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Formation of patterns induced by thermocapillarity and gravity

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Abstract. — Consider a liquid film on a slightly inclined plane driven by gravity and thermocapillarity. We derive an equation describing the nonlinear evolution of the interface between the liquid and the atmosphere. In line with an approach introduced by us elsewhere we preserve the full impact of the curvature that contributes toward formation of bubbles and together with convection induced by the inclined plane contributes toward either breaking of waves or prevention of rupture that otherwise will always occur. Travelling waves are also possible. A variety of possible equilibrium states is also discussed.

1. Introduction.

A liquid film at rest bounded on one side by a heated flat plane and on the other open to the atmosphere exhibits under certain conditions an unstable behavior. One of the sources of that instability is the temperature variation of the interfacial tension which leads to surface tractions. This so-called Marangoni, or the thermocapillary effect [1, 2], is important when the liquid film is thin or when the gravity is very low, such as in the extra-terrestrial conditions.

Experimental studies of the thermocapillary effect are well documented, references [2-4] and the references therein. The linear analysis of the Marangoni effect was presented in references [1, 5, 6] and the conditions for the onset of the instability were derived. Weakly nonlinear aspects of the phenomenon based on amplitude expansion were studied in references [7-12]. Those studies differ each other in asymptotic representations of the independent and the dependent variables lead to different evolution equations describing the behavior of the interface in different parametric regimes. However, no numerical studies of those equations were reported except for the overly simplified problem in reference [7].

The three-dimensional problem of a thermocapillary flow was treated in reference [10] using what might be best described as a boundary layer approach. An equation describing the
nonlinear evolution of the interface of the film supported by the cool plane was derived there
in the form
\[ h_r - \nabla \cdot \left[ \frac{Bh^2 \nabla h}{2(1+Bh)^2} + \frac{h^3}{3} \nabla (Gh - s \nabla^2 h) \right] = 0 \]  
(1)

where \( h \) is the location of the interface, \( B \) is the Biot number, \( G \) is the Bond number
characterizing the relation between the effects of gravity and the thermocapillarity and \( s \) is the
surface tension number measuring the relative importance of the surface tension and the
thermocapillarity. The derivation of equation (1) given in reference [10] implicitly uses the
assumption that in addition to the amplitude its spatial gradients are also small. However, it
turns out that the solutions of equation (1) do not necessarily obey that assumption.

While in this work we follow reference [10] with respect to the asymptotic scaling we extend
it in two ways. First, we allow for a slight plane inclination. This adds a convective part into the
dynamics. Second, we preserve the full effect of the interfacial curvature on surface
tension to derive an evolution equation
\[ h_r + \frac{G\beta_0}{3} (h^3)_x - \nabla \cdot \left[ \frac{Bmh^2 \nabla h}{2(1+Bh)^2} + \frac{h^3}{3} \nabla (Gh - sN \nabla^2 h) \right] = 0 \]  
(2)

being \( N = [1 + \epsilon^2 (\nabla h)^2]^{-3/2} \) and \( m = \pm 1 \). The parameter \( G \) is positive when the film is
supported from below and negative when it is bounded by a rigid plane from above and by the
ambient air flow below (negative gravity). These modifications will be shown to be important: each of them can contribute toward prevention of rupture which otherwise will
always occur.

We also carry out the numerical studies of equations (1), (2) to study the impact of different
physical mechanisms on the behavior of the interface.

Note the presence of \( \epsilon \), the perturbation scale, in \( N \). Though formally of higher order, \( (h_\epsilon = o(\epsilon^{-1}) \), see below), it enters into a sensitive balance between backward diffusion
induced by the destabilizing effect of the gravity (for negative gravity) and the stabilization
due to the surface tension. As shown later, preserving the full effect of curvature is under
certain conditions crucial and results in a dramatic change in the forming patterns. It will be
also shown that equation (2) exhibits a rich variety of possible patterns. That includes the film
rupture and formation of dry areas, travelling waves and the waves propelled toward breaking
similar to the shallow water waves. More about the asymptotic approach used here is said in
section 4.

A similar asymptotic approach to the problem of the thermocapillary flow albeit using a
formally different scaling of the independent and the dependent variables was presented in
references [9, 13]. The point of bringing it up here is that their scaling leads to an equation
identical to ours though they assumed the fluid layer to be of thickness \( O(1) \). This is
summarized in appendix where using scale invariance it is also shown that the two approaches
are identical. The main difference worth noting is that due to a different scale length for
normalization the resulting dimensionless quantities are of different order. This effectively
renders equation (2) of interest for a wide parametric range.

It is of interest to note that the problem of the Marangoni flow was usually investigated for
the case of positive gravity (a film supported from below). In the numerical studies we stress
the case of negative gravity where the surface tension is essential to balance the destabilizing
effect of gravity.

The plan of the paper is as follows. In section 2, we formulate the problem and derive the
evolution equation using the regularization approach [14-19]. In section 3 we present the
essence of our numerical studies and the emergence of a rich variety of patterns. This is followed by summary and concluding remarks in section 4. In the appendix, we present an evolution equation for a film of a finite thickness for the combined case of Benard-Marangoni flow.

2. Derivation of the interfacial equation.

Consider a thin liquid film on a slightly inclined plane. We assume the surface tension being a function of the temperature (the Marangoni effect). The thermal conductivity, the viscosity and the density of the fluid are assumed to be temperature-independent and the phenomenon of the buoyancy is neglected.

