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Asymptotics of Simple Branching Populations

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Résumé. — On considère l'itération d'une structure déterministe arborescente selon laquelle un ancêtre engendre un nombre fini de descendants dont la durée de vie (à valeurs entières) est donnée. Dans un premier temps on s'intéresse aux trois distributions asymptotiques suivantes : répartition des générations, aptitude à engendrer des descendants et répartition selon l'âge. Ensuite nous développons le formalisme thermodynamique pour mettre en évidence le caractère multifractal de la scission d'une masse unitaire associée à cette arborescence.

Abstract. — In this paper we study a simple deterministic tree structure: an initial individual generates a finite number of offspring, each of which has given integer valued lifetime, iterating the same procedure when dying. Three asymptotic distributions of this asynchronous deterministic branching procedure are considered: the generation distribution, the ability of individuals to generate offspring and the age distribution. Thermodynamic formalism is then developped to reveal the multifractal nature of the mass splitting associated to our process.

1. Introduction

The subject of this paper is the study of a very simple deterministic tree structure, according to which an ancestor generates a given (finite) number of offspring, each of which has a given lifetime with values in \( \mathbb{N} \) and iterates the same procedure. By the appropriate choice of the unit of lifetime this assumption seems to be more natural than the continuity (of lifetime) and it simplifies essentially our model. The fact that the individuals have distinct lifetimes implies a great difference between asymptotical properties of our population and the one of equal lifetimes. Our study has already been initiated in [1], following in this some previous works [2–5] in population dynamics. The purpose of the preceding work [1] was to give a probabilistic version of the studied structure and to underline the differences and analogies with some problems of the theory of age-dependent branching processes [6,7]. It happens that the deterministic structure deserves a more detailed study from the asymptotic point of view;
this is the object of the present work. Three asymptotic distributions of such populations are therefore considered, relative to the generation distribution, the ability of individuals to generate offspring and the age distribution, respectively. We extend the quoted results [1] to a mass-splitting procedure. Multifractal analysis of self-similar measures has recently emerged as an important concept in various fields [8-12], when an unit mass can be spread over a region in such a way that its distribution varies widely. The example of Bernoulli cascade is currently used [12]. In our paper we study a richer multiplicative phenomena, i.e., one associated to an asynchronous deterministic branching process. Our results can be included in the theory developed by Michon and Peyrière [13,14] on self-similar structures. By the simplicity of our model most computations are now quite explicit.

In Section 2 we define our model and we recall our results concerning the generation distribution [1]. Section 3 deals with the ability of individuals to generate offspring. In Section 4 we consider the asymptotic age distribution. The asynchronous mass splitting is discussed in Section 5. We study the thermodynamic limit of the mass partition function and we show its existence and analyticity. We compute explicitly the Legendre transform of the renormalized limit partition function and we give its interpretation as the dimension spectrum. The next section is devoted to some geometrical interpretation of the previously obtained facts. In the last part of this paper we present an application of the method of large deviations, i.e., the limit theorem for Hölder exponents.

2. Definitions

Let us consider the branching model defined in reference [1]. An initial individual at time $n = 0$ generates $M$ offspring having respective lifetimes (not random) $T_i \in \mathbb{N}; 1 \leq i \leq M$. $M$ is fixed and the same for each individual. The parent dies upon giving birth. Each descendant iterates then the same procedure. The offspring are numbered by increased age; this implies a natural way of numbering the branches of the induced tree. Our results do not depend of the method of numbering. Among these values, let us choose all distinct $K$ values $T_j^* \in \mathbb{N}; 1 \leq j \leq K$, in increasing order and let

$$\kappa_- \overset{def}{=} T_1^*,$$
$$\kappa_+ \overset{def}{=} T_K^*.$$

For all $1 \leq k \leq \kappa_+$ we let $a_k$ be the number of offspring with lifetime $k$, if this value figures among the numbers $T_j^*; 1 \leq j \leq K$, and zero, otherwise. One has therefore

$$\sum_{k=1}^{\kappa_+} a_k = M.$$

The trivial case $a_k = 0, 1 \leq k < \kappa_+, a_{\kappa_+} \neq 0$, is excluded in the sequel.

If one gets interested into the number $N_n, n \geq 0$, of individuals living at time $n$ then one has

$$N_0 = 1, N_n = \sum_{i=1}^{M} \{1_{[T_i \geq n]} + N_{n-i} 1_{[T_i < n]}\}, n \in \mathbb{N}.$$

In the preceding notations (with $a_k = 0$ for $k \neq T_i^*, T_K^*$) this is also:

$$N_n = \sum_{k=1}^{\kappa_+} \{a_k 1_{[k \geq n]} + a_k N_{n-k} 1_{[k < n]}\}, n \in \mathbb{N}.$$

This renewal equation defines the generator of the reproduction tree. In particular, the self-similarity of the tree should be pointed out, since this is the key to the recursion formula.
A closer look at this structure leads to consider a variable \( N_n(p) \) which represents the number of individuals at time \( n \) of the \( p^{th} \) generation, which means, possessing exactly \( p \) ascendants. It satisfies the equations

\[
N_0(0) = 1;
\]

\[
N_n(p) = \sum_{i=1}^{M} \{ 1_{[T_i \geq n]} 1_{[p=1]} + N_{n-T_i}(p-1)1_{[T_i < n]} \}, \quad n \in \mathbb{N}, \quad p^-(n) \leq p \leq p^+(n);
\]

with

\[
p^+(n) \overset{df}{=} \begin{cases} \frac{n}{\kappa_-} & \text{if } \frac{n}{\kappa_-} \text{ integer,} \\ \left\lfloor \frac{n}{\kappa_-} \right\rfloor + 1 & \text{otherwise;} \end{cases}
\]

\[
p^-(n) \overset{df}{=} \begin{cases} \frac{n}{\kappa_+} & \text{if } \frac{n}{\kappa_+} \text{ integer,} \\ \left\lfloor \frac{n}{\kappa_+} \right\rfloor + 1 & \text{otherwise.} \end{cases}
\]

