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Perfect plasticity in a random medium

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Abstract. — We study numerically a simple network model which is an electrical analog of a perfectly plastic disordered medium. For a uniform random distribution of local maximum current, (e.g. yield stress), we study the evolution of the properties of the system as a function of the number of elements which reached the plastic limit. It is shown that the macroscopic yield point can be interpreted as a critical point.

1. Introduction.

In the recent years, much attention has been paid to the role of heterogeneities in the properties of materials. In particular, when the local behavior is non-linear, disorder can lead to a macroscopic behavior different from those of the elementary constituents of the material. Brittle fracture in particular is naturally controlled by the presence of disorder, and new approaches, based on a statistical analysis, have lead to new results and new tools. A recent survey of this field is presented in reference [1].

The purpose of this paper is to analyse the role of disorder in a material where the local behavior can be assumed to be perfectly plastic. The simplicity of this characteristic, and piece-wise linear character allows to decompose the problem in a series of elementary linear problems. Those ‘tangent’ problems are very close to some brittle fracture models studied earlier. This correspondence will be discussed in more details in the following.

The point where the entire medium reaches a perfectly plastic behavior can also be mapped in two dimensions (resp. three dimensions) onto the problem of finding the optimum conformation of a polymer (resp. membrane) in a random medium [2, 3]. Most results originating from this field, the “minimal interface problem”, can thus be transposed to the question of plasticity. This correspondence has already been used to account for the evolution of the strength of porous plastic materials with the porosity [4]. A similar relation between the interface problem

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and brittle fracture has also been suggested in order to forecast the fracture path prior to fracture by Jeulin [5].

This problem is at the interface between the fracture problem where some results are mostly based on computer simulations, and the minimal interface problem in a random medium where in two dimensions a few exact results exist. Thus, it is a good candidate for being a key to a better understanding of these problems.

In particular, it has been suggested recently that cracks are self-affine [6]. This property is well known for directed polymers and using the correspondance (recalled hereafter) with plasticity, it is appealing to search for a relation.

However, generally speaking, these problems are very time-consuming when studied numerically, and thus, the first step in this approach is to simplify the problem inasmuch as the important scaling properties are unaffected by this simplification. In this spirit, we deal here exclusively with a two-dimensional network model which is the electrical analog of the perfect plasticity case. We recall in table I the correspondence used to define the problem. We will use freely in the following the terminology resulting from this correspondance.

Table I. — *Correspondence used in the definition of an electrical analog of a mechanical problem.*

Mechanical	Electrical	Notations
Force	Current	$I(\text{global})$ or $i(\text{local})$
Stress	Current density	I/L
Displacement	Voltage	$V(\text{global})$ or $v(\text{local})$
Strain	Electric field	V/L
Elastic modulus	Conductance	$G(\text{global})$ or $g(\text{local})$

Apart from the relation with other problems, this model is interesting *per se*. In particular, the macroscopic yield limit is the point where a continuous cluster of plastic bonds percolates for the first time across the network, (percolation is to be understood here in a loose sense, and not as a reference to percolation theory, since the distribution of plastic bonds in the network is far from being random). At this point, the tangent conductivity vanishes. Thus it is appealing to describe the evolution of the network within the framework of critical phenomena, with for instance, the voltage or the density of plastic element as a control parameter, and the tangent conductivity as an order parameter. We will present below a few results along these lines.

2. Model.

We consider a regular square lattice consisting of elements having the characteristic shown in figure 1. For a small voltage drop across the element, its behavior is ohmic with a constant conductance $g = 1$ (chosen to be identical for all bonds in the network). As the voltage reaches a threshold value, v_c , the current saturates to a value $i_c = gv_c$. The thresholds are distributed at random from a uniform distribution $p(v_c) = 1$ for $0 < v_c < 1$. The bonds of the lattice are oriented at $\pi/4$ with respect to the mean current direction. The lattice is of size $L \times L$. The voltage is imposed along two opposite borders, and periodic boundary conditions are set in the perpendicular direction. Figure 2 shows the geometry of the problem, as well as an example of the distribution of "plastic elements" at the very final stage of evolution.

Fig. 1. — Schematic characteristic of the elementary components of the lattice. For a small voltage drop v , the bond is ohmic, whereas for voltages larger than a given threshold the current saturates to a constant value i_g . The conductance of the ohmic regime is chosen to be identical for all elements in the lattice, but the threshold v_g is distributed at random from a uniform distribution between zero and one. This behavior constitutes the electrical analog of perfect plasticity.

