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Elastic Properties of Tenuous Networks

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Abstract. — A disordered cubic network of springs is introduced as a model of 3-dimensional tenuous networks. It presents an unusual elastic behavior since it does not resist shear but only shear coupled to compression by a type of bending rigidity. Numerical simulations of the elastic behavior of random packings of hard spheres and of disordered cubic networks show a similar stress distribution, which can be partially understood by considering the appearance of buckling.

1. Introduction

Since the work of Dantu [1] it is known that stresses in deformed granular materials are heterogeneous. Although the reason of this behavior is not known, it is thought to be related to different characteristics of random packings such as (a) the topological or positional disorder of the underlying network of spheres in contact, (b) the non-linear (Hertzian) between force-law curved bodies in contact or (c) the transmission of a couple between two spheres in contact. In the present paper we consider the elasticity of different disordered networks in the limit of very small displacements where we assume a linear force law and no couple between spheres in contact. We show some evidence that even in this very simple case, we get a very unusual elastic behavior. It comes out that the fragility of the disordered network, i.e., the network is at a rigidity threshold, leads to a quite peculiar elastic behavior for which we give a tentative explanation by considering that linear chains of particles are transversally stabilized by bending.

In three dimensions, we construct packings of hard spheres according to the balistic deposition algorithm [2, 3]. Packings are characterized by an average coordination number of \( \langle N_c \rangle = 6 \) and a packing fraction of \( \rho = 0.5820 \). Although these packings are homogeneous at length scales above 5\( D \), where \( D \) is the diameter of spheres, they do exhibit a short range order below that length: the distribution of coordinations shows that a sphere \( i \) may be in contact with \( N_c \) spheres, where \( 3 \leq N_c \leq 9 \) and the centers correlation function has pronounced oscillations [4].

Counting the degrees of freedom, we get for a sequentially constructed packing of \( N_p \) spheres \( 3N_p - 6 \) degrees of freedom (3 translations of each sphere minus rigid translations and rotations of the packing) and \( 3(N_p - 2) \) constraints corresponding to contacts between spheres (for 3 initial

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spheres forming an equilateral triangle, there are 3 contacts and each new sphere brings 3 new contacts for a sequential packing). We see that the number of degrees of freedom is equal to the number of constraints. When packings are constructed using other boundary conditions, the equality will be true asymptotically for $N_v \gg 1$.

Concerning the rigidity of the packing, this implies that the removal of a single constraint, \textit{i.e.} allowing for 2 spheres to separate, should introduce a degree of freedom consisting in a displacement of all or some of the spheres keeping the distances between neighbors unchanged. The multiplication of such degrees of freedom will eventually result in a loss of rigidity for the whole packing. Of course it depends on the detailed structure of these new degrees of freedom and also that they are not constrained by the boundary conditions. We will identify this situation with some kind of rigidity threshold.

In the next part we will consider a simple model network with the same coordination $(N_v) = 6$ for which we can calculate the elastic energy. In the third part, we will give the results of numerical simulations of both the model network and the random packing of hard spheres. The last part will consist in a discussion of the results.

2. Elastic Energy

In order to understand the elasticity of random packings of hard spheres, let us first consider a simpler 3-dimensional understood fragile network. Let us consider a lattice with the same coordination as the disordered random packing. The simplest example consists in a cubic lattice. However it is known that a cubic lattice with springs between first neighbors does not resist shear. Let us consider a disordered version thereof. The disorder consists in a small random displacement $\epsilon_i$ of each vertex $i$ such that in the resulting configuration the springs are not stressed. Each component of $\{\epsilon_i\}$ has a normal distribution with 0 mean and variance $\epsilon_0^2$.

The elastic energy is

$$E(\{\mathbf{u}_i, \epsilon_i\}) = \sum_i \sum_j^{(i)} [|\mathbf{r}_i - \mathbf{r}_j| - |\mathbf{r}_i^0 - \mathbf{r}_j^0|]^2,$$

where $\mathbf{r}_i = \mathbf{R}_i + \epsilon_i + \mathbf{u}_i$ and $\mathbf{r}_j^0 = \mathbf{R}_i + \epsilon_i$. Here $i$ and $j$ are two sites linked by a bond, $\mathbf{u}_i$ is the displacement of site $i$ from the equilibrium position defined by $\mathbf{R}_i + \epsilon_i$, where $\mathbf{R}_i$ is a cubic lattice point. Development to 2-nd order in the components of $\epsilon$ and $\mathbf{u}$ leads to