We suppose that \( N_n(p) = 0 \) for \( p < p^-(n) \) or \( p > p^+(n) \). Its generating function

\[
\psi_n(\gamma) \overset{df}{=} \sum_{p=p^-(n)}^{p^+(n)} \gamma^p N_n(p), \quad \gamma \in [0, 1],
\]

satisfies the equation

\[
\psi_n(\gamma) = \gamma \sum_{i=1}^{M} \{ 1_{[T_i \geq n]} + \psi_{n-T_i}(\gamma)1_{[T_i < n]} \}, \quad n \in \mathbb{N},
\]

with initial condition

\[
\psi_0(\gamma) = 1.
\]

Let us observe that \( p^-(n) = 1 \) for \( T_i \geq n \), which explains the first term on the right side of equation (2). Moreover, the second term on the right side of equation (2) is obtained since \( p^-(n-T_i) \geq p^-(n) - 1 \) and \( p^+(n-T_i) \leq p^+(n) - 1 \).

In an equivalent manner, (2) can be written

\[
\psi_n(\gamma) = \gamma \sum_{k=1}^{\kappa_+} \{ a_k 1_{[k \geq n]} + a_k \psi_{n-k}(\gamma)1_{[k < n]} \}, \quad n \in \mathbb{N}.
\]

One may check that

\[
\psi_n(1) = N_n = \sum_{p=p^-(n)}^{p^+(n)} N_n(p), \quad n \in \mathbb{N},
\]

and

\[
\psi_n(M^{-1}) = \sum_{p=p^-(n)}^{p^+(n)} M^{-p} N_n(p) = 1, \quad n \in \mathbb{N},
\]

which means the conservation of a unit mass diffused in the tree.
3. Ability to Generate Offspring

Now we are going to consider the limit behaviour of our population from the point of view of the ability to generate offspring.

Let \( X_n \in \mathbb{N}^{\kappa_+} \) be a vector whose \( t \)th component \( X_{n,t} \) represents the number of individuals living at time \( n \), which produce at least one descendant at step \( n + t - 1 \).

It should be clear that
\[
X_0^T = (1, 0, \ldots, 0)
\]
and that the vectors \( X_n, n \geq 0 \), satisfy the following relation:
\[
X_{n+1} = AX_n, \quad n \geq 0,
\]
where the \( \kappa_+ \times \kappa_+ \) companion matrix \( A \) is given by:
\[
A \overset{\text{df}}{=} \begin{pmatrix}
  a_1 & 1 & 0 & \ldots & 0 \\
  a_2 & 0 & 1 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{\kappa_+ - 1} & 0 & 0 & \ldots & 1 \\
  a_{\kappa_+} & 0 & 0 & \ldots & 0
\end{pmatrix}.
\]

In particular, the first component \( X_{n,1} \) represents the number of individuals branching at time \( n \). Moreover, the number of individuals in the population at time \( n \) is given by:
\[
N_n = C^T X_n = \| X_n \|_1 \overset{\text{df}}{=} \sum_{i=1}^{\kappa_+} | X_{n,i} |, \quad n \geq 0,
\]
where
\[
C^T \overset{\text{df}}{=} (1, 1, \ldots, 1),
\]
therefore
\[
N_n = \| A^n X_0 \|_1 = \| A^n \|_{[1]} \overset{\text{df}}{=} \max_{1 \leq j \leq \kappa_+} \sum_{i=1}^{\kappa_+} | (A^n)_{ij} |, \quad n \in \mathbb{N}.
\]

The non-negative matrix \( A \) is primitive and invertible, its characteristic polynomial \( \chi \) is
\[
\chi(\lambda) \overset{\text{df}}{=} \lambda^{\kappa_+} - \sum_{k=1}^{\kappa_+} a_k \lambda^{\kappa_+ - k}, \quad \lambda \in \mathbb{C}.
\]

The roots of \( \chi(\lambda) \) are all distinct; those with non-negative real part are all complex and conjugated inside the unit circle except exactly one, \( \alpha \), which is real and \( > 1 \); those with a negative real part are complex or real of modulus \( < \alpha \) [15]. Then \( \alpha = \rho(A) \), where \( \rho(A) \) is the spectral radius of \( A \).

It follows that
\[
\lim_{n \to \infty} N_n^{\frac{1}{n}} = \lim_{n \to \infty} \| A^n \|_{[1]}^{\frac{1}{n}} = \rho(A) = \alpha > 1,
\]
therefore
\[
\lim_{n \to \infty} \frac{1}{n} \log N_n = \log \alpha.
\]
4. Asymptotic Age Distribution

Let us consider the age structure of the population. More precisely, let us introduce the vector

\[ \mathbf{Y}_n^T \stackrel{df}{=} (Y_{n,1}, \ldots, Y_{n,2}, \ldots, Y_{n,\kappa_+}), \]

where \( Y_{n,i} \) gives the number of individuals at time \( n \) of age \( i \), supposed \textit{a priori} bounded by a maximal age \( \kappa_+ \).

If an individual of age \( j \) is then supposed to have the \textit{fertility coefficient}

\[ f_j = \frac{\text{number of offspring born of individuals of age } j}{\text{number of individuals of age } j} \]

and if

\[ s_j = \frac{\text{number of survivors among the individuals of age } j}{\text{number of individuals of age } j} \]

is its \textit{survival factor} from age \( j \) to \( j + 1 \), it is easy to deduce the recurrence on \( \mathbf{Y} \):

\[ \mathbf{Y}_{n+1} = M \mathbf{Y}_n, \quad n \geq 1, \tag{5} \]

where

\[ M \stackrel{df}{=} \begin{pmatrix} f_1 & f_2 & f_{\kappa_+ - 1} & f_{\kappa_+} \\ s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ \vdots \\ 0 & 0 & s_{\kappa_+ - 1} & 0 \end{pmatrix}. \tag{6} \]

