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Exact Solution and Multifractal Analysis of a Multivariable Fragmentation Model

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Abstract. — Most theoretical kinetic approaches proposed so far to describe fragmentation processes rely on the assumption that the fragments can be characterized by a unique relevant variable, for example their size or mass $h$. We investigate the consequences of introducing additional variables by considering a fragmentation model in which the fragments are described by two internal variables, $h$, $w$, and a breakup rate proportional to $h^\alpha w^\beta$. The exact solution, already partly presented in reference [1], is derived and discussed in details with an emphasis on the fragment distribution function. It is shown that, because of the random multiplicative nature of the process, no reduction to an effective one-variable problem is usually possible. The fragment mass distribution has no simple scaling properties; it rather exhibits multiscaling, a feature that is interpreted in the framework of a multifractal formalism.

Résumé. — La plupart des approches théoriques cinétiques proposées jusqu’ici pour décrire les processus de fragmentation reposent sur l’hypothèse que les fragments peuvent être caractérisés par une seule variable pertinente, par exemple leur taille ou leur masse $h$. Nous examinons les conséquences de l’introduction de variables supplémentaires en considérant un modèle de fragmentation, dans lequel les fragments sont décrits par deux variables internes, $h$, $w$, et un taux de brisure proportionnel à $h^\alpha w^\beta$. La solution exacte, déjà partiellement présentée dans la référence [1], est dérivée et discutée en détails en mettant l’accent sur la fonction de distribution des fragments. On montre à cause de la nature aléatoire multiplicative du processus, qu’aucune réduction à un problème effectif à une variable n’est possible. La distribution de masse des fragments n’a pas de propriétés de lois d’échelles simples; il apparaît plutôt des échelles multiples, une caractéristique qui est interprétée dans le cadre du formalisme des échelles multiples.

1. Introduction

Fragmentation is a usually irreversible process which occurs in a variety of physical situations like polymer degradation [2–5], vaporization of droplets [6], breaking of compact macroscopic objects [7,8] or of hot nuclei [9,10]. These phenomena have recently stimulated increasing experimental work and various theoretical approaches. Although they may involve very different physical systems, some generic features usually come out of the experiments, such as power-law

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behaviors of the fragment mass distribution function in the small-mass region [11]. Some studies have drawn a parallel with critical phenomena in equilibrium systems in order to understand the final distributions observed in some nuclear collisions [12]. For similar problems the use of percolation theory happened to be adequate as well [9,13-16]; it may also allow to discuss the intermittency in the fragment distributions [15,16]. Other approaches used probabilistic and combinatorial techniques [17,18].

Fragmentation can also be investigated via a kinetic description. We will adopt here this point of view and will devote particular attention to the long time behavior of the distribution functions. The approach consists in modeling the phenomena as sequential processes in which fragments break up in cascades. This may represent an oversimplication for fragmentation processes that produce a large number of fragments very rapidly and in a very complex way, but it can be taken as a first step towards more microscopic description (see e.g. Ref. [19]). Moreover, fragmentation occurring over relatively large time scales (of the order of one minute) have also been experimentally observed in the case of silica colloidal aggregates [20]. For such systems, each fragment breaks up independently from the others and the process can indeed be taken as sequential. A kinetic formalism, in which the time evolution is governed by a linear rate equation accounting for creation and destruction of the fragments, was first introduced by Filippov [21] and later developed by Ziff and McGrady [22,23] and Cheng and Redner [26]. Additional mechanisms like inactivation [17] or mass loss [25] were also considered.

A particle or fragment issued from an initial system is generally characterized by variables like its mass (size), geometry (elongation), charge, excitation, or kinetic energy. We will restrict ourselves to systems for which the break up is driven by external causes. The kinetics then depend on the coupling between the external conditions and the physical properties of the fragments. Collisions and recombinations between fragments are neglected, although they may be important in some cases [26]. Introducing a set of internal variables \( \{h_i\} \) to describe of a given fragment, the fragment distribution function \( g(\{h_i\},t) \) at time \( t \) is assumed to follow a linear rate equation:

\[
\frac{\partial g(\{h_i\},t)}{\partial t} = -a(\{h_i\})g(\{h_i\},t) + \int .. \int a(\{h_i'\}) \ g(\{h_i'\},t) \ f(\{h_i'\} | \{h_i\}) \ \Pi dh_i'.
\]  (1)

The quantity \( a(\{h_i\}) \) represents the overall breakup rate of the fragments characterized by \( \{h_i\} \) and \( f(\{h_i'\} | \{h_i\}) \) is the conditional probability that a fragment \( \{h_i\} \) is produced from a fragment \( \{h_i'\} \). The first term in equation (1) takes into account the loss of the fragments due to their scission and the second term describes their creation from larger ones.

Each internal variable can play a role or not in the breakup of a particle. Nearly all previous studies assumed that a single internal parameter is sufficient to describe a fragment, usually its mass or size. Equations similar to equation (1), but reduced to a one-variable description, were used to model atomic collision cascades [27], problems of degradation of polymers [2,28], or aggregate fragmentation [20]. However, such a restriction may not be legitimate for many physical systems. For instance, the fragment mass distribution obtained from the crush of macroscopic objects made of gypsum or glass [7,8] depends on the initial geometry of the objects (spherical, cylindrical, ellipsoidal...). In this case, the breakup rate can be function of both mass and shape. Other examples include degradation of a polyelectrolyte that can \textit{a priori} be controlled by its length and its charge and disintegration of heavy ions and atomic aggregates that may depend upon both their mass and their excitation energy, or their velocity and their kinetic energy.

A description in terms of a unique variable, \textit{e.g.} the mass \( h \) of the fragment usually contains the implicit or explicit [10] assumption that all other variables can be replaced by their mean value. It means that \( a(h, h_i) \approx a(h, \langle h_i \rangle_h) \) and \( f(h', h_i | h, h_i) \approx f(h', \langle h_i \rangle_h | h, \langle h_i \rangle_h) \) where
the averages are taken over all fragments of same mass. In the long-time limit, the number of fragments increases and their mean size goes to zero; the asymptotic evolution of a mass-dependent process is thus controlled by the form of the breakup rate \( a \) in the small-mass region. If \( a(h) \) goes to a nonzero constant, the scission is random. The other cases contained in the more general homogeneous form

\[
   a(h) \sim h^\alpha
\]

have been studied by Ziff and McGrady [22,23], Cheng and Redner [26], and Ernst and Szamel [24].

The purpose of this article is to study how the main characteristics of linear fragmentation processes are modified by the introduction of additional variables. A preliminary account of our work was published in reference [1]. We consider for simplicity a two-variable description and assume that both variables, denoted \( h \) and \( w \) and arbitrary called “mass” and “energy”, are conserved at each breakup event. If \( a(h, w) \) becomes independent of \( w \) when \( w \) goes to zero, the fragments can be asymptotically described by a unique variable \( h \). More generally, we consider the form

\[
   a(h, w) \sim h^\alpha w^\beta,
\]

which represents a natural extension of equation (2). If \( \beta \neq 0 \), two fragments of same mass may have quite different probabilities of breaking up. When \( \alpha \) and \( \beta \) have opposite signs, one expects competing effects which are not contained in the one-variable case.

We consider the binary scission of a fragment \( \{h', w'\} \) into \( \{h' - h, w' - w\} \) and \( \{h, w\} \), where \( h \) and \( w \) are randomly and uniformly chosen in the intervals \([0, h']\) and \([0, w']\), respectively. This process is thus conservative for the mass \( h \) and the energy \( w \) and is ruled by the following kinetic equation:

\[
   \frac{\partial g(h, w, t)}{\partial t} = -h^\alpha w^\beta g(h, w, t) + 2 \int_{h}^{\infty} \int_{w}^{\infty} h'^{\alpha-1} w'^{\beta-1} g(h', w', t) \, dh' \, dw'.
\]

The choice of a uniform conditional probability \( f \) simplifies the description without imposing too much restriction. It is worth noting that equation (4) has a simple geometrical interpretation. It describes the uniform binary fragmentation of an initial rectangle in smaller ones (see Fig. 1a). The breakup rate of a given rectangle (of length \( h \) and width \( w \)) hence depends on its area \( hw \) and its aspect ratio, \( h/w \), as \( (hw)^{\alpha\beta}(h/w)^{\alpha\beta} \). The process is anisotropic since
2. One-Variable Fragmentation. A Summary

We briefly summarize the main features of the standard one-variable process proposed by Ziff and McGrady [22]. In this case, equation (1) with \( a(h) = h^\alpha \) becomes

\[
\frac{\partial g(h,t)}{\partial t} = -h^\alpha g(h,t) + \int_h^\infty h'^\alpha f(h|h')g(h',t) \, dh'.
\]

(5)

An exact solution was derived for a binary uniform breakup i.e., \( f(h|h') = 2/h' \). Cheng and Redner [26] further considered cases where the breakup conditional probability \( f \) has the more general form \( f(h|h') = b(h/h')/h' \). In the large-time limit, the mass distribution function shows interesting scaling law behavior. For \( \alpha > 0 \), the function \( g(h,t) \) can indeed be written in the form

\[
g(h,t) \simeq s(t)^{-2}\phi(h/s(t)),
\]

(6)

where \( s(t) \) is a typical particle mass, and the exponent \(-2\) is a consequence of the mass conservation. The scaling ansatz, equation (6), reduces the description of the process with variables \( h \) and \( t \) to a single variable \( h/s(t) \). It also requires that \( s(t) \) be proportional to the inverse number of fragments, \( \int_0^\infty dh \, g(h,t) \). Therefore, \( s(t) \) is the only characteristic size of the system and it behaves asymptotically like \( t^{-1/\alpha} \) For large values of \( x = h/s(t) \), \( \phi(x) \sim x^{\beta(1-\alpha^2) - \alpha x^0} \) [26].