Fig. 2. — A voltage drop is applied across the network from two opposite borders (shown in grey). In the horizontal direction periodic boundary conditions are implemented. The lattice is here shown twice so as to make clear those conditions. The bold lines represent the set of bonds which will support a larger current if the external voltage is increased. The thin lines schematize the bonds which are screened by the bonds which have reached their plastic limit. The latter are omitted on the picture for clarity. This state is the last stage before the network becomes completely plastic.

We start our simulation by applying a very small voltage to the system, in such a way that the entire network remains ohmic. At this stage, it is easy to compute which bond in the lattice will be the first to reach its threshold current. Due to the fact that the geometry of the lattice is regular, and that the conductances are all identical, this first element k_1 is the one for which the threshold v_c is minimum. Let us call $V^{(1)}$ the external voltage needed to reach this point. When the voltage is increased above $V^{(1)}$, the current flowing through the bond k_1 is now constant and thus independent of the external voltage increment above $V^{(1)}$. For all the other bonds, the conductance is still one. Thus before a second bond reaches its plastic limit, the *relative* variation of local voltages with respect to their value when $V = V^{(1)}$

is linear in $V - V^{(1)}$. This defines a "tangent" problem which is easy to understand: we have to solve for the potential distribution in the lattice where all ohmic bonds have a conductance of one, and all plastic bonds have a zero conductance (perfect insulators). In this tangent problem, plastic bonds can be seen as cracks. Let us call u_k the voltage drop across bond k in the latter problem, for an external voltage $V = 1$, and $v_k^{(1)}$ the voltage drop in the original network for $V = V^{(1)}$. As long as no other bonds reach their plastic limit, i.e., in the range $V^{(1)} < V < V^{(2)}$, we can write:

$$v_k = v_k^{(1)} + u_k (V - V^{(1)}) \quad (1)$$

As the voltage is increased, the second bond which reach its plastic limit satisfies $v_{kc} = v_k^{(1)} + u_k (V^{(2)} - V^{(1)})$. Since it should be the next to plastify, $V^{(2)}$ is thus the minimum over all bonds, of the quantity

$$V^{(2)} = \min_k \left\{ V^{(1)} + (v_{kc} - v_k^{(1)})/u_k \right\} \quad (2)$$

The evolution after V has reached this second point, $V^{(2)}$, is similar. The new tangent problem has an additional bond missing. The fact that the elementary characteristic is piecewise linear allows to define a series of tangent linear problems which may simply be superposed to construct the actual solution. The evolution is stopped as soon as the tangent conductivity has reached zero, i.e., when the network has reached a perfectly plastic stage. At this point there exist a continuous structure across the lattice which consists of plastic bonds. We will come back to the stucture of the network at this stage.

Each time a new bond becomes plastic, we have to solve Kirchhoff laws to find out the current distribution in the tangent lattice. This is done using a conjugate gradient algorithm with a precision of 10^{-8} . The lattice sizes used in these simulations range from 10 to 80, and the corresponding number of realizations are reported in table II.

Table II. — *Number of lattices generated in the simulations for different sizes. All data were averaged at a given lattice size, by keeping the number of plastic bonds fixed, or by selecting only the final stage of evolution.*

Size	Number of lattices
10	1000
15	500
20	500
30	50
40	50
60	10
80	2

The final stage of the network, where it reaches a perfectly plastic stage is of special interest, and thus we recorded a few global properties of the network at this stage and averaged over realizations. We also recorded the distribution of current in the entire network for the tangent problem, at the final stage. This distribution is averaged over all networks of a given size.

During the evolution, global as well as local quantities are explicitly computed as a function of the number of bonds which are plastic. All those quantities are averaged over different

realizations. The averaging is done by keeping fixed the number of plastic bonds. The recorded quantities are the voltage $V^{(n)}$ and the current $I^{(n)}$ flowing through the network at the point where the n th bond is becoming plastic, the tangent conductivity at this stage, as well as the number of bonds which carried a non-zero current (this set of bonds will be referred to as the "backbone" in the following). It has to be noted that the number of lattices over which each data is averaged is not constant: only those which are not plastic yet are taken into account, in contrast with the previous averaging procedure.

3. Results.

3.1 THE GLOBAL YIELD POINT. — The most important features of the problem are the properties of the network at the final stage of the network were it reaches the perfectly plastic state. At this point the current is I^* and the voltage V^* . Figure 3 shows the evolution of the current density and the electric field with the system size to the power $-2/3$ (this form will be justified below). We see clearly that these quantities converge toward a finite value when the system size increases.

