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A Guide to the Symmetry Structure of Quasiperiodic Tiling Classes

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Abstract. — Based upon the torus parametrization which was introduced recently, we present a recipe allowing for a complete analysis of the symmetry structure of quasiperiodic local isomorphism classes in one and two dimensions. A number of results is provided explicitly, including some widely used planar tiling classes with 8-, 10- and 12-fold rotational symmetry.

1. Introduction

Given a (periodic) lattice in $\mathbb{R}^n$, one defines its (special) *Wyckoff positions* to be the sets of symmetrically equivalent points that are mapped onto itself by a non-trivial symmetry operation of the space group [1]. The explicit determination of Wyckoff positions and the corresponding symmetries is a classic problem of crystallography [1].

A naive extension of this problem to non-crystallographic structures with trivial translation group does not lead to an interesting question. To reach a more relevant situation, however, one can reformulate the problem in terms of so-called local isomorphism classes (LI-class for short).

The LI-class of some discrete structure $\mathcal{T}$ consists of all patterns $\mathcal{T}'$ that are locally indistinguishable from $\mathcal{T}$ in the sense that arbitrarily large patches of $\mathcal{T}$ also appear in $\mathcal{T}'$ and *vice versa*. For the comparison of patches, only translations are allowed [2]. As LI-class elements, we also distinguish global translates, so that the patterns are equipped with an origin in a natural way. Using this concept, one defines an isometry g to be a *generalized point symmetry* [2] of $\mathcal{T}$ iff g stabilizes LI[$\mathcal{T}$], *i.e.* iff $g(\mathcal{T}') \in$ LI[$\mathcal{T}$] for all $\mathcal{T}' \in$ LI[$\mathcal{T}$]. While generalized symmetry is thus a property of the entire LI-class, there may well be single patterns with *exact* symmetries: $g(\mathcal{T}) = \mathcal{T}$. Of course, all exact symmetries of LI-class elements form subgroups of the (total) generalized symmetry group, but this is not a sufficient condition. The question arises whether one can determine all these symmetries for a given LI-class as well as the patterns that preserve them.

Applying this concept to the crystallographic case, we see that the problem here is indeed reduced to the determination of the Wyckoff positions: The LI-class of a periodic structure

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consists of a single translation class, the generalized symmetry of a lattice coincides with its holohedry and an LI-class element shows (exact) symmetry if and only if its origin is in a Wyckoff position.

In general, the classification problem of symmetric LI-class elements turns out to be rather non-trivial and is in fact still unsolved. However, LI-classes of quasiperiodic cut-and-project patterns allow for a complete solution by means of the recently introduced torus parametrization [3]. Here, not only geometric point symmetries but also rescaling symmetries, typical of quasiperiodic patterns, can be included in the classification.

For this article to be self-contained, we start with a short presentation of the method, illustrated with the 1D case of silver mean chains. We then give the results for the most relevant 2D LI-classes (quasiperiodic tilings of the plane) which are widely used in literature. For related work, we also refer to [4] where some of the geometric point symmetries have been determined, however not in the context of a classification of LI-class elements.

2. The Torus Parametrization

The LI-class of a *crystallographic* pattern $\mathcal{T}$ (where the periods of $\mathcal{T}$ span the entire space) consists of a single translation class. It can thus be written as:

$$\mathrm{LI}(\mathcal{T}) = \left\{ \mathcal{T} + \mathbf{x} \mid \mathbf{x} \in \mathrm{FD}(\Gamma) \right\} \tag{1}$$

where $\mathrm{FD}(\Gamma)$ is a fundamental domain of the underlying translation lattice Γ. This way, we obtain a one-to-one correspondence between the elements of the LI-class and the points of $\mathrm{FD}(\Gamma)$. The translation vector $\mathbf{x}$ hereby marks the origin of space relative to the lattice and, for standardization, the pattern with a lattice point coinciding with the origin shall be parametrized by zero. As $\mathrm{FD}(\Gamma)$ forms an n-dimensional torus $\mathbb{T}^n$ after identification of opposite facets, this is called the *torus parametrization.*

The key in generalizing this concept to LI-classes of quasiperiodic patterns is that these patterns can be described as sections through periodic structures of higher dimension. So by reversing the cut-and-project mechanism, the LI-class can be parametrized by the points of a fundamental domain of the embedding lattice:

Let $\Lambda \subset \mathbb{R}^n$ be the embedding lattice and E and $\mathrm{E_{int}}$ the physical resp. internal space, $\mathrm{E} + \mathrm{E_{int}}$ spanning $\mathbb{R}^n$. Then a quasiperiodic pattern $\mathcal{T} \subset \mathrm{E}$ is obtained by projecting all those lattice points into E (parallel to $\mathrm{E_{int}}$) which project, parallel to E, into a compact subset W of $\mathrm{E_{int}}$, called the projection window. (This can easily be extended to the situation with several translation classes of points and hence a system of windows, compare [5]). For simplicity, we shall also assume that W is the closure of its interior and that ∂W is of zero Lebesgue measure. Translating the window by a vector $\mathbf{t} \in \mathrm{E_{int}}$ typically results in new, but locally isomorphic patterns [5]. Now let $\mathbf{t}$ be given as the projection image of some point $\mathbf{x} \in \mathrm{FD}(\Lambda)$ while the projection of $\mathbf{x}$ on E shall define the origin of the physical space. This way, a function $f : \mathrm{FD}(\Lambda) \to \mathrm{LI}[\mathcal{T}]$ is defined. We now assume that the dimension of the embedding lattice Λ is minimal (Λ projects densely into $\mathrm{E_{int}}$). Then the function f is *injective* since we restricted its domain to $\mathrm{FD}(\Lambda)$. On the other hand, f is also *surjective* – with one subtlety as exception: whenever the projection of some lattice point into $\mathrm{E_{int}}$ coincides with a boundary point of the window, we have to make a choice whether to include this point in the pattern or not, corresponding to limiting processes of window shifts from different sides. Each such limiting process results in a so-called *singular* projection pattern (all other patterns are called regular). Since singular patterns also belong to the LI-class, several elements here correspond to a common parameter. Put in other words, this means that mutually singular patterns are identified by the torus parametrization. As a matter of fact, if ∂W has Lebesgue measure zero, almost all patterns of the LI-class are regular and mutually singular patterns differ only in mismatches of zero density [5]. Two regular patterns, on the other hand, are either identical or differ in mismatches of positive density. So using the torus parametrization just means to identify patterns that are identical *almost everywhere* which is also reasonable physically.

