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On Low-Velocity Collisions of Viscoelastic Particles

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Résumé. — La théorie du contact élastique de deux corps developpée par Hertz [1] est généralisée tenant compte de la contribution des effects visqueux à la tension totale Une équation différentielle nonlinéaire est derivée pour des particles dont les surfaces ont une courbure arbitraire. Elle est résolue numériquement dans le cas des particles sphériques. La dépendence du coefficient de restitution normale de la vitesse d'impact est calculée et comparée avec des données expérimentales obtenues pour la glace aux températures basses [2,3]. Un bon accord est trouvé qui permet l'estimation des constantes du matériel dans certains cas. Une application astrophysique de nos résultats est discutée brèvement dans un cas d'interêt particulier: des particles de glace dans des anneaux planétaires

Abstract. — The theory of the elastic contact of two bodies developed by Hertz [1] is generalized including the contribution of viscous effects to the total stress A nonlinear differential equation for the compression is derived for particles with arbitrary curvature of their surfaces and is solved numerically for spherical particles The resulting dependence of the normal restitution coefficient on the impact velocity is calculated and compared with experimental data for ice at low temperatures [2,3]. A good agreement is found which allows to estimate unknown material constants in certain cases An astrophysical application of the results is briefly discussed for the especially interesting case of icy particles in planetary rings

1. Introduction

Hertz's theory of the contact of elastic bodies [1] has been widely used in contact mechanics. Unfortunately, up to now no satisfying extension of this theory seems to exist taking into account in a realistic way such effects like viscosity, although inelasticity effects in collisions play an important role in physical processes and can affect seriously the development of many-particle systems. One case of particular interest are planetary rings, e.g. those of Saturn, which are composed of particles mainly consisting of water ice [4]. The lifetimes of these systems are influenced by several factors, especially the collisional properties of the particles. Wiesel [5] had already found that the stability of planetary rings is not only strongly dependent on their velocity distribution, but also on the restitution coefficient of the material, i.e the ratio of the postcollisional relative velocity of two particles to the precollisional one. In these simulations a constant value was assumed, but experiments by Bridges, Hatzes *et al.* [2, 3, 6] have shown a strong dependence of the restitution coefficient on the impact velocity Therefore the form of this dependence is liable to affect the stability and thus the evolution of a planetary ring Thus, we have started detailed investigations of the collision mechanism which seems to play an important role in structures related to the gravitational action of satellites [7].

As one of the first attempts to generalize the theory of collisions Poschl [8] proposed the introduction of a dissipative term proportional to a power of the velocity of the compression. In the special case of a quadratic dependency he was able to express this velocity in form of a power series in the compression. Although he achieved qualitatively right results, his proposition had the disadvantage that the choice of the exponent of the velocity was motivated by mathematical convenience rather than by physical arguments. Thus, no relation of the coefficients in the equation to the elastic and viscous constants of the material of the colliding bodies could be derived from his model.

Other theoretical approaches [9, 10] were discussed in a recent article by Dilley [11]. He found that they do not describe well the above mentioned experimental results and presented a new collision model based on the assumption of viscous dissipation of energy. First, the particles are treated as point masses Both the deformation and its time derivative appear in the first power, and the coefficient of the deformation velocity is made dependent on the impact velocity via a power law The spherical shape of the particles is taken into account by introducing some effects of the Hertz theory "by hand". This procedure allows for a good fit to results of experiments with clean ice spheres [3] but requires the right choice of three constants (coefficients of actual deformation and velocity, exponent of the impact velocity) which cannot be found *a priori*.

In this paper which is part of more general investigations on granular material [12] we propose an extension of Hertz's theory starting with the consideration of the stresses exerted on the bodies in contact due to their elastic and viscous properties. Following the algorithm used by Hertz [1, 13, 14] a differential equation for the time dependence of the compression is derived Its coefficients depend on the material properties of the colliding bodies and also on the curvature of their surfaces. The equation is solved numerically for the simple case of spheres where experimentally obtained data are available for ice at low temperatures [2, 3], an example of importance for further applications of models of inelastic collisions for the dynamics of planetary rings [15]. A good fit of the theoretical curves for the velocity dependent restitution coefficient with the experimental ones with different dependencies of the restitution coefficient on the impact velocity is achieved. Only two coefficients have to be chosen for this purpose Because these are directly related to the material properties of the particles, this allows the estimation of the order of the magnitude of the yet unknown viscosity. This ability of our model is an advantage in comparison to the theory of inelastic impact discussed in [14] which was intended for the treatment of collisions of metallic bodies. It was based on a static analysis and the assumption of a constant mean pressure and led to a restitution coefficient proportional to the power $-1/4$ of the impact velocity.

Our model is also compared with the solutions of other differential equations for the compression, e.g. that one of Pöschl's model [8]. Our rigorous model is found to fit certain experimental results better than the other ones.

Finally, we discuss possible extensions of the model and experimental investigations which are required to check our results.

2. Collision Theory

2.1 STRESSES AND DISPLACEMENTS — The elastic stress acting on a solid body is given by the following expression:

$$\sigma_{ik}^{\mathrm{el}} = K u_{ll}\delta_{ik} + 2\mu^*\left(u_{ik} - \frac{1}{3}\delta_{ik}u_{ll}\right) \tag{1}$$

with $u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$ being the tensor of the displacements which is derived from the deformation $\mathbf{u}$, δ_{ik} the Kronecker symbol, K the compression modulus and μ^* the shear modulus. The latter two are related to the Young modulus E and the Poisson ratio ν respectively and the Lamé constants which we will denote here with λ_{I} and λ_{II} via

$$K = \frac{E}{3(1-2\nu)} = \lambda_{\mathrm{I}} + \frac{2}{3}\lambda_{\mathrm{II}}$$
$$\mu^* = \frac{E}{2(1-\nu)} = \lambda_{\mathrm{II}} \tag{2}$$

