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To cite this version:
Thierrey Huillet, Bernard Jeannet. Finitely Generated Multifractals Can Display Phase Transitions. Journal de Physique I, EDP Sciences, 1996, 6 (2), pp.245-256. <10.1051/jp1:1996146>. <jpa-00247181>

HAL Id: jpa-00247181
https://hal.archives-ouvertes.fr/jpa-00247181
Submitted on 1 Jan 1996
Finitely Generated Multifractals Can Display Phase Transitions

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(Received 11 September 1995, received in final form 10 October 1995 and accepted 25 October 1995)

PACS.64.60.Ak – Renormalization-group, fractal, and percolation studies of phase transitions
PACS.05.50.+q – Lattice theory and statistics; Ising problems
PACS.75.10.-b – General theory and models of magnetic ordering

Abstract. — A new class of multifractal objects (“skewed” multifractals) is introduced, the multiplicative generator of which has a finite number of branches of different real-valued depths. Both microscopic and macroscopic scales are represented by such objects, each of these corresponding to a specific thermodynamical regime. In the “diluted” regime, the partition function $Z_t$ is exactly renormalizable which means in the sequel, as is the case in the general multifractal theory, that $t^{-1} \log Z_t$ as a non trivial limit as $t$ tends to infinity. In the “condensed” one the partition function converges. Details about the transition between these two regimes are given.

Résumé. — Une nouvelle classe de “multifractales” est introduite, pour laquelle le générateur présente un nombre fini de branches de longueur variable à valeurs réelles. Les échelles macroscopiques et microscopiques sont représentables par de tels objets, chacune d’elles correspondant à un régime thermodynamique spécifique. Dans la phase “diluée”, la fonction de partition $Z_t$ est exactement renormalisable, en ce sens (classique) que la limite quand $t \rightarrow \infty$ de $t^{-1} \log Z_t$ est non triviale. Dans la phase “condensée” la fonction de partition converge. Les détails thermodynamiques concernant cette transition de phase sont fournis.

1. Introduction

Inhomogeneous natural objects can generally be considered as assemblies of fragments or “atoms”, the sizes, weights and spatial organization of which cover an extremely wide range of orders of magnitude, ranging from the microscopic scale to the macroscopic one. A number of common examples of such composite objects, such as beaches, forests, galaxies etc..., have already been widely discussed in the literature about fractals. When only microscopic arbitrarily small fragments are considered, then, a complete description of the assembly is not accessible and a statistical description is the key point of the multifractal theory.

In most cases however both microscopic and macroscopic fragments coexist in the object. Therefore the first question that arises is: what kind of model would be capable of describing

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such a situation? The second question is: how does one handle or understand the transition between these two phases, for instance in a way similar to that used to understand the initial process of condensation, i.e. the clustering of molecules into droplets when the temperature of a gas is lowered. One of the goals of this paper is to work out this analogy and see its limitations. According to an hypothesis commonly used in natural sciences the structures investigated here will be assumed to present a character of self-similarity.

The purpose of this paper is i) to construct a simple mathematical model able to describe composite objects as those described previously, ii) to use the statistical properties of this model in order to reproduce the main features of these objects, namely: coexistence of both diluted and condensed phases and phase transition.

In order to exploit the analogy with a condensing fluid mentioned previously, the language adopted here will be naturally that of thermodynamics. A purely geometrical model has been chosen. It is based upon a special partition of the unit-interval $I \equiv \{0 \leq x \leq 1\}$ into a set of sub-intervals, the lengths of which are either “observable” or not. This set is derived from a special kind of “skewed” non-random finitely-generated multiplicative cascade. In this cascade, the partitioning of a finite number of the initial sub-intervals is stopped, whereas it is continued for the other sub-intervals. Under these conditions the limiting distribution of the intervals presents at the same time both microscopic and macroscopic scales. Some critical value of an external parameter (separation point or line) separates these two phases. We then show that two different thermodynamical limits exist for these two phases. In the diluted phase, fragmentation is complete, entropy diverges but remains renormalizable, or self-similar, as it is in a standard multifractal theory; this phase is dominated by chaos in this sense. In the condensed phase, the limit entropy exists; we shall conversely say that order dominates. The statistical physics of such hierarchical systems is then developed. Particular attention is paid to the divergence mode of the thermodynamical functions near the critical point or line. From this view point the present model exhibits special structures which are worth examining in some detail.

