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Brownian Motion Ensembles and Parametric Correlations of the Transmission Eigenvalues: Application to Coupled Quantum Billiards and to Disordered Wires

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Abstract. — The parametric correlations of the transmission eigenvalues $T_i$ of a $N$-channel quantum scatterer are calculated assuming two different Brownian motion ensembles. The first one is the original ensemble introduced by Dyson and assumes an isotropic diffusion for the $S$-matrix. We derive the corresponding Fokker-Planck equation for the transmission eigenvalues, which can be mapped for the unitary case onto an exactly solvable problem of $N$ non-interacting fermions in one dimension with imaginary time. We recover for the $T_i$ the same universal parametric correlation than the ones recently obtained for the energy levels, within certain limits. As an application, we consider transmission through two chaotic cavities weakly coupled by a $n$-channel point contact when a magnetic field is applied. The $S$-matrix of each chaotic cavity is assumed to belong to the Dyson circular unitary ensemble (CUE) and one has a $2\times$ CUE → one CUE crossover when $n$ increases. We calculate all types of correlation functions for the transmission eigenvalues $T_i$ and we get exact finite $N$ results for the averaged conductance ($g$) and its variance ($\langle g^2 \rangle$), as a function of the parameter $n$. The second Brownian motion ensemble assumes for the transfer matrix $M$ an isotropic diffusion yielded by a multiplicative combination law. This model is known to describe a disordered wire of length $L$ and gives another Fokker-Planck equation which describes the $L$-dependence of the $T_i$. An exact solution of this equation in the unitary case has recently been obtained by Beenakker and Rejaie, which gives their $L$-dependent joint probability distribution. Using this result, we show how to calculate all types of correlation functions, for arbitrary $L$ and $N$. This allows us to get an integral expression for the average conductance which coincides in the limit $N \to \infty$ with the microscopic non linear $\sigma$-model results obtained by Zirnbauer et al., establishing the equivalence of the two approaches. We review the qualitative differences between transmission through two weakly coupled quantum dots and through a disordered line and we discuss the mathematical analogies between the Fokker-Planck equations of the two Brownian motion models.

1. Introduction

It is now rather well established that the quantum energy levels of classically chaotic billiards have Wigner–Dyson correlations [1,2]. These universal correlations are closely associated with the concept of “quantum chaos” and have first been obtained in random matrix theory [3].
(RMT) in a different context (statistical description of the spectral properties of complex nuclei and small metallic particles). Futhermore, many works [4-10] have considered the case where the system Hamiltonian $H$ depends on an external parameter $t$. Instead of a single matrix ensemble defined by a certain probability density, a continuous family of ensembles, defined by a $t$-dependent probability density, is considered. The universality of the level correlations can then be extended to a broader domain, which involves not only (small) level separations at a given $t$, but also (small) external parameter separation $\delta t$. This broader universality has been numerically checked in very different physical systems: e.g., disordered conductors [6], quantum billiards [11], the hydrogen atom in a magnetic field [12] and correlated electron systems [13, 14]. Analytical derivations use perturbation theory [6] or the non linear $\sigma$-model [7]. Another fruitful way to calculate the universal expressions of those parametric level correlations is based on Brownian motion ensembles of random matrices, first introduced by Dyson. The idea is to assume that the $N \times N$ Hamiltonian matrix diffuses when one varies the parameter $t$ in the available manifold given its symmetries. The $t$-dependence of the level distribution is then determined by a Fokker-Planck equation [8, 9], which can be solved in the large $N$-limit for arbitrary symmetries and for finite $N$ in the unitary case (Sutherland method).

For quantum transport in the mesoscopic regime, the system is not closed, but open to electron reservoirs, and becomes a $N$-channel scatterer with transmission eigenvalues $T_i$. Many transport properties [15] (conductance, quantum shot noise...) are linear statistics of the $N$ eigenvalues $T_i$, which are the appropriate levels for a scattering problem. The scattering matrix $S$ or transfer matrix $M$, suitably parametrized by the $N$ transmission eigenvalues $T_i$ and certain auxiliary unitary matrices, are the relevant matrices for which RMT approaches have been formulated. For instance, if the unitary $S$-matrix is distributed according to one of Dyson's classical circular ensembles [22], the corresponding transmission eigenvalue distributions have been recently obtained [15, 16], exhibiting the same logarithmic pairwise repulsion than for the energy levels, and hence giving essentially the same Wigner-Dyson correlations for the transmission eigenvalues. This RMT description is mainly relevant for quantum transport through ballistic chaotic cavities, which is a recent field of significant experimental investigations. Such cavities can be made with semiconductor nanostructures known as quantum dots [17] with a few channel contacts to two electron reservoirs. The validity of this RMT description is confirmed by microscopic semiclassical approaches [18] and by numerical quantum calculations [16, 19].

The universality of the transmission eigenvalue correlations are established for a given $t$, and therefore we want to know if it can be extended to their parametric dependence, too, as for the energy levels. Futhermore, we want to see if these parametric correlations obey a similar behavior than that of the energy levels. This is one of the issues which we address in this work, using two Brownian motion ensembles for the transmission eigenvalues, which result from two different assumptions concerning the $t$-dependence of $S$ ($S$-Brownian motion ensemble) or of $M$ ($M$-Brownian motion ensemble). This issue has also been recently considered [20, 21].

The $S$-Brownian motion ensemble is simply the original ensemble introduced by Dyson [4, 5] for the scattering matrix $S$

$$S = \begin{pmatrix} t & t' \\ t & t' \\ \end{pmatrix}$$

The idea is to assume that $S$ diffuses in the $S$-matrix space with respect to a fictitious time $t$ of the Brownian motion, and converges in the limit $t \to \infty$ to a stationary probability distribution given by one of the well known circular ensembles: circular unitary ensemble (CUE, $\beta = 2$) when there is no time reversal symmetry (applied magnetic field); circular orthogonal ensemble (COE, $\beta = 1$) when there are time reversal and spin rotation symmetries (no spin–orbit coupling or applied magnetic field) and circular symplectic ensemble (CSE,
\[ \beta = 4 \] otherwise. Choosing an initial probability distribution at \( t = 0 \) of higher symmetry than the stationary limit, e.g., two independent circular ensembles diffusing towards a single one, one can perform with the parameter \( t \) the crossover from the initial ensemble to the stationary ensemble. Actually, the Brownian motion ensembles COE \( \rightarrow \) CUE, CSE \( \rightarrow \) CUE, and \( 2 \times \text{CUE} \rightarrow \) CUE have been investigated \([24, 25]\) in terms of the scattering phase shifts \( \theta_n \). Semiclassical and numerical justifications of these models are given in reference \([25]\). The decisive difference between our approach and references \([24, 25]\) is that we do not consider the usual eigenvector–eigenvalue parametrization of \( S \), but the parametrization which explicitly uses the \( N \) transmission eigenvalues \( T_n \) of \( tt^\dagger \) as previously introduced \([15]\). Since there exists neither a simple mathematical method, nor an intuitive way to relate the \( 2N \) eigenvalues \( e^{i\theta_n} \) of \( S \) with the \( N \) transmission eigenvalues \( T_n \), it is justified to reconsider the Brownian motion in terms of the latter. This parametrization is suitable to determine the “time” dependence of the \( T_n \) by a Fokker–Planck equation which we map, using a transformation introduced by Sutherland, onto a Schrödinger equation with imaginary time. In the case of the stationary CUE limit, one has a problem of \( N \) non interacting fermions which we solve exactly for arbitrary \( N \). One obtains for the corresponding parametric correlations of the transmission eigenvalues the same universal form than for the energy levels, after an appropriate rescaling and within certain limits.

