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Short Communication

A Dual 2D Model for the Quantum Hall Fluid (*)

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Abstract. — We present a dual 2D statistical model to describe the physical properties of a Quantum Hall Fluid. Such a model depends on a coupling constant $g$ and an angular variable $\theta$, which couples the electric and the magnetic charges. We show that it has topologically non-trivial vacua (corresponding to rational values of the filling), which are infrared stable fixed points of the renormalization group. Moreover its partition function has a dual infinite discrete symmetry, $\text{SL}(2,\mathbb{Z})$, which reproduces the phenomenological laws of corresponding states. Such a symmetry allows for an unified description of its fixed points in terms of a 2D Conformal Field Theory with central charge $c = 1$.

Our understanding of the conduction properties of a Quantum Hall Fluid at the plateaux has greatly improved recently by means of the two-dimensional Conformal Field Theory (2D CFT) [1]. In particular a theoretical framework has been proposed, which basically substantiates Laughlin physical idea of associating a magnetic flux to a charge identifying the anyons as basic excitations [2]. Furthermore both the Integer Quantum Hall Effect and the Fractional one now appear strictly related to each other, sharing similar many-body properties. Then we can ask the question of the form of the phase diagram and its critical points [3,4]. In this context there have been theoretical as well as phenomenological attempts [3,5,6] to uncover the physical properties of such a diagram: its dependence on two parameters (and their physical meaning), its nested character and the universality shared by the fixed points (attractive or repulsive) describing the Hall fluid at different fillings. In this paper we give an unified description of the attractive fixed points in terms of a 2D statistical model, which, besides the usual $g$-dependence of the kinetic term and a coupling between the electric and magnetic charges, parametrized by the angle $\theta$, [7,8], contains newly generated electric and magnetic background terms.

We extend the Renormalization Group (RG) analysis to such a model sorting out its new interesting properties, as, for example, the existence of Infrared (IR) stable fixed points which are non-trivial because of the presence of both electric and magnetic backgrounds, allowing for a non-neutral charges distribution (see for example Eq. (17)). These are interpreted as topologically non-trivial vacua of the field theory in the continuum which takes the form of a generalized dual Coulomb Gas with background, reproducing, for rational values of $\theta/2\pi$, the phenomenologically observed properties of a Hall fluid at the plateaux at corresponding fillings.

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Also we observe the existence of an infinite discrete symmetry SL(2, Z) (generalizing the well known Kramers-Wannier duality) which acts as in reference [8], mapping those non trivial fixed points one into another. This suggests a unified picture of the IR fixed points in terms of a 2D Conformal Field Theory with central charge $c = 1$. We stress that there is a simple connection between the duality transformations of the model and the phenomenological “laws of corresponding states” advocated by the authors of reference [5]. For clarity sake, we should make an obvious comment: being two-dimensional our model is suitable for a description of the equilibrium properties of the Hall system. In particular at filling $f = 1/\bar{\mu}$, $\bar{\mu}$ integer, our partition function (see Eq. (22)) reproduces the square modulus of the Laughlin wave function (at that filling), and the particles which condense (see Eq. (20)) carry the electric charge of the electron and the “magnetic” charge equal to $\bar{\mu}$. Furthermore, in tracing such a correspondence $g$ and $\theta/2\pi$ acquire the physical meaning of “stiffness” and filling (correspondingly) of the Hall fluid. Presently we are working on the conduction properties so to get a closer connection of the present work to the 2+1 Chern-Simons effective Lagrangian approach [9]. Let us first remind how Cardy and Rabinovici build their 2D model starting from a U(1) gauge theory with both electric and magnetic matter coupled by a surface $\theta$ term [7]. Then we will search for a non trivial extension of such a model in which a fixed background is generated. The explicit action is given by:

$$S[A, S, n] = \frac{1}{2g} \int d^2r (\partial_\beta A - S_\beta)^2 - i \int d^2r nA - \frac{\theta}{2\pi} \int d^2r \epsilon_{\beta\gamma} S_\beta \partial_\gamma A,$$

(1)

where $\epsilon_{\beta\gamma}$ is the antisymmetric tensor in 2D, $A$ is a real scalar field and $S_\beta(r)$ is the “magnetic frustration field”, defined by the constraint:

$$\epsilon_{\beta\gamma} \partial_\beta S_\gamma(r) - m(r) = 0,$$

(2)

while $n(r)$ and $m(r)$ are the electric and magnetic charge densities.