SYMMETRY ANALYSIS. — Considering the periodic case, the torus parametrization can be used to determine the Wyckoff positions: given an element g of the holohedry, represented as a lattice automorphism $\hat{g}$, the corresponding Wyckoff position appears as the set of all fixed points of $\hat{g}$ on the torus $\mathbb{T}$:

$$g(\mathcal{T}(\mathbf{x})) = \mathcal{T}(\mathbf{x}) \quad \Longleftrightarrow \quad \hat{g}(\mathbf{x}) = \mathbf{x}\ (\mathrm{mod}\,\mathbb{T}). \tag{2}$$

Furthermore, the total number $N(g)$ of torus points in a Wyckoff position can easily be calculated. If $\hat{g} - \mathbb{1}$ is not singular, there are

$$N(g) = |\det(\hat{g} - \mathbb{1})| \tag{3}$$

distinct solutions of (2) since $(\hat{g} - \mathbb{1})\mathbb{T}$ is a $N(g)$-fold cover of $\mathbb{T}$. If $\hat{g} - \mathbb{1}$ is singular (so there are fixed spaces of $\hat{g}$ of dimension ≥ 1), (2) allows for entire solution manifolds or subtori of $\mathbb{T}$ (see [3] for details on the structure of subtori).

How does this generalize to the determination of symmetric elements of quasiperiodic LI-classes? It turns out that there are two conditions in this case [3]:

1) The corresponding torus parameter must be in a *Wyckoff position* of the symmetry considered as a symmetry *of the embedding lattice* Λ.

2) The symmetry must belong to the generalized symmetry group of the LI-class.

While 1) trivially implies 2) for crystallographic LI-classes (each lattice coincides with its embedding lattice of minimal dimension), the symmetry must be compatible with the projection procedure in the quasiperiodic case: The generalized symmetry is just the maximal subgroup of the holohedry of Λ that stabilizes E and $\mathrm{E}_{\mathrm{int}}$ together with the projection window.

At this point, let us compare our approach with the different notion of special points for quasilattices given in [4]. While both approaches use the Wyckoff positions of the higher dimensional embedding lattice as a starting point, in [4] special points of single quasiperiodic LI-class members are defined by projecting the higher dimensional ones in a special way. Within our framework, on the other hand, Wyckoff positions parametrize symmetric quasilattices and lead to a symmetry classification of the entire LI-class.

Furthermore, another type of symmetries can be included in our scheme, namely (invertible) inflation/deflation symmetries. They appear as affine transformations on the torus which can be characterized as follows: Their linear part is a lattice automorphism $\hat{I}$ on the embedding lattice leaving E and $\mathrm{E}_{\mathrm{int}}$ invariant, but is not an isometry like in the case of geometric point symmetries. In general, a non-vanishing translational part is needed to make the inflation/deflation *local* [2]. This gives rise to a simple shift of the torus parameters of inflation symmetric patterns but has no effect on the total numbers of symmetric solutions. We call $\hat{I}$ an *inflation* if, restricted to E, its determinant is bigger than 1 in absolute value. In most cases known, an inflation just means a refinement followed by a rescaling of the original pattern corresponding to $\hat{I}$ being diagonal on E, but there are also other types, like rotation-dilations, see below. Point or inflation symmetric patterns then are determined as in the periodic case by equations (2, 3).

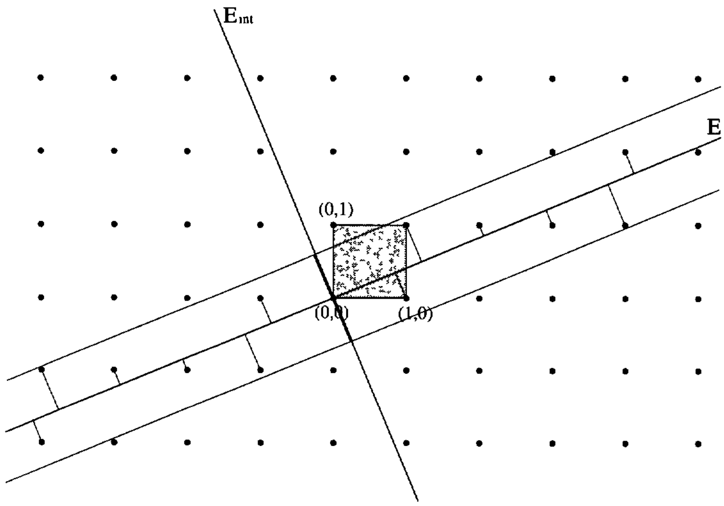


Fig. 1. — Projection scheme for silver mean chains. One fundamental domain is marked as the torus region used for the parametrization.

3. Silver Mean Chains

Of course, the geometric symmetries of quasiperiodic chains are well known in the quasicrystal community, as well as the fact that infinitely many members of a given LI-class exhibit inflation symmetries. In particular, for the Fibonacci chain, the latter is shown in [6]. This section shall serve as an illustrative introduction of our method, but also leads to a classification of all symmetries for the first time, and to a much simpler approach than commonly found.

The LI-class of the silver mean chains LI[SM] can be defined by the following inflation rule on a two-letter alphabet:

$$\varrho : \begin{cases} \mathrm{a} & \rightarrow & \mathrm{aba} \\ \mathrm{b} & \rightarrow & \mathrm{a} \end{cases} \tag{4}$$

Starting with the two-letter word aa by continued inflation we get a bi-infinite inflation invariant word that determines LI[SM]:

$$\mathrm{a|a} \quad \rightarrow \quad \mathrm{aba|aba} \quad \rightarrow \quad \mathrm{abaaaba|abaaaba} \quad \rightarrow \tag{5}$$

On the other hand, the silver mean chains can be obtained from the square lattice $\mathbb{Z}^2$ by the standard strip projection method as shown in Figure 1.

The slope of the physical space E fixes the proportion of the a's to the b's in the chains to the *silver number* $\lambda = 1 + \sqrt{2} = [2; 2, 2, 2, . \; .]$. The chain shown corresponds to the torus parameter (0, 0), the other chains are obtained by moving the origin of the 2D space $\mathrm{E} + \mathrm{E}_{\mathrm{int}}$ (together with the window) inside the shaded torus region.