We have shown in [1] that there exists an age distribution of the tree defined by (5) with the initial condition

\[ \mathbf{Y}_1^T = (M, 0, \ldots, 0), \]

where the matrix \( M \) is given by (6) with for fertility coefficients:

\[ f_j = \frac{a_j M}{\sum_{k=j}^{\kappa_+} a_k} \in [0, M], \quad j = 1, \ldots, \kappa_+, \tag{7} \]

and for survival factors:

\[ s_j = \frac{\sum_{k=j+1}^{\kappa_+} a_k}{\sum_{k=j}^{\kappa_+} a_k} \in [0, 1], \quad j = 1, \ldots, \kappa_+ - 1. \tag{8} \]

Moreover, these coefficients are related by

\[ s_j = 1 - \frac{f_j}{M}, \quad j = 1, \ldots, \kappa_+ - 1. \tag{9} \]

The asymptotic behaviour is now given by the Perron-Frobenius theorem [15, 16]. More precisely, let \( \mathbf{p} \stackrel{df}{=} (1, p_2, \ldots)^T \) and \( \mathbf{q} \stackrel{df}{=} (1, q_2, \ldots)^T \) the positive left and right Perron-Frobenius eigenvectors of the matrix \( M \), which means, the particular solutions of

\[ M \mathbf{p} = \alpha \mathbf{p}, \tag{10} \]
$M^T q = \alpha q$,

where $\alpha = \rho(M)(= \rho(A))$ is the spectral radius of $M$.

Let

$$p^* \overset{df}{=} \frac{P}{\|P\|_1},$$

$$q^* \overset{df}{=} \frac{q}{\|q\|_1},$$

be the normalized vectors so that $S^* \overset{df}{=} p^* q^{*T}$ is the orthonormal projector on the eigenvector $p^*$.

One checks that in the large time

$$M^n \sim \alpha_M^n S^*,$$

therefore

$$Y_n = M^{n-1} Y_1 \sim \alpha_M^{n-1} S^* Y_1 = c \alpha_M^{n-1} p^*,$$

where $c \overset{df}{=} q^{*T} Y_1 = M$. Then

$$\frac{Y_n}{\|Y_n\|_1} \sim p^*.$$ 

Let therefore

$$\mu_M \overset{df}{=} \sum_{k=1}^{K^+} p_k \delta(k),$$

$$\mu_M^* \overset{df}{=} \sum_{k=1}^{K^+} p_k^* \delta(k),$$

be the asymptotic age distribution measures. Using equations (7), (8) and (10), the explicit expression of $p_k$ is:

$$p_k = \alpha_M^{1-k} \prod_{l=1}^{k-1} s_l = \frac{\alpha_M^{1-k}}{M} \sum_{l=k}^{K^+} a_l, \quad 1 \leq k \leq K^+.$$

An asymptotic average age $\overline{T}_M$ can be introduced:

$$\overline{T}_M \overset{df}{=} \sum_{k=1}^{K^+} k p_k^*.$$

Let then

$$L_M(\lambda) \overset{df}{=} \int_{\mathbb{R}} e^{-\lambda x} d\mu_M^* = \sum_{k=1}^{K^+} e^{-\lambda^k} p_k^*, \quad \lambda \in \mathbb{R},$$

with values in $[0, \infty)$. The cumulant function $K_M(\lambda) \overset{df}{=} -\log L_M(\lambda)$ is concave and analytic on $\mathbb{R}$. Moreover,

$$K'_M(0) = \overline{T}_M.$$

Let

$$M^* \overset{df}{=} \begin{pmatrix}
1 & 1 & 0 & 0 \\
p_2 & 0 & 1 & 0 \\
p_{K^+ - 1} & 0 & 0 & 1 \\
p_{K^+} & 0 & 0 & 0
\end{pmatrix}.$$
The matrix $\tilde{M}$ is non-negative and primitive, so it has a simple real dominant eigenvalue which coincides with its spectral radius denoted by $\alpha_{\tilde{M}}$. Also, $\alpha_{\tilde{M}} > 1$. It should be clear that

$$K_M(\log \alpha_{\tilde{M}}) = \log \|p\|_1.$$ 

and it is therefore possible to introduce the number

$$D_M \triangleq \frac{1}{T_M} \cdot \frac{\log \|p\|_1}{\log \alpha_{\tilde{M}}} \in ]0, 1[.$$

Let us consider the propagation system of the weights:

$$\tilde{Y}_0^T = (1, 0, \ldots, 0);$$

$$\tilde{Y}_{n+1} = \tilde{M} \tilde{Y}_n, \quad n \geq 0.$$

The numbers

$$M_n \triangleq \|\tilde{Y}_n\|_1, \quad n \in \mathbb{N},$$

represent then the weights of the population at time $n$.

Let us consider now the variable $\tilde{Y}_n(p)$ which represents the weight of the individuals at time $n$ of the $p$th generation, that is, possessing exactly $p$ ancestors. It satisfies the equation

$$\tilde{Y}_0(0) = 1;$$

$$\tilde{Y}_n(p) = \sum_{i=1}^{M} \left\{ 1_{[T_i \geq n]}1_{[p=1]} + \tilde{Y}_{n-T_i}(p-1)1_{[T_i<n]} \right\}, \quad n \in \mathbb{N}, \quad p \geq 1.$$