$$E(\{\mathbf{u}_i, \epsilon_i\}) = \sum_i \sum_j^{(i)} |u_{ij}(1 - \epsilon_{ij}^2/2) + (1 - \epsilon_{ij}) (u_{ij\perp} + \epsilon_{ij\perp})|^2 + O(\epsilon^2),$$

where $\|$ means the component along $\mathbf{R}_i - \mathbf{R}_j = \mathbf{R}_{ij}$ and $\perp$ is the vector perpendicular to $\mathbf{R}_{ij}$. In equation (2), $\mathbf{u}_{ij}$ is related to the displacement field $\mathbf{u}(\mathbf{r})$ by

$$\mathbf{u}_{ij} \sim [(\mathbf{R}_{ij} + \epsilon_{ij}) \cdot \nabla + \frac{1}{2}((\mathbf{R}_{ij} + \epsilon_{ij}) \cdot \nabla)^2 + ..] \mathbf{u}(\mathbf{r}).$$

In the presence of exterior forces, $\{\mathbf{f}_i\}$, the free energy $F$ is obtained by first minimizing $1/2E(\{\mathbf{u}_i, \epsilon_i\}) - \sum_i \mathbf{f}_i \cdot \mathbf{u}_i$ over $\{\mathbf{u}_i\}$ and then averaging over the disorder $\{\epsilon_i\}$ (quenched average), \textit{i.e.}, $F = \frac{1}{N_v} \min_{\{\mathbf{u}_i\}} (1/2E(\{\mathbf{u}_i, \epsilon_i\}) - \sum_i \mathbf{f}_i \cdot \mathbf{u}_i)$. Since $E$ is quadratic the minimization over $\{\mathbf{u}_i\}$ leads to $-1/2f \cdot N^{-1}f$, where $N$ is a matrix such that $E = \mathbf{u} N \mathbf{u}$. Writing $N = D + M$, where the matrix $M$ is linear in $\{\epsilon_i\}$ and the matrix $D$ includes a term quadratic in $\{\epsilon_i\}$, we
get \( N^{-1} = (1 + D^{-1}M)^{-1}D^{-1} \). The development gives \( N^{-1} = (1 - D^{-1}M + D^{-1}MD^{-1}M + ... D^{-1}) = [D^{-1}]_d + [D^{-1}MD^{-1}M]_d + ... \), which corresponds to an effective elastic energy obtained by replacing \( N \) by \([D - MD^{-1}M]_d\).

The calculation is better performed in Fourier space. To second order in \( q \), the part of the elastic energy of equation (2) corresponding to \([D]_d\) reads

\[
E_1^{(2)}(\{u(q), \epsilon(q)\}) = \sum_{\alpha, \beta} q_\alpha^2 u_\alpha(q) u_\beta(-q) \left[ 1 - \sum_{\beta \neq \alpha} L_{\alpha\beta} + L_{\alpha\alpha} \right] + \sum_{\alpha \neq \beta, q} |q_\alpha u_\beta(q) + q_\beta u_\alpha(q)|^2 L_{\alpha\beta} + 2 \sum_{\alpha \neq \beta, q} q_\alpha u_\alpha(q) q_\beta u_\beta(-q) L_{\alpha\beta}, \tag{4}
\]

where \( L_{\alpha\beta} = \sum_k k^2 \epsilon_\beta(k) \epsilon_\alpha(-k) \). And the part of elastic energy of equation (2) corresponding to \([-MD^{-1}M]_d\) is

\[
E_2^{(2)}(\{u(q), \epsilon(q)\}) = -\sum_{\alpha, q} q_\alpha^2 u_\alpha(q) u_\alpha(-q) \left[ 4L_{\alpha\alpha} + \sum_{\alpha \neq \beta} (L_{\beta\beta} + M_{\alpha\beta\gamma}) \right] - 2 \sum_{\alpha \neq \beta, q} q_\alpha u_\alpha(q) q_\beta u_\beta(-q) L_{\alpha\beta} - \sum_{\gamma \neq \alpha \neq \beta, q} q_\alpha u_\alpha(q) q_\beta u_\beta(-q) M_{\alpha\beta\gamma} - \sum_{\alpha \neq \beta, q} |q_\alpha u_\beta(q) + q_\beta u_\alpha(q)|^2 L_{\alpha\beta}, \tag{5}
\]