The only geometric point symmetry possible in 1D is *inversion symmetry* $S : \mathbf{x} \mapsto -\mathbf{x}$. According to (2) the inversion symmetric chains correspond to the parameters with $2\mathbf{x} = \mathbf{0}$ on the torus, forming (within a fundamental domain) the Wyckoff position for inversion symmetry on $\mathbb{Z}^2$. According to (3) there are $\det(2\cdot\mathbb{1}) = 4$ solutions (in $\mathbb{R}^2$), namely:

$$\begin{aligned}
\mathcal{S}_{(0,0)} &= \quad \mathrm{abaaabaabaabaaab\underline{aa}baaabaabaabaaaba}\,.. \quad , \\
\mathcal{S}_{(\frac{1}{2},0)} &= \quad \mathrm{aabaabaabaaabaaba\underline{a}abaabaaabaabaabaa}\,.. \quad , \\
\mathcal{S}_{(0,\frac{1}{2})} &= \quad \mathrm{baabaaabaabaaabaa\underline{b}aabaaabaabaaabaab}\,. \quad , \\
\mathcal{S}_{(\frac{1}{2},\frac{1}{2})} &= \quad \mathrm{aabaabaaabaabaa}\{\tfrac{\mathrm{ab}}{\mathrm{ba}}\}\mathrm{aabaabaaabaabaa}\,..
\end{aligned} \tag{6}$$

Hence we obtain three regular and one singular solutions. The first chain is just the one generated by inflation in (5). Let us take a closer look at inflation symmetry in the projection picture. Inflation, as defined in (4), is represented by the lattice automorphism

$$\hat{I} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \tag{7}$$

on $\mathbb{Z}^2$. It has the eigenvalues λ and $-1/\lambda$ in the directions of E and $\mathrm{E_{int}}$, respectively. Using equation (3), we can immediately determine the number a_n of silver mean chains that are invariant under n-fold inflation (called I^n-symmetric from now on):

$$a_n = |\det(\hat{I} - \mathbb{1})|. \tag{8}$$

The a_n's obey the following recursion relation:

$$a_n = 2a_{n-1} + a_{n-2} + 2(1-(-1)^n)\ ; \quad a_0 = 0\,,\, a_1 = 2\ . \tag{9}$$

We can now avoid multiple countings and calculate the number b_n of I^n-symmetric chains that are not I^m-symmetric for any m dividing n, and finally, since the chains counted by b_n form closed n-cycles under inflation, we obtain the number c_n of *inflation orbits* of length n as:

$$b_n = a_n - \sum_{\substack{m|n \\ m<n}} b_m, \qquad c_n = \frac{b_n}{n} \tag{10}$$

Up to $n = 12$ the values of a_n, b_n and c_n are listed in Table I.

Table I. — *Counts of inflation symmetric silver mean chains.*

n	1	2	3	4	5	6	7	8	9	10	11	12
a_n	2	4	14	32	82	196	478	1152	2786	6724	16238	39200
b_n	2	2	12	28	80	180	476	1120	2772	6640	16236	38976
c_n	2	1	4	7	16	30	68	140	308	664	1476	3248

It is possible to encapsulate the statistics into a generating function. The best choice is the so-called dynamical or Artin-Mazur ξ-function [3, 7]. It is for our case

$$Z(x) = \frac{x^2 - 1}{x^2 + 2x - 1} \tag{11}$$

and gives the a_n's through the series

$$\log\left(Z(x)\right) = \sum_{n=1}^{\infty} a_n \frac{x^n}{n} \tag{12}$$

while the c_n's appear in its Euler product expansion as follows:

$$\frac{1}{Z(x)} = \prod_{n=1}^{\infty} (1-x^n)^{c_n} = (1-x^1)^2 (1-x^2)^1 (1-x^3)^4 \tag{13}$$

Finally, the closest pole of (11) to the origin is $x_0 = \sqrt{2} - 1 = 1/\lambda$ which means that the a_n grow asymptotically like λ^n and the c_n like $\frac{1}{n}\lambda^n$.

JOINT CLASSIFICATION OF INFLATION AND INVERSION SYMMETRIC CHAINS. — In order to obtain a complete symmetry classification of LI[SM], we have to study the interplay between the two basic symmetries, inversion and inflation.

Since the two symmetry transformations commute on the torus $\mathbb{T}^2$, we immediately can conclude that the set of inversion symmetric chains dissects into inflation orbits just like the entire LI-class: We find that two of the inversion invariant chains are also inflation symmetric ($\mathcal{S}_{(0,0)}$ and $\mathcal{S}_{(\frac{1}{2},\frac{1}{2})}$ in (6)), while the other two form an inflation 2-cycle.

Finally, we have to look for chains that transform under n-fold inflation into a space inverted copy. Since these chains are of course I^{2n}-symmetric, they already appear among the numbers of Table I, but rather have a full symmetry that we shall call $-I^n$ (generated by n-fold inflation followed by inversion). Their numbers $\tilde{a}_n = |\det(-\hat{I}^n - \mathbb{1})|$, $\tilde{b}_n$ and $\tilde{c}_n = \tilde{b}_n/2n$ can be calculated in analogy to the case above.

We now are in the position to determine the complete symmetry structure of the LI-class: The full generalized symmetry group of LI[SM] is $S \times I$, generated by an inversion (S) and an inflation (I). We now give the numbers of symmetry orbits (with respect to $S \times I$) of chains that show exact symmetries:

- There are two chains (each its own orbit) with the maximum symmetry possible, $S \times I$, namely $\mathcal{S}_{(0,0)}$ and $\mathcal{S}_{(\frac{1}{2},\frac{1}{2})}$ in (6).
- There is one 2-cycle of chains with $S \times I^2$-symmetry ($\mathcal{S}_{(\frac{1}{2},0)}$ and $\mathcal{S}_{(0,\frac{1}{2})}$ of (6)).
- The numbers d_n and $\tilde{d}_n$ of symmetry orbits that contain chains with I^n- and $(-I^n)$-symmetry, respectively, are shown in Table II up to $n = 15$. (Here $\tilde{d}_n = \tilde{c}_n$ for $n > 2$ while $d_n = (c_n - \tilde{d}_{n/2})/2$ for n even and $d_n = c_n/2$ for n odd.)

Table II. — *Symmetry orbit counts of LI[SM]: There are d_n orbits that contain chains with I^n as exact symmetry and $\tilde{d}_n$ orbits of $(-I^n)$-symmetric chains.*

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
d_n	0	0	2	3	8	14	34	68	154	328	738	1616	3640	8126	18384
$\tilde{d}_n$	0	1	2	4	8	16	34	72	154	336	738	1632	3640	8160	18384

4. Remarks on the Generality of the Approach

In principle, each LI-class that can be described within the projection scheme can be analyzed in a similar fashion as described above. One might ask how general results of this kind are.

Obviously, the symmetry structure is not changed by local, symmetry preserving decorations of the patterns: it is thus a common feature of entire S-MLD classes (they consist of all patterns that are equivalent with respect to Symmetry preserving Mutual Local Derivability [2,8]). But there is more to be said.