Fig. 3. — Evolution of the current density I^*/L (×), and of the electric field V^*/L (□), at the yield point, as a function of the system size raised to the power $-2/3$, $L^{-2/3}$. Both of these quantities converge toward a constant as L goes to infinity.

$$\begin{aligned} I^*/L &\rightarrow j_\infty \\ V^*/L &\rightarrow v_\infty \end{aligned} \quad (3)$$

where $j_\infty \approx 0.42$ and $v_\infty \approx 0.52$.

It has been observed in brittle fracture using the same distribution of thresholds, that the density of broken bonds at the end of the process followed a power-law decrease with the system size with an exponent close to -0.25 [7]. A least squares regression on the log-log plot of the density of plastic bonds at the final state d_{plast}^* as a function of L provides the raw estimate $d_{\text{plast}}^* \propto L^{-0.07}$. A slight curvature in the data indicates however, that most likely, the density of plastic bonds converges toward a constant $d_{\text{plast}}^* \rightarrow 0.27 \pm 0.01$ as L increases. This result is markedly different from the brittle fracture case (see Fig. 4).

The distribution of plastic bonds within the lattice is far from being random. There are very strong shielding or screening effects, which result from the fact that a bond which is surrounded by plastic bonds cannot see its current increase. In the tangent lattice, such a bond will appear as a dead-end. Figure 2 exemplifies this situation clearly. We have drawn the lattice at the end of the process omitting the plastic bonds, and drawing in thin lines the screened bonds. The bonds in bold lines are the conducting part of the tangent lattice, the backbone. The density of the backbone converges to a constant as the system size increases, as shown in figure 4. This asymptotic density can be estimated to be approximately 0.45.

Fig. 4. — Density of plastic bonds, d_{plast} , ($\square$) and backbone density, d_{bb} , ($\blacksquare$) at the yield point of the lattice as a function of the inverse of the system size, $1/L$. These two densities converge toward a non-zero limit as the system size increases.

We also studied the mean tangent conductivity $G = dI/dV$ of the network just before the plastic point. Figure 5 shows in a log-log scale the evolution of this quantity. A straight line of slope -0.65 is reported on the graph for comparison. We see that such a power-law $G \propto L^{-0.65}$ provides a good fit to the numerical data.

Fig. 5. — Tangent conductance of the network at the yield point as a function of the system size in a log-log plot. The slope of the reference line is -0.65 .

An important observation in brittle fracture problems is the fact that the current distribution at the final stage is multifractal [7]. Thus the entire statistical distribution of current in the network can be described for any system size, through a size-independent function, $f(\alpha)$, the “multifractal spectrum”, where f and α are related to the distribution of current $\mathcal{N}(j)$ through $\alpha \equiv \log(j)/\log(L)$ and $f \equiv \log(\mathcal{N}(j))/\log(L)$ [8]. We have seen in the definition of the model that the quantity which should be compared to a local current in the fracture problem is the local current in the *tangent* plasticity problem, or the relative increase in the local current, j , due to an increase of the external current dj/dI . We calculated the distribution of these tangent local currents just before the yield point and then used the definition of the reduced variables introduced above, f and α for various system sizes. Figure 6 shows the resulting plot of $f(\alpha)$ for sizes 10 to 40. We clearly see from the figure that all data point collapse onto a single master curve, and thus these reduced variables do account for the size effect. We conclude that the tangent currents are distributed according to a multifractal distribution.

Fig. 6. — Multifractal spectrum of the tangent currents di/dI . Distributions were computed at the final stage of evolution, for different system sizes ($L = 10$ ($\diamond$); $L = 20$ ($+$); $L = 30$ ($\square$); $L = 40$ ($\times$)). The way to account for the change of this distribution due to the the system size is to introduce reduced variables as explained in the text.

3.2 DEVELOPMENT OF PLASTICITY. — After having given the size dependence of the global properties of the networks at the yield point, let us now consider the intermediate stages where the networks have not yet reached the perfectly plastic stage.

As can be seen from table II, the number of samples studied were smaller and smaller as the size of the system increases. Thus for large lattices, fluctuations are large, and thus in order to have a reasonable averaging, we will mainly consider in this section the data collected for $L = 40$.

As mentioned earlier, an average over different realizations was performed by keeping the number of plastic bond fixed. We will thus use the density of those bonds d_{plast} as the control parameter.