Similar problems were first investigated in [1] by using a reductible transfer matrix. As it will be seen later our conclusions are also in agreement with other recent investigations [2]. The loss of analyticity of the thermodynamical functions has also been introduced in multifractal models describing diffusion-limited aggregates [3,4]. It is shown by these Authors that the anomalous behaviour of multiplicatively generated measures is compatible with the self-similarity of these measures which is a direct consequence of the fact that the multifractals considered there are not finitely generated. This leads the Authors to define left-sided multifractals. Such a situation is known to be a signature of a phase transition, which requires some generalization of classical fractals (see for instance the multinomial measures in [5–7]), keeping analytical the thermodynamical functions. Similar treatments and conclusions can be found in the field of dynamical systems [8].

It is worth mentioning however that these approaches should not be confused with the present one, where it is shown that finitely generated multifractals may fail to be renormalizable.

2. The Model

Now we come to a brief description of the phenomena we are dealing with here, i.e. a hierarchical multiplicatively generated system of intervals on: $I = \{0 \leq x \leq 1\}$. At time zero, this initial interval, generates offsprings in finite number (say $M$). There are two categories of “sons” in our model, namely the “productive” ones that will repeat their ancestors’ “growth program”, and the “sterile” ones that do not split anymore.
Each of the productive first-generation intervals possesses its own real-valued lifetime before it generates a second-generation in the same ratio of sizes. Thus, the length of the intervals is multiplicatively propagated in the cascade. As multiplicative phenomena are quite difficult to handle, one can in an equivalent manner work with an additive structure, rather considering the logarithm of the length process.

At a given generation the lifetimes are not all the same. The different lifetimes of the first generation intervals are assumed to take their values into a finite set of $K$ real distinct values: $(t_k)^K_{k=1}$. It is convenient to represent this first generation of intervals by a generator tree, as in Figure 1, for which each branch symbolizes an interval. The length of each branch indicates the time at which the interval is cut into sub-intervals.

The different lengths of the first generation intervals are assumed to take their values into a finite set of $L$ real distinct values: $(x_l)_{l=1}^L$.

The "sterile" fragments remain forever. They have infinite lifetimes. What only matters is to keep track of their distinct length, taking their different possible values in a finite set of $\tilde{L}$ real distinct values: $(\tilde{x}_l)_{l=1}^{\tilde{L}}$.

Let $(a_{i,k})_{l,k=1}^{L,K}$ denote the number of first-generation sons of length $x_l$ among those that will split at $t_k$, and $(\tilde{a}_{l,\infty})_{l=1}^{\tilde{L}}$ denote the number of sterile first-generation sons of length $\tilde{x}_l$.

There are thus: $a_{i,k} = \sum_{l=1}^{L} a_{l,k}$ productive individuals whose lifetime is $t_k, k = 1...K$ and $a_l, = \sum_{k=1}^{K} a_{l,k}$ whose length is $x_l, l = 1,..,L$. We shall speak of degeneracy to describe this fact. Let also $A = \sum_{k=1}^{K} a_{i,k}$ be the total number of productive sons, and $D = \sum_{l=1}^{\tilde{L}} \tilde{a}_{l,\infty}$ the total number of sterile sons. We get of course, $A + D = M$. "Length conservation" is assumed.

which means: $\sum_{l=1}^{L} a_{l} x_l + \sum_{l=1}^{\tilde{L}} \tilde{a}_{l,\infty} \tilde{x}_l = 1$. Figure 1 is an example of such a generator for which, $L = 3, K = 2, a_{3,1} = 1, a_{3,2} = 1; a_{1,\infty} = 1, a_{2,\infty} = 1, a_{1,1} = 1, a_{1,2} = 1, a_{2,1} = 1, a_{2,2} = 1, A = 3$ and $D = 2$.