The densities ($n, m$) are constrained by the neutrality condition (required to make the system infrared stable):

$$\int d^2r \ n(r) = \int d^2r \ m(r) = 0.$$

(3)

In the following we need two representations of our model, one as a Coulomb gas where the role of the electric and magnetic charges is emphasized, and the gauge representation, where both charges have gauge interaction.

The Coulomb gas representation is defined by:

$$e^{-S_{CG}(n, m)} = \int \mathcal{D}A \int \prod_{\alpha=1,2} \mathcal{D}S_\alpha \delta(\epsilon_{\beta\gamma} \partial_\beta S_\gamma - m) e^{-S[A, S, n]}$$

(4)

It is straightforward to evaluate the path integrals in equation (4) and the result is given by:

$$S_{CG}[n, m] = \frac{g}{2} \int d^2rd^2r'(n(r) + \frac{\theta}{2\pi} m(r))(n(r') + \frac{\theta}{2\pi} m(r'))G(r - r')$$

$$+ \frac{1}{2g} \int d^2rd^2r' m(r)m(r')G(r - r') + i \int d^2rd^2r' n(r)m(r')\varphi(r - r'),$$

(5)

where $G(r)$ and $\varphi(r)$ are the “longitudinal” and “transverse” Green-Feynman functions in 2D given by:

$$G(r) = \ln \left( \frac{|r|}{\alpha} \right), \quad \varphi(r) = \arctan \left( \frac{y}{x} \right),$$

(6)
where \( a \) is a cutoff. The last term in equation (5) is the (imaginary) Bohm-Aharonov term [10]. Also notice that equation (5) defines for \( \theta = 0 \) the standard Coulomb gas for both electric and magnetic charges (see [11]). To obtain the gauge representation we solve the constraint (2) as:

\[
\delta(\epsilon_{\beta\gamma} \partial_\beta S_{\gamma} - m) = \int \mathcal{D}A_D \exp \left( -i \int d^2r \ A_D(r)(\epsilon_{\beta\gamma} \partial_\beta S_{\gamma} - m) \right).
\]

(7)

Then by evaluating explicitly the functional integral in equation (4) one gets the action:

\[
S[A, A_D] = \frac{g}{2} \int d^2r (\partial_\beta A_D)^2 - i \int d^2r \epsilon_{\beta\gamma} \partial_\beta A_\gamma A_D.
\]

(8)

It turns out that the relevant 2-point functions are:

\[
\langle A(r)A(r') \rangle = gG(r-r'),
\]

(9a)

\[
\langle A_D(r)A_D(r') \rangle = \frac{1}{g}G(r-r'),
\]

(9b)

\[
\langle A(r)A_D(r') \rangle = i\varphi(r-r').
\]

(9c)

The dual form of the Green functions in equations (9a, b) should be noticed. Naturally one can recover the Coulomb gas representation equation (5) by choosing for the charge densities:

\[
n(r) = \sum_i n_i \delta(r-r_i); \quad m(r) = \sum_i m_i \delta(r-r_i).
\]

(10)

We now search for solutions where the charge densities can be splitted in two parts: a ‘background’ one and a “fluctuating” one, as:

\[
n(r) = \bar{n} + \nu(r) \quad m(r) = \bar{m} + \mu(r)
\]

