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HAL Id: jpa-00247551
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Submitted on 1 Jan 1991

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Short Communication

Noether symmetries and the Swinging Atwood Machine

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(Received 13 February 1991, revised 13 May 1991, accepted 17 May 1991)

Abstract. — In this work we apply the Noether theorem with generalised symmetries for discussing the integrability of the Swinging Atwood Machine (SAM) model. We analyse also the limitations of this procedure and compare it with the Yoshida method.

In the last years several analytical procedures have been applied in the study of the integrability of dynamical systems: the direct method for the identification of invariants, the Painlevé test, Melnikov's method, Ziglin's theorem, the analysis of the symmetries of the system, etc. In a recent paper published in this journal, Tufillaro et al. [1] used some results obtained by Yoshida [2] for discussing the domain of integrability for a two-dimensional Hamiltonian system: the so-called Swinging Atwood Machine (SAM). By the Liouville theorem, a two-dimensional Hamiltonian system will be integrable if exists a second conserved quantity (invariant) in addition to the energy.

In this letter we apply a method based on the generalised Noetherian symmetries [3] for finding the integrable case and its associated symmetries, for the SAM problem. We compare the two methods, Yoshida's theorem and the symmetry analysis, and discuss their joint utilisation.

1. Noether theorem.

The procedure that we employ here starts from the generalised formulation of the Noether theorem [4] where we suppose the existence of symmetries with an explicit dependence on the velocities. If we impose that the two-dimensional Hamiltonian system is invariant under a symmetry transformation we get an overdetermined system of partial differential equations. This system of equations, if solved, permits us the identification of the symmetries and invariants for the dynamical system.

If the action functional for the \(n\)-dimensional system

\[ S = \int L(t, x_1, x_1') \, dt \]  

(1)
is invariant (up to a constant) under the infinitesimal point transformation with the form
\[ t' = t + \varepsilon \xi (t, x_1, x_2) \]
\[ x'_i = x_i + \varepsilon \eta_i (t, x_1, x_2). \]  
(2)

then, by the Noether theorem, it exists a conserved quantity for the system given by
\[ I = \partial L / \partial \dot{x}_i (\eta_i - \xi \dot{x}_i) + \xi L - f (t, x_1, \dot{x}_1). \]  
(3)

The conditions for the invariance of \( S \) are
\[ L \partial \xi / \partial t + \eta_i \partial L / \partial x_i + L (\partial \xi / \partial t + x_i \partial \xi / \partial x_i) + \]
\[ + \partial L / \partial x_i [\partial \eta_i / \partial t + x_i \partial \eta_i / \partial x_i - x_i (\partial \xi / \partial t + \]
\[ + x_i \partial \xi / \partial x_i)] = \partial f / \partial t + x_i \partial f / \partial x_i. \]  
(4)

We can choose \( \xi \) equal to zero. Furthermore, the symmetry which is related to the invariant \( I \) is given by
\[ \eta_i = -g^{ij} \partial I / \partial x_j \]  
(6)

where \( g_{ij} \) is defined by the relation
\[ g^{ij} \partial^2 L / \partial x_i \partial x_j = \delta_i^j \]  
(7)

For a two-dimensional system, we take the following Lagrangian in polar coordinates:
\[ L = Ar^2 / 2 + Br^2 \theta^2 / 2 - V(r, \theta) \]  
(8)

Conditions (4) and (5) when applied to this Lagrangian lead to the following system of equations:
\[ (n + 1)F_{n+1} A^{-1} \partial V / \partial r + (Br^2)^{-1} (\partial V / \partial \theta) \partial F_n / \partial \theta - \partial F_{n-1} / \partial r - \]
\[ - A^{-1} Br \theta^2 (n + 1)F_{n+1} + 2r^{-1} \delta \partial F_{n-1} / \partial \theta - \theta \partial F_n / \partial \theta = 0, \]  
(9)

where we have made the following expansion for the invariant \( I \).
\[ I = \sum_{n=0}^{N} F_n \left( r, \theta, \theta \right) r^n, \]  
(10)

and \( n = 0, 1, 2, \ldots, N \). We have therefore \( N + 2 \) equations to be satisfied.