We will further use the notation in terms of the Lamé constants which allows us to benefit from the analogy between the formulae describing elastic and viscous phenomena. Then the elastic stress tensor can be written in the form:

$$\sigma_{ik}^{\mathrm{el}} = \lambda_{\mathrm{I}} u_{ll}\delta_{ik} + 2\lambda_{\mathrm{II}}u_{ik} \tag{3}$$

The viscous stress tensor is expressed in terms of the displacement velocity $\dot{u}_{ik}$:

$$\sigma_{ik}^{\mathrm{vi}} = \eta_{\mathrm{I}} u_{ll}\delta_{ik} + 2\eta_{\mathrm{II}}u_{ik} \tag{4}$$

with the bulk viscosity η_{I} and the shear viscosity η_{II}. Thus, one has for the total stress

$$\sigma_{ik} = \sigma_{ik}^{\mathrm{el}} + \sigma_{ik}^{\mathrm{vi}} = \lambda_{\mathrm{I}} u_{ll}\delta_{ik} + 2\lambda_{\mathrm{II}}u_{ik} + \eta_{\mathrm{I}}\dot{u}_{ll}\delta_{ik} + 2\eta_{\mathrm{II}}\dot{u}_{ik} \tag{5}$$

which yields (remember that the force can be expressed as $\partial\sigma_{ik}/\partial x_k$) the following dynamical equation for the continuum medium:

$$(\lambda_{\mathrm{I}} + \lambda_{\mathrm{II}})\ \nabla\ (\nabla\ \cdot \mathbf{u}) + \lambda_{\mathrm{II}}\ \Delta\mathbf{u} + (\eta_{\mathrm{I}} + \eta_{\mathrm{II}})\ \nabla\ (\nabla\ \cdot \dot{\mathbf{u}}) + \eta_{\mathrm{II}}\ \Delta\mathbf{u} = \rho\ddot{\mathbf{u}} \tag{6}$$

We introduce as the characteristic scale R and the characteristic velocity v_0 of the problem the particle's radius and velocity, respectively Then $\tau = R/v_0$ will be the characteristic time. Taking into account that $\lambda_{\mathrm{II}}/\rho = c_{\mathrm{t}}^2$ and $(\lambda_{\mathrm{I}} + 2\lambda_{\mathrm{II}})/\rho = c_{\mathrm{l}}^2$, where c_{l} and c_{t} are the longitudinal and transversal sound speed in the material, one can rescale the variables and rewrite the equation in the following way (for simplicity we keep the same notations for the rescaled variables as for the original ones).

$$\left(\nabla^2 - \nabla\nabla\cdot\right)\mathbf{u} + \left(\frac{c_{\mathrm{l}}}{c_{\mathrm{t}}}\right)^2(\nabla\nabla\ )\mathbf{u} + \varepsilon\left(\beta\left(\nabla^2 - \nabla\nabla\ \right)\mathbf{u} + \gamma(\nabla\nabla\ )\mathbf{u}\right) = \varepsilon^2\ddot{\mathbf{u}} \tag{7}$$

where $\varepsilon = (v_0/c_{\mathrm{t}})$, and $\beta = \eta_{\mathrm{I}}/(\rho R c_{\mathrm{t}})$, $\gamma = (\eta_{\mathrm{I}} + 2\eta_{\mathrm{II}})/(\rho R c_{\mathrm{t}})$. One can see that if the characteristic velocity of the problem is much less than the speed of the sound in the material, i e. if $\varepsilon \ll 1$, and if the dissipation in the bulk is low, that means that the coefficients β, γ are of the order of unity, one can use the quasistatic approximation

$$\left(\nabla^2 - \nabla\nabla\cdot\right)\mathbf{u} + \left(\frac{c_{\mathrm{l}}}{c_{\mathrm{t}}}\right)^2(\nabla\nabla\ )\mathbf{u} = 0 \tag{8}$$

Thus, in the quasistatic approximation the displacement field $\mathbf{u}(\mathbf{r},t)$ in the material coincides with the one for the static problem $\mathbf{u}(\mathbf{r})$ Note that in the static case only the elastic stress is present. The static (elastic) contact problem had already been solved by Heinrich Hertz in 1882 [1] Before we deal with the generalization on the viscoelastic case, we will sketch briefly the main results of this classical theory (for details see e.g. [13])

We assume for simplicity that only normal forces with respect to the contact area act between two solid bodies labeled 1 and 2 Their surfaces in the contact region will be flattened Using a coordinate system centered in the middle of the contact region where we set $z=0$, one can write the following equation

$$Mx^2 + Ny^2 + u_{z1} + u_{z2} = h \tag{9}$$

where $u_{z1} = u_{z1}(x,y)$ and $u_{z2} = u_{z2}(x,y)$ are z-components of the displacements in the material of the bodies on the plane $z=0$, h is the sum of the compressions of both bodies in the centre of the contact area while the constants M and N are related to the radii of curvature of the surfaces in contact via relations (cf. [1,13]).

$$\begin{aligned}
2(M+N) &= \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_1'} + \frac{1}{R_2'} \\
4(M-N)^2 &= \left(\frac{1}{R_1} - \frac{1}{R_2}\right)^2 + \left(\frac{1}{R_1'} + \frac{1}{R_2'}\right)^2 \\
&\quad + 2\cos 2\varphi \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\frac{1}{R_1'} + \frac{1}{R_2'}\right)
\end{aligned} \tag{10}$$

where R_1, R_2 and R_1', R_2' are the principal radii of curvature of the two bodies and φ is the angle between the planes corresponding to the curvature radii R_1 and R_1' The values of u_{z1} and u_{z2} may be expressed in terms of the normal pressure $P_z(x,y)$, that acts between the bodies in the plane $z=0$

$$\begin{aligned}
u_{z1}(x,y,0) &= \frac{1}{\pi}\Lambda_1 \int\int \frac{P_z(x',y')}{r}\,\mathrm{d}x'\mathrm{d}y' \\
u_{z2}(x',y') &= \frac{1}{\pi}\Lambda_2 \int\int \frac{P_z(x',y')}{r}\,\mathrm{d}x'\mathrm{d}y'
\end{aligned} \tag{11}$$