We start with the governing equations of the incompressible flow:

\[
\begin{align*}
v_t + (v \cdot \nabla) v &= -\frac{1}{\rho} \nabla p + v \nabla^2 v + F \\
\text{div } v &= 0 \\
T_t + v \cdot \nabla T &= \eta \nabla^2 T
\end{align*}
\]

wherein \( v = \{u, v, w\} \), \( p \) are the fluid flow velocities and the pressure, respectively, \( T \) is the temperature, \( \nu \) and \( \eta \) are the kinematic viscosity and the thermal conductivity of the fluid, respectively, and \( F \) is the body force with components \( \{g \sin \beta, 0, -g \cos \beta\} \) in the coordinate system shown in figure 1, \( \beta \) being the inclination angle of the plane and \( g \) the gravitational acceleration.

The boundary conditions are:

1) at the plane \( x_3 = 0 \), we use the non-slip conditions for the velocities and the specified temperature \( T_1 \).

\[
v = 0, \quad T = T_1.
\]

2) at the free surface \( x_3 = H(x_1, x_2, t) \) we employ:

(a) the stresses balance:

\[
-(p - p_a) \mathbf{n} + 2 \mu \mathbf{D} \cdot \mathbf{n} = 2 \sigma K \mathbf{n} + \nabla \cdot \sigma
\]

(b) the heat transfer condition:

\[
\frac{\partial T}{\partial n} + q(T - T_2) = 0
\]
(c) the kinematic condition:

$$\frac{\partial H}{\partial t} + \mathbf{v} \cdot \nabla H = 0$$  \hspace{1cm} (9)$$

where $T_2$ and $p_s$ are the given, uniform temperature and the pressure of the ambient air.

$\mathcal{D}$ is stress tensor, $\mu$ is the fluid viscosity, $\sigma$ is the temperature-dependent surface tension, $\sigma = \sigma_0 - \kappa (T - T_0)$, $T_0$ being a reference temperature, $K$ is the mean interfacial curvature, $q$ is the ratio between the heat transfer rates by convection and conduction, $\mathbf{n}$ and $\nabla s$ are the unit normal vector and the surface gradient at the interface, respectively.

Introducing the longitudinal (transverse) length scale $l(a)$ and a characteristic velocity $U$ we define the following dimensionless quantities:

$$x = x / l, \quad y = y / l, \quad z = z / a, \quad \tau = Ut / l, \quad \varepsilon = a / l$$

$$\theta = \frac{T - T_L}{T_H - T_L}, \quad p = (p - p_s) \frac{a^2}{Ut / \mu}, \quad h = \frac{H}{a},$$

$$u = \frac{v_1}{U}, \quad v = \frac{v_2}{U}, \quad w = \frac{v_3}{(Ua)} l(Ua)$$

wherein $T_L, T_H$ are the low and the high temperatures at the boundaries.

Projecting equation (7) on the interface $\Gamma$ and comparing the order of magnitude of various terms one obtains

$$U = \varepsilon \frac{\kappa (T_H - T_L)}{\mu} \quad \text{where} \quad \kappa = - \frac{\partial \sigma}{\partial T} .$$  \hspace{1cm} (10)$$

Using $\varepsilon$ equations (10) and (11) read

$$x_1 = x \ell, \quad x_2 = y \ell, \quad x_3 = \varepsilon z \ell, \quad \tau = \varepsilon^{-1} \tau \ell / U_0 ,$$

$$\varepsilon u U_0, \quad \varepsilon v U_0, \quad \varepsilon w U_0,$$

$$\theta = \frac{T - T_L}{T_H - T_L}, \quad p - p_s = \varepsilon \frac{U_0 \mu}{\ell} P , \quad H = \varepsilon h \ell$$

where the reference velocity $U_0$ is given by

$$U_0 = \frac{\kappa (T_H - T_L)}{\mu}$$

Introducing equations (12) into the equations of motion one obtains

$$u_{zz} - P_x + G \varepsilon^{-1} \sin \beta = \varepsilon^2 \text{Re} \left( u_x + uu_x + vu_y + wu_z \right) - \varepsilon^2 (u_{xx} + u_{yy})$$

$$v_{zz} - P_y = \varepsilon^2 \text{Re} \left( v_x + uu_y + vv_y + wv_z \right) - \varepsilon^2 (v_{xx} + v_{yy})$$

$$- P_z - G \cos \beta = \varepsilon^4 \text{Re} \left( w_x + uw_x + vw_y + ww_z \right) - \varepsilon^4 (w_{xx} + w_{yy}) - \varepsilon^2 w_{zz}$$

$$u_x + v_y + w_z = 0$$

$$\theta_{zz} = \varepsilon^2 \text{Re} \text{Pr} \left( \theta_x + uu_x + uu_y + wu_z \right) - \varepsilon^2 (\theta_{xx} + \theta_{yy}) .$$