(11)

correspondingly the gauge fields will be splitted as:

\[
A(r) = \bar{A}(r) + a(r), \quad \partial^2 \bar{A}(r) = -ig(\bar{n} + \frac{\theta}{2\pi}\bar{m})
\]

\[
A_D(r) = \bar{A}_D(r) + a_D(r), \quad \partial^2 \bar{A}_D(r) = \frac{i}{g}\bar{m}
\]

\[
S_\beta(r) = \bar{S}_\beta(r) + s_\beta(r), \quad \epsilon_{\beta\gamma} \partial_\beta \bar{S}_\gamma(r) = \bar{m}.
\]

(12)

The equations above can be easily solved obtaining for the background gauge fields:

\[
\bar{A}(r) = -\frac{ig}{4}(\bar{n} + \frac{\theta}{2\pi}\bar{m})r^2; \quad \bar{A}_D(r) = \frac{i}{4g}\bar{m}r^2,
\]

(13)

which reproduce the usual harmonic form for the neutralizing background.

We can now integrate over the fluctuating field \( s_\beta \), equation (12), by taking into account the constraint given by equation (7), and obtain:

\[
\mathcal{Z}[\bar{n}, \bar{m}; \mu, \nu] = \int \mathcal{D}a \int \mathcal{D}a_D e^{-S_\theta[a, a_D]} \exp(i \int d^2r(\nu(r)
\]

\[
+ \frac{\theta}{2\pi}\mu(r)(\bar{A}(r) + a(r)) - i \int d^2r \mu(r)(\bar{A}_D(r) + a_D(r))),
\]

(14)

where \( \mu(r), \nu(r) \) are the fluctuating charge densities,

\[
S_\theta[a, a_D] = \frac{g}{2} \int d^2r(\partial_\beta(a_D))^2 - i\epsilon_{\beta\gamma} \int d^2r(\partial_\beta(a)\partial_\gamma(a_D))
\]

(15)
and we define the background term $S_B$ as:

$$S_B = -i \int d^2 r [\nu(r) + \frac{\theta}{2\pi} \mu(r)] \bar{A}(r) + i \int d^2 r \mu(r) \bar{A}_D(r)$$
$$= -\frac{\theta}{4} (\bar{n} + \frac{\theta}{2\pi} \bar{m}) \int d^2 r r^2 (\nu(r) + \frac{\theta}{2\pi} \mu(r)) - \frac{1}{4g} \bar{m} \int d^2 r r^2 \mu(r).$$  \hspace{1cm} (16)$$

Equation (16) gives a "background term" whose meaning may be derived by rewriting the partition function at fixed backgrounds $\bar{N} = \int d^2 r \bar{n}(r)$ and $\bar{M} = \int d^2 r \bar{m}(r)$ where the fluctuating charges are given by:

$$\mu(r) = \sum_i \mu_i \delta(r - r_i), \quad \nu(r) = \sum_i \nu_i \delta(r - r_i).$$

We then get:

$$Z[\bar{N}, \bar{M}, \{\mu\}, \{\nu\}] = Z_\text{f} e^{-S_B} \prod_i \exp \left( i (\nu_i + \frac{\theta}{2\pi} \mu_i) a(r_i) - \mu_i a_D(r_i) \right),$$  \hspace{1cm} (17)$$

where

$$Z_\text{f} \equiv \int \mathcal{D} a \mathcal{D} a_D e^{-S_f[a,a_D]}$$

and $\langle \rangle$ denotes the averaged value with respect to the "weight" $\exp(-S_f)$. In equation (17) a non neutral correlator between vertices appears. In fact, the neutrality condition looks like:

$$\bar{M} = -\sum_{i=1}^{N_p} \mu_i, \quad \bar{N} = -\sum_{i=1}^{N_p} \nu_i.$$

(18)

In other words the background charges neutralize the Coulomb charges; therefore the "splitting" of the charge densities in an uniform and a "fluctuating" part has allowed us to describe a non neutral Coulomb gas which generalizes the usual Coulomb gas largely analyzed in the literature [7, 11].