Here $r = \sqrt{(x-x')^2 + (y-y')^2}$, and $\Lambda_i = \dfrac{\lambda_{\mathrm{I}}^i + 2\lambda_{\mathrm{II}}^i}{4\lambda_{\mathrm{II}}^i\left(\lambda_{\mathrm{I}}^i + \lambda_{\mathrm{II}}^i\right)}$, $i = 1,2$ The normal pressure P_z is simply related to the total normal force F^{el}

$$P_z(x,y) = \frac{3F^{\mathrm{el}}}{2\pi ab}\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \tag{12}$$

where a and b are the semiaxes of the contact ellipse. The latter values as well as the compression h may be found from the set of equations [1,13,14]

$$\begin{aligned}
h &= \frac{F^{\mathrm{el}}D}{\pi}\int_0^\infty \frac{\mathrm{d}q}{\sqrt{(a^2+q)(b^2+q)q}} = \frac{2F^{\mathrm{el}}D}{\pi}\frac{1}{b}K(k) \\
M &= \frac{F^{\mathrm{el}}D}{\pi}\int_0^\infty \frac{\mathrm{d}q}{(a^2+q)\sqrt{(a^2+q)(b^2+q)q}} = \frac{2F^{\mathrm{el}}D}{\pi}\frac{b^2E(k) - a^2K(k)}{a^2b(b^2-a^2)} \\
N &= \frac{F^{\mathrm{el}}D}{\pi}\int_0^\infty \frac{\mathrm{d}q}{(b^2+q)\sqrt{(a^2+q)(b^2+q)q}} = \frac{2F^{\mathrm{el}}D}{\pi}\frac{K(k) - E(k)}{b(b^2-a^2)}
\end{aligned} \tag{13}$$

with $D = (3/4)(\Lambda_1 + \Lambda_2)$ and $E(k)$ and $K(k)$ being the Jacobian elliptic functions in usual notation [16, 17]. $k = \sqrt{b^2 - a^2}/b$ is the eccentricity of the contact ellipse ($b > a$ without restrictions for generality). The size of this ellipse depends on the normal force, its semiaxes a and b are related to $F^{\rm el}$ via the second and third of the above equations. Using this dependence, from the first of equations (13) Hertz's famous solution of the elastic contact problem can be derived [1, 13, 14]: for all bodies in contact the total elastic force and the compression are related by a power law

$$F^{\rm el}(h) = {\rm const}\ h^{3/2} \tag{14}$$

with a constant depending on the elastic constants of the materials and on the local curvatures of the surfaces of the colliding bodies. If we assume that both bodies consist of the same material, in the case of spherical particles the above equation reads:

$$F^{\rm el}(h) = \frac{2E\sqrt{\overline{R}}}{3(1-\nu^2)}\, h^{3/2} \tag{15}$$

where $\overline{R} = R_1R_2/(R_1 + R_2)$ and R_1, R_2 are radii of particles in contact.

The most important property of the solution of the elastic contact problem is that the displacement fields $\mathbf{u}_1(\mathbf{r})$ and $\mathbf{u}_2(\mathbf{r})$ are completely defined by the value of $F^{\rm el}$, and thus by the value of the compression h To emphasize this we write $\mathbf{u}(\mathbf{r}) = \mathbf{u}(\mathbf{r}, h)$ so that the displacement field parametrically depends on the compression. We obtain for the field of the displacement velocities in the quasistatic approximation:

$$\dot{\mathbf{u}}(\mathbf{r},\ t) = \dot{h}\ \frac{\partial \mathbf{u}(\mathbf{r},\ h)}{\partial h} \tag{16}$$

Now we turn to the calculation of the dissipative part of the stress. We assume for simplicity that the colliding particles are of the same material. The general case may also be considered using an appropriate coordinate rescaling.

Using (16) we can write for the dissipative part of the stress tensor ·

$$\sigma_{ik}^{\rm vi} = \dot{h}\ \frac{\partial}{\partial h}\left\{\eta_{\rm I} u_{ll}\delta_{ik} + 2\eta_{\rm II} u_{ik}\right\} = \dot{h}\ \frac{\partial}{\partial h}\sigma_{ik}^{\rm el}\left(\lambda_{\rm I} \leftrightarrow \eta_{\rm I},\ \lambda_{\rm II} \leftrightarrow \eta_{\rm II}\right) \tag{17}$$

Here we emphasize that the expression in the curled brackets in the right-hand side of the above equation is the same one as for the elastic stress with the only difference that the viscous constants are substituted by the elastic ones. Note that the component $\sigma_{zz}^{\rm el}$ of the elastic stress is equal to the normal pressure P_z at the plane $z = 0$, which is given above Namely, we have .

$$\sigma_{zz}^{\rm el}(x, y, 0) = \lambda_{\rm I}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2\lambda_{\rm II}\frac{\partial u_z}{\partial z} = \frac{3F^{\rm el}}{2\pi ab}\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \tag{18}$$