A physical application of such a parametric ensemble can for example be realized by two weakly coupled ballistic (irregular) cavities, each of them connected to one electron reservoir (Fig. 1). The crossover between two CUE towards a single CUE is performed by changing the strength of the coupling from nearly uncoupled cavities (two CUE) to ideally coupled cavities (one CUE), when a small constant magnetic field is applied. The transmission eigenvalue parametrization is very well adapted to this case since the initial condition is easily realized by the requirement \( T_n = 0 \). Another application of the Dyson Brownian motion ensemble, concerning the crossover COE \( \rightarrow \) CUE in terms of the \( T_n \), is presented elsewhere \([26]\).

Outside the problem of two weakly coupled quantum dots, we are also motivated by numerical calculations \([23]\) showing that the scattering matrix of a quasi one-dimensional disordered wire exhibits also some properties of the circular ensembles, in the sense that the correlation of the
scattering phase shifts $\theta$, were found to be approximately described by the universal correlations functions of these ensembles. For the disordered wire, a crossover from one circular ensemble towards two independent circular ensembles was observed as the length of the wire exceeds its localization length. The appropriate statistical description \cite{27-31} of the transmission eigenvalues of disordered wires is given by a different Brownian motion ensemble than the original Dyson model, and results from the multiplicative combination law of the transfer matrix $M$. This is what we call the $M$-Brownian motion ensemble which yields another Fokker-Planck equation taking into account the quasi one-dimensional structure. This Fokker-Planck equation has been solved by Beenakker and Rejaei \cite{33} in the unitary case, using the Sutherland transformation. A remarkable result which they found is that the transmission eigenvalue pairwise interaction is universal, in the sense that it does not contain any adjustable parameter. It coincides with the standard logarithmic RMT repulsion for small eigenvalue separations, and it is halved for larger separations. The physical interpretation of this halving is unclear, but must be somewhat related to the statistical decoupling of the reflection properties of the two opposite system edges. If it is right, the Dyson Brownian motion ensemble must also exhibit in the crossover regime $2\times \text{CUE} \rightarrow 1\times \text{CUE}$ a somewhat similar pairwise interaction for weak transmission. It is therefore an interesting idea to compare those two different Brownian motion ensembles for the transmission eigenvalues. Let us mention the obvious difference between them: their initial and stationary limits are inverse of each other. For the $S$-matrix model, we start from no transmission towards a good CUE transmission. For the $M$-matrix model on the contrary, the initial condition corresponds to perfect transmission, and the system evolves with the time $t$ (here to be identified with conductor length $L$) to the zero transmission localized limit.

This paper is composed of four main sections. First, we introduce the transmission eigenvalue parametrization of $S$ and define the $S$-Brownian motion ensemble (Section 2.1). The corresponding Fokker-Planck equation is derived in Appendix A. In Section 2.2, the Fokker-Planck equation is solved for an arbitrary initial condition in terms of fermionic one particle Green's functions, using the method introduced by Beenakker and Rejaei for the $M$-Brownian motion ensemble. In Section 2.3, we show how to recover for the transmission eigenvalues the same universal parametric correlations than for the energy levels, after rescaling and within certain limits.

In Section 3, we treat completely the system of two coupled ballistic cavities by the $S$-Brownian motion ensemble for the unitary case. The time of the Brownian motion is given in terms of the numbers of channels in the contacts to the electron reservoirs ($N$) and in the contact between the two cavities ($n$). In Section 3.1, the crossover $2\times \text{CUE} \rightarrow \text{CUE}$ is considered. The method of orthogonal polynomials, well-known in the framework of random matrix theory \cite{3}, is extended to a more general crossover situation and yields the correlation functions as determinants (Eq. (3.18)) containing a function $K_N(x,y;t)$ given as a sum of Legendre polynomials (Eq. (3.21)). This Section contains with equations (3.27-3.30) also exact results for the average and the fluctuations of the conductance. In Section 3.2, we show that the transmission eigenvalue interaction deviates from the pure RMT logarithmic interaction, in the limit of weakly coupled cavities. One can note some interesting similarities with the pairwise interaction of the $M$-Brownian motion model given in reference \cite{33}.

The third part of this paper (Section 4) reconsiders the $M$-Brownian motion ensemble of reference \cite{28} for disordered wires in the unitary case. We use the exact expression of the joint probability distribution \cite{33} and calculate the corresponding correlation functions, using a method very similar to the one used in Section 3.1. This is possible because, from a mathematical point of view, the Brownian motion models for $S$ and $M$ are very similar. We get an expression of the $m$-point correlations as a determinant (Eq. (4.20)) with a function
$K_N(\lambda, \tilde{\lambda}; t)$ given as a sum and an integral with Legendre polynomials and Legendre functions (Eq. (4.21)). This allows us to prove the equivalence between the $M$–Brownian motion ensemble and a more microscopic approach, based on a non linear $\sigma$-model formulation and supersymmetry, as far as the behavior of the first two moments of the conductance $\langle g \rangle$ and $\langle g^2 \rangle$ is concerned (Section 4.2). We emphasize that the solution of the Brownian motion ensemble is now complete, contrary to the sigma model approach where one only knows $\langle g \rangle$ and $\langle g^2 \rangle$ in the large $N$-limit. In Section 4.3, some further simplifications valid in the localized limit lead to asymptotic expressions for the density and the two point function for the logarithm of the transmission eigenvalues.

In Section 5, we review some essential differences between the transmission properties characterizing two weakly coupled ballistic chaotic cavities ($S$–Brownian motion ensemble) on one side and an homogeneous disordered wire ($M$–Brownian motion ensemble) on the other side. We give some concluding remarks in Section 6, notably on the mathematical similarities between the $S$ and $M$ Brownian motion ensembles.

