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Orbital Magnetism in Two-Dimensional Integrable Systems

E. Gurevich and B. Shapiro (*)

Department of Physics, Technion-Israel Institute of Technology, Haifa 32000, Israel

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Abstract. — We study orbital magnetism of a degenerate electron gas in a number of two-dimensional integrable systems, within linear response theory. There are three relevant energy scales: typical level spacing $\Delta$, the energy $\Gamma$, related to the inverse time of flight across the system, and the Fermi energy $\varepsilon_F$. Correspondingly, there are three distinct temperature regimes: microscopic ($T \ll \Delta$), mesoscopic ($\Delta \ll T \lesssim \Gamma$) and macroscopic ($\Gamma \ll T \ll \varepsilon_F$). In the first two regimes there are large finite-size effects in the magnetic susceptibility $\chi$, whereas in the third regime $\chi$ approaches its macroscopic value. In some cases, such as a quasi-one-dimensional strip or a harmonic confining potential, it is possible to obtain analytic expressions for $\chi$ in the entire temperature range.

1. Introduction

A degenerate electron gas, in the presence of a weak magnetic field, exhibits weak orbital magnetism [1] (the Landau diamagnetism). For a two-dimensional gas the value of the orbital magnetic susceptibility is given by $\chi_o = -e^2/12\pi M c^2$, where $M$ is the electron mass. (Double degeneracy with respect to spin is implied in this expression, as well as in all subsequent formulae.) This result applies to a macroscopic system.

Whether a sample of a given size $L$ can be considered as macroscopic depends on the temperature $T$. Namely, $T$ (in energy units) should be compared with the characteristic size-dependent energies such as the mean level spacing, $\Delta \simeq 2\pi h^2/ML^2$, or the inverse time of flight across the sample, $\Gamma \simeq h v_F/L \simeq k_F L \Delta/2\pi$, where $v_F$ and $k_F$ are the Fermi velocity and wave number. Generally, one should distinguish between three temperature regimes:

(i) For $T < \Delta$ (the “microscopic” regime) discreteness of the energy levels comes into play and the sample can be viewed as a giant atom. The magnetic response in this case can be very strong and includes such exotic possibilities as perfect diamagnetism and Meissner effect [2].

(ii) For higher temperatures, $\Delta \ll T \ll \Gamma$, the system enters the mesoscopic regime (for recent reviews see Refs. [3,4]). Here the typical value of the magnetic susceptibility is

(*) Author for correspondence (e-mail: boris@physics.technion.ac.il)

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of order \((k_FL)^\alpha |\chi_L|\) and can have either sign. The exponent \(\alpha\) depends on the sample geometry. For most two-dimensional integrable systems \(\alpha = 3/2\), although other values are also possible in some special cases (see below). For completely chaotic two-dimensional systems \(\alpha = 1\).

(iii) For still higher temperatures, when \(T \gg \Gamma\) (but smaller than the Fermi energy \(\varepsilon_F\)), the system can be considered as macroscopic and its magnetic susceptibility, up to small corrections, is given by the Landau value \(\chi_L\).

Thus, at present there is a good qualitative understanding of the phenomenon of orbital magnetism in various temperature regimes and for various geometries. However, reliable quantitative results are scarce. Most of such results refer to the mesoscopic regime [4–6] and are based on a semiclassical approximation for the density of states. This approximation becomes inadequate both at very low temperatures, \(T < \Delta\), and at high temperatures, \(T > \Gamma\). It, therefore, seems useful to consider a few simple systems, starting with an exact expression for the susceptibility \(\chi\), and to observe the behaviour of \(\chi\) in the entire temperature range. This is done in the present paper by using a linear response expression for \(\chi\).

In Section 2 we present several equivalent expressions for the orbital magnetic susceptibility within the linear response theory. In Sections 3, 4, 5 we consider specific examples of a strip, disc, and square-geometries. Section 6 is devoted to an electron gas confined by a two-dimensional parabolic potential.

2. Linear Response Theory for the Magnetic Susceptibility

We consider an electron gas, confined to some domain in the \(xy\)-plane and subjected to a weak magnetic field \(B\) in the \(z\)-direction. The grand-canonical potential

\[
\Omega = -\frac{1}{\beta} \int dE \rho(E) \ln[1 + e^{\beta(\mu - E)}],
\]

where \(\mu\) is the chemical potential, \(\beta = 1/T\) and \(\rho(E)\) is the single-particle density of states. Sometimes it is useful, by integrating by parts twice, to rewrite equation (1) as:

\[
\Omega = -\int dE \Omega(E) \frac{\partial f}{\partial E}
\]

where \(f(E) = \{1 + \exp[\beta(\varepsilon - \mu)]\}^{-1}\) is the Fermi function and the quantity

\[
\Omega(E) = -\int_{-\infty}^{E} dE' \int_{-\infty}^{E''} dE'' \rho(E'')
\]

has the meaning of a grand-canonical potential, for the same system, at zero temperature and with the chemical potential equal to \(E\).

The density of states can be written as

\[
\rho(E) = -\frac{1}{\pi} \text{Im} \; \text{Tr} \; G(E) = -\frac{1}{\pi} \text{Im} \; \text{Tr} \; (E + i\eta - H)^{-1},
\]

where \(G(E)\) is the retarded Green's function, at energy \(E\). The full (single-particle) Hamiltonian \(H\) is split into the unperturbed part, \(H_0\), and the perturbation

\[
V = -\frac{e^2}{2Mc^2} A^2,
\]
which describes the effect of the (static) magnetic field. It is assumed that the vector potential \( A \) satisfies the condition \( \text{div} \; A = 0 \).

