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HAL Id: jpa-00247116
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Submitted on 1 Jan 1995

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Microscopic Simulation of the Stock Market: the Effect of Microscopic Diversity

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(Received 10 March 1995, received in final form 3 April 1995, accepted 10 April 1995)

Abstract. — Although the representative individual framework has been shown to be generally illegitimate and erroneous, it continues to be widely used for lack of a more suitable approach. In this paper, we present an alternative methodology for economic study — the microscopic simulation (MS) approach. We employ the MS methodology to study a basic model of the stock market formulated at the microscopic level — the level of the individual investor. We contrast macroscopic market behavior arising from heterogeneous as opposed to homogeneous (representative) investors. We study the macroscopic effects of different forms of the microscopic heterogeneity: heterogeneity of preference, expectation, strategy, investor-specific noise, and their combinations. We show that representative investor models lead to unrealistic market phenomena such as periodic booms and crashes. These periodic booms and crashes persist even when we relax the assumption of homogeneous preference. Only when heterogeneous expectations are introduced does the market behavior become realistic.

Introduction

Very few financial analysts would deny that agents operating in the market are diverse in their preferences and in their frameworks of forming expectations. Yet, in order to obtain analytical results, many macroeconomic models assume a “representative” agent. This approach is analogous to the mean field approximation in statistical physics. The “representative” agent models often yield results that are very different from observed market behavior. For instance, the representative individual framework leads to the unrealistic scenario of no trade in the market. The implication of such a result negates the existence of a meaningful stock market [1–4].

Another erroneous result emanating from the representative agent assumption is reflected in the Sharpe-Lintner capital asset pricing model (CAPM). In this model, which derives equilibrium prices of risky assets, all agents are assumed to be risk-averse. Heterogeneous preference is allowed, yet, all agents are assumed to agree on the expected returns and on the variance-covariance matrix (i.e., they have homogeneous expectations). Based on these assumptions the CAPM is derived where “beta” is the risk measure of the individual asset. However, once

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the agents are diverse with respect to their expectations or the information at their disposal, or when they incur different transaction costs, the whole model collapses, beta becomes meaningless, and a much more complex measure of risk is called for. Thus, when the assumption of representative investors (with respect to expectation) is relaxed, the derivation of the risk-return relationship leads to a result that is very different to Sharpe-Lintner’s CAPM(1).

Kirman, in his review of the problematic nature of the representative individual concept, concludes:

“In particular, I will argue that heterogeneity of agents may, in fact, help to save the standard model... it is clear that the “representative” agent deserves a decent burial, as an approach to economic analysis that is not only primitive but fundamentally erroneous” [9].

In this study, we suggest microscopic simulation as an alternative to the representative individual framework. Microscopic simulation (MS) is a methodology that is used to study the macroscopic behavior of a system composed of many basic interacting elements acting according to a known microscopic rule. In microscopic simulation, the microscopic elements and their actions are simulated by computer according to the microscopic rule.

The simulation can be thought of as an acting out of reality by the computer. For a concrete example consider an ideal gas in a box. The microscopic elements are the gas atoms, and the microscopic rule states that when an atom collides with another atom, or with the wall, it will bounce off like a billiard ball; otherwise, it will simply continue to move at the same velocity. One can write a computer program that will accept as input the initial positions and velocities of all the atoms, and will simulate the development of the system through time. In the case of the ideal gas, the macroscopic properties of the system can also be derived analytically. However, if the microscopic algorithm is somewhat more complex, analytic methods fail. In contrast, this additional complexity does not constitute a problem in the simulation method. Any system with any ”wild” microscopic algorithm can be simulated. MS is a powerful tool because it is not restricted by the constraints of analytical treatment.

MS is widely used in physics [10, 11]. More recently, MS has been implemented in biology (see, for example, [12–16]) and in sociology [17]. MS has proven to be a very fruitful approach in these fields, producing new understanding and insight into the behavior of complex systems. We believe that MS will be as fruitful in finance and in economics as it has been in physics, biology and sociology. The purpose of this paper is to apply MS to the study of the stock market. We construct a microeconomic model of the stock market and we analyze the resulting macroeconomic market behavior via MS. In particular we analyze and contrast the macroeconomic behavior produced by two microeconomic models: one with homogeneous agents (the representative agent approach), and the other with diverse agents. In both models we assume investors to be expected utility maximizers.

MS allows us to analyze the effects of agent heterogeneity on the macroeconomic results and the errors obtained by assuming the unrealistic assumption of the representative investor. We then go on to analyze various sources of heterogeneity, and show that some heterogeneity assumptions are not crucial to the macroeconomic results but others completely change the results relative to the non-homogeneous case.

Agents can be heterogenous in several respects. The main heterogeneity factors analyzed in this paper are:

1) Preference (risk aversion measure).

2) Agent-specific noise (i.e., the agent is an expected utility maximizer but may deviate from the optimal investment strategy due to random noise).

