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Short Communication

Singular instabilities (collapses) and multifractality of structure functions in turbulence

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Abstract. — It is argued that the multifractality of higher order structure functions arises as a result of local singular instabilities (collapses) in 3D-turbulence. The modified Landau approach (with poles) is used for studying supercritical instability in this case. For the simplest situation (pole of order two), the asymptotic behavior of the structure functions is investigated. The arguments are supported by comparison with experimental data from laboratory and geophysical experiments.

1. Introduction.

The internal intermittency of turbulent flow leads to the existence of subregions with self-generation of turbulence. In the subregions quasi-laminar motion becomes unstable and goes to chaos by one or another scenario. The properties of this chaos depend on the type of scenario. This phenomenon is one of the sources of multifractal behavior of turbulence [1, 2]. The well-known Landau approach [3] consists in the description of turbulence near the critical Reynolds number with taking into account the most unstable mode with complex frequency $\omega = \omega_1 + i\alpha_1/2$, here $\omega_1 \gg \alpha_1$. Fluctuations of the velocity field are decomposed as

$$u = A(t)g(x,t) \tag{1}$$

with amplitude

$$A(t) \sim \exp(\alpha_1 t/2 - i\omega_1 t) \tag{2}$$

When $\alpha_1 > 0$, representation (2) is valid only for small values of $t$. To analyze the nonlinear stability problem Landau proposed the following amplitude equation

$$\frac{d|A|^2}{dt} = \sum_{n=1}^{\infty} \alpha_n |A|^{2n} \tag{3}$$
under the assumption that the average over the time intervals smaller than $\alpha_1^{-1}$ and larger than $\omega_1^{-1}$ was taken. The simplest nonlinear situation, for small $|A|$, gives us

$$\frac{d|A|^2}{dt} = \alpha_1 |A|^2 + \alpha_2 |A|^4$$

(4)

with a nontrivial result for asymptotic amplitude ($t \to \infty$):

$$|A|^2 \to -\alpha_1/\alpha_2.$$  

(5)

Here $\alpha_1 > 0$ and $\alpha_2 < 0$ (supercritical regime). We investigate the local instabilities in stochastic media and rewrite the amplitude equation as

$$\frac{dA}{dt} = [...] + f(t),$$

(6)

where [...] denotes the Landau expansion in powers $A$ and $A^*$, ($^*$ is the complex conjugate), $f(t)$ is the external stochastic force from the turbulent environment. Let us assume that the correlation radius of "external" (turbulent) forces is smaller than typical times of dynamical problem, then (in the first approximation) $f(t)$ and $f^*(t)$ are Gaussian random functions with a mean value equal to zero:

$$\langle f(t)f^*(t') \rangle = 2\sigma^2 \delta(t-t')$$

(7)

We define a probability density for $I = |A|^2$ as

$$P_t = \langle \delta(|A|^2 - I) \rangle.$$  

(8)

After averaging over the times order of $\omega_1^{-1}$ and allowing equation (4) we obtain the following Fokker-Plank equation for $P_t$:

$$\frac{\partial P_t(I)}{\partial t} = -\frac{\partial}{\partial I}[(\alpha_1 I + \alpha_2 I^2)P_t(I)] + 2\sigma^2 \frac{\partial}{\partial I} \left[ I \frac{\partial P_t}{\partial I} \right]$$

(9)

The stationary solution has the form

$$P_\infty \propto \exp \left[ \frac{(\alpha_1 I + \alpha_2 I^2)}{2\sigma^2} \right] \Theta(I),$$

(10)

where $\Theta(I)$ is the Heaviside function.

The maxima of $P_\infty(I)$ correspond to the local stable states while the minima to unstable ones (see [4] and [5]). Figure 1 (continuous line) demonstrates the schematic dependence of $P_\infty$ on $I$. The original state, $I = 0$, is statistically unstable whereas the second state, $I_1 = -\alpha_1/\alpha_2$, stable (cf. with dynamical result (5)). Similarly, one can obtain the Fokker-Plank equation and its stationary solution for the general case, equation (3).

2. Singular instabilities (collapses).

It is known that the three-dimensional Euler equation has solutions which blow up within a finite time (see, for example, [6-12]). This collapse can be described as a singular case of supercritical instability. Let us use the singular expansion in amplitude equation (6) (and (3)), instead of the regular one. The simplest nontrivial case is the expansion with the pole of order two:

$$\frac{dI}{dt} = \sum_{n=1}^{\infty} \alpha_n I^n + \beta \frac{I}{I^2} + \gamma \frac{I}{I},$$

(11)
Indeed, in the vicinity of $I = 0$, we obtain (cf. with regular case (10))

$$P_\infty \propto \exp\left[-\frac{1}{2\sigma^2}(\beta I^{-1} + \gamma I^{-2}/2)\right] \Theta(I).$$

(12)

Figure 1 (dotes) shows the dependence of $P_\infty$ on $I$ for supercritical situation ($\beta < 0$ and $\gamma > 0$). In this case, $P_\infty(0) = 0$, this corresponds to the collapse of the original solution. In the regular supercritical situation, $P_\infty(0) \neq 0$, and function $P_\infty(I)$ has a local minimum at point $I = 0$ that corresponds to instability of the original solution ($I = 0$). This solution coexists with the stable secondary one ($I_1$) in the regular supercritical case (Fig. 1, continuous line), while in the singular supercritical situation we have the secondary solution only (since $P_\infty(0) = 0$). This solution ($I_1 = -\gamma/\beta$) is stable because $P_\infty(I)$ has its local maximum at $I = I_1$ (see condition (13)). If the poles of the orders more than 2 are taken into account more than one equilibrium states emerge ($a_0 = 0$ because of $P_\infty(I)$ divergency).