For closer inspection, let us impose two conditions on the projection window. In order not to bring along any restriction to the geometrical symmetries we shall assume the window to be invariant under the maximal point group that stabilizes E_{int}. Furthermore, in order to attain a well defined global inflation in the tiling space, it must be possible to reconstruct the inflated window by countably many unions and intersections from the original one, its complement, and shifts of them by projected lattice vectors. (It can be shown that *local* inflation exists iff the reconstruction can be done with only *finite* unions and intersections [8].) As a consequence of these conditions, the symmetry transformations do not mix regular and singular members of the LI-class.

It is remarkable that under these conditions, the detailed shape of the projection window has no influence on the total number of the symmetric LI-class elements, and has to be taken into account only if we want to decide whether a symmetry is realized by singular or regular members of the LI-class under consideration, *i.e.* if the torus coordinate projects onto the projection window boundary or not.

Consider for that reason the projection window extended to the whole internal space, thus projecting the entire embedding lattice. For LI-classes obtained by cut-and-project techniques with minimal embedding this projection image in the physical space is exactly the Limit Translation Module (LTM for short) [5]. So we can alternatively say that, under the above assumptions, the derived results are valid for all LI-classes of "cut-and-project-type" which share the same LTM. Regarding the silver mean chains, our results apply to *all* LI-classes of quasiperiodic chains with λ-inflation (and inversion symmetric window).

To a second generalization we are led by the one-to-one correspondence between the inversion symmetric chains and the twofold inflation symmetric chains. This result can be read off from the inflation matrix $\hat{I}$ which fulfils the identity $\hat{I}^2 - 1 = 2\hat{I}$. But this equation does not depend on details of the projection scheme, so the above correspondence is valid for all LI-classes with λ-inflation *in any dimension.* The number theoretic reason for this is evident from Pleasants' approach [9]. So results obtained by symmetry analysis generalize to LI-classes whose LTM contains the LTM of the special given LI-class.

It is thus natural to reformulate the symmetry analysis without explicit reference to the cut-and-project formalism and without fixing a specific representation in form of the embedding lattice. Let us explain this in more detail for LTM's of dimension two.

SYMMETRY ANALYSIS FROM A NUMBER THEORETIC POINT OF VIEW. — Given a special cut-and-project tiling class in 2D, the total numbers of its symmetric members are perhaps most easily determined by calculating determinants according to (3). The advantage of the number theoretic approach consists in the fact that symmetries can be determined even if the particular embedding is unknown and that the generality of the derived results is clarified. The usefulness of number theoretic tools in the field of quasicrystals was first realized by Pleasants in 1984 [10], the importance of cyclotomic fields for a classification of symmetries of quasilattices has been pointed out by Niizeki [11].

Let an LI-class of tilings in 2D be given with n-fold (generalized) rotation symmetry. In the minimal rank case, the corresponding LTM is (up to a similarity transformation) the ring of cyclotomic integers [12]

$$\mathbb{Z}[\xi] = \{a_1 + a_2\xi + a_3\xi^2 + \cdots + a_{\phi(n)}\xi^{\phi(n)-1} \mid \xi = e^{2\pi i/n},\, a_k \in \mathbb{Z},\, k = 1, \ldots, \phi(n)\},$$

where ϕ denotes Euler's totient function from number theory. Regarding $\mathbb{Z}[\xi]$ as projection image of a periodic lattice of higher dimension, it is natural to identify the generalized symmetries with the module automorphisms on $\mathbb{Z}[\xi]$, *i.e.* those preserving the structure of $\mathbb{Z}[\xi]$ as a module of rank $\phi(n)$, but not necessarily its additional ring structure. The inner automorphisms constitute the unit group $U_n = \mathbb{Z}[\xi]^\times$ of $\mathbb{Z}[\xi]$. It indeed contains the rotation group C_n together with the inflation symmetries [11]. An important outer automorphism is complex conjugation by which, when combined with the rotations, one obtains the reflection symmetries.

If we want to count symmetries, we have to look for the number theoretic counterpart of the torus equation (2) in the cut-and-project scheme. It translates to a linear equation in the cyclotomic field $\mathbb{Q}(\xi)$. The determinant (3) equals the ***absolute algebraic norm*** in the corresponding ring of cyclotomic integers, which we now define (see *e.g.* [12,13]):

The ring $\mathbb{Z}[\xi]$ (the integer polynomials in ξ) consists of *all* algebraic integers of the cyclotomic field $\mathbb{Q}(\xi)$, the rational functions in ξ. Now, ξ is a root of the n-th cyclotomic polynomial, $P_n(x)$, which is integral, irreducible over $\mathbb{Z}$ and of degree $\phi(n)$. The other solutions are the $\phi(n)-1$ algebraic conjugates of ξ in $\mathbb{Z}[\xi]$. The $\phi(n)$ roots of $P_n(x)$, the primitive n-th roots of unity, are obtained from ξ by the action of the Galois group $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$ on ξ, where $|\mathcal{G}| = \phi(n)$. Any $\sigma \in \mathcal{G}$ is already uniquely determined by its action on ξ because it then has a *unique* extension to a field automorphism of $\mathbb{Q}(\xi)$ which fixes the elements of $\mathbb{Q}$. So, if $\mathcal{G} = \{\sigma_0, \sigma_1, \ldots, \sigma_{\phi(n)-1}\}$, with $\sigma_0 = Id$, we can define the norm of *any* number $\alpha \in \mathbb{Q}(\xi)$ as

$$N_{\phi(n)}[\alpha] := \alpha \cdot \sigma_1(\alpha) \cdot \ldots \cdot \sigma_{\phi(n)-1}(\alpha). \tag{14}$$

The correspondence to determinants can be seen by representing a linear transformation $x \mapsto \alpha x$ with $\alpha \in \mathbb{Q}(\xi)$ as rational matrix over the basis $\{1, \xi, \xi^2, \ldots, \xi^{\phi(n)-1}\}$, say. Its characteristic polynomial then has the constant term $(-1)^{\phi(n)} N_{\phi(n)}[\alpha]$.

Let us illustrate how to compute abstract algebraic norms for the eightfold module which underlies the Ammann-Beenker LI-class. The unit group U_8 is generated by two elements, namely by a 45°-rotation $\xi = e^{2\pi i/8}$ and the silver mean dilation $\lambda = 1+\sqrt{2} = 1+\xi+\bar{\xi}$. The cyclotomic polynomial of the eightfold module reads $P_8(x) = x^4+1$ with solutions ξ, $\sigma_1(\xi) = \xi^3 = -\bar{\xi}$, $\sigma_2(\xi) = \xi^5 = -\xi$, $\sigma_3(\xi) = \xi^7 = \bar{\xi}$. Here, $\mathcal{G} = \{Id, \sigma_1, \sigma_2, \sigma_3\} \simeq C_2 \times C_2$, and σ_3 is complex conjugation. For $\alpha = a + b\xi + c\xi^2 + d\xi^3 \in \mathbb{Q}[\xi]$ we then have $\sigma_k(\alpha) = a + b\sigma_k(\xi) + c(\sigma_k(\xi))^2 + d(\sigma_k(\xi))^3, k = 0, \ldots, 3$.