2. $S$–Brownian Motion Ensemble for Transmission Eigenvalues

2.1. Parametrization, Definition and Fokker–Planck Equation. — The scattering matrix $S$ of a system connected to two electron reservoirs by two $N$-channel contacts can be described in the transmission eigenvalue parametrization by

$$S = \begin{pmatrix} v_1 & 0 \\ 0 & u_1 \end{pmatrix} \begin{pmatrix} \sqrt{1-T} & i\sqrt{T} \\ i\sqrt{T} & \sqrt{1-T} \end{pmatrix} \begin{pmatrix} v_2 & 0 \\ 0 & u_2 \end{pmatrix} \quad (2.1)$$

where $v_{1,2}, u_{1,2}$ are $N \times N$ unitary matrices and $T$ is a diagonal $N \times N$-matrix with real entries $0 \leq T_{jj} \leq 1$. The transmission matrix is $t$, but we mean by “transmission eigenvalues” $T_j$ the eigenvalues of $t^*t$. This parametrization has recently [15,16] been used to calculate the conductance properties of a ballistic cavity, which is known to be suitably described by one of Dyson’s circular ensembles [22]. In the unitary case ($\beta = 2$) all unitary matrices $v_{1,2}, u_{1,2}$ are independent, whereas for the orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) symmetry classes, one has $v_2 = v_2^D$ and $u_2 = u_2^D$ (with $M^D = M^T$ in the orthogonal case and $M^D = JMTJ^T$, $J = -i\sigma_y$, in the symplectic case). In the following, the transmission eigenvalues $T_j$ are described by angles $\varphi_j \in [0, \pi/2]$ via $T_j = \sin^2 \varphi_j$ and therefore

$$\begin{pmatrix} \sqrt{1-T} & i\sqrt{T} \\ i\sqrt{T} & \sqrt{1-T} \end{pmatrix} = \begin{pmatrix} \cos \varphi & i\sin \varphi \\ i\sin \varphi & \cos \varphi \end{pmatrix} = \exp \begin{pmatrix} 0 & i\varphi \\ i\varphi & 0 \end{pmatrix} \quad (2.2)$$

where $\varphi$ is a diagonal matrix with entries $\varphi_j$. The invariant measure for $S$ in the parametrization (2.1) and the change of variables (2.2) is given by [15,16]

$$\mu(dS) = F_\beta(\varphi) \prod_j d\varphi_j \mu(dw_1) \mu(dw_2) \mu(dv_1) \mu(dv_2) \quad (2.3)$$

with

$$F_\beta(\varphi) = C_\beta \prod_{j<k} |\sin^2 \varphi_j - \sin^2 \varphi_k|^\beta \prod_j (\sin^\beta \varphi_j \cos \varphi_j), \quad \beta = 1, 2, 4 \quad (2.4)$$

In the cases $\beta = 1, 4$ only two products $\mu(dw_1) \mu(dw_2)$ appear.

The Brownian motion ensemble which we consider (cp. Dyson [4]) describes a unitary random matrix which depends on a fictitious time. A small change on the time variable $t \rightarrow t + \delta t$ yields (in the unitary case) for $S$ the change:

$$S(t + \delta t) = S(t)e^{i\delta X}, \quad (2.5)$$
where $\delta X$ is an infinitesimal Hermitian random matrix with averages

$$
\langle \delta X_{ij} \rangle = 0 , \quad \langle \delta X_{ij} \delta X_{kl} \rangle = D \delta t \delta_{ik} \delta_{jl}
$$

and that is independently distributed of $S(t)$. The diffusion constant $D$ will be precised afterwards, depending on the considered physical problem. We will only consider the unitary case. The Brownian motion ensembles for the other cases are more involved and their precise definition can be found elsewhere [4]. Let $p(\varphi, t)$ be the joint probability density for the angles $\varphi_j$ of $S$ at the time $t$, obtained after integration over the unitary matrices $v_{1,2}, u_{1,2}$. This Brownian motion is then characterized by the Fokker-Planck equation:

$$
\frac{\partial p(\varphi, t)}{\partial t} = \frac{D}{4} \sum_j \frac{\partial}{\partial \varphi_j} \left( F_B(\varphi) \frac{\partial}{\partial \varphi_j} \left( F_B(\varphi)^{-1} p(\varphi, t) \right) \right)
$$

(2.7)

with $F_B(\varphi)$ given by equation (2.4). In the Appendix this Fokker-Planck equation is derived for the unitary case $\beta = 2$ which is our main concern.

The details of the derivation in the Appendix show that the Fokker-Planck equation (2.7) remains valid even if the statistics of the matrix $\delta X$ is arbitrarily chosen (including the case of a constant matrix $\delta X$) instead of (2.6). For this to be true, one then needs instead of equation (2.6) the non-trivial assumption that the unitary matrices $v_2$ and $u_2$ that appear in the parametrization (2.1) of $S(t)$ are for each $t$ uniformly distributed on the space of $N \times N$ unitary matrices (i.e., they have always two independent $N \times N$ CUE distributions). The time step $\delta t$ is then determined by equation (A.8) of the Appendix. It is interesting to note that this assumption concerning $u_2$ and $v_2$ is fulfilled at least at $t = 0$ for the two applications concerning the crossovers $2 \times \text{CUE} \rightarrow \text{CUE}$ or $\text{COE} \rightarrow \text{CUE}$ which are treated here or in reference [26], respectively. We assume that this holds also for arbitrary $t > 0$ and the Brownian motion model is applicable to rather general physical situations.

2.2. Solution of the Fokker-Planck Equation for Arbitrary Initial Condition

We want now to solve the Fokker-Planck equation (2.7) for an arbitrary initial condition $p(\varphi, t = 0) = \tilde{p}(\varphi)$ and for $\beta = 2$. We set $D = 4$ for convenience. We proceed in a similar way as in reference [33] and apply the so-called Sutherland transformation [36]:

$$
\tilde{p}(\varphi, t) = (F_B(\varphi))^{1/2} p(\varphi, t) .
$$

(2.8)

The function $\tilde{p}(\varphi, t)$ fulfills a Schrödinger equation with imaginary time:

$$
\frac{\partial \tilde{p}(\varphi, t)}{\partial t} = -\mathcal{H} \tilde{p}(\varphi, t)
$$

(2.9)

where $\mathcal{H}$ is a many-particle Hamilton operator

$$
\mathcal{H} = -\sum_j \left( \frac{\partial}{\partial \varphi_j} \right)^2 + V(\varphi)
$$

(2.10)

with a potential $V(\varphi)$ having the form

$$
V(\varphi) = -\sum_j \frac{1}{\sin^2(2\varphi_j)} \left( \sin^2 \varphi_j + (\beta - 1)(3 - \beta) \cos^2 \varphi_j \right) + \frac{\beta(\beta - 2)}{8} \sum_{j \neq k} \left( \frac{\sin^2(2\varphi_j) + \sin^2(2\varphi_k)}{\sin^2 \varphi_j - \sin^2 \varphi_k} \right) + C_N
$$

(2.11)
and the constant
\[ C_N = -\frac{1}{4}\beta^2 N - \frac{\beta^2}{3} N(N-1)(N-2) - \frac{1}{2} \beta(\beta + 2) N(N-1) \] (2.12)

One can see that the unitary case \( \beta = 2 \) is apparently much simpler, since the complicated many particle Hamilton operator reduces in this case to a sum of independent one-particle Hamiltonians, i.e., \( \mathcal{H} = \sum_j h(\varphi_j) + C_N \) with
\[ h(\varphi_j) = -\left( \frac{\partial}{\partial \varphi_j} \right)^2 - \frac{1}{\sin^2(2\varphi_j)} . \] (2.13)