Expanding \( G(E) \) in terms of the unperturbed Green’s functions, \( G_0(E) = (E + i\eta - H_0)^{-1} \), one obtains, up to second order,

\[
G = G_0 + G_0 V G_0 + G_0 V G_0 V G_0,
\]

which, on substitution into equation (4), leads to the following expression for the correction \( \delta \rho(E) \) to the density of states:

\[
\delta \rho = -\frac{1}{2\pi} \left( \frac{e}{M c} \right)^2 \frac{\partial}{\partial E} \text{Im} \; \text{Tr} \{ [G_0(A \cdot p)]^2 + M(G_0 A^2) \}.
\]

The first and the second term describe, respectively, the para- and diamagnetic corrections. Plugging equation (7) into (1) and integrating by parts gives:

\[
\delta \Omega = -\frac{1}{2\pi} \left( \frac{e}{M c} \right)^2 \int dE f(E) \text{Im} \; \text{Tr} \{ [G_0(A \cdot p)]^2 + M(G_0 A^2) \}.
\]

This expression is proportional to \( B^2 \), so that the susceptibility, in the \( B \to 0 \) limit, is

\[
\chi = -2\delta \Omega / B^2 A,
\]

where \( A \) is the sample area.

For further calculations it is useful to have an expression for \( \chi \) in terms of \( \chi_0(E) \) which is the susceptibility of the system at \( T = 0, \mu = E \) (compare with Eq. (2)):

\[
\chi = -\int dE \chi_0(E) \frac{\partial f}{\partial E}.
\]

Equation (8) is written in an abstract operator form. For practical calculations it is useful to use a particular representation. For example, if one computes the traces using as a basis the eigenstates of \( H_0 \) (with the appropriate boundary conditions), one obtains, after some algebra:

\[
\chi = \frac{1}{AB^2} \sum_i \left\{ 2 f(\varepsilon_i) \varepsilon_i'' + \frac{\partial f}{\partial \varepsilon_i} (\varepsilon_i')^2 \right\}.
\]

Here the summation is over all states of the unperturbed Hamiltonian \( H_0 \), with eigenenergies \( \varepsilon_i \). The energies \( \varepsilon_i' \) and \( \varepsilon_i'' \) denote the first (i.e. proportional to \( B \)) and the second (proportional to \( B^2 \)) corrections to the unperturbed energies \( \varepsilon_i \). The first correction can exist only for levels which are degenerate in the absence of \( B \).

Another useful expression for \( \chi \) is obtained by writing equation (8) in the coordinate representation. This results in:

\[
\chi = \frac{1}{\pi AB^2} \left( \frac{e}{M c} \right)^2 \text{Im} \int dE \; f(E) \int d^4r \int d^4r' \times \left\{ -\hbar^2 \left[ A(r) \cdot \frac{\partial}{\partial r} G_0(r,r';E) \right] \left[ A(r') \cdot \frac{\partial}{\partial r'} G_0(r',r;E) \right] + M \delta(r - r') G_0(r,r';E) A^2(r) \right\},
\]

where \( G_0(r,r';E) \) is the unperturbed (retarded) Green’s function in the coordinate representation. Since this representation is usually the most appropriate for making various approximations, equation (11) is a good starting point in many cases. It was used, for instance, in the study of mesoscopic effects in disordered systems [7] (in this case \( G_0 \) includes the random potential of impurities). It also helps to prove in the most direct way that, for \( T > \Gamma \), the susceptibility approaches its macroscopic value \( \chi_L \), independently of the sample geometry. Indeed,
the Fermi function \( f(E) \) has poles at values \( E_n = \mu + i(\pi/\beta)(2n + 1) \). Therefore the integral over \( E \) in equation (11) can be replaced by a sum containing \( G_0(\mathbf{r}, \mathbf{r}'; E_n) \). This function in an infinite system decays with distance as \( \exp(-k_F |\mathbf{r} - \mathbf{r}'| \pi(2n + 1)/\beta\mu) \). It is therefore clear that for a system with size \( L \) larger than \( \beta \mu/k_F \), i.e. for \( T > \Gamma \), the susceptibility ceases to depend on sample size or on its geometry. Therefore, for \( T > \Gamma \) linear response theory is applicable as long as the cyclotron energy \( \hbar \omega_c \) is smaller than \( T \). However, for \( T < \Gamma \) the susceptibility \( \chi \) does depend on sample size and its geometry, and the linear response condition requires that \( \hbar \omega_c \) is smaller than the level spacing \( \Delta \) (i.e. the magnetic flux \( \Phi \) through the sample is smaller than the flux quantum \( \Phi_0 = 2\pi \hbar c/e \)).

A useful approximate expression for \( \chi \) is obtained upon using the semi-classical approximation [8] for the Green’s functions in equation (11). Let us briefly outline the derivation (details are presented elsewhere [9]). First, one derives a semi-classical approximation for the function \( \chi_0(E) \). This is done by rewriting the Green’s functions in terms of the propagators \( K(\mathbf{r}, \mathbf{r}', t) \), approximating the propagators by their semi-classical expressions [8] and performing the integrals within a saddle-point approximation. Then one substitutes \( \chi_0(E) \) into equation (9) and integrates over \( E \), making use of the approximation

\[
- \int dE E^\alpha \cos(F(E)) \frac{\partial f}{\partial E} \approx \mu^\alpha \cos(F(\mu)) \mathcal{R}(\pi F'(\mu)T),
\]

where \( F(E) \) is some smooth function of \( E \) (i.e. does not change appreciably within an interval of order \( T \)) and \( \mathcal{R}(x) \equiv x/\sinh x \). The resulting semi-classical expression for \( \chi \) is

\[
\frac{\chi}{|\chi_L|} = 24\pi AM \sum_{\lambda, r} \frac{1}{\tau^2_\lambda} \frac{\langle A^2_{\lambda} \rangle}{A^2} d_{\lambda, r}(\mu) \mathcal{R} \left( \frac{\pi T}{\tau} \right).
\]