where \( M_{\alpha\beta\gamma} = \sum_k k^2 \epsilon_\beta(k) \epsilon_\gamma(k) \epsilon_\alpha(-k) \), and \( \gamma \neq \alpha \neq \beta \) means \( \gamma \neq \alpha \), \( \gamma \neq \beta \) and \( \alpha \neq \beta \). Combining equations (4) and (5) we get an effective elastic energy to second order in \( q \)

\[
E(\{u(q), \epsilon(q)\}) = \sum_{\alpha, q} q_\alpha^2 u_\alpha(q) u_\alpha(-q) \left[ 1 - 3L_{\alpha\alpha} - \sum_{\beta \neq \alpha} (L_{\alpha\beta} + L_{\beta\beta} + M_{\alpha\beta\gamma}) \right] - \sum_{\gamma \neq \alpha \neq \beta, q} q_\alpha u_\alpha(q) q_\beta u_\beta(-q) M_{\alpha\beta\gamma}, \tag{6}
\]

which shows that the terms in \( |q_\alpha u_\beta(q) + q_\beta u_\alpha(q)|^2 \) with \( \alpha \neq \beta \) cancel. This shows that to second order in \( q \), the 3-dimensional disordered cubic network has no resistance to shear.

This result is not obvious since an annealed average of equation (2) would lead to a term in \( \sum_{\alpha \neq \beta} q_\alpha u_\alpha(q) q_\beta u_\beta(-q) \) and this will result in an usual elastic behavior. However there might be higher order terms depending on shear. To fourth order, the resistance to shear depends on the terms \( q_\alpha q_\beta u_\alpha(q) q_\gamma q_\beta u_\gamma(-q) \) with \( \beta \neq \alpha \) or \( \gamma \neq \alpha \). Performing the gradient expansion and averaging further, we get this terms as (see [5])

\[
E^{(4)}(\{u(q), \epsilon(q)\}) = -\frac{3}{2} \sum_{\alpha \neq \beta, q} q_\alpha q_\beta u_\alpha(q) q_\alpha q_\beta u_\alpha(-q) L_{\alpha\beta} + \sum_{\alpha \neq \gamma, q} q_\alpha q_\gamma u_\alpha(q) q_\alpha q_\gamma u_\alpha(-q) [N_{\gamma\gamma\gamma} - 3N_{\alpha\alpha\gamma}] - \sum_{\alpha \neq \beta \neq \gamma, q} 3q_\alpha q_\gamma u_\alpha(q) q_\beta q_\gamma u_\beta(-q) N_{\alpha\beta\gamma}, \tag{7}
\]

where \( N_{\alpha\beta\gamma} = \sum_k k^2 k^2 k^{-4} \epsilon_\gamma(k) \epsilon_\alpha(-k) \). An estimation of the prefactors for a normal disorder defined by the probability \( P(\{\epsilon(r)\}) \sim \exp(-\sum_i \frac{\epsilon_i^2}{2\epsilon_0^2}) \) gives \( L_{\alpha\beta} = \frac{\epsilon_0^2}{3}, \ N_{\alpha\alpha\gamma} = \frac{\epsilon_0^2}{15} \) and \( N_{\gamma\gamma\gamma} = \epsilon_0^2 \) leading to a positive prefactor for the term \( q_\alpha q_\gamma u_\alpha(q) q_\alpha q_\gamma u_\alpha(-q) \). Since the resistance to connected springs acquire a sort of bending rigidity.
Indeed, let us now consider a strut of bending modulus $B$ parallel to the $YY'$ axis. Bending corresponds to the change of the angle $\alpha$ between the strut and the vertical axis $YY'$, which is written in term of the displacement field $u(y)$ as $\delta \alpha = u_x(y + 1) - u_x(y)$. The corresponding contribution to the elastic energy writes $(\nabla_y \nabla_x u_y(r_i))^2 = (\nabla_y \alpha)^2$.

The contribution of $(\nabla_y \nabla_x u_y(r_i))^2$ (see Eq. (7)) to the elastic energy can be interpreted in a similar way: Let us consider a spring parallel to the horizontal axis $XX'$ between the points $(x, y)$ and $(x + 1, y)$. The displacement field $u_y(x, y)$ introduces a change of angle $\theta$ between the spring and the horizontal axis $XX'$, with $\delta \theta = u_y(x + 1, y) - u_y(x, y)$. Then the contribution $(\nabla_y \nabla_x u_y(r_i))^2$ to the elastic energy can be written as $(\nabla_y \theta)^2$. This contribution can be interpreted by noticing that it corresponds to a vertical stripe of horizontal springs which tend to become parallel. It is to be compared with the bended strut where a vertical line of vertical segments who tend to become parallel.

