H_\perp = 0$, $\mu_P \rightarrow \pi/2$ (corresponding to the $yz$ Mode) In Figures 4d-4f, the quantities are plotted against the magnetic tilt, $\alpha$, for $r_H = (1)$ 10 (2) 6 While the $xz$ Mode is generally unfavorable, $\mu_P$ attains $\pi/2$ (corresponding to the $yz$ Mode) when $\alpha = 0, \pi/2$ and $\pi$ (Sect 4).

Using (24), $R_o = 0.972$ and $Q_o = 1.243$ in agreement with Figures 4a, 4b. The closeness of $R_o$ to unity suggests that small changes in the parameters (24) should be able to suppress the PD.

The remaining limits are similarly treated Putting $\alpha = 0$ ($H_\perp$ along $x$) for $\sigma_H \neq 0$, and extremizing $\sigma_E$ (25) with respect to $\mu$, one again finds that $\mu = 0$ or $\mu = \pi/2$. The $xyz$ Mode may degenerate into either the $xz$ or the $yz$ Mode. While the $yz$ Mode can exist for any $\sigma_H$, the $xz$ Mode exists for $H_\perp > H_C$ (27). The elastic constant associated with the $x$ variation of the $xz$ Mode perturbations is $K_1$, $K_2$, which is less than $K_1$ plays the corresponding role for the $yz$ Mode. Naturally, the $yz$ Mode has lower threshold than the $xz$ Mode for all $\sigma_H$ and hence, $\mu_P \rightarrow \pi/2$ when $\alpha \rightarrow 0$ (Fig. 4d-4f).

When $\alpha = \pi/2$ ($H_\perp$ is along $y$), the argument is simpler. For this value of $\alpha$, extremization of (25) with respect to $\mu$ again leads to the same pair of solutions. But the $xz$ Mode ($\mu = 0$) cannot exist at all; it can be shown from (26) that the $xz$ Mode cannot exist when

$$\sin^2 \alpha > \sin^2 \alpha'_X,$$

$$\sin^2 \alpha'_X = \frac{K_3^2 \varepsilon_A^2}{K_3^2 \varepsilon_A^2 + 4K_2 \varepsilon ||(K_1 \varepsilon || - K_3 \varepsilon_A)}$$

(29)

for any value of $\sigma_H$. For CCH-7 (24), $\alpha'_X = 1.01$ radian. As the $xz$ Mode cannot exist in the interval $\alpha'_X < \alpha < \pi - \alpha'_X$, the $xyz$ Mode again degenerates into the $yz$ Mode (Figs. 4d-4f).
The fact that at $\alpha = \pi/2$ the reduced threshold is independent of $r_H$ and given by (28) can be seen from the observation that in this configuration, only $\phi$ couples with the stabilizing $H_L$ and damps out; $\theta$ does not couple with $H_L$ and hence the electric threshold is unaffected by $r_H$.

The limits attained for high $r_H$ can be obtained by retaining terms in (25) proportional to $\sigma_H$ and neglecting the others. This causes $\sigma_H$ to factor out leaving (for $\alpha \neq 0$).

$$\sigma_E = \frac{\beta_1(\alpha - \mu)Q^4 + q^2(\beta_1 + j(\alpha - \mu))Q^2 + q^4}{S^2(\alpha + q^2)}$$  \hspace{1cm} (30)

Minimizing (30) with respect to $Q$ and $\mu$ it is found that either $\mu = \alpha$ (wavevector along $H_L$) or $\mu = \alpha + \pi/2$ (wavevector normal to $H_L$). As $K_1 > K_2$, the case $\mu = \alpha$ has lower threshold. The reduced electric threshold and dimensionless wavevector are again found to be $r_H$ and $Q_0$, respectively, of (28). For low $\alpha$ (for instance, curves 1 in Figs. 4a-4c), this limit is achieved for $r_H \approx 40$; when $\alpha$ is higher (curves 2 in Figs. 4a-4c), this limit is achieved at lower $r_H$.

A brief account of the effect of parameter variation is necessary. The discussion after (26) showed that the $xyz$ Mode can exist even in the absence of a stabilizing $H_L$. With $\sigma_H = 0$, minimization of $\Omega_{XY}$ with respect to $\mu$ implies that the quantity $\beta_1 + \beta_2 - 1$ be negative for the $yz$ Mode to set in with wavevector along $y$. If this quantity becomes positive, the $yz$ Mode will not have lower threshold than HD when $H_L = 0$. This can be achieved by increasing either $\beta_1$ (to 0.7) or $\beta_2$ (to 0.54). Increase of $\beta_1$ implies increase of $\epsilon_L$ with respect to $\epsilon_H$; this may be possible in materials exhibiting dielectric relaxation for high frequency electric fields [3,4]. The increase of $\beta_2$ occurs when $K_2$ increases relative to $K_3$; in some materials, this may occur close to the nematic-smectic A transition point [1,2].