The field $\mathbb{Q}[\xi]$ contains $\mathbb{Q}[\sqrt{2}] = \mathbb{Q}[\lambda]$ as its maximal real subfield. Its Galois group is $\{Id, \sigma_1\}$, a subgroup of the previous one. Now, the norm is defined as follows: for $\gamma = r + s\lambda \in \mathbb{Q}[\lambda]$ we have (with $\sigma_1(\sqrt{2}) = -\sqrt{2}$, and hence $\sigma_1(\lambda) = -1/\lambda$)

$$N_2[\gamma] = \gamma \cdot \sigma_1(\gamma) = (r+s\lambda)\cdot(r+s\sigma_1(\lambda)) = r^2 + 2rs - s^2. \tag{15}$$

We then have an alternative expression for the N_4-norm:

$$N_4[\alpha] = \alpha\bar{\alpha} \cdot \sigma_1(\alpha\bar{\alpha}) = N_2[|\alpha|^2].$$

Note that the definition of N_2 is usually given as $N_2[a+b\sqrt{2}] = a^2 - 2b^2$. Though this is equivalent to (15), the latter is more suitable for our needs. These rules can be used to simplify norm calculations and are helpful to relate results in different dimensions.

5. The Symmetry Structure of Quasiperiodic Tiling Classes

TILINGS WITH 8-FOLD ROTATIONAL SYMMETRY. — The symmetry of the underlying module and generalized symmetry of the LI-classes under consideration is $D_8 \times I$. Leaving mirror

Table III. — *Total numbers of tilings invariant under n-fold inflation followed by k-fold 45° rotation in LI-classes with* ($D_8 \times I$) *generalized symmetry.*

$a_{n,k}$	0	1	2	3	4	5	6	7
0	0	2	4	2	16	2	4	2
1	4	2	8	18	4	18	8	2
2	16	34	36	34	64	34	36	34
3	196	162	200	242	196	242	200	162
4	1024	1154	1156	1154	1296	1154	1156	1154
5	6724	6498	6728	6962	6724	6962	6728	6498
6	38416	39202	39204	39202	40000	39202	39204	39202
7	228484	227138	228488	229842	228484	229842	228488	227138
8	1327104	1331714	1331716	1331714	1336336	1331714	1331716	1331714

symmetries aside for the moment, we obtain as total number of tilings that are invariant under k-fold rotation through 45°, followed by n-fold λ-inflation:

$$a_{n,k} = N_4[\lambda^n \xi^k - 1]. \tag{16}$$

The results are given in Table III. The first row agrees *mutatis mutandis* with the corresponding results of [4], but note that their meaning is now rather different.

As the next step in our classification, we determine the tilings with multiple (inflation and rotation) symmetries. Like in the case of LI[SM] considered above, tilings with geometric symmetries have to dissect into inflation orbits. In the number theoretic approach, multiple symmetries are reflected by identities (up to units) within the module:

$$\begin{aligned} \lambda - 1 &= \xi^5(\xi^2 - 1) &(17)\\ \lambda^2 - 1 &= \lambda\xi^4(\xi^4 - 1)\ (= 2\lambda). &(18) \end{aligned}$$

Looking at Table III with these identities in mind, we conclude that all four tilings with 90° rotational symmetry (ξ^2-symmetry) are also inflation invariant (two of them are in fact symmetric under 45° rotation). Furthermore (as already stated above), inversion (or ξ^4-) symmetry coincides with invariance under twofold λ-inflation. In fact, 8 of the 16 inversion symmetric patterns show even higher symmetry: Besides the four inflation invariant tilings there are another four that inflate to a 90° rotated copy (this can be seen from the identity $(\xi^6\lambda - 1) = \xi^3(\xi^2\lambda - 1)$).

We now can eliminate all multiple countings of the same patterns from Table III. This is done recursively:

$$b_{n,k} = a_{n,k} - \sum_{m,j} b_{m,j}\ , \text{ where } m|n \text{ and } \frac{nj}{m} \equiv k\ (\mathrm{mod}\, d). \tag{19}$$

Here, $b_{m,j}$ counts ξ^d-symmetric tilings. Table IV shows the number $c_{n,k}$ of symmetry orbits with respect to inflation and rotation which is defined *via*

$$c_{n,k} = \frac{b_{n,k} d}{8n}. \tag{20}$$

Table IV. — *Orbit counts with respect to inflation and rotation for LI-classes with generalized $D_8 \times I$-symmetry.*

$c_{n,k}$	0	1	2	3	4	5	6	7
1	0	0	0	2	0	2	0	0
2	0	2	1	2	3	2	1	2
3	8	6	8	10	8	10	8	6
4	30	36	34	36	38	36	34	36
5	168	162	168	174	168	174	168	162
6	792	816	808	816	824	816	808	816
7	4080	4056	4080	4104	4080	4104	4080	4056
8	20700	20808	20772	20808	20844	20808	20772	20808

INCLUDING MIRROR SYMMETRY. — In order to obtain the complete symmetry struture, we have to include mirror reflection in our analysis. We observe that the mirror symmetry in each row of Tables III and IV is due to the fact that the tilings (or orbits) in columns $k = 5$, 6 and 7 are mirror images of the ones in columns $k = 3$, 2 and 1, respectively. These tilings are thus mirror symmetric if and only if they are also invariant under some rotation – but the rotation symmetric patterns are just those with multiple symmetries determined above.

The situation is somewhat more involved for the tilings of columns $k = 0$ and 4. Since their symmetry, generated by inflation and inversion, commutes with reflection, we find entire inflation-orbits of mirror symmetric patterns for each reflection axis. Also, inflation orbits of order $2n$ may rather contain tilings that transform under n-fold inflation to a mirror reflected copy: $(I^n \quad S)$-symmetry.