The equations (4) and (5) also furnish
\[ \eta_1 = - A^{-1} \sum_{n=0}^{N} nF_n r^{n-1} \]  
(11)
\[ \eta_2 = - (Br^2)^{-1} \sum_{n=0}^{N} r^n \partial F_n / \partial \theta \]  
(12)
\[ f = - \sum_{n=0}^{N} \left[ \theta \partial F_n / \partial \theta + (n - 1)F_n \right] r^n \]  
(13)
2. The Swinging Atwood Machine.

The Lagrangian for this system, studied by Tufillaro et al. [1], is

$$L = (1/2)(M + m)r^2 + (1/2)m\theta^2 - gr[M - m\cos(\theta)]$$

(14)

This Lagrangian is a particular case of (8), with $A = M + m$, $B = m$ and $V = gr[M - m\cos(\theta)]$

If we assume that $N = 1$, i.e., if $I$ has a linear dependence on $r$, the system (9) is reduced to

$$2r^{-1}\theta \frac{\partial F_1}{\partial \theta} - \frac{\partial F_1}{\partial r} = 0,$$

(15)

$$(Br^2)^{-1} \partial V/\partial \theta \left( \partial F_1/\partial \theta \right) + 2r^{-1}\theta \frac{\partial F_0}{\partial \theta} - \theta \frac{\partial F_1}{\partial \theta} - \theta \frac{\partial F_0}{\partial \theta} = 0,$$

(16)

$$A^{-1}F_1\partial V/\partial r + (Br^2)^{-1} \partial V/\partial \theta \left( \partial F_0/\partial \theta \right) - A^{-1}Br\theta F_1 - \theta \frac{\partial F_0}{\partial \theta} = 0$$

(17)

Solving (15) we get

$$F_1 = F_1(\theta, r^2\theta)$$

(18)

By supposing a linear dependence of $F_1$ on $r^2\theta$

$$F_1 = P_0(\theta) + P_1(\theta)r^2\theta$$

(19)

we find, from (14) and (16),

$$F_0 = q_0(\theta, r) + q_1(\theta, r)\theta + q_2(\theta, r)\theta^2$$

(20)

and

$$4r^{-1}q_2 - \partial q_2/\partial r - r^2dP_1/d\theta = 0$$

(21)

$$2r^{-1}q_1 - \partial q_1/\partial r - dp_0/d\theta = 0$$

(22)

$$\partial q_0/\partial r - rP_1 \sin(\theta) = 0$$

(23)

Equation (17) leads to the following additional conditions:

$$\partial q_2/\partial \theta + \lambda^2r^3p_1 = 0,$$

(24)

$$\partial q_1/\partial \theta + \lambda^2rp_0 = 0;$$

(25)

$$\partial q_0/\partial \theta - \lambda^2r^2P_1g[\mu - \cos(\theta)] - 2q_2r^{-1}g\sin(\theta) = 0,$$

(26)

$$q_1\sin(\theta) - \lambda^2[\mu - \cos(\theta)]rp_0 = 0,$$

(27)

where $\lambda = [m/(m + M)]^{1/2}$ and $\mu = M/m$

The compatibility of equations (22), (25) and (27) imposes $q_1 = p_0 = 0$ The solution for equations (23), (21) and (24) is

$$p_1 = C_1 \cos(\lambda\theta) + C_2 \sin(\lambda\theta),$$

$$q_2 = -\lambda^3C_1 \sin(\lambda\theta) + \lambda C_2r^3 \cos(\lambda\theta),$$

$$q_0 = (g/2)r^2 \sin(\theta) [C_1 \cos(\lambda\theta) + C_2 \sin(\lambda\theta)],$$