No special ordering is chosen in the arrangement of lengths of intervals. It is however useful, for notational convenience only, to introduce some particular values of these numbers, namely: $\frac{-\log x^*}{k^*} \equiv \max_{l,k=1,..,L,K} \left( -\frac{\log x_l}{t_k} \right)$. Let $(l^*, k^*) \equiv \text{Arg} \max_{l,k=1,..,L,K} \left( -\frac{\log x_l}{t_k} \right)$.
Fig. 2. — Generation and fragmentation of the unit-interval at any time $t$.

an associated value, 
\[
\log a^\ast \equiv \left. \frac{\log a_{l,k}}{t_k} \right|_{l^*, k^*}
\]

In a similar fashion, if: 
\[
- \log \tilde{x}^\ast \equiv \max_{l=1...\tilde{L}} ( - \log \tilde{x}_l ) , \quad \text{and} \quad t^* \equiv \text{Arg} \ \max_{l=1...\tilde{L}} ( - \log \tilde{x}_l ) ,
\]

we define 
\[
\log \tilde{a}^\ast \text{ the associated log-number value of such sterile branches: } \log \tilde{a}^* \equiv \log \tilde{a}_{l, \infty} \big|_{l^*} .
\]

At any time $t$, (see Fig. 2), we thus have constructed a system of intervals embedded in the unit interval. We shall call $X$, the limit set of points obtained by considering the endpoints of these intervals when $t \to \infty$. Observe that, due to our construction including sterile individuals, there is an infinite number of adjacent sites separated by a measurable distance, and of course a large amount of "infinitely" close pairs.

We have adopted here the "population theory" language because the presentation of our model appears to be easier this way. We might have described our model in terms of "contact processes" in a very similar fashion. So, let $X_\varepsilon \equiv \bigcup_{x \in X} B_\varepsilon (x)$ stand for the "coarse-grained" set obtained from $X$ by centering balls, of radius $\varepsilon$ on each point $x$ of this set, and considering the reunion of these balls. When the resolution is very low, say when $\varepsilon$ exceeds some value $0 < \varepsilon_0 < 1$, $X_\varepsilon$ is limited to a single connected component. By increasing resolution, that is to say by positioning $\varepsilon$ below $\varepsilon_0$, this component is split into $M$ sub-components within which each point is $\varepsilon$-indiscernible. Part of these $M$ sub-components ($A$ of them) will then split into sub-sub-components when $\varepsilon$ crosses respectively the value $(\varepsilon_k)_{k=1...K}$, (not including degeneracy). Part of these $(D$ of them) will not split, which means $\varepsilon_l = 0$, $l = 1...\tilde{L}$, in this case. Thus, each sub-component splits at different levels of resolution, if it splits at all. Increasing the resolution ad infinitum (that is to say, taking the limit $\varepsilon \to 0^+$) subcomponents shrink therefore into the continuum of points $X$.

Thus, considering the change of variables: 
\[
\varepsilon = \varepsilon_0 e^{-t},
\]

according to which the sequence $(\varepsilon_k)_{k=1...K}$ can be obtained from $(t_k)_{k=1...K}$ by: 
\[
\varepsilon_k = \varepsilon_0 e^{-t_k},
\]

these two models appear very much alike. Thus, more and more details are discovered from $X$ as resolution goes to infinity $(\varepsilon \to 0^+$ or $t \to \infty$). Reversing the arrow of time, details amalgamate until we get a final blur, just as epidemics eventually contaminates a given population.
3. Evidence of a Phase Transition in the Thermodynamic Limit

3.1. Statistical Physics of the Cascade. — From the statistical physics point of view, all information on inter-distance distribution is enclosed in the partition function: $Z_t(\beta) \overset{\text{def}}{=} \sum_{i=1}^{N_t} x_t(i)^\beta$, $\beta \in \mathbb{R}$, $t \geq 0$ where $N_t$ stands for the total number of individuals available at time $t$, both sterile and productive ones, and $x_t(i)$ the distance separating the two consecutive sites $i-1$ and $i$, $i = 1, \ldots, N_t$, at time $t$. We define the local partition functions of the sterile and productive offsprings, $a_\infty(\beta) \overset{\text{def}}{=} \sum_{i=1}^L \tilde{a}_i$, $a_k(\beta) \overset{\text{def}}{=} \sum_{i=1}^L a_i k x_i^\beta$, respectively. One can then easily check that $Z_t(\beta)$ meets the renewal equation:

$$Z_t(\beta) = a_\infty(\beta) + \sum_{k=1}^K [a_k(\beta) 1_{t_k > t} + a_k(\beta) Z_{t-t_k}(\beta) 1_{t_k \leq t}], \quad t \geq 0, \quad Z_0(\beta) = 1,$$

which basically describes the "self-similar" character of our model (see Fig. 2). In this equation, "1" represents the indicator which is 1 if its argument is true, and 0 otherwise.