Here, $\text{Re} = U_0 a / v$ is the Reynolds number, $\text{Pr} = v / \eta$ is the Prandtl number and $G = ga^2(U_0 v)$ is the Bond number characterizing the relative importance of gravitational vs. surface tension effects.
We now rescale the boundary conditions to obtain
\begin{equation}
    u = v = w = 0 \text{ at } z = 0 \tag{18}
\end{equation}
\begin{equation}
    \theta = \theta_1 \text{ at } z = 0 \tag{19}
\end{equation}
\begin{equation}
    \theta_z - \varepsilon^2(\theta_x h_x + \theta_y h_y) + a q (\theta - \theta_z) N^{-1/3} = 0 \text{ at } z = h(x, y, \tau) \tag{20}
\end{equation}
\begin{equation}
    P = 2[\varepsilon^4(u_x h_x^2 + u_y h_y^2 - w_x - w_y h_y) + \varepsilon^3 h_x h_y (u_y + v_x) + \varepsilon^2(w_z - u_z h_x - v_z h_y)] N^{2/3} - \frac{\sigma}{U_0 \mu} [\varepsilon^2 h_{xx}(1 + \varepsilon^2 h_x^2) - 2 \varepsilon^4 h_x h_y h_{xy} + \varepsilon^2 h_{yy}(1 + \varepsilon^2 h_y^2)] N \text{ at } z = h(x, y, \tau) \tag{21}
\end{equation}
\begin{equation}
    \left[ \varepsilon^2 u_x h_x - \varepsilon^2 h_x w_z + \frac{1}{2} \varepsilon^2 h_x (u_y + v_x) + \frac{1}{2} \varepsilon^2 h_x h_y (v_z + \varepsilon^2 w_y) + \right. \left. \frac{1}{2} (u_z + \varepsilon^2 w_x) (\varepsilon^2 h_x^2 - 1) \right] \frac{1}{2} (\theta_x + h_x \theta_z) N^{-1/3} \text{ at } z = h(x, y, \tau) \tag{22}
\end{equation}
\begin{equation}
    \left[ \varepsilon^2 v_y h_y - \varepsilon^2 h_y w_z + \frac{1}{2} \varepsilon^2 h_y (u_y + v_x) + \frac{1}{2} \varepsilon^2 h_x h_y (u_z + \varepsilon^2 w_x) + \right. \left. \frac{1}{2} (v_z + \varepsilon^2 w_y) (\varepsilon^2 h_y^2 - 1) \right] \frac{1}{2} (\theta_y + h_y \theta_z) N^{-1/3} \text{ at } z = h(x, y, \tau) \tag{23}
\end{equation}
\begin{equation}
    h_x + uh_x + vh_y - w = 0 \text{ at } z = h(x, y, \tau) \tag{24}
\end{equation}
where \( N = [1 + \varepsilon^2 (\nabla h)^2]^{-3/2} \) is the metrics that measures the interfacial distortion.

Next, take \( \varepsilon \ll 1 \), and assume the angle \( \beta \) to be small, \( \beta = \varepsilon \beta_0 \). Alternatively to keep \( \beta = O(1) \) one has to assume \( G = O(\varepsilon) \) which is proper for microgravity conditions. Dropping higher order terms in equations (13)-(24) leads to the following, simplified set of equations and boundary conditions
\begin{equation}
    u_{zz} + P_x + G \beta_0 = 0 \tag{25}
\end{equation}
\begin{equation}
    v_{zz} - P_y = 0 \tag{26}
\end{equation}
\begin{equation}
    P_z = -G \tag{27}
\end{equation}
\begin{equation}
    u_x + v_y + w_x = 0 \tag{28}
\end{equation}
\begin{equation}
    \theta_z = 0 \tag{29}
\end{equation}
\begin{equation}
    u = v = w = 0 \text{ at } z = 0 \tag{30}
\end{equation}
\begin{equation}
    \theta = \theta_1 \tag{31}
\end{equation}
\begin{equation}
    \theta_z + \text{Bi} (\theta - \theta_z) N^{-1/3} = 0 \tag{32}
\end{equation}
\begin{equation}
    P = -s N \nabla^2 h \tag{33}
\end{equation}
\begin{equation}
    u_z = - (\theta_x + h_x \theta_z) N^{-1/3} \tag{34}
\end{equation}
\begin{equation}
    v_z = - (\theta_y + h_y \theta_z) N^{-1/3} \tag{35}
\end{equation}
\begin{equation}
    h_x + uh_x + vh_y - w = 0 \tag{36}
\end{equation}
wherein \( s = \sigma_0 \varepsilon^2 \mu U_0 \) (\( \sigma/\mu U_0 = O(\varepsilon^{-2}) \)) and \( \text{Bi} = aq \) is the Biot number. Here we have also assumed that the values \( \varepsilon^2 \text{Re} \) and \( \varepsilon^2 \text{Ma} \) are of higher order of magnitude, \( \text{Ma} = \text{RePr} \) being the Marangoni number.

In carrying out the asymptotic expansion we have allowed \( h_x \), the derivative of \( h \) in the direction of the flow along the plate to be as large as \( o(\varepsilon^{-1}) \). While it allows us to account for possible large gradients [14-19], it also ensures that all the higher order terms are rightfully neglected.

Using the boundary conditions (31), (32) integration of equation (29) yields the temperature profile

\[
\theta = \frac{m\text{Bi}N^{-1/3}}{1 + \text{Bi}N^{-1/3} h} \cdot z + \theta_1 \equiv \bar{\theta} + \theta_1
\]  

(37)

where \( m = -1 \), \( \theta_1 = 1 \) in the case of a cool ambient air and a hot plane. In the opposite case of the hot ambient air and the cool plane \( m = 1 \) and \( \theta_1 = 0 \). The surface temperature \( \theta_R \) is therefore given via

\[
\theta_R = \theta (z = h) = \frac{\text{Bi}mN^{-1/3}}{1 + \text{Bi}N^{-1/3} h} + \theta_1.
\]  

(38)

Note that as \( |h_x| \to \infty \), \( \theta_R \to \theta_2 \).

Integrating equation (27) and using the boundary condition (33) one finds that the pressure is distributed according to

\[
P = G (h - z) - sN \nabla^2 h.
\]  

(39)

The first term corresponds to the hydrostatic part. The second one accounts for the effects of the distorted interface.