The current density-electric field characteristic is shown in figure 7. As expected, we see a progressive decrease of the tangent conductivity until the current saturates to a constant value. The tangent conductivity can be used as an order parameter so as to recast this model

Fig. 7. — Average current density *vs.* electric field characteristic of lattices of size 40.

in the framework of critical phenomena. Figure 8 gives this evolution as a function of d_{plast} . We expect that this behavior can be described by a critical behavior of the form

$$G \propto (d_{\text{plast}}^* - d_{\text{plast}})^t \quad (4)$$

where t is a critical exponent. Using the data relative to size 40, we search for such a relation. Considering the threshold d_{plast}^* as a free parameter, we looked for the best value of this parameter which allowed to recover the behavior of equation (4). The optimum values were found to be $d_{\text{plast}}^* \approx 0.285$ and $t \approx 0.7$. Figure 9 shows the log-log plot of G versus $(d_{\text{plast}}^* - d_{\text{plast}})$ for this choice of d_{plast}^* . The value of this threshold differs from the asymptotic estimate (0.27) as a result of finite size corrections. A very general finite size scaling argument can be used to obtain an additional piece of information. If the correlation length diverges at threshold as $\xi \propto (d_{\text{plast}}^* - d_{\text{plast}})^{-\nu}$, then the size evolution of the conductivity with L is

$$G \propto L^{-t/\nu} \quad (5)$$

at threshold. We have reported such a result above, with the estimate $t/\nu \approx 0.65$. Thus we deduce the correlation length exponent to be $\nu \approx 1.1$.

The number of screened bonds increases with d_{plast} . Figure 10 shows that this increase can be described by a power-law as $d_{\text{screen}} \propto d_{\text{plast}}^{2.3}$. Such a law can be justified if one assumes (1) that the distribution of clusters of plastic bonds (in the dual space) is a power-law, and (2)

Fig. 8. — Tangent conductivity G as a function of the density of plastic bonds d_{plast} for lattices of size 40.

that the clusters are fractal. However, in the course of the present study, we did not investigate those two hypothesis, and thus we simply point out this property.

3.3 CONNECTION WITH THE MINIMAL INTERFACE PROBLEM. — Let us recall the connection between the minimal interface problem and the plasticity model studied here.

Let us consider a d dimensional lattice, where each bond is assigned a random weight. To each $d - 1$ surface S which cut the lattice into two pieces, we can associate an 'energy' $E(S)$ as the sum of the random weight of the bonds which intersect the surface. The minimal interface problem consists in finding the surface S^* which minimizes the energy over all possible interfaces. In two dimensions, this problem has been solved exactly through a mapping onto a Burgers' equation with noise. Since the interface in this case is one-dimensional, the problem is often considered in the context of the conformation of a polymer in a random medium. The energy of the minimum interface per unit length converges to a constant as the length L increases, with a non-analytic correction term:

$$E(S^*)/L = e_0 + e_1 L^{-2/3} \quad (6)$$

for most distributions of weight (in particular all distributions where the weight is only positive). Other properties of this minimal interface are known, such as the self-affinity of the interface geometry characterized by a universal roughness exponent, .

In order to connect this problem to the plasticity case, let us consider an interface which parts the lattice into two pieces. The maximum current flowing through this interface is

Fig. 9. — Tangent conductivity as a function of the distance to threshold $d_{plast}^* - d_{plast}$ in a log-log plot. The straight line of slope -0.7 is the least squares linear regression over the intermediate range.

obviously limited by the sum of the maximum current of each bond cut by the surface, thus the 'energy' of the surface. Since the same inequality holds for any surface, the total current flowing through the network at the global yield point is less than the minimum $E(S^*)$.

We will show now that this upper bound is also a lower bound. Indeed, if we assume that I^* is strictly less than $E(S^*)$, for $I = I^*$, there does not exist any continuous interface which only crosses plastic bonds. Thus there exists at least one continuous path joining the two electrodes, which consists in bonds which have not yet reached their plastic limit. The tangent conductance is then larger than the conductance of this single path, and thus it is non-zero. The latter result is in contradiction with the starting hypothesis, $I = I^*$. This concludes the proof that $I^* = E(S^*)$. As a result we obtain a direct estimate of the asymptotic current density e_0 which had been estimated to be $e_0 \approx 0.43$ [3] in agreement with the present estimate (see Fig. 3). Moreover, figure 3 shows that the leading correction due to the finite size of the network obeys equation (6).

It is also possible to see the properties of these random interfaces as critical by studying the set of all interfaces whose energy is smaller than a given parameter. The critical point corresponding then to the value of the parameter for which one single interface is obtained. As the parameter approaches the critical value, it is straightforward to define correlation lengths which diverges. The critical exponents of these correlation length are dependent on their orientation; either parallel or perpendicular to the mean orientation of the interface. However,

Fig. 10. — Fraction of screened bonds d_{screen} as a function of the density of plastic bonds d_{plast} in a log-log plot. The slope of the straight line is 2.3.

away from threshold, it is no longer possible to find a one-to-one correspondence with the plasticity problem.