(1) For more details on the effect of agent’ heterogeneity on equilibrium results, see references [5–8]
3) Expectation — Even if all investors have the same preference (utility function), they may have heterogeneous expectations regarding the future distribution of the random variable (return in the stock market). For example, all agents estimate the future distribution of returns by looking at the historical rates of return, but they are heterogeneous regarding their memory spans; some look back at the last five years' returns whereas others may look back at the last ten years' returns.

4) Strategy — Agents may have different investment strategies. Some may adjust their expectations and their investment policies frequently, whereas others, with higher transaction costs, may have a longer run approach.

The microeconomic model that we present here does not yield to analytical treatment even in the most simple case where agents are homogeneous. MS allows us not only to study the resulting macroscopic behavior in the homogeneous and heterogeneous cases, but also to study the effects of each of the above factors of heterogeneity separately, as well as in combination. Like Kirman, we come to the conclusion that the introduction of heterogeneity is necessary in order to produce results that conform to the stock market's observed behavior. Relying on representative agents leads to the absurd result of a no trade market. The introduction of agent-specific noise does not have a mitigating effect: unacceptable results, such as sharp booms and crashes in the stock market, are obtained; moreover, these booms and crashes are periodic and therefore predictable, which grossly contradicts the concept of an efficient market. Only when heterogeneous expectation is introduced do we obtain more realistic stock price behavior.

The structure of the paper is as follows: Section 1 provides the model. Section 2 provides the data and the results, and Section 3 concludes the paper.

1. The Model

The microscopic "element" of our model is the individual investor. Individual investors interact via the buying and selling of stocks and bonds.

The model presented here is the most basic model attainable in which all the crucial elements of the stock market are included. We have consciously made certain simplifying assumptions and omitted some of the features of the real market. This was done in order to focus on the main elements of the market and not because of any limitations of the MS method. As mentioned before, any model can be microscopically simulated.

We begin by describing the basic model where investors are homogeneous in all respects. Later we will describe how different forms of diversity are introduced.

1.1. The Homogeneous Investor Model. — Our stock market consists of two investment options: a stock (or index of stocks) and a bond. The bond is assumed to be a riskless asset, and the stock is a risky asset. The stock serves as a proxy for the market portfolio, (e.g., the Standard & Poor's index). The extension from one risky asset to many risky assets is straightforward. However, one stock (the index) is sufficient for our present analysis because we restrict ourselves to global market phenomena and do not deal with distributions across several risky assets. The investors are allowed to revise their portfolio at given time points, i.e., we discuss a discrete time model.

The bond is assumed to be a riskless investment yielding a constant return at the end of each time period. The bond is exogenous and investors can buy from it as much as they wish at a given rate. We denote this riskless rate of return by $r$. Thus, an investment of $W$ dollars at time $t$ will yield $W(1 + r)$ at time $t + 1$. 
The return on the stock is composed of two elements:

(i) Capital gain (loss): The price of the stock is determined collectively by all investors by the law of supply and demand. If an investor holds a stock any rise (fall) in the price of the stock contributes to an increase (decrease) in the investors' wealth.

(ii) Dividends: The company earns income \( \bar{Y} \) (a random variable) and distributes dividends. We assume that the firm pays a dividend of \( D_t \) per share at time \( t \). We will elaborate on \( D_t \) in the next section.

Thus, the overall return on stock in period \( t \), denoted by \( H_t \), is given by:

\[
H_t = \frac{P_t - P_{t-1} + D_t}{P_{t-1}}
\]

(1)

where \( P_t \) is the stock price at time \( t \).

In order to decide on the optimal diversification between the risky and the riskless asset, one should consider the \textit{ex-ante} returns. However, since in practice these returns are generally not available, we assume that the \textit{ex-post} distribution of returns is employed as an estimate of the \textit{ex-ante} distribution.

In our model investors keep track of the last \( k \) returns on the stock which we call the stock's History. We assume that investors have a bounded recall in that they believe that each of the last \( k \) history elements at time \( t \) \( H_{t,j}=t-1,...,t-k+1 \) has an equal probability of \( 1/k \) to reoccur in the next time period \( (t+1) \)\(^2\). The bounded recall framework has been employed in other game theory analyses [18–20].

Investors derive "well being" or "utility" from their wealth. Each investor is characterized by a utility function, \( U(W) \), reflecting his/her personal preference. The utility is a monotonically increasing function of wealth, denoted by \( W \), and is also usually considered concave: \( U'(W) > 0, U''(W) < 0 \). In the homogenic model we assume the same utility function for all investors, and we take this function to be \( \log W \). Later on, when we introduce heterogeneity, we allow diverse preferences. In a situation with uncertainty the objective of each investor is to maximize the expected value of his/her utility. Investors divide their money between the two investment options in the optimal way which maximizes their expected utility. We will elaborate on this point below.

1.1.1. The Dynamics. — To illustrate the dynamics of our model consider the state of the market at some arbitrary time \( t \). We denote the price of the stock at this time by \( P_t \). The stock's history at this time is a set of the last \( k \) returns on the stock, \( H_{t,j}=t-1,...,t-k+1 \). We denote the wealth of the \( i \)th investor at time \( t \) by \( W_t(i) \), and the number of shares held by this investor by \( N_t(i) \).

