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Short Communication

An Alternative Understanding of the Fractional Quantum Hall Effect

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Abstract. — We show that, in the presence of a strong magnetic field, a two-dimensional interacting electron gas can effectively be described by a Hamiltonian of ‘free’ hard-core anions subject to the magnetic field. According to a new understanding proposed for a set of specific anionic systems, we explain the fractional quantum Hall effect in a very simple way.

The fractional quantum Hall effect (FQHE) [1] has been a subject of great interest in condensed matter physics from both the theoretical and experimental point of view [2]. A complete understanding of the fractional quantization of the Hall conductance observed in heterostructures in the presence of an external uniform magnetic field perpendicular to the system demands a great deal of theoretical sophistication. A very nice and convincing explanation for the existence of a family of fractional states at the filling factor of the Landau level \( \nu = 1/(2n + 1) \) \( (n = 0, 1, 2 \ldots) \) was given in the seminal paper by Laughlin [3], in which trial wavefunctions were proposed to treat the interactions of two-dimensional electrons in a strong magnetic field using a variational procedure. The Laughlin state gives rise to a set of topologically interesting vortex-like quasiparticle excitations. As analogies of electrons and holes, these excitations have non-fluctuating fractional charges and, more intriguingly, obey fractional statistics [4, 5]. The Laughlin wave function also admits of a number of generalizations which describe other more complicated rational fractions. These are the hierarchy schemes of Haldane [6] and Halperin [7]. On the other hand, motivated by analogies between the FQHE and a superfluid [8], as well as the existence of large correlated-ring-exchanges on large length scales [9], Girvin and MacDonald [10] demonstrated that the Laughlin state has a hidden form of off-diagonal long-range order. They further suggested a non-local order parameter for the Laughlin state. Interestingly, such an observation is closely related to a novel picture of anions or fractional statistics [5, 11]. As a matter of fact, substantial efforts have been devoted to a theoretical realization of fractional quantum statistics and to an effective field-theory description of the FQHE [5, 12–15]. Although the Chern-Simons field-theory approach with bosons or fermions can successfully account for most of the FQHE phenomena, a clear justification for mapping a system of interacting electrons in two-dimensions in the presence of a strong magnetic field into a system
of hard-core bosons or fermions coupled to a Chern-Simons gauge field, i.e., into a system of ‘free’ hard-core anions subject to the same magnetic field, is still lacking, despite its obvious fundamental importance. In this paper, we address this crucial issue in the Hamiltonian formalism. First a physically reasonable understanding of the mapping is presented. Then we show the mathematical validity of this mapping. Moreover, using a new result that for a set of specific anion systems a single-particle quantum state can effectively be occupied by an integer number of anions \([16,17]\), we are able to explain the FQHE in a very straightforward manner.

As is well known, a hard-core anion can be considered as a charged fermion attached with a certain ‘fictitious’ magnetic flux. When these ‘composite fermions’ are subject to a very strong magnetic field in the two-dimensional \(xy\)-plane, all flux lines are nearly parallel to the \(z\)-axis. In this case, apart from the energy of the pure fermions in the magnetic field, there are also, in analogy to the magnetic flux lines in type-II superconductors \([18]\), two-dimensional repulsive interactions among the flux lines. It is quite reasonable to expect this kind of interaction potential to have effectively the same form as that for two-dimensional charged fermions. Therefore, it seems quite suggestive that the Hamiltonian describing a system of hard-core anions subject to a strong magnetic field may map, in principle, into that describing a two-dimensional electron gas with the coulomb repulsion, as long as the ‘fictitious’ flux is suitably chosen.

We now proceed to justify the mapping mathematically within some reasonable approximation. In the presence of a strong magnetic field along the \(z\)-direction, the Hamiltonian of a two-dimensional electron gas can be written as

\[
H_e = H_0 + H_{\text{int}}
\]

\[
= \frac{\hbar^2}{2m} \int \text{d}x \ n(x) \left( \frac{\nabla}{i} + \frac{2\pi}{\Phi_0} \mathbf{A} \right)^2 + \frac{e^2}{2\epsilon} \int \text{d}x \text{d}x' \frac{n(x)n(x')}{|x - x'|},
\]

(1)

where \(n(x) = \sum \delta(x - x_i)\) is the density of electrons, \(\mathbf{A}\) is the vector potential which satisfies the relation: \(\nabla \times \mathbf{A} = B\hat{z}\) with \(B\) the magnetic field, \(\Phi_0 = hc/e\) is the flux quantum, and other quantities have their usual meaning. Considering a sufficiently strong magnetic field, we need only concentrate on the case where all electrons occupy the lowest Laudau level (LLL) with spin aligned.

