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Dry Friction as an Elasto-Plastic Response: Effect of Compressive Plasticity

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Abstract. — Unlike the well known Amontons-Coulomb laws of dry friction, the recent precise experimental studies of low velocity friction dynamics provide sensitive tests for a theoretical description. We introduce a model, in which the friction force originates from multistability induced by interactions of individual pairs of asperities whose properties undergo a slow time evolution due to plastic creep induced by the compressive forces acting on the real contacts. Even a crudely schematic form of the model reproduces the time strengthening of the static friction force and the velocity weakening of the steady motion dynamic one. Instability with respect to the onset of stick slip is found, but open questions remain concerning the features of the corresponding bifurcation, pointing towards the need to extend the model to take into account shear-induced plastic effects.

1. Introduction

Recent experimental advances in experiments on friction have put to the fore again the idea that, as was already argued long ago by Rabinowicz [1], it is by studying the velocity (resp. time) variations of dynamic (resp. static) friction coefficients, and, more generally, the details of frictional dynamics that one can get the best information about the nature of the relevant physical processes and about their dependence on material properties.

Indeed, the work of Greenwood et al. [2] has made clear that the most salient feature of dry friction — namely the fact that the friction force $F$ is independent of the apparent area of contact and proportional to the normal load $N$ (Amontons-Coulomb laws) — is primarily a geometrical effect. That is, they have shown that, whether the real (micro) contacts between the two surfaces involved are completely elastic (Hertzian) or plastic [3], as soon as one takes into account the fact that real surface profiles present asperities with random sizes and heights, the real area of contact is (quasi) proportional to $N$. The only condition for this to hold

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is that the distributions of heights and sizes do not exhibit slowly decreasing tails. Hence the fact that Amontons law is essentially material independent and does not yield any information on the details of the physical processes at work.

Absolute values of friction coefficients $\mu = F/N$ are not very informative either, as they do not vary very significantly nor systematically between different classes of materials [3].

In order to go further in the discussion, it is important to recall experimental facts which are still now very often overlooked, in spite of the emphasis put upon them by Rabinowicz and by several rock mechanicians [4]. The “high school model” of friction states that:

- there is a static friction coefficient $\mu_s$, corresponding to the minimum force needed to set a solid into sliding motion on top of another solid,
- the dynamic friction coefficient is velocity-independent and $\mu_d < \mu_s$. $\mu_s$ and $\mu_d$ thus appear as two numbers characterizing a given couple of solids.

This is in fact not true. Well controlled experiments on a wide variety of materials have shown that $\mu_s$ and $\mu_d$ exhibit small but measurable variations, respectively with the time of contact prior to sliding (waiting time $t_w$) and with the sliding velocity.

Moreover, the dynamics of sliding is complex. While, in some ranges of values of the control parameters of the dynamical system formed by the slider and the pulling machine (pulling velocity $V$ and stiffness of the pulling system $K$), stationary motion at velocity $V$ is observed, in other regions of parameter space, the stationary regime is unstable against periodic stick-slip. Intermittencies have also been observed.

That important physical information can be gained from the study of this dynamics has been proved by Dieterich [5] whose pioneer work on transients following velocity jumps in the stationary regime showed, in agreement with previous suggestions of Rabinowicz [1], that the dynamics of dry friction involves a memory length $D_0$, typically on the order of micrometers. He has proposed, and recently illustrated by direct optical experiments [6], that this length is directly related with the size of the microcontacts.

This work, together with the recent systematic exploration by Heslot et al. [7,8] of the dynamical phase diagram of a paper-on-paper system, has given renewed impetus to the formulation of phenomenological equations describing the dynamics of friction in various physical situations. We have proposed and used a phenomenology inspired from previous work by Ruina and Rice [9] to analyze the stick-slip bifurcation of paper, as well as other dynamical features of this system. Carlson and Batista [10] have set up a dynamical description of lubricated friction between atomically planar surfaces, while recently Persson [11] has formulated a model for the same kind of system in the case where the lubricant is a grafted film (which therefore does not exhibit a shear induced melting transition).

We will specialize from now on to the case of dry (unlubricated) friction. The success of the analysis of the paper system based on the phenomenological equations immediately leads to a further question: if the phenomenological equations we have used are, at least to some level of approximation, correct, it must be possible to derive, rather than merely assume them on the basis of plausibility arguments, from a more “microscopic” theoretical approach.

Also, the analysis of experimental data leads to a puzzling question. The form of the assumed expression of the dynamic friction force implies that:

- the dynamics can be reduced to the translational motion of the center of mass,
- this occurs via noise activated motion across potential barriers whose height increases with the “age” of the microcontacts (i.e. the time needed to slide a characteristic distance $D_0$).

Such an analysis of experimental data yields a value of the strength of the noise acting on that single degree of freedom several orders of magnitude larger than the most “optimistic”
direct estimate. This casts a doubt at least on the physical interpretation of this heuristic model, and possibly on the form of the model itself \(^1\).

These remarks point to the need for building a theoretical frame which would permit to incorporate, at least in a partly phenomenological way, the qualitative knowledge about the main physical mechanisms at work in dry friction which emerges from experiments. The recent results of Baumberger et al. on paper and PMMA \([12, 13]\) confirm that, as first proposed by Tabor \([3]\), for most materials under usual loading conditions

- solid/solid contact is associated with a finite elastic contact stiffness,
- the slow increase with waiting time of \(\mu_s\) and the corresponding velocity weakening of \(\mu_d\) can be interpreted as due to the plastic (or viscoplastic) evolution of the microcontacts, which results in their slow strengthening. This is also confirmed by the recent observation by Dieterich and Kilgore \([6]\), on various transparent materials, of the (quasi logarithmic) slow growth of contact sizes.

That is, dry friction must be modelled as an elasto-plastic response.

2. Hysteresis Elastic Model

As a first step towards this goal, Nozieres and one of us formulated a model \([14]\) in which friction results from the purely elastic response of the asperities involved in the microcontacts. We first briefly outline here the basic features of this model, as well as its conclusions and limitations.

Asperities are schematically modelled as dilute pinning centers, randomly distributed on the two solid surfaces, which are swept through as the slider moves. They are constrained to move in the sliding direction only (quasi-1D model). Two asperities interact (i.e. a microcontact or active trap is created), when the distance \(r\) between their centers is smaller than their radius \(a\), via a pinning potential \(\Phi(r)\). This potential, which remains at this stage phenomenological, can be understood as resulting from the energy of elastic compression necessary for the prongs to retract in order to allow crossing. It may be either repulsive or attractive, depending on the local relative geometry of the two surface profiles. If the asperity height is comparable to their width, its maximum strength is of order \(Ea^3\), with \(E\) the elastic modulus of the slider \(^2\).

Let \(\rho_i\) be the configuration coordinate of a given trap \(i\), i.e. the corresponding intersasperity distance in the system in the absence of shear elasticity. In the real, deformable, system, it undergoes, under the effect of the pinning force, a horizontal displacement \(u_i\), measured from a reference point at infinity on the scale \(a\), for example the point where the external pulling force is applied. The total energy of the system is then the sum of the pinning energy of the contacts between the sheared surfaces and of an elastic term:

\[
U_{\text{tot}} = \sum \Phi(\rho_i + u_i) + \frac{1}{2} \sum \lambda_{ij} u_i u_j. \tag{1}
\]

The \(\lambda_{ij}\)'s define an elastic stiffness matrix. The elements of its inverse, \((\lambda^{-1})_{ij}\), measure the horizontal displacement \(u_i\) at the site of contact \(i\) resulting from a unit horizontal force applied at contact \(j\). So, \((\lambda^{-1})_{ii} \approx 1/Ea\), while off diagonal elements \((\lambda^{-1})_{ij} \approx 1/Ed_{ij}\), where \(d_{ij}\) is the distance between contacts \(i\) and \(j\). Since contacts are sparse, \(d_{ij} \gg a\), so that \(\lambda_{ii} \approx Ea\), \(\lambda_{ij} \approx Ea^2/d_{ij} \ll \lambda_{ii}\).

\(^1\) Indeed, the behavior close to a bifurcation yields only limited information on a dynamical system, so that the corresponding fits should be considered as a plausibility test only.