(28)
where \( C_1, C_2 \) are constants. From (28) and (26) we find

\[
C_1 \cos(\theta) \cos(\lambda \theta) \left[ \lambda^2 + 1/2 \right] + C_2 \cos(\theta) \sin(\lambda \theta) \left[ \lambda^2 + 1/2 \right] + \\
+ (3 \lambda / 2) C_1 \sin(\theta) \sin(\lambda \theta) - (3 \lambda / 2) C_2 \sin(\theta) \cos(\lambda \theta) - \\
- \left[ 1 - \lambda^2 \right] C_1 \cos(\lambda \theta) - \left[ 1 - \lambda^2 \right] C_2 \sin(\lambda \theta) = 0
\]  

(29)

This is a transcendental equation for \( \lambda \) : it must be satisfied for any value of \( \theta \). If we make \( \theta = 0 \) (29) leads to

i) \( C_1 = 0 \) or ii) \( \lambda = 1/2 \).

In the first case the equation (29) reduces to

\[
C_2 \cos(\theta) \sin(\lambda \theta) \left[ \lambda^2 + 1/2 \right] - (3 \lambda / 2) C_2 \sin(\theta) \cos(\lambda \theta) - \\
- \left[ 1 - \lambda^2 \right] C_2 \sin(\lambda \theta) = 0
\]  

(30)

Making \( \theta = \pm \pi / 2 \) we get that the unique solution occurs for the trivial cases: \( \lambda = 0 \), corresponding to \( m = 0 \), and \( \lambda = 1 \), corresponding to \( M = 0 \). In the second case we find, from (29), that \( \lambda = 1/2 \) is a solution for any \( \theta \) if \( C_2 = 0 \) This solution corresponds to \( \mu = M / m = 3 \). From (19), (20) and (28) we get

\[
F_1 = r^2 \theta \cos(\theta / 2) \\
F_0 = - (r^3 / 2) \theta^2 \sin(\theta / 2) + gr^2 \sin(\theta / 2) \cos^2(\theta / 2)
\]  

(31)

Equations (10), (11) and (12) give us the invariant [5] and the associated symmetries for the integrable case (\( \mu = 3 \))

\[
I = - (r^3 / 2) \theta^2 \sin(\theta / 2) + gr^2 \sin(\theta / 2) \cos^2(\theta / 2) + \\
+ r^2 r \theta \cos(\theta / 2),
\]  

(32)

\[
\eta_1 = - (1/4) r^2 \theta \cos(\theta / 2), \\
\eta_2 = r \theta \sin(\theta / 2) - r \cos(\theta / 2)
\]  

(33)

The symmetry analysis permits us to conclude that there is no second invariant, with a linear dependence on \( r \), except for the case \( \mu = 3 \) and for the trivial cases where \( M \) or \( m \) are equal to zero. We can now compare this result with the analysis made by Tufillaro, Casasayas and Nunes [1]. They applied Yoshida's theorem [2] to the SAM potential; the integrability coefficient is given by

\[
\beta = 2M / (M - m)
\]  

(34)

As the degree of the system is 1, the systems where

\[
\beta \neq j(j + 1)/2,
\]

with \( j = 0, 1, 2, \ldots \), are non-integrable systems.

Consequently, Yoshida's theorem defines the following domain of possible integrability for the SAM potential:

\[
\mu \in [1, 3], \text{ in the points where } \mu = j(j + 1) / [j(j + 1) - 4]
\]  

(35)
We note that, if $\mu = M/m = 1$, Yoshida’s theorem does not apply. We showed that, in this case, there is no second invariant with a linear dependence on $\hat{r}$, but we do not have a definitive answer about the integrability (or not) of this system. There are, however, some numerical results which suggest the integrability of the system in this case [6].

The symmetry analysis enables us to find invariants and their associated symmetries for $n$-dimensional Hamiltonian systems. The main deficiency of this procedure lies on the necessity of a polynomial expansion on the velocities and on the resolution of the system of differential equations coming from this expansion. On the other hand, Yoshida’s theorem is a direct and powerful method for the analysis of the integrability domain for two-dimensional homogeneous Hamiltonian systems. However, it gives us only the necessary condition for a system being an integrable one and does not furnish the form of the second invariant. The joint application of the two methods, as proposed here for the SAM potential, helps us to contour some of these difficulties.

We would like to thank the referee for some useful comments.

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