We observe that $Z_t(0) = N_t$ is accurate with: $N_t = D + \sum_{k=1}^K [a_k \cdot 1_{t_k > 1} + a_k \cdot N_{t-t_k} \cdot 1_{t_k \leq t}]$, $t \geq 0$, giving the number of intervals $N_t$, in the cascade, at time $t$. Hence, $Z_t(1) = 1$, which means that it is a fragmentation of the interval $[0, 1]$, at any time.

Letting now: $\tilde{Z}(\beta, s) \overset{\text{def}}{=} \int_0^\infty e^{-st} Z_t(\beta) dt$, one gets, taking the Laplace transform of (1), and after elementary computations:

$$\tilde{Z}(\beta, s) = \frac{a_\infty(\beta) + \sum_{k=1}^K a_k(\beta) (1 - e^{-st_k})}{s \left(1 - \sum_{k=1}^K a_k(\beta) e^{-st_k}\right)},$$

provided $s > \max(0, \log \gamma(\beta)) \overset{\text{def}}{=} \log z(\beta)$, considering the two singularities of $\tilde{Z}(\beta, s)$.

Those singularities are $s = 0$ and the largest solution $s = \log \gamma(\beta)$ to the "generating equation": $\sum_{l,k=1}^{L,K} a_l k x_l^\beta e^{-st_k} = \sum_{l,k=1}^{L,K} a_l k x_l^\beta \gamma(\beta)^{-t_k} = 1$. Moreover, $\gamma(\beta)$ is analytic on $\mathbb{R}$. It should be observed that, on the contrary, function $-\log z(\beta)$, as defined, is not itself analytic. Indeed, $-\log z(\beta) = 0$ for $\beta > \beta_c$, with $0 < \beta_c < 1$, the unique critical value solution to $\gamma(\beta) = 1$; hence: $\sum_{l=1}^L a_l x_l^{\alpha_l} = 1$. Moreover, $-\log z(\beta)$ is finite, non-decreasing and concave (see Fig. 3). It should be noted that: $-\log z(0) = -\frac{\log A}{T}$, where $T$ is solution of the equation:

$$\sum_{k=1}^K a_k A^{\frac{1}{k}} = 1.$$ (Note that if all productive branches have the same length $\tau$, solution of the last equation is $T = \tau$).
Fig. 3. — Function $-\log z(\beta)$: the “renormalized partition function”.

The existence of a phase transition in our model, crossing the critical point, has thus been proved. Note that if $a_\infty(\beta) = 0$, $\hat{Z}(\beta, s)$ has only one singularity. In this case there is no phase transition.

It therefore follows from the study of singularities that the “thermodynamic limit” holds:

$$\lim_{t \to \infty} \frac{1}{t} \log Z_t(\beta) = -\log z(\beta),$$

where function $-\log z(\beta)$ is the “renormalized partition function”.

At $\beta = 0$, formula (3) reads:

$$\lim_{t \to \infty} Z_t(0) \frac{1}{t} \stackrel{\text{def}}{=} \lim_{t \to \infty} \left( N_t \right)^{\frac{1}{t}} = A^\frac{1}{t} \stackrel{\text{def}}{=} x(0) > 1$$

Thus, the total number of intervals of the cascade grows like: $A^\frac{1}{t}$.

Let now $N_t(e) \stackrel{\text{def}}{=} \left\{ i = 1, \ldots, N_t / \left( -\frac{1}{t} \log x_t(i) > e \right) \right\}$ denote the number, among these $N_t$ intervals at time $t$, sharing the property within the bracket: $x_t(i) < \exp(-et)$. It follows from the large deviation theorem on general random processes [9] that, provided $e > -\frac{\text{d} \log z(\beta)}{\text{d} \beta} \bigg|_{\beta=0}$

$$\lim_{t \to \infty} N_t(e)^{1/t} = \exp s(e)$$

Formula (5), thus gives the asymptotic shape of the distribution of interval length. In case $e < -\frac{\text{d} \log z(\beta)}{\text{d} \beta} \bigg|_{\beta=0}$, change > into < in the formula defining $N_t(e)$.