\(^2\) The track is assumed to be non deformable.
Finally, low velocity sliding is described as quasistatic motion of the asperities. Indeed, the time scale for sweeping through a trap potential \( \approx a/V \) is considerably larger than the asperity acoustic relaxation time (typically, the time for propagating sound out of a region of scale \( a \)).

The analysis of this schematic model leads to conclusions which can be summarized as follows:

(i) As already pointed out long ago by Tomlinson [15], and even earlier by Brillouin [16], dry friction (*i.e.* the existence of a finite drag force in the low velocity limit and of a finite static threshold) exists only if the elastic trap response is multistable. This can occur only if the trap potential is strong enough (or, equivalently, the bulk rigidity of the slider weak enough) for the condition

\[
\lambda_{ij} < |\Phi''_{m}| = |\min(q^{2}\Phi(r)/dr^{2})|
\]

to be fulfilled. The friction coefficient is then proportional to the area of the corresponding hysteresis cycle.

(ii) Elastic interactions between contacts play a negligible role, as long as one focuses on the average friction force (they certainly have a major influence on the noise spectrum). This behavior, which contrasts with the case of vortices or charge density waves, where even weak traps can give rise to pinning, results from the dimensionality of the dry friction problem. Here we are dealing with a 2D random set of contacts coupled *via* a 3D elastic *medium*, hence the above mentioned decrease of the off diagonal \( \lambda_{ij} \) with intercontact distance. The so-called Larkin length, which measures the space scale beyond which elastic interactions become "relevant" (*i.e.* induce non perturbative cooperative pinning however small the trap potentials) is in the present case exponentially large [17], so that it can be considered for all practical purposes infinite as compared with real sample dimensions in laboratory friction experiments.

(iii) The dynamic friction coefficient which results is velocity independent. The approximation of quasi static asperity motion and zero temperature leads to describing contact breaking as the instantaneous jump of asperities out of their trap potential when the spinodal bifurcation corresponding to the metastability limit is reached. This can be improved upon by taking into account

- departures from quasistatic equilibrium due to the finite value of the asperity relaxation time, which result in a finite delay of the spinodal jumps. The corresponding correction to \( \mu_{d} \) is velocity strenghtening, but negligibly small in the small-\( V \) regime. It can only become important at large velocities, when friction is dominated by "viscous" effects,

- depinning due to early jumps above the metastability barrier, activated by thermal noise. This gives rise to a very small quasi logarithmic correction to \( \mu_{d} \), which is velocity strenghtening, contrary to what is observed experimentally.

So, it appears that such a purely elastic model, although it is certainly worth studying in more depth in order to try and understand the effect of elastic interactions on the noise spectrum, in particular *via* cascade jumps, cannot account for the observed time and velocity variations of \( \mu_{s} \) and \( \mu_{d} \), nor for the instability of steady sliding against stick-slip.

Clearly, what is missing in such a description is the above mentioned fact deduced from experiments that the deformation response of asperities to the large local stresses they bear is not purely elastic, but at least partly plastic. In this article we propose a first, schematic version of such an extended *elasto-plastic model.*
3. Elasto-Plastic Model

This extended model is based on the idea that the active contacts undergo an irreversible ageing process which is very slow as compared with the quasi instantaneous elastic processes. The elasto-plastic model is then defined as the elastic one, in which the characteristics of the elastic traps undergo a slow evolution. Their instantaneous values are used to describe the quasi static trap equilibrium, and hence the trap dynamics.

We therefore first specify more precisely the basic elastic description.

3.1. Elastic Contact Model

(i) We assume that asperities can only move along the sliding direction, thus making the problem quasi one dimensional. They are randomly distributed on the surfaces of the slider and of the track.

(ii) We neglect elastic interactions between traps (see above).

(iii) Traps are identical.

(iv) The trap potential is taken to be repulsive and of the form:

\[ \Phi(x) = \Phi_0 \Psi(x/a) \quad |x| < a \]
\[ = 0 \quad |x| > a. \]

The total energy of a trap is then:

\[ U = \Phi(\rho + u) + \lambda u^2 / 2 \]

where \( \rho \) is the configuration coordinate of the trap (see Sect. 2) and \( u \) its elastic displacement. We assume the trap to displace rigidly over its width \( a \) — i.e. \( u \) to be constant on its width \( a \) — that is, we neglect intra-asperity elastic strains.

At adiabatic equilibrium, \( u \) assumes the value \( u^* \) which minimizes \( U \), i.e.:

\[ \Psi'(z^*) + \beta(z^* - (\rho/a)) = 0 \]

where \( z^* = z^*(\rho/a) = (\rho + u^*)/a \) and the reciprocal \( \beta^{-1} \) of the dimensionless parameter

\[ \beta = \lambda a^2 / \Phi_0 \]

measures the trap pinning strength in terms of the elastic energy needed for displacing the asperity by an amount of the order of its size, i.e. for “breaking” the contact.

(v) For the numerical work, we take \( \Psi \) to be of the form:

\[ \Psi(x) = 1 + 2|x|^3 - 3x^2. \]

The overall shape of \( \Psi \) is reasonable and its use is justified at this stage, where the model remains schematic. It is convenient, as it makes the trap equilibrium equation (4) solvable analytically. This is shown in the appendix, where analytic expressions for the pinning force, the spinodal threshold, etc. are also derived. It is convenient, for \( \Psi \) given by (6), to define \( \alpha = \beta/6 \), as the system exhibits multistability if

\[ \alpha = \beta/6 < 1. \]
Fig. 1. — a) Pinning force $f_p(\rho)$ for a single elastic trap with potential given by equation (2); $\alpha = 0.13$. The stable branches are labelled (+) and (−), the metastable one is dotted. C and C’ are the spinodal points. The hysteretic area $H_o$ (Eq. (12)) is the sum of areas marked (I) and (II), or, equivalently, of (I) and (III). b) Distribution of asperities with slider at rest. The full lines represent the occupied parts of the (+) and (−) branches - under zero tangential load: the discontinuity of the distribution lies at the Maxwell plateau MM′; the total force $F = 0$, under a small finite tangential load $\delta F < F_b$: the distribution is rigidly shifted by $\delta \rho$; $\delta F$ (Eq. (13)) corresponds to the hatched area.

Moreover, two cases are possible, corresponding to different physical situations:
- if the slider is very soft (small $\alpha$) (or, equivalently, if trap pinning is strong), the bifurcation out of the trapped state (point C in Fig. 1a) takes place at $y_c = (\rho/a)_c > 1$. In this case, the asperities directly jump out of contact; moving traps die when $y = \rho/a$ reaches $y_c$. This is the case, for the form of $\Psi$ given by equation (6), when

$$\alpha < \alpha_o = 3 - 2\sqrt{2}$$

(8)

- if $\alpha_o < \alpha < 1$, the bifurcation occurs at $y_c < 1$, where traps jump into a state where $\Phi \neq 0$, they only die when $y$ reaches 1.

We assume from now on that pinning is “very strong”, i.e. that $\alpha < \alpha_o$. We have checked that this assumption, while simplifying to some extent numerical calculations, is qualitatively unimportant.

The corresponding net pinning force exerted on the track by the trap:

$$f_p(\rho) = \Phi'(\rho + u^*(\rho))$$

(9)
Fig. 2. — a) Pinning force curve $f_p(t_w)$ for the system aged at rest under zero tangential load, for $t_w=25t_o; \varepsilon=0.04$. Dashed line: elastic curve of Figure 1. Full and dotted lines: occupied and empty states for the aged system. b) Distribution at $t=t_{w+}$, immediately after the system of Figure 2a has been rapidly pulled up to its static threshold. Full line: occupied states. The upper edge of the distribution reaches $\rho_c(t_w)$.

is represented in Figure 1a, where the broken line corresponds to unstable equilibrium of the trap. It was shown in reference [14] that the total friction force can be written as:

$$F_{el} = \sum_r \int d\rho P^{(r)}(\rho)f_p(\rho)$$

(10)

$r = (+, -)$ labels the upper and lower stable branches of the hysteresis cycle (see Fig. 1a). $P^{(r)}(\rho)$ is the statistical trap population appropriate to a specific experimental situation. When the system is sliding towards positive $\rho$'s at the imposed velocity $V$, the (+) branch only is populated, and this homogeneously (due to the randomness of asperity distributions on both the slider and track) up to the spinodal limit. Then [14]:

$$P^{(+)}(\rho) = An_p(2a)^{-1} \quad P^{(-)}(\rho) = 0$$

(11)

where $A$ is the (apparent) area of the slider, $n_p$ the density of active traps $^3$. Introducing

\footnote{$^3n_p = nn_o(2a)^2$, where $n$ and $n_o$ are the densities of asperities on the slider and \textit{on the track}.}
As mentioned in Section 2, it is velocity independent. It was also shown in reference [14] that, in such a model, the static threshold force is equal to the dynamic one.