We transform the coordinates in the following way

$$x = \alpha x'\ ,\ y = \alpha y'\ ,\ z = z' \tag{19}$$

with

$$\alpha = \frac{\eta_{\rm I}\left(\lambda_{\rm I} + 2\lambda_{\rm II}\right)}{\lambda_{\rm I}\left(\eta_{\rm I} + 2\eta_{\rm II}\right)} \tag{20}$$

and obtain·

$$\begin{aligned} \eta_{\rm I}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2\eta_{\rm II}\frac{\partial u_z}{\partial z} &= \frac{\eta_{\rm I}}{\alpha\lambda_{\rm I}}\left(\lambda_{\rm I}\left(\frac{\partial u_x}{\partial x'} + \frac{\partial u_y}{\partial y'} + \frac{\partial u_z}{\partial z'}\right) + 2\lambda_{\rm II}\frac{\partial u_z}{\partial z'}\right) \\ = \frac{\eta_{\rm I}}{\alpha\lambda_{\rm I}}\frac{3F^{\rm el}}{2\pi ab}\sqrt{1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2}} &= \frac{\eta_{\rm I}}{\alpha\lambda_{\rm I}}\frac{3F^{\rm el}}{2\pi ab}\sqrt{1 - \frac{x^2}{\alpha^2 a^2} - \frac{y^2}{\alpha^2 b^2}} \end{aligned} \tag{21}$$

Applying the operator $\dot{h}\,\partial/\partial h$ on the last expression in the preceding equation we obtain the viscous stress .

$$\sigma_{zz}^{\text{vi}}(x,y,0) = \dot{h}\,\frac{\partial}{\partial h}\,\frac{\eta_{\text{I}}}{\alpha\lambda_{\text{I}}}\,\frac{3F^{\text{el}}}{2\pi ab}\sqrt{1-\frac{x^2}{\alpha^2a^2}-\frac{y^2}{\alpha^2b^2}} \tag{22}$$

The total viscous force may be obtained by integrating the viscous stress over the contact area, yielding the following result·

$$F^{\text{vis}} = A\,\dot{h}\,\frac{\partial}{\partial h}F^{\text{el}}(h) \tag{23}$$

where

$$A = \frac{2}{3}\left(\frac{\eta_{\text{I}}}{\lambda_{\text{I}}}\right)^2\frac{\lambda_{\text{I}}+2\lambda_{\text{II}}}{\eta_{\text{I}}+2\eta_{\text{II}}} = \frac{2}{3}\,\frac{\eta_{\text{I}}^2}{\eta_{\text{I}}+2\eta_{\text{II}}}\,\frac{(1-\nu^2)(1-2\nu)}{E\nu^2} \tag{24}$$

From the last equation one can find the general relation for the total force which acts between two viscoelastic bodies colliding in the quasistatic regime·

$$F = \text{const}\,\left(h^{3/2}+\frac{3}{2}A\,h^{1/2}\dot{h}\right) \tag{25}$$

The constant in this equation coincides with that one for the elastic force so that equation (25) reads e.g for spherical particles:

$$F = \frac{2E\sqrt{R}}{3(1-\nu^2)}\,\left(h^{3/2}+\frac{3}{2}A\,h^{1/2}\dot{h}\right) \tag{26}$$

As one can see from the above equations, the total force between the colliding particles exceeds the elastic force in the first stage of the collision when $\dot{h} > 0$, and the particles decelerate more efficiently than in the elastic case. The maximal compression is thus less than in an elastic collision On the other hand, when the particles move away from each other and $\dot{h} < 0$, the force is less than in the elastic case and yields a lower acceleration. As a result the postcollisional velocity is lower than the precollisional one. Their ratio defines the normal restitution coefficient (t_c is the duration of the collision).

$$\epsilon_N = \dot{h}(t_c)/\,\dot{h}(0) \tag{27}$$

For an interpretation of the constant A we notice that the viscous constants $\eta_{\text{I/II}}$ can be written as follows:

$$\eta_{\text{I/II}} = \tau_{\text{vis}}\lambda_{\text{I/II}} \tag{28}$$

where τ_{vis} is the relaxation time for the dissipative processes in the material accompanying the time-dependent deformation. If we assume for simplicity that the relaxation times are the same for both viscous coefficients, one can see from this equation as well as from the definition of A that $A \approx \tau_{\text{vis}}$. Furthermore, the order of magnitude of $\dot{h}$ is h/t_c Thus, one can write

$$A\,h^{1/2}\dot{h} \approx Ah^{3/2}/t_c \approx (\tau_{\text{vis}}/t_c)\,h^{3/2} \tag{29}$$

If the viscous relaxation time is much shorter than the duration of the collision· $(\tau_{\text{vis}}/t_c) \ll 1$, one can write the total force in the following form:

$$F = \text{const}\,\left(h + A\,\dot{h}\right)^{3/2} \tag{30}$$

The familiarity between equations (25) and (30) can also be demonstrated by an expansion in a power series of the latter. We note that the approximative formula (30) can be obtained by considering $\mathbf{u}$ and $\dot{\mathbf{u}}$ in equation (6) as independent variables [7], which is in reality not the case.

We conclude that in the quasistatic regime in the case of moderate bulk viscosity all the results for the elastic contact theory may be used if instead of the compression h the renormalized one $h + A\dot{h}$ is substituted in the relations. In Section 3 we will compare the solutions for both equations

With the viscous relaxation time one can also estimate the coefficients β and γ introduced above. We obtain e g. $\beta = \eta_{\mathrm{I}}/(\rho R c_{\mathrm{t}}) \sim (\tau_{\mathrm{vis}} c_{\mathrm{t}})/R$ (assuming $\lambda_{\mathrm{I}} \approx \lambda_{\mathrm{II}}$). That means that the case of moderate viscosity corresponds to the condition $(\tau_{\mathrm{vis}} c_{\mathrm{t}})/R \sim 1$

2.2. INFLUENCE OF THE SHAPE OF THE COLLIDING PARTICLES. — Particles in granular gases have a wide distribution of masses, radii, and curvatures at the point of contact. Therefore, for practical simulations of the evolution of the granular gases one should know how the shape of the particles influences their collisional behaviour and how the shapes of the particles are distributed