The eigenvalue problem of this operator can be solved using Legendre polynomials. The properly normalized eigenfunctions \( \psi_n(\varphi_j) \) with eigenvalues \( \varepsilon_n, n = 0, 1, 2, \ldots \), are given by
\[ \psi_n(\varphi_j) = \sqrt{(1 + 2n)\sin(2\varphi_j)} P_n(\cos(2\varphi_j)), \quad \varepsilon_n = (1 + 2n)^2 \] (2.14)

The solution of the Fokker-Planck equation can be expressed by the many particle fermionic Green’s function \( G(\varphi, \varphi; t) \), as already done [33] via
\[ p(\varphi, t) = \int d^N \bar{\varphi} \bar{\rho}(\bar{\varphi}) G(\bar{\varphi}, \varphi; t) \] (2.15)
where
\[ G(\varphi, \varphi; t) = \frac{1}{N!} \rho(\varphi)^{-1} \rho(\varphi) \det(g(\varphi_i, \varphi_j; t)) e^{-C_N t} , \] (2.16)
\[ \rho(\varphi) = \prod_{i<j} (\sin^2 \varphi_i - \sin^2 \varphi_j) \prod_j \sqrt{\sin(2\varphi_j)} , \] (2.17)
\[ g(\varphi_i, \varphi_j; t) = \sum_{n=0}^{\infty} \psi_n(\varphi_i) \psi_n(\varphi_j) e^{-\varepsilon_n t} \] (2.18)

Equation (2.18) describes the one particle Green’s function, defined by
\[ \left( \frac{\partial}{\partial t} + h(\varphi_i) \right) g(\varphi_i, \varphi_j; t) = 0 , \quad g(\varphi_i, \varphi_j; 0) = \delta(\varphi_i - \varphi_j) \] (2.19)

For technical reasons, we will now switch to new coordinates \( x_j = \cos(2\varphi_j) \in [-1, 1] \), which are just the arguments of the Legendre polynomials. Let \( p(x, t) \) be the probability density in terms of the \( x_j \)'s taking into account the Jacobian of this transformation, i.e., \( \prod_j (4 \sin(2\varphi_j)) p(\varphi, t) \rightarrow p(x, t) \). Equations (2.15-2.18) are then replaced by
\[ p(x, t) = \int d^N \bar{x} \bar{\rho}(\bar{x}) G(\bar{x}, x; t) , \] (2.20)
\[ G(\bar{x}, x; t) = \frac{1}{N!} \rho(\bar{x})^{-1} \rho(x) \det(g(x_i, \bar{x}_j; t)) e^{-C_N t} , \] (2.21)
\[ \rho(x) = \prod_{i>j} (x_i - x_j) , \] (2.22)
\[ g(x_i, \bar{x}_j; t) = \sum_{n=0}^{\infty} \frac{1}{2} (1 + 2n) P_n(x_i) P_n(\bar{x}_j) e^{-\varepsilon_n t} \] (2.23)
We have used in (2.20-2.23) the same symbols $p$, $\hat{p}$, $G$, $g$ and $\rho$ for objects which are related with the corresponding objects of (2.15-2.18) via suitable transformations including the corresponding jacobians, and some factors $\sqrt{\sin(2\varphi_j)}$.

One can easily check that $p(x,t)$ in the limit $t \to \infty$ corresponds to the CUE distribution. For this, we need to consider only the first $N$ contributions, i.e., $n = 0, 1, \ldots, N - 1$, in the sum of (2.23). The matrix $g(x_i, x_j; t)$ is then the matrix product of two matrices $P_n(x_i)$, $P_n(x_j)$ and a diagonal matrix with entries $(\frac{1}{2} + n)e^{-\varepsilon_nt}$. Then the determinant of equation (2.21) becomes proportional to the two Vandermond determinants $\rho(x), \rho(\hat{x})$ and to the factor $\exp(-\sum_{n=0}^{N-1} \varepsilon_n t)$ which is just canceled by the contribution $e^{-CNt}$ in equation (2.21). This gives:

$$\lim_{t \to \infty} p(x, t) \propto \rho^2(x) \cdot \int d^N \hat{x} \hat{p}(\hat{x})$$

(2.24)

which corresponds to a stationary solution equal to $F_2(\varphi)$, once the Jacobian of the transformation $x_j = \cos(2\varphi_j)$ has been taken into account. The first correction to the limit (2.24) is proportional to $\exp(-\varepsilon_N - \varepsilon_{N-1})t) = \exp(-8Nt)$, giving the typical time scale $t_c = 1/(8N)$ for the Brownian motion to reach the stationary solution.

2.3. Universal Parametric Correlations and Their Limits. — The universality of the energy level parametric correlation has recently attracted a considerable interest. In mesoscopic quantum physics, this remarkable property obtained after an appropriate rescaling of the variables, was proven by diagrammatic methods [6]. An alternative derivation of this result can be directly applied to our problem: Beenakker and Rejaei have indeed shown [8] how to recover this universal behavior from a Brownian motion model for the Hamiltonian ($H$-Brownian motion ensemble). This derivation can be easily adapted to the $S$-Brownian motion ensemble and to the transmission eigenvalue parametric correlation. As in references [6,8], we do not consider here a crossover regime (i.e., $2 \times$ CUE $\to$ CUE or COE $\to$ CUE, for instance) but just a CUE $\to$ CUE parametric dependence, which reduces to the large $t$-behavior of the crossover of two decoupled $N \times N$ CUE towards a single $2N \times 2N$ CUE considered in Section 3. Our discussion follows very closely the derivation of reference [8], where a parametric dependence on a quantity $X$ was considered, related to the Brownian motion time through the relation $X^2 = 8Nt = t/t_c$. We measure $t$ in units of $8N = t_c^{-1}$ in order to have a stationary distribution proportional to $e^{-x^2}$, and we use again the variables $\varphi$ (\footnote{We think that they are the appropriate coordinates since they have a uniform density in the limit $N \to \infty$.}).

Usually, one considers [6,8] two types of correlation functions which are the density correlation function

$$S(\varphi, X, \varphi', X') = \sum_{i,j} \langle \delta(\varphi - \varphi_i(X)) \delta(\varphi' - \varphi_j(X')) \rangle$$

$$- \sum_i \langle \delta(\varphi - \varphi_i(X)) \sum_j \langle \delta(\varphi' - \varphi_j(X')) \rangle \rangle$$

(2.25)

and the current correlation function

$$C(\varphi, X, \varphi', X') = \sum_{i,j} \langle \hat{\varphi}_i(X) \hat{\varphi}_j(X') \delta(\varphi - \varphi_i(X)) \delta(\varphi' - \varphi_j(X')) \rangle$$

(2.26)

For an arbitrary linear statistic (e.g., conductance, shot noise power, etc.) of the form

$$A(X) = \sum_i a(\varphi_i(X))$$

(2.27)
we can calculate the correlator via

$$\langle \delta A(X) \delta A(X') \rangle = \int_0^{\pi/2} d\varphi \int_0^{\pi/2} d\varphi' a(\varphi) a(\varphi') S(\varphi, X, \varphi', X').$$

(2.28)