The elastic contact stiffness can also be evaluated, with the help of expression (10), as follows: when the horizontal pulling force is switched off and the two solids are left in contact under the normal load only, the system adjusts from its previous state of motion to the condition of zero horizontal force by undergoing a global elastic recoil. That is, the trap distribution gets shifted towards negative \( \rho \)'s until its discontinuity reaches the Maxwell plateau of the \( F_p (\rho) \) curve (see Fig. 2b).

Imposing a small horizontal displacement of the slider, \( \delta \rho \ll a \), from this equilibrium state, corresponds to shifting rigidly the trap population by \( \delta \rho \). The force \( \delta F \) necessary to perform this displacement corresponds to the hatched area shown in Figure 2b. So:

\[
\delta F \approx \frac{An_p}{2a} (\Delta f_p) \delta \rho
\]  

where \( \Delta f_p \) is the discontinuity of \( f_p \) across the Maxwell plateau. Therefore, the shear contact stiffness:

\[
\kappa \approx \frac{An_p}{2a} (\Delta f_p).
\]

3.2. Compressive Plastic Evolution of the Contacts. — We now want to modify the above elastic model in order to take into account the fact that, since the real area of contact (the total trap area) is much smaller than the apparent one, when the traps start their active life (either because the two solids have just been put into non moving contact or because motion is proceeding), they are submitted to normal stresses much above the yield stress \( Y \) of the material. This gives rise to slow irreversible plastic deformation of the asperities: as observed by Dieterich and Kilgore [6], contact sizes grow slowly — on average, quasi logarithmically.

In this perspective, we base our model on a simplified description of the evolution of contacts in a non moving system recently proposed by Brechet and Estrin [18]. They model the asperities as cylinders of equal initial radius and height with a flat horizontal base. Their scenario of plastic evolution then involves two phases. In a very short "pre-initial" phase, fast plastic flow results in an increase of the asperity radius up to a value \( a_o \) such that the normal stress borne by each asperity is reduced to about the compressive yield stress \( Y \).

Plastic evolution then proceeds through a second, slow phase of thermally activated creep across energy barriers biased by the applied stress \( \sigma \), the dynamics of which obeys an Eyring-Nabarro law (\(^4\)):

\[
2a^{-1} (da/ dt) = \dot{\varepsilon}_o \exp (\sigma / S)
\]  

where

\[
\sigma = Y (a/a_o)^2.
\]  

\(^4\)Note that the thermally activated processes underlying expression (15) are concerned with structural rearrangements on the atomic scale (e.g. vacancy or interstitial diffusive jumps, local changes of conformation of long molecules, or of small glassy clusters...). That is, thermal noise at room temperature is usually sufficient to drive them quite efficiently while, as mentioned in Section 1, it is much too small to activate jumps of the center of mass "macroscopic" degree of freedom.
Equations (15, 16) immediately yield:

$$a^2 = a_o^2[1 + \epsilon \ln(1 + t/t_o)]$$  \hspace{1cm} (17)

$\varepsilon$ and $t_o$ are related to the material parameters $\dot{\varepsilon}_o$ and $S$ (the strain rate sensitivity) by:

$$\varepsilon = S/Y; \quad t_o^{-1} = \dot{\varepsilon}_o Y/S \exp(Y/S).$$  \hspace{1cm} (18)

These considerations will now be extended phenomenologically in order to define a model, which, though incomplete and crude, yet incorporates a number of important physical ingredients in a consistent manner.

Before proceeding to describe this in more detail, let us briefly explain the general point of view which we adopt here. The authors of reference [18] had in mind creep due to the compressive plasticity of metals as the underlying microscopic mechanism, but the derivation given is, in essence, phenomenological, and applies equally well to a variety of materials, e.g. polymeric and/or glassy, where quasi logarithmic slow ageing $a$ is also known to take place, although via different microscopic processes such as local conformational changes or cluster rearrangement. Equations (15-18) represent the phenomenological basis of our mesoscopic model, which, though crude, has the virtue of being consistent as far as we are concerned here with the frictional dynamics and statics of a solid-solid multicontact interface under the combined effect of elastic forces and of creep caused by compressive forces alone. We expect this model to be relevant, as far as such effects are concerned, to a wide variety of materials, which only have in common the phenomenological logarithmic ageing law.

As pointed out in Section 1, we exclude from consideration plastic evolution of the contacts due to shear stress. We will comment on this strong assumption in the last section.

Our model of elasto-plastic contact is then specified by the following assumptions:

(i) The ageing law (12) is assumed to hold both in the static case and for motion at low sliding velocities. We define the plastic age of each active trap $i$ as $\phi_i = t - t_i$, where $t_i$ is the time at which the corresponding pair of asperities first came into contact. The characteristic radius of an active trap then increases with time as:

$$a_i(t) = a_o[1 + \varepsilon \ln(1 + \phi_i(t)/t_o)]^{1/2} = a_o a_r(\phi_i(t))$$  \hspace{1cm} (19)

where $a_o$ is the characteristic initial size of the trap, while $a_r$ is a dimensionless ageing function depending on the two phenomenological parameters defined in equation (18). $\varepsilon$ measures the creep rate. Its typical order of magnitude is 0.01 [18, 12]. The characteristic time $t_o$ defines an extrinsic time scale for the friction process, which should be compared with the dwell time for static friction, or with the trap sweeping time in the dynamic case. In view of its exponential dependence on the ratio $Y/S$, and of the material, pressure and temperature dependence of $\dot{\varepsilon}_o$, no typical order of magnitude can be safely assigned to it a priori.

(ii) We assume the elastic compliance $\lambda$ to scale as $a(t)$, that is

$$\lambda_i(t) = \lambda_o(a_i(t)/a_o),$$  \hspace{1cm} (20)

while the potential of each active trap $i$ at time $t$ retains the form (2), with the scaling laws

$$a \rightarrow a(t), \quad \Phi_o \rightarrow \Phi_o(t) = \Phi_{o0}(a(t)/a_o)^3.$$  \hspace{1cm} (21)

These assumptions are made in correspondence with the scaling estimates given in Section 2. For the elastic compliance, the scaling is a purely geometrical effect, while for the potential
strength, the scaling is chosen to reflect the creep induced age strengthening of an elastically compressed contact which has been initially crushed to size $a_0$ and is therefore not purely hertzian. With these assumptions, the dimensionless pinning strength is not affected by the creep process:

$$\alpha = \text{const},$$

(22)

remains time independent and provides a signature of a given trap.

The above scaling assumption is at least qualitatively consistent with the above estimates ($\lambda \approx a$, $\Phi_0 \approx a^3$) for the elastic response of asperities which have been initially crushed plastically to size $a_0$, and are therefore not purely hertzian.

4. Static Ageing and Static Friction

Let us now consider what appears (at least at first sight) the most simple ageing situation, namely the case where the slider and track are put into non moving contact at time $0$. At time $t > 0$ all the active traps have the same age: $\phi_i(t) = t$, i.e. the same size $a(t)$. Since, under the assumption of constant $\alpha$, the equilibrium elastic displacements $u^*_i$ scale as $a(t)$, the net pinning force at time $t$ on a trap of configuration coordinate $\rho$, age $\phi_i(t)$, is:

$$f_p(\rho_i, \phi_i(t)) = \left( \frac{a^2(t)}{a_0^2} \right) \Phi_{00} \Phi' \left( z^* \left( \frac{\rho_i}{a(t)} \right) \right)$$

(23)

that is, the hysteretic curve of Figure 1 grows in size while retaining its shape (5).