We will now briefly discuss the influence of the curvature of the particle surfaces in the vicinity of the point of contact in analogy to the elastic case already discussed in [14] From equations (13) follows:

$$\frac{M}{N} = \frac{E(k) - (1-k^2)K(k)}{(1-k^2)\,(K(k) - E(k))} \tag{31}$$

We see that the eccentricity of the contact ellipse depends only on the ratio M/N, i e finally on the curvature of the two bodies in the point of contact. The force acting between them and their material will only affect the absolute size of the contact figure From equations (13) also the relation between force and compression

$$h = \left(\frac{2F_z D}{\pi}\right)^{2/3} \left(\frac{Nk^2}{K(k) - E(k)}\right)^{1/3} K(k) \tag{32}$$

can be obtained, and equation (25) for the viscoelastic force can be rewritten.

$$F = \frac{\pi}{2D}\left(\frac{K(k) - E(k)}{Nk^2}\right)^{1/2} (K(k))^{-3/2} \left(h^{3/2} + \frac{3}{2}Ah^{1/2}\dot{h}\right) \tag{33}$$

With the reduced mass $\mu = m_1 m_2/(m_1 + m_2)$ we can write the following dynamical equation for the collision of nonspherical viscoelastic particles:

$$\ddot{h} + \frac{\pi}{2\mu D}\left(\frac{K(k) - E(k)}{Nk^2}\right)^{1/2} (K(k))^{-3/2} \left(h^{3/2} + \frac{3}{2}Ah^{1/2}\dot{h}\right) = 0 \tag{34}$$

with the initial conditions $\dot{h}(0) = g_N$ and $h(0) = 0$ g_N is the precollisional velocity. The last equation (34) and the dynamical equation with the force (26) have both the same form except for the factor

$$\zeta = \frac{\pi}{2\mu D}\left(\frac{K(k) - E(k)}{Nk^2}\right)^{1/2} (K(k))^{-3/2} \tag{35}$$

which takes the value $\zeta_{\mathrm{sph}} = \dfrac{2E\sqrt{R}}{3(1-\nu^2)}$ in the case of spherical particles This shows that the problem of the collision of particles with non-spherically curved surfaces can successfully be mapped onto the corresponding collision problem for spherical particles after an appropriate rescaling of the initial particle velocities. We see that the collision problem for the spherical particles is actually a "benchmark" problem for the treatment of binary collision processes.

We also see that for simulations of the evolution of granular gases the distribution of the shapes of the particles can be characterized by the distribution of the value of ζ, which may be considered as a stochastic variable The restitution coefficient can then be calculated as a function not only of the impact velocity, but also of the stochastic variable ζ Therefore for the dynamical description of granular material one has to solve the twofold problem

(i) calculation of the restitution coefficient $\epsilon_N(g_N, \zeta)$

(ii) evaluation of the distribution function $f(\zeta)$.

As we have just noticed, the first problem can be reduced after an appropriate rescaling to the calculation of the restitution coefficient for colliding spherical particles as a function of the impact velocity, i e of $\epsilon_N(g_N)$

To find the distribution function for the variable ζ, an approximation for the elliptic integrals by means of elementary functions is useful. Because the usual approximations of high precision for the Jacobian elliptic function [17] also contain logarithmic terms they cannot be easily solved in k Thus, we tried to express the factor in k in equation (34) in terms of the ratio M/N in equation (31) by a simple numerical fit. We found:

$$\frac{K(k)-E(k)}{k^2} \approx \frac{k}{1\ 125(1-k^2)^{0\ 665}} \tag{36}$$

The eccentricity k of the contact ellipse can be estimated from the ratio M/N approximately as

$$k \approx \sqrt{1-\left(\frac{N}{M}\right)^{1\ 315}} \tag{37}$$

We can now express the factor in k in terms of this ratio as follows

$$\frac{K(k)-E(k)}{k^2} \approx \frac{\sqrt{1-\left(\frac{N}{M}\right)^{1\ 315}}}{1\ 125\left(\frac{N}{M}\right)^{0\ 874}} \tag{38}$$

The deviations of our fits (36) and (37) from the values obtained by using the "polynomial" approximations [17] turn out to be less than 1 %. This justifies the use of these rather simple fits for the further treatment of collisions of non-spherical bodies

Finally we give one of the "polynomial" approximations for $K(k)$ (cf. [17]): $K(k) = (a_0 + a_1(1-k^2) + a_2(1-k^2)^2) + (b_0 + b_1(1-k^2) + b_2(1-k^2)^2)\ln\frac{1}{1-k^2}$ with $a_0 = 1\ 3862944$, $a_1 = 0\ 1119723$, $a_2 = 0.0725296$ and $b_0 = 0.5$, $b_1 = 0.1213478$, $b_2 = 0\ 0288729$. Substituting the above expressions into the dynamic equation, one can solve it numerically, and find the restitution coefficient

In the following we will only deal with spherical particles because this is a "benchmark" model, and besides, it is the only case where up to now experimental data are available which are of interest for our applications.

3. Comparison of Collision Models with Experimental Results

We have solved numerically the differential equation

$$\dot{h} = \frac{1}{\mu}F_N \tag{39}$$

for forces F_N given by equations (25) and (30) and with the initial conditions $h(0) = 0$ and $h(0) = g_N$, i.e for zero initial deformation and a given impact velocity. From the value g'_N of the relative velocity at the end of the contact between the two bodies the normal restitution coefficient has been calculated Its dependence $\epsilon_N(g_N)$ on the impact velocity has been compared with the results of experiments [2,3] In these experiments, spherical ice particles with a diameter of some centimeters hit a plane ice block which can be considered to be a sphere of infinite radius. The reduced radius $\overline{R}$ is then equal to the radius of the ice sphere. The conditions were chosen to reflect those typical for planetary rings. the temperature was $T \approx 150$ K, the impact velocities were in the order of some centimeters per second or less, the apparatus was mounted in an evacuated container.