As a result of explicit RG analysis for this generalized Coulomb gas (at first order in the relevant parameters) we find that the IR stable fixed points correspond to the maximum value of the critical exponents as it should be.

Being $\bar{M}$ the magnetic background and $\bar{N}$ the electric one we assume that the most probable condensate of $N_p$ particles is the one which corresponds to the maximum value of the critical exponent which, in the presence of magnetic and electric background, is given by:

$$x(\nu, \mu) = 2 + g \left[ \nu(\bar{N} - \nu) + \frac{\theta}{2\pi} \mu(\bar{M} - \mu) \right] + \frac{1}{g} \mu(\bar{M} - \mu).$$

(19)

Then we maximize the above exponent with respect to $\mu$ and $\nu$ by imposing the double constraint, given by equation (18), obtaining:

$$\mu_0 = \frac{\bar{M}}{N_p} \quad \forall k = 1, \ldots, N_p; \quad \nu_0 = \frac{\bar{N}}{N_p} \quad \forall k = 1, \ldots, N_p.$$  \hspace{1cm} (20)$$

We see that the charges of the condensate are the same for each particle and their values given above are fixed by the background only.

We can now make a closer connection of the physical content of our model with Laughlin plasma description of the Hall fluid by evaluating the partition function $Z_R$ defined as $Z_R \equiv Z Z^*$. 
By using equation (20) we obtain:

\[
Z_R = Z_f^2 \exp \left[ 2 \left( g + \frac{\theta}{2\pi} \right) \frac{\mu^2}{a} \sum_{i\neq j} \ln \left| \frac{r_i - r_j}{a} \right| \right] \\
\times \exp \left[ -\frac{1}{2} \left( g + \frac{\theta}{2\pi} \right) \left( \bar{\mu} + \frac{\theta}{2\pi} \bar{\mu} \right) \left( \bar{n} + \frac{\theta}{2\pi} \bar{m} \right) + \frac{1}{g} \mu \bar{m} \sum_k r_k^2 \right]
\]

which, when the condition \( Q_e = f Q_m \) is satisfied, becomes:

\[
Z_R = Z_f^2 \exp \left[ 2 \left( g + \frac{\theta}{2\pi} \right)^2 \frac{\mu^2}{a} \sum_{i\neq j} \ln \left| \frac{r_i - r_j}{a} \right| \right] \\
\times \exp \left[ -\frac{1}{2} \left( g + \frac{\theta}{2\pi} \right)^2 \frac{1}{g} \mu \bar{m} \sum_k r_k^2 \right]
\]  

(21)

If we identify \( g (f + \frac{\theta}{2\pi}) + \frac{1}{g} = f = 1/\bar{\mu} \) equation (22) gives the square modulus of the Laughlin wave function at filling \( f = 1/\bar{\mu} \) which has been interpreted in [2] as the partition function for the Hall fluid at that filling.

By using the non-linear part of the Renormalization Group, that is by considering vortex-antivortex pair creation and annihilation processes, the "stiffness" \( g \) appears to go to infinity (in agreement with the incompressibility property of the Hall fluid at the plateaux) and \( |\theta/(2\pi)| \) flows to the rational value of the filling \( 1/\bar{\mu} \). However it is easy to show that the model is invariant under the following discrete \( \text{SL}(2,\mathbb{Z}) \) transformations defined in terms of the complex variable \( \zeta = 1/g + i\theta/2\pi \) as:

\[
\begin{align*}
S & \quad \zeta \rightarrow \zeta - i; \quad \bar{N} \rightarrow \bar{N} + \bar{M}, \quad \bar{M} \rightarrow \bar{M} \\
T & \quad \zeta \rightarrow -\frac{1}{\zeta}; \quad \bar{N} \rightarrow \bar{M}, \quad \bar{M} \rightarrow -\bar{N} \\
C & \quad \zeta \rightarrow \zeta^*; \quad \bar{N} \rightarrow -\bar{N}, \quad \bar{M} \rightarrow \bar{M}.
\end{align*}
\]