Its generating function

$$\psi_n(\gamma) \triangleq \sum_{p=p^-(n)}^{p^+(n)} \gamma^p \tilde{Y}_n(p), \quad \gamma \in [0, 1],$$

with

$$p^+(n) \triangleq n;$$

$$p^-(n) \triangleq \begin{cases} \left\lfloor \frac{n}{\kappa_+} \right\rfloor + 1, & \text{if } \frac{n}{\kappa_+} \text{ integer,} \\ [\left\lfloor \frac{n}{\kappa_+} \right\rfloor], & \text{otherwise,} \end{cases}$$

satisfies the equation

$$\psi_n(\gamma) = \gamma \sum_{k=1}^{\kappa_+} \left\{ p_k 1_{[k \geq n]} + p_k \psi_{n-k}(\gamma)1_{[k<n]} \right\}, \quad n \in \mathbb{N},$$

with the initial condition

$$\psi_0(\gamma) = 1.$$ 

One may check that

$$\psi_n(1) = M_n = \sum_{p=p^-(n)}^{p^+(n)} \tilde{Y}_n(p), \quad n \in \mathbb{N},$$
and
\[ \psi_n(\|p\|_1^{-1}) = \sum_{p = p^-(n)}^{p^+(n)} \|p\|_1^{-p} \tilde{Y}_n(p) = 1, \ n \in \mathbb{N}, \]
which means the unit mass conservation.
A direct extension of results of [1] gives:

**Theorem 1:**

If \( D_M(n) \equiv -\sum_{i=1}^{\lfloor M \rfloor} \|p\|_1^{-p_n(i)} \log_{M_n} \|p\|_1^{-p_n(i)}, \ n \in \mathbb{N}, \) where \( p_n(i) \) denotes the number of ancestors of the \( i \)th individual at time \( n \), then the limit \( s_M \equiv \lim_{n \to \infty} D_M(n) \) always exists and one has the equality: \( s_M = D_M \).

**Example.** — For the Fibonacci tree:

\( M = 2, \ \kappa_+ = 2, \ a_1 = a_2 = 1, \)

one has \( r_1 = 1, \ r_2 = \frac{1}{\alpha}, \) where \( \alpha = 1.618... \). Moreover, \( \|r\|_1 = \alpha \) so that \( \overline{r}_A = \frac{2\alpha + 1}{2\alpha + 1} \approx 1.381964... \). One has \( p_1 = 1, \ p_2 = \frac{1}{2\alpha}, \) where \( \alpha = 1.618... \). Moreover, \( \|p\|_1 = \frac{2\alpha + 1}{2\alpha} \) so that \( \overline{r}_M = \frac{2(\alpha + 1)}{2\alpha + 1} \approx 1.239718... \). This gives

\[ D_M = \frac{2\alpha + 1}{2(\alpha + 1)} \cdot \frac{\log \frac{2\alpha + 1}{\alpha}}{\log \alpha}, \]

where \( \alpha_M \) is the largest real solution of the equation \( x^{-1} + \frac{1}{2\alpha} x^{-2} = 1 \).

5. Mass Splitting

We turn now to the description of singular mass distributions on the interval \([0, 1]\), exploiting the peculiarities of the branching tree structure defined in the preceding sections. In this set-up, we compute the corresponding partition function and we show that it is exactly renormalizable. Next, we develop some thermodynamic formalism, in particular we show that the free energy function is analytic.

Besicovitch has considered a repartition of the unit mass in the interval \([0, 1]\). His method firstly consists to distribute a unit mass in the interval \([0, 1]\) in such a way that the mass \( \pi_i > 0 \) is uniformly distributed on the interval \( \left[ \frac{i}{M}, \frac{i + 1}{M} \right], \ 0 \leq i \leq M - 1, \sum_{i=0}^{M-1} \pi_i = 1 \). The following steps iterate this procedure for each sub-interval.

The asymptotic distributions are then concentrated on everywhere dense sets in \([0, 1]\), whose Hausdorff dimension is equal to the entropy \([17]\):

\[ d \equiv \frac{M - 1}{\sum_{i=0}^{M-1} \pi_i \log M \pi_i}. \]

One can consider the following situation adapted to our case: each segment \( \left[ \frac{i}{N_n}, \frac{i + 1}{N_n} \right]; \ 0 \leq i \leq N_n - 1, \) at step \( n \) receives the mass \( M^{-p_n(i)}, \ 1 \leq i \leq N_n \), uniformly distributed,
where $p_n(i)$ denotes the number of ancestors of the $i^{\text{th}}$ individual (with respect to the natural ordering defined before) at time $n$.

This method naturally leads to consider the entropy

$$D_G(n) \overset{df}{=} - \sum_{i=1}^{N_n} M^{-p_n(i)} \log_{N_n} M^{-p_n(i)}, \ n \in \mathbb{N}.$$ 

We have proved that:

**Theorem 2 [1]:**

The limit $D_G \overset{df}{=} \lim_{n \to \infty} D_G(n)$ always exists and one has the following equality:

$$D_G = \frac{M}{\sum_{1 \leq k \leq \kappa^+} k a_k} \cdot \frac{\log M}{\log \alpha} = \frac{1}{\overline{T}} \cdot \frac{\log M}{\log \alpha} \in [0, 1],$$

where

$$\overline{T} \overset{df}{=} \sum_{1 \leq k \leq \kappa^+} k a^*_k, \quad a^*_k \overset{df}{=} \frac{a_k}{\sum_{1 \leq k \leq \kappa^+} a_k}.$$

**Example (continued).** — For the Fibonacci tree one finds $D_G = \frac{2 \log 2}{3 \log \alpha} = 0.9603...$, where $\alpha = 1.618...$ is the golden number. Moreover, $\overline{T} = \frac{3}{2}$.

Assume now a unit mass is being split into $M$ branches, $a_k$ of which are being of length $k$; $k = 1, \ldots, \kappa^+$. Each branch receives the mass $\pi_l > 0$, $l = 1, 2, \ldots, M$; $\sum_{l=1}^{M} \pi_l = 1$. The splitting is then iterated as before. A particular case of this model when $\pi_l = \frac{1}{M}$, $l = 1, 2, \ldots, M$, has already been considered above. Now we shall consider the situation when the masses $\pi_l > 0$, $l = 1, 2, \ldots, M$, are not necessarily equal. We want to derive some statistical limit properties of the mass repartition within each branch as $n \to \infty$.

At step $n$, it is here also of interest to compute the partition function of the mass distribution along the cylinders. Usually in the literature on infinite trees, the word “cylinder $i$” denotes the subtree of all the descendants of particle $i$ in the infinite tree. In our model it can be identified, in the equivalent manner, to the set of the ancestors of particle $i$.