Substituting equation (39) into equations (25), (26) and using the boundary conditions (30), (34), (35) yields the velocity field components \( u, v \):

\[
u = [\text{G}h_x - s(N \nabla^2 h)_x - G\beta_0] \cdot (z^3/2 - hz) - zN^{-1/3} \theta_{R_x}
\]  

(40)

\[
v = [\text{G}h_y - s(N \nabla^2 h)_y] \cdot (z^3/2 - hz) - zN^{-1/3} \theta_{R_y}.
\]  

(41)

The component \( w \) of the velocity is found using equations (28), (40), (41) and the boundary condition (30):

\[
w = \frac{z^2}{2} \nabla h \cdot \nabla [\text{G}h - sN \nabla^2 h] = (z^3/6 - hz^2/2) \nabla^2 [\text{G}h - sN \nabla^2 h] +
\]

\[
+ G\beta_0 \frac{z^2}{2} \cdot h_x + \frac{z^2}{2} [(N^{-1/3} \theta_{R_x})_x + (N^{-1/3} \theta_{R_y})_y].
\]  

(42)

Introducing equations (40)-(42) and (38) into the kinematic condition (36), leads to an equation describing the evolution of the liquid-air interface

\[
h_x - \frac{\gamma\beta_0}{3} (h^3)_x + \nabla \cdot \left[ \left( \frac{\gamma h^3}{3} - \frac{Bm h^2}{2(1 + Bh)^2} \right) \nabla h \right] + \frac{s}{3} \nabla \cdot [h^3 \nabla (N \nabla^2 h)] = 0.
\]  

(43)

Here \( \gamma = -G \) and we have also introduced \( B \) as an effective Biot number. As before \( N = [1 + \varepsilon^2(Nh^2)]^{-1/2} \).

The second term of equation (43) represents the convective effect due to the plane inclination. The next two terms are of diffusive nature. The first is due to the gravity while the
second one is due to the Marangoni phenomenon. The gravity stabilizes the evolution of the interface when the film is supported from below, \( \gamma < 0 \) (the film flowing on a floor) and destabilizes it when the film is supported from above, \( \gamma > 0 \) (flow on a ceiling). If the free surface is exposed to the air cooler then the rigid plate \( (m = 1) \) the Marangoni effect has a destabilizing impact. In the contrary case of the air hotter than the plate \( (m = 1) \) the Marangoni effect tends to stabilize the forming pattern. The last term in equation (43) is due to the surface tension. It always has a stabilizing effect.

In using an effective Biot number, \( B \), we can safely omit its dependence on the metric term \( N \) due to the very mild variation of \( N \). However, as past experience has taught us [14-19], this variation may have a crucial impact on the balance between the surface tension and the driving instability because it affects the width of the band of unstable modes.

In the one-dimensional case \( (\delta_i = 0) \)

\[
h_{x} = \frac{\gamma \beta_0}{3} (h_1)^i \, + \, \left[ \left( \frac{\gamma h_1^3}{3} - \frac{B n h_1^2}{2(1 + B h_1^3)} \right) h_1 \right]_{i}, + \frac{\lambda}{3} [h_{i}^2 (h_1, N)]_{i}, = 0 \tag{44}
\]

now \( N = [1 + \lambda^2 h_1^2]^{-\frac{3}{2}} \).

Equations (43) or (44) are our key result. Equation (43) reduces to equation (2.18) of reference [10] if the parameters \( \lambda \) and \( \beta_0 \) are set to be zero.

Unless one deals in conditions of microgravity it is clear that the gravity is the most important mechanism of stabilization. The impact of the heat transfer at the interface, described by Biot number \( B \) is minor for large and positive \( \gamma \), corresponding to thicker films or smaller temperature differences between the boundaries. Comparison of the two effects shows that thermocapillarity dominates gravity only if the average height \( h_M \) satisfies

\[
h_M \leq \left( \frac{3}{8 \gamma} \right)^{1/2} \equiv h^* \]

which is the equivalent to the condition \( a \leq \sqrt{\frac{3 U_0 \nu}{8 g}} \) and if the Biot number ranges in some interval encompassing the value \( B = \sqrt{8 \gamma / 3} \).

As an illustration, assume a water layer of 1 mm average thickness at 20 °C at the terrestrial value of gravity and the temperature difference between the boundaries \( \Delta T \) being 1 °C. For this situation \( \gamma \approx 61 \) and \( h^* \approx 0.08 \). Therefore, the heat transfer at the interface can significantly affect only those parts of the interface for which \( h < 0.08 \) mm. Unless this is the case heat transfer at the free surface can be disregarded. Equation (44) then becomes

\[
h_{x} = \frac{\gamma \beta_0}{3} (h_1)^i \, + \, \left[ \frac{h_1^3}{3} \left( \gamma h_1 + s h_{i}^1, N \right) \right]_{i}, = 0 \tag{45}
\]

which corresponds to the evolution of an effectively thermally insulated interface.

Note that equation (43), and its descendants as well, equations (44) and (45), preserve the total mass, i.e.

\[
\int_{\Omega} h \, d\Omega = \tilde{h}_0 = \text{const}. \tag{46}
\]

where \( \Omega \) is either infinite or periodic domain. For equation (45) (the 1-D case) another conservation law is available, namely

\[
\frac{d}{d\tau} \int_{\Omega} \frac{dx}{h} = \frac{2}{3} \int_{\Omega} d x \left( \gamma h_1^3 - s N h_{i}^2 \right) \equiv I (\tau) \tag{47}
\]
Equation (47) has been derived from equation (45) by multiplying the latter by $h^{-2}$ and integrating that over $\Omega$. (For $B \neq 0$ in equation (47) $\gamma \rightarrow \gamma - Bm/2 \ h (1 + Bh)^2$.) A similar conservation law is also valid for the 2-D case under substitution $h_x \rightarrow \nabla h$, $h_{xx} \rightarrow \nabla^2 h$.