Let us denote $P_t(\rho)$ the normalized statistical distribution of trap configuration coordinates at time $t$. The total horizontal force on the slider reads:

$$F(t) = N_p \int d\rho \sum_r P_t^{(r)}(\rho) f_p^{(r)}(\rho, \phi_i(t))$$

(24)

where $N_p = An_p$ is the total number of active traps.

Assume from now on that the static friction coefficient is measured according to the following (realistic) protocol: the system is put into contact and kept under zero horizontal load during a waiting time $t_w$. At this instant, a rapidly increasing horizontal force is switched on, and slider motion is recorded. Let us call $F(t)$ the corresponding static threshold.

At the time of first contact, $a = a_0$, $F = 0$. The system adjusts instantaneously by elastic recoil so as to satisfy the zero horizontal loading condition. We assume asperities to be distributed homogeneously in the absence of contact. The distribution $P_t^{(r)}(\rho)$ is then necessarily of the form

$$P_t^{(+)}(\rho) = C\Theta[(\rho_M - \rho)(\rho + a_0)]$$

(25a)

$$P_t^{(-)}(\rho) = C\Theta[(\rho - \rho_M)(a_0 - \rho)]$$

(25b)

with $\Theta$ the Heaviside function and $\rho_M$ the position of the Maxwell plateau. $\rho_M = 0$ for our $x$-symmetric potential $\Phi$. The normalizing constant $C = [2a_0]^{-1}$

As time elapses, asperities which are not engaged into contact ($|\rho| > a_0$) do not evolve, while the potentials of the active traps evolve according to (21). Meanwhile, $P(\rho)$ remains

(5) Note that expression (15) implies that we consider that the duration of the "preinitial" stage of Brechet and Estrin is negligible on the scale of the times of interest (typically, waiting times in static friction experiments are at least seconds, while trap crossing times in the low velocity dynamic regime of interest here are larger than $10^{-2}$s).
unchanged, so that, for \( t < t_w \), the distribution of net pinning forces has the structure shown in Figure 2a.

When the pulling force is switched on, the slider moves globally (say, to the right) by some finite \( \delta \rho \), which results in a rigid translation of \( P \) by the same amount. Irreversible motion starts (i.e. the static threshold is reached) when the discontinuity of \( P \) reaches the spinodal bifurcation point, i.e. when \( \delta \rho = \rho_c(t_w) \) determined by:

\[
p_c(t_w) = a(t_w) \gamma_c.
\]

As the pulling force is increased further, the (+) contacts with \( \rho > \rho_c \) start snapping off down to the (-) branch of the hysteresis curve. Note that the asperities which have been thus pulled into contact \((-a_o < \rho < -a_o + \rho_c(t_w))\) feel the “fresh” potential \( \Phi(\rho, a = a_o) \). The static friction force is given by equation (24), with:

\[
F_s(t_w) = P^0(\rho + \rho_c(t_w)).
\]

This distribution is shown in Figure 2b. Then:

\[
F_s(t_w) = \frac{N_p \Phi_0}{2a_o^2} \left\{ \alpha^2(t_w) \int_{-a_o + \rho_c(t_w)}^{\rho_c(t_w)} d\rho \Psi'\left(\frac{z(\rho)}{a(t_w)}\right) + \int_{-a_o}^{-a_o + \rho_c(t_w)} d\rho \Psi'\left(\frac{z(\rho)}{a(t_w)}\right) \right\}.
\]

Given expression (17) for \( a(t) \), \( F_s \) only depends on the reduced time variable \( \tau = t_w/t_o \). The reduced static force \( F_s(\tau) \), measured in units of \( F_{el} \) (Eq. (12)), depends on the creep parameter \( \varepsilon \) and on the reduced pinning strength \( a \). In all the numerical illustrations to follow, we have chosen \( a = 0.13 \). Figure 3 shows a plot of \( F_s(\tau) \), for several values of \( \varepsilon \). It does not simply scale as \( f_p \) (i.e. as \( a_o^2(t) \)). This stems from two facts:

(i) at the instant when sliding starts, the set of active traps is composed for part of contacts which have experienced plastic ageing, for part of newborn ones which got engaged when the pulling force was applied,

(ii) the relevant spinodal limit is that for the aged traps, increased by ageing from the elastic value \( \gamma_c a_o \) to \( \gamma_c a_o \gamma_a(t_w) \).
To first order in $\varepsilon$, expression (27) reduces to:

$$\delta \tilde{F}_s(\tau) = \tilde{F}_s(\tau) - 1 \approx \varepsilon \frac{3H_1}{2H_0} \ln(1 + \tau)$$  (28a)

where $H_0$ is the total area of the elastic hysteresis cycle (Eq. (12)) and:

$$H_1 = \int_{-1 + y_c}^{y_c} dy \Psi'(\varepsilon^+(y)).$$  (28b)

The experimental results on $\delta \tilde{F}_s$ obtained by Heslot et al. [7] for paper and by Baumberger et al. [12] for PMMA are well fitted by linear $\ln \tau$ laws. This, compared with equation (28a), entails that the plastic time $t_o$ for these materials is smaller than the shortest $t_w$, of order 1 s, in the experiments. The measured $d \ln \delta \tilde{F}_s/d \ln \tau$ are on the order of a few percent, which confirms the fact that $\varepsilon$ indeed lies in the $10^{-2}$ range.

It should now be clear that the expression for the static force derived above is valid only for the assumed experimental protocol (waiting under zero horizontal load). Had we for example assumed that, as soon as contact between the slider and track is established, a horizontal force is applied, with a value such that the system remains in the reversible elastic regime, the initial distribution $P_0(\rho)$ would have had its discontinuity at a non zero value of $\rho$ defined by the contact elastic stiffness. The translated distribution corresponding to incipient sliding would therefore contain more ripe contacts and less newborn ones, both $\mu_p$ and $d \mu_p/d \ln t_w$ would therefore be larger than those measured for ripening under zero horizontal loading. Such a trend is indeed observed experimentally [19]. However, this qualitative agreement has to be considered with extreme care. Indeed, our crude model of plastic evolution does not take into account the possible contribution of shear induced plasticity.

The above remark leads us to emphasize that the static friction coefficient is not a uniquely defined characteristic of a couple of materials. It depends on the experimental protocol used to measure it. This fact is often overlooked because the relative variations of $\mu_p$ are small. It is nevertheless of conceptual importance since, as already appears, it is precisely these variations which contain most of the information about the physical processes at play in friction. We will come back to this point later.

5. Stationary Dynamic Friction

Consider now the case where the slider is moving to the right at the constant velocity $V$. In this stationary regime, asperities get engaged at a constant rate into active traps and dragged across the trap potentials up till when they jump out of contact (bifurcate down) at the same rate. In the absence of any ageing process, this means that they populate uniformly all the states of branch (+) of the elastic hysteresis curve.

In the presence of plastic ageing, the contacts evolve as the system is moving. Trap $i$ becomes active (contact $i$ is created) at time $t_i$ when: $\rho_i(t_i) = -a_o$. At time $t_i$ $\rho_i = -a_o + V(t - t_i)$. The age of a trap $\phi_i(t) = t - t_i$ is therefore a linear function of its configuration coordinate:

$$\phi_i(\rho_i) = (\rho_i + a_o)/V$$  (29)

corresponding to a trap radius

$$a(\rho) = a_o[1 + \varepsilon \ln(1 + (\rho + a_o)/Vt_o)]^{1/2}$$  (30)
The set of active traps now populates uniformly the (+) states of instantaneous equilibrium of the continuously deforming potential $U(\rho, u|a(\rho))$. So, a trap with configuration coordinate $\rho$ experiences a net pinning force

$$f_p^{(st)}(\rho) = (a(\rho)/a_o)^2 \Phi_{00} a_o^{-1} \Psi'(z^+(\rho/a(\rho))).$$

(31)

The corresponding hysteresis curve is shown in Figure 4. Contrary to what was the case for static ageing, it cannot be deduced from the purely elastic curve by a simple scaling, due to the fact that, in steady motion, contact age continuously increases with $\rho$.

The bifurcation takes place for $\rho = \rho_c(V)$, with $\rho_c$ defined by

$$\rho_c = y_c$$

where, due to our scaling assumption (20, 21), $y_c$ is a fixed number.