Here \( \lambda \) labels families of primitive periodic orbits for integrable systems or isolated orbits for chaotic ones. \( r \) is the winding number, \( \tau_\lambda \) is the period of a primitive periodic orbit and \( \tau_t = \hbar/\pi T \). Factors \( d_{\lambda, r}(\mu) \) are related to the oscillating part of the (unperturbed) semi-classical density of states [8, 10]:

\[
\rho_{osc}(\mu) = \sum_{\lambda, r} d_{\lambda, r}(\mu).
\]

For integrable system \( \langle A^2_\lambda \rangle \) is an orbit area squared and averaged over the family \( \lambda \). For a chaotic system it is simply the squared area of an isolated orbit. Averaging over a family amounts to integration over one of the angle variables [4]

\[
\langle A^2_\lambda \rangle = \frac{1}{2\pi} \int_0^{2\pi} A^2(\theta)d\theta.
\]

The semi-classical expression for \( \chi \), equation (13), coincides with the one derived in reference [4], in the \( B \to 0 \) limit. This fact demonstrates that it does not matter which of the two approximations, i.e. linear response or semi-classics. is done first.

3. Strip Geometry

In this section we consider electrons confined to a strip \(-L_z/2 < x < L_z/2, -L_y/2 < y < L_y/2 \). Periodic and zero boundary conditions are assumed along \( x \) and \( y \) directions respectively, and the limit \( L_z \to \infty \) is taken. It is convenient to choose the Landau gauge: \( A_x = -By \),
\( A_y = A_z = 0 \). Stationary states are labeled by two quantum numbers, \(-\infty < k < +\infty \) and \( n = 1, 2, \ldots \). The eigenfunctions \( \psi_{n,k}(x,y) = e^{ikx} u_{n,k}(y) \), where \( u_{n,k} \) satisfies:

\[
\left\{ -\frac{\hbar^2}{2M} \frac{\partial^2}{\partial y^2} + \frac{1}{2M}(\hbar k + \frac{e}{c} By)^2 \right\} u_{n,k}(y) = \epsilon u_{n,k}(y).
\]

Treating the magnetic field as a perturbation, one obtains [11] that the first order correction \( \epsilon'_{nk} = 0 \) and the second order correction

\[
\epsilon''_{nk} = \frac{L_y^2}{24M} \left( \frac{eB}{c} \right)^2 \left[ 1 - \frac{6}{\pi^2 n^2} + \frac{k^2 L_y^2}{\pi^4} \left( \frac{\pi^2}{n^2} - \frac{15}{n^4} \right) \right].
\]

Thus, only the first term in equation (10) is present, and the zero-temperature susceptibility (including spin degeneracy) \( \chi_0(E) \) is given by

\[
\chi_0(E) = \frac{4}{B^2 L_y} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \sum_{n=1}^{\infty} \epsilon''_{nk} \theta(E - \epsilon_{nk}),
\]

where \( \epsilon_{nk} = (\hbar^2 k^2/2M) + (\hbar^2 \pi^2 n^2/2ML_y^2) \) are the unperturbed energy levels. Next, we integrate over \( k \), apply the Poisson summation formula [1] to the sum over \( n \), and insert the resulting expression for \( \chi_0(E) \) into equation (9). The integral over \( E \) is then performed, using the approximation (12), which amounts to neglecting small terms of order \( T/\mu \). The final expression for \( \chi \) is:

\[
\chi = 1 + \sqrt{\frac{k_F L_y}{\pi}} \sum_{\ell=1}^{\infty} \cos \left( \frac{2\ell k_F L_y - \frac{3\pi}{4}}{\ell^{3/2}} \right) \mathcal{R} \left( \frac{2\pi \ell T}{\Gamma} \right) + O(1/\sqrt{k_F L_y}),
\]

where \( k_F = \sqrt{2M \mu/\hbar^2} \), \( \Gamma = \hbar^2 k_F / \hbar \mu = \hbar v_F / L_y \) and the function \( \mathcal{R}(x) \) has been defined above. In equation (18) we wrote down only the leading oscillating term, of order \( \sqrt{k_F L_y} \), and the term 1, responsible for the Landau diamagnetism. There exist also small oscillating corrections, of order \( (k_F L_y)^{-1/2} \) and smaller, which are not written down explicitly, even though they are calculable [9]. Let us only mention that, in addition to oscillating corrections, there is also a small non-oscillating paramagnetic correction, 9 \( |\chi_L|/8k_F L_y \), to the Landau value \( \chi_L \).

Thus, the strip geometry provides a rare example for which it is possible to obtain an essentially exact (up to small corrections of order \( T/\mu \)) expression for the linear susceptibility \( \chi \), including all size-dependent terms. One can observe the change of \( \chi \) with \( T \) in the entire temperature range, from \( T = 0 \) and up to \( T \gg \Gamma \) when \( \chi \) becomes equal to its macroscopic value \( \chi_L \). Equation (18) is quite similar to the corresponding expression for the case of a parabolic confinement [12]. This fact demonstrates that the nature of confinement (i.e. hard walls or “soft” confinement) is immaterial for the phenomenon of size-dependent fluctuations.

In fact, the leading oscillating term in \( \chi \) can be obtained, within a semi-classical approximation, for any confining potential and for arbitrary magnetic field [9]. For small fields, the oscillations are given by an expression similar to equation (18), up to an overall factor of order unity and an extra phase in the argument of the cosine. \( L_y \) should be understood as some effective width of the confining potential.
Fig. 1. — Linear magnetic susceptibility in a disc geometry, calculated at $T = 0.1\Delta$ and normalized to $|\chi_L|/(k_FR)^2$.