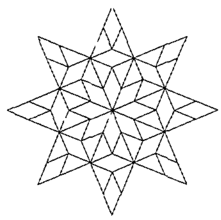
For the latter case, we obtain the numbers as $|N_2[\lambda^n - 1] \cdot N_2[-\lambda^n - 1]|$ (for each reflection axis), but for counting reflection symmetric tilings with additional inflation symmetry, a closer look at the structure of the solution manifolds of reflection symmetric patterns on the torus is needed. While for details we refer to [3], the situation, in summary, is like this: The solution manifolds form subtori of lower dimension (2D in this case) within the original torus. Symmetries that commute with mirror reflection will stabilize these subtori. Thus we can restrict our analysis to these subtori. But in our case, restricting inversion and λ-inflation to 2D, we are back to the situation of the silver mean chains! The symmetry counts of LI[SM] (times the number of subtori) thus give us the numbers of mirror symmetric tilings (for each reflection axes) in the LI-classes under consideration. Since we find one and two subtori respectively (for each reflection axes) in the two different conjugation classes of D_8, there are altogether $3d_n$ orbits of mirror symmetric inflation n-cycles and $3\tilde{d}_n$ orbits of tilings with $(-I^n \times S)$-symmetry (d_n and $\tilde{d}_n$ of Tab. II).

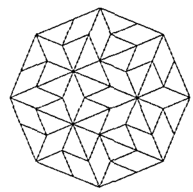
We give the results of the now complete symmetry analysis together with examples from the quasiperiodic Ammann-Beenker tiling class LI[AB]. The tilings (resp. tiling seeds) below are constructed through iterated inflation. For the inflation rules see [14].

Table V. — *Symmetry orbit counts for LI-classes with $D_8 \times I$ generalized symmetry. If multiple entries in the same column occur, the second refers to orbits of mirror symmetric tilings and the third (if present) to orbits of tilings with $(I^{n/2}\ S)$-symmetry.*

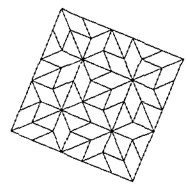
$c_{n,k}$	0	1	2	3	4
1	0	0	0	2	0
2	0	2	1	2	0 + 3
3	1 + 6 + 0	6	8	10	1 + 6
4	8 + 9 + 5	36	34	36	13 + 12
5	72 + 24 + 0	162	168	174	72 + 24
6	362 + 42 + 26	816	808	816	388 + 48
7	1989 + 102 + 0	4056	4080	4104	1989 + 102
8	10182 + 204 + 132	20808	20772	20808	10314 + 216

- There are two tilings with maximum possible symmetry, $D_8 \times I$. In LI[AB] one of them is regular, the other corresponds to a set of eight singular tilings.

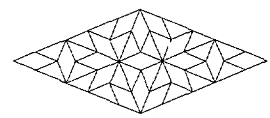




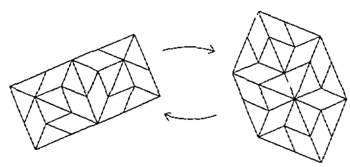
- We obtain one 2-cycle of $D_4 \times I$ symmetric tilings. They are singular in LI[AB].



- There is one orbit of four tilings with $D_2 \times (I \cdot C_4)$ symmetry, hence the tilings are transformed into 90° rotated copies by inflation. They are regular in LI[AB].



- Finally there is one orbit containing eight (in LI[AB] singular) tilings with $(D_2 \times I^2)$-symmetry.



- The orbit counts of tilings with no rotational symmetry are given in Table V.

TILINGS WITH 10-FOLD ROTATIONAL SYMMETRY. — The most prominent tiling classes with $(D_{10} \times I)$-symmetry are the triangle and Penrose patterns. They have been studied in detail in [3]. Here, we give the results in a completed (but concise) form.

The unit group of the cyclotomic field $\mathbb{Q}(\xi)$ is generated by the 36°-rotation $\xi = \exp(\pi i/5)$ and the golden mean inflation factor $\tau = (1+\sqrt{5})/2 = \xi + \bar{\xi}$. The 10th cyclotomic polynomial reads $P_{10}(x) = x^4 - x^3 + x^2 - x + 1$ [11,13] and the Galois group, $\mathcal{G} \simeq C_4$, is cyclic. Note that one could equally well use $P_5(x) = x^4 + x^3 + x^2 + x + 1$ because it has the same splitting field

Table VI. — *Symmetry orbit counts for LI-classes with $(D_{10} \times I)$ generalized symmetry. If multiple entries occur in one column, the second refers to orbits containing mirror symmetric tilings and the third gives the number of orbits with $(I^{n/2}\ S)$-symmetric tilings.*

$c_{n,k}$	0	1	2	3	4	5
1	0	0	0	1	1	0
2	0	0	0	0	0	0+1
3	0	0	0	1	1	0
4	0	1	1	1	1	0+2
5	0+2+0	2	2	3	3	0+2
6	0+2+2	5	5	5	5	2+2
7	4+4+0	11	11	13	13	4+4
8	8+4+4	27	27	27	27	12+6
9	26+8+0	62	62	66	66	28+8
10	62+10+10	150	150	150	150	72+12

as $P_{10}(x)$ [12]. Tilings with rotation symmetries again have inflation symmetries, due to the identities: $\xi - 1 = \xi^3(\tau - 1)$, $\xi^2 - 1 = \xi^2\tau^{-1}(\tau^2\xi - 1)$ and $\xi^5 - 1 = \tau^{-1}(\tau^3 - 1)$. On restriction to mirror symmetric tilings we get to the symmetry structure of one-dimensional Fibonacci chains (there is one subtorus for each reflection axes in this case). In summary, we find:

- There is one tiling with full symmetry, $D_{10} \times I$. It is singular both in the Penrose and in the triangle LI-class.
- There is one orbit containing 4 tilings with $(D_5 \times (C_2 \cdot I^2))$-symmetry. The triangle tilings are singular, but the patterns of the Penrose LI-class are regular.
- Finally, there is an orbit of 15 tilings with $(D_2 \times I^3)$-symmetry, singular in both cases.
- The orbit counts for tilings without rotational symmetry are shown in Table VI.

12-Fold Rotational Symmetric Tilings. — The underlying module belonging to twelvefold symmetric tilings, like the square-triangle tiling, the Stampfli tiling or the Socolar tiling, is the module $\mathbb{Z}[\xi]$ with $\xi = e^{2\pi i/12}$. The corresponding cyclotomic field is the splitting field of the 12th cyclotomic polynomial, $P_{12}(x) = x^4 - x^2 + 1$ [11,13]. The primitive 12th roots of unity are $\xi, \xi^5, \xi^7, \xi^{11}$, and the Galois group is $\mathcal{G} \simeq C_2 \times C_2$.