Before closing, it seems worth dealing briefly with $H$ along $z$. When $H_L$ is applied along $z$ (which is also one of the symmetry directions), the initial homeotropic orientation, it has destabilizing influence. As a stabilizing field tends to encourage the formation of PD, it is intuitively clear that $H_L$ will tend to suppress PD. With $H_L$, the $xyz$ Mode electric threshold is given by $\sigma_E$ of (25) with the following definitions:

$$\sigma'_H = \frac{\chi_A h^2 H_L^2}{K_3}; \quad u'_1 = j(\mu), \quad u'_2 = q^2[1 + j(\mu)] + \sigma'_H;$$
$$u'_3 = q^2(q^2 + \sigma'_H); \quad t_1 = \beta_1 \beta_2 \gamma_1;$$
$$t'_2 = \frac{q^2 \gamma_2 + \beta_1(\gamma_1 + \beta_2)u'_3}{q^2}; \quad t'_3 = \frac{q^2(q^2 + \sigma'_H)^2}{\gamma_1 + \beta_2}u'_3 + \beta_1 t'_4/q^2;$$

Extremizing $\sigma_E$ with respect to $\mu$ results in the solution $\mu = \pi/2$ ($yz$ Mode) showing that PD will continue to have periodicity along $y$ in the presence of $H_L$. The pair $\theta, \psi$ give rise to the $yz$ Mode which ceases to exist when

$$H_L > H_G; H_G^2 = \frac{\epsilon_H^2 K_3}{4\epsilon_A h^2 \chi_A} \left[ \frac{\epsilon_A}{\epsilon_H} - \frac{K_3}{K_3} \right]$$

(the expression resembles closely that of $H_C$ (27); the change in sign of $\chi_A$ is due to the change in sign of the magnetic torque). For $H_L > H_G$, the deformation above the electrical threshold is HD involving $\theta, \psi$. But $\phi$ gets decoupled and can develop as HD above the bend Fredericiksz threshold $H_B = (\pi/2h)/\sqrt{K_3/(\chi_A)}$. If $H_G < H_B$, there will exist a true critical point between the $yz$ Mode and HD involving $\theta, \psi$. For CCH-7 (24), $H_G \approx 0.66H_B$. Hence, for $H_G < H_L < H_B$, HD with $\theta$ and $\psi$ should occur in the $xz$ plane; for $H_L > H_B$, HD with $\phi$ should form in the $yz$ plane.
A tentative explanation for the occurrence of the $xyz$ Mode in preference to HD (Fig. 4) will be briefly presented. As $\varepsilon_A > 0$, $E'_z$ tends to stabilize the initial orientation, is in phase with $\theta$ and $\phi$ and is also symmetric relative to $z = 0$; hence, its stabilizing action is strong. On the other hand, $E'_x$ and $E'_y$ have destabilizing influence. By Maxwell’s equation, $E'_{z,y} = E'_{y,z}$ and $E'_{z,x} = E'_{x,z}$; hence, $E'_y$ and $E'_x$ are not only antisymmetric relative to $z = 0$ but also out of phase with respect to $\theta$ and $\phi$ along $y$ and $x$, respectively. Because of this reason, the destabilizing influence of $E'$ is not strong. This, as well as the need to counter stabilizing elastic torques associated with $\theta$ and $\phi$ accounts for the $xyz$ Mode threshold being so close to the HD threshold.

The $yz$ Mode is not associated with $\phi$ and to this extent, its threshold should be expected to be lower than that of the $xyz$ Mode. But we note that the $yz$ Mode benefits from the destabilizing influence of $E'_y$ as $E'_z = 0$. It is possible that the partial destabilizing influence available from $E'$ is responsible for making the $yz$ Mode threshold slightly higher than that of the $xyz$ Mode.

5. Conclusions; Limitations of the Mathematical Model

The linear perturbation theory has been employed to discuss periodic and aperiodic deformations in nematics with different kinds of electric and magnetic susceptibility anisotropies by the action of crossed $E_0$ and $H_\perp$ fields imposed in the sample plane. While $E_0$ acts along $x$, the tilt of $H_\perp$ relative to $E_0$ can be varied in the $xy$ plane. Results for PD have been presented for initial homeotropic director alignment ($n_0$ along $z$). The stabilizing field is initially imposed on the uniformly aligned sample. Subsequently, the destabilizing field is made to act causing either HD or PD. The calculations reduce to the solution of an eigenvalue problem in which the eigenvalue yields the threshold but the perturbation amplitudes which form components of an eigenvector are not completely determinable. Hence, these calculations cannot predict (non-linear) effects above threshold; in addition, thresholds studied in this work pertain only to second order transitions. As real materials with the required properties are available [23,24], most of the conclusions of this work can be checked by experimental detection of thresholds. The main conclusions of this work are as follows.