The two correlation functions (2.25) and (2.26) are related [6, 8] through

$$\frac{\partial^2}{\partial \varphi \partial \varphi'} C(\varphi, X, \varphi', X') = \frac{\partial^2}{\partial X \partial X'} S(\varphi, X, \varphi', X')$$

(2.29)

In Subsection 3.1, we calculate the probability density $R_{1,1}(x, t; y, t + \tau)$ to find the correlation function at time $t$ and the eigenvalue $y$ at time $t + \tau$. For a given $\tau$ and in the limit $t \to \infty$, the particular information of the initial condition will be lost and we have the density correlation function for the CUE $\to$ CUE parametric dependence:

$$S_N(\varphi, 0, \varphi', X) = \lim_{t \to \infty} 4 \sin(2\varphi) \sin(2\varphi')$$

$$\times \left( R_{1,1}(\cos(2\varphi), t; \cos(2\varphi'), t + \tau) - R_{1,1}(\cos(2\varphi); t) R_{1,1}(\cos(2\varphi'); t + \tau) \right)$$

(2.30)

The parameter $X$ is now related with $\tau$ through $X^2 = \tau/t_c$ and the sinus-prefactors are just the Jacobians due the variable transformation $x = \cos(2\varphi)$. The subscript means that $N$ can have arbitrary values and that the limit $N \to \infty$ has still not been taken. Using the results (3.25), (3.26) and (3.12) derived in the next section, we get directly the exact expression:

$$S_N(\varphi, 0, \varphi', X) = 4 \sin(2\varphi) \sin(2\varphi')$$

$$\times \left( \sum_{n=N}^{\infty} (n + \frac{1}{2}) P_n(\cos(2\varphi)) P_n(\cos(2\varphi')) e^{-\varepsilon_n \tau} \right)$$

$$\times \left( \sum_{k=0}^{N-1} (k + \frac{1}{2}) P_k(\cos(2\varphi)) P_k(\cos(2\varphi')) e^{+\varepsilon_k \tau} \right)$$

(2.31)

where $\varepsilon_n = (1 + 2n)^2$ is the eigenvalue of the one particle Hamiltonian $h(\varphi)$ given by (2.13). In another work [26], we have calculated the function $R_{1,1}$ for the crossover COE $\to$ CUE which differs at small $t$ from the result (3.25) we have used here. However, in the limit $t \to \infty$ both functions become identical and give the expression (2.31), which has in fact the same structure as in equation (5.27) of reference [8]. One has just to replace the harmonic oscillator eigenfunctions used in [8] by the Legendre polynomials appearing in the $S$-Brownian motion.

First, we consider the limit $N \to \infty$ and differences $\delta \varphi = \varphi' - \varphi$ of the order $1/N$, in order to use the large $n$ expansion of the Legendre polynomials:

$$P_n(\cos(2\varphi)) \simeq \sqrt{\frac{2}{\pi n}} \frac{1}{\sqrt{\sin(2\varphi)}} \cos \left( (2n + 1)\varphi - \frac{\pi}{4} \right)$$

(2.32)

and to replace the sums by integrals. After some smoothing over the fast oscillating terms, the correlation function depends only on the difference $\delta \varphi$, i.e.,

$$S(\delta \varphi, X) = \lim_{N \to \infty} S_N(\varphi, 0, \varphi + \delta \varphi, X)$$

$$= \rho_0^2 \int_0^1 ds \int_1^\infty ds' \cos(\rho_0 \pi s \delta \varphi) \cos(\rho_0 \pi s' \delta \varphi) \exp \left( \alpha X^2 (s^2 - s'^2) \right)$$

(2.33)
where \( \rho_0 = 2N/\pi \) is the average density of the variable \( \varphi \) and \( \alpha = N/2 = \pi \rho_0/4 \). The expression (2.33), which is valid in the large \( N \)- and small \( \delta \varphi \)-limits, coincides exactly with the result of reference [8] given by a Brownian motion for the Hamiltonian \( H \) and also with a supersymmetric calculation for the unitary case [7]. The calculations of the current correlation function (2.26) and of the correlator (2.28) are then straightforward and can be found in reference [8]. We note that (2.33) yields an algebraic decay for the correlator of an arbitrary linear statistic as \( X \to \infty \) [8]

\[
\langle \delta A(0) \delta A(X) \rangle \sim \frac{1}{X^4},
\]

which is also found [20] in a simplified Brownian motion for the transmission eigenvalues.

However, we cannot directly use the expression (2.33) to evaluate the integrals in (2.28), due to the restriction \( \delta \varphi \ll 1 \). In fact, we will see in the next section that the conductance correlator decays exponentially in the \( S \)-Brownian motion ensemble. In order to understand this behavior, we consider again the limit \( N \to \infty \), but now without any restriction on \( \varphi \) and \( \varphi' \). We replace in the exact expression (2.31) again the Legendre Polynomials by the asymptotic formula (2.32) but now we keep the discrete sums instead of replacing them by integrals. One can see that this step is very crucial: the variable \( \varphi \) varies between 0 and \( \pi/2 \) and the correlation function should rather be expanded in a discrete Fourier-series instead of a continuous Fourier-integral, being responsible [8] for the algebraic decay (2.34). In the limit \( N \to \infty \) at fixed ratio \( \tau/t_c = X^2 \), the main contributions comes from the terms with \( n - N \ll N \) and \( N - k \ll N \) so that we can linearize the energy difference \( \epsilon_k - \epsilon_n = 4(1 + k + n)(k - n) \approx 8N(k - n) \). In addition, all fast oscillating terms of the type \( \cos(4N(\cdots) + \cdots) \) are omitted. Then, one of the sums is easily done and we obtain for the density correlation function the expression

\[
\lim_{N \to \infty} S_N(\varphi, 0, \varphi', X) = \tilde{S}(\varphi + \varphi', X) + \tilde{S}(\varphi - \varphi', X)
\]

(2.35)

where the function \( \tilde{S}(\varphi, X) \) is given as a Fourier series

\[
\tilde{S}(\varphi, X) = 2 \sum_{n=1}^{\infty} S_n(X) \cos(2n\varphi) = \sum_{n=-\infty}^{\infty} S_n(X) e^{2in\varphi}
\]

(2.36)

with Fourier coefficients

\[
S_n(X) = \frac{2|n|}{\beta \pi^2} e^{-|n|X^2}, \quad \beta = 1, 2, 4
\]

(2.37)

Our derivation accounts of course only for the unitary case \( \beta = 2 \) but equation (2.37) is valid for the orthogonal \( (\beta = 1) \) and symplectic case \( (\beta = 4) \) too [34]. The derivation for all \( \beta \) can be done by an asymptotic expansion of the Fokker-Planck equation [34] similarly as in Section 4 of reference [8] and equation (2.37) is indeed the discrete version of equation (4.14) in reference [8].