Now, let us see what happens at the next trade point, time \( t+1 \).

Income Gain

First, note that the investor accumulates wealth in the interval between time \( t \) and time \( t+1 \). He/she receives \( N_t(i)D_t \) in dividends and \( (W_t(i) - N_t(i)P_t)r \) in interest. \( (W_t(i) - N_t(i)P_t \) is the money held in bonds as \( W_t(i) \) is the total wealth and \( N_t(i)P_t \) is the wealth held in stocks). Thus, before the trade at time \( t+1 \), the wealth of investor \( i \) is:

\(^2\) We also consider models where the equal probability expectation is replaced with the expectation of exponentially decaying reoccurrence probability.
\[ W_t(i) + N_t(i)D_t + (W_t(i) - N_t(i)P_t)r \]  

During the interval between time \( t \) and time \( t + 1 \) there is no trade, therefore the share price does not change and there is no capital gain or loss. However, at the next trade, at the time \( t + 1 \), capital gain or loss can occur, as explained below.

The Demand Function for Stocks

We derive the aggregate demand function for various hypothetical prices \( P_h \), and then find \( P^*_h = P_{h+1} \), the equilibrium price at time \( t + 1 \).

Suppose that at the trade at time \( t + 1 \), the price of the stock is set at a hypothetical price \( P_h \). How many shares will investor \( i \) want to hold at this price? First, let us observe that immediately after the trade the wealth of investor \( i \) will change by the amount \( N_t(i)(P_h - P_t) \) due to capital gain or loss. Note that there is capital gain or loss only on the \( N_t(i) \) shares held before the trade, and not on shares bought or sold at the time \( t + 1 \) trade. Thus, if the hypothetical price is \( P_h \), the hypothetical wealth of investor \( i \) after the \( t + 1 \) trade, \( W_h(i) \), will be:

\[ W_h(i) = W_t(i) + N(i)D_t + (W_t(i) - N_t(i)P_t)r + N_t(i)(P_h - P_t), \]  

where the first three terms are from equation (2).

The investor has to decide at time \( t + 1 \) how to invest this wealth. He/she will attempt to maximize his/her expected utility at the next period, time \( t + 2 \). As explained before, the \( \text{ex-post} \) distribution of returns is employed as an estimate for the \( \text{ex-ante} \) distribution. If investor \( i \) invests at time \( t + 1 \) a proportion \( X(i) \) of his/her wealth in the stock, his/her expected utility at time \( t + 2 \) will be given by:

\[ EU(X(i)) = 1/k \sum_{j=t}^{t+k+1} \log \left[ (1 - X(i))W_h(i)(1+r) + X(i)W_h(i)(1+H_j) \right], \]  

where the first term in the square brackets is the bond’s contribution to his/her wealth and the second term is the stock’s contribution.

The investor will choose the investment proportion, \( X(i) \), that maximizes his/her expected utility(3) As the optimum investment proportion is generally dependent on \( P_h \), (through \( W_h \)), let us denote this proportion by \( X_h(i) \).

The amount of wealth that investor \( i \) will hold in stocks at the hypothetical price \( P_h \) is given by \( X_h(i)W_h(i) \). Therefore, the number of shares that investor \( i \) will want to hold at the hypothetical price \( P_h \) will be:

\[ N_h(i, P_h) = \frac{X_h(i)W_h(i)}{P_h}. \]  

This constitutes the personal demand curve of investor \( i \). Summing the personal demand functions of all investors, we obtain the following collective demand function:

(3) As no borrowing or short-selling is allowed, we have \( 0 \leq X \leq 1 \). However, we introduce a constraint asserting that \( 0 < a < X_i \leq b < 1 \) where \( b \) is very close to 1, (e.g., 0.99) and \( a \) is very close to zero. Thus, the assumption is that, even if the pure mathematical solution advocates 100% investment in stocks, to guarantee some money for emergency needs, investors will not invest more than \( b \) in stock, and they will keep some money in (riskless) bonds. If we were to introduce borrowing, we would still have an upper bound on \( X \), set by the bank. In this case we would have \( b > 1 \).
Equilibrium

As the number of shares in the market, denoted by \( N \), is assumed to be fixed, the collective demand function determines the equilibrium price \( P_h^* \). \( P_h^* \) is given by the intersection point of the aggregate demand function and the supply function, which is a vertical line\(^4\). Thus, the equilibrium price of the stock at time \( t + 1 \), denoted by \( P_{t+1} \), will be \( P_h^* \).

History Update

The new stock price, \( P_{t+1} \) and dividend \( D_{t+1} \), give us a new return on the stock, \( H_{t+1} \).

\[
H_{t+1} = \frac{P_{t+1} - P_t + D_{t+1}}{P_t}
\]

We update the stock's history by including this most recent return, and eliminating the oldest return \( H_{t-k+1} \) from the history.

This completes one time cycle. By repeating this cycle, we simulate the evolution of the stock market through time.