In the last formula, $s(e) \stackrel{\text{def}}{=} \inf_{\beta \in \mathbb{R}} (e\beta + \log z(\beta))$ is the concave Legendre transform of $-\log z(\beta)$, i.e. “renormalized entropy”. We shall call $e(\beta) \stackrel{\text{def}}{=} -\frac{\text{d} \log z(\beta)}{\text{d} \beta}$, the “renormalized internal energy”. Thus, formula (5) is valid provided $e > e(0)$.

3.2. THE RENORMALIZED ENTROPY SHAPE. — What follows is a brief description of the shape of $s(e)$, which essentially differs from the left-sided ones of [3,4] (see Fig. 4). Taking
the Legendre transform of $-\log z(\beta)$, as defined by (3), one gets: $s(\epsilon)$ is defined, continuous, nonnegative and concave when $0 \leq \epsilon \leq \epsilon(-\infty) < \infty$. On the "renormalized internal energy" interval: $0 \leq \epsilon \leq \epsilon(-\infty)$, entropy is strictly concave, with a linear part $s(\epsilon) = \beta_c \epsilon$ for all $\epsilon \in [0, \epsilon(\beta_c)]$. Here, $\epsilon(-\infty) = \frac{\log x^*}{t^*}$, $s(\epsilon(-\infty)) = \frac{\log a^*}{t^*}$ (definitions of these numbers are given in Section 2) and $s'(\epsilon(-\infty)) = \infty$. $s$ attains its maximum, $\frac{1}{T} \log A$, at $\epsilon(0)$.

Remarks

1) One can show that $\beta_c$ has a natural interpretation, in terms of box counting dimension, namely: $\dim X \overset{\text{def}}{=} \limsup_{\delta \to 0^+} \frac{\log N(\delta)}{\log \frac{1}{\delta}}$, where $N(\delta)$ is the minimal number of intervals necessary to cover $X$.

2) Up to now, we have only been dealing with the statistical properties of set $X$, through the Laplace parameter $\beta$. At this step, indeed, $\beta$ has only been used as a distortion parameter, allowing to study the family of distributions: $\{x_t(i)^\beta, i = 1, \ldots, N_t\}$. For example, negative values of $\beta$ underline small intervals in this distribution, whereas at $\beta = 0$, it is not possible to discriminate the small ones from the large ones at all. The statistical physics language can be adopted, provided one gives a richer meaning to this control parameter $\beta$. To do so, observe that our formulation amounts to consider two repelling particles whose potential energy is $E = -\log x$, that is to say, the logarithm of the distance between them ( [2]). Suppose that the only reachable energy levels, at time $t$, are precisely: $E_t(i) \overset{\text{def}}{=} -\log x_t(i), i = 1, \ldots, N_t$. This pair of particles, at "temperature" $T = \frac{1}{\beta}$ (related to its own vibrational energy), may then be used to observe the different energy levels. Indeed, one of the axioms of statistical physics tells us that the probability of configuration $i$ is: $Z_t(\beta)^{-1} e^{-\beta E_t(i)} = Z_t(\beta)^{-1} x_t(i)^\beta$. The statistical physics of this pair can thus be derived from the study of the partition function $Z_t(\beta)$, where the control parameter $\beta$ is now interpreted as the inverse of the temperature.

This analogy allows us to skip from the statistical language to the one of statistical physics. Any allusion to statistical physics in the sequel should be understood in this physical context, only.
3) It should be emphasized that the presence of a phase transition is a direct consequence of our structure presenting macroscopic scales, due to the presence of sterile individuals. Thus, $\beta_c$ separates two phases: the "cold" one (i.e. when $\beta > \beta_c$) for which macroscopic order prevails, and the "hot" one for which disorder is the rule and which is identified with the exact "renormalization" of the partition function.

3.3. Behaviour of the Thermodynamic Functions near the Critical Point

Going deeper into the analogy with thermodynamics, mentioned above, it is possible to introduce the following quantities: $\beta$ can thus be interpreted as the inverse of some "temperature". Occurrence of a phase transition appears in the limit: $t \to \infty$.