We are now able to calculate the correlator for the linear statistics (2.27). If we expand the function \( a(\varphi) \) in a Fourier series

\[
a(\varphi) = \sum_{n=-\infty}^{\infty} a_n e^{2in\varphi}, \quad a_n = a_{-n},
\]

(2.38)

we find

\[
\langle \delta A(0) \delta A(X) \rangle = \pi^2 \sum_{n=1}^{\infty} a_n^2 S_n(X) = \frac{2}{\beta} \sum_{n=1}^{\infty} a_n^2 n e^{-nX^2}
\]

(2.39)
In (2.38), we have extended $a(\varphi)$ to an even function on the interval $-\pi/2 \leq \varphi \leq \pi/2$ so that the Fourier-coefficients are even in $n$ (one should imagine that $a$ is a well defined function of $x = \cos(2\varphi)$).

As an example we consider the conductance $g = A$ with $a(\varphi) = \sin^2(\varphi)$, i.e. $a_0 = 1/2$, $a_{\pm 1} = -1/4$ and $a_n = 0$ if $|n| \geq 2$,

$$
\langle \delta g(0) \delta g(X) \rangle = \frac{1}{8\beta} e^{-X^2} = \frac{1}{8\beta} e^{-t/\tau_c} \tag{2.40}
$$

which is (in the limit $t \to \infty$) consistent with equation (3.30) of the next Section.

The current correlation function is calculated from (2.29) and the assumption that $S$ and $C$ depend only on the difference $X' - X$. We obtain

$$
C(\varphi, 0, \varphi', X) = -\tilde{C}(\varphi + \varphi', X) + \tilde{C}(\varphi - \varphi', X) \tag{2.41}
$$

where $\tilde{C}(\varphi, X)$ has the Fourier coefficients

$$
C_n(X) = -\frac{1}{4n^2} \frac{\partial^2}{\partial X^2} S_n(X) = \frac{1}{\beta \pi^2} \left( 1 - 2|n|X^2 \right) e^{-|n|X^2} \quad \text{for } n \neq 0 \tag{2.42}
$$

The coefficient $C_0(X)$ is not determined and not needed due to (2.41). The modified sign in the first contribution in (2.41) is important because the current correlation function is odd as a function of $\varphi$ (or $\varphi'$) if $\varphi'$ (or $\varphi$) is fixed.

In summary, we have found that the parametric correlations of the $S$-Brownian motion are only locally (on a range $\delta \varphi = \varphi' - \varphi \ll 1$) identical with the universal correlations found by a $H$-Brownian motion [8] or by a microscopic approach in the framework of the zero-dimensional supersymmetric non linear $\sigma$-model [7]. This agreement does not hold if we consider the whole range of $\varphi, \varphi'$. In particular, quantities integrated over the whole spectrum like the correlator of a linear statistic must be treated with more care and indeed the conductance correlator shows an exponential instead of an algebraic decay, in the $S$-Brownian motion ensemble.

3. Application to Two Weakly Coupled Chaotic Dots

The $S$-Brownian motion ensemble with CUE stationary limit can be applied to a system made by two quantum dots in series under an applied magnetic field, for describing how its transmission eigenvalue distribution varies as a function of the strength of their coupling. We assume that the two cavities are coupled between them by a wire with $n$ conducting channels (see Fig. 1), and each of them to one electron reservoir through $N(\neq n)$ channels. The underlying classical dynamics is supposed to be fully chaotic in each dots.

For this particular application, we need to express the fictitious time $t$ of the Brownian motion, which has been introduced in a rather abstract mathematical way, in terms of the physical parameters $N$ and $n$. It was shown [32] that a closed system where the classical motion remains (nearly) in several separated chaotic regions of the classical phase space can be modelled by a random matrix approach similar to Dyson's Gaussian Brownian motion ensemble discussed in reference [4]. The time parameter of the latter can then be related to the particle flux between the nearly separated regions of phase space. In principle, one can relate [26] the $H$-Brownian motion (for a closed system) with the $S$-Brownian motion (for an open system) by some microscopic assumptions about the $S(H)$-dependence and then apply the results of reference [32] in order to obtain the physical meaning of the $S$-Brownian motion time.

For the scope of this paper, we will only give a simple heuristic derivation. Therefore, we assume that the $(N + n)$-dimensional $S$-matrix of each cavity has a CUE distribution when an
external parameter is varied (Fermi energy, magnetic field, shape of the dot, etc.) and has the form shown in equation (1.1), excepted that $r$ is $N \times N$-matrix, whereas the coupling gives rise to a $n \times N$-matrix $t$. When one electron is injected from a reservoir into one of the cavities by the $i^{th}$ of the $N$ incoming channels of the corresponding lead, one can assume that the probability that it goes directly back to reservoir is just

$$p_{\text{out}} = \sum_{j=1}^{N} |t_{ji}|^2 = \frac{N}{N+n}$$

and that the probability to go through the $n$ channel wire into the other cavity is

$$p_{\text{trans}} = \sum_{j=1}^{n} |t_{ji}|^2 = \frac{n}{N+n} = 1 - p_{\text{out}}$$

This results from the assumption that the chaotic motion inside the dot uniformly explores the dot boundaries. Furthermore, one can assume that the typical time for the electron to stay in one cavity is very long compared to the time needed to cross the connection, i.e., the electron looses completely the memory from which reservoir it was injected to the actual cavity. In this case, the total probability $p_{\text{total}}$ for the electron to be transmitted from one reservoir to the other is given by $p_{\text{total}} = p_{\text{trans}}p_{\text{out}} + p_{\text{trans}}p_{\text{total}}$, a relation which accounts for all type of processes where the electron can move between the cavities an arbitrary number of times before it leaves the system into the other reservoir. The solution of this equation is given by

$$p_{\text{total}} = \frac{p_{\text{trans}}}{1 + p_{\text{trans}}} = \frac{n}{N+2n}$$

Then, the average conductance can be estimated by $\langle g \rangle = Np_{\text{total}}$ which leads with the exact result equation (3.27) of Section 3.1 to the identification

$$t \simeq \frac{1}{8N} \ln \left(1 + \frac{2n}{N}\right)$$

where $t$ the Brownian motion time. The prefactor in (3.4) accounts for the typical time scale $t_c = 1/(8N)$ which is found in the end of Section 2.2. In the weak coupling limit $n \ll N$, we have the simple proportionality $t/t_c \simeq (2n)/N$.