So, the normalized distribution

$$P^{(+)}(\rho) = (2a_o)^{-1}; \quad P^{(-)}(\rho) = 0.$$  

(33)

The reduced steady dynamic friction force then finally reads:

$$\tilde{F}_d^{(st)}(V) = F_d^{(st)}/F_{el} = H(V)/H_o$$  

(34a)

with:

$$H(V) = \int_{-a_o}^{\rho_c} \frac{d\rho}{a_o} a_{\tau}^2(\rho) \Psi' \left[ z^{(+)} \left( \frac{\rho}{a_o a_{\tau}(\rho)} \right) \right]$$

(34b)

which can be rewritten, when setting $\rho/a_{\tau}(\rho) = y$, as:

$$H(V) = \int_{-1}^{y_c} dy \Psi' \left[ z^{(+)}(y) a_{\tau}^2[\rho(y)] \left( \frac{d(\rho/a_o)}{dy} \right) \right].$$

(34c)

Expanding this expression to 1st order in $\varepsilon$, one finds, in the low velocity limit $V \ll V_o = a_o/t_o$, that:

$$\tilde{F}_d^{(st)} - 1 \approx -\frac{3}{2} \varepsilon \ln(V/V_o).$$

(35)
Fig. 5. — Dimensionless steady dynamic friction force $\tilde{F}_d^{(st)}$ versus reduced velocity $v = V/V_o$, for $\varepsilon = 0.01$ (1); 0.02 (2)...; 0.05 (5).

$\tilde{F}_d^{(st)}(V)$ as computed from expressions (34a, b) is plotted in Figure 5 for several values of the creep parameter $\varepsilon$. As is intuitively expected, it exhibits a quasi logarithmic velocity weakening behavior, which saturates to 1 for $V \gg V_o$, the characteristic velocity at which the time for sweeping across a trap becomes smaller than the plastic time $t_o$. Such a saturating behavior has been observed on paper [7] for sliding velocities on the order of a few 10 $\mu$m s$^{-1}$. If the characteristic trap size $a_o$ is, as seems natural, identified with the “memory length” $D_o$ (see below), of the order of 1 $\mu$m for this material, a rough fit of experimental data on the steady dynamic friction coefficient yields a value of $t_o$ on the order of $10^{-1}$ s. No saturation regime has been observed on glassy PMMA at room temperature for $V \leq 100$ $\mu$m s$^{-1}$ [20], which points to the fact that, in this case, $t_o < 10^{-2}$ s $^6$.

6. Comparison between Static and Steady Dynamic Friction

Following Dieterich, several authors have proposed that the dry friction dynamics is characterized by a memory length $D_o$, which can be interpreted as the sliding length necessary to destroy a set of contacts and replace them by fresh ones. This has led Dieterich [4] and Scholtz [5] to propose that the static and stationary dynamic friction coefficients are related by the “scaling law”:

$$\mu_d(V) = \mu_s(D_o/V)$$  \hspace{1cm} (36)

which has been reasonably well confirmed by low velocity experimental data on very different classes of materials, ranging from rocks to glassy polymers. “Reasonably” here means that, within experimental accuracy, the slopes $d\mu_s/d\ln t_w$ and $d\mu_d/d\ln V$ are roughly equal. Fits performed according to (36) then yield values of $D_o$. These range, as already mentioned,

$^6$ Preliminary results [19] on PMMA at temperatures of order 90 $^\circ$C, close below the glass transition, seem to show a trend towards saturation of $\mu_d$ in the range $V \approx 100$ $\mu$m s$^{-1}$, which would suggest an increase of $t_o$ in these regime. However, this must be considered only as an indication which should be confirmed by further study.
Fig. 6. — Plot of \( F_0^{(st)}(v) \) (full line) and of \( F_s(a_o C/t_w V_o) \) for various values of the fitting parameter \( C \): (---) \( C = 1 \); (---) \( C = 2, 3, 4 \); (-----) \( C = 2.45 \).

in the micrometer range — \( i.e. \) are roughly of the same order of magnitude as the contact sizes observed optically by Dieterich [6].

The qualitative physical basis of statement (36) is, we think, unquestionable: dry friction dynamics is essentially controlled by the interrupted slow irreversible evolution of a sparse random set of contacts. This dynamical birth and death process is characterized by a length on the order of the average contact size. Since these physical ingredients are precisely the basis of our model, we are now in a position to evaluate its quantitative validity in more detail.

A first, important point, is the following. We have seen in Section 4 that the rate of strengthening of static friction depends on the ageing conditions during the waiting time (\( i.e. \) on the experimental protocol), while \( \mu_d \) is a uniquely defined, intrinsic, quantity. This remark suffices to prove that (36) cannot be a quantitatively exact relation.

In order to test its heuristic value for analysis of experimental data, we have compared \( F_0^{(st)}\) (Eq. (34)) and \( F_s \) (Eq. (27)) corresponding to waiting under zero horizontal load, following the fitting procedure used for experimental data. That is, we plot in Figure 6 \( F_0^{(st)} \) and \( F_s \) for \( \varepsilon = 0.04 \), \( versus \), respectively, \( \log(V/V_o) \) and \( \log(a_o C/t_w V_o) \). \( C \) is a floating constant with respect to which one performs a one parameter fit. The Dieterich length is then \( D_o = C a_o \).

It is seen in Figure 6 that a global fit is not possible. However, the two curves can be brought into asymptotic coincidence in the high \( V \)/small \( t_w \) limit, by the choice \( D_o \approx 2.45 a_o \), which is the traversal length \( (1 + y_c) a_o \) for the pinning strength \( (\alpha = 0.13) \) which we have used.

In practice, as already mentioned, high \( V \) saturation has been clearly observed, up to now, only on the paper-on-paper system. The other existing data, which are reasonably log-linear, seem to be ascribable to the low \( V \)/large \( t_w \) limit. In this range we expect, on the basis of equations (28a) and (35), the slopes of the \( F_0^{(st)} \) and \( F_s \) curves to be different, although of comparable magnitudes. This, together with the fact that the corresponding variations of \( \mu_s \) and \( \mu_d \) are numerically small, means that very precise experiments are needed for this difference to be evidenced. It has indeed been observed recently in the recent experiments of Baumerger and Berthoud [20] on the temperature dependence of friction of PMMA-on-PMMA.
In summary, we contend that, if (36) is undoubtedly useful to extract from experimental data an order of magnitude of a memory length, it is not an exact scaling law. This results from the fact that, even though the nature of the ageing process is common to all situations, the distribution of ages among the set of contacts relevant to measurements under different conditions depends on the history of the ageing conditions, i.e. on the dynamical and loading history of the system.

Finally, we would like to stress one more point, concerning the physical meaning of the memory length \( D_0 \). It is, as intuitively expected, of the order of the contact diameter. This characteristic length emerges in many contexts related to friction: it is directly observable [6], it enters the interface roughness statistics [2], and more recently it has been identified [13] as a characteristic of the static elastic stiffness of multicontact interfaces. However, only in the memory length can we identify the asperity size as related to a length characteristic of an irreversible dissipative process.

7. Linear Stability of the Stationary Motion

We have studied in the previous two sections the effect of compressive plasticity on dynamical friction in steady motion at constant velocity \( V \). In actual experimental situations, motion is imposed to the slider through a pulling machine of finite stiffness \( K \). If this is small enough, the slider does not move steadily at the externally imposed pulling velocity \( V \), but exhibits stick-slip motion (a periodic alternation of fast slips at the end of which it “sticks” again to the track for a finite time).

As mentioned in Section 1, the recent detailed studies of the corresponding dynamical phase diagram of slow frictional motion in the space of the control parameters \((V,K)\) have proved to be very fruitful in terms of both qualitative and quantitative information. In particular, the position in parameter space of the bifurcation curve \( K = K_c(V) \) which separates the steady sliding regime \((K > K_c(V))\) from that where motion occurs via periodic stick-slip has been shown to provide much more detailed and accurate information about the \( \mu_d(V) \) curve than do direct measurements of the friction coefficient itself.