For \( \alpha \leq 0 \), the scaling description breaks down and a striking behavior appears: the total mass \( M_1 = \int_0^\infty dh \, h g(h,t) \) decreases exponentially with time. It is the signature of a "shattering" transition that takes place in parameter space at \( \alpha = 0 \) [22]. A finite fraction of the initial mass is lost in an infinite number of zero-size particles, called the dust phase. The mass apparently decreases because equation (5) only describes the evolution of fragments of nonzero mass, that do not belong to the dust phase. The total mass \( M_1 + M_{\text{dust}} \) is actually conserved. In the long-time and small-mass regime, the distribution function behaves as [26]

\[
g(h,t) \sim e^{-t \, h^{1-\alpha}/-2}
\]

(7)
3. Two-Variable Model

The model is defined by equation (4), to which we add monodisperse initial conditions, 
\[ g(h, w, t = 0) = \delta(h - h_0)\delta(w - w_0). \]  
(Because of the linearity of the rate equation, any general initial condition is reducible to this monodisperse condition). We further simplify the problem by rescaling the variables, 
\[ h/h_0 \rightarrow h, \quad w/w_0 \rightarrow w, \quad th_0^\beta w_0^\delta \rightarrow t, \]  
so that one can set 
\[ h_0 = w_0 = 1 \]  
in what follows. It is convenient to introduce the moments of the fragment distribution,
\[ M(\lambda, \mu, t) = \int_0^\infty dh \int_0^\infty dw \ h^\lambda w^\mu g(h, w, t). \]  
(8)

\( M(\lambda, \mu, t) \) also represents the double Mellin transform of the function \( g(h, w, t) \). In physical terms, \( M(1,0,t) \) is the total mass of the fragments, \( M(0,1,t) \) the total energy, and \( M(0,0,t) \) the number of fragments created at time \( t \). With the monodisperse initial condition, the moments must satisfy:
\[ M(\lambda, \mu, t = 0) = 1. \]  
(9)

Combining equation (8) with equation (4), one obtains the following equation for the moments:
\[ \frac{\partial M(\lambda, \mu, t)}{\partial t} = \left( \frac{2}{(\lambda + 1)(\mu + 1)} - 1 \right) M(\lambda + \alpha, \mu + \beta, t). \]  
(10)

The time derivative of a given moment is thus proportional to the moment obtained by translation by a vector with components \((\alpha, \beta)\) in the \((\lambda, \mu)\)-plane. We stress that equation (10) is only valid if the moments involved are defined or finite. Different cases will have to be considered, depending on the signs of the parameters \(\alpha\) and \(\beta\). We will particularly discuss the long-time limit, since it is physically the most interesting regime.

The knowledge of all the moments leads the to the distribution function itself via two successive inverse Mellin transforms:
\[ g(h, w, t) = \frac{1}{(2\pi i)^2} \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\mu \ M(\lambda - 1, \mu - 1, t) \ h^\lambda \ w^\mu. \]  
(11)

Since it is not always easy to calculate nor study such integrals, we also introduce various mass distribution functions, \( g_\mu(h, t) \), as
\[ g_\mu(h, t) = \int_0^\infty dw \ w^\mu g(h, w, t). \]  
(12)

The function \( g_0(h, t) \) simply represents the density of fragments having a mass between \( h \) and \( h + dh \), and it is the only distribution that is encountered in the one-variable model. In the case where \( \beta = 0 \), it is easy to check that equation (4) reduces to the one-variable equation for \( g_0(h, t) \). The behavior of the functions \( g_\mu(h, t) \) in the small-mass region can be inferred from a study of the domain of definition of the moments. At fixed parameters \(\alpha\) and \(\beta\), for any value \(\mu\), there is a value \(\lambda_c(\mu)\) such that
\[ \begin{cases} 
M(\lambda, \mu, t) = \infty \text{ for } \lambda \leq \lambda_c(\mu), \\
M(\lambda, \mu, t) < \infty \text{ for } \lambda > \lambda_c(\mu). 
\end{cases} \]  
(13)

As \( M(\lambda, \mu, t) = \int_0^\infty h^\lambda g_\mu(h, t) \ dh \), the leading behavior of \( g_\mu \) for small \( h \) is then given by
\[ g_\mu(h, t) \sim h^{-(\lambda_c(\mu) + 1)} \]  
(14)

This technique has also been used in the one-variable model [26]. More details will be given below.
4. Solution for the Case $\alpha > 0, \beta > 0$

4.1. Solution for the Moments. — We first consider this case because it does not exhibit any shattering phenomenon. Equation (10) can be solved by following the method that Charlesby used in the framework of a one-variable model for polymer degradation [28]. It consists in expanding the moments in power of $t$:

$$M(\lambda, \mu, t) = 1 + \sum_{k=1}^{\infty} \frac{\partial^k M(\lambda, \mu, 0)}{\partial t^k} \frac{t^k}{k!}.$$  \hspace{1cm} (15)

When $\alpha$ and $\beta$ are positive, $\lambda^{\alpha+k\beta} w^{\mu+k\beta} < \lambda^\alpha w^\mu$, so that if $M(\lambda, \mu, t)$ is finite, then $M(\lambda + \alpha, \mu + \beta, t)$ is also finite. Each partial derivative at $t = 0$ can be computed recursively from equation (10) and the initial condition equation (9). It yields

$$\frac{\partial^k M(\lambda, \mu, 0)}{\partial t^k} = \left(\frac{2}{(\lambda + 1)(\mu + 1)} - 1\right) \left(\frac{2}{(\lambda + (k-1)\alpha + 1)(\mu + (k-1)\beta + 1) - 1}\right).$$  \hspace{1cm} (16)

With the usual notation $(a)_k = a(a+1)...(a+k-1)$ and $(a)_0 = 1$, one thus obtains

$$M(\lambda, \mu, t) = \sum_{k=0}^{\infty} \frac{(A)_k(B)_k}{((\lambda + 1)/\alpha)_k((\mu + 1)/\beta)_k} \frac{(-t)^k}{k!}.$$  \hspace{1cm} (17)

where the coefficients $A$ and $B$ are given by

$$A = \frac{\alpha(\mu + 1) + \beta(\lambda + 1) + \sqrt{[\alpha(\mu + 1) - \beta(\lambda + 1)]^2 + 8\alpha\beta}}{2\alpha\beta}$$  \hspace{1cm} (18)

and

$$B = \frac{\alpha(\mu + 1) + \beta(\lambda + 1) - \sqrt{[\alpha(\mu + 1) - \beta(\lambda + 1)]^2 + 8\alpha\beta}}{2\alpha\beta}$$  \hspace{1cm} (19)

The solution can be reexpressed in terms of a generalized hypergeometric function [30],

$$M(\lambda, \mu, t) = _2F_2(A, B, \frac{\lambda + 1}{\alpha}, \frac{\mu + 1}{\beta}, -t).$$  \hspace{1cm} (20)

This result, already given in reference [1], is a generalization of the solution obtained by Ziff and McGrady for the one-variable problem. This latter involves a generalized hypergeometric function of lower order: $M(\lambda, t) = _1F_1((\lambda - 1)/\alpha, (\lambda + 1)/\alpha, -t)$. Taking the limit $\beta \to 0$ in equation (20) gives this result back.

It is easy to see from equation (10) that all defined moments $M(\lambda, \mu, t)$ which are such that $(\lambda + 1)(\mu + 1) = 2$ are independent of time. Contrary to the one-variable model there is an infinite number of conservation laws for $\alpha > 0$ and $\beta > 0$: not only the total mass and the total energy are conserved, but also an infinity of other nontrivial moments of the distribution.