### 3.1. Crossover of Two CUE's to One CUE for the Transmission Eigenvalues.

In the parametrization (2.1), the case of two independent CUE's of dimension $N$ corresponds to the requirement that all $T_j = 0$ or $x_j = 1$, respectively. We have therefore $p(x) = \prod_{j} \delta(1 - x_j)$ for the initial condition. The evaluation of equation (2.20) leads to the limit $p(x, t) = \lim_{x_j \to 1} G(x, x; t)$. We note that the Legendre polynomials are particular Hypergeometric functions, i.e., $P_n(x) = F(-n, n+1; 1; \frac{1}{2}(1-x))$, which can be expanded as

$$P_n(x) = \sum_{k} q_k \left(\frac{1}{2} (n+\frac{1}{2}) \right)^k (x-1)^k$$

with the polynomials

$$q_k(u) = \frac{1}{k!^2 2^k} \sum_{\nu=0}^{k-1} (u - (\nu + \frac{1}{2})^2)$$
of degree k. We now expand the one particle propagator \( g(x, \tilde{x}_j; t) \) with respect to \((1 - \tilde{x}_j)^{N-1}\). Again we can identify a matrix product in the determinant of equation (2.20) which yields a factor \( \rho(\tilde{x}) \) just cancelling the inverse contribution in (2.20). For the further calculation we use the notation \( P_n(x, t) = P_n(x) e^{\varepsilon_n t} \) and find from (2.20), (2.21)

\[
p(x, t) \propto \det(g_k(x, t)) \det(g_k(x, t))
\]

with

\[
g_k(x, t) = \sum_{n=0}^{\infty} (n + \frac{1}{2}) q_k \left( (n + \frac{1}{2})^2 \right) P_n(x) e^{\varepsilon_n t}
\]

Because of the determinant in equation (3.7), we note that one can replace in equation (3.8) the polynomials \( q_k(u), k = 0, 1, \ldots, N - 1 \) by an arbitrary set of linear independent polynomials of (maximal) degree \( N - 1 \). Such a substitution affects only the normalization constant. In the following, we will use two different substitutions for two different purposes: first to calculate exactly all kind of correlation functions and second to determine the effective interaction between the \( x_j \) (or \( \varphi_j \)) variables. We begin with the calculation of the correlation functions defined by

\[
R_k(x_1, \ldots, x_k; t) = \frac{N!}{(N-k)!} \int dx_{k+1} \cdot dx_N p(x, t) , \quad k = 1, 2, \ldots, N
\]

We put \( u_l = (l + \frac{1}{2})^2 = \frac{1}{2} \varepsilon_l, l = 0, 1, \ldots, N - 1 \) and replace the polynomials \( q_k(u) \) by the Lagrangian interpolation polynomials \( L_k(u) \) for the \( u_k \), i.e.

\[
L_k(u) = \prod_{l=0, l \neq k}^{N-1} \frac{u - u_l}{u_k - u_l} \Rightarrow L_k(u_l) = \delta_{kl}
\]

where \( l, k = 0, 1, \ldots, N - 1 \). We find now for the joint probability density the result

\[
p(x, t) = \frac{1}{N!} \det(P_n(x, t)) \det(h_k(x, t))
\]

with

\[
h_k(x, t) = (k + \frac{1}{2}) P_k(x) e^{-\varepsilon_k t} + \sum_{n=N}^{\infty} (n + \frac{1}{2}) L_k \left( (n + \frac{1}{2})^2 \right) P_n(x) e^{-\varepsilon_n t}
\]

The normalization in (3.11) follows directly from the obvious property

\[
\int_{-1}^{1} dx P_n(x, t) h_k(x, t) = \delta_{nk}
\]

Actually, this “orthogonality” relation enables us to calculate all correlation functions (3.9). We define the function

\[
K_N(x, y; t) = \sum_{n=0}^{N-1} P_n(x, t) h_n(y, t)
\]

and find the following three properties

\[
p(x, t) = \frac{1}{N!} \det(K_N(x, y; t)_{1 \leq i, j \leq N}) , \quad \int_{-1}^{1} dx K_N(x, x; t) = N , \quad \int_{-1}^{1} dy K_N(x, y; t) K_N(y, z; t) = K_N(x, z; t)
\]
From these properties and a well-known theorem in the theory of random matrices \([3,37]\) (cp. Theorem 5.2.1. of Ref. [3]), we find directly the result for the correlation functions

\[
R_k(x_1, \ldots, x_k; t) = \det (K_N(x_i, x_j; t))_{1 \leq i, j \leq k} \quad .
\]  

(3.18)

For the cases \(k = 1, 2\), we have

\[
\begin{align*}
R_1(x; t) &= K_N(x, x; t) \\
R_2(x, y; t) &= R_1(x; t) R_1(y; t) - K_N(x, y; t) K_N(y, x; t)
\end{align*}
\]

(3.19) \hspace{1cm} (3.20)

For practical applications, we give the more explicit expression for (3.14)

\[
K_N(x, y; t) = \sum_{k=0}^{N-1} \frac{1}{2} (1 + 2k) P_k(x) P_k(y) + \sum_{k=0}^{N-1} \sum_{n=N}^{\infty} (n + \frac{1}{2}) D_{kn} P_k(x) P_n(y) e^{\varepsilon_t - \varepsilon_n} t
\]

(3.21)

with coefficients

\[
D_{kn} = L_k \left( (n + \frac{1}{2})^2 \right) = \prod_{l=0, l \neq k}^{N-1} \frac{(n + l + 1)(n - l)}{(k + l + 1)(k - l)}
\]

\[= \frac{(-1)^{N-1-k} (n + N)!}{(n - N)! (k + N)! (N - k)!} \cdot \frac{(2k + 1)(N - k)}{(n + k + 1)(n - k)} \]

(3.22) \hspace{1cm} (3.23)

One might also consider correlation functions at two different times, i.e.,

\[
R_{k,l}(x_1, \ldots, x_k; t_1; y_1, \ldots, y_l; t + \tau) = \frac{N!^2}{(N - k)! (N - l)!} \cdot \int dx_{k+1} \cdot dx_N \int dy_{l+1} \cdot dy_N \cdot p(x, t) p(y, \tau) G(x, y; t)
\]

(3.24)

Without going into technical details, we mention that one can calculate the function \(R_{1,1}(x; t; y, t)\) with the functional derivative technique [3]. The result is then

\[
R_{1,1}(x; t; y, t + \tau) = R_1(x; t) R_1(y; t + \tau) + \left( g(x, y; \tau) - K_N(x, y; t + \tau) \right) K_N(y, x; t; t, t)
\]

(3.25)

where

\[
K_N(x, t_1; y, t_2) = \sum_{k=0}^{N-1} P_k(x, t_1) h_k(y, t_2)
\]

(3.26)

is a generalization of (3.14) for different times. The limit \(\tau \to 0\) is given by \(R_{1,1}(x; t; y, t) = R_2(x, y; t) + \delta(x - y) R_1(x)\).

The knowledge of the correlations functions (3.19, 3.25) enables us to calculate exact expressions for the average of the conductance \(g = \sum_j T_j = \frac{1}{2} \sum_j (1 - x_j)\) and the auto correlation of its variance \(\delta g = g - \langle g \rangle\). Using the orthogonality of the Legendre polynomials and their recursion relation we find

\[
\langle g(t) \rangle = \frac{N}{2} \left( 1 - e^{-8 N t} \right)
\]

(3.27)

\[
\langle \delta g(t) \delta g(t + \tau) \rangle = \frac{N^2}{4} e^{-8 N t} \left\{ \frac{1}{4 N^2 - 1} + \frac{N + 1}{2 N + 1} e^{-8 (2N+1) t} \right. \\
+ \frac{N - 1}{2 N - 1} e^{-8 (2N-1) t} - 1 \cdot e^{-16 N t} \left\}
\]