Since the elastic energy of equations (6, 7) combines terms related to dilation and terms related to bending, there appears a possibility of buckling, i.e., a coupling between compression (or dilation) and bending.

3. Numerical Simulations

In order to check the relevance the model above, we have resort to numerical simulations with disordered cubic networks and random packings of hard spheres. The elastic energy is taken to be the development of equation (1) to second order with respect to $u_i = r_i - r_i^0$,

$$E(\{u_i\}) = \sum_i \sum_j [(u_i - u_j) \cdot n_{ij}]^2,$$

where $n_{ij}$ is the unit vector from $r_i^0$ to $r_j^0$. This energy corresponds to the model presented above assuming that the development over the disorder $\epsilon(r)$ is not performed. We expect it to be valid for very small displacements. The minimization of $E(\{u_i\})$ results in a set of linear equations from which $\{u_i\}$ are determined, for given boundary conditions.

Numerical simulations were carried by constructing (a) disordered cubic networks with springs between first neighbors and (b) random packings of hard spheres of size $L \times L \times 3L$, with $L$ from 4 to 24. Periodic boundary conditions were used in the 2 horizontal directions. In the vertical direction, spheres whose centers were in a bottom and a top slices, of widths equal to the diameter of spheres, were given a fixed displacement, $u_i = \{-r_i \cdot z\}$. The minimization of $E(\{u_i\})$ was performed using the conjugate-gradient procedure.

A special attention has to be given to the boundary conditions. With periodic boundary conditions in the 2 horizontal direction, the deformation could be imposed as (i) a fixed vertical stress or as (ii) a fixed vertical displacement of the upper slice. In the case (i), we have to minimize $F(\{u_i\}) = E(\{u_i\}) - f_i \cdot u_i$ where $f_i \neq 0$ for $i$ in the upper slice. There appears an unconstrained degree of freedom: the system is unstable versus shear leading the an infinite negative $F(\{u_i\})$. This degree of freedom can be constrained by the boundary conditions by either imposing a zero shear with a Lagrange multiplier or by using a fixed vertical displacement of the upper slice (ii). In the case (ii), we then have to minimize $E(\{u_i\})$ keeping $u_i$ fixed for i corresponding to the upper slice. Both approaches result in equilibrium configurations. This is the illustration of the sensitivity of these systems to boundary conditions and is characteristic of a rigidity threshold.

The results are the following:

(a) In the cubic lattices, the disorder consists in small random displacements $\epsilon_i$ of each vertex such that in the resulting configuration the springs are not stressed. For the numerical simulations each component of $\{\epsilon_i\}$ is uniformly distributed between $-\epsilon_0$ and $\epsilon_0$. It appears that
the disordered cubic network is stable for the given boundary conditions. The resulting picture of stress distribution \( t_{ij} = n_{ij} \cdot (u_i - u_j) \), is shown in Figure 1, where blue links correspond to \( t_{ij} > 0.25 \ t_{\text{max}} \) and yellow links correspond to \( t < 0.25 \ t_{\text{min}} \). It appears that stresses are localized along vertical and horizontal lines. In Figure 2 we show the springs whose rotation defined as \( r_{ij} = |n_{ij} \wedge (u_i - u_j)| \) is above the threshold 0.5 \( r_{\text{max}} \). It appears that these springs
are aligned in horizontal directions as expected from equation (7). Furthermore the packing was analyzed by plotting the histogram of the distribution of springs' tensions $t_{ij}$, such that $N(t)dt$ is the number of springs whose tension $t_{ij}$ is between $t$ and $t + dt$. Figure 3 shows $N(t)$ for the disordered cubic packing for different values of $\epsilon_0$.

(b) The random packings of hard spheres is represented in Figure 4 with the same color conditions as Figure 1. The figure shows that the distribution of tension is very inhomogeneous. The picture of rotated links is represented in Figure 5. It appears that the rotated springs are concentrated in the center of the simulation cell. The histograms of the springs tensions $N(t)$ are represented in Figure 6 for $L = 12$ and $L = 24$.