In M1 (17), PD with $y$ modulation ($yz$ Mode) should occur (stripes parallel to $x$, normal to the electrodes) when $E_0$ and $H_\perp$ are along $x$. If $H_\perp$ is tilted sufficiently away from $E_0$, PD with $x$ modulation ($xz$ Mode) becomes favorable (stripes along $y$ and parallel to the electrodes). HD alone will set in when the magnetic tilt exceeds a limit which is determined by material parameters. A given PD Mode occurs only over a limited range of $E_{x0}$. Thus, HD will be favorable for weak as well as strong $E_0$. Transitions between the two PD Modes should be accompanied by changes in the magnitude and direction of the periodicity wavevector, provided that the more general $xyz$ Mode does not appear; this seems to be true in the limit of small perturbations. The $xyz$ Mode may become observable in a (hypothetical) material having $K_1 < K_2$, say near the nematic-smectic A transition. In a real situation, the $xz$ or $yz$ Mode that develops at threshold may become unstable above threshold and go over to the $xyz$ Mode. This possibility has to be investigated using nonlinear perturbations.

In a nematic such as CCH-7 (24), the $xyz$ Mode seems more favorable than the other two PD Modes. The amplitude and direction of the wavevector vary continuously as functions of magnetic strength and tilt. When $H_\perp$ is strong, the wavevector is parallel to $H_\perp$. When $H$ is absent or when it is applied along $x$ (parallel to $E_0$) or $y$, the $xyz$ Mode degenerates into the $yz$ Mode with direction of periodicity along $y$.

PD seems to have lower threshold than HD mainly due to the modification of $E$ inside the sample caused by director perturbations. Minimizing this effect increases the PD threshold...
relative to the HD threshold; PD can even be suppressed. In M1, PD can be suppressed by causing additional elastic deformation and diminishing the effect of dielectric interaction (achieved by rotating $\mathbf{H}_\perp$ away from $\mathbf{E}_o$). In CCH-7, PD can be suppressed by a destabilizing $\mathbf{H}_\parallel$ along $\mathbf{n}_o$. In a hypothetical material with the same properties as CCH-7 (24) but positive $\chi_A$, $\mathbf{H}_\perp$ acts as a destabilizing field. At a given magnetic tilt $\alpha$, PD can be suppressed if $\mathbf{H}_\perp$ is strong enough with critical value of $H_L$ being a function of $\alpha$. This case will be studied separately.

From the viewpoint of experiments, the following statements can be made. The studies of field induced biaxiality [19], effect of rotating magnetic fields [29,30] and field induced transient convective patterns (see [31] for a review) appear to offer interesting possibilities involving crossed $\mathbf{E}_o$ and $\mathbf{H}$ fields imposed on homeotropic samples of materials such as M1 and CCH-7. For instance, the rotating $\mathbf{H}$ may be stabilizing (in CCH-7) or destabilizing (in M1). The transient patterns may be induced by the destabilizing $\mathbf{H}$ (in M1) or the destabilizing $\mathbf{E}_o$ (in CCH-7). The nature of field induced biaxiality can be studied as a function of the magnetic tilt with respect to $\mathbf{E}_o$.

Consider now initial homeotropic alignment along $z$. In M1, no deformation of any kind will occur when $\mathbf{H}_\parallel$ acts along $z$ and $\mathbf{E}_o$ along $x$ because both stabilize the initial orientation. But with $\mathbf{H}_\perp$ in the $xy$ plane, PD may result (Sect. 3). Hence, it is intuitively clear that if $\mathbf{H}$ is tilted sufficiently out of the $xy$ plane, PD may be suppressed. The experiment would involve starting with an aligned sample with $\mathbf{E}_o$ along $z$. Then $\mathbf{H}$ is imposed at some tilt $\alpha$ in the $xy$ plane. In subsequent experiments, $\alpha$ is kept fixed but $\mathbf{H}$ is tilted out of the $xy$ plane. The consequences of this cannot be worked out using the present model which considers only linear perturbations.