(3.28)
Of particular interest is the large $N$ limit. In order to obtain a non-trivial crossover, we have to measure the time variable in units of the typical crossover scale $t_c = 1/(8N)$ and to keep the $t/t_c$ ratio fixed in the limit $N \gg 1$. We find then up to corrections of order $1/N$

$$\langle g(t) \rangle = \frac{N}{2} \left( 1 - e^{-t/t_c} \right),$$

$$\langle \delta g(t) \delta g(t + \tau) \rangle = \frac{1}{16} e^{-\tau/t_c} \left\{ 1 + \left( \frac{2t^2}{t_c^2} - \frac{2t}{t_c} - 1 \right) e^{-2t/t_c} \right\}.$$  

3.2. **Effective Pairwise Interaction in the Crossover Regime.** — We want to evaluate (3.7) in a suitable way for having the effective pairwise interaction between the $x_j$- or $\varphi_j$-variables. We now replace in equation (3.8) the polynomials $q_k((n + \frac{1}{2})^2)$ by $4^k(n + \frac{1}{2})^{2k} = \varepsilon_n^k$ and find

$$p(x, t) \propto \det(P_n(x, t)) \det(\tilde{g}_k(x, t))$$

with

$$\tilde{g}_k(x, t) = \sum_{n=0}^{\infty} (n + \frac{1}{2}) \varepsilon_n^k P_n(x) e^{-\varepsilon_n t} = (-1)^k \left( \frac{d}{dt} \right)^k \tilde{g}_0(x, t)$$

$$= \frac{1}{t^k} \left( \frac{d}{ds} \right)^k \tilde{g}_0(x, t(1-s)) \bigg|_{s=0}$$

The quantity $\tilde{g}_0(x, t)$ can be related to the heat kernel on the unit sphere $S_2$ with initial condition at the origin, which yields a useful approximation in the limit $t \ll 1$

$$\tilde{g}_0(\cos(2\varphi), t) \approx \frac{1}{8t} e^{-\varphi^2/(4t)}$$

From the expansion

$$\frac{1}{1-s} e^{-xs/(1-s)} = \sum_{k=0}^{\infty} \frac{L_k(x)}{k!} s^k$$

(here $L_k(x)$ denote the Laguerre polynomials) and (3.33), (3.32) we find

$$\tilde{g}_k(\cos(2\varphi), t) = \frac{1}{8^k+1} e^{-\varphi^2/(4t)} L_k \left( \frac{\varphi^2}{4t} \right), \quad t \ll 1.$$  

This results leads to a probability density (now for the variables $\varphi_j$)

$$\tilde{p}(\varphi, t) \simeq C(t) \prod_j \left( \sin(2\varphi_j) e^{-\varphi_j^2/(4t)} \right) \prod_{i<j} \left\{ (\sin^2 \varphi_i - \sin^2 \varphi_j) (\varphi_i^2 - \varphi_j^2) \right\}$$

where $C(t)$ is a normalization constant which depends on $t$. If we write (3.36) under the form of a "Gibbs factor" $\tilde{p}(\varphi, t) \sim \exp[-\beta(\sum_{i<j} u(\varphi_i, \varphi_j) + \sum_i V(\varphi_i, t))]$, we find that the fictitious corresponding Hamiltonian has the pairwise interaction and the one-body confining potential:

$$u(\varphi_i, \varphi_j) = -\frac{1}{2} \left( \ln | \sin^2 \varphi_i - \sin^2 \varphi_j | + \ln | \varphi_i^2 - \varphi_j^2 | \right),$$

$$V(\varphi_i, t) = \frac{\varphi_i^2}{8t} - \frac{1}{2} \ln | \sin(2\varphi_i) |$$

This can be simplified when $t$ is very small, i.e., $t \ll t_c = 1/(8N)$ since $\sin^2 \varphi_i \approx \varphi_i^2$ in this limit. One gets a joint probability distribution (3.36) identical to the so-called "Laguerre
ensemble" [38], with a logarithmic repulsion between the $\varphi^2$ [36]. For $N \gg 1$, the density reduces to a quarter-circle law:

$$\tilde{R}_1(\varphi; t) \simeq \frac{1}{4\pi t} \sqrt{16tN - \varphi^2}$$  \hspace{1cm} (3.39)$$

if the argument of the square root is not negative (otherwise $\tilde{R}_1(\varphi; t) = 0$). Using $\varphi^2 \simeq T = \cosh^{-2} \nu$, one gets

$$\tilde{R}_1^{(\nu)}(\nu; t) = \frac{2\sinh \nu}{\cosh^3 \nu} \tilde{R}_1(\cosh^{-1} \nu; t)$$  \hspace{1cm} (3.40)$$

for the variable $\nu \in [0, \infty]$.

One can note that the exact pairwise interaction for the $\varphi_j$ variables (3.37) differs for $1/N \ll t \ll 1$ from the standard RMT interaction given by (2.4) for $\beta = 2$. However, this concerns essentially the quantitative difference between $\sin^2 \varphi$ and $\varphi^2$ whereas for the quasi one-dimensional disordered wire considered in reference [33] this effect was much more stronger due to the appearance of $\sinh^2(\ldots)$ instead of $\sin^2(\ldots)$

In Sections 5 and 6, we will return to this point and compare the results of equations (3.19), (3.40) with the corresponding expressions obtained for disordered lines, assuming the $M$-Brownian motion ensemble.

4. $M$–Brownian Motion Ensemble For the Transmission Eigenvalues

The $S$-Brownian motion ensemble was built on the evolution law $S(t + \delta t) = S(t)\exp(i\delta X)$ where $\delta X$ was an infinitesimal Hermitian random matrix (Eq. (2.5)). A Fokker-Planck equation was deduced, giving the $t$-dependence of the transmission eigenvalues. A different Fokker-Planck equation is more natural when one builds a quasi one-dimensional wire, adding in series small diffusive blocs [27–30]. The evolution law of the matrices $u$ and $v$ (“angular part”) and of the matrix $T$ (“radial part”) defined in Eq. (2.1) can indeed be deduced from the multiplicative transfer matrix $M$, expressed [23] in the same parametrization. The law of $M$ describing the addition in series of the scatterers is now:

$$M(t + \delta t) = M(t)M(\delta t)$$  \hspace{1cm} (4.1)$$

The Brownian motion time $t$ is in this case related to the number of scatterers in series (i.e., the length $L$ of the wire). $M$ can be parametrized with the same coordinates used for $S$ in equation 2.1 [15]. Assuming that the scatterers are isotropically distributed, the length dependence of the transmission eigenvalues for the $M$-Brownian motion ensemble is given by a Fokker-Planck equation [27–30], which presents interesting analogies and differences with equation (2.7). The comparison of those two Brownian motions, assuming two different evolution laws, will enable us to describe in details the similarities and the differences between quantum transmission through two weakly coupled cavities ($S$-ensemble) and through a disordered wire ($M$-ensemble). Before this, having used the Beenakker-Rejaei method to solve the $S$-Brownian motion ensemble, we have been able to improve this method and we show in return how to complete the solution of the transmission problem through a disordered wire. Indeed, we explain how to calculate exactly the length dependence of $n$ order correlation functions of the transmission eigenvalues, for arbitrary values of $n$. This allows