1.1.2. Noise. — The model described so far is deterministic. The decision making process is conducted by maximizing the expected utility. \( \) It is a bounded rational, predictable decision making [18–20]. In more realistic situations, investors are influenced by many factors other than rational utility maximization [21]. The net effect of a large number of uncorrelated random influences is a normally distributed random influence or "noise". Hence, we take into account all the unknown factors influencing decision making by adding a normal random variable to the optimal investment proportion\(^5\). To be more specific, we replace \( X(i) \) with \( X^*(i) \) where

\[
X^*(i) = X(i) + \varepsilon(i)
\]

and \( \varepsilon(i) \) is drawn at random from a normal distribution with standard deviation \( \sigma \). We should emphasize that \( X(i) \) is the same for all investors, but \( X^*(i) \) is not because \( \varepsilon(i) \) is drawn separately for each investor. Thus, the noise, which is investor-specific, is the first factor introduced to induce investor heterogeneity.

1.2. HETEROGENEOUS INVESTOR MODELS. — We contrast the homogeneous investor model to models in which the investors may be diverse. First, we study the case where investors differ in their expectations. We consider investors with different memory spans of past returns leading to different expectations regarding future returns. Next we consider diversity with respect to preference. We take investors with the following utility function:

\(^{\text(4)\text{The model can easily be modified to incorporate new stock issues. Note that also it is assumed that accumulated dividends and interest is reinvested. This is not necessary and the model can be easily modified to allow consumption or additional saving which is invested in the market.}}\)

\(^{\text(5)\text{Investor-specific noise can be introduced into the system in a variety of other ways. One possibility is to assume asymmetric information in the market, (i.e., a different interpretations of the stock’s history by different individuals).}}\)

\(^{\text(6)\text{If } X^*(i) < a \text{ or } X^*(i) > b, \text{ we disregard this choice of } \varepsilon(i), \text{ and we draw another } \varepsilon(i) \text{ instead. Thus } X^*(i) \text{ is normally distributed around } X(i) \text{ with standard deviation } \sigma, \text{ but it is truncated at } a \text{ and at } b.}\)
\[ U_i(W) = \frac{W^{1-\alpha_i}}{1 - \alpha_i}, \]

where the \( i \)th investor is characterized by the risk aversion measure \( \alpha_i \). This function, as well as the log function, are myopic utility functions which would dictate a fixed investment proportion in stocks for a given distribution of rates of return. This is not the case in our model as the distribution (or History) changes every trading round, hence the investment proportion also changes.

We also consider the case of differences in the strategies employed by investors. The investors considered so far update their expectations and investments at every trade point. We now introduce investors who go for the long run and disregard trends. We take the extreme case of the investors who keep their investment proportion in the stock constant, regardless of the history.

In the next section we describe the results of our simulations, and the effects of these different types of heterogeneity on market dynamics.

2. Data and Results

In the simulations described in this paper, we choose the time period between each trade to be one year. Accordingly, we chose the rest of the parameters realistically. We take the annual interest rate to be 4%. The initial History, consists of a discrete distribution of returns with a mean of 4.15% and a standard deviation of 0.3%. With these parameters, the investment proportion in the risky asset in the first round is about 50%, thus the bond and stock are more or less compatible in the initial stage.

The number of investors is 100 and the number of outstanding shares is 10,000. The initial wealth of each investor is $1,000. The initial share price is $4.00(^7). The initial dividend is taken to be $0.20 (dividend yield 5%). We increase the dividend by 5% annually(^8). This is close to the long run average dividend growth rate of the S&P.

We should stress that our results are general and that there was no fine-tuning of the parameters. The initial conditions do not affect the main features of the dynamics.

Let us first examine the dynamics of the homogeneous investor market. Figure 1 plots the price of the stock as a function of time (in years) in a market of homogeneous investors with memory span \( k = 15 \) and logarithmic utility functions. Two cases are shown: one with perfectly rational investors (i.e., no noise, \( \sigma = 0 \)) and the other with a moderate degree of noise or irrationality (i.e., \( \sigma = 0.2 \)). We begin by discussing the no-noise dynamics.

In this case all investors are completely homogeneous and will therefore have identical demand functions. As a result, the shares will always be divided equally between all investors. Investors always end up holding the same number of shares: the total number of shares in the

(^7) In the initial stage we ensure that the total wealth is greater than the market value of the stock. As we do not allow borrowing, we preclude a situation where the market value of the stocks (at $4.00 per share) is greater than the total wealth of all investors which would contradict equilibrium. The initial price is determined arbitrarily as if there were no history at all to determine the price, as in the case of an initial public offering. However, we can reach the same results by assuming that the initial price is determined by some history, and then induce some initial shock in the system. The shock can take the form of a change in the wealth, a change in the interest rate, etc. After the shock, a new history is created and the process is exactly as given in this paper.

(^8) We also studied the effects of fluctuating dividends. We do not report these results in this paper for the sake of brevity.
Fig. 1. — Stock price as a function of time with and without investor-specific noise (100 investors with logarithmic utility functions and memory span 15).

