Fully Developed Turbulence: A Unifying Point of View

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J. Phys II France **5** (1995) 895–899

Classification
Physics Abstracts
47.27Gs

Short Communication

Fully Developed Turbulence: A Unifying Point of View

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(Received 9 February, accepted 5 April 1995)

Abstract. — Various approaches have been proposed for understanding velocity statistics in fully developed turbulence. Two recent papers focused on the distribution of dissipation. This note shows the connection between these ideas and previous work and their complementary character.

Recently several letters [1–3] appeared on a new development in isotropic 3D turbulence. She and Levêque [1] postulated the existence of a hierarchical relation between the moments of the averaged dissipation on scale ℓ, ε_ℓ:

$$\frac{< \varepsilon_\ell^{p+1} >}{\varepsilon\ell\infty < \varepsilon_\ell^{p} >} = A_p \left(\frac{< \varepsilon_\ell^{p} >}{\varepsilon\ell\infty < \varepsilon_\ell^{p-1} >} \right)^{\beta'} \quad (1)$$

where $\varepsilon_{\ell\infty}$ is the "maximum" value of ε_ℓ, and the A_p are some non-universal constants (β' is called β in Refs. [1–3] but we have to avoid a confusion later in the paper). They postulated this hierarchy to be due to a "hidden symmetry" of the Navier-Stokes equation. Dubrulle [2] and She and Waymire [3] later showed that hypothesis (1) is equivalent to assuming a special shape for the distribution of ε_ℓ. Namely, introducing $Y = \ln(\varepsilon_\ell/\varepsilon_{\ell\infty})/\ln\beta'$, the distribution $F_\ell(Y)$ is:

$$F_\ell(Y) = \int \Psi_\ell(Z) G(Y-Z) \mathrm{d}Z \quad (2)$$

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where $\Psi_\ell(Z)$ is a Poisson distribution whose parameter $\lambda(\ell) =< Z >$ contains the whole dependence with ℓ, and G is an ℓ independent probability distribution, which fixes the constants A_p.

The purpose of this note is to put these ideas in the context of more ancient works to stress the progress. In a series of papers [4–6] the probability density functions of velocity differences δv_ℓ at scale ℓ have been shown to follow:

$$\frac{1}{\sigma_\ell} P_\ell \left(\frac{\delta v_\ell}{\sigma_\ell} \right) = \int T_{\ell,L} \left(\ln \frac{\sigma}{\sigma_\ell} P_L \right) \left(\frac{\delta v_\ell}{\sigma} \right) \mathrm{d} \ln \sigma \tag{3}$$

where σ_ℓ is a normalising velocity, and L the large (integral) scale.

Defining $Y_\ell = \dfrac{3 \ln |\delta v_\ell|}{\ln \beta'}$, the distribution F_ℓ of Y_ℓ is:

$$F_\ell(Y_\ell) = \int T_{\ell,L}(Z) F_L(Y_\ell - Z) \mathrm{d}Z \tag{4}$$

where

$$Z = \frac{3 \ln(\sigma/\sigma_\ell)}{\ln \beta'}.$$

If the statistical equivalence between ε_ℓ and $|\delta v_\ell^3|$ is assumed, like in [2] and [1], there is therefore equivalence between equations (4) and (3) and equation (2) provided the distribution G is identified as F_L and $T_{\ell,L}$ is taken as a Poisson distribution. Conversely equation (3) has the great advantage to avoid any hypothesis on the statistical equivalence between $|\delta v_\ell^3|$ and ε_ℓ. It even avoids the use of the quantity ε_ℓ which is ill-defined: local dissipation or local energy transfer rate depending on the authors. Recent experimental work [6] which gives some hint for a correct definition of ε_ℓ, also shows that previous approximate definition were wrong.