We have also estimated the time dependence of the deformation and of the relative velocity during the collision.

The same was done with differential equations for other collision models which we will briefly discuss here As already mentioned, Pöschl [8] proposed to include the dissipative effects due to the nonelasticity of the material in form of a power law of the relative velocity, i.e. to formulate an equation of the form

$$h + A_1 h^{3/2} + A_2 \dot{h}^{\alpha} = 0 \tag{40}$$

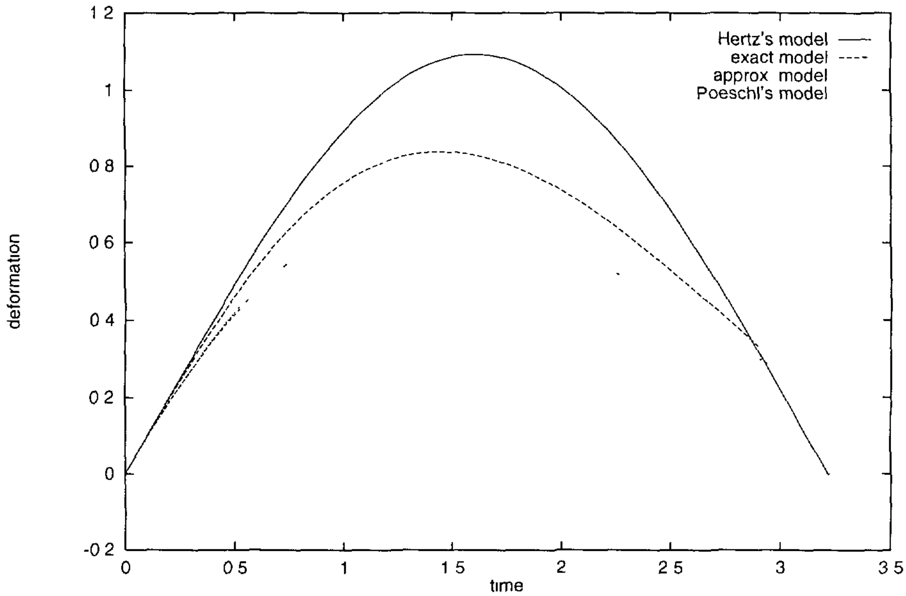
with a constant A_2 and an exponent α as fitting parameters. As above, h denotes the compression We have abbreviated $A_1 = \dfrac{2}{3}\dfrac{\sqrt{\overline{R}}E}{\mu(1-\nu^2)}$ With respect to the analytical integrability of this equation, thus only motivated by mathematical convenience, Pöschl took $\alpha = 2$ and obtained a dependence of h (h) in form of a power series. We have examined also the behaviour of the solution of equation (40) for other exponents, namely $\alpha = 3/2$ [15]. This choice can be motivated heuristically by arguing that the curved shape of the surfaces in contact would affect the dependence of the force on the time derivative of the deformation in a similar way as the one on the deformation itself.

For numerical convenience we scaled the differential equation (25) by a characteristic length l_0 and a characteristic time t_0 according to $h = l_0 x$ and $t = t_0 \hat{t}$. The scaling values l_0 and t_0 have been determined using the results of Hertz's theory for a non–dissipative collision of two spheres with the Young modulus $E = E_1 = E_2$ and the Poisson ratio $\nu = \nu_1 = \nu_2$. For these calculations we have taken $g_N = 1$ cm/s, which has been a characteristic impact velocity in the experiments [2,3] In the following the scaled velocities are labeled by V_N

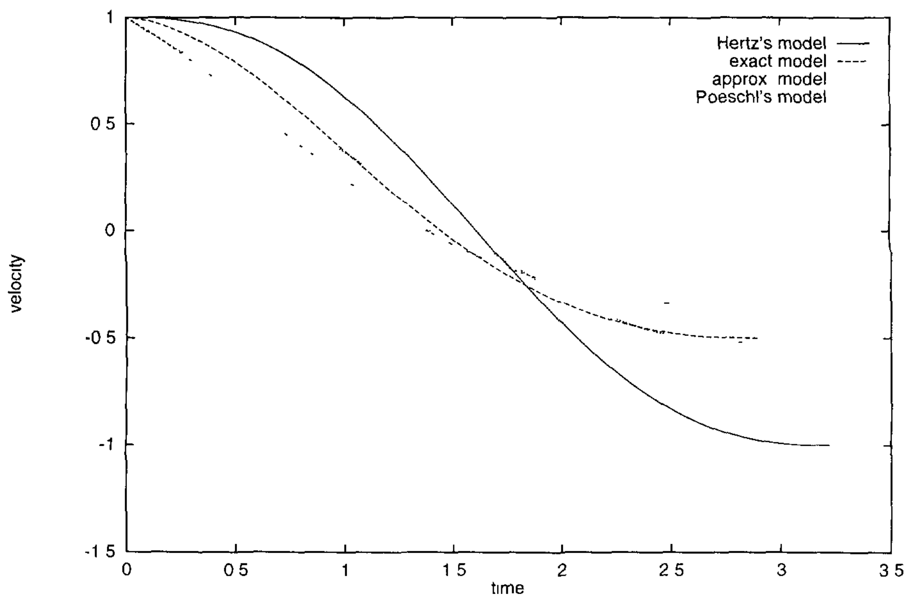
With these scalings and assuming typical values for ice at low temperatures [18]. the Young modulus $E \approx 10$ GPa, the mean density $\langle\rho\rangle \approx 10^3$ kg m^{-3}, a Poisson ratio $\nu \approx 0\,3$, and a typical particle size $\bar{R} \approx 10^{-2}$ m, one obtains the scaling values $l_0 \approx 10^{-4}\bar{R}$ and $t_0 \approx 10^{-4}$ s and the constant $A_1 \approx 1$. The above values of the material constants are valid for temperatures of about -35 °C which is quite too high compared with the conditions in planetary rings This raises the necessity of extended investigations of the material properties of ice at low temperatures, and thus, the above values have to be considered as approximations