Market divided by the number of investors. Thus, shares do not change hands, and we have a no trade market.

We see that the stock price increases sharply at first and, thereafter, it increases at a steady exponential rate (a straight line on the semi-logarithmic scale). Let us first focus on the sharp initial increase. The rate of return on the stock from the first trade is relatively high. This creates a distribution of returns that is "better" than the initial History, where "better" means that investors are willing to increase their investment proportion in equity. Changes in the investment proportions, especially near the maximum investment proportion allowed, induce dramatic changes in the stock price, as demonstrated in Appendix A(9).

After this sharp initial increase, the stock's History of returns becomes very attractive, and the investment proportions in the risky asset are fixed at the maximum. As shown in Appendix B, under the condition of homogeneous investors and fixed investment proportions, the long run capital appreciation becomes:

\[
P_{t+1} = \begin{cases} 
1 + g & \text{if } g > r \\
1 + r & \text{if } g \leq r
\end{cases}
\]

where \( g \) is the constant growth rate of the dividend. This explains the constant slope of the exponential climb. In our case \( g = 5\% \), \( r = 4\% \) and the slope becomes 1.05. The 5% annual growth rate of the stock price that we obtain is realistic. The compound annual growth rate of the S&P due to capital appreciation during the years 1925-1992 was 5.4% (source: Ibbotson

(9) If the rate of return on the stock from the first trade is low, the process is opposite and we observe on a sharp initial decrease in stock price. As will become evident, the initial behavior is irrelevant to the long run dynamics (see reference [22]).
The reason we reach very high stock prices is that we are looking at a very long time horizon (200 years). See also Figure 13.

Now let us turn to the case with noise. Again we see a sharp initial increase in the stock price. However, now the sharp increase stops at a lower price\(^{(10)}\). After the sharp increase the price increases at the same steady exponential rate as in the zero noise case. However, as opposed to the smooth increase of the zero noise case, now the price fluctuates around the trend. This is a result of the noise. The noise induces fluctuations in the investment proportions which in turn induce these price fluctuations. These fluctuations are also the cause of the crash that occurs around time 60. In order to understand this crash we must first note that one of the consequences of the results of Appendix B is, that in no noise case the History of returns on the stock becomes homogeneous:

$$H_t = \frac{P_t - P_{t-1} + D_t}{P_{t-1}} \rightarrow g + (1 + g) \frac{D_0}{P_0} = g + \frac{D_1}{P_0} \quad (g > r)$$

where \(D_1\) is the first year dividend. When a moderate degree of noise is introduced the returns will fluctuate around this value, but will remain relatively homogeneous. Because of this homogeneity, a small change in one of the returns can induce a dramatic change in the investment proportions.

Let us elaborate. The investor's terminal wealth, \(W\), is given by:

$$W = X(1 + \tilde{H}) + (1 - X)(1 + r) = 1 + X(\tilde{H} - r) + r$$

where \(\tilde{H}\) is the random return on the stock. The expected utility is given by:

$$EU(W) = EU \left[ 1 + X(\tilde{H} - r) + r \right].$$

Expand \(EU(W)\) by the Taylor series around the mean \((1 + X(E(\tilde{H}) - r) + r) \equiv EW\) to obtain:

$$EU(W) = U \left[ 1 + X(E(\tilde{H}) - r) + r \right] + U'(EW)E[W - E(W)] + \frac{U''}{2}(EW)\sigma_w^2 + ..$$

Because the second term vanishes, we have:

$$EU(W) = U \left[ 1 + r + X(E(\tilde{H}) - r) \right] + \frac{U''}{2}\sigma_w^2 + ..$$

We have seen that \(H_t \rightarrow g + (1 + g)\frac{D_0}{P_0}\) (a constant number). Therefore \(\tilde{H} \rightarrow E(\tilde{H})\) and \(\sigma_w^2\) (not to mention higher moments), will approach zero. Thus, if \(E(\tilde{H}) > r\) the optimum solution will be \(X = 1\) (or in our case \(X = b\)), and, for \(E(\tilde{H}) < r\), the optimum solution will be \(X = 0\) (or \(X = a\)). To see this "discontinuity" property, consider the following example: \(r = 10\%\),

\(^{(10)}\)When there is no noise, the sharp initial increase stops when the investment proportions in the stock reach the maximum (i.e., \(X = 0.99\)). The noise generally widens the distribution of investment proportions. When the investment proportions are near the maximum, the noise can only drive them down (see footnote (6)). Thus, when optimal diversification advocates maximal investment in stocks, the noise has the effect of lowering the average investment proportion. This is the reason the sharp initial increase stops at a lower price when noise is introduced.
and the return on the stock has two possible values, 8% and 12%, with equal probability. In this case, maximization of the expected logarithmic utility yields \( X = 0 \). A slight increase in one of the possible returns on the stock from 12% to 12.1% is enough to push the proportion of investment to the other extreme, \( X = 1 \). Thus, a small change in the investment proportion (due to the noise) may create a small change in the return on the stock which will change the history so that \( E(\dot{H}) \) becomes smaller than \( r \). This will cause a sharp decrease in \( X \) resulting in a crash. This is similar to the market discontinuity described in reference [23]. This result is robust and does not depend on the assumed preference.