At given (finite) $t$, the statistical properties of this model are enclosed into the "free energy" function: $F_t(\beta) \equiv \frac{1}{\beta} \log Z_t(\beta)$.

Entropy $S_t(E)$, the Legendre transform of $-\log Z_t(\beta)$, in the variable $\beta$, satisfies:

$$S_t(E)|_{E_t(\beta)} = \frac{d \log Z_t(\beta)}{d \beta} = \beta E_t(\beta) + \log Z_t(\beta) = -\sum_{t \geq 1} \frac{e^{-\beta E_t(t)}}{Z_t(\beta)} \log \frac{e^{\beta E_t(t)}}{Z_t(\beta)}$$

where $E_t(\beta)$ is the average of the "energy levels": $-\log x_t(t)$, as a function of $\beta$.

Looking at $t$ as the resolution power, its dual variable, in the sense of Legendre, $P$, will represent "thermodynamical pressure" [10]. This allows to define the thermodynamical "free enthalpy": $G(\beta, P)$, the Legendre transform of which, in the variable $P$, is nothing but $F_t(\beta)$.

Pressure as a function of $\beta$ and $t$ is: $P_t(\beta) \equiv \frac{\partial F_t(\beta)}{\partial t}$

As $t \to \infty$, $\lim_{t \to \infty} \frac{1}{t} S_t(E)|_{E_t(\beta)} = s(e)|_{e(\beta)} = -\frac{d \log z(\beta)}{d \beta} = \beta e(\beta) + \log z(\beta)$, where the "renormalized internal energy" $e(\beta) \equiv \lim_{t \to \infty} \frac{1}{t} E_t(\beta) = -\frac{d \log z(\beta)}{d \beta}$ is the limit ($t \to \infty$) average of the "normalized energy levels": $-\frac{1}{t} \log x_t(t)$. One can show that pressure: $P(\beta) \equiv \lim_{t \to \infty} P_t(\beta) = \frac{\log z(\beta)}{\beta} (\geq 0)$. Moreover, the "specific heat per unit": $\frac{d e(\beta)}{d(\beta^{-1})} = \beta^2 \frac{d^2 (-\log z(\beta))}{d \beta^2}$ can be defined.

We shall now briefly discuss the behaviour of these functions as $\beta \to \beta_c^-$. From the definition of $-\log z(\beta)$ and $P(\beta)$, it follows that "pressure" vanishes for $\beta > \beta_c$, is regular and decreasing as $\beta < \beta_c P(\beta_c) = 0$. Thus "pressure" is a non-increasing, continuous function of $\beta$.

On the contrary, "internal energy" $e(\beta)$ is a non-increasing function of $\beta$, vanishing when $\beta > \beta_c$ but with a jump at $\beta_c$, whose amplitude can easily be computed. Occurrence of a jump at $\beta = \beta_c$ is shared by the "specific heat per unit".

3.4. Including the Generation Number. — Additional information on the hierarchical structure under study is enclosed in the bi-partition (grand canonical) function: $\Xi_t(\lambda, \beta) \equiv \sum_{p=1}^{P(t)} e^{-\lambda p} \sum_{i=1}^{N_t(p)} x_t(i)^\beta$, $\lambda, \beta \in \mathbb{R}$, $t \geq 0$, where $N_t(p)$ stands for the number of individuals at $t$ in the cascade with exactly $p$ ancestors (the generation number).

$p^+(t) \equiv \text{Integer part} \left( \frac{t}{\min_{k=1...K}(t_k)} \right)$ indicates the maximal number of such ancestors at $t$. 
Indeed, both marginals: \( \Xi_t(0, \beta) = Z_t(\beta) \), and \( \Xi_t(\lambda, 0) = \sum_{p=1}^{p_+(n)} e^{-\lambda p} N_t(p) \) are of interest, in this case. This is one of the main interest of the “skewed” character of our multifractal family.