3. Higher order structure functions and local collapses.

The equilibrium states, defined above, are different from the Kolmogorov one [13]. The equilibrium states are determined by the condition that the right hand side of (3) or (11) equals to zero. Here the term $\Delta A^2/\Delta t$ plays the role of the mean input of the energy of fluctuations ($\varepsilon$), but in the Kolmogorov approach it is a governing parameter. In the vicinity of equilibrium $I_1$, $\varepsilon$ leads to zero which means that the Kolmogorov approach does not work here. There are two parameters, $\beta$ and $\gamma$, which can be candidates for the role of the governing parameter in this case. The condition for statistical stability of the equilibrium state $I_1$ is

$$\frac{d^2P}{dI^2} \propto -\beta^4/\gamma^3 < 0.$$  

(13)

As one can see from (13), statistical stability of the equilibrium state $I_1$ does not depend on the sign of parameter $\beta$ and is strongly dependent on parameter $\gamma$. i.e., parameter $\gamma$ controls statistical stability of equilibrium $I_1$. For this reason, it is reasonable to assume that $\gamma$ is the governing parameter for this equilibrium state.
Fig. 2. — Scaling exponent $\zeta_p$ of structure functions: ($\times$) - jet ($Re_\lambda = 852$, [14]) and ($\bullet$) - large wind tunnel in Modane, ($Re_\lambda = 2720$, [15]).

In the Kolmogorov approach, dimensional considerations lead to estimate [3, 13]

$$\Delta u \propto \varepsilon^{1/3} r^{1/3},$$

(14)

where $\Delta u = u(x + r) - u(x)$, whereas in the vicinity of equilibrium $I_1$ the same considerations lead to

$$\Delta u \propto \gamma^{1/7} r^{1/7}$$

(15)

(the dimension of parameter $\gamma$ is obtained from Eq. (11): $\gamma \propto [L]^6[T]^{-7}$).

Numerical experiments show that the collapses are localized in the space [7-11]. Thus scaling (15), unlike the Kolmogorov one (14), can be observed only in the local subregions. One can separate the subregions with the help of higher order structure functions. Indeed, the structure functions of order $p$ are

$$\langle (\Delta u)^p \rangle = \frac{\sum_{i=1}^{N} (\Delta_i u)^p}{N},$$

(16)

where $\Delta_i$ corresponds to the subregion with number $i$ (and scale $r$), while $N$ is the total number of these $r$-subregions. If, in the subregions with relatively large values of $\Delta u$, scaling (15) takes place, then for large $p$,

$$\langle (\Delta u)^p \rangle \propto r^{d+p/7},$$

(17)

where $d$ is the dimension of the space (or of the experimental signal) and $N \propto r^{-d}$. For usual experimental signals $d = 1$ (see below and Fig. 2). By introducing the exponent $\zeta_p$ via the relation [1]

$$\langle (\Delta u)^p \rangle \propto r^{\zeta_p}$$

(18)

and using relation (17), it is straightforward to obtain the following asymptotic representation

$$\zeta_p \approx d + p/7.$$  

(19)

The values of $\zeta_p$, obtained from two laboratory experiments [14] and [15] (at rather large Reynolds numbers), are shown in figure 2, where the straight lines are drawn for comparison with (19). As we can see, the asymptotic slopes of the experimental lines are approximately equal to $1/7$, and $d \approx 1$ (the signals are one-dimensional in these experiments).
4. Discussion.

There is a number of factors which prevent the collapses. The first one is, naturally, the viscous dissipation ([16] and [17]). The dimension of motion can also be one of these factors. The conservation laws prohibit the collapses in strictly 2D-turbulence. It leads to an interesting phenomenon in real quasi-2D-turbulence (as is known, strictly 2D-turbulence is unstable to 3D-disturbances). The 3D-instability can be realized as the regular topological instability (see, for example, [18]) and as the singular instability (the local collapse) because the prohibitive 2D-conservation laws are invalid for 3D-disturbances. These local 3D-instabilities lead to the appearance of blobs characterized by active mixing. If there are conditions for numerous appearances of such blobs, then the phenomenon controls large-scale diffusion of passive scalar. Then, in full analogy with the Kolmogorov approach [13], it follows from dimensional consideration that diffusivity, \( K_* \), has the following form

\[
K_* = \frac{1}{6} \frac{\text{d}l^2}{\text{d}t} \propto \gamma^{3/7} t^{8/7},
\]

where \( l \) is the characteristic scale of a cloud of the passive scalar (in the Kolmogorov case - \( K_* \propto \varepsilon^{1/3} l^{4/3} \) [13]). Expression (20) is different from both the case of 3D-turbulence (Richardson-Kolmogorov law) and from the case of purely 2D-turbulence.

From (20), we obtain

\[
l \propto t^{7/6}
\]

Relations (20) and (21) are in good agreement with the results of the experiments on diffusion of passive scalar in the troposphere [19] and ocean [20], as can be seen from figures 3 and 4.

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Fig. 3. — Observations of widths (horizontal standard deviation) of diffusing tracer as a function of downwind travel time. Different symbols correspond to the results of different authors. The figure is adapted from [19].

Fig. 4 — Diffusivity versus scale \( l \) in the ocean. Adapted from [20].
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