The long-time behavior of the moments is obtained from the study of the $_2F_2$ functions [30]. Their asymptotic behavior in time is algebraic, which leads to [1]

$$M(\lambda, \mu, t) \simeq \frac{\Gamma\left(\frac{\lambda + 1}{\alpha}\right) \Gamma\left(\frac{\mu + 1}{\beta}\right) \Gamma(A - B)}{\Gamma(A) \Gamma\left(\frac{\lambda + 1}{\alpha} - B\right) \Gamma\left(\frac{\mu + 1}{\beta} - B\right)} t^{-B},$$  \hspace{1cm} (21)
where $\Gamma$ denotes the gamma function; $A$ and $B$ are defined by equations (18,19). In the $(\lambda, \mu)$-plane, the moments are defined when the arguments in the gamma functions are positive, i.e., in the region $\{\lambda + 1 > 0, \mu + 1 > 0\}$. Figure 2 displays the hyperbola of the conserved moments that separates the domain of definition of the moments in two parts: in the lower left one the moments (like the number of fragments) increase with time and they decrease in the upper right one. A remarkable characteristic of this model is the irrational dependence of the exponent $B$ on $\alpha$ and $\beta$. This results from the correlations between the two variables $h$ and $w$. As an illustration, the total number of fragments increases like

$$N(t) = M(0,0,t) \sim t^{\frac{\sqrt{(\alpha-\beta)^2 + 6\alpha\beta - \alpha - \beta}}{2\alpha\beta}},$$

whereas in the one-variable model it goes as

$$N(t) \sim t^{1/\alpha}$$

4.2. **Absence of Scaling Solution in the Two-Variable Model.** — We now show that the function $g(h,w,t)$ does not obey a scaling form, contrary to the Ziff-McGrady model.
Consider a two-variable scaling ansatz,

\[ g(h, w, t) = k(t) \phi \left( \frac{h}{s(t)} \frac{w}{\sigma(t)} \right), \tag{24} \]

where \( s(t) \) and \( \sigma(t) \) would represent the typical mass and the typical energy of the fragments at time \( t \), respectively. Whereas the mass is the only conserved variable when \( \alpha > 0 \) and \( \beta = 0 \), an infinite number of moments \( M(\lambda, \mu, t) \) are conserved for \( \alpha > 0 \) and \( \beta > 0 \). It is straightforward to derive that the scaling form, equation (24), combined with the conservation of all moments such that \( (\lambda + 1)(\mu + 1) = 2 \) would imply that the functions \( k(t), s(t), \) and \( \sigma(t) \) satisfy an infinite number of conditions:

\[ k(t)s(t)^{\lambda+1}\sigma(t)^{\frac{1}{\lambda+1}} = C_\lambda, \tag{25} \]

for any value of \( \lambda > 0 \). As \( C_\lambda \) is independent of time, the above condition is only fulfilled when \( k(t), s(t) \) and \( \sigma(t) \) are constant: from equation (10), one can check that this corresponds to the unacceptable solution \( g(h, w, t) = 0 \). Consequently, the system cannot be described by a single pair of mass or energy scales. It rather requires an infinity of scales. We will further investigate the absence of scaling in Section 7 by studying the multifractal properties of the fragment distribution.

As done by Krapivsky et al. [29] in their model of scission of a rectangle, one can also calculate the correlations between the variables \( h \) and \( w \). We thus consider the averages \( \langle h^m w^n \rangle \) which are taken over the fragment distribution and which can be derived from the knowledge of the moments according to

\[ \langle h^m w^n \rangle = \frac{M(n, m, t)}{M(0, 0, t)}. \tag{26} \]

Comparing the average \( \langle h^m w^n \rangle \) with the product of averages \( \langle h^m \rangle \langle w^n \rangle \) illustrates how the fragment distribution deviates from the scaling form given by equation (24). By using equations (19, 21, 26), one obtains for instance for the case \( m = n \) and \( \alpha = \beta \):

\[ \frac{\langle h^m w^n \rangle}{\langle h^m \rangle \langle w^n \rangle} \sim t^{-\frac{\sqrt{n^2 + 8\alpha} - \sqrt{8}}{\alpha}} \tag{27} \]

The above ratio goes asymptotically to zero for \( n > 0 \), whereas it would reach a fixed nonzero value in the case of a distribution obeying a scaling form. The same feature holds for the mass distribution, \( g_0(h, t) \equiv \int_0^h dw \ g(h, w, t) \), which has no scaling form either. The ratio \( \langle h^n \rangle / \langle h \rangle^n \) increases unboundedly with time for \( n \geq 1 \),

\[ \frac{\langle h^n \rangle}{\langle h \rangle^n} \approx t^{\lambda_n/2\alpha\beta}, \tag{28} \]

with

\[ \lambda_n = (n - 1)\sqrt{(\alpha - \beta)^2 + 8\alpha\beta} + \sqrt{(n + 1)(\alpha - \beta)^2 + 8\alpha\beta} - n(2\alpha + \beta). \tag{29} \]

which is the signature of a broad distribution of masses. The one-variable fragmentation model does not produce large fluctuations, since the ratio \( \langle h^n \rangle / \langle h \rangle^n \) reaches a constant nonzero value at large times. A unique mass scale is not sufficient to describe the distribution. As time passes, more and more mass scales appear in the system: it is the signal of the "multiscaling" property.
4.3. Fragment Distribution Function. — To explain these large fluctuations, deeper insight into the fragment distribution is required. We now derive the exact expression of the distribution function \( g(h, w, t) \) by taking the inverse Mellin transform of the solution that we have already obtained for the moments. The factor associated with each power of \( t \) in equation (17) contains a finite number of poles. With the help of the residue theorem, one can express \( g(h, w, t) \) as a uniformly convergent series in powers of \( h^\alpha \), \( w^\beta \) and \( t \):

\[
g(h, w, t) = \delta(w - 1)\delta(h - 1) \ e^{-t} + 2t \\
+ 2\sum_{k=2}^{\infty} \left[ \sum_{j=1}^{k} \prod_{i=1, i \neq j}^{k} \left( \frac{2}{\alpha^2(i-j)^2} - 1 \right) h^{\alpha(j-1)} w^{\beta(j-1)} \right] \frac{t^k}{k!} \\
+ \sum_{k=2}^{\infty} \left[ \sum_{j=1}^{k} \sum_{i \neq j}^{k} \frac{-4}{\alpha^2(i-j)^2} \prod_{l=1, l \neq i, l \neq j}^{k} \left( \frac{2}{\alpha^2(l-i)(l-j)} - 1 \right) h^{\alpha(l-1)} w^{\beta(l-1)} \right] \frac{t^k}{k!}.
\]

The function \( g(h, w, t) \) reaches a finite value when \( h \) or \( w \) goes to zero. We thus check that the moments are finite for \( \lambda + 1 > 0 \) and \( \mu + 1 > 0 \). We recall that equations (17, 20) for the moments are only valid for positive values of the parameters \( \alpha \) and \( \beta \). Nevertheless, equation (30) is always the proper solution of the rate equation, equation (4), whose structure is independent of the signs of \( \alpha \) and \( \beta \). It is valid whatever the signs of \( \alpha \) and \( \beta \), but, unfortunately, it is only tractable numerically for positive \( \alpha \) and \( \beta \). In the case where at least one of the parameters is negative, the series converge slowly, and this situation requires a special study which is done in the following sections.

For \( \alpha > 0 \) and \( \beta > 0 \), we paid particular attention to the energy distribution among fragments of fixed mass \( h \). Figure 3a displays the energy probability distribution function for three different values of the fragment mass at a given time, while Figure 3b shows the same function at two different times for a given mass. The rather flat shape of these curves indicates large fluctuations in energy among fragments of same mass. Two fragments of same mass may have quite different energies with comparable probabilities and may thus evolve very differently. Without a typical energy, the process cannot be reduced to a one-variable model, since it would be incorrect to replace the energy \( w \) of each fragment of mass \( h \) by the mean value \( \langle w \rangle_h \), assuming that fluctuations around this mean value are negligible. The multiscaling property revealed by equations (28,29) can thus be viewed as resulting from the fluctuations of the second variable \( w \).