This suggests that the study of accelerated motion should provide a much finer probe of friction models than does that of the constant velocity regime. A natural first step in this direction is to study the linear stability of this dynamical regime. This is what we do in this section, by studying the time evolution of small amplitude fluctuations of the slider velocity about steady motion at the external constant pulling velocity \( V \).

For this purpose, we first derive the expression for the frequencies of the corresponding eigenmodes. We then show that, although the present model does give rise to an instability with respect to oscillatory motion, the nature of the corresponding bifurcation does not reproduce satisfactorily all the experimentally observed features of this instability, even when we take into account the statistical distribution of contact sizes.

7.1. The Linear Fluctuation Spectrum. — We consider the system represented schematically in Figure 7. Let us call \( X(t) = Vt \) the (externally imposed) position of the pulling point (right end of the “pulling spring” of stiffness \( K \), equilibrium length \( L \)) in the frame of the track. For the sake of definiteness, we assume that the spring is attached at point \( O \) (position \( X_o \)) to a rigid non deformable solid layer covering the upper surface of the slider. \( X_s \) is the real position of the active asperity (i) in the elastically deformed system, \( \xi \) the position of the center of the trap of which (i) is part (\( \xi \) can be viewed as the position of the active counter-asperity on the track), \( d \) the equilibrium distance separating this asperity from \( O \) in the unstressed solid along the pulling direction.
Again, we assume here that motion is slow enough for inertial effects to be negligible, i.e. that the typical trap sweeping time is much larger than the inertial period:

$$2a_o/V \gg (M/K)^{1/2},$$

where $M$ is the slider (plus rigid overlayer) mass.

The system then obeys the conditions of instantaneous mechanical equilibrium, namely:

- global equilibrium of the slider, which reads:

$$F_d = K[X(t) - X_o - L] = \sum_i \partial \phi(X_i - \xi_i)/\partial X_i,$$

where $F_d$ is the dynamical friction force on the slider,

- asperity equilibrium:

$$-\lambda[X_i - d_i - X_o] - \partial \phi(X_i - \xi_i)/\partial X_i = 0.$$  

Note that $(d_i + X_o - \xi_i)$ is precisely the configuration coordinate $\rho_i(t)$ of the $i$ trap, while $(X_i - d_i - X_o)$ is its elastic displacement $u_i$. So, equation (39) is identical with equations (3, 4), with:

$$\rho_i(t) = X(t) - L - (F_d/K) + d_i - \xi_i$$  

and its solution(s) read(s):

$$X_i - \xi_i = z^{\tau}[\rho_i(t)].$$

In equations (39, 40) we have neglected the shear compressibility of the slider as compared with that of the pulling machine, according to usual experimental conditions.

In order to specify the trap potentials in our plastic system, we must know what is the age of a trap which has, at time $t$, a given value of $\rho$. We now consider situations where the slider motion is slightly perturbed with respect to the stationary regime, where $\rho_{i,\text{st}}(t) = -a_o + V(t - t_i)$. That is, now:

$$\rho_i(t) + a_o = V(t - t_i) + \delta \rho(t) - \delta \rho(t_i).$$

To first order in $\delta \rho$, the perturbed value of the age is:

$$\phi_i(t, \rho) = (\rho + a_o)/V - [\delta \rho(t) - \delta \rho(t - (\rho + a_o))] / V.$$

from which one obtains from equation (19) the corresponding value of the asperity radius $a(\rho, t) = a_{\text{st}}(\rho) + \delta a(\rho, t)$. The unperturbed $a_{\text{st}}(\rho)$ is given by equation (30).

This explicit dependence on time of the age of a trap in a given configuration $\rho$ expresses the fact that, when the motion is accelerated, the time needed to sweep the distance $(\rho + a_o)$...
depends not only on the distance itself, but on when the trap was born (or, equivalently, arrives at \( \rho \)). It results in the fact that the hysteresis curve becomes explicitly time dependent.

Following the same steps as in the previous section, we get for the reduced friction force:

\[
\tilde{F}_d(t) = \frac{F_d(t)}{F_\mathrm{el}} = \int_{-\infty}^{\rho_e(t)} d\rho \Psi' \left[ z^{(+)} \left( \frac{\rho}{a(\rho, t)} \right) \right] a_\rho^2(\rho, t). \tag{43}
\]

Setting \( \rho/a(\rho, t) = y \), linearizing expression (43) in \( \delta \rho \), and using the variation of equation (40a), namely: \( \delta \rho(t) = -\left( \delta F_d / K \right) \), we obtain:

\[
-K \delta \rho(t)/F_\mathrm{el} = \delta H(t)/H_\circ \tag{44a}
\]

with:

\[
\delta H(t) = \int_{-1}^{y_c} dy \Psi'[z^{(+)}(y)] \frac{a_\rho^2(\rho, t)}{1 - y(d\rho/d\rho)} \left[ \frac{3 \delta a(\rho, t)}{a_\rho(\rho)} + y \frac{d(\delta a(\rho, t)/dt)}{1 - y(d\rho/d\rho)} \right]. \tag{44b}
\]

Since \( \delta a \) can be expressed as a linear functional of \( \delta \rho(t) \), the linear homogeneous set of equations (44) provides the dispersion relation of our frictional dynamical system, the eigenfunctions of which describe the time evolution of linear fluctuations about the steady sliding state. Since this is time invariant, the linear eigenmodes have, as can be checked directly, the form

\[
\delta \rho(t) = \eta \exp(i\nu t). \tag{45}
\]

When expression (35) is plugged into (44), it yields an algebraic equation for the discrete set of \( (a \text{ priori complex}) \) eigenfrequencies \( \nu_n \), of the form:

\[
K a_\circ/F_\mathrm{el} = Y(\nu). \tag{46}
\]

The non dimensional frequency \( \omega = \nu a_\circ/V \), is measured in units of the natural frequency of our steady problem, namely the inverse of the trap sweeping time.

We have checked numerically that, for \( \varepsilon \) in the range of interest \( (\approx 10^{-2}) \), one can safely approximate the adimensional response function \( Y(\nu) \) by its first order \( \varepsilon \)-expansion, which reads:

\[
Y(\nu) = \varepsilon \frac{2}{H_\circ} \int_0^{1+y_c} d\eta \Psi'[z^{(+)}(\eta - 1)] \left\{ i \omega (\eta - 1) e^{-\omega \eta} + (1 - e^{-\omega \eta}) \left( 3 - \frac{\eta - 1}{\eta + v} \right) \right\}. \tag{47}
\]

For a given value of the reduced velocity \( v = V/V_\circ \), if, in some range of \( K \), all \( \omega_n \) have positive imaginary parts, all fluctuations decay, stationary motion is linearly stable. Instability occurs at the value \( K_c(V) \) for which the solution \( \omega_1 \) with smallest imaginary part crosses the imaginary \( \omega \)-axis.

7.2. Looking for the Stick-Slip Bifurcation. — At the bifurcation, if it exists, equation (46) must therefore have a real solution: \( \omega_1 = \Omega \). Setting

\[
Y(\Omega) = Y_R(\Omega) + iY_I(\Omega) \tag{48}
\]

we can rewrite equation (47) as a set of two real equations:

\[
\begin{align*}
Y_I(\Omega) &= 0 \tag{49a} \\
Y_R(\Omega) &= K_c a_\circ/F^{(st)}_d(V). \tag{49b}
\end{align*}
\]
Fig. 8.—(a) Real and (b) imaginary parts $Y_R$, $Y_I$ of the response function $Y$ (Eq. (47)) versus reduced frequency $\omega$, for $\varepsilon = 0.04$, $\nu = 0.25$.

In order to locate the bifurcation we must first find the solutions $\Omega_n$ of equation (48a). The bifurcation curve is then given by $K_c(\nu)a_o = F_0 \max\{Y_R(\Omega_n)\}$.

Note that $Y(\Omega)$ depends on the two dimensionless parameters $\varepsilon, \alpha$, and on the reduced velocity $\nu$. Due to the steadiness of the unperturbed state, $Y(0) = 0$. $Y_I(\Omega)$ and $Y_R(\Omega)$ are, respectively, odd and even functions of $\Omega$.