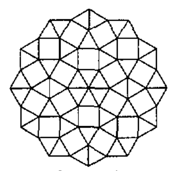
Its unit group U_{12} is generated by a 30°-rotation ξ together with a rotation-inflation $\eta = \sqrt{\rho}\xi$ where $\rho = 2 + \sqrt{3}$. Note the contrast to the eight- and tenfold cases where the elementary inflation were pure dilations. The full symmetry group therefore is a *semidirect* product between the reflection group S and the rotation-inflation group U_{12}. As in the above cases, every tiling with a geometric symmetry also shows some inflation symmetry, indicated by the identities $(\xi^3 - 1) = (\xi\eta - 1)\xi^2$, $(\xi^4 - 1)\xi^9\eta = \xi\eta^2 - 1$ and $(\xi^6 - 1)\xi^2\eta = (\xi\eta)^2 - 1$. The symmetry structure of the mirror symmetric tilings can be read from the maximal real submodule $\mathbb{Z}[\rho]$ which corresponds to the so-called Platinum number chains. Tilings with one single reflection axis split into $\pm\rho^n$-orbits. Again, we find one subtorus for each reflection axis.

Table VII. — *Symmetry orbit counts for LI-classes with $(C_{12} \times I) \times_s S$ generalized symmetry. In columns with several entries, the second corresponds to orbits of mirror symmetric tilings, the third entry corresponds to orbits with $I_\eta^{n/2}$ S symmetry.*

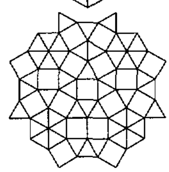
$c_{n,k}$	0	1	2	3	4	5	6	7	8	9	10	11
1	0	0	0	1	0	1	1	0	1	0	0	0
2	0	0	0	1	0	0+1	0	1	0	0	0	0
3	1	1	2	1	2	2	1	2	1	1	1	1
4	3	4	4	4	2+3	4	4	4	3	4	0+1+1	4
5	11	13	12	13	13	12	13	11	12	11	11	12
6	36	39	36	32+8	36	39	36	36	36	18+8+8	36	36
7	123	120	123	123	120	123	117	120	117	117	120	117
8	392	392	372+24	392	392	392	384	392	338+21+21	392	384	392
9	1299	1309	1309	1299	1309	1292	1299	1292	1292	1299	1292	1309
10	4356	4320+72	4356	4380	4356	4356	4356	4200+72+72	4356	4356	4356	4380

We illustrate our results with square-triangle tilings. The precise definition of our class LI[ST] is given in the appendix. Patterns are constructed through iterated inflation.

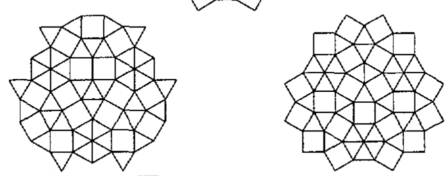
- There is one single tiling with full symmetry $(C_{12} \times I) \times_s S$. It is singular in LI[ST].



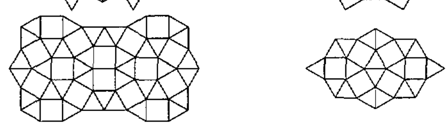
- There is one orbit containing three tilings with $(C_4 \times I \cdot C_{12}) \times_s S$ symmetry. It is singular in LI[ST].



- There is one orbit of eight tilings with $(C_3 \times I^2 \cdot C_{12}) \times_s S$ symmetry. The members of LI[ST] are regular.



- Finally, there is one orbit of twelve tilings with $(C_2 \times I^2 \; C_6) \times_s S$ symmetry. The tilings in are regular in LI[ST].



- The orbit structure for tilings without rotational symmetry is shown in Table VII.

14-Fold and Higher Rotational Symmetry. — Up to this day no quasicrystal has been found in nature with higher rotational symmetry than 12-fold. On the other hand, it is no problem to extend the theoretical description to rotation symmetries of arbitrary order. However, the situation becomes more complex on going beyond 12-fold rotation: While there is no inflation symmetry for crystallographic LI-classes and one inflation scale for quasicrystallographic

LI-classes with D_8, D_{10} or D_{12} generalized symmetry, there are two or more independent inflation scales for LI-classes with higher rotational symmetry [11]. This corresponds to the fact that in the projection picture we need to start with an embedding lattice of dimension higher than four.

Here, we shortly present how to deal with these cases within the torus parametrization. The case of LI-classes with $D_{14} \times I_1 \times I_2$ generalized symmetry shall serve as an example.

The unit group of the cyclotomic field $\mathbb{Q}(\xi)$, $\xi = \exp(\pi i/7)$, is generated by the $\pi/7$ rotation and the two inflation scales $\tau_1 = 1 + \xi^2 + \bar{\xi}^2 \approx 2.247$ and $\tau_2 = \tau_1^2 - 1 \approx 4.049$ [11,13]. The 14th cyclotomic polynomial has degree 6, $P_{14}(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$, with roots ξ, ξ^3, ξ^5 and their complex conjugates. Its Galois group is cyclic: $\mathcal{G} \simeq C_6$, the same as that of $P_7(x) = P_{14}(-x)$. (Hence 6 is also the minimum dimension of the embedding lattice). The total numbers of symmetric tilings are then given again by the corresponding N_6-norms as in (14). Multiple symmetries for special tilings are indicated through identities within $\mathbb{Q}(\xi)$ like in the cases discussed above.

Here, we give a classification of all tilings with nontrivial rotation symmetry.

- There is one tiling with maximum possible symmetry: $D_{14} \times I_1 \times I_2$.
- One symmetry orbit contains 6 tilings with $(C_7 \times (I_1^3 \cdot C_{14}) \times I_2) \times_s S$ symmetry.
- There is one symmetry orbit of 14 tilings with $C_2 \times (I_1 \cdot C_{14}) \times (I_1 \cdot I_2)$ symmetry. So the two inflations act as mutually inverse $\pi/7$-rotations on these tilings. Mind that the tilings have no mirror axes.
- Finally, there is one orbit containing 49 tilings with $D_2 \times I_1^7 \times (I_1 \cdot I_2)$ symmetry.

6. Concluding Remarks

We have presented an approach allowing for a complete classification of the symmetry structure of "cut-and-project" LI-classes. Within the framework of the torus parametrization, symmetric LI-class elements correspond to Wyckoff positions of the embedding lattice. Inflation symmetries can be included in the classification. The method can be extended to higher dimensions, see [3] for a full discussion of quasiperiodic LI-classes with icosahedral symmetry.

Results have been presented explicitly for 2D tiling classes with 8, 10, 12 and 14-fold rotational symmetry. While it is straightforward to apply our method to other "cut-and-project" LI-classes of interest, nothing can be said here about symmetries in LI-classes that can not be obtained *via* projection. So this remains an open question for the future.