Analogous scalings were made for the differential equations of the type (40). The problem is that it is not always easy to relate the constant A_2 to the physical properties of the material except in the case of $\alpha = 3/2$ where this is possible at least concerning the dimension

In Figure 1a the results for the time dependence of the deformation are shown for an initial (impact) velocity of $V_N = 1$ for these different models and compared with the results of Hertz's model. Using the above material parameters, the constants take the values of $A_1 = 1$ and $A_2 = 0\,7$ Figure 1b shows the time dependence of the relative velocity. We remember our discussion



a)



b)

Fig 1 — a) Time dependence of the deformation for an impact velocity of 1 cm/s Solid line Hertz's model (elastic collision) Long dashes Solution of equation (25) Short dashes Solution of equation (30) Dots Poschl's model b) Time dependence of the deformation velocity for an impact velocity of 1 cm/s Solid line Hertz's model (elastic collision). Long dashes Solution of equation (25) Short dashes Solution of equation (30) Dots Poschl's model of the collision process in the context of equation (26) which is very well illustrated by these figures It can easily be seen that the presence of the dissipative terms leads to a more efficient deceleration in the first and to a lower acceleration in the second stage of the collision This is the case for all three models which have been examined At the end of an inelastic collision which is reached when the acceleration has dropped to zero, a certain permanent deformation remains. These, of course, are common features of all three models Differences are visible especially in the first stage of the collision, in particular in the time dependence of the relative velocity (Fig 1b). The curve which we have obtained from equation (25) shows a behaviour similar to the one of Hertz's model: the deceleration is very low at the beginning of the collision, but increases later In contrary, equations (40) (with $\alpha = 2$) and (30) lead to curves where the deceleration already has a significant value at the beginning of the collision and then undergoes only little changes over a quite long time range This indicates that different models might be suitable for different types of material of the colliding bodies although the behaviour in the latter case seems to be quite unconventional for most materials

The dependence of the restitution coefficient ϵ_N on the impact velocity g_N is shown in Figures 2a-c where we compare our numerical results with the experimental ones [2, 3]. Here the constants have been chosen in a way that the best fit to the experimental curves is achieved.

We find that the results of our model - with the forces given by equations (25) and (30) - agree very well with the experimental curves obtained for frost covered ice spheres, notwithstanding the abovementioned different behaviour in the first stage of the collision. Also the model with $\alpha = 3/2$ shows a quite good agreement [15]. Pöschl's model ($\alpha = 2$) gives decreasing values of the restitution coefficient with increasing velocity, which is qualitatively right, but doesn't provide a good agreement with these experimental curves. We have found that the choice $\alpha = 1$ leads to results in complete contradiction to the experiments However, in a previous work [15] we had already demonstrated that no matter how the constants are chosen the best fit for an equation of the type (40) is not reached for both exponents being equal to 3/2 but for somewhat different values This might indicate the presence of mixed terms in both h and h. Thus, our model seems to be preferable It shows the best agreement with the experimental results for frost covered objects, and it has the advantage that only the two constants A_1 and A_2 have to be estimated.

One problem in this comparison (already mentioned in [11]) is the following: since in the experiments the ice balls were mounted on an apparatus, the dependence of ϵ_N on the particle mass could not be estimated If this dependence is known, we are able to estimate unknown material constants, in particular the viscosity, from fitting our curves to the experimental ones. Here we can only estimate the order of the magnitude With the above values of A_1 and A_2 we are led to viscosities of about 10 MPa s [15]. This quite low value indicates a really soft surface like a frost layer Apparently, the main processes of the collision take place in this relatively thin layer This is plausible regarding the low collision velocities and small masses of the bodies in the experiments An improved knowledge of the material constants which can only be obtained by new measurements will allow more conclusive statements

Surprisingly, our model does not seem to work for ice spheres which were designed to have very clean surfaces where an exponential fit instead of a power law had been found [3]. This indicates that other than purely viscoelastic phenomena (on which our model is based) play a role in these collisions However, a curve obtained from Pöschl's model with $\alpha = 2$ shows in this case not a perfect, but a better agreement than the ones we got from our model

In the experiments [3] a compression of the initial frost layer was observed, which caused the surface of the particles to smooth after some collisions, and a saturation effect was found in the dependence of ϵ_N on the number of collisions a particle had survived. Because in these