Fluctuations in the investment proportions induce price fluctuations, and in turn, fluctuations in the return on the stock. One of the fluctuations is big enough to cause a discontinuous crash. This happens first at around time 60. Once the price is down, the dividends become significant again in their contribution to the returns on the stock, and the returns are high. After 15 periods, when the catastrophic return resulting from the crash is beyond the memory span of the investors, the stock price jumps back up. This cycle repeats itself very regularly with a period of 30 trades (15 highs, 15 lows).

Due to the investor-specific nature of the noise, not all investors behave in the same way in spite of the homogeneity of preference, memory span, and strategy. Hence, there will be trade in the market and investors will become diverse in respect to their wealth.

How is the wealth distributed across investors throughout the run? The answer to this question is shown in Figure 2. The horizontal axis depicts the wealth as a percentage of the average wealth (for a given time period), and the vertical axis depicts the number of investors with that wealth. As can be seen, at time 1, the distribution is very narrow — almost all investors have the same wealth — the average wealth. Over time, the distribution widens — the random element in the decision making process, \( \varepsilon \), affects individual investors differently.

![Wealth Distribution](image)

**Fig. 2.** Wealth distributions at different times. This data corresponds to the simulation depicted in Figure 1 (100 investors with logarithmic utility functions and memory span 15. \( \sigma = 0.2 \)).
and they become richer or poorer than the average (see time 20). Some investors are "lucky" — they happen to buy stocks before the booms and sell before the crashes whereas the "unlucky" investors do the opposite. As would be expected, the wealth distribution in this case assumes the form of a normal distribution with a standard deviation that increases over time.

How is the behavior of the market affected by the memory span of the investors? The dynamics of a market of homogeneous investors with logarithmic utility functions and memory span 5 is shown in Figure 3. The dynamics is very similar to that depicted in Figure 1, except that the cycle is reduced to 10 time periods because of the shorter memory span (5 highs, 5 lows).

We have also studied the case where the investors' expectation, that each of the last $k$ returns on the stock has an equal probability of $1/k$ to reoccur, is replaced with the expectation that the probability of reoccurrence is exponentially decaying with time, i.e.:

$$P(H_{t+1} = H_j) \propto e^{-\frac{(t+1-j)}{\tau}} \quad j = t, t - 1, \ldots, t - k + 1.$$  \hspace{1cm} (15)

where $\tau$ is a time scale constant. The results of this simulation with $\tau = 4.0$ are depicted in Figure 4. The cycles are shorter because of the shorter effective memory, however, the cyclic booms and crashes persist.

What can we learn from Figures 1, 3 and 4? First, with perfectly rational homogeneous investors, we obtain very unrealistic results: the stock price increases continuously at a constant rate (the straight line in Fig. 1). Furthermore, as all investors are assumed to be identical in all respects, they all have identical demand functions and we have a no trade market.

When we introduce investor-specific noise, there will be trade but we still obtain unrealistic results: booms and crashes with cyclical regularity occur, hence they are fully, or almost fully, predictable. This is in complete contradiction to market efficiency. Moreover, changing the
memory span to five periods \((k = 5)\) does not change anything except for the length of the cycle. The introduction of exponentially decaying reoccurrence of returns probability leads to shorter cycles of booms and crashes, but does not make the dynamics more realistic.

Thus, the assumption of a representative investor leads from reasonable microfoundations to totally unreasonable macroeconomic results. The introduction of investor-specific noise does not help (for further details on the effects of varying degrees of noise see [22]). We now turn to analyze the effects of other sources of heterogeneity.

The first case of non-homogeneous investors studied was that of a market with two types of investors: investors with memory span 15 and investors with memory span 5. All investors were assumed to have logarithmic utility functions and are divided equally between these two groups. The results are shown in Figure 5. As can be seen, what we have here is clearly not a simple superposition of Figure 1 and Figure 3. The dynamics resulting from the interaction between these two investor types is very complex, perhaps even chaotic. We still see booms and crashes, however, they are much less predictable. Thus, heterogeneous memory spans induce more realistic results than those observed in Figures 1 and 3. It is interesting to note that around time 100, there is a transition from cycles of about 10 periods (corresponding to a 5-period memory span) to longer cycles of about 30 periods (corresponding to a 15-period memory span). This can be partially understood from Figure 6. In Figure 6 we see that although the two investor groups start with the same wealth distribution, after a number of time periods they form two distinct wealth classes. The population with the larger memory span dominates most of the money in the market. This is probably the reason for the transition to longer cycles.

We continued to make investors more heterogeneous and for each investor in the market, we chose a random memory span between 1 and 20 (utility functions are still all logarithmic). The
Fig. 5. — Stock price as a function of time resulting from two equal investor groups: one of investors with memory span 5 and the other investors with memory span 15 (A total of 100 investors all with logarithmic functions, \( \sigma = 0.2 \)).