The histograms (Fig. 3 for $\epsilon_0 = 1.0$ and Fig. 6) are similar. They are like Laplace distributions, $N(t) \sim \exp(-|t - a|/b)$, with the difference that the two branches are not symmetrical. The variance is much bigger than the maximum which is very close to $t = 0$. It shows that, while the average tension is very small, most of the elastic energy is contained inside fluctuations around this average tension. Also, the form of the queues of the curve for large $|t|$ is close to exponentials, suggesting that very few of the links support most of the load. Figures 1 and 4 show some striking similarities: in both pictures there appears lines of strongly extended (yellow) springs and strongly compressed (blue) springs. These similarities lead us to suppose that both networks present the same elastic behavior.

For the packing of hard spheres we have calculated the finite-size scaling of the first and second moments of the distribution of tensions which exhibits a power-law scaling: see Figure 7.
Fig. 4. — A picture of stressed springs in the packing of hard spheres for size $L = 16$. Same legend as in Figure 1.

Fig. 5. — A picture of rotated springs in the packing of hard spheres for size $L = 16$. Same legend as in Figure 2. The threshold value of $r$ is $0.25\ r_{\text{max}}$.

We get

$$t_1(L) = \frac{\sum_{ij} u_{ij} \cdot n_{ij}}{\sum_{ij}} = 0.61\ L^{-\gamma_1}, \gamma_1 = 1.44 \pm 0.1,$$
Fig. 6. — Histogram of the tension distribution for a packing of hard spheres of size $L = 12$ and $L = 24$.

\[ t_2(L) = \frac{\sum_{ij} (u_{ij} n_{ij})^2}{\sum_{ij}} = 0.29 \ L^{-\gamma_2}, \ \gamma_2 = 1.48 \pm 0.1. \quad (9) \]

However these power laws are to be taken with caution since Figure 5 show that the influence of the boundary conditions is very significant. We have also performed calculation of $t_2(L)$ after a percentage $p$ of bonds chosen at random were removed. It appears that the removal of 0.5 percent of bonds decreases $t_2(L)$ by an order of magnitude indicating that the packing of hard spheres is indeed at some rigidity threshold.

4. Discussion

In the field of percolation the rigidity of similar elastic systems has been intensively studied in 2-dimensional triangular lattices. It has been assumed that either (i) the flexible-rigid transition is similar to percolation transition i.e., the rigidity correlation length diverges at the transition [6] or more recently assuming that (ii) the rigidity correlation length stays finite at the transition [7]. If we define the elastic modulus as scaling like $\gamma_2$, the numerical value of $\gamma_2$ is different from the exponents predicted by (i) [8], while it seems to be consistent with the mean field exponent of model (ii) [7].
Fig. 7. — Finite-size scaling: \( t_1(L) \) and \( t_2(L) \) as functions of the size of the packing of hard spheres for \( L \) from \( L = 4 \) to \( L = 24 \).

If we compare the histograms of tensions to the histograms of the 2-dimensional triangular lattices, the most striking feature in the present case is that the fluctuations are much higher than the mean tension. This is also reflected by finite scaling of equation (9) which shows that \( t_2 \sim t_1 \gg t_1^2 \). A macroscopic compression goes together with a random deformation field of higher magnitude. For each highly extended spring, there is a compressed spring with about equal magnitudes of deformations. We believe that this behavior is related to the appearance of bending rigidity and the possibility of buckling.

The most basic example of buckling is the buckling of a strut, discovered by Euler [9]. When an increasing force \( F \) is applied at the ends of a straight strut, it starts to contract. In minimizing the stretching energy and, above a threshold force \( F_E \), it bends and contracts, minimizing the sum of bending and stretching energies. In the study of buckling one part consists of finding the threshold after which the instability occurs, the buckling threshold, and another part consists of studying the behavior after the instability, the so-called post-buckling behavior.

Let us consider a strut of length \( L \) and bending modulus \( B \) with a force of modulus \( F \) applied to one end while the other is kept fixed. The strain tensor writes \( u_{xx} = \partial_x u + \frac{1}{2} (\partial_x \nu)^2 \), where \( u(x) \) is the displacement parallel and \( \nu(x) \) is the displacement perpendicular to the strut. The total elastic energy becomes

\[
U = \frac{1}{2} \int_0^L dx \ E \omega \left[ \partial_x u + \frac{1}{2} (\partial_x \nu)^2 \right]^2 + \frac{1}{2} \int_0^L dx \ B (\partial_x^2 \nu)^2 - F u(L),
\]