For the case of a horizontal plane, $\beta_0 = 0$, equation (44) can be rewritten in the conservative form

$$h_r = - \left \{ h^3 \left[ \frac{Q(h)}{3} + \frac{\xi}{3} h_{xx} N \right] \right \}_x$$

(48)

where

$$Q(h) = \frac{\gamma h}{3} - \frac{Bm}{2} \left[ \ln \frac{h}{1 + Bh} + \frac{1}{(1 + Bh)} \right] - \frac{A}{6}$$

(50)

where $A$ is an integration constant.

Multiplying equation (48) by $Q(h) + (s/3) \ h_{xx} N$ and integrating it over $\Omega$ by parts using periodic boundary conditions one has

$$\frac{dF}{d\tau} = \frac{d}{d\tau} \int_\Omega \left[ P(h) + \frac{s}{3} \left( \sqrt{1 + e^2 h_x^2} - 1 \right) \right] dx = - \int_\Omega h^3 \left[ \left( \frac{Q(h)}{3} + \frac{\xi}{3} h_{xx} N \right) \right]_x^2 dx$$

(49)

where

$$F = \int_\Omega \left[ P(h) + s \left( \sqrt{1 + e^2 h_x^2} - 1 \right) / 3 \right] dx$$

$$P(h) = - \int_\Omega Q(h) \ dh = - \frac{\gamma h^2}{6} + \frac{Bm}{2} h \ln \frac{h}{1 + Bh} + \frac{A}{6} h$$

The right-hand side of equation (49) is always non-negative, therefore

$$\frac{dF}{d\tau} \leq 0 \ .$$

(50)

Note also that equation (48) can be written as

$$h_r = \frac{\partial}{\partial x} \left[ h^3 \frac{\partial}{\partial x} \left( \delta F \delta h \right) \right] .$$

(51)

This form is reminiscent of the Cahn-Hilliard equation with $F$ playing the role of free energy and $h^3$ that of mobility coefficient.

Relations (46)-(51) provide a start for any meaning full analysis of equation (44).


Our numerical investigation was limited to one-dimensional study, equation (44) with periodic boundary conditions. Nevertheless, we have found equation (44) to predict a rich variety of patterns which are discussed next.

We consider first equation (45) (i.e. $B = 0$) in the interval $0 \leq x \leq 2 \pi$ amended with the periodic boundary conditions and initial conditions of the form

$$h(x, 0) = h_0(x) = h_M + r \sin kx$$

(52)
where $k$ is integer, $h_m$, $r$ are real constants. The initial condition (52) represents the interface perturbed around the quiescent state $h = h_m$. Impact of thermocapillarity will be considered later.

Since for $\gamma < 0$ no nontrivial patterns can form, the case of primary interest is that of $\gamma > 0$ (negative gravity). This corresponds to a liquid film flowing on a ceiling.

Integrals (46) and (47) are the key to understand the patterns that emerge during the evolution. In particular an important role is played by the sign of $I(0)$. Observe that the right hand side of equation (47) (with $N = 1$) is variationally related to the linearized part of the last two terms in equation (45) and thus to the linear spectrum of equation (45). This one has a long wave instability with a cutoff at $k = k_c = \sqrt{\gamma/s}$. Thus in the case of $I(0) < 0$ for all $k$'s all the modes are linearly stable and therefore a superposition of these modes is stable as well. Hence $I(\tau) \ll 0$ for all $\tau$ and equation (46) evolves toward a homogeneous equilibrium. On the other hand if initially $I(0) > 0$ but there is at least one unstable mode, though initially we expect decay to the average state, the available nonlinearities and noise ultimately provide a possible channel for a nonlinear excitation of that mode. If at the outset $I(0) < 0$, the unstable mode(s) start to develop immediately. Observing equation (45) one notes that the long wave instability whether present initially or excited via coupling will grow explosively due to the backward diffusion unless the available nonlinearity channels enough energy into higher wavelengths where it is damped by surface tension. While we were unable to prove it to be so, all our numerical simulations indicate that whatever the complexity of the emerging pattern may be, the amplitude saturates at a finite level for any choice of parameters $\gamma$, $s$ and $\beta_0$.

Before we continue to describe in further detail our results, note the invariance of equation (45) under

$$
h \rightarrow h_m h_1, \quad x \rightarrow Lx, \quad \tau \rightarrow \tau_1 h_m^{-3} L^4, \quad \epsilon \rightarrow \epsilon_1 = \epsilon h_m^3 L, \quad \gamma \rightarrow \gamma_1 = \gamma L^2, \quad \beta_0 \rightarrow \beta_1 = \beta_0 L^4 h_m, \quad s \rightarrow \tilde{s}
$$

which clearly displays the change of (de)stabilizing and convective effects with mass and length scales. In particular, note the suppression of convective effects with the increase of the average height of the film and the existence of at least one linearly unstable mode for an adequately large domain. To avoid discussing trivial structures from now on we shall assume that the equilibrium has at least one linearly unstable mode.

It is now convenient to separate our discussion and consider first

A : flow on a horizontal plate ($\beta_0 = 0$).

A1) $\epsilon = 0$. Irrespective of the initial value of $I(0)$, since at least one mode can be linearly unstable, the interface evolves into a multimodal pattern with the number of the nonlinear modes roughly correlated to the wavelength of the fastest growing mode, see figure 2 for $\gamma = 20$ and two humps. For $\gamma = 5$ but otherwise the parameters only one hump emerges (not shown). The most notable point about these modes is the emergence of ruptured-like zones.