3.4 CONNECTION WITH FRACTURE. — We have seen the exact relation between the plasticity model and the minimal interface model. We now turn to a more qualitative similarity between the plasticity model and brittle fracture models.

The starting point of this analogy is to consider the evolution of the tangent problem. The latter can be seen as a network of fuses, in which cracks develop. The criterion for deciding which element will break next is very close to the one used for a fracture problem. To make the correspondence explicit we may define an “effective fracture threshold” for bond k as $u_{kc}^{(n)} = (v_{kc} - v_k^{(n)})$ after n bonds have been broken. The macroscopic voltage increase, $U^{(n+1)} = V^{(n+1)} - V^{(n)}$, is comparable to a breaking voltage, as result from the rewriting of equation (2)

$$U^{(n+1)} = \min_k \left\{ \frac{u_{kc}^{(n)}}{u_k} \right\} \quad (7)$$

Figure 11 shows a fracture characteristic constructed from the plastic ($I-V$) characteristic, by plotting the increment of current *versus* the increment of voltage between two consecutive stages. The shape of this characteristic is quite reminiscent from usual fracture problems.

The main difference with a usual fracture model, is the fact that the “effective fracture thresholds” of the bonds, $u_{kc}^{(n)}$, evolves as the fracture proceeds (it depends on n). This difference is important. Indeed, suppose that we study the fracture of a fuse network with a uniform

distribution of strength between 0 and 1. After a few bonds have been broken, the smallest threshold in the lattice gets larger and larger, (depletion effect), and thus the current and the voltage have to increase in order to break an additional bond. In our analogy, this is no longer true. Since the voltage reached at the previous step is subtracted from the plastic threshold, the smallest threshold in the lattice does not change drastically in the first steps of the process. This is apparent from the scale of the fracture characteristic shown in figure 11. The current and the voltage at the peak are of the order of the inverse system size.

Fig. 11. — Reconstructed fracture characteristic obtained through the increments of external current *vs.* the increments of global potential for lattices of size 40.

There exists one limit case for fracture where this depletion effect becomes negligible. It is in the unphysical limit where the distribution of initial thresholds approaches $p(u) \rightarrow 1/u$ close to zero. This limit can be described using percolation theory [9]. We recover in this limit that the density of broken bonds at the fracture point is finite, in agreement with the result obtained for plasticity.

The fact that the local voltage has to be subtracted from the yield threshold to give the plastic threshold implies a strong enhancement of the disorder as compared to the fracture case.

3.5 GENERAL DISTRIBUTION. — All along this work, we focused on a particular distribution of thresholds, i.e. a uniform distribution between zero and one. In this last section, we would

like to point out that the scaling results we have presented are in fact valid for much more general distributions.

This can be seen through the algorithm used to determine step by step which element will reach its yield threshold next. As developed in the previous section, we can only focus on the tangent problem, and see it as a fracture problem with a distribution of fracture threshold given by the difference between the yield threshold and the current flowing through the element at the previous time step. If the initial distribution of yield currents is continuous and non-zero over its support, then at any stage of the process, the distribution of "effective failure thresholds" will have a non-zero density close to zero. This property is enough to determine the scaling property of the fracture process and by integration, that of the plasticity problem.

Consequently, the existence of global yield limit which is independent of the system size, with a non-analytic finite size correction is a generic property of the problem. Similarly, the yield point is reached after a finite fraction of bonds have reached their thresholds.

It is interesting to note that the result concerning the non-zero macroscopic threshold can also be obtained using the correspondence with the random interface problem. Indeed in this framework, it has been shown by real space renormalization techniques, that for any distribution having a finite lower-bound, the energy of the minimal interface asymptotically scales as reported above (Eq. (6)). [10] The exceptions to this result are to be found for distributions extending to minus infinity as a power law [11]. Such cases are obviously irrelevant for our purpose.

4. Conclusions.

We have studied the development of plasticity in a disordered medium, through a numerical simulation of an electrical network analog. We have shown that the macroscopic yield current tends to a size independent limit, after a finite fraction of bonds have reached their plastic threshold. Close to the macroscopic yield current, the system can be described as approaching a critical point: the tangent conductivity of the system vanishes as a power-law of the system size, the tangent local currents are distributed according to a multifractal distribution. The threshold can be put in correspondence with a minimal interface problem. Finally, the relation with a brittle fracture problem and the case of more general distributions of local thresholds have been discussed.

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