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A Probabilistic Cellular Automaton for Evolution

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Abstract. --- We study two variants of a binary model for Darwinian Coevolution in an ecosystem of interacting species based on a model recently proposed by Bak and Sneppen [1]. We present results from a computer simulation and an improved mean-field calculation, including exponents, critical points and probability of certain spin configurations.

Recently a model for Darwinian Coevolution in a dynamical ecosystem of interacting species was proposed by Bak & Sneppen (BS-model) [1]. The only mechanisms which are considered in this model are mutation and natural selection. In this note, we will study a simplified version of the BS-model and we will not discuss questions concerning the applicability to biological evolution. The BS-model consists of $L^d$ sites (species) arranged on a $d$-dimensional hypercubic lattice. Here, we set $d=1$ and use periodic boundary conditions. To each species $i$ a random number (barrier) $B_i$, $0 \leq B_i \leq 1$ (equally distributed) is assigned in the beginning. The evaluation proceeds as follows: choose the site $i$ with the lowest value $B_i < B_j$ for all $j \neq i$ and assign new random numbers to $i$ and its left and right neighbors. This choice of the interaction between the species is quite arbitrary. One can, for example, also study a only “one-sided”-version of this model (assign new random numbers only to $i$ and, say, $i+1$) [2]. After a transient time the system organizes itself into a stationary state with critical properties. The distribution $p(B)$ of the barriers converges towards a step-function in the limit of an infinite system ($L \to \infty$):

$$p(B) = \begin{cases} 
1/(1-p_c) & \text{for } B > p_c \\
0 & \text{for } B < p_c 
\end{cases}$$

We now define a probabilistic binary cellular automaton (PCA-model) based on this picture of the stationary state of the BS-model [3, 4]. All barriers below the threshold are set to 1 (spin 1), all remaining barriers are set to 0 (spin 0). The evaluation proceeds as follows: choose randomly a site $i$ with spin $n_i = 1$ and assign independently three new spins $n_i, n_{i-1}$ and $n_{i+1}$ to $i, i-1$ and $i+1$ (1 with probability $c$, 0 with probability $1-c$), irrespective of their former values. We also study the one-sided version of this model.

Note that the global feature of the BS-update-rule disappears but is reflected in the new control parameter $c$.

The one-sided case with the constraint to leave the spin 1 site which was chosen for update
unchanged and only to assign a new spin to its one neighbor was solved in reference [5]. There the stationary state is trivial:

\[ \langle n_{11} n_{i_2} \ldots n_{i_r} \rangle = c^r \]

for all \( c \) and it is possible to calculate the equilibrium spin autocorrelation function of the time-dependent occupation number fluctuations at a certain site. In the case of our PCA-models exists a critical probability \( c^* \) below which the system converges towards the absorbing state \( n_i = 0 \). Above \( c^* \) the density \( p_1 \) of sites with spin 1 converges to some \( c \)-dependent value. Figure 1 shows \( p_1 \) as a function of time for a system of \( 3 \times 10^5 \) sites. The simulation was performed on a SUN workstation. The unit of time is simply the total number of updates. At \( c^* \) we find

\[ p_1 \sim t^\tau, \quad \tau < 0. \]

Our results for the PCA’s are \( c^* = 0.6353 \pm 0.0001 \) (two-sided, in agreement with [4]) and \( c^* = 0.768 \pm 0.0001 \) (one-sided). The exponent \( \tau \) turned out to be the same for both models: \( \tau = -0.185 \pm 0.004 \). The two-sided PCA with parallel dynamics was studied in [6] and is known to be in the same universality class as directed percolation. Kinzel finds a different \( c^* \).

In order to compare his exponent \( \tau' = 0.157 \pm 0.002 \) with our data one has to rescale the time properly, since our rate of update is coupled to \( p_1(t) \). Doing this, one obtains \( \tau' = \tau/(\tau + 1) \), which is in good agreement with our value of \( \tau \).

We turn to an improved mean-field description of the PCA’s in the stationary state. We denote the probability to find a configuration \( n_i n_{i+1} \ldots n_{i+r} \) of \( r+1 \) neighbored spins with \( p_{n_i n_{i+1} \ldots n_{i+r}} \), for example \( p_{11} = \langle n_i n_{i+1} \rangle \).

![Fig. 1.](image)

**Fig. 1.** — The density of 1-spins in a system of \( 3 \times 10^5 \) sites with two-side update versus time. The time unit is the total number of updates and the parameter is the probability \( c \).
We now write down the flow equation for $p_1$ (two-sided):

$$\dot{p}_1 = \gamma \{(3c-1)p_1 - 2p_{11}\}.$$ 

$\gamma$ characterizes the rate of the update. This equation yields $p_{11} = \frac{(3c-1)}{2}p_1$ in the stationary state. A simple mean-field ansatz would factorize $p_{11}$ into $p_1 \cdot p_1$ and therefore lead to $p_1 = (3c-1)/2$ with a critical point at $c = 1/3$. This is the same result as for the original BS-model. Figure 2 shows $p_1(c), p_{11}(c), \ldots, p_{i+i}(c)$ (note that $p_{i+1}\ldots i$ sites is equal to $<n_i \cdot n_{i+1} \ldots n_{i+r-1}>$). The mean-field solution is dotted. It is clear that this mean-field solution becomes exact for $c \to 1$.

We now discuss the special case $c = 1$. The stationary solution in this case is trivial and the transient can be exactly solved. To do this, it is convenient to examine strings of $l$ sites with $n_i = 0$ and to denote the probability for such a string with $z_l$. The case $z_0$ corresponds to $p_{11}$.