The mass distribution function itself can be obtained directly by a simple Mellin transform of the moment \( M(\lambda - 1, 0, t) \). We get

\[
g_0(h, t) = \delta(h - 1) \ e^{-t} + 2t \\
+ \sum_{k=2}^{\infty} \left[ \sum_{j=1}^{k} \prod_{i=1, i \neq j}^{k} \left( \frac{2}{\alpha(i-j)^2} \right) \left( \frac{2}{\alpha(i-j)(1+i-j)} - 1 \right) h^{\alpha(j-1)} \right] \frac{t^k}{k!}.
\]

The asymptotic behavior of the above expression can also be derived by an inverse transform of the asymptotic expression of the moments:

\[
g(h, t) \approx \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(1/\beta)\Gamma(A_n - B_n)}{\Gamma(A_n)\Gamma(-n - B_n)\Gamma(1/\beta - B_n)} \ h^{n\alpha} \ e^{-B_n}.
\]

with

\[
A_n = \frac{1}{2} \left[ \frac{1}{\beta} - n \pm \sqrt{(1/\beta + n)^2 + 8/(\alpha\beta)} \right]
\]

\[
B_n = \frac{1}{2} \left[ \frac{1}{\beta} - n \pm \sqrt{(1/\beta + n)^2 + 8/(\alpha\beta)} \right].
\]
As the exponent $-B_n$ increases indefinitely with $n$, the mass distribution function $g_0$ has no simple asymptotic form when $t \to \infty$. Note that equation (31) is still valid for $\alpha < 0$, provided $\beta > 0$. When $\alpha = 0$ and $\beta > 0$, the scission is random as far as the mass is concerned and $g_0(h,t)$ does not depend on $h$. When $\alpha > 0$, the distribution is characterized by a finite value at $h = 0$, as it is shown in Figure 4.

5. Solution for the Cases $\alpha \beta < 0$

When at least one of the parameters is negative, according to the equation (8), the inequality $M(\lambda + \alpha, \mu + \beta, t) < M(\lambda, \mu, t)$ is not always valid since $h^{\lambda+\alpha}$ diverges more rapidly than $h^\lambda$ when $h$ goes to zero. Charlesby's method can no longer be used. We rewrite the moment equation (10) in the form

$$\frac{\partial M(\lambda, \mu, t)}{\partial t} = -\frac{AB}{\left(\frac{\lambda + 1}{\alpha}\right)} \left(\frac{\mu + 1}{\beta}\right) M(\lambda + \alpha, \mu + \beta, t),$$

(34)

where the coefficients $A$ and $B$ depend on $\lambda, \mu, \alpha, \beta$ as given by equations (18,19). Equation (34) is only valid for well-defined moments and may not be applicable everywhere in the region $\{(\lambda + 1) > 0, (\mu + 1) > 0\}$. The divergence of some moments is linked to a shattering phenomenon. If one assumes that there exists values of $\alpha$ and $\beta$ such that the "total" mass
Fig. 4. — Mass distribution function $g_0(h, t)$ versus $h$ for two given times $t$ ($t = 1, 10$) for $\alpha > 0$ and $\beta > 0$.

$M(1, 0, t)$ of finite size particles decreases, it means that $M(1 + \alpha, \beta, t)$ is infinite. More generally, if some of the moments $M(\lambda, \mu, t)$ such that $(\lambda + 1)(\mu + 1) = 2$ decrease, the corresponding moments $M(\lambda + \alpha, \mu + \beta, t)$ must be infinite.

We emphasize that, when shattering occurs, equation (4) cannot describe the time evolution of the dust phase composed by particles of zero mass or zero energy. As the moments defined by equation (8) are derived from the equation (4), they do not include contributions from the singular part of the distribution function, i.e. from the dust phase.

5.1. SOLUTION FOR THE MOMENTS. — To get an idea of the long time behavior of the moments, it is useful to consider their Laplace transform, $\tilde{M}(\lambda, \mu, s) = \int_0^\infty dt e^{-st} M(\lambda, \mu, t)$, for the value $s = 0$. The expression is readily obtained and reads:

$$\tilde{M}(\lambda, \mu, 0) = \frac{\left(\frac{\lambda + 1}{\alpha} - 1\right) \left(\frac{\mu + 1}{\beta} - 1\right)}{(A - 1)(B - 1)} \quad (35)$$

When $\tilde{M}(\lambda, \mu, 0)$ is infinite, $M(\lambda, \mu, t)$ is infinite or else decreases more slowly than $1/t$. The solution in equation (20) does satisfy equations (10) or (34) and the proper initial condition $M(\lambda, \mu, 0) = 1$, but it has an unacceptable asymptotic behavior as soon as $\alpha$ or $\beta$ is negative: it behaves as $t^{-A}$ when $\alpha < 0$ and $\beta > 0$ and as $t^{-B}$ when $\alpha > 0$ and $\beta < 0$, which is not compatible with equation (35). Actually, provided that the moments still possess a power-law asymptotic behavior, one must have

$$M(\lambda, \mu, t) \sim t^{-B} \quad \text{if } \alpha < 0 \text{ and } \beta > 0$$

$$\sim t^{-A} \quad \text{if } \alpha > 0 \text{ and } \beta < 0, \quad (36)$$

with $A$ and $B$ defined in equations (18,19). The method of resolution relies then on developing the moments on a set of functions than include the $2F_2$ given in equation (20) and have similar
asymptotic behaviors \((t^{-A}, t^{-B}, \ldots)\). Such a set of functions is given by all the solutions of the hypergeometric differential equation satisfied by \(2F_2(A, B, (\lambda + 1)/\alpha, (\mu + 1)/\beta; -t)\). An equivalent formulation is obtained by expressing the moments with Meijer G-functions, which are extensions of the generalized hypergeometric functions [30]. Details of the resolution are given in the appendix. The solution for \(\alpha < 0\) and \(\beta > 0\) can be written as follows:

\[
M(\lambda, \mu, t) = \frac{\Gamma(1-A)\Gamma \left( \frac{\mu + 1}{\beta} \right)}{\Gamma(B)\Gamma \left( 1 - \frac{\lambda + 1}{\alpha} \right)} \text{G}_{23}^{21} \left( \begin{array}{c} 1-B, 1-A \alpha \lambda + 1, 1 - \mu + 1 \beta \end{array} \left| t, 0, 1 - \frac{1-B, 1-A}{\lambda + 1, 1 - \mu + 1} \right. \right),
\]

(37)

or, equivalently,

\[
M(\lambda, \mu, t) = \frac{\Gamma(1-B)\Gamma \left( \frac{\lambda + 1}{\alpha} \right)}{\Gamma(A)\Gamma \left( 1 - \frac{\mu + 1}{\beta} \right)} \text{G}_{23}^{21} \left( \begin{array}{c} 1-A, 1-B \alpha \lambda + 1, 1 - \mu + 1 \beta \end{array} \left| t, 0, 1 - \frac{1-A, 1-B}{\lambda + 1, 1 - \mu + 1} \right. \right),
\]

(39)

with \(u = (\lambda + 1)/\alpha\) and \(v = (\mu + 1)/\beta\). In the limit \(\beta \to 0\), one recovers the formula given by Ernst et al. [24] for the moments of the one-variable problem. When \(\alpha > 0\) and \(\beta < 0\), the solution is deduced from equation (37) by permuting \(A\) with \(B\) and \((\lambda + 1)/\alpha\) with \((\mu + 1)/\beta\):

\[
M(\lambda, \mu, t) = \frac{\Gamma(1-B)\Gamma \left( \frac{\lambda + 1}{\alpha} \right)}{\Gamma(A)\Gamma \left( 1 - \frac{\mu + 1}{\beta} \right)} \text{G}_{23}^{21} \left( \begin{array}{c} 1-A, 1-B \alpha \lambda + 1, 1 - \mu + 1 \beta \end{array} \left| t, 0, 1 - \frac{1-A, 1-B}{\lambda + 1, 1 - \mu + 1} \right. \right),
\]

(39)

5.2. Shattering Transition. — With the exact solutions in hand, one can study the main features of the shattering transition. Using the properties of the Meijer G-functions [30], the asymptotic behavior of the solution for \(\alpha < 0\) and \(\beta > 0\) is shown to have the form [1]

\[
M(\lambda, \mu, t) \simeq \frac{\Gamma \left( \frac{\mu + 1}{\beta} \right) \Gamma(1-A)\Gamma \left( 1+B - \frac{\lambda + 1}{\alpha} \right)}{\Gamma \left( 1 - \frac{\lambda + 1}{\alpha} \right) \Gamma(1+B-A)\Gamma \left( \frac{\mu + 1}{\beta} - B \right)} t^{-B}.
\]

(41)

which corresponds to the expected result (Eq. (36)). When \(\alpha\) and \(\beta\) have opposite signs, only part of the hyperbola \((\lambda + 1)(\mu + 1) = 2\) in the \((\lambda, \mu)\)-plane corresponds to conserved moments. For \(\alpha < 0\) and \(\beta > 0\), equation (19) shows that the exponent \(B\) in the above asymptotic formula is equal to zero on the hyperbola only if

\[
\lambda + 1 \geq \sqrt{\frac{2|\alpha|}{\beta}}.
\]

(42)
Fig. 5. — Domain of definition of the moments in the \((\lambda + 1, \mu + 1)\)-space for \(\alpha < 0\) and \(\beta > 0\). The conserved moments belong to that part of the hyperbola for which \(\lambda \geq \lambda_0\) (full curve). The moments decrease in the upper right region, including the left part of the hyperbola (dashed curve); they are not defined in the hatched region and increase with time in the remaining.