One can easily check that \( \Xi_t(\lambda, \beta) \) meets the extended renewal equation:

\[
\Xi_t(\lambda, \beta) = e^{-\lambda} a_{\infty}(\beta) + e^{-\lambda} \sum_{k=1}^{K} [a_k(\beta) 1_{t_k > t} + a_k(\beta) \Xi_{t-t_k}(\lambda, \beta) 1_{t_k \leq t}], \quad t \geq 0,
\]

\[
\Xi_0(\lambda, \beta) = 1
\]  

(6)

This follows from the straightforward recurrence formula defining \( N_t(p) \):

\[
N_t(p) = D \cdot 1_{p=1} + \sum_{k=1}^{K} [a_k \cdot 1_{p=1} \cdot 1_{t_k > t} + a_{p-k} \cdot N_{t-t_k}(p-1) \cdot 1_{t_k \leq t}], \quad t \geq 0, \quad p = 1, \ldots, p_+(t).
\]

Letting now, \( \tilde{\Xi}(\lambda, \beta, s) \overset{\text{def}}{=} \int_0^\infty e^{-st} \Xi_t(\lambda, \beta) dt \), one gets, taking the Laplace transform of (6):

\[
\tilde{\Xi}(\lambda, \beta, s) = \frac{e^{-\lambda} \left( a_{\infty}(\beta) + \sum_{k=1}^{K} a_k(\beta)(1 - e^{-st_k}) \right)}{s \left( 1 - e^{-\lambda} \sum_{k=1}^{K} a_k(\beta)e^{-st_k} \right)},
\]

(7)

provided \( s > \max(0, \log \gamma(\lambda, \beta)) \overset{\text{def}}{=} \log \xi(\lambda, \beta) \). Here \( s = \gamma(\lambda, \beta) \) is (for each \((\lambda, \beta)\)) the largest solution to the “generating equation”: \( e^{-\lambda} \sum_{k=1}^{K} a_k(\beta)e^{-st_k} = e^{-\lambda} \sum_{l,k=1}^{L,K} a_l,k x_l^\beta e^{-st_k} = 1 \) that is, satisfying:

\[
\sum_{l,k=1}^{L,K} a_l,k x_l^\beta \gamma(\lambda, \beta)^{-t_k} = e^\lambda.
\]

Function \( -\log \xi(\lambda, \beta) \) thus fails to be analytic, which is the signature of a phase transition. This function is finite, non-decreasing and concave. Note that \( \xi(0, \beta) = z(\beta) \). It satisfies \( -\log \xi(0,0) = -\log A^\frac{1}{\beta} \), with \( A^\frac{1}{\beta} \overset{\text{def}}{=} \xi(0,0) > 1 \), and \( T \) defined as in Section 3.1. Moreover, \( -\log \xi(\lambda, \beta) = 0 \), when \( \lambda > \tau(\beta) \), with \( \tau(\beta) \) the strictly convex critical line of equation:

\[
\tau(\beta) = \log \sum_{l,k=1}^{L,K} a_l,k x_l^\beta,
\]

(8)

for which: \( \tau(0) = \log A, \tau(\beta_c) = 0 \). One has thus proved the existence of a phase transition, crossing the critical line this time, the equation of which is \( \lambda = \tau(\beta) \), in the \((\beta, \lambda)\) plane.

It also follows that the following thermodynamic limit holds: \( \lim_{t \to \infty} -\frac{1}{t} \log \Xi_t(\lambda, \beta) = -\log \xi(\lambda, \beta) \), and that the behaviour of all thermodynamic variables can of course be derived from this more general situation.

The behaviour of the thermodynamic functions, as \( \beta \to \beta^- \) can easily be obtained in a similar way.
4. Thermodynamics of the Condensed Phase

4.1. Convergence of the Partition Function. — In case $\beta > \beta_c$, it should now be clear that the limit of the partition function $Z(\beta) \overset{\text{def}}{=} \lim_{t \to \infty} Z_t(\beta)$ itself exists and is:

$$Z(\beta) = \lim_{s \to 0+} s \cdot \tilde{Z}(s, \beta) = \frac{a_\infty(\beta)}{1 - \sum_{k=1}^{K} a_k(\beta)} = \frac{1}{1 - \sum_{k=1}^{L} a_l, z^\beta_l}$$

(9)

from the initial value theorem of Laplace. Figure 5 gives the shape of the function $-\log Z(\beta)$.

