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Short Communication

Fully Developed Turbulence: A Unifying Point of View

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Abstract. — Various approaches have been proposed for understanding velocity statistics in fully developed turbulence. Two recent papers focused on the distribution of dissipation. This note shows the connection between these ideas and previous work and their complementary character.

Recently several letters [1–3] appeared on a new development in isotropic 3D turbulence. She and Levêque [1] postulated the existence of a hierarchical relation between the moments of the averaged dissipation on scale \( \ell , \varepsilon_\ell \):

\[
\frac{<\varepsilon_\ell^{p+1}>}{\varepsilon_{\ell \infty} <\varepsilon_\ell^p>} = A_p \left( \frac{<\varepsilon_\ell^p>}{\varepsilon_{\ell \infty} <\varepsilon_\ell^{p-1}>} \right)^{\beta^*} \tag{1}
\]

where \( \varepsilon_{\ell \infty} \) is the “maximum” value of \( \varepsilon_\ell \), and the \( A_p \) are some non-universal constants (\( \beta^* \) is called \( \beta \) in Refs. [1–3] but we have to avoid a confusion later in the paper). They postulated this hierarchy to be due to a “hidden symmetry” of the Navier-Stokes equation. Dubrulle [2] and She and Waymire [3] later showed that hypothesis (1) is equivalent to assuming a special shape for the distribution of \( \varepsilon_\ell \). Namely, introducing \( Y = \ln(\varepsilon_\ell/\varepsilon_{\ell \infty})/\ln \beta^* \), the distribution \( F_\ell(Y) \) is:

\[
F_\ell(Y) = \int \Psi_\ell(Z)G(Y - Z)dZ \tag{2}
\]
where $\Psi_\ell(Z)$ is a Poisson distribution whose parameter $\lambda(\ell) = \langle Z \rangle$ contains the whole dependence with $\ell$, and $G$ is an $\ell$ independent probability distribution, which fixes the constants $A_p$.

The purpose of this note is to put these ideas in the context of more ancient works to stress the progress. In a series of papers [4–6] the probability density functions of velocity differences $\delta v_\ell$ at scale $\ell$ have been shown to follow:

$$\frac{1}{\sigma_\ell} P_\ell \left( \frac{\delta v_\ell}{\sigma_\ell} \right) = \int T_{\ell,L} \left( \ln \frac{\sigma}{\sigma_\ell} P_L \right) \left( \frac{\delta v_\ell}{\sigma_\ell} \right) d \ln \sigma$$  \hspace{1cm} (3)

where $\sigma_\ell$ is a normalising velocity, and $L$ the large (integral) scale.

Defining $Y_\ell = \frac{3 \ln |\delta v_\ell|}{\ln \beta'}$, the distribution $F_\ell$ of $Y_\ell$ is:

$$F_\ell(Y_\ell) = \int T_{\ell,L}(Z) F_L(Y_\ell - Z) dZ$$  \hspace{1cm} (4)

where

$$Z = \frac{3 \ln (\sigma/\sigma_\ell)}{\ln \beta'}.$$

If the statistical equivalence between $\epsilon_\ell$ and $|\delta v_\ell^2|$ is assumed, like in [2] and [1], there is therefore equivalence between equations (4) and (3) and equation (2) provided the distribution $G$ is identified as $F_L$ and $T_{\ell,L}$ is taken as a Poisson distribution. Conversely equation (3) has the great advantage to avoid any hypothesis on the statistical equivalence between $|\delta v_\ell^2|$ and $\epsilon_\ell$. It even avoids the use of the quantity $\epsilon_\ell$ which is ill-defined: local dissipation or local energy transfer rate depending on the authors. Recent experimental work [6] which gives some hint for a correct definition of $\epsilon_\ell$, also shows that previous approximate definition were wrong.

As pointed out in reference [7], the fact that equation (3) is a linear relation between $P_\ell$ and $P_L$ is of fundamental importance. Linearity is a necessary condition for the relation to be universal. Following again reference [7] it implies that the scale $L$ should play no peculiar role and that, for any scale $\ell_1 > \ell$:

$$\frac{1}{\sigma_\ell} P_\ell \left( \frac{\delta v_\ell}{\sigma_\ell} \right) = P_{\ell_1} \left( \ln \frac{\sigma_1}{\sigma_\ell} \right) \frac{1}{\sigma_1} P_{\ell_1} \left( \frac{\delta v_\ell}{\sigma_1} \right) d \ln \sigma_1$$  \hspace{1cm} (5)

Using for $P_{\ell_1}$ its expression given by equation (3), we see that:

$$T_{\ell L} = T_{\ell_1} \otimes T_{\ell_1 L}$$

where $\otimes$ stands for the convolution product. Then for any series of scales $\ell_0 = L > \ell_1 > \ldots, > \ell_n$ we have:

$$T_{\ell_n L} = T_{\ell_n \ell_{n-1}} \otimes \ldots \otimes T_{\ell_1 L}$$

This defines a kind of cascade without any direct reference to energy or to scale invariance, and with an obvious arbitrariness in the number of steps $n$. Now, we can say that the cascade is self-similar if it is possible to choose the sequence of $\ell$, such that all the distributions $T_{\ell_\ell, \ell_{-1}}$ are equal to the same distribution $H$ [7]. This could be impossible if the physics of the correspondence between scales were completely different for small and large scales. However, we do not assume scale invariance and the ratio $\ell_{-1}/\ell$, have not to be equal.