On the other hand, in the fermion representation, the Hamiltonian for ‘free’ hard-core anions in a magnetic field can be represented as \([5,14]\)

\[
H_a = \frac{\hbar^2}{2m} \int \text{d}x \ n(x) \left[ \frac{\nabla}{i} + \frac{2\pi}{\Phi_0} (\mathbf{A} + \mathbf{a}) \right]^2,
\]

(2)

where \(\mathbf{a}\) is the ‘fictitious’ vector potential characterizing the anion’s property, it satisfies the relation \(\nabla \times \mathbf{a} = \Phi n(x)\hat{z} \) (\(2k < \Phi/\Phi_0 < 2k + 1\) with \(k\) an integer). When the gauge condition \(\nabla \cdot \mathbf{a} = 0\) is imposed, it is straightforward to write

\[
\mathbf{a} = \left( \frac{\Phi}{2\pi} \right) \int \text{d}x' \left[ \frac{n(x')}{|x - x'|} \mathbf{e}_\phi \right],
\]

where \(\mathbf{e}_\phi(x - x') = \hat{z} \times (x - x')/|x - x'|\) is the azimuthal unit vector with respect to \((x - x')\). In view of that \(n(x')/|x - x'|\) is non-negative everywhere, we can apply the mean theorem of
the integral twice and rewrite $a$ as

$$a = \left[ \frac{\Phi}{2\pi} \right] \int \frac{d\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^2} n(\mathbf{x}') |\mathbf{x} - \mathbf{x}_0| e^\phi$$

$$= \left[ \frac{\Phi}{2\pi} \right] \int \frac{d\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^2} \kappa e^\phi$$

$$= a_\phi e^\phi , \quad (3)$$

where $e^\phi(\mathbf{x} - \mathbf{x}_0)$ is the azimuthal unit vector with respect to $(\mathbf{x} - \mathbf{x}_0)$, $\kappa = |\mathbf{x} - \mathbf{x}_0| / |\mathbf{x} - \mathbf{x}_1|$, $\mathbf{x}_0[n(\mathbf{x})]$ and $\mathbf{x}_1[n(\mathbf{x})]$ are functional of $n(\mathbf{x})$. After some algebra, equation (2) can be reformulated,

$$H_a = H_0 + \frac{2\pi\hbar^2}{m\Phi_0} \int d\mathbf{n} n(\mathbf{x}) a_\phi F_\phi , \quad (4)$$

where $F_\phi = e^\phi \cdot [\nabla/i + (2\pi/\Phi_0)A] + \pi a_\phi / \Phi_0$. In the derivation of the above equation, we have also used the relation $\nabla \cdot a = 0$. The LLL constraint implies [19,20] that $\nabla/i + (2\pi/\Phi_0)A \approx 0$. Thus, we can have $F_\phi \approx \pi a_\phi / \Phi_0$. More importantly, considering that $n(\mathbf{x}) a_\phi$ is also non-negative everywhere, and using again the mean theorem of the integral, we can write

$$H_a = H_0 + \frac{2\pi^2\hbar^2}{m\Phi_0} a_\phi \kappa \int \int d\mathbf{x} d\mathbf{x}' \frac{n(\mathbf{x}) n(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} , \quad (5)$$

where $a_\phi \kappa$ denotes the weighted mean value. When the LLL is fully filled, $n(\mathbf{x})$ is nearly uniquely determined and $a_\phi \kappa$ depends essentially on the magnetic field $B$ and the number of electrons $N$. Comparing equation (5) with equation (1) one sees that the Hamiltonian $H_a$ can effectively describe the two-dimensional interacting electron gas, particularly in the fully-filled LLL. It is worthwhile to emphasize that to obtain equation (5) the approximation $\nabla/i + (2\pi/\Phi_0)A \approx 0$ has been used, which is reasonable when the magnetic field is sufficiently strong so that only the LLL is occupied by electrons. We also expect that when several low-lying Landau levels are involved, this approximation is fulfilled and our results are valid.

So far, we have justified that the two-dimensional Coulomb interacting electron gas in the presence of a strong magnetic field can be described effectively in terms of a ‘free’ anion Hamiltonian, equation (2). After a gauge transformation, we can write a ‘free’ anion Hamiltonian in the anionic representation directly [5],

$$H_a = \frac{\hbar^2}{2m} \int d\mathbf{x} n_\alpha(\mathbf{x}) \left[ \frac{\nabla}{i} + \frac{e}{c} A \right]^2 , \quad (6)$$

where $n_\alpha(\mathbf{x})$ is the density of anions. Although there have been several theoretical interpretations of the FQHE in terms of anions [7,12], we here would like to account for it simply making use of a key speculative understanding recently proposed that each usual single-particle state is allowed to be effectively occupied by an integer number ($M$) of hard-core anions [16,17]. Let us recapitulate some results of quantum mechanics. In the presence of a magnetic field, the energy spectrum of a charged particle in two dimensions is discretized and highly degenerate. The number of states in each level is $n_0 = (eB/c)h$ per unit area. When the LLL is fully filled, which gives rise to the plateau in Hall conductance against $B$ [2], the total number of electrons and anions per unit area are respectively $n_0$ and $MBn_0$. Since the total charge per unit area is $n_0 e$, the effective charge of an anion must be $q' = e/M$. Correspondingly, the effective magnetic field for the charge-$q'$ anions becomes $B' = MB$ because the Hamiltonian, equation (6), of these non-interacting anions depends on $(qB)$ as a product and is thus invariant under the
transformation: \((e, B) \rightarrow (q', B')\). As the textbooks teach us, the Hall conductance of 'free' anions is now calculated as \(\frac{n_0 q e}{B'} = \frac{1}{M} \frac{e^2}{h}\), which clearly indicates the remarkable fractional quantization of Hall conductance.

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