To this purpose let us introduce the concave partition function [10]:

$$\psi_n(\lambda) \overset{df}{=} \sum_{i=1}^{N_n} \mu(i)^\lambda, \ \lambda \in \mathbb{R},$$

where $\mu(i)$ denotes the mass attached to the cylinder $i$; $i = 1, 2, \ldots, N_n$.

Let us remark that $\psi_n(\lambda)$ is also:

$$\psi_n(\lambda) \overset{df}{=} \sum_{p \in p^+(n)} \sum_{i=1}^{N_n(p)} \mu(i)^\lambda, \ \lambda \in \mathbb{R},$$

so that, $\psi_n(\lambda)$ clearly reduces to the analogous partition function given by (1), when $\pi_l = \frac{1}{M}$, $l = 1, 2, \ldots, M$. 
As before, $\psi_n(0) = N_n$ is the number conservation equation and $\psi_n(1) = 1$ is the mass conservation equation.

It follows clearly from the preceding that:

**Lemma 1:**

*For each $\lambda \in \mathbb{R}$ we have the equality*

$$\psi_n(\lambda) = \|X_n(\lambda)\|_1,$$

*where $X_n(\lambda)$ is the solution of:*

$$X_{n+1}(\lambda) = A\lambda X_n(\lambda), \quad X_0(\lambda) = (1, 0, \ldots 0)^T,$$

*with*

$$A_\lambda \overset{\text{df}}{=} \begin{pmatrix}
    a_1(\lambda) & 1 & 0 & \ldots & 0 \\
    a_2(\lambda) & 0 & 1 & \ldots & 0 \\
    a_{k+1}(\lambda) & 0 & 0 & \ldots & 1 \\
    a_k(\lambda) & 0 & 0 & \ldots & 0
\end{pmatrix}, \quad (12)$$

*where the last sum is performed over all branches of depth $d(l) = k$ of the generator.*

It should now be recalled that, due to the primitivity of $A_\lambda$, the matrix $A_\lambda$ has a unique maximal absolute eigenvalue, say $\alpha(\lambda) > 0$, of algebraic (and hence geometric) multiplicity one. $\alpha(\lambda)$ satisfies the characteristic equation of (8) as an eigenvalue of $A_\lambda$, namely:

$$\sum_{k=1}^{\kappa^+} \left\{ \sum_{l=1}^{M} \pi_i^\lambda \right\} \frac{1}{\alpha(\lambda)^k} = 1, \quad \lambda \in \mathbb{R}. \quad (13)$$

Moreover, $\alpha : \mathbb{R} \to \mathbb{R}^+$ is strictly decreasing and goes to zero as $\lambda \to +\infty$ [15].

Also, due to the structure of $A_\lambda$ and from the preceding sections: $\alpha(0) = \alpha > 1$ and $\alpha(1) = 1$.

From this situation, one can conclude:

**Theorem 3:**

*Let us define:*

$$F_n(\lambda) \overset{\text{df}}{=} -\log_{N_n} \psi_n(\lambda), \quad \lambda \in \mathbb{R}, \quad n \in \mathbb{N}.$$

*The pointwise "thermodynamic limit"*

$$\lim_{n \to \infty} F_n(\lambda) \overset{\text{df}}{=} F(\lambda) = -\log_\alpha \alpha(\lambda), \quad \lambda \in \mathbb{R}, \quad (14)$$

*is concave and analytic. Moreover*

$$F(0) = -1, \quad F(1) = 0.$$
Proof:
From Lemma 1,
\[ \psi_n(\lambda) = \|X_n(\lambda)\|_1 = \|A_\lambda^n\|_1, \]
therefore
\[ \lim_{n \to \infty} \frac{1}{n} \log \psi_n(\lambda) = \lim_{n \to \infty} \frac{1}{n} \log \|A_\lambda^n\|_1 = -\log \alpha(\lambda), \lambda \in \mathbb{R}. \]

\( F(\lambda) \) is thus concave as a simple limit of concave functions.

The analyticity results from the fact that \( \alpha(\lambda) \) is itself analytic, as it was kindly brought to our attention by Michon. Indeed, let
\[ P(\lambda, x) \overset{df}{=} \det(A_\lambda - xI), \lambda \in \mathbb{R}, x \in \mathbb{R}. \]

By the Perron-Frobenius theorem, \( P(\lambda, \alpha(\lambda)) = 0 \) for all \( \lambda \in \mathbb{R} \).
Suppose that \( P(\lambda_0, \alpha(\lambda_0)) = 0 \) for fixed \( \lambda_0 \). Since \( \alpha(\lambda_0) \) is a simple root of \( P \):
\[ \frac{\partial P}{\partial \lambda}(\lambda_0, \alpha(\lambda_0)) \neq 0, \]
therefore by the theorem of implicit analytic functions there exist an analytic function \( r : \mathbb{R} \to \mathbb{R} \) and \( \beta > 0 \) such that \( [\lambda_0 - \alpha, \lambda_0 + \alpha] \cap \mathbb{R} \) is the graph of \( r(\lambda) \).

Moreover, letting \( x_1 < x_2 < \ldots < \alpha(t_0) \) denote all real roots of \( P \), the continuity of roots of an equation depending continuously on a parameter reads as follows:
For \( \varepsilon > 0 \) there exists \( \eta \) such that, if \( |\lambda - \lambda_0| < \eta \) and if \( P(\lambda, x) = 0 \), then
\[ x \in [x_1 - \varepsilon, x_1 + \varepsilon] \cup [x_2 - \varepsilon, x_2 + \varepsilon] \cup \ldots \cup [\alpha(\lambda_0) - \varepsilon, \alpha(\lambda_0) + \varepsilon]. \]

If \( \varepsilon \) is well chosen, then those intervals are disjoint and for \( \lambda < \eta \), the greatest root is \( r(\lambda) \). This proves that \( r(\lambda) = \alpha(\lambda) \) at some neighborhood of \( \lambda_0 \).