As pointed out in reference [7], the fact that equation (3) is a linear relation between P_ℓ and P_L is of fundamental importance. Linearity is a necessary condition for the relation to be universal. Following again reference [7] it implies that the scale L should play no peculiar role and that, for any scale $\ell_1 > \ell$:

$$\frac{1}{\sigma_\ell} P_\ell \left(\frac{\delta v_\ell}{\sigma_\ell} \right) = P_{\ell \ell_1} \left(\ln \frac{\sigma'}{\sigma_\ell} \right) \frac{1}{\sigma'} P_{\ell_1} \left(\frac{\delta v_\ell}{\sigma'} \right) \mathrm{d} \ln \sigma' \tag{5}$$

Using for P_{ℓ_1} its expression given by equation (3), we see that:

$$T_{\ell L} = T_{\ell \ell_1} \otimes T_{\ell_1 L}$$

where $\otimes$ stands for the convolution product. Then for any series of scales $\ell_0 = L > \ell_1 > ..., > \ell_n$ we have:

$$T_{\ell_n L} = T_{\ell_n \ell_{n-1}} \otimes ... \otimes T_{\ell_1 L}$$

This defines a kind of cascade without any direct reference to energy or to scale invariance, and with an obvious arbitrariness in the number of steps n. Now, we can say that the cascade is self-similar if it is possible to choose the sequence of ℓ_i such that all the distributions $T_{\ell_i \ell_i - 1}$ are equal to the same distribution H [7]. This could be impossible if the physics of the correspondence between scales were completely different for small and large scales. However we do not assume scale invariance and the ratio l_{i-1}/ℓ_i have not to be equal.

Then $T_{\ell_n L} = H^{\otimes n}$ and the distribution T are infinitely divisible as remarked by She and Waymire [3]. The sequence ℓ_i defines a function $n(\ell)$ ($n(\ell_i) = i$) which acts as the parameter of the infinitely divisible distributions T. We can write :

$$\int e^{px} T_{\ell,L}(x) dx = \left(\int e^{px} H(x) dx\right)^{n(\ell)}$$
$$= \exp\left(n(\ell)\xi(p)\right)$$

Therefore, the function $\xi(p)$ contains all the information about the shape of $T_{\ell,L}$ and is the real object of the exciting ideas raised by She *et al.* [1,3], and Dubrulle [2]. For instance, defining $S_p(\ell) =< |\delta v\ell|^{3p} >$ and:

$$y(p) = \ln \frac{S_p(\ell) S_{p-1}(\ell')}{S_{p-1}(\ell) S_p(\ell')}$$

one can look, like Ruiz Chavarria *et al.*, [8], at the slope of $y(p+1)$ *versus* $y(p)$. It is:

$$\beta'(p) = \frac{\partial}{\partial p}\left(\xi(3p+3) - \xi(3p)\right) / \frac{\partial}{p}\left(\xi(3p) - \xi(3p-3)\right)$$

If $T_{\ell L}$ is a Gaussian in $\ln\sigma, \beta'(p) = 1$. If β' is constant $T_{\ell L}$ is a Poisson distribution. In any case $\beta'(p)$ is independent from ℓ if the cascade is self-similar in the above sense. This is part of the "hidden symmetry" mentioned by She *et al.* The second part (constancy of β' with p) could be linked with a scale covariance of the Navier-Stokes equations [2,9].

On the other hand, all the dependence of the distribution of δv_ℓ with ℓ is contained in the function $n(\ell)$. It can be measured, within a constant factor, by looking at the variance of $T_{\ell,L}$ [5,10,11]. This variance is proportional to $n(\ell)$ (as any cumulant of $T_{\ell,L}$) and thus measures the depth of the cascade. This function $n(\ell)$ is the real scaling parameter, playing the role of $\ln \frac{L}{\ell}$ when the Reynolds number is finite and the scale invariance is not ensured. In reference [12] it is suggested that $\ln < |\delta v_\ell^3| >$ can also play this role down to the dissipative scale. It would imply a proportionality between these two quantities. Let us examine this possibility.

In references [5,9–11] it is shown that $n(\ell)$ behaves as a power law on ℓ. It can be approximated by the relation:

$$n(\ell) \propto \left(\frac{L}{\ell}\right)^\beta - 1 \tag{6}$$

where the exponent β, introduced in references [10] and [5] must not be confused with the quantity introduced recently by She and Lévêque, which is called β' in this paper.