Some limitations of this work will be discussed. All results have been obtained by solving the governing equations (6)-(8) with (10). Solving for HD poses no problems as the only condition on the potential ($\psi$) relevant at the boundaries is that $\psi_x = 0$. Thus, for HD only the derivative of $\psi$ vanishes at the boundaries, but $\psi$ itself remains. In Sections 3 and 4, equations for PD are solved using the same boundary conditions as for HD. This causes the expression for the PD threshold to reduce continuously to that of the HD threshold as the periodicity wavevector is reduced to zero. In principle [26], the additional conditions on $\psi$ should also be used.

When we start with $\mathbf{n}_o$ normal to the destabilizing field, the net torque exerted on the director field is zero. In a real situation, perturbations in alignment exist; these may be static, caused by non-uniformity in surface treatment etc. or dynamic caused by thermal fluctuations. These perturbations are taken into account as purely static in the present theory which is non-stochastic. The thermal origin of perturbations can be explicitly taken into account [32] and is of importance in understanding the dynamics of the deformation above threshold. Even the explanation of pretransitional biaxiality depends on the presence of these fluctuations.

The rigid anchoring hypothesis has been employed. The effect of finite anchoring becomes perceptible when $\Sigma = Bh/K \sim 1$ where $B$ is the anchoring strength (dyne cm$^{-1}$) and $K$ a curvature elastic constant. In an experiment involving $\mathbf{E}_o$ parallel to the sample planes, it is not convenient to use very thin samples. With $K \sim 10^{-7}$, $h \sim 10^{-2}$ cm and $B \sim 10^{-2}$, we find $\Sigma \sim 10^3$; hence, the rigid anchoring hypothesis seems to be applicable except in cases where the anchoring strengths at the surfaces are extremely weak. The effects of surface elastic constants [1-4] may also not become important in thick samples. However, studies on the effect of anchoring strength are interesting from a pedagogical viewpoint as is clear from earlier work [8,9].

The uniformity of $\mathbf{E}$ in the absence of perturbations may be achievable only for large electrode gaps, $2g$; then higher voltages become necessary. This may not be convenient in an experiment; generally the electrode gap is not even an order of magnitude greater than the sample thickness.
A realistic calculation for such a set up must include boundary conditions not only at the sample planes (interface between two dielectrics) but also the electrodes (interface between a dielectric and a conductor).

Neglect of flexoelectricity and director pretilt (away from the homeotropic) has made possible presentation of analytical results via a study of spatially symmetric Modes. Though a continuous variation of director tilt ($\phi_0$) is experimentally difficult, it is of pedagogical importance. When $\phi_0 \neq 0$, there exists no cylindrical symmetry about $z$ axis (the sample normal). The nature of the elastic torque (7, 8) as well as the dielectric interaction (6) become such that no perturbation can be assumed to have the simple form given in Sections 3 and 4. For a closed solution of (6)-(8), every perturbation is the sum of two terms, one symmetric and the other antisymmetric relative to $z = 0$. This mode mixing increases the elastic energy associated with PD causing the PD threshold to increase with respect to the HD threshold. As HD retains modal purity, PD may even be suppressed for high enough $\phi_0$. The shape of phase boundaries separating different PD Modes may also change. These calculations will be reported separately.

Physically, the neglect of flexoelectricity is not unrealistic if $ac$ electric fields are employed; then the $rms$ value of the voltage is taken. With a $dc$ or low frequency field, flexoelectricity cannot be ignored. Even under the rigid anchoring hypothesis, taking account of flexoelectricity leads to mode mixing. With the assumption of finite anchoring energy at the boundaries, the results of this work will be of only academic importance for $dc$ fields as a distortion may set in without threshold [33]. This offers the following possibility. If the predictions of this work are found to be true for $ac$ fields, then the frequency can be diminished to check at what point distortions of the type predicted in [33] will result.

The assumption about the nematic being an insulator must be borne in mind as a variety of electrohydrodynamic (EHD) instabilities can be caused in nematics by the presence of ionic impurities (for a recent review, see [34]); such instabilities may occur when $E_0$ is impressed parallel to the sample planes. If the threshold for this instability becomes less than that of the PD or HD thresholds calculated above, results of this work will not be of real interest. This points towards the need to use pure samples for studying static effects. While experiments will have the ultimate say in this matter, it appears necessary to calculate exactly the EHD thresholds using appropriate material parameters.

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References

[26] There are three coupled equations of second order; with boundaries at z = ±h, only six conditions can be accommodated. It appears, therefore, that the six conditions (10) are sufficient. But electromagnetic theory demands that the components of the E field parallel to the boundaries should also be continuous. As (10) offer sufficient number of conditions, these additional conditions are being ignored except in places where they are automatically satisfied.