The moments associated with the upper part of the curve decrease; see Figure 5. An apparent loss of mass is observed if the point \(\{\lambda + 1 = 2, \mu + 1 = 1\}\) belongs to this upper part; it is the case when

\[
\beta < \frac{|\alpha|}{2}
\]

Similarly, when \(\alpha > 0\) and \(\beta < 0\), the moments of the hyperbola \((\lambda + 1)(\mu + 1) = 2\) decrease if \(\lambda + 1 > \sqrt{2\alpha}/|\beta|\). The total mass is then decreasing with time when \(|\beta| > \alpha/2\). So, to summarize, the total mass decreases algebraically in the region \(\beta < -\alpha/2\), according to

\[
M(1, 0, t) \sim t^{-\left(\frac{\lambda}{2} + \frac{3}{2}\right)}
\]

A new type of behavior is thus observed: contrary to what occurs in the one-variable model (and to the present model when \(\alpha < 0\) and \(\beta < 0\)), the moments that are associated with the shattering transition (like \(M(1, 0, t)\) when \(\beta < \alpha/2\)) do not decay exponentially with time but rather algebraically. This results from the competing effects that are generated by the opposite
Fig. 6. — a) Phase diagram of the shattering transition in the \((\alpha, \beta)\) parameter space. The total mass \(M(1,0)\) is conserved in the hatched region, decreases algebraically in the lower right sector (bounded by the two straight lines \((\beta = 0, \beta = -\alpha/2)\)) and in the upper left sector (bounded by the two straight lines \((\alpha = 0, \beta = -\alpha/2)\)), and decreases exponentially for \(\alpha \leq 0, \beta \leq 0\). b) Similar to (a), except that the shattering transition is now defined by a decay of the total energy \(M(0,1)\).

The signs of \(\alpha\) and \(\beta\) in the overall breakup rate and lead to a slow (or “critical”) evolution. If \(\alpha < 0\) and \(\beta > 0\), the disintegration cascade primarily affects the variable \(h\). A singular distribution of fragments appears, but the fragmentation involving \(w\) makes the whole process much less effective than when \(\alpha\) and \(\beta\) are both negative. There is thus some ambiguity in defining the shattering transition. One could choose as its definite signature either the appearance of a singular distribution of fragments or the decay of the moment \(M(1,0,t)\) associated with the total mass (or else the decay of another normally conserved moment, like that associated with the energy, \(M(0,1,t)\)). The latter choice is more restrictive and is not equivalent to the former since a singular distribution of fragments in \(h = 0\) appears for all pairs \((\alpha < 0, \beta > 0)\). Nevertheless, this singularity may not be strong enough to affect the conservation of the total mass. Retaining now the decay of the mass to define shattering, we can draw the phase diagram of the shattering transition in the \((\alpha, \beta)\)-plane. It is shown in Figure 6a. Note that, although the model is symmetric in \(h\) and \(w\), the phase diagram is not symmetric because of the choice of \(M(1,0,t)\) to characterize shattering. If one chooses instead the total energy \(M(0,1,t)\), one obtains a different phase diagram displayed in Figure 6b.

It is quite remarkable that the total mass is conserved when \(\alpha < 0\), provided \(\beta > -\alpha/2\). Such a feature is totally absent from one-variable models. Symmetrically, a decay of the mass is observed for \(\alpha > 0\), provided \(\beta < -\alpha/2\). The shattering transition is then triggered by the disintegration cascade primarily involving the energy \(w\) via the correlation between \(h\) and \(w\). The part of the diagram corresponding to the case \(\alpha < 0\) and \(\beta < 0\) will be discuss later on.

5.3. Mass Distribution Function. — When \(\alpha < 0\) and \(\beta > 0\), the moments are defined (i.e., finite and positive) in the domain of the \((\lambda, \mu)\)-plane where the arguments of the gamma functions in equation (41) are all positive and the exponent \(B\) is real. \(M(\lambda, \mu, t)\) exists in the
following conditions:

\[
\begin{align*}
\alpha < 0, \beta > 0 & \quad \left\{ \begin{array}{l}
(\lambda + 1 + |\alpha|)(\mu + 1 - \beta) \geq 2 \quad \text{if} \quad \lambda + 1 \leq |\alpha| \left(-1 + \sqrt{\frac{2}{|\alpha\beta|}}\right) \\
\frac{\lambda + 1}{|\alpha|} + \frac{\mu + 1}{\beta} \geq 2 \sqrt{\frac{2}{|\alpha\beta|}} \quad \text{if} \quad \lambda + 1 \geq |\alpha| \left(-1 + \sqrt{\frac{2}{|\alpha\beta|}}\right)
\end{array} \right. \\
\alpha > 0, \beta < 0 & \quad \left\{ \begin{array}{l}
(\lambda + 1 - \alpha)(\mu + 1 + |\beta|) \geq 2 \quad \text{if} \quad \lambda + 1 \geq \alpha \left(1 + \sqrt{\frac{2}{|\alpha\beta|}}\right) \\
\frac{\lambda + 1}{\alpha} + \frac{\mu + 1}{|\beta|} \geq 2 \sqrt{\frac{2}{|\alpha\beta|}} \quad \text{if} \quad \lambda + 1 \leq \alpha \left(1 + \sqrt{\frac{2}{|\alpha\beta|}}\right)
\end{array} \right.
\end{align*}
\]

(45) (46)

This is illustrated in Figure 5, where the hatched zone denotes the region where the moments are not defined. This region is delimited by a portion of hyperbola and a portion of straight line. With the property discussed in and around equations (13,14), we deduce the leading behavior of the mass distribution function in the small-mass region. For \(\alpha\beta < 0\),

\[
g_0(h,t) \sim h^{\frac{\alpha}{|\beta|} - \sqrt{8|\alpha/\beta|}} \quad \text{if} \quad \beta > 1 \quad \text{or} \quad \alpha > -\frac{2\beta}{(1-\beta)^2}
\]

(47) and

\[
g_0(h,t) \sim h^{-\alpha - \frac{2}{1-\beta}} \quad \text{if} \quad \beta < 1 \quad \text{and} \quad \alpha < -\frac{2\beta}{(1-\beta)^2}.
\]

(48)

The value of the exponent of the algebraic law can be either positive or negative. For strong disintegrations (\(\alpha\) sufficiently negative, for example) the initial fragment disappears almost instantaneously to produce dust particles: the mass distribution vanishes as \(h \to 0\) because small (but finite) fragments occurs with very low probability. Conversely, in disintegrations of intermediate strength, the initial fragment has the opportunity to break up in numerous small (but finite) ones, while dust is produced at the same time. It is important to notice that the mass exponent is always larger than \(-2\), insuring that the corresponding total mass is always finite, which is not the case of the number of fragments for instance. Figure 7 sums up the different behaviors encountered in the \((\alpha, \beta)\)-plane. On the \(\beta = 0\) axis, we recover the results of the one-variable model [22,26] for which a transition between two kinds of mass distribution occurs at \(\alpha_c = -2\). The introduction of a second variable modifies the scission of the small fragments and logically shifts the value \(\alpha_c\) to smaller values when \(\beta > 0\) and to larger values when \(\beta < 0\).

It is also worth noting that contrary to the one-variable model, the present one may lead to a power-law divergence of the mass distribution in the small-mass regime, even when shattering does not occur, as is indeed observed experimentally. This requires the presence of a second variable which, when \(\alpha > 0\), boosts the production of the small but finite fragments (case \(\beta < -\alpha/8\)).

6. Solution for the Case \(\alpha < 0\) and \(\beta < 0\)

6.1. Shattering Transition. — We first show by heuristic arguments that the system necessarily produces a dust phase for all pairs \((\alpha < 0, \beta < 0)\), and that the (apparent) mass loss is exponential in time. Consider the moment equation, equation (10), in which one sets \(\lambda = 1 + |\alpha|\) and \(\mu = |\beta|\):

\[
\frac{\partial M(1 + |\alpha|,|\beta|,t)}{\partial t} = \left(\frac{2}{(2 + |\alpha|)(1 + |\beta|)} - 1\right) M(1,0,t).
\]

(49)
As the prefactor in the right-hand side of the above equation is negative, assuming that $M(1,0,t)$ remains constant during the process implies that $M(1+|\alpha|,|\beta|,t)$ decreases linearly with time. This moment then becomes negative after a finite time, which is not acceptable. The consistency of equation (49) requires that the mass must not be conserved along the process.