(23a, b, c)

The above infinite discrete symmetry \( \text{SL}(2,\mathbb{Z}) \) allows us to generate all non trivial vacua (IR fixed points) starting from a given one. In fact the system defined by the background charges \( \bar{M}, \bar{N} \) transforms in a new one as given by equations (23a, b, c). At this stage we can compare the previous discrete symmetry \( \text{SL}(2,\mathbb{Z}) \) of our model with the phenomenological "laws of corresponding states" introduced in reference [5]: 1) \( f \rightarrow f + 1 \); 2) \( 1/f \rightarrow 1/f + 2 \); 3) \( f \rightarrow 1 - f \) (for \( f < 1 \)), where the filling factor \( f = N_e/N_o \) rewritten in our units is given by \( f = \bar{N}/\bar{M} \). We then easily prove that the above laws can be expressed in terms of the duality transformations, equations (23a, b, c) as \( 1) = \text{S}, \ 2) = \text{TST}, \ 3) = \text{SC} \).

To be precise the above transformation laws generate only a subgroup of the duality transformations, which preserves both the oddness of the denominator in the filling \( f \) and the sign of the condensed electric and magnetic charges( [5]).

We now discuss some properties of the Infrared fixed points. For a subset of them the charges \( Q_e, Q_m \) satisfy the "chirality" condition: \( Q_e = f Q_m \), i.e. the field theory is chiral and describes the plateaux at filling \( f = 1/\bar{\mu} \). Then the relevant vertex operators of the left (chiral) sector associated to these charges take the simple form

\[
\hat{V}_\mu(z) = \exp(i \mu \sqrt{\bar{f}} \phi_L(z))
\]

(24)

when expressed in terms of a scalar field \( \phi_L \) depending only on \( z \) ( [1]).

Furthermore for these plateaux it is known that \( \sigma_H \) is a topological invariant which takes the values \( \sigma_H = 1/\bar{\mu} \) if one imposes periodic boundary conditions( [13]).

These points are then described by a CFT, whose primary fields are the vertices of equation (24) realized in terms of a chiral field \( \phi_L(z) \) compactified on a radius \( R \) such that \( R^2 = \bar{\mu} \), with central charge of Virasoro algebra \( c = 1 \).
As observed above, by using the duality transformations given by equations (23a, b, c) we can get all the other non trivial IR fixed points associated to the fillings \( f = p/\bar{\mu} \) (with \( p \) and \( \bar{\mu} \) prime factors). Then all these fixed points are in the same universality class of the \( c = 1 \) CFT just mentioned.

Summarizing we proposed an extension of Cardy\'s two-dimensional model with “electric” and “magnetic” background terms, which has non-trivial fixed points of the Renormalization Group. At such points the Coulomb Gas description given above reproduces Laughlin\’s plasma for filling \( f = 1/\bar{\mu} \) and the \( \text{SL}(2, \mathbb{Z}) \) symmetry present in the model allows for an unified description, for general filling \( f = p/\bar{\mu} \), in terms of a 2D Conformal Field Theory with central charge \( c = 1 \).

On one hand it would be interesting, by considering the conduction properties, to make a connection with the 2+1 Chern-Simons theory as given in [9] and in particular with the effective description given in reference [14].

On the other hand by using the non-linear part of the Renormalization Group it would be nice to study the phase diagram of the model and appreciate more the usefulness of the \( \text{SL}(2, \mathbb{Z}) \) invariance in describing its features. A central point in this matter would be to give a more complete description comprehensive of the so-called “repulsive fixed points” and its connection with the middle points of the slope between plateaux where the longitudinal conductance is different from zero [15].

It seems also quite reasonable that the phase diagram, in our case, has many features similar to the ones found in [8] being it a strong consequence of the duality symmetry \( \text{SL}(2, \mathbb{Z}) \).

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