It is easy to find the flow equation for $z_l$ by considering the possible gains and losses in one time-step, here done for the one-sided model:

$$\dot{z}_l = \gamma \{ z_{l-1} - z_l \} \quad (l \neq 0)$$

and

$$\dot{z}_0 = \gamma \{ z_1 \}.$$ 

The rate of update $\gamma$ in our model is given by $1/(Lp_1)$ ($L$=system size) and therefore time-dependent. The strategy is now to solve these equations for a system consisting only of strings $z_l$ with one certain $l = l_0$ in the beginning. The solution for a general starting configuration is then obtained by taking the average over the string-configuration in the beginning since the strings are not spatially correlated. The solution for the mentioned special starting configuration is

$$z_l(l_0) = \frac{\Gamma(l_0 - l)}{(l_0 - l)!} e^{-\Gamma(t)}$$

with

$$\Gamma(t) = \int_0^t \gamma(t')dt'$$

As an example, one can calculate $p_1(t)$ for a starting configuration which is given by $p_1(t = 0) = p^0$ and $p_0(t = 0) = 1 - p^0$. One has then

$$p_1(t) = \sum_{l_0=0}^{\infty} p^0(1-p^0)^{l_0} \left(1 - \sum_{l=0}^{l_0-1} z_l(l_0-1)\right),$$

which yields

$$p_1(t) = 1 - (1 - p^0)e^{-p^0\Gamma(t)}$$

This is an integro-differential equation for $p_1(t)$. The corresponding differential equation is directly solvable and one obtains finally

$$p_1(t) + \ln(1 - p_1(t)) = -\frac{tp^0}{L} + p^0 + \ln(1 - p^0).$$
Fig. 2. — a) The stationary values of $p_1, p_{11}, ..., p_{11111}$ as a function of $c$ for a system of $3 \times 10^5$ sites (two-sided). The simple mean-field solution for $p_1$ is dotted, the improved mean-field approximation is drawn in full lines; b) The same for the one-sided variant of this model.

We now return to our general model $c \leq 1$ and apply the (2,1)-cluster approximation, as described in reference [7]. This means that we decompose all clusters into two-site-clusters.
overlapping just one site:

\[ p_{n_1 n_1+1 \ldots n_r+\varepsilon} \approx \prod_{j=1}^{i+r-2} \frac{p_{n_j n_j+1} p_{n_j+1 n_j+2}}{p_{n_j+1}} \]

The stationary flow equation for \( p_{00} \),

\[ 0 = (1 - c)^2 p_1 + (1 - c) p_{011} - c p_{001} \]

becomes within this approximation

\[ 0 = (1 - c)^2 p_1 + (1 - c) \frac{p_{01} p_{11}}{p_1} - c p_{01} + c \frac{p_{01}^2}{p_0}, \]

which implies that, using relations like \( p_{01} = p_1 - p_{11} \),

\[ p_1 = \frac{9c^3 - 2c - 1}{7c - 1} \quad (c > c^*). \]

This leads to \( c^* = \frac{1}{9}(1 + \sqrt{10}) \approx 0.46 \), which is an improvement compared to the simple mean-field-approximation. Figure 2 shows this improved solution. Data and calculation agree very nicely for higher values of \( c \).

Figure 2 also shows that the mean-field cluster approximation is very good for calculating expectation values like \( < n_t \cdot n_{t+1} \ldots n_{t+r-1} > \). The data of the simulation coincide nicely with the law

\[ p_{11\ldots1}(r \text{ sites}) = \left( \frac{3c - 1}{2} \right)^{r-1} p_1. \]

We give the corresponding results for the one-sided PCA:

\[ p_1 = \frac{4c^3 - c - 1}{3c - 1} \]

and

\[ c^* = \frac{1}{8}(1 + \sqrt{7}) \approx 0.6404, \]

and the numerical law

\[ p_{11\ldots1}(r \text{ sites}) = (2c - 1)^{r-1} p_1. \]

Again, the simple mean-field solution yields the same result as for the one-sided BS-model, which is \( c = 2/3 \).

To calculate the critical point more precisely, one could simply try a \((3,2)\)-cluster approximation and so forth, but unfortunately the dimension of the system of equations grows exponentially, besides the difficulties arising in writing down the correct flow equations. We tried another ansatz: one can imagine to map a system of \( L \) spins in the stationary state of a PCA-model onto a system of \( L' = L/3 \) spins by performing a sort of block-spin-transformation with majority-rule \( p_1' = p_{111} + 2p_{011} + p_{101} \). One expects \( p_1' \approx p_1 \) in the vicinity of \( c^* \). This ansatz together with the flow equation of \( p_{00} \) and the \((2,1)\)-type approximation \( p_{111} = \frac{p_{111}^2}{p_1} \) yields \( c^* = 1 \), the trivial critical point, and \( c^* = \frac{1}{36}(11 + \sqrt{193}) \approx 0.69 \), which is much better than the \((2,1)\)-cluster-approximation result.
The same method used for the one-sided PCA leads to a non-trivial critical point 
\[ c^* = \frac{1}{16}(5 + \sqrt{57}) \approx 0.784. \]

In summary, we did a computer simulation of two closely related simple cellular automata for coevolution. We find the same exponent of the time decay of the density of sites with spin 1 as for directed percolation but different critical points. The critical points can be approximated by means of a simple renormalization argument. We find a numerical law for the higher cluster probabilities and give the results of an improved mean-field theory which is very good for these probabilities.

References