4. Circular Geometry

The electron gas is confined to a disc of radius $R$. The unperturbed, i.e. zero-field stationary states are given, up to a normalization factor, by $\exp(i m \theta)J_m(\lambda_{mn} r/R)$, where $\lambda_{mn}$ is the $n$-th zero of a Bessel function $J_m(x)$. The unperturbed energies are $\varepsilon_{mn} = (\hbar^2/2MR^2)\lambda_{mn}^2$. A pair of states $|m,n\rangle$ and $|-m,\bar{n}\rangle$ have the same energy.

The perturbation term in the Hamiltonian, due to the magnetic field, is:

$$V = i\hbar eB \frac{\partial}{\partial \theta} + \frac{e^2B^2}{8Mc^2} r^2,$$

and the first and second-order energy corrections are:

$$\varepsilon'_mn = \frac{\hbar eB}{2Mc} m, \quad \varepsilon''mn = \frac{e^2B^2}{8Mc^2}(mn|r^2|mn).$$

Note that the first-order term in $V$ does not contribute to the correction $\varepsilon''mn$ which is therefore purely diamagnetic. It now follows from equation (10) that

$$\frac{\chi}{|\chi_L|} = -\frac{6}{R^2} \left\{ \sum_{mn} (mn|r^2|mn)f(\varepsilon_{mn}) + \frac{\hbar^2}{M} \sum_{mn} m^2 \frac{\partial f}{\partial \varepsilon_{mn}} \right\}.$$  

We analyze first the low-temperature regime, $T \ll \Delta \equiv 2\hbar^2/2MR^2$. Since $(mn|r^2|mn) \approx R^2$, the first (diamagnetic) term is of order of the total number of electrons, $N \simeq (k_F R)^2 = 4\mu/\Delta$. The second (paramagnetic) term exhibits sharp peaks every time when the chemical potential $\mu$ coincides with an energy level $\varepsilon_{mn}$. Indeed, for $\mu = \varepsilon_{mn}$, the function $\partial f/\partial \varepsilon_{mn}$ is equal to $-1/4T$, and it rapidly decreases when $\mu$ deviates from $\varepsilon_{mn}$ by a few $T$. Since the quantum number $m$, for states near $\mu$, is typically of order $k_FR$, the height of the paramagnetic peaks is of order $(k_F R)^2 \Delta/T \simeq \mu/T$. Thus, in the low-temperature regime the susceptibility $\chi$ (normalized to the Landau value $|\chi_L|$) plotted as a function of $\mu$, displays a diamagnetic background, of order $\mu/\Delta$, with sharp paramagnetic peaks of height $\mu/T$ and width $T$. An exact numerical computation of the expression (21) confirms this qualitative picture (Fig. 1).

Next, we consider temperatures in the range $\Delta \ll T \sim h\nu_F/2R$. For such temperatures the paramagnetic peaks get smeared and the diamagnetic background, of order $(k_F R)^2$, cancels
with the corresponding paramagnetic term. The net effect is an oscillating term, of order \((k_F R)^{3/2}\). The calculation is based on a Poisson summation of the double sum in equation (21) and on a semi-classical approximation for the unperturbed energies, or \(\lambda_{mn}\), which satisfy [13]:
\[
\sqrt{\lambda_{mn}^2 - m^2} - m \arccos(m/\lambda_{mn}) = \pi (n + \frac{3}{4}).
\] (22)
Let us outline the calculation of the paramagnetic term
\[
\chi^{(p)}/|\chi_L| = -(6h^2/MR^2) \sum m^2 (\partial f/\partial \epsilon_{mn}).
\]
as usual, it is simpler to consider first the zero-temperature case and then to use equation (9). At \(T = 0\), \(\partial f/\partial \epsilon = -\delta(\mu - \epsilon)\) and
\[
\chi_0^{(p)}(E) = 12 |\chi_L| \sum_{m,n} m^2 \delta(\lambda_{mn}^2 - \gamma^2),
\] (23)
where \(\gamma^2 = 4\mu/\Delta\). After applying the Poisson summation formula, \(m\) and \(n\) go over into continuous variables: \(m \rightarrow x\), \((n + \frac{3}{4}) \rightarrow y\). The integral over \(y\) is immediate, due to the \(\delta\)-function. The integral over \(x\) is done in the saddle-point approximation, using the large parameter \(\gamma\) and the continuous version of equation (22). The resulting expression for \(\chi_0^{(p)}(E)\) is:
\[
\chi_0^{(p)}(E) = |\chi_L| \left\{ \frac{3}{4} \gamma^2 + \gamma^{3/2} \sum_{K_x,K_y} \phi(K_x,K_y) + O(\gamma) \right\},
\] (24)
where the sum runs over \(1 \leq K_y < \infty\) and \(0 \leq 2K_x \leq K_y\), and
\[
\phi(K_x,K_y) = \frac{24}{\sqrt{\pi K_y}} \cos^2 \left( \frac{\pi K_x}{K_y} \right) \sin^{3/2} \left( \frac{\pi K_x}{K_y} \right) \cos \left[ 2\gamma K_y \sin \left( \frac{\pi K_x}{K_y} \right) - \frac{3}{2} \pi K_y + \frac{\pi}{4} \right].
\] (25)
A similar treatment of the diamagnetic term results in a term \(-3\gamma^2/4\) plus oscillating corrections of order \(\sqrt{\gamma}\). Thus, the large non-oscillating terms cancel and the net result for \(\chi_0(E)\) is given by the second term in equation (24). Finally, using equations (9, 12), we find:
\[
\chi = |\chi_L| \gamma^{3/2} \sum_{K_x,K_y} \phi(K_x,K_y) \mathcal{R} \left( \frac{\pi K_y T}{\Gamma} \sin \left( \frac{\pi K_x}{K_y} \right) \right).
\] (26)
Thus, in the temperature range \(\Delta \ll T \ll \Gamma\), the susceptibility \(\chi\) is, typically, of order \((k_F R)^{3/2}\) and can have either sign. It oscillates, as a function of \(\mu\), with a period of order \((\mu \Delta)^{1/2} \approx \Gamma\).