In reference [12] it is observed that all quantities $\ln < |\delta v_\ell|^p >$ behave as linear functions of each other down to the dissipative scale. Assuming that $< |\delta v_\ell^3| >$ is proportional to $< \delta v_\ell^3 >$, the Kolmogorov relation:

$$< \delta v_\ell^3 >= -\frac{4}{5} < \varepsilon > \ell + 6v\frac{\partial}{\partial \ell} < \delta v_\ell^2 > \tag{7}$$

allows us to determine $< |\delta v_\ell^3| >$ with

$$< |\delta v_\ell^3| >= \left(\frac{< \delta v_\ell^2 >}{C_2}\right)^{1/\zeta_2} \tag{8}$$

Then we have a contradiction as $\ln[< |\delta v_\ell^3| > / < |\delta v_L^3| >]$ cannot be proportional to $n(\ell)$ given by equation (8) while solving equations (9) and (10). However this contradiction cannot

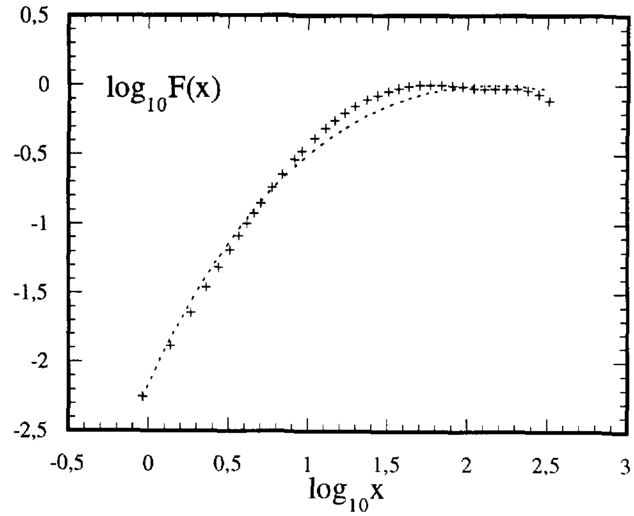

Fig. 1. — Comparison between the function $F(x) = L < |\delta v_\ell^3| > / \ell < |\delta v_L^3| > (x = \ell/\eta)$ measured in [12] for a turbulence behind a cylinder at $Re = 18000$ (crosses) and computed from formula (9), with $\beta = 1\ 2/\ln(L/\eta)$ (dotted line).

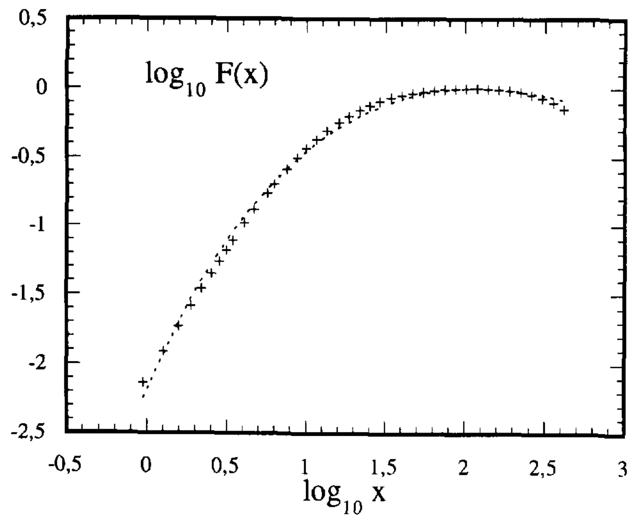

Fig. 2. — Same as in Figure 1 for an experiment at $Re = 9000$.

be taken too seriously: first it is obtained through several approximations. Second $< |\delta v_\ell^3| >$ obtained through equation (8) provides a fit to lower Reynolds number experiments of [12] as good as that obtained using Kolmogorov equation (Figs. 1 and 2).

In conclusion this note shows that the problem of the so-called intermittency in developed turbulence can be split into two distinct ones. The first is the evolution of the energy cascade by the behaviour of its "depth" or "number of steps" $n(\ell)$ *versus* the scale ℓ. This is well documented and raises interesting developments [5–7, 10, 11]. The second is the shape of the "elementary" process which connects a scale with the following one, the information on which is contained in $\xi(p)$. This only begins to be studied for itself.

Acknowledgments

The authors thank G. Ruiz-Chavaria and S. Ciliberto for communicating results prior to publication.

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