Acknowledgments

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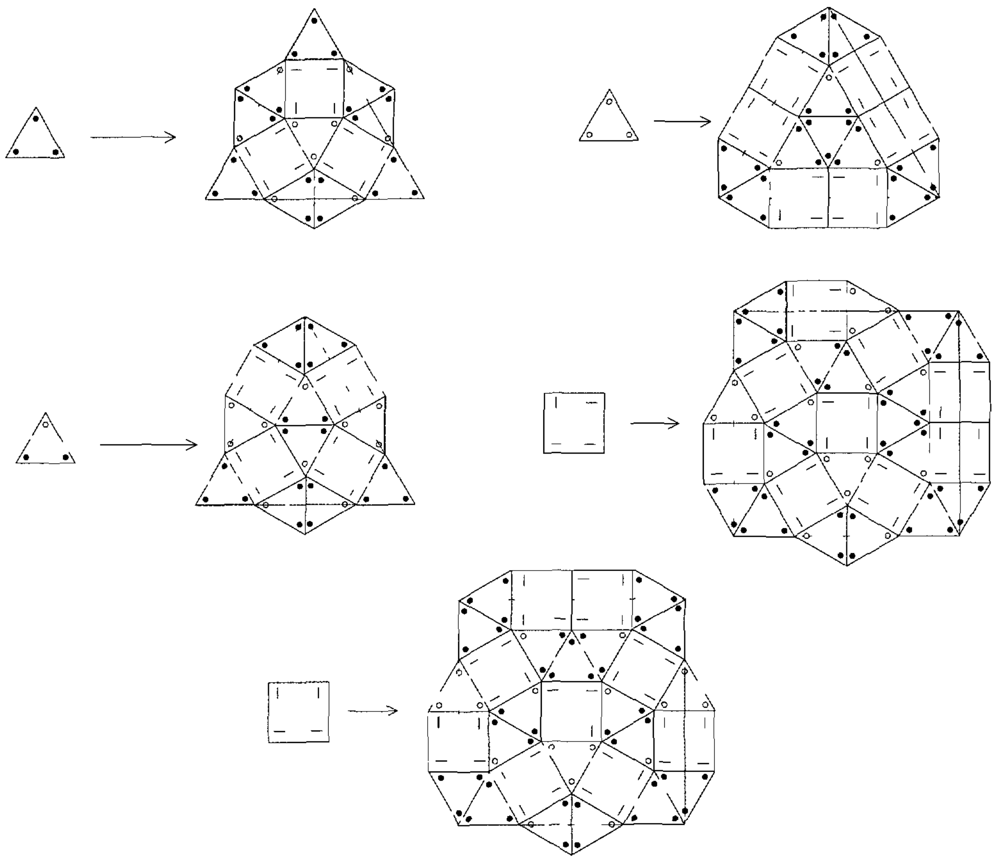

Fig. 2. — Inflation rules for the square-triangle tiling class LI[ST].

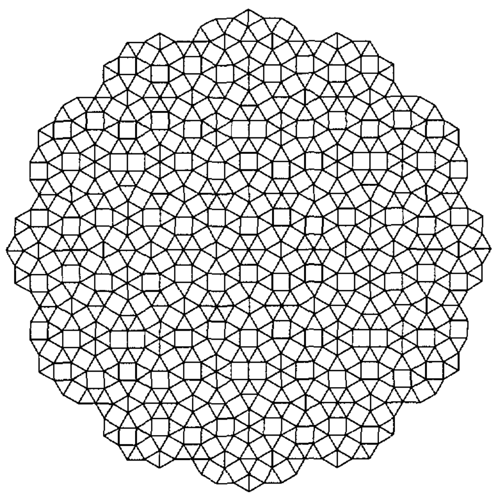

Fig. 3. — A finite patch of the maximally symmetric tiling from LI[ST].

Appendix: Inflation Rule for LI[ST]

A method to construct twelvefold symmetric square-triangle tilings was first presented by Stampfli [15]. His tilings, however, do not possess local inflation. Here, we use an inflation rule due to Martin Schlottmann [16], see Figure 2. The resulting tilings are indeed of cut-and-project type, as can be proved directly and follows from more general arguments outlined in [17]. The windows have fractal boundaries (of zero Lebesgue measure) which of course does not affect the symmetry analysis. A larger patch is shown in Figure 3.

References

[1] International Tables for Crystallography , Volume A, 4th edition, T. Hahn, Ed. (Kluwer Academic Publishers, 1983).
[2] Baake M. and Schlottmann M., Geometric aspects of tilings and equivalence concepts, in: Proceedings of the 5th International Conference on Quasicrystals, C. Janot and R. Mosseri, Eds. (World Scientific, Singapore, 1995) pp. 15-21.
[3] Baake M., Hermisson J. and Pleasants P., *J. Phys. A* **30** (1997) 3029-3056.
[4] Niizeki K.,*J. Phys. A* **22** (1989) 4281-4293.
[5] Schlottmann M., Geometrische Eigenschaften quasiperiodischer Strukturen, Dissertation (Univ. Tübingen, 1993).
[6] Goda M. and Kubo H., *J. Phys. Soc. Jpn* **58** (1989), 3624-3634.
[7] Degli Esposti M. and Isola S.,*Nonlinearity* **8** (1995) 827-842.
[8] Baake M., Schlottmann M. and Jarvis P.D., *J. Phys. A* **24** (1991) 4637-4654.
[9] Pleasants P.A.B., Quasicrystals with arbitrary symmetry group, in: Proceedings of the 5th International Conference on Quasicrystals, C. Janot and R. Mosseri, Eds. (World Scientific, Singapore, 1995) pp. 22-30.
[10] Pleasants P.A.B., Quasicrystallography: some interesting new patterns, in: Elementary and analytic theory of numbers (PWN, Warsaw, 1984) pp. 439-61.
[11] Niizeki K., *J. Phys.A* **22** (1989) 193-204.
[12] Washington L.C., Introduction to Cyclotomic Fields (Springer, New York, 2nd edition, 1997).
[13] Lidl R. and Pilz G., Applied Abstract Algebra (Springer, New York, 1984).
[14] Ammann R., Grünbaum B. and Shephard G.C., *Discr. Comput. Geom.* **8** (1992) 1-25.
[15] Stampfli P., *Helv. Phys. Acta* **59** (1986) 1260-1263.
[16] Schlottmann M., private communication (1991).
[17] Gähler F., Self-Affine Quasiperiodic Tilings, preprint (1996), to appear in "Proc. of ICGTMP XXI, W. Scherer *et al.*, Eds. (World Scientific, Singapore) in press.