The argumentation of Cheng and Redner [26] for the one-variable model can be applied here to show that the time dependence of the function $g(h,w,t)$ is controlled by $e^{-t}$ for all values of $h$ and $w$. The demonstration uses equation (4) and is based on the fact that $e^{-t}$ is both a lower and upper bound for $g(h,w,t)$. The total mass then decays as

$$M(1,0,t) \sim e^{-t},$$

(50)

as well as any other defined moment. This result is reminiscent of the behavior in the one-variable model [22, 26].

6.2. MOMENTS AND DISTRIBUTION FUNCTIONS. — Since their long time behavior is not algebraic in time, but rather exponential, the moments cannot be expressed with the solutions,
equations (20), (37), or (39) derived in the previous sections. We then look for a solution involving a different Meijer function \( G^{\alpha \beta}_{23} \) and taking into account the conditions \( M(\lambda, \mu, 0) = 1 \) and equation (50). The derivation is given in the appendix and we obtain, as given without proof in reference [1],

\[
M(\lambda, \mu, t) = \frac{\Gamma(1-A)\Gamma(1-B)}{\Gamma \left(1 - \frac{\lambda + 1}{\alpha} \right) \Gamma \left(1 - \frac{\mu + 1}{\beta} \right)} G^{30}_{23} \left( \begin{array}{c} 1-B, 1-A \\ 0, 1-\frac{\lambda + 1}{\alpha}, 1-\frac{\mu + 1}{\beta} \end{array} \right),
\]

(51)

or, equivalently,

\[
M(\lambda, \mu, t) = 2 F_2(\alpha, \beta, u, v; -t) + \frac{\Gamma(u - 1)\Gamma(u - v)\Gamma(1 - B)\Gamma(1 - A)}{\Gamma(u - B)\Gamma(u - A)\Gamma(1 - u)\Gamma(1 - v)} t^{1-u} 2 F_2(1 + A - u, 1 + B - u, 2 - u, 1 + v - u; -t) + \frac{\Gamma(v - 1)\Gamma(v - u)\Gamma(1 - B)\Gamma(1 - A)}{\Gamma(v - B)\Gamma(v - A)\Gamma(1 - v)\Gamma(1 - u)} t^{1-v} 2 F_2(1 + A - v, 1 + B - v, 2 - v, 1 + u - v; -t).
\]

(52)

As required, the asymptotic behavior of \( G^{30}_{23} \) [30] is exponential,

\[
M(\lambda, \mu, t) \approx \frac{\Gamma(1-A)\Gamma(1-B)}{\Gamma \left(1 - \frac{\lambda + 1}{\alpha} \right) \Gamma \left(1 - \frac{\mu + 1}{\beta} \right)} e^{-t}
\]

(53)

The moments are defined provided that the arguments of the gamma functions in the above formula are strictly positive. It is the case when

\[
(\lambda + 1 + |\alpha|)(\mu + 1 + |\beta|) > 2.
\]

(54)

Figure 8 displays the domain of definition for \( \alpha = -1 \) and \( \beta = -0.5 \). The behavior of the mass distribution functions \( g_\mu(h,t) \) is derived from equations (53,54). For \( h \to 0^+ \) and \( t \to +\infty \),

\[
g_\mu(h,t) \sim h^{|\alpha| - \frac{2}{\mu + 1 + |\beta|}} e^{-t},
\]

(55)

so that the mass distribution function \( g_0(h,t) \) diverges in the small-mass region when

\[
|\alpha| < \frac{2}{1 + |\beta|}.
\]

(56)

For \( \beta = 0 \) and \( \mu = 0 \), one thus recovers the one-variable solution, \( g_0(h,t) \sim e^{-t h^{|\alpha| - 2}} \) [22,26].

7. Multifractality

We have seen that the absence of scaling solution for the fragment distribution in the case \( (\alpha > 0, \beta > 0) \) is a consequence of the infinite number of scales generated by the fragmentation process. Conversely, a unique typical size characterizes the distribution when \( \beta = 0 \) (or \( \alpha = 0 \)). The multiscaling property can be interpreted in terms of the multifractal structure of the mass distribution function. Multifractality is often encountered in random multiplicative processes like fully developed turbulence [32,33] or strange Cantor sets [34]. We adopt here a point of view slightly different from the one we used until now. Instead of studying the asymptotic kinetics \( t \to +\infty \) of the distribution functions, we choose a given time \( t_c \). The multifractal analysis that we present consists in studying how the fragment mass distribution at a fixed time \( t_c \) varies when the initial conditions are modified. This is an application to fragmentation of the multifractal formalism established by Amitrano et al. [35] in the context of growth processes.
Fig. 8. — Domain of definition of the moments in the \((\lambda + 1, \mu + 1)\)-space for \(\alpha < 0\) and \(\beta < 0\). The moments decrease in the upper right region, including the hyperbola (dashed curve). The moments are not defined in the hatched region.

7.1. THE MULTIFRACTAL FORMALISM. — Consider a system composed of fragments that are characterized by their mass, \(h\). The quantity \(L\) denotes a characteristic variable of the initial fragment. If it is a relevant variable for the fragmentation process, the mass distribution \(g_0(h)\) function should depend on \(L\). The aim is to determine how the distribution depends on \(L\), all other initial characteristics remaining fixed. Let the \(q\)th moment of the mass distribution be defined as

\[
Z_q(L) = \int h^q g_0(h, L) \, dh
\]

\[
= \int h^{q-1} g_m(h, L) \, dh
\]

(57)

where \(g_m(h, L) \, dh = h g_0(h, L) \, dh\) represents the total mass of the fragments which have their mass in the interval \([h, h + dh]\). When the scaling law

\[
Z_q(L) \sim L^{\xi(q)}
\]

(58)

is encountered for large \(L\), the system is fractal [36]. The asymptotic behavior of \(g_m(h, L)\) with \(L\) can then be deduced from the set of exponents \(\xi(q)\) [35]. With the change of variable \(y = \ln h\), equation (57) becomes

\[
Z_q(L) = \int dy \, \exp [qy + \ln \tilde{g}_m(y, L)],
\]

(59)
and can be evaluated by the steepest-descent method (for \( L \) large enough). The integrand is maximum for a value \( y^* \) that is given by

\[
\frac{\partial \ln \hat{g}_m(y, L)}{\partial y}
\bigg|_{y=y^*} = -q.
\] (60)

Thus, to each value of \( q \) corresponds a value \( h^* = e^{y^*} \), which gives the dominant contribution to the \( q \)th moment. Assuming the following scaling ansatz,

\[
h^* = A(q)L^{-\tilde{\alpha}(q)},
\]

\[
g_m(h^*, L) = B(q)L^f(q),
\]

one deduces from equations (58,59):

\[
Z_q \sim L^{f(q)-q\tilde{\alpha}(q)},
\]

\[
f(q) = \xi(q) + q\tilde{\alpha}(q).
\]

Moreover, the condition given by equation (60) yields

\[
\tilde{\alpha}(q) = -\frac{d\xi(q)}{dq}
\]

(65)

Since \( h^* \) may take a large number of possible values for \(-\infty < q < \infty\), it can be considered as an independent variable and it is simply denoted \( h \). In addition, the function \( \tilde{\alpha}(q) \) given by equation (65) is monotonous and it can be inverted to express \( q \) as a function of \( \tilde{\alpha} \). One then has

\[
g_m(h, L) = C(\tilde{\alpha})L^{\phi(\tilde{\alpha})},
\]

(66)

with

\[
\phi(\tilde{\alpha}) = f(q(\tilde{\alpha})), \quad C(\tilde{\alpha}) = B(q(\tilde{\alpha})).
\]

(67)

The system of fragments can thus be interpreted as a partition into subsystems characterized by a value of the singularity exponent \( \tilde{\alpha} = -\ln h/\ln L \). The function \( f \) (or \( \phi(\tilde{\alpha}) \)) is the fractal dimension of the subset made of particles with same \( \tilde{\alpha} \). If \( \xi(q) \) is a linear function of \( q \), equations (64,65) imply that \( \tilde{\alpha}(q) \) and \( f(q) \) are constant: the fractal system is homogeneous. If \( \xi(q) \) is non linear, \( f(q) \) varies with \( q \) and an infinite number of exponents characterizes the scaling properties of the mass distribution function. In this case, the distribution has a "multifractal" structure.

This formalism was applied to diffusion limited aggregation (DLA) [35,37]. In growth processes, \( L \) represents the system size (or the observation window) and \( h \) the growth probability. Multifractality has also been observed on the voltage distribution of a resistive network at the percolation threshold [36].