We have computed them for values of $\varepsilon$ from 0.01 to 0.05, values of $\alpha$ ranging between $\alpha_o$ and 0.10, and for $\nu$ between $10^{-2}$ and $10^2$. Figure 8 shows a typical plot. Both functions are seen to exhibit very slowly decreasing oscillations with periods of the order of a few units. This behavior is robust against parameter variations.

This entails that the $\{\Omega_n\}$ set is very large — if finite. Then, which particular $\Omega_n$, $\Omega_{bif}$, corresponds to the largest $Y_R$, depends on the details of the relative location of the oscillations of the two functions, and so does the value of $K_c(\nu)$. Also, in such a situation, there is a large probability that several different $\Omega_n$ yield values of $Y_R$ very close below $Y(\Omega_{bif})$, which signals,
correspondingly, the existence of several slowly decaying modes besides the one which gives rise to the instability (7).

In other words, we do find a stick-slip bifurcation, but its characteristics do not exhibit the robustness which can be inferred from the experimentally observed behavior — namely, the stick-slip reduced frequency \( \Omega_{\text{bif}} \) is, in the cases which have been investigated, stably on the order of a few units, and, in the vicinity of the bifurcation, all the observed dynamical features are very well described in terms of a single slow mode.

This leads us to reexamine critically the assumptions of our model, in the light of the following remark: experimental results seem to point towards response functions with smooth shapes and at most only a few oscillations. On the other hand, the oscillatory behavior of \( Y_{\text{R},1} \) can be traced back to our description of the trap instability. Indeed, they appear (see Eq. (47)) as Fourier transforms of functions which are non zero only on the finite range \((-1, y_c)\) of the trap. While these functions go smoothly to 0 for \( \eta = 1 + (\rho / a) = 0 \), they exhibit at the other end of the interval discontinuities related to the fact that, in our description, asperities jump instantaneously out of the trap at a common spinodal limit. This gives rise to the well-known Gibbs oscillation phenomenon. More precisely, one easily shows, when analyzing the contribution of the step discontinuity of \( \Psi' \) to the integral (47), that, while the last two terms in the curly brackets produce, for large \( \Omega \), a term \( \approx \Omega^{-1}\text{osc}(\Omega) \), the first one results in a completely undamped oscillatory term.

We are therefore led to reconsider at least two of our previous assumptions.

The first one is the approximation of quasi static equilibrium of the traps, which leads to neglecting the finite, though small, duration of the spinodal jump. This is determined by asperity inertia and by the friction term which account for the fact that the elastic energy of asperities decays via radiation of elastic waves into the bulk of the slider. As was done in reference [14], such effects can be estimated by assuming that asperity motion is essentially dissipative. Equation (4) then becomes

\[
m\gamma \ddot{u} = -\Phi'(\rho + u) - \lambda u
\]

(50)

where the friction coefficient \( \gamma \approx Ea^2 / mc_a \), with \( c_a \) the sound velocity.

The resulting delay of the spinodal jump translates into a smoothing of the discontinuity of the hysteresis curve on a width \( \Delta \rho \approx a_0 (V/c_a)^{2/3} \) [14], which induces a decay of the response functions \( Y_i \) on a range \( \Delta \Omega \approx (c_a/V)^{2/3} \). Since we are interested here in the low velocity regime where, typically, \( V \) is of the order of a few ten \( \mu m/s \) at most, \( \Delta \Omega > 10^4 \).

That is, taking into account the non instantaneity of trap dynamics does not solve, for all practical purposes, the questions raised above.

The next natural improvement on our schematic model is concerned with trap \( i.e. \) asperity) statistics.

7.3. Effect of Trap Size Statistics. — Up to now, we have assumed for the sake of simplicity all traps to be identical, which is obviously a caricatural representation of contact between two real surface profiles. Indeed, as studied in detail in particular by Greenwood, such profiles exhibit random distributions of asperity heights. The corresponding statistical distribution functions are in general well peaked about an average value \( \overline{\sigma}_c \), and have rapidly decreasing tails. Such a structure has for example recently been observed and characterized

(7) Worse, when parameters are varied, the \( Y_R \) value corresponding to some different \( \Omega_a \) may degenerate with \( Y_R(\Omega_{\text{bif}}) \), leading, if several of these frequencies are commensurate, to a more complicated bifurcation dynamics.
by Berthoud et al. [13] on surfaces of ground PMMA. One may reasonably approximate such a distribution by a truncated Gaussian — namely, setting \( a_o = \bar{a}_o x \), by:

\[
g(x)dx = \exp \left( - \frac{(x - 1)^2}{\sigma^2} \right) dx \quad (x > 0).
\]

(51)

In the numerical calculations to follow, we have chosen, on the basis of semi quantitative estimates from reference [13], \( \sigma = 10^{-1} \)

Assuming that each trap still evolves, from its particular initial radius \( a_o \), as specified by equations (19–22), friction forces become statistical averages, weighted according to (51), of those calculated above. That is, the friction force is now given by:

\[
F_d(t) = \frac{N_d}{2a_o} \int_0^{+\infty} dx x^{-1} g(x) \int d\rho f(\rho, t; a_o)
\]

(52)

where \( f \) is the pinning force appropriate to a given ageing history.

We have checked that, as can be expected, the smearing of the trap size distribution only results in minute changes of the numerical values of the friction force in the steady sliding regime, but does not affect the shape of its velocity dependence.

This is to be contrasted with the effect on the response function \( Y(\omega) \). Indeed, the trap size distribution results in smearing the position \( \rho_c \) of the spinodal jump, hence in a corresponding damping of the oscillations of the response \( \langle Y \rangle \) of the system of randomly sized contacts. Equations (46, 47) for the single sized traps are easily extended, with the help of (52), into

\[
\langle Y(\bar{\omega}, \bar{v}) \rangle = \frac{H_o}{2} \int d\eta \frac{1}{\eta + \bar{\omega}} \left\{ \langle x^2 \Psi' \rangle \left[ \frac{i\omega \gamma e^{-i\bar{\omega} \eta}}{\eta + \bar{\omega}} + \frac{2 \eta + 3 \bar{\omega}}{\eta + \bar{\omega}} \left( 1 - e^{-i\bar{\omega} \eta} \right) \right] + \langle x^3 \Psi' \rangle \left[ -i\omega e^{-i\bar{\omega} \eta} + \frac{1 - e^{-i\bar{\omega} \eta}}{\eta + \bar{\omega}} \right] \right\}
\]

(53)

\( \langle \rangle \) designates averaging with weight (51), \( \Psi' \) stands for \( \Psi'[z(+) (\frac{u}{x} - 1)] \), and the reduced frequency and velocity are now referred to \( \bar{a}_o \):

\[
\bar{\omega} = \nu \bar{a}_o / V; \quad v = \bar{v} / x.
\]

(54)

The condition for the appearance of an oscillatory instability of steady sliding correspondingly reads, in the same notations:

\[
K \bar{a}_o / F_{cl} = \langle Y(\bar{\Omega}, \bar{v}) \rangle.
\]

(55)

Figure 9 shows a plot of the real and imaginary parts of \( \langle Y(\bar{\Omega}) \rangle \) for various values of \( \bar{v} \). It is seen that trap size smearing results in the fact that \( \langle Y \rangle_1 \) now has only a finite number of zeros (the larger the width \( \sigma \), the smaller this number). However, it can be shown that \( \langle Y \rangle_1 \) is always positive both at very small and large frequencies, so that this number is always even. As \( \bar{v} \) is decreased, \( \langle Y \rangle_1 \) increases, so that pairs of zeros, starting with the largest ones, gradually merge and disappear. This qualitative result is robust against variations of the width parameter \( \sigma \). That is, we find that the stick-slip instability disappears at velocities somewhat smaller than \( V_o \). The lower \( V \) limit of the bifurcation range increases slowly with \( \sigma \).

We also calculate the values of \( K \) corresponding to the various zeros of \( \langle Y \rangle_1 \) (see Eqs. (49b, 55)), the largest of which determines the critical \( K_c \). Figure 10 shows that, for \( \sigma = 0.10 \), at the larger velocities, it is the lowest of these frequencies, \( \bar{\Omega}_1 \), which determines \( K_c \). However, as \( V \) is decreased, the two branches \( K_1 \) and \( K_3 \) cross. This indicates that the stick-slip frequency
at the bifurcation should correspondingly exhibit a rapid variation with velocity from $\bar{\Omega}_1$ to $\bar{\Omega}_3$, while, in the cross over region, the bifurcation dynamics should appear bimodal.