In Figure 2 we present \(\chi\) as a function of \(\gamma\), as obtained from equation (26) (solid line). For comparison, dots represent the result of a numerical computation, based on the exact equation (10), with \(\epsilon'_i\) and \(\epsilon''_i\) given by equation (20). These oscillations reflect the structure of the density of states smoothed over energy intervals \(\Delta E \ll \Gamma\), in the same way as the sharp peaks of the low-temperature regime reflected the exact (discrete) spectrum of the system.

Equation (26) can be obtained from the corresponding expression of reference [4], in the \(B \rightarrow 0\) limit. The derivation in reference [4] was based on a semi-classical approximation for the density of states [8,10], with a subsequent introduction of the magnetic field via the classical action. In contrast, we have started with an exact expression for the linear susceptibility and used (in the mesoscopic temperature regime) a semi-classical approximation for the energy levels of the system. In this respect our derivation is similar to that of Dingle [13]. He calculated the orbital magnetization of a thin cylinder and a small sphere, and obtained an expression for \(\chi\), which contained oscillating terms as well as a large steady term. His result for the oscillating terms appears to be correct, while the steady term is due to an error.
5. Square Geometry

Here we discuss electrons within a square of size $L$. The unperturbed energies are $\epsilon_{nm} = (\pi^2 \hbar^2 / 2ML^2)(n^2 + m^2)$. A state $|n,m\rangle$ is degenerate with the state $|m,n\rangle$ (there can be, in addition, accidental degeneracies if a pair $n',m'$ has the same sum of squares as the pair $n,m$).

Let us first consider low temperatures, $T \ll \Delta = 2\pi \hbar^2 / ML^2$, and discuss the paramagnetic peaks due to the second term in equation (10). The first order correction $\epsilon'_{nm}$ is due to lifting of the double degeneracies by the magnetic field. (We do not consider accidental degeneracies, although the treatment is readily extended to include that case as well.) The degeneracy is lifted only if $n$ and $m$ have different parity and in that case

$$\epsilon'_{nm} = -\epsilon'_{mn} = \frac{32 \hbar B}{\pi^2} \frac{n^2 m^2}{M c (n^2 - m^2)^{3}}. \tag{27}$$

Fig. 2. — Linear magnetic susceptibility in a disc geometry, calculated at $T = 5\Delta$ and normalized to $|\chi_L|(k_F R)^{3/2}$. Analytic result obtained from equation (26) (solid line) is compared to numerical one based on equation (35) (dots).

Fig. 3. — Linear magnetic susceptibility in a square geometry, calculated at $T = 0.1\Delta$ and normalized to $|\chi_L|(k_F R)^{3/2}$.
The largest corrections occur when \( n = m \pm 1 \). In such cases \( \epsilon_{nm}^r \approx (\hbar B/Mc)k_F L \) and the height of the corresponding peak in susceptibility is \( \chi_{\text{max}} \approx |\chi_L| (k_F L)^2 \Delta/T \), just as in the case of the disc. Note, however, that in the square geometry such large peaks are much more rare than in a disc. For a disc large peaks were separated by a distance of order \( \Delta \). For a square such peaks occur, roughly, for each pair \( (n,n+1) \), i.e. are separated by a distance of order \( n\Delta \approx k_F L \Delta \approx \Gamma = h\nu T/L \). The difference between a square and a disc is clearly seen, if one compares Figure 3 to Figure 1. Except for the large paramagnetic peaks in Figure 3 one can see an oscillating background. This background comes form the first term in equation (10). It will be shown below that this term can be either para- or diamagnetic and its typical value is of the order \( (k_F L)^{3/2} \).

In the mesoscopic temperature regime, \( \Delta < T < \Gamma \), the susceptibility \( \chi \) is described by the semi-classical expression (13). This case was studied in detail in reference [6] and particularly in reference [4] where expressions for the factors \( d_{\lambda_1} \) and \( \langle A_{\lambda_1}^2 \rangle \) can be found. The resulting expression for \( \chi \) is:

\[
\frac{\chi}{|\chi_L|} = \frac{8}{5\sqrt{\pi}} (k_F L)^{3/2} \sum_{r=1}^{\infty} \sum_{u_x u_y \text{ odd}} \sin \left( \frac{2r \sqrt{u_x^2 + u_y^2} k_F L + \pi/4}{r^{1/2} (u_x^2 + u_y^2)^{5/4} u_x u_y} \right) \\
\times R \left( \frac{2\pi r \sqrt{u_x^2 + u_y^2} T}{\Gamma} \right),
\]

where \( u_x \) and \( u_y \) are positive coprime integers, which label primitive orbits. Only odd \( u_x \) and \( u_y \) enter the sum in equation (28), since otherwise the area enclosed by an orbit is exactly zero [4].

As an example, in Figure 4 we present \( \chi/|\chi_L| \) as calculated from equation (28) (solid line). We chose the same ratio \( T/\Delta = 5 \) as in reference [4]. The result is practically indistinguishable from the numerical one (dots), based on equation (10) (Let us note, that in reference [4] some disagreement between equation (28) and numerics was observed). There are some qualitative differences between the mesoscopic oscillations in the square geometry (Fig. 4) and those for a circle (Fig. 2): in the circle oscillations are modulated on an energy scale which exceeds \( \Gamma \) by an order of magnitude.