Remark 1:
A more detailed study of \( F \) shows that when \( \lambda \to -\infty \), \( F(\lambda) \) behaves like the line of equation
\[ y = t_{\max} \lambda, \]
where \( t_{\max} \overset{df}{=} -\frac{1}{\kappa^-} \log \alpha_*, \pi_* \overset{df}{=} \inf_{\frac{1}{\lambda(t)} = \kappa^-} \pi_t. \]

Also when \( \lambda \to +\infty \), \( F(\lambda) \) behaves like the line of equation
\[ y = t_{\min} \lambda, \]
where \( t_{\min} \overset{df}{=} -\frac{1}{\kappa^+} \log \alpha_*, \pi_* \overset{df}{=} \sup_{\frac{1}{\lambda(t)} = \kappa^+} \pi_t. \]

We are now in the position to introduce the Legendre transform of \( F(\lambda) \):
\[ f(t) \overset{df}{=} \inf_{\lambda \in \mathbb{R}} \{ \lambda t - F(\lambda) \}, t \in [t_{\min}, t_{\max}]. \] (15)

The function \( f(t) \) is concave and analytic on \([t_{\min}, t_{\max}] \). Moreover, it is involutive:
\[ F(\lambda) = \inf_{t \in [t_{\min}, t_{\max}]} \{ \lambda t - f(t) \}, \lambda \in \mathbb{R}. \]

\( f(t) \) also reads
\[ f(t) = \lambda^*(t) t - F(\lambda^*(t)), t \in [t_{\min}, t_{\max}], \]
with \( \lambda^* \) defined by
\[ \frac{dF}{d\lambda}(\lambda^*(t)) = t. \]
It has the usual bell shape \[10\]; see Figure 1. It follows from the above that:

\[ f(F'(0)) = -F'(0) = 1, \]

\[ f(F'(1)) = F'(1) - F(1) = F'(1) \frac{df}{D(1)}, \]

and from Theorem 1 and equation (14) that

\[
F'(0) = \frac{-\sum_{k=1}^{\kappa^+} \alpha^{-k} \sum_{1 \leq t \leq M} \log_{\alpha} \pi_t}{\sum_{k=1}^{\kappa^+} ka_k \alpha^{-k}} > 1,
\]

\[
F'(1) = \frac{-\sum_{l=1}^{M} \pi_l \log_{\alpha} \pi_l}{\sum_{k=1}^{\kappa^+} k \sum_{1 \leq t \leq M} \pi_t} < 1.
\]

We are not able at this step to derive more properties of both \( f(t) \) and \( F(\lambda) \) as defined by equations (13) and (15), in particular, their explicit expression. Nevertheless, some additional information can be extracted.

To do this, we let:

\[ t^*(\lambda) = F'(\lambda) \]
be the inverse of $\lambda^*(t)$, so that one can define the function

$$\tilde{f}(\lambda) \overset{df}{=} f(t^*(\lambda)) = \lambda F'(\lambda) - F(\lambda), \ \lambda \in \mathbb{R}.$$  \hfill (16)

Let us define a one-parameter family of probability measures:

$$\mu_{\lambda}(i) \overset{df}{=} \frac{\mu(i)^\lambda}{\sum_{i=1}^{N_n} \mu(i)^\lambda}, \ 1 \leq i \leq N_n, \ \lambda \in \mathbb{R},$$

with $\mu_1(i) = \mu(i), \ 1 \leq i \leq N_n$.

The *imprecision function* [18]:

$$I_n(\lambda) \overset{df}{=} F'_n(\lambda) = -\sum_{i=1}^{N_n} \mu_{\lambda}(i) \log N_n \mu(i), \ \lambda \in \mathbb{R},$$

is a measure of the lack of information of an observer who, ignoring the true distribution $\mu_{\lambda}(i)$ decides to affect his own guess $\mu(i)$. Also

$$I^*_n(\lambda) \overset{df}{=} -\sum_{i=1}^{N_n} \mu(i) \log N_n \mu_{\lambda}(i), \ \lambda \in \mathbb{R},$$

is the dual imprecision function, reversing the roles of $\mu(i)$ and $\mu_{\lambda}(i)$.

Another interesting function is the *Renyi’s entropy* of of $\mu(i)$:

$$D_n(\lambda) \overset{df}{=} \frac{1}{\lambda - 1} F_n(\lambda), \ \lambda \neq 1.$$  

Clearly,

$$D_n(\lambda) \xrightarrow{n \to \infty} \frac{1}{\lambda - 1} F(\lambda) \overset{df}{=} D(\lambda) \hfill (17)$$

for all $\lambda \neq 1$ and

$$D_n(1) \xrightarrow{n \to \infty} D(1) \overset{df}{=} D'(1). \hfill (18)$$

Let us define the *Shannon’s entropy* of of $\mu_{\lambda}(i)$:

$$S_n(\lambda) \overset{df}{=} -\sum_{i=1}^{N_n} \mu_{\lambda}(i) \log N_n \mu_{\lambda}(i), \ \lambda \in \mathbb{R}.$$  

An immediate extension of Theorem 1 is:

**Theorem 4:**

For each $\lambda \in \mathbb{R}$ the following convergence holds:

$$S_n(\lambda) \xrightarrow{n \to \infty} \tilde{f}(\lambda),$$

with

$$\tilde{f}(\lambda) \overset{df}{=} \frac{-\sum_{i=1}^{M} \pi_i(\lambda) \log_{T_i} \pi_i(\lambda)}{\sum_{k=1}^{M} \sum_{1 \leq i \leq M} \pi_i(\lambda) \delta(i) = k}$$
Letting Kullback's information gain be

\[ G_n(\lambda) \triangleq I_n(\lambda) - S_n(\lambda) = -\sum_{i=1}^{N_n} \mu(i) \log N_n \frac{\mu(i)}{\mu(\lambda)}, \quad \lambda \in \mathbb{R}, \]

and its dual expression:

\[ G_n^*(\lambda) \triangleq I_n^*(\lambda) - S_n(1) = -\sum_{i=1}^{N_n} \mu(i) \log N_n \frac{\mu(i)}{\mu(\lambda)}, \quad \lambda \in \mathbb{R}, \]

after an easy computation, one has

\[ S_n(\lambda) = D_n(\lambda) - \frac{\lambda}{\lambda - 1} G_n(\lambda), \quad \lambda \neq 1. \quad (19) \]