Fig. 6. — Wealth distribution of two investor groups \((k = 5, k = 15)\) at different times. This data correspond to the simulation presented in Figure 5.
Fig. 7. — Stock price as a function of time. The memory span of each investor was chosen randomly, $1 \leq k \leq 20$ (100 investors with logarithmic utility functions, $\sigma = 0.2$).

Fig. 8. — Stock price as a function of time resulting from two equal investor groups: one of investors with $\alpha = 0.5$ and the other of investors with $\alpha = 3.5$. (A total of 100 investors, all with memory span 15, $\sigma = 0.2$.)
simulation results are presented in Figure 7. The general trend of transition to longer cycles persists but it occurs later and becomes more complex.

We then studied a second form of heterogeneity — diversity of preference. We took two types of investors: investors with risk aversion measure $\alpha = 0.5$, and investors with $\alpha = 3.5$. (see Eq. (8)). All investors had memory span $15$. Figure 8 shows that the ordered cycles are still evident, with period $30$, as expected. Thus, heterogeneity of preference alone does not lead to reasonable market behavior. Although it looks as if we have the dynamics of a homogeneous investor market, Figure 9 shows that there is a power struggle between the two investor groups. Investors who are less risk averse ($\alpha = 0.5$) became on average wealthier than the risk averters ($\alpha = 3.5$).

In Figure 10 we show the results of a simulation on a market with twofold diversity for each investor: the memory span was chosen randomly ($1 \leq k \leq 20$) and the preference was chosen randomly ($0 < \alpha_t < 4$). It would appear that the more diverse the investors, the more complex and realistic the market.

We further diversified our market model by introducing investors with "long run" investment strategies, keeping their investment proportion in the stock fixed at $20\%$. The rest of the investors we took as in Figure 10 with memory span and preference chosen randomly. The simulation results are presented in Figure 11. We see that the "long runners" have a stabilizing effect on the market, in that the fluctuations are smaller compared with Figure 10. Figures 12 and 13 show the underlying tug-of-war between the "trend followers" and the "long runners". For a general impression we compared the total returns in the simulation presented in Figure 11 with historical data on the S&P 500 index. This is presented in Figure 14.
Fig. 10. — Stock price as a function of time in a market with two-fold heterogeneity; \( k \) and \( \alpha \) are chosen randomly for each investor, \( 1 \leq k \leq 20, \, 0 < \alpha < 4; \, \sigma = 0.2 \).

Fig. 11. — Stock price as a function of time in a market with 50% of investors holding a constant investment proportion in the stock \((X = 0.2)\), and the rest of the investors with random \( k \) and \( \alpha \), \( 1 \leq k \leq 20, \, 0 < \alpha < 4. \, \sigma = 0.2 \).
Fig. 12. — Wealth distribution corresponding to the simulation presented in Figure 11, Time 40.

Fig. 13. — Wealth distribution corresponding to the simulation presented in Figure 11, Time 140.

3. Summary

In this paper we suggest microscopic models of the stock market. We employ microscopic simulation to study the macroscopic market behavior resulting from different microscopic models.
Fig. 14. — Comparison of the accumulated total return on the S&P 500 index during 1926-1980 to the accumulated total return on the stock in the simulation presented in Figure 11.

We focus on the difference between the macroeconomic behavior resulting from representative investor models and heterogeneous investor models.

We analyzed the following forms of investor heterogeneity: investor-specific noise, heterogeneous preference, heterogeneous expectations and heterogeneous investment strategies. The main findings of this paper are:

1) With homogeneous investors and no investor-specific noise the stock price increases at a constant rate and is fully predictable. There is no trade taking place in the market.

2) When investor-specific noise is introduced, periodic (and therefore predictable) booms and crashes in stock price are obtained, in gross contradiction to market efficiency.

3) The introduction of heterogeneous preference (investors differ in their risk aversion) does not affect the periodic booms and crashes.

4) Only when heterogeneous expectations are introduced, the periodic booms and crashes disappear, and more realistic results are obtained. Combinations of the different forms of heterogeneity produce the most realistic macroscopic results. The more diverse the investors, the more complex and realistic the resulting market behavior.

Our results are in sharp contrast to the results implied by models that rely on the representative investor assumption. Microscopic simulation allows us to relax this assumption and to study the stock market realistically. We believe that microscopic simulation holds great potential for new insights into the complex behavior of the stock market.
Acknowledgments

The authors acknowledge helpful comments by Jacob Bergman. H.L. is grateful to the Krueger center for Finance for financial support.

Appendix A

In this appendix we show that a moderate change in the investment proportion in the stock dramatically affects the stock price. For simplicity, let us assume only one investor with an unchanging number of stocks ($N$). This is identical to a market with many homogeneous investors and no noise because, in such a market, all investors act identically. Let us denote the initial wealth of the investor by $W_0$ and his/her initial investment proportions by $X_0$.