Fig. 4. — Linear magnetic susceptibility in a square geometry, calculated at \( T = 5 \Delta \) and normalized to \( |\chi_L| \). Analytic result obtained from equation (28) (solid line) is compared to numerical one based on equation (35) (dots).
Comparison with an exact numerical computation demonstrates that expression (28) is valid down to temperatures $T \approx \Delta$, but fails for $T \ll \Delta$. Nevertheless, it can be used for estimating the aforementioned background, due to the first term in equation (10). The point is that this term ceases to change when temperature is lowered from $T \approx \Delta$ down to $T = 0$. To obtain an estimate of expression (28) at low temperatures, we square it and average out the fast oscillations. This gives, for the typical value of $\chi^2$ in the background,

$$\chi^2_{\text{back}} \approx \chi^2_L \frac{32}{25\pi} (k_F L)^3 \sum_{r, u_x, u_y} R^2 \left( \frac{2\pi r \sqrt{u_x^2 + u_y^2 T / \Gamma}}{r (u_x^2 + u_y^2)^{3/2}} u_x^4 u_y^4 \right)$$  \hspace{1cm} (29)

The sum over repetitions $r$ is estimated by replacing it with an integral, with an upper cutoff provided by the function $R$. This gives a logarithmic factor, so that

$$|\chi_{\text{back}}| \approx |\chi_L| (k_F L)^{3/2} \sqrt{\ln(k_F L)}.$$  \hspace{1cm} (30)

Let us mention that a similar logarithmic factor appears in chaotic billiards, at low temperatures [5]. The behaviour of $\chi$ in that case is, of course, quite different from the square geometry: the enhancement factor is $k_F L$, instead of $(k_F L)^{3/2}$, and the large paramagnetic peaks disappear.

6. Harmonic Confinement

Our last example is a degenerate electron gas confined by a harmonic potential [14–17]. The problem of orbital magnetism in this case was considered previously by a number of authors, and some analytical and numerical results have been obtained for various temperatures and fields. Below we shall derive an analytical expression for the linear susceptibility $\chi$, as a function of temperature [18]. This expression demonstrates the crossover from strong magnetic effects towards the weak Landau diamagnetism, under increase of temperature.

We consider a slightly anisotropic harmonic potential, $U(x, y) = (M/2)(\Omega_1 x^2 + \Omega_2 y^2)$, with $\Omega_1$ close to $\Omega_2$, namely $|\Omega_1 - \Omega_2| \equiv \Delta \Omega < \hbar \Omega_1^2 / \mu$. In the absence of a magnetic field the spectrum is given by $\varepsilon_{nm} = \hbar \Omega_1 (n + \frac{1}{2}) + \hbar \Omega_2 (m + \frac{1}{2})$, where $n, m = 1, 2, \ldots$. For the isotropic case, $\Omega_1 = \Omega_2 = \Omega$, energy levels can be labeled by a single integer $\ell = 1, 2, \ldots$, and the $\ell$-th level is $\ell$-fold degenerate. A small anisotropy $\Delta \Omega$ splits the degenerate levels into narrow “multiplets”. The width of the $n$-th multiplet is of order $\ell \hbar \Delta \Omega$, which is $\mu \Delta \Omega / \Omega$ for multiplets near the energy $\mu$. The above formulated condition, $\Delta \Omega < \hbar \Omega^2 / \mu$, is just the requirement for having well defined multiplets. It is clear on physical ground, and is verified by the calculation below, that such weak anisotropy can affect the physical properties of the system only at temperatures, $T < \mu \Delta \Omega / \Omega$.

When a weak magnetic field is applied, energy levels acquire a second order correction

$$\varepsilon_{nm}'' = \frac{1}{2} \hbar \left( \frac{e B}{M c} \right)^2 \frac{(n + \frac{1}{2})\Omega_1 - (m + \frac{1}{2})\Omega_2}{\Omega_1^2 - \Omega_2^2}$$  \hspace{1cm} (31)

Since for any finite anisotropy the first order correction is zero, equation (10) gives:

$$\frac{\chi}{|\chi_L|} = -\frac{24 \hbar}{MR^2} \frac{1}{\Omega_1^2 - \Omega_2^2} \sum_{n, m} \left[ (n + \frac{1}{2})\Omega_1 - (m + \frac{1}{2})\Omega_2 \right] f(\varepsilon_{nm}).$$  \hspace{1cm} (32)
The effective radius $R$ is determined from $M \Omega^2 R^2 / 2 = \mu$, where $\Omega = (\Omega_1 + \Omega_2) / 2 \approx \Omega_1$. Applying to the double sum in equation (32) the Poisson summation formula, we obtain for the zero-temperature susceptibility $\chi_0(E)$:

$$\frac{\chi_0(E)}{|\chi_L|} = -\frac{12}{\gamma(1-\lambda^2)} \sum_{l,k=0}^{\infty} (-1)^{l+k} \int_{0}^{\gamma/\lambda} \int_{0}^{\gamma-\lambda y} dx(x - \lambda y)e^{2\pi(\lambda y + kx)}$$

where $\lambda = \Omega_2 / \Omega_1$ and $\gamma = \mu/\hbar \Omega = k_T R/2$.