As \( n \to \infty \), by Theorem 3, (17) and (19) we have

\[ G_n(\lambda) \xrightarrow{n \to \infty} G(\lambda). \]

By (16), (17), (19),

\[ G(\lambda) = (\tilde{f}(\lambda) - D(\lambda)) \frac{1 - \lambda}{\lambda} = (1 - \lambda) F'(\lambda) + F(\lambda) \]

for all \( \lambda \neq 1 \) and

\[ \tilde{f}(1) = D(1), \]

due to \( G(1) = 0 \), \( G'(1) = 0 \).

**Corollary 1:**

The limit entropy of the mass distribution along our branching tree always exists and:

\[ \lim_{n \to \infty} -\sum_{i=1}^{N_n} \mu(i) \log N_n \mu(i) = D(1). \]

6. Geometrical Interpretation

The preceding considerations suggest to introduce more general computations relative to the discrete measures

\[ \mu \triangleq \sum_{k=1}^{\kappa_+} a_k \delta_{(k)}, \quad \mu^* \triangleq \sum_{k=1}^{\kappa_+} a_k^* \delta_{(k)}. \]

Let

\[ L(\lambda) \triangleq \int_{\mathbb{R}} e^{-\lambda x} d\mu^* = \sum_{k=1}^{\kappa_+} e^{-\lambda k} a_k^*, \quad \lambda \in \mathbb{R}, \]
with values in \( [0, \infty) \), be the real Laplace transform of \( \mu^* \).

Then the cumulant function \( K(\lambda) \overset{\text{df}}{=} -\log L(\lambda) \) is concave and analytic on \( \mathbb{R} \).
Moreover, one has \( K(\log \alpha) = \log M \) and \( K'(0) = \bar{T} \).

This gives an interesting geometrical interpretation of the entropy dimension and "shows" that \( D_G \in [0, 1] \).

It is quite natural to introduce now the Legendre transform of the function \( K(\lambda) \).
Let:

\[
s(t) \overset{\text{df}}{=} \inf_{\lambda \in \mathbb{R}} \{\lambda t - K(\lambda)\}, \; t \in [\kappa_-, \kappa_+].
\]

The function \( s(t) \) is concave and analytic on \( [\kappa_-, \kappa_+] \). Moreover, it is involutive:

\[
K(\lambda) = \sup_{t \in [\kappa_-, \kappa_+]} \{\lambda t - s(t)\}, \; \lambda \in \mathbb{R}.
\]

\( s(t) \) also reads \( s(t) = \lambda^*(t)t - K(\lambda^*(t)) \), \( t \in [\kappa_-, \kappa_+] \),
with \( \lambda^* \) defined by

\[
\frac{dK}{d\lambda}(\lambda^*(t)) = t.
\]

If \( t^* \) denotes the inverse of \( \lambda^* \), one can in an equivalent manner work on the entropy

\[
S(\lambda) \overset{\text{df}}{=} s(t^*(\lambda)) = \lambda K'(\lambda) - K(\lambda); \; \lambda \in \mathbb{R}.
\]
Therefore \( E(\lambda) \overset{\text{df}}{=} K'(\lambda) \) can be interpreted as an internal energy and \( F(\lambda) \overset{\text{df}}{=} \frac{1}{\lambda} K(\lambda) \) as a free energy, \( \lambda \) being the inverse of the temperature.

Fig. 2. — The geometrical interpretation of \( D_G \).
In a Fig. 3. The internal energy.

The evaluation of these quantities at point $\lambda = \log \alpha$ gives

$$F(\log \alpha) \overset{df}{=} \frac{1}{\log \alpha} K(\log \alpha) = D_G \overline{T},$$

$$E(\log \alpha) \overset{df}{=} K'(\log \alpha) \overset{df}{=} d_G \overline{T},$$

with $d_G < D_G$ and

$$S(\log \alpha) = \overline{T}(d_G - D_G) \log \alpha < 0.$$

7. Statistics of the Hölder Exponents

In this section, we give first the heuristics of the multifractal study of the mass distribution along the tree defined in Section 2. A rigorous approach to this problem is then given which essentially is an application of the large deviation theorem for a non-Markov renewal process that we exhibit.

At step $n$ of the above branching procedure, $N_n \sim \alpha^n$ individuals are present in the population. Let $i(n) \in \{1, \ldots, N_n\}$ denote one of them. The mass associated to $i(n)$ is $\mu(i(n))$. The quantity

$$t_{i(n)} = \frac{\log \mu(i(n))}{\log N_n}$$

is called the coarse Hölder exponent of $i(n)$ [12].

For each value of $t_{i(n)}$, it is of interest to evaluate the number, say $N_n(t_{i(n)})$, of individuals taking approximatively this coarse Holder exponent, i.e.,

$$N_n(t_{i(n)}) \overset{df}{=} \#\{i \in \{1, \ldots, N_n\}; \frac{\log \mu(i)}{\log N_n} \sim t_{i(n)}\}.$$
Let us fix $n \in \mathbb{N}$ and suppose that there exists the functions
\begin{equation}
    f(t_{i(n)}) \sim \frac{\log N_n(t_{i(n)})}{\log N_n},
\end{equation}
and
\begin{equation}
    C(t_{i(n)}) \sim \frac{\log p_n(t_{i(n)})}{\log N_n} = f(t_{i(n)}) - 1,
\end{equation}
with
\begin{equation}
    p_n(t_{i(n)}) \equiv \frac{N_n(t_{i(n)})}{N_n} \sim \alpha^{C(t_{i(n)})},
\end{equation}
(the "probability" to take the value $t_{i(n)}$ for some individual).