7.2. APPLICATION TO FRAGMENTATION. — We now apply the above formalism to the two-variable fragmentation process. In this section, the initial energy \( w_0 \) of a fragment of fixed mass \( h_0 \) is considered as the tuning parameter \( L \). The \( q \)th moment of the mass distribution is given by

\[
Z_q = \int_0^{h_0} dh \int_0^{w_0} dw \, g(h, w, t_c)
\]

\[
= \int_0^{h_0} dh \, h^q g_0(h, w_0, t_c)
\]

(68)
where \( t_c \) is fixed. The moment \( Z_q \) can be expressed in terms of the expression that we have previously obtained for the moments \( M(q,0,t) \) with normalized initial conditions \( h_0 = w_0 = 1 \):

\[
Z_q = h_0^q M(q,0,t_c h_0^\alpha w_0^\beta).
\]

(69)

In the case \( \alpha > 0 \) and \( \beta > 0 \), using equation (21) for large \( w_0 \), we obtain:

\[
Z_q \sim h_0^{\alpha B + q} w_0^{-\beta B},
\]

(70)

where \( B \) is given by equation (19) with \( \lambda = q \) and \( \mu = 0 \). As the initial size \( h_0 \) is fixed, the moments \( Z_q \) depend algebraically on the initial energy,

\[
Z_q(w_0) \sim w_0^{(-\alpha - \beta(q+1)+\sqrt{(\alpha-\beta(q+1))^2+8\alpha\beta})/(2\alpha)} \sim w_0^\xi(q)
\]

(71)

The irrational dependence of \( \xi(q) \) on \( q \) is a signature of the multifractal structure the mass distribution function. Inserting equation (71) in equations (64,65) and introducing \( r = \beta/\alpha \) yield

\[
\tilde{\alpha}(q) = \frac{1}{2} r \left[ 1 + \frac{1 - r(q + 1)}{\sqrt{(1-r(q+1))^2 + 8r}} \right],
\]

(72)

\[
f(q) = \frac{1}{2} \left[ -(r + 1) + \frac{1 + r^2 + 6r + rq(r-1)}{\sqrt{(1-r(q+1))^2 + 8r}} \right].
\]

(73)

When \( \beta = 0 \) (one-variable case), \( h_0 \) is the only initial variable and one has

\[
Z_q(h_0) \sim t_c^{(q-1)/\alpha h_0}
\]

(74)

for \( \alpha > 0 \). Therefore, \( f(q) = 1 \). Note also that, in the case \( \alpha < 0 \) and \( \beta < 0 \), the mass distribution does not have the scaling form, equation (58), and no exponent \( \xi(q) \) can be defined since

\[
Z_q \sim h_0^\alpha \exp(-t_c h_0^\alpha w_0^\beta).
\]

(75)

Figure 9 shows the typical shape of the \( q \)-dependence of \( \xi(q) \) as given by equation (71). Although moments are not defined for negative values of \( q \), the function is extended over the whole interval \( -\infty, \infty \). A strong deviation from the linear behavior is observed for values of \( q \) around zero. Figure 10a displays the spectrum \( f(q) \) as a function of \( q \). This function has some general features that hold whatever the value of \( r = \alpha/\beta \): \( f(q) \rightarrow -1 \) when \( q \rightarrow \infty \) and is maximal for \( q = 0 \). We also plot the fractal dimension \( f \) as a function of the singularity exponent \( \tilde{\alpha} \) (Fig. 10b). These curves are convex, as predicted by the general multifractal formalism.

With the help of the exact solution for the mass distribution function, equation (31), we can check the validity of the scaling properties, equations (61,62). We replace in equation (31) the time \( t \) by \( t_c w_0^\beta h_0^\alpha \) (with \( t_c = 1 \) and \( h_0 = 1 \)). According to the multifractal formalism, the curve \( \ln(g(h,w_0)) \) as a function of \( \ln w_0 \), at a fixed value of \( \tilde{\alpha} = -\ln h/\ln w_0 \), must converge asymptotically to a straight line with a slope that depends on \( \tilde{\alpha} \). When \( q \rightarrow \infty \), the largest fragments give the maximum contribution to the moments \( Z_q \). In this region of the mass distribution function, the scaling exponent is given by \( f(\infty) = -1 \) and the singularity exponent by \( \tilde{\alpha}(\infty) = 0 \). In Figure 11 the curve \( \ln(g(h,w_0)) - \ln w_0 \) is plotted for \( \tilde{\alpha} = 0 \) (which corresponds to \( h = h_0 \)), and one indeed obtains a straight line with a slope around \( -1 \). When \( \tilde{\alpha} \) is larger, the slope increases. We have thus checked two features already discussed: (i) \( f(\tilde{\alpha} = 0) = -1 \) and (ii) \( f(\tilde{\alpha}) \) rapidly increases with \( \tilde{\alpha} \) for small value of \( \tilde{\alpha} \) (cf. Fig. 10b).
Fig. 9. — The exponent $\xi(q)$ versus $q$ for different values of the ratio $r = \beta/\alpha$.

Fig. 10. — The exponent $f$ as a function of $q$ (a) and as a function of $\tilde{\alpha}$ (b) for different values of $r = \beta/\alpha$.

The preceding study can be repeated by considering the energy $w_0$ as fixed and the mass $h_0$ as the tuning parameter. Similar curves are then obtained for the scaling exponents.

The shape of the $f(q)$ or $f(\alpha)$ curves often has some universal features. The above multifractal formalism has recently been applied to the free flight paths in self-ion collisional cascades [38] and the $f(q)$ curves obtained from the numerical simulations are close to those presented in
Fig. 11. — Log-log plot of $g(h, w_0)$ versus $w_0$ for three different values of $\alpha$. For large $w_0$, a linear behavior is asymptotically attained; the slopes as obtained from least-square fitting are $-1.06$, $-0.93$, and $-0.56$ for $\alpha = 0$, $0.01$, and $0.05$, respectively.

Figure 10b. The functions $\xi(q)$ and $f(\alpha)$ that we have obtained for the two-variable fragmentation process are also quite similar to those corresponding to DLA [35,37] or to Cantor constructions [34].

An alternative multifractal analysis is based on the box-counting method [34], in which one studies how some moments (or various averages) defined within a small subsystem go to zero when the size $\epsilon$ of the subsystem goes to zero. This method has been extensively used in fully developed turbulence and the appearance of an infinite set of exponents $\xi(q)$ is called "intermittency" [32,33]. This formalism has been extended to fragmentation, by subdividing the whole range of possible fragment sizes into boxes of size $\epsilon$ [15–17]. An intermittent behavior that is formally analogous to the one observed in turbulence was encountered in nuclear multifragmentation [9,15]. However, this kind of multifractal analysis is not equivalent to the one that we have previously discussed. For the two-variable fragmentation model, in the case $\alpha > 0$ and $\beta > 0$, the system is multifractal according to the definition of Amitrano et al. [35], but the numerical simulations that we have performed, but do not reproduce here, show that the system is not intermittent. The specificity of the method that we have proposed in this section is to focus on the moments of the whole mass distribution function. This may be useful to interpret fragmentation experiments in which the influence of an initial macroscopic variable ($L$) can be analyzed.
8. Conclusion

We have studied a sequential fragmentation model in which the fragments are described by two internal variables, mass $h$ and energy $w$. Both of these variables have an influence on the process through a breakup rate that is proportional to $h^\alpha w^\beta$. When the exponents $\alpha$ and $\beta$ are both positive or with opposite signs, there is an infinite number of conserved moments during the process. Therefore, the exact solution obtained for the fragment distribution cannot be written in a scaling form, contrary to one-variable models. The large fluctuations of the variable $w$ among fragments of same mass $h$ make impossible a description of the fragments with a single internal variable. This shows the limits of the previously proposed fragmentation models that neglect such fluctuations. In addition, we also observe a new kind of shattering transition that is characterized by a power-law ("critical") decay in time in place of the usual exponential decay. It is worth noting that these features are not specific to the two-variable case, and still hold when the model is generalized to include three (or more) variables. We have also discussed the behavior of the mass distribution function. When $\alpha > 0$ and $\beta > 0$, we have further investigated how the fragment distribution depends on the initial conditions, with the help of a multifractal formalism. We show that the mass distribution function has a multifractal structure, analogous to that of several multiplicative processes. Again, this property is not restricted to the two-variable case considered here, but can be shown to be valid for more general forms of the conditional breaking probability and larger sets of relevant variables. The multifractal analysis could motivate new experimental approaches, in particular to determine the number of relevant variables by studying the influence of initial macroscopic quantities on fragmentation processes. One potential example is the breaking of macroscopic objects already mentioned [7,8], for which the moments of the fragment size distribution could be computed for systems of varying initial mass but same elongation and geometry, or vice-versa.