(10)

where \( B \) is the bending modulus of the strut, \( E \) is Young’s modulus and \( \omega \) the surface of the section. For \( F < \pi^2 B/L^2 = F_E \), we have \( u(x) = -\frac{F}{E \omega} x \) and \( \nu(x) = 0 \), while for \( F > F_E \), \( u(x) = -\frac{F}{E \omega} x - 2(\frac{F - F_E}{E \omega}) x \) and \( \nu(x) = (\frac{8}{E \omega})^{1/2} \frac{x}{\pi}(F - F_E)^{1/2} \sin(\frac{\pi x}{L}) \) [10]. The maximum deviations are \( u(L) \sim F/(E \omega) \) and \( \nu \sim L(F/(E \omega))^{1/2} \), assuming that \( F_E \ll F \). The derivatives
\[ \partial_z u \sim F/(E\omega) \text{ and } \partial_z \nu \sim (F/(E\omega))^{1/2} \text{ obey the relation} \]

\[ -\partial_z u \sim (\partial_z \nu)^2, \tag{11} \]

which is quite general and comes from the extremization of the first term of equation (10): \[ \int [\partial_z u + \frac{1}{2} (\partial_z \nu)^2]^2. \] Its importance has been stressed by Alexander [11].

When the strut is inextensional, the calculations are quite different: assuming \( F > F_E \) and \( \nu(x) = A \sin(x/a) \), where \( a^2 = \pi^2 B/F \), we get \( A \) from the initial length of the strut by \( L = \int_0^\pi \nu(x)^2 dx \). We have then \( \nu(L/2) \sim (a(L-a))^{1/2} \) and \( u(L) \sim -(L-a) \) [12,13]. We still have \( \partial_z u \sim u(L)/a \sim -L-a \) and \( \partial_z \nu \sim \nu(L/2) \sim (L-a)^{1/2} \), i.e., \( -\partial_z u \sim (\partial_z \nu)^2 \).

In both cases, extensional and inextensional strut, the post buckling behavior is characterized by equation (11). We have considered the case of a straight strut for which the buckling threshold and post-buckling behavior are easily described; as soon as the strut is not initially straight, the calculation are much more complicated and might lead to different results. In the field of elastic stability many system are subject to buckling and there appears a great variety of behaviors: tensile or compressive buckling, with or without threshold, etc. [14, 15].

Equation (9) seems to indicate that a coupling between compression and bending has taken place leading to phenomena similar to buckling. Moreover since the calculations of the model and of the numerical simulations are made to second order in the displacement field (see Eqs. (2, 8)), there seems to be no threshold for the appearance of buckling.

To summarize the numerical simulations of the random packing of hard spheres show an unusual elastic behavior, which is similar to the elastic behavior of the disordered cubic network of springs. For the latter, we calculate the elastic energy to second order derivatives of the displacement field and showed that it leads to zero resistance versus shear and the appearance of a kind of bending rigidity, whose expression is confirmed by Figure 2. An elastic system with resistance to compression and bending might show buckling whose manifestation is the fact that \( \gamma_2 \sim \gamma_1 \) (see Eqs. (9, 11)).

### 5. Conclusion

In conclusion we have introduced a possible model accounting for the elasticity of disordered tenuous networks. Since the velocity of propagation of bending waves in a strut is proportional to the wave vector [16], the buckling of chains of particles should be considered when trying to understand the very unusual feature of sound propagation in sand such as the dependence of the speed of sound on hydrostatic pressure [17] and the difference between the phase and group velocities [18]. Also, the possibility of buckling should be considered when describing the elasticity of ordered lattices with low coordination at the rigidity threshold.

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[5] In equation (7), the factor of $N_{\gamma\gamma\gamma}(\epsilon)$ comes from the development of 
$\sum_k (-\frac{2q\cdot q_k^2}{(k\cdot q)^2}) \cdot \epsilon_k \cdot \epsilon_{\alpha\beta} \cdot (-q\cdot q_k)$, where we are interested in 
the fourth order elastic energy, we should take the third-order development of 
equation (3). However the calculations are becoming quite intricate. Consequently we will 
consider the first order derivatives. We have checked that the inclusion of higher order 
terms does not modify the obtained behavior, but only the values of the precoefficients.
[9] See the Additamentum ‘De curvis elasticis’ in Euler L., Methodus inveniendi lineas curvas 
maximis minimis proprietate graudentibus (Lausanne, 1744).