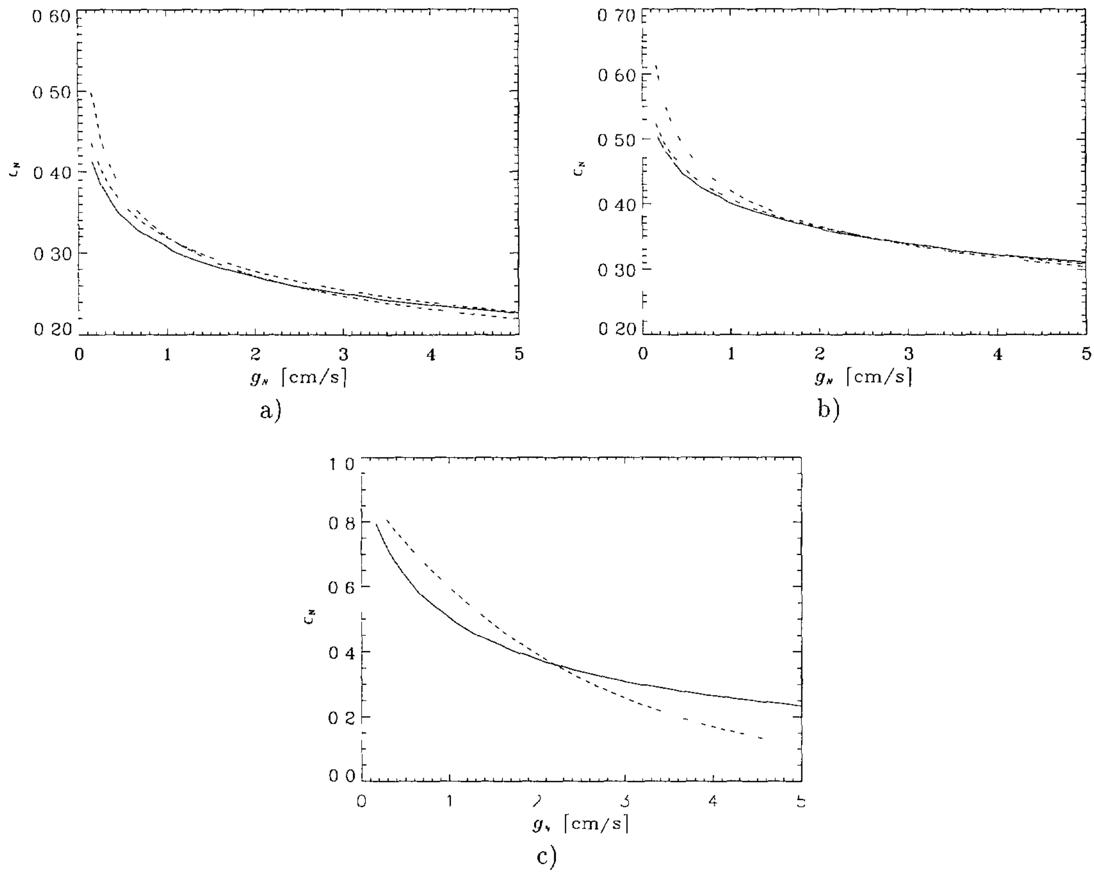


Fig 2 — a) Restitution coefficient ϵ_N *vs.* impact velocity g_N - comparison of our model with the experiments made with frost covered spheres [2] Dash-dotted experimentally obtained curve Dashes curve obtained using equation (30). Solid line curve obtained using equation (25) b) Restitution coefficient ϵ_N *vs* impact velocity g_N - comparison of our model with the experiments made with frost covered spheres [3] Dash-dotted experimentally obtained curve Dashes curve obtained using equation (30) Solid line curve obtained using equation (25) c) Restitution coefficient ϵ_N *vs* impact velocity g_N - comparison of Poschl's model with the experiments performed with clean, smooth spheres [3] Dashed experimentally obtained curve Solid line Poschl's model with $A_2 = 0\ 75$

experiments the particles hit the ice brick only with one and the same side, we believe that this behaviour is in reality more complicated. In particular, continued brittle fracture is likely to produce a quite soft surface layer which, in turn, can protect the bulk material from further destruction Thus, we do not consider this case to be relevant for our applications.

In addition, the surfaces of particles in natural systems are usually not as clean as it is possible to produce in a laboratory. They may be covered with mineral dust or other material. In particular, particles in planetary rings which have already survived many collisions from several sides during the long existence of the system are unlikely to have smooth surfaces.

4. Conclusions

In the present study we have developed a theoretical model for the collision of viscoelastic particles. We have used a quasistatic approximation which corresponds to the case where the collision velocity is much lower than the sound speed in the material The expression for the total viscoelastic force is then a generalization of the well-known Hertz relation [1, 13] for the elastic contact problem. An explicit relation has been obtained for the force acting between spherical colliding particles. The general case of the collision of particles of arbitrary shape may be mapped onto the collision problem for spherical particles The theoretical results for the restitution coefficient have been compared with experimental data for spherical icy particles [2, 3].

Our model is based on simple viscoelastic phenomena It provides a very satisfying agreement with experimental estimations of the restitution coefficient of frost covered ice spheres The low value of the viscosity which could be estimated approximately from our theory agrees well with the assumption, that the main processes of the collision take place in this quite soft layer, whose properties may resemble to a fluid. We assume that this case is most relevant for applications towards the evolution of planetary rings, because during the long time of the formation of these systems the particles have already survived many collisions which may have fractured their surfaces. This and the fact, that the ring particles seem to be covered with dust, will contribute to surface properties similar to those of the frost layer in the experiments.

The worse agreement with the experimental results for spheres which were designed to have very clean surfaces can be explained by the experience that ice - especially at very low temperatures - is a very brittle material. Consequently, instead of viscoelasticity, fracturing processes will govern these collisions even at low impact velocities. This is likely to produce fractal surfaces Preliminary investigations [19] have shown that the model according to equation (40) is more suitable for this type of surfaces

The comparison with the results of the different experiments shows that the processes governing the collisions of icy objects found e.g in planetary rings seem to be more complicated than pure viscoelasticity We can therefore state that the collisional behaviour in normal direction is influenced not only by the properties of the bulk material, but also by the properties of the surfaces. This should be proved by a microscopic examination of ice spheres used in such experiments.

It is also necessary to estimate the influence of the mass of the particles on the outcome of the collisions, and we think that new experiments, perhaps using particles falling in an evacuated tube, can help to clarify these problems.

A comparison should also be made with experiments with bodies consisting of other material (e g. metal) with known viscous and elastic properties. Furthermore, the material constants of ice at low temperatures as in the experiments should be estimated

Topics of future theoretical interest are the physical meaning of the constants and exponents in equations of the type (40) which may be more suitable for the treatment of bodies consisting of special materials or having particular surface geometries and the solution of the collision problem without the assumption of quasistationarity.

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