Calculating the elementary integrals and making use of equations (9, 12), one can obtain a final expression for the susceptibility $\chi(T)$. This expression is rather cumbersome and will not be given here [19]. For temperatures $T \gg \gamma \hbar \Delta \Omega$ and anisotropy $\Delta \Omega \ll \Omega / \gamma$ it simplifies to:

$$\frac{\chi}{|\chi_L|} = -1 + 2\gamma^2 \sum_{r=1}^{\infty} \cos(2\pi \gamma r) R \left( \frac{\pi^2 r T}{\Gamma} \right)$$

where $\Gamma = \hbar v_F / 2R = \hbar \Omega / 2$. Except for the leading oscillating term, of order $\gamma^2$, there are smaller oscillating terms which are not written in equation (34). This equation does not contain anisotropy, which demonstrates that, for $T$ much larger than the multiplet width $\gamma \hbar \Delta \Omega$, the anisotropy does not come into play (up to exponentially small corrections). For $T \gg \Gamma$, the oscillations in equation (34) are negligible and $\chi = \chi_L$. For $T \lesssim \Gamma$, one can keep only the first term in the sum, which results in an oscillating term of order $\gamma^2 \approx (k_T R)^2$. For $\gamma \hbar \Delta \Omega \ll T \ll \Gamma$, many terms contribute to the sum. The resulting expression exhibits paramagnetic peaks, of height $(k_T R)^2 \Gamma / 16T$ and width $T$, on a diamagnetic background of order $-(k_T R)^2$.

For $T < \gamma \hbar \Delta \Omega$, equation (34) ceases to be applicable. An analysis of the full expression [19] for $\chi(T)$ shows that it matches the equation (34) at $T \approx \gamma \hbar \Delta \Omega$ and that only minor changes in $\chi(T)$ occur when $T$ is lowered below $\gamma \hbar \Delta \Omega$. This means that at low temperatures, $T < \gamma \hbar \Delta \Omega$, the width of the paramagnetic peaks becomes $\gamma \hbar \Delta \Omega$ and their height is of order $k_T R \Gamma / \hbar \Delta \Omega$.

This result is a manifestation of a general rule. If there is a cluster of nearly degenerate levels about some energy $\varepsilon_i$ (the width of the cluster $\delta$ is much smaller than the typical level spacing $\Delta$), then for $T > \delta$ the cluster behaves as a single degenerate level: it gives rise to a paramagnetic peak of height $g(k_T L)^2 \Delta / T$, where $g$ is the number of non-zero eigenvalues of the matrix $\langle \mathbf{i}|\mathbf{L}_z|\mathbf{j} \rangle$ in the subspace of the cluster. For $T < \delta$, the width of the cluster becomes relevant and the peak saturates at a value of order $g(k_T L)^2 \Delta / \delta$. This is best seen if $\varepsilon'_i$ and $\varepsilon''_i$ in equation (10) are written explicitly using the symmetric gauge. This leads to:

$$\frac{\chi}{|\chi_L|} = -\frac{3\pi}{A} \left\{ \sum_i (\varepsilon_i \langle \mathbf{i}|\mathbf{\Delta}_i|\mathbf{f} (\varepsilon_i) + \frac{1}{M} \sum_{ij} |\langle i|\mathbf{L}_z|j \rangle|^2 \int_0^1 dx \ f' (\varepsilon_i - x) \right\}.$$  \hspace{1cm} (35)

When $\delta < T < \Delta$ the paramagnetic contribution of the cluster, as given by the second term in equation (35), is approximately $-(3\pi / AM) f'(\varepsilon_i) \mathbf{L}_z^2$, where the trace is within the subspace of the cluster. Estimating the trace as $g\hbar^2 (k_T L)^2$ leads to the expression $g(k_T L)^2 \Delta / T$ for the paramagnetic peak.

Let us, finally, mention that when the anisotropy $\Delta \Omega$ approaches the value $\Omega / \gamma$ (i.e. neighbouring multiplets start to overlap), the well pronounced paramagnetic peaks disappear, although the oscillations are still of order $(k_T R)^2$.
7. Conclusions

We have calculated the linear magnetic susceptibility $\chi(T)$ for several two-dimensional integrable systems. Generally, there are three distinct energy scales: level spacing $\Delta$, inverse time of flight $\Gamma$ and the Fermi energy $\epsilon_F$.

For $T < \Delta$, the susceptibility is sensitive to the detailed structure of the energy spectrum as well as the matrix elements of the angular momentum operator. In particular, degeneracies lead to large paramagnetic peaks of order $g(k_F L)^2 \Delta / T$, $g$ being the level degeneracy. At such low temperatures the sample dimensionality is of no importance and similar peaks exist also in three-dimensional systems with degenerate levels [20]. For a pair of nearly degenerate levels, when the level separation $\delta \ll \Delta$, the height of the corresponding peak saturates at $T \approx \delta$. The typical value of $\chi$ between the peaks is also non-universal: for systems with rotational symmetry it is of order $\chi_L(k_F L)^2$ whereas for a square it is of order $|\chi_L| (k_F L)^3/2 \sqrt{\ln(k_F L)}$ and can be either para- or diamagnetic.

For the mesoscopic temperatures, $\Delta \ll T \ll \Gamma$, the orbital magnetic susceptibility remains large. In units of $|\chi_L|$, it is of order $(k_F L)^\alpha$ and oscillates, as a function of the chemical potential $\mu$, with a period $\Delta \mu \approx \Gamma$. For generic integrable systems $\alpha = 3/2$ (compare to $\alpha = 1$ for chaotic systems [3]). For quasi-one-dimensional systems (a strip) $\alpha = 1/2$ and for a harmonic confining potential $\alpha = 2$. Thus, the harmonic potential is a very special case even among the integrable systems. The point is that, for the isotropic case and at zero magnetic field, the two-dimensional harmonic potential exhibits “accidental” degeneracies, which are not related to the rotational symmetry (alike the “accidental” degeneracies in the hydrogen atom).