Thus, in the “cold” phase ($\beta > \beta_c$), order prevails and the partition function has itself a thermodynamic limit. This phase thus deserves its own statistical physics. For example, defining the Legendre transform: $S(E) = \inf_{\beta > \beta_c} (E\beta + \log Z(\beta))$, the entropy, it should be clear that the whole “machinery” of statistical physics can be defined. In particular, the “internal energy” $E(\beta) \overset{\text{def}}{=} -\frac{d\log Z(\beta)}{d\beta}$, the “free energy” $F(\beta) \overset{\text{def}}{=} -\frac{1}{\beta} \log Z(\beta)$ and the “specific heat” $C(\beta) \overset{\text{def}}{=} -\beta^2 \frac{d^2(-\log Z(\beta))}{d\beta^2}$ can be of interest, in the condensed phase.

4.2. The Shape of the Entropy. — We first focus on the shape of entropy, as a function of the internal energy (see Fig. 6). For obvious reasons, this function is concave, non-negative, defined for $E \geq -\log \tilde{x}^*$, where $\tilde{x}^*$ is the largest value among the $(\tilde{x}_i)_{i=1,...,L}$. Moreover, $S(-\log \tilde{x}^*) = \log \tilde{a}^*$ with $\tilde{a}^*$ the number of individuals attached to this value. $S(E(1)) = E(1) > 0$ where $E(1)$ is the bounded derivative at $\beta = 1$ of $-\log Z(\beta)$. The slope of $S(E)$ is infinite at $E = -\log \tilde{x}^*$, $(\beta \to +\infty)$, one at $E = E(1)$.

The behaviour of $S(E)$ near $E = \infty$ is: $S(E) \overset{E \to \infty}{\approx} \beta_c E + \log E + \text{cste}$, traducing the fact that $E(\beta)$ diverges at $\beta = \beta_c$. Thus, this entropy linearly diverges, as a function of $E$. This result is in accordance with the linear part of the renormalized entropy $s(e)$, defined in Section 3.2. Divergence of $E$ is due to the divergence of the average value of the $-\log x_t(i)$, when $\beta \geq \beta_c$.

In this case $E_t(\beta) = -\frac{d\log Z_t(\beta)}{d\beta} \overset{t \to \infty}{\approx} -t \frac{d\log z(\beta)}{d\beta} = te(\beta)$ provided $e(\beta) \leq e(\beta_c)$. Thus, $E$
diverges like $te$, where $t \to \infty$ with $e \leq e(\beta_c)$ and $s(e) = \lim_{t \to \infty} \frac{S(et)}{t} = \beta_c e$, under hypothesis $e \leq e(\beta_c)$.

4.3. **ASYMPTOTIC BEHAVIOUR NEAR THE CRITICAL REGION.** — We briefly stress here how it can be obtained as $\beta \to \beta_c^\pm$. Observe from (9) that the denominator of $Z(\beta)$ is also:

$$\sum_{l=1}^{L} a_l (x_i^{\beta_c} - x_i^\beta),$$

referring to the equation defining the critical value. Thus:

$$Z(\beta) \approx \frac{K(\beta - \beta_c)^{-1}}{\beta = \beta_c^\pm},$$

In a similar way, one can show that the “internal energy” behaves like:

$$E(\beta) \approx K' + (\beta - \beta_c)^{-1},$$

that the “specific heat” behaves like:

$$c(\beta) \approx \left(1 - \frac{\beta_c}{\beta}\right)^{-2},$$

and that the “free energy” grows like:

$$F(\beta) \approx \frac{1}{\beta = \beta_c^\pm} K''(\beta - \beta_c),$$

where $K'$, $K''$ are positive known constants. Finally, the entropy $S(\beta) \overset{\text{def}}{=} S(E) \Big|_{E(\beta)} = \beta E(\beta) + \log Z(\beta)$, as a function of temperature grows like:

$$S(\beta) \approx K_1(\beta - \beta_c)^{-1} - K_2 \log(\beta - \beta_c),$$

where $K_1$ and $K_2$ are positive known constants.

5. **Concluding Remarks**

In this article, we emphasize a basic property of nonrandom multifractals, under which phase transition occurs at finite temperature, namely the coexistence of micro and macroscopic features. It is indeed the presence of macroscopic characteristics within them which induces loss of analyticity of the thermodynamic functions. We illustrate these ideas on a continuous-time (resolution) family of skewed multifractals, which in itself deserves interest. Part of the “thermodynamics” of such structures is studied in some detail, including behaviour near the critical point (or line).
References