Then $T_{\ell_n L} = H^{\otimes n}$ and the distribution $T$ are infinitely divisible as remarked by She and Waymire [3]. The sequence $\ell_i$ defines a function $n(\ell) \ (n(\ell_i) = i)$ which acts as the parameter of the infinitely divisible distributions $T$. We can write:
\[
\int e^{px} T_{l,L}(x) dx = \left( \int e^{px} H(x) dx \right)^{n(\ell)} = \exp(n(\ell) \xi(p))
\]

Therefore, the function \(\xi(p)\) contains all the information about the shape of \(T_{l,L}\) and is the real object of the exciting ideas raised by She et al. [1,3], and Dubrulle [2]. For instance, defining \(S_p(\ell) = \langle |\delta u|^p \rangle\) and:

\[
y(p) = \ln \frac{S_p(\ell) S_{p-1}(\ell')}{S_{p-1}(\ell) S_p(\ell')}
\]

one can look, like Ruiz Chavarria et al., [8], at the slope of \(y(p+1)\) versus \(y(p)\). It is:

\[
\beta'(p) = \frac{\partial}{\partial p} (\xi(3p + 3) - \xi(3p)) / \left( \frac{\partial}{\partial p} (\xi(3p) - \xi(3p - 3)) \right)
\]

If \(T_{l,L}\) is a Gaussian in \(\ln \sigma, \beta'(p) = 1\). If \(\beta'\) is constant \(T_{l,L}\) is a Poisson distribution. In any case \(\beta'(p)\) is independent from \(\ell\) if the cascade is self-similar in the above sense. This is part of the “hidden symmetry” mentioned by She et al. The second part (constancy of \(\beta'\) with \(p\)) could be linked with a scale covariance of the Navier-Stokes equations [2,9].

On the other hand, all the dependence of the distribution of \(\delta v_t\) with \(\ell\) is contained in the function \(n(\ell)\). It can be measured, within a constant factor, by looking at the variance of \(T_{l,L}\) [5,10,11]. This variance is proportional to \(n(\ell)\) (as any cumulant of \(T_{l,L}\)) and thus measures the depth of the cascade. This function \(n(\ell)\) is the real scaling parameter, playing the role of \(\ln \frac{L}{\ell}\) when the Reynolds number is finite and the scale invariance is not ensured. In reference [12] it is suggested that \(\ln \langle |\delta v_t|^p \rangle\) can also play this role down to the dissipative scale. It would imply a proportionality between these two quantities. Let us examine this possibility.

In references [5,9–11] it is shown that \(n(\ell)\) behaves as a power law on \(\ell\). It can be approximated by the relation:

\[
n(\ell) \propto \left( \frac{L}{\ell} \right)^\beta - 1
\]

where the exponent \(\beta\), introduced in references [10] and [5] must not be confused with the quantity introduced recently by She and Lévéque, which is called \(\beta'\) in this paper.

In reference [12] it is observed that all quantities \(\ln \langle |\delta v_t|^p \rangle\) behave as linear functions of each other down to the dissipative scale. Assuming that \(\langle |\delta v_t|^3 \rangle\) is proportional to \(\langle |\delta v_t|^2 \rangle\), the Kolmogorov relation:

\[
\langle |\delta v_t|^2 \rangle = -\frac{4}{5} \varepsilon + 6\nu \frac{\partial}{\partial \ell} |\delta v_t|^2
\]

allows us to determine \(\langle |\delta v_t|^2 \rangle\) with

\[
\langle |\delta v_t|^2 \rangle = \left( \frac{\langle |\delta v_t|^2 \rangle}{C_2} \right)^{1/\zeta_2}
\]

Then we have a contradiction as \(\ln[\langle |\delta v_t|^2 \rangle / \langle |\delta v_t|^3 \rangle]\) cannot be proportional to \(n(\ell)\) given by equation (8) while solving equations (9) and (10). However this contradiction cannot
Fig. 1. — Comparison between the function $F(x) = L < |\delta v_1^2| > / \ell < |\delta v_2^2| > (x = \ell/\eta)$ measured in [12] for a turbulence behind a cylinder at $Re = 18000$ (crosses) and computed from formula (9), with $\beta = 1.2 / \ln(L/\eta)$ (dotted line).

Fig. 2. — Same as in Figure 1 for an experiment at $Re = 9000$.

be taken too seriously: first it is obtained through several approximations. Second $< |\delta v_2^2| >$ obtained through equation (8) provides a fit to lower Reynolds number experiments of [12] as good as that obtained using Kolmogorov equation (Figs. 1 and 2).

In conclusion this note shows that the problem of the so-called intermittency in developed turbulence can be split into two distinct ones. The first is the evolution of the energy cascade by the behaviour of its “depth” or “number of steps” $n(\ell)$ versus the scale $\ell$. This is well documented and raises interesting developments [5–7,10,11]. The second is the shape of the “elementary” process which connects a scale with the following one, the information on which
is contained in $\xi(p)$. This only begins to be studied for itself.

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**References**