Appendix

Solution of the Moment Equation

In order to simplify the expressions, we note $u = (\lambda + 1)/\alpha$ and $v = (\mu + 1)/\beta$. The generalized hypergeometric function $_2F_2(A, B; u, v; -t)$, which is always solution of equation (10) can be expressed for large values of $t$ as the sum of three contributions having $t^{-A}$, $t^{-B}$ and $e^{-t}$ asymptotic behaviors, respectively [31]:

\[
_2F_2(A, B; u, v; -t) = \frac{\Gamma(A)\Gamma(B-A)t^{-A}}{M_{22}\Gamma(u-A)\Gamma(v-A)} \ {}_2F_1(A, 1+A-u, 1+A-v; 1+A-B; \frac{1}{t}) \ \\
+ \frac{\Gamma(B)\Gamma(A-B)t^{-B}}{M_{22}\Gamma(u-B)\Gamma(v-B)} \ {}_2F_1(B, 1+B-u, 1+B-v; 1+B-A; \frac{1}{t}) \ \\
+ e^{-t} \frac{1}{M_{22}} t^{A+B-u-v} \left( 1 + \sum_{k=1}^{\infty} c_k (\frac{t}{2})^{-k} \right), \tag{A.1}
\]

with $M_{22} = \Gamma(A)\Gamma(B)/\Gamma(u)\Gamma(v)$. When $\alpha > 0$ and $\beta > 0$, $t^{-A}$ and $e^{-t}$ are always negligible compared to $t^{-B}$ as $t \to \infty$. This result is no longer valid if $\alpha$ or $\beta$ is negative. Since we expect that the moments behave like $t^{-B}$ for $\alpha < 0$ and $\beta > 0$, like $t^{-A}$ for $\alpha > 0$ and $\beta < 0$ and like $e^{-t}$ for $\alpha < 0$ and $\beta < 0$, we look for a solution that can be expressed in terms of functions including the generalized hypergeometric $_2F_2$ function and other functions whose asymptotic
time behavior is a linear combination of the three terms present in equation (A.1). Such functions are provided by the solutions of the hypergeometric differential equation satisfied by 
\[ _2F_2(A,B,u,v; -t) \] [30]; they include, besides the latter,

\[ t^{1-u} _2F_2(1 + A - u, 1 + B - u, 2 - u, 1 + v - u; -t) \] (A.2)

and

\[ t^{1-v} _2F_2(1 + A - v, 1 + B - v, 2 - v, 1 + u - v; -t). \] (A.3)

At this point it is convenient to introduce the Meijer G-functions, which are generalizations of the generalized hypergeometric functions [30]. The present case requires the Meijer functions \( G_{23}^{mn} \) (0 \( \leq m \leq 3, \ 0 \leq n \leq 2 \) that are linear combinations of the three solutions, equations (A.1-A.3), discussed above. For example, the generalized hypergeometric function \( _2F_2 \), which is proper solution of the moment equation when \( \alpha > 0 \) and \( \beta > 0 \) can be expressed as

\[ _2F_2(A,B,u,v; -t) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(A)\Gamma(B)} G_{23}^{12} \left( \begin{array}{c}
1 - B, 1 - A \\
0, 1 - u, 1 - v
\end{array} \right). \] (A.4)

The G-functions have the following properties [30] which will prove useful in solving the moment equation:

\[ \frac{d}{dt} G_{pq}^{mn} \left( t \left| \begin{array}{c}
a_1, \ldots, a_p \\
b_1, \ldots, b_q
\end{array} \right) \right) = -\frac{1}{t} G_{pq}^{mn} \left( t \left| \begin{array}{c}
a_1, \ldots, a_p \\
1 + b_1, b_2, \ldots, b_q
\end{array} \right) \right), \] (A.5)

and

\[ t^\sigma G_{pq}^{mn} \left( t \left| \begin{array}{c}
a_1, \ldots, a_p \\
b_1, \ldots, b_q
\end{array} \right) \right) = G_{pq}^{mn} \left( t \left| \begin{array}{c}
a_1 + \sigma, \ldots, a_p + \sigma \\
b_1 + \sigma, \ldots, b_q + \sigma
\end{array} \right) \right). \] (A.6)

With \( \sigma = -1 \), this leads to

\[ \frac{d}{dt} G_{pq}^{mn} \left( t \left| \begin{array}{c}
a_1, \ldots, a_p \\
b_1, \ldots, b_q
\end{array} \right) \right) = -G_{pq}^{mn} \left( t \left| \begin{array}{c}
a_1 - 1, \ldots, a_p - 1 \\
b_1, b_2 - 1, \ldots, b_q - 1
\end{array} \right) \right). \] (A.7)

By specifying the above relation to case of the \( G_{23}^{mn} \) functions with same parameters as those in equation (A.4), but with \( 0 \leq m \leq 3, \ 0 \leq n \leq 2 \), one obtains

\[ \frac{d}{dt} G_{23}^{mn} \left( t \left| \begin{array}{c}
1 - B, 1 - A \\
0, 1 - u, 1 - v
\end{array} \right) \right) = -G_{23}^{mn} \left( t \left| \begin{array}{c}
-B, -A \\
0, -u, -v
\end{array} \right) \right). \] (A.8)

Let now define

\[ f_{\lambda,\mu} = G_{23}^{mn} \left( t \left| \begin{array}{c}
1 - B, 1 - A \\
0, 1 - u, 1 - v
\end{array} \right) \right). \] (A.9)

Since the translation \( \lambda \rightarrow \lambda + \alpha, \ \mu \rightarrow \mu + \beta \) implies that \( u \rightarrow u + 1, \ v \rightarrow v + 1, \ A \rightarrow A + 1, \) and \( B \rightarrow B + 1 \), the combination of equations (A.8) and (A.9) leads to

\[ \frac{df_{\lambda,\mu}(t)}{dt} = -f_{\lambda+\alpha,\mu+\beta}(t). \] (A.10)

The above relation is similar to the moment equation, equation (10), except for a missing factor of \( 1 - 2/(\lambda + \alpha)(\mu + \beta) = AB/(uv) \) in its right-hand side. We thus consider the following ansatz for the moments:

\[ M(\lambda, \mu, t) = k_{\lambda,\mu} f_{\lambda,\mu}(t), \] (A.11)

where \( k_{\lambda,\mu} \) and \( f_{\lambda,\mu}(t) \) must satisfy

\[ k_{\lambda,\mu} = \frac{AB}{uv} k_{\lambda+\alpha,\mu+\beta}. \] (A.12)
and
\[ k_{\lambda,\mu} f_{\lambda,\mu}(0) = 1 \] (A.13)

The coefficient \( k_{\lambda,\mu} \) and the values of the indices \((m, n)\) in the definition of \( f_{\lambda,\mu}(t) \), equation (A.9) depend on the region of the \((\alpha, \beta)\)-plane that is considered.

a) For \( \alpha < 0 \) and \( \beta > 0 \), the condition, equation (A.13), implies that the term in equation (A.3) must not be not present in \( f_{\lambda,\mu}(t) \) because \( t^{1-\nu} \equiv t^{1-(\mu+1)/\beta} \) is infinite at \( t = 0 \) for large values of \( \mu \) for which the moments should be defined. The unique \( G_{23}^{21} \) function that is a linear combination of equations (A.1,A.2) is
\[ G_{23}^{21} \left( t \left| \begin{array}{c} 1 - B, 1 - A \\ 0, 1 - u, 1 - v \end{array} \right. \right) , \] (A.14)
which we thus take as \( f_{\lambda,\mu}(t) \). It is easy to check that the solution to equations (A.12,A.13) is then given by
\[ k_{\lambda,\mu} = \frac{\Gamma(1 - A)\Gamma(v)}{\Gamma(B)\Gamma(1 - u)} , \] (A.15)
which leads to equation (37). The solution for \( \alpha > 0 \) and \( \beta < 0 \) is simply obtained by permuting \( A \) with \( B \) and \( u \) with \( v \) in the preceding formulas which gives equation (39).

b) For \( \alpha < 0 \) and \( \beta < 0 \), both the \( 2F2 \) function and the two expressions in equations (A.2,A.3) are finite at \( t = 0 \) for large values of \( \mu \) and \( \lambda \). The only linear combination of these three functions that has the expected \( e^{-t} \) asymptotic behavior is the function
\[ G_{23}^{30} \left( t \left| \begin{array}{c} 1 - B, 1 - A \\ 0, 1 - u, 1 - v \end{array} \right. \right) , \] (A.16)
which we take as \( f_{\lambda,\mu}(t) \). Noting that the above function at \( t = 0 \) is equal to \( \frac{\Gamma(1-u)\Gamma(1-v)}{\Gamma(1-B)\Gamma(1-A)} \), we obtain that the solution to equations (A.12,A.13) is given by
\[ k_{\lambda,\mu} = \frac{\Gamma(1-B)\Gamma(1-A)}{\Gamma(1-u)\Gamma(1-v)} , \] (A.17)
which leads to equation (51).

References