Indeed, if this were the case, using the definition of the partition function (11), one gets, regrouping all the individuals at step $n$ with the same exponent $t_{i(n)}$ and recalling $N_n \sim \alpha^n$,
\begin{equation}
    \psi_n(\lambda) \sim \sum_{i(n)} \alpha^{-n(\lambda t_{i(n)} - f(t_{i(n)})}, \lambda \in \mathbb{R},
\end{equation}
where the last sum is performed over all individuals having distinct Holder exponents. For $n$ large the effective contribution to this sum is given by the term which minimizes $\lambda t_{i(n)} - f(t_{i(n)})$.

We may therefore expect
\begin{equation}
    -\frac{1}{n} \log \psi_n(\lambda) \sim \inf_{t_{i(n)}} (\lambda t_{i(n)} - f(t_{i(n)}), \lambda \in \mathbb{R},
\end{equation}
for large $n$, as required.

So we define
\begin{equation}
    F(\lambda) \equiv \inf_{t} (\lambda t - f(t)), \lambda \in \mathbb{R}.
\end{equation}

The measure carried by those individuals whose coarse Holder exponent is $t_{i(n)}$ is:
\begin{equation}
    \mu(t_{i(n)}) \equiv N_n(t_{i(n)}) N_n^{-t_{i(n)}} \sim -n(t_{i(n)} - f(t_{i(n)}))
\end{equation}
In the case the limit exists, the subset of those individuals for which
\begin{equation}
    \lim_{n \to \infty} t_{i(n)} = D(1) = F'(1)
\end{equation}
(i.e., the unique value which minimizes $t - f(t)$) carries all the measure, i.e.,
\begin{equation}
    \lim_{n \to \infty} \mu(t_{i(n)}) = 1.
\end{equation}

If $N_n(t_{i(n)})$ attains its maximum at the value(s) $t_{i(n)}$, then
\begin{equation}
    \lim_{n \to \infty} t_{i(n)} = F'(0)
\end{equation}
for which
\begin{equation}
    f(F'(0)) = -F'(0) = D(0) = 1
\end{equation}
is maximum. Therefore, $D(0) = 1$ is the dimension of the support of this measure [9,12].

Let us discuss now the rigorous approach to the above heuristics.

Let $T$ be a random variable with probability distribution
\begin{equation}
    \mu \equiv \sum_{k=1}^{\infty} \frac{a_k}{M} \delta_k.
\end{equation}
Let $\Delta X_T$ be a random variable with probability distribution

$$
\sum_{k=1}^{\kappa^+} \frac{a_k}{M} \sum_{l=1}^{M} \frac{1}{a_k} \delta\{ - \log \pi_l \} = \frac{1}{M} \sum_{l=1}^{M} \delta\{ - \log \pi_l \}.
$$

Thus, given $[T = k]$, $\Delta X_T$ expresses a random equidistributed choice of one among the $a_k$ values $-\log \pi_l$; $1 \leq l \leq M$ of depth $d(l) = k$.

$\Delta X_T$ will be considered as a random increment of a process defined as follows. We start by the following equation

$$
X_n = \Delta X_T 1_{\{ T > n \}} + (\tilde{X}_{n-T} + \Delta X_T) 1_{\{ T \leq n \}}; \quad n \geq 0,
$$

where the equality is considered in law and $\tilde{X}_{n-k}$ given $[T = k \leq n]$ has the same probability distribution as $X_{n-k}$. In this equation the only required hypothesis on $X_n$ is that after some regeneration time $T$ the future of this process is supposed to be a statistical copy of the initial process. Let us check now that this process is well defined in law.

Let

$$
\Phi_n(\lambda) \triangleq E e^{-\lambda X_n} = \frac{1}{N_n} \psi_n(\lambda); \quad n \geq 0.
$$

It follows from (20) that

$$
\Phi_n(\lambda) = \frac{1}{M} \sum_{k=1}^{\kappa^+} \{ a_k(\lambda) 1_{\{ T > n \}} + \Phi_{n-k}(\lambda) a_k(\lambda) 1_{\{ T \leq n \}} \}; \quad n \geq 0.
$$

Also, letting

$$
\Phi(s, \lambda) \triangleq \sum_{n=0}^{\infty} e^{-sn} \Phi_n(\lambda)
$$

one gets

$$
\Phi(s, \lambda) = \frac{1}{M} \sum_{k=1}^{\kappa^+} a_k(\lambda) \{ 1 - e^{-sk} \} / s \{ 1 - \frac{1}{M} \sum_{k=1}^{\kappa^+} e^{-sk} a_k(\lambda) \}; \quad \lambda \in \mathbb{R}^1, \quad s > \log \alpha_N(\lambda),
$$

where $\alpha_N(\lambda)$ is the the largest zero of the denominator of (21), i.e.,

$$
\frac{1}{M} \sum_{k=1}^{\kappa^+} a_k(\lambda) \frac{1}{\alpha_N(\lambda)^k} = 1,
$$

which is a normalized version of (13). It is well known that

$$
\lim_{n \to \infty} (\Phi_n(\lambda))^\frac{1}{n} = \alpha_N(\lambda).
$$

Letting now

$$
F_N(\lambda) \triangleq - \log \alpha_N(\lambda),
$$

and $f_N(t)$ its Legendre transform, it follows from Theorem II.6.1. of reference [19] that

$$
\lim_{n \to \infty} P[ - \frac{1}{n} \log \alpha M_n \geq t ]^\frac{1}{n} = \exp f_N(t) = \alpha^f(t)^{-1},
$$
provided $t > F'(0)$, with $f$ defined by (11) and $X_n \overset{df}{=} -\log M_n$. Alternatively,

$$\lim_{n \to \infty} \#\{1 \leq i \leq N_n; \frac{\log \mu(i)}{\log N_n} \geq t\}^{\frac{1}{n}} = \alpha^{f(t)}.$$ 

References