Finally, we find that, when the bifurcation exists, $K_c$ is a slowly decreasing function of $V$, with values stably on the order of $d(\ln F_d^{\text{st}})/d(\ln V)$ — two features which agree qualitatively with observations. But, on the other hand, neither a rapid variation of the frequency along the stick-slip bifurcation curve, nor the existence of a low velocity regime where steady sliding would be absolutely stable are observed experimentally.

That is, including trap size statistics effects into our model of compressive plastic contact evolution does result in a stick-slip bifurcation with characteristics reasonably close to those actually observed, but still presenting features which do not appear wholly satisfactory in the light of presently available experimental data.
Fig. 10. — Values \((Y_R(\vec{\Omega}_n))\) of the real part of the response function for the frequencies \(\vec{\Omega}_n(\nu)\) of the various zeros of \((Y_R(\vec{\Omega}))\), plotted versus the reduced velocity \(\vec{\nu}\). The largest \((Y_R)\) defines the critical stiffness \(K_c\). Curves are labelled according to the natural order of the zeros of \((Y_1)\).

8. Discussion

In summary, we have presented here what we believe to be the simplest possible version of an elasto-plastic model of dry friction.

Its basic physical ingredients are:

(i) a set of sparse real contacts (traps) distributed randomly on the two solid surfaces involved in the frictional process,

(ii) these traps are elastically multistable. The friction force originates primarily from the irreversible dissipation of energy associated with the corresponding spinodal instability (which provides a schematic representation of a contact breaking event),

(iii) they evolve (age) slowly, due to Nabarro-like plastic creep. This evolution is taken into account in an essentially phenomenological way, in terms of a quasi-logarithmic increase of their width and strength.

The net pinning force on a contact thus depends on its age, which is determined by the sliding dynamics itself. The resulting change in the basic elastic hysteretic force curve thus depends on the nature of the dynamical regime.

This enables us to account with semi quantitative accuracy for the observed quasi-logarithmic increase of the static friction threshold with waiting time before sliding, and for the corresponding velocity weakening of the dynamic friction force in the regime of steady motion. The memory length which emerges from experimental studies of the sliding dynamics naturally appears as related with the basic trap size.

However, we show that the scaling law relating the static and dynamic friction coefficients first proposed by Dieterich, although it provides a useful semi quantitative guide, is not an exact relation.

Our analysis of static friction also shows that, in as much as ageing effects are important, the static threshold is not an intrinsic property of a dry friction couple — it depends on the details of the experimental protocol. A meaningful analysis of waiting time effects must specify the horizontal loading conditions.
The study of the linear stability of stationary motion against stick-slip for this basic version of the model opens a whole set of questions while suggesting an important qualitative remark. Trap strengthening ageing effects resulting from the plastic creep induced by compressive forces are able to destabilize steady sliding and result into stick-slip. This result may at first sight appear surprising in view of the usual phenomenological description of this instability in terms of constitutive equations à la Ruina-Rice. Indeed, our present model only accounts for the mechanism corresponding, in the constitutive equations description, to the age dependent contribution to $\mu_k$, which is velocity-weakening. In that approach, this term needs to be balanced by a counteracting $V$-strengthening (or, equivalently, time weakening) one for an oscillatory instability to result.

Constitutive equations assume that contact strengthening effects can be described by an average age variable, the heuristic definition of which does not properly capture the details of the dependence of the ageing process on the history of the motion of the set of contacts, while our more "microscopic" model deals with the full statistical distribution of ages. This is why we find that stick-slip may occur even in the absence of any mechanism leading to traps weakening with waiting time.

However, as discussed in the previous section, our model does not correctly account for all of the robust features of the stick-slip bifurcation. We may then think of two basic directions for improving it.

On the one hand, while preserving the present overall framework of description of compressive plastic effects, one could conceive of several improvements. For example, the asperity geometry could be made 2D [21]. A more fundamental extension could be to incorporate explicitly the elastic interactions between asperities (for a qualitative discussion, see [14]).

However, we contend that such improvements are of secondary importance with respect to a major physical effect which we have neglected here, namely the fact that, in the sliding system, creep is occurring under the combined effect of the normal and tangential stresses. Such effects have been documented already long ago, and were recently measured by Berthoud et al. [20] on PMMA. As discussed in particular in the work of Johnson et al. on the failure of sheared adhesive contacts [22], and in that of Savkoor on viscoelastic materials [23], they may originate from various dissipative mechanisms, the relative importance of which is certainly material-dependent — for example internal plastic flow of the asperities, dissipative yield of a very thin layer at the contact interface or, more generally, "microscopic friction" on the inner contact scale. They result in slow relaxation of the slider under static horizontal loading, i.e. in a time-weakening of the friction force. As this, as is described in the phenomenological models, will counteract to some extent the compressive strengthening effects considered here, we expect that inclusion of shear yield into our description will affect qualitatively the structure of the dynamic response function. Such an extension is presently under study.

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Appendix

With the choice of dimensionless trap potential (Eq. (6)):

$$
\Psi(x) = \begin{cases} 
1 + 2|x|^3 - 3x^2 & |x| < 1 \\
0 & |x| > 1.
\end{cases}
$$

(A.1)
The trap equilibrium equation (4) reads (with $\beta = 6\alpha$, see Eq. (7)):

$$\Psi'(z) + 6\alpha(z - y) = 0$$  \hspace{1cm} (A.2)

where $y = \rho/a$. For large values of $\alpha$, equation (A.2) has a single solution:

$$z^*(y) = \frac{\text{sgn} y} {2} \left[ 1 - \alpha + \sqrt{(1 - \alpha)^2 + 4\alpha|y|} \right] \quad |y| < 1$$

$$z^*(y) = \frac{\text{sgn} y} {2} \left[ 1 - \alpha + \sqrt{(1 - \alpha)^2 + 4\alpha|y|} \right] \quad |y| > 1.$$  \hspace{1cm} (A.3)

This is the case as long as: $\alpha > 1$. For $\alpha < 1$, equation (A.2) develops three branches of solutions, which read, for $|y| < 1$:

$$z^{(+)} = -\frac{1}{2} \left[ 1 - \alpha + \sqrt{(1 - \alpha)^2 - 4\alpha y} \right]$$

$$z^{(u)} = -\frac{\text{sgn} y} {2} \left[ 1 - \alpha + \sqrt{(1 - \alpha)^2 - 4\alpha|y|} \right]$$

$$z^{(-)} = \frac{1}{2} \left[ 1 - \alpha + \sqrt{(1 - \alpha)^2 + 4\alpha y} \right].$$  \hspace{1cm} (A.4)

The trap is multistable; $z^{(\pm)}$ (resp. $z^{(u)}$) correspond to locally stable (resp. unstable) equilibria. Branches $(+)$ (resp. $(-)$) and $(u)$ terminate by merging at the spinodal limit point $C$ (resp. $C'$) with:

$$y_c = -y'_c = \frac{(1 - \alpha)^2}{4\alpha}.$$  \hspace{1cm} (A.5)

For $y = y_c$:

$$z^{(+)}(y_c) = \frac{\alpha - 1}{2}.$$  \hspace{1cm} (A.6)

If $y_c < 1$, after the spinodal jump, $z^{(-)}(y_c) = (1 - \alpha)\frac{1 + \sqrt{2}}{2}$, and the pinning force is non zero. If $y_c > 1$, the asperity jumps directly out of the trap and the pinning force vanishes. This occurs, from (A.5), when

$$\alpha < \alpha_c = 3 - 2\sqrt{2} \approx 0.17.$$  \hspace{1cm} (A.7)

In this latter case, which we have assumed to hold, the dimensionless area of the elastic hysteresis cycle is:

$$H_0(\alpha) = \Phi^{-1}_o U^{(+)}(u_c, y_c)$$  \hspace{1cm} (A.8)

with $u_c = z^{(+)}(y_c) - y_c$. Then:

$$H_0(\alpha) = 1 + \frac{(1 - \alpha)^2(\alpha + 3)}{16\alpha}.$$  \hspace{1cm} (A.9)

References