A2) $\epsilon > 0$. There exists a critical $\epsilon_c(h_m, L)$ that depends on interval length and the average height of the initial data such that if $\epsilon < \epsilon_c$ (subcritical domain) then the evolution is qualitatively the same as in the $\epsilon = 0$ case. Thus in the subcritical regime the actual value of $\epsilon$ is of no importance. However for $\epsilon > \epsilon_c$, the employment of the full curvature causes the appearance of vertically walled domains (finger shapes) and at that moment the numerical experiment is terminated. Clearly at this parametric domain the impact of the fully retained curvature is crucial. In this, supercritical, domain, as a law we observed the formation of fingers prior to the appearance of ruptured zones. Qualitatively similar bubble shapes of the interface were obtained numerically in reference [20] for a related problem of Rayleigh-
Fig. 2. — The spatio-temporal evolution of the thermally effectively insulated interface on a horizontal ceiling: $B = 0, \beta_0 = 0, \gamma = 20, s = 4, \epsilon = 0$ and $h_0 = 2 + 0.01 \sin x. I(0) > 0$. $g$ points to the direction of gravity.

Fig. 3. — The spatio-temporal evolution of liquid interface on a ceiling for the same parameters as in figure 2. However, now the full expression for the curvature is used ($\epsilon = 0.25$) and this is a supercritical case. Finger shapes with infinite gradients form in a finite time. $I(0) > 0$. The arrows point out the emergence of a vertical wall.

Taylor instability. Conversely, if ruptured zones appeared first, fingers did not form and the $\epsilon$-domain was essentially subcritical, see figures 2 and 3.

$B : \beta_0 \neq 0$. Flow on a tilted plane.

A tilted plane generates convection and as before the emerging pattern is related to the unstable linearized spectrum. For the later phase of evolution one observes the possible existence of $\beta_0$-dependent domains.

If $0 < \beta_0 < \beta_*$ then the structure is qualitatively the same as in the $\beta_0 = 0$ case. The only difference is that the structure drifts to the left due to convection. Depending on the value of $\epsilon$ either ruptured (subcritical convected case) or fingered domains form (supercritical convected case).

To describe steeper deflections of the plane assume first that $I(0) > 0$. Then for any $\beta_0$ such that $\beta_* < \beta_0$ the convection appears to be sufficiently strong to avoid the formation of
ruptures zones. The patterns have a genuinely travelling structure. If $\varepsilon$ is supercritical fingers will form. If $\varepsilon$ is subcritical then in the bimodal case, often these modes do not coexist for too long. Though the two modes are initially of the same amplitude, later the second mode grows at the expense of the first and overtake it to form a mainly unimodal structure with a number of smaller lobes. (See Figs. 4 and 5). This symmetry breaking between the two modes happened at times at an early stage of evolution and occasionally after a substantial travelling time during which the two modes travelled unchanged through the periodic system. Then suddenly the second mode grew at the expense of the first and overtook it. Whether this overtaking always occurs is unknown to us.

Assume now that $I(0) < 0$ and the existence of two linearly unstable modes. Example: $\gamma = 20, s = 4$ and $h_0 = 1 + r \sin 4x$. Then at first the perturbation decays to an almost average state. To describe structure that emerge later the $\beta_0 > \beta_0$ domain is further subdivided as follows. For $\beta < \beta_0 < \beta_a$ a travelling bimodal structure emerges. Those modes coexist and do not seem to overtake each other. In the next, narrow, zone of $\beta_a < \beta_0 < \beta_b$ out of the averaged state a mainly unimodal structure emerges, with the dominant mode and number of much smaller lobes. Finally, for any $\beta_0$ such that $\beta_0 > \beta_b$ the initial perturbation after collapsing to the average state, remains there indefinitely. We would like to note that while we are confident about the existence of the outlined domains we do not know whether the boundaries between these domains, as symbolized by $\beta_*$, $\beta_a$ and $\beta_b$, are really sharp. For our example we have found $\beta_* \approx 0.08, \beta_a \approx 0.4, \beta_b \approx 0.48$.

Returning to discuss the Marangoni effect ($B > 0$) we reasserte horizontal plane ($\beta_0 = 0$). We shall concentrate on two situations, namely that of warm floor or cold ceiling where a parametric window exists in which the Marangoni effect plays a meaningful role. (In the cases of warm ceiling (cold floor) thermocapillarity only slightly enhances the destabilizing (stabilizing) effect of gravity). It enables the existence of interesting non-trivial steady states. We assume that $\varepsilon = 0$ and integrate equation (44) three times to bring it into a potential form

$$\left( \frac{dh}{dx} \right)^2 + V(h) = V_0,$$

where the potential $V$ is given as

$$sV(h) = -6P(h).$$

The freedom in the choice of $A$ (see Eq. (49)) may be related to the freedom to choose the maximal height of the equilibrium pattern. In figure 6 relevant potentials are displayed. Figure 6a corresponds to a warm floor ($\gamma < 0, m = -1$) and figure 6b to a cold ceiling ($\gamma > 0, m = 1$). In both situations for $|A| > 500$ periodic equilibrium solutions are possible. Note that while in the case of a warm floor the equilibrium «touches the floor» (i.e. $h = 0$) and thus assumes a drop-like shape, the solution on the ceiling represents a periodic finger type equilibrium that blends into a constant finite width film (see Fig. 7 for equilibrium that evolved out of non-equilibrium data. If however the average value $h_M$ is increased to $h_M = 0.25$, no equilibrium emerges). While the evolution of equilibrium in figure 7 is indicative of partial stability of equilibria on the ceiling, the extent of stability of equilibrium on the floor is unclear.

The equilibrium structures on the ceiling become even more interesting in the case of microgravity and Biot numbers slightly under unity. In figure 8, example of such potentials is shown for $B = 0.9$ and 0.95, respectively. In both cases but particularly for $B < 0.9$ two classes of equilibria are possible. For $V_0 < V_{0c}$ (see Fig. 8) in addition to a periodic solution that centers around $h_p$ there exist drop-like solutions with maximal amplitude $h < h_c$. At $V_0 = V_{0c}$ these two structures may coexist and for bigger $V_0$ they merge into a large drop-like
Fig. 4. — The interfacial shape at $\tau = 1$. The parameters are as in figure 2 but the ceiling is tilted: $\beta_0 = 1$. Note that for $\tau \geq 0.7$ the interface behaves as a travelling wave. The double arrow points out the travel direction.