We have:

$$P_0 = \frac{X_0 W_0}{N}$$

(A.1)

or

$$W_0 = \frac{N P_0}{X_0}$$

(A.2)

Let $P_1$, $X_1$, $W_1$ be the new values after the trade. We have:

$$W_1 = W_0 + N(P_1 - P_0)$$

(A.3)

(For simplicity, disregard accumulated dividends and interest) and

$$W_1 = \frac{N P_1}{X_1}$$

(A.4)

substituting $W_1$, we obtain:

$$\frac{N P_1}{X_1} = \frac{N P_0}{X_0} + N(P_1 - P_0)$$

(A.5)

or

$$\frac{P_1}{P_0} = \frac{(1/X_0 - 1)}{(1/X_1 - 1)}.$$  

(A.6)

Take, for example, $X_0 = 0.50$, $X_1 = 0.60$ to obtain:

$$\frac{P_1}{P_0} = 1.50.$$

namely, an increase of 0.1 in the investment proportion (from 0.5 to 0.6) leads to a 50% increase in the stock price. If we take $X_0 = 0.85$, $X_1 = 0.95$:

$$\frac{P_1}{P_0} = 3.34$$

When the initial investment proportion is higher, the same increase in the investment proportion leads to a much sharper price increase.
Appendix B

In this Appendix we show that for homogeneous investors holding fixed investment proportions in the bond and in stocks, the following holds in the long run: if \( g > r \) the stock price grows at \( g\% \) a year, if \( r \geq g \), it grows at \( r\% \) a year, where \( g \) is the annual growth in dividends and \( r \) is the riskless interest rate.

First, recall that in the homogeneous case there is no trade. If an investor receives an additional amount of money (cash flow in the form of interest and dividends), and if he/she wishes to diversify this cash flow between stocks and bonds at some fixed proportions, he/she can not buy stock, because all the investors wish to buy the stocks and no one wishes to sell them. Hence, if money is not withdrawn for consumption (which is the assumption in our model), all the additional cash flow can be invested only in bonds, which are exogenous to our model. However, to keep the constant investment proportions in the bond and the stock, there will be a demand for the stock, its price will go up until in equilibrium the proportion of investment in the bond and in the stock is restored. Thus, if we are able to calculate the growth rate of the money held in bonds, we can deduce the growth rate of the stock price, which must be the same rate, otherwise the proportions held do not remain constant.

As no stocks are bought or sold, all income gains are invested in the bond. The value of the bond in the investors’ portfolio at time \( t \), denoted by \( B_t \), is given by:

\[
B_t = B_0(1 + r)^t + \sum_{j=1}^{t} D_0(1 + g)^{j}(1 + r)^{t-j}
\]  

(B.1)

where \( B_0 \) is the value of the bond at time 0, and \( D_0 \) is the dividend at time 0. The first term represents the compounded interest on the bond. The second term represents dividends that are reinvested in the bond.

The annual growth rate of bonds held in the portfolio is:

\[
\frac{B_{t+1} - B_t}{B_t} = B_0(1 + r)^{t+1} - B_0(1 + r)^{t} + \frac{(1 + g)D_0}{(1 + r)^{2}(r - g)} [(1 + r)^{t+1} - (1 + g)^{t+1} - (1 + r)^{t} + (1 + g)^{t}]
\]

\[
B_0(1 + r)^{t} + \frac{(1 + g)D_0}{(1 + r)^{2}(r - g)} [(1 + r)^{t} - (1 + g)^{t}]
\]

Let us first assume that \( g > r \). Divide all terms by \( (1 + g)^{t} \) to obtain:

\[
\lim_{t \to \infty} \frac{B_{t+1} - B_t}{B_t} = \frac{-(1 + g) + 1}{-1} = g
\]

In the case that \( r > g \), divide all terms by \( (1 + r)^{t} \) to obtain:

\[
\lim_{t \to \infty} \frac{B_{t+1} - B_t}{B_t} = \frac{B_0r + \frac{(1 + g)D_0}{(1 + r)^{2}(r - g)} \times r}{B_0 + \frac{(1 + g)D_0}{(1 + r)^{2}(r - g)}} = r
\]

Finally, if \( r = g \) the cash flow from dividends becomes:
\[
\sum_{j=1}^{t} D_0 (1 + g)^j (1 + r)^{t-j} = t[D_0 (1 + r)^t]
\]

and the growth rate:
\[
\frac{B_{t+1} - B_t}{B_t} = \frac{B_0 (1 + r)^{t+1} - B_0 (1 + r)^t + D_0 (t+1)(1 + r)^{t+1} - D_0 (1 + r)^t}{B_0 (1 + r)^t + D_0 t(1 + r)^t}
\]

divide all terms by \(t(1 + r)^t\) to obtain the long run growth rate:
\[
\lim_{t \to \infty} \frac{B_{t+1} - B_t}{B_t} = \frac{(1 + r) - 1}{1} = r
\]

With the assumption of fixed investment proportions the growth rate of stock price must be the same as the growth rate of the bond’s value in the portfolio. Thus, in the long run, we obtain:
\[
\frac{P_{t+1} - P_t}{P_t} = \begin{cases} 
g & \text{if } g > r \\
r & \text{if } g \leq r \end{cases}
\]

(B.2)

References