Finally, for $\Gamma \ll T \ll \mu$, all large orbital magnetic effects disappear and $\chi$ becomes equal to the Landau value $\chi_L$. Thus, for the macroscopic limit to be achieved, it is not sufficient to have $T \gg \Delta$, as one might naively expect. A much more stringent condition, $T \gg \Gamma$, is needed. We close the paper with the following remarks:

(i) Although our calculations have been done for the grand-canonical ensemble, one can immediately infer about the picture for the canonical ensemble. For $T \ll \Delta$, the susceptibility can be very sensitive to the exact number of particles in the system, provided that the last occupied level is degenerate. If such a level is partially occupied, the system is paramagnetic with $\chi \approx c|\chi_L|(k_F L)^2 \Delta / T$, where the factor $c$ accounts for the degeneracy and occupancy of the level. For a fully occupied level, and in the presence of rotational symmetry, the response is diamagnetic, with $\chi \approx \chi_L(k_F L)^2$. (The discussion can be generalized to the case of a cluster of nearly degenerate levels, as was done above for the grand-canonical ensemble.) For mesoscopic temperatures, $\Delta \ll T \ll \Gamma$, the standard transition from the grand-canonical ensemble to the canonical one applies, i.e. $\chi_{\text{can}}(N) = \chi_{\text{gr}}(\mu(N))$. Indeed, the thermal fluctuation $\delta N \approx \sqrt{T/\Delta}$ in the number of particles is much smaller than the change in the number of particles, $(\Delta N) \approx (\Delta \mu)(\partial N / \partial \mu) \approx T / \Delta$, corresponding to a significant change in $\chi$.

(ii) So far we have discussed only the orbital magnetic susceptibility. The Zeeman splitting, due to the electron spin, also contributes. Within the linear response its contribution $\chi_s$ is simply added to the orbital part of the susceptibility. This is clearly seen from equation (10), if the Zeeman splitting $\pm \mu_e B$ is included into the first order energy correction $\epsilon'$. This gives $\chi_s = A^{-1} \mu_e^2 (\partial N / \partial \mu)$, where $N = \sum_i f(\epsilon_i)$. At low temperatures, $T \ll \Delta$, $\chi_s$ exhibits paramagnetic peaks $\chi_s \approx |\chi_L| \Delta / T$, which is just the Curie paramagnetism due to the last occupied level. For temperatures $T \gg \Delta$, $\chi_s$ is given by the Pauli paramagnetism, $\chi_s = 3 |\chi_L|$, up to small corrections. Thus, in this case, there is no appreciable mesoscopic effects due to the electron spin.
(iii) The magnetic field in this paper was treated as a given (homogeneous) external field, $B_{\text{ext}}$. For sufficiently large $\chi$, however, the orbital magnetic currents flowing in the sample produce a field $B_{\text{curr}}$ which is comparable to $B_{\text{ext}}$. In three-dimensional sample this happens when the magnetization $M$ becomes comparable to $B_{\text{ext}}/4\pi$. or $|\chi| \approx 1/4\pi$. In two dimensions the magnetization is defined as the magnetic moment per unit area (rather than per unit volume), so that $\chi$ has units of length. Also, since the thickness of the sample (height in $z$-direction) is much smaller than its size $L$, $B_{\text{curr}}$ differs very much from the volume magnetization (demagnetization effect). So one has to estimate the field $B_{\text{curr}}$ and compare it to $B_{\text{ext}}$. The most stringent limitation is set by the requirement, that the largest possible value the local field $B_{\text{curr}}$ can assume should be much smaller than $B_{\text{ext}}$. This yields [9] $|\chi| k_F \ln(k_F L) \ll 1$, or $|\chi/\chi_L| \ll \ln(k_F L)/k_F r_e$, where $r_e = e^2/Mc^2$ is the classical electron radius. (Though, in some cases, e.g. in a square geometry, the condition is less severe, namely $|\chi/\chi_L| \ll L \ln(k_F L)/r_e$.) As an example consider a disc geometry, where at paramagnetic peaks $\chi/|\chi_L| \simeq (k_F L)^2 \Delta/T$. If $\lambda_F \simeq 10^{-6}$ cm and $k_F L \approx 100$, then, as long as $T > 10^{-3} \Delta$, the self-consistent treatment is not necessary. If the condition is not satisfied, one cannot assume anymore a given external field, but should solve the entire problem self-consistently. A self-consistent treatment leads to such possibilities as Meissner effect and orbital ferromagnetism, both in bulk samples and rings [2, 3, 20–23]. An interesting possibility appears to exist in the strip geometry, considered in Section 3. Here the linear susceptibility, equation (18), is $\chi_0 \simeq |\chi_L| \sqrt{k_F L_y}$. However, the differential magnetic susceptibility, $\chi_d$, in a finite field $B \lesssim M c v_F / e L_y$, turns out to be of order $|\chi_L| (k_F L_y)^{3/2}$ (see Ref. [12], where a strip with harmonic confinement was analyzed) [24]. If $|\chi_0| \ll \lambda_F/4\pi \ln(k_F L)$ but $|\chi_0| > \lambda_F/4\pi \ln(k_F L)$, then one encounters a situation similar to the one which can occur in the de Haas-van-Alphen effect and leads to a break up of the sample into magnetic domains [25].

Acknowledgments

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References


[18] The analysis can be extended to an arbitrary magnetic field (Ref. [19]).


[24] This can be explained in terms of the semi-classical picture. In the strip geometry the leading contribution to $\chi_{osc}$ comes from the “bouncing-ball” trajectories (at $B = 0$ these are self-retracing ones). At zero magnetic field an area enclosed by the trajectories is exactly zero. This area increases along with $B$ and becomes of order $L_y^2$ at $B \approx mc_{F}/eL_y$, what yields $\chi_d \approx |\chi_L|(k_{F}L)^{3/2}$ at such a field.