Fig. 5. — The parameters are as in figure 4 but $\epsilon > 0$. Using $\epsilon = 0.25$ one is in a supercritical zone. The evolution of the interface in this case is very much like that of the Rayleigh-Taylor waves studied in reference [18]. The arrows point out the emergence of a vertical wall. The double arrow points out the travel direction.

Fig. 6. — The potential $sV(h)$ for various values of the constant $A$. (a) warm floor; (b) cold ceiling.
Fig. 7. — Emergence of steady state pattern for $B = 50$, $\beta = 0$, $\gamma = 32$, $m = 1$, $\varepsilon = 0$ and initial perturbation $h_0 = h_M + 0.01 \sin x$, $h_M = 0.1$. Note that for $h_M \geq 0.25$ the initial perturbation does not evolve into a steady state but rather into a ruptured-like structure described in figure 2.

Fig. 8. — The potential $sV(h)$ for microgravity conditions, now $\gamma = 0.02$, $A = 0.1$, $B = 0.9$ and 0.95. Note that in the latter case the almost flat valley points to a very limited domain of existence of periodic solutions.

structure. One notes that in the $B = 0.95$ case the periodic solution has a very limited domain of existence. In the parametric regime above $B = 0.95$ one has only drop-like solutions. Below $B = 0.9$, the maximum height, $h_c$ of the possible drop-like equilibrium decreases rapidly with $B$:

$$
\begin{array}{cccc}
B & 0.9 & 0.8 & 0.6 & 0.2 \\
h_c & 1.4 & 1.0 & 0.7 & 0.4 \\
\end{array}
$$

with the major mode of equilibrium being again that of periodic solution(s). Since $B$, $\gamma$ and $A$ are equilibrium controlled parameters, it is thus clear that in the discussed parametric domain of microgravity, a slight change in equilibrium conditions will cause a significant change in equilibrium structure, which in turn suggests that dynamical changes are to be expected to occur frequently as the system tries to settle from one equilibrium to another.

It should be clear that the drop-like shapes must be supported by walls. The present model does not allow for self-supported equilibrium structures of finite extent. That should be evident from the shape of the potential $sV(h)$ near $h = 0$, which does not allow for the flux on the contact line to vanish. On the contact line the dominant force is that due to the surface tension. This is easily shown by contradiction assuming first that the Marangoni effect is the dominant one and thus looking at the dynamics of

$$
\frac{\partial h}{\partial t} = (h^2 h_r)_t
$$

where for studying $h \ll 1$ the effect due to $Bh$ is neglected. Equation (53) has a similarity solution of the form

$$
h = \tau^{-1/4} f_M(\eta), \quad \eta = \tau^{-1/4}
$$
which near the front \( h = (\eta_f - \eta)^{1/2} \) leading to Marangonian flux \( h^2 h_x \propto (\eta_f - \eta)^{1/2} \rightarrow 0 \). But then flux due to the presumably small surface tension is \( q_s \propto h^3 h_{xxx} \propto (\eta_f - \eta)^{-1} \rightarrow \infty \). Contradiction. We thus assume the surface tension to dominate on the front and look at

\[
h_r + (h^3 h_{xxx})_x = 0.
\]

Its similarity solution has the form

\[
h = \tau^{-1/7} f_s(\eta), \quad \eta = x \tau^{-1/7},
\]

which near the front behaves as \( h \propto (\eta_f - \eta)^{4/3} \) Thus \( q_s \propto (\eta_f - \eta)^{7/3} \) and while \( q_M \propto (\eta_f - \eta)^3 \) and thus decays faster than \( q_s \). Consequently, independently of whether the Marangoni effect is stabilizing or destabilizing, the «flux» due to surface tension dominates near the front.

### 4. Summary and concluding remarks.

Using an asymptotic approach similar to that of the lubrication theory we have derived a nonlinear evolution equation, equation (43), describing the behavior of the interface of a thin film flowing on a slightly inclined plane, under the influence of gravity and thermocapillarity. The derivation employs the regularization introduced in earlier works [14-19] which takes into consideration the possible appearance of large spatial gradients.

Numerical studies of the one-dimensional version of equation (43), equation (44), reveal that in the case of a horizontal plane two operational domains are possible. In the first, subcritical, domain the evolution results in emergence of ruptured-like zones. The second, supercritical one, is characterized by the emergence of vertically walled finger shapes prior to the film rupture. The nonlinear convection due to the plane inclination enriches the variety of the forming patterns by inducing different types of travelling waves, which break, in the supercritical domain, very much like the Rayleigh-Taylor waves described elsewhere [18]. The thermocapillarity, albeit relatively small under regular gravitational conditions, enables the emergence of equilibrium structures.

Several methodological remarks are now in order. In a way our approach may be best described as an analytical continuation. We start with adequately small \( \varepsilon \) such that the properties of the evolving structure are direct continuation of the \( \varepsilon = 0 \) case. With the change of the \( \varepsilon \) the emerging properties change continuously, leading at some \( \varepsilon \) to accumulation of a sizeable quantitative change in the resulting pattern. The most relevant feature for our discussion is the gradual increase with \( \varepsilon \) of the maximal gradient. (The amplitude is less sensitive to these changes). This change is continuous with \( \varepsilon \). A qualitative change emerges at that \( \varepsilon \) for which sharp gradient appears for the fi rst time and the evolution comes to halt. Our equation in its present form cannot treat multivalued interfaces. And while the moment of breaking is formally outside of the expansion validity, the approach to it both in time (for \( \varepsilon \gg \varepsilon_c \)) and for all \( t \) for \( \varepsilon \rightarrow \varepsilon_c \) is perfectly acceptable making it clear that the limiting phenomenon of breaking is a real one. It is this «analytical continuation» that gives credibility to presented results.

Beyond the generalizations introduced perhaps the more original part of this work is the unfolding of the critical threshold for the formation of the bubbles. As shown in the text, \( \varepsilon \) has to cross some critical threshold for this to occur. The threshold is \( \gamma \)-dependent. The higher the gravity parameter the lower becomes the critical threshold for breaking. A detailed study of the dependence of the threshold on equilibrium parameters is given for a related problem in reference [18]. This effect appears to be beyond all orders in expansion; going
next order in a direct expansion will not bring in this effect. Indeed expansion of the metric term yields an infinite tail that starts two orders down. In principle to unfold the criticality the standard expansion would have to be continued indefinitely. Realizing the source of the critical behavior we preserved the original expression which is like a partial summing of the whole expansion. Actually most amplitude expansions of similar problems cannot be directly extended beyond the present level of closure. If attempted at the next level a term with sixth order derivative appears with a «wrong» sign rendering the whole problem ill-posed.

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Appendix.

Here we derive an equation describing the evolution of a film interface in the case of the combined thermocapillary and thermogravitational flow (Benard-Marangoni flow) on a plane hotter than the ambient air. The derivation procedure is essentially similar to that in the main text and a brief outline will suffice.

Start with equations (3)-(5) and boundary conditions (6)-(9). To incorporate the buoyancy effect introduce the body force $F = \{0, 0, -g - gaT_1\}$, where $\alpha$ is the volumetric expansion coefficient. Assume the film thickness to be $O(1)$ and apply the scaling used in references [9, 13]

$$
\begin{align*}
    x &= \epsilon^{-1} \xi, \quad y = \epsilon^{-1} \eta, \quad z = z, \quad t = \epsilon^{-1} \tau, \\
    (u, v) &= (U_0, V_0)(\xi, \eta, z, \tau) + O(\epsilon), \\
    w &= \epsilon W_0(\xi, \eta, z, \tau) + O(\epsilon^2), \\
    P &= \frac{1}{\epsilon} P_0(\xi, \eta, z, \tau) + O(1), \\
    \theta &= \theta_0(\xi, \eta, z, \tau) + O(\epsilon).
\end{align*}
$$

Assuming also that $Ma = \epsilon^{-1} M$, $Ra = \epsilon^{-1} R$, $Ga = \epsilon^{-1} G$, $s = \epsilon^{-3} S$ one obtains for the location of the interface $h(\xi, \eta, \tau)$ (remember: $h$ is now $O(1)$)

$$h_{\tau} + \nabla \cdot \left[ \left( - \frac{G}{3} h^3 + \frac{12MBh^2 + PrR(8 + 5Bh)h^3}{24Pr(1 + Bh)^2} \right) \nabla h \right] + \frac{S}{3} \nabla \cdot [h^3 \nabla (N \nabla^2 h)] = 0.
$$

Here $Ra$ and $Ga$ are the Rayleigh and the Galileo numbers,

$$
Ra = g \alpha (T_1 - T_2) a^2 \nu \eta, \quad Ga = ga^2 \nu^2
$$

and $s = \sigma a^2 \rho \nu^2$ is the surface tension number. $\nabla = \{\partial_\xi, \partial_\eta\}$, $N = \left[1 + \epsilon^2(\nabla h)^2\right]^{-3/2}$ and as before $B$ is the effective Biot number. Note $\tau \to \tau Pr$.

In the case of $R = 0$, equation (A1) reduces exactly to equation (43). The fact that equation (43) is common for two different scalings of the thermocapillary flow elevates its importance. The scaling used in the main text assumed that the film is thin, $H = O(\epsilon)$, and yields a slow fluid flow $v_1$, $v_2 = O(\epsilon^2)$, low pressure $p = O(\epsilon)$ and a large surface tension parameter, $s = O(\epsilon^{-2})$. In the contrast, the scaling introduced here assumed finite film thickness, $H = O(1)$, and leads to larger velocities, $v_1$, $v_2 = O(1)$, $v_3 = O(\epsilon)$, much higher pressure...
\[ p = O(\epsilon^{-1}) \] and even larger surface tension parameter, \( s = O(\epsilon^{-3}) \). Both derivations, however, lead to the same evolution equation written in terms of \( h \), the rescaled film thickness.

Assume now that in equation (A1) \( R = 0 \) and scale the spatial parameters as \((x, y, z, h) \rightarrow \epsilon(x, y, z, h)\) and the dimensionless parameters as \((B\epsilon, M\epsilon, Ge^3, Se^{-1}) \rightarrow (B, M, G, S)\), then one finds that equation (A1) transforms to equation (43) (for \( \beta_0 = 0 \)) and vice versa. Thus solutions transform one into the other. Note that in a film of a finite thickness it makes sense to study the thermogravitational effect \((R \neq 0)\) as well.

Finally equation (A1) can be recasted in the form (51) \((\partial_x \leftrightarrow \nabla)\) where \( P(h) \) is now

\[
P(h) = \frac{G}{6} h^2 + \frac{5 R}{24 B} h \ln (1 + Bh) - 1 + \frac{MB}{2 Pr} h \ln \frac{h}{1 + Bh} + \left( \frac{R}{12 B^2} - \frac{M}{2 Pr} \right) \ln (1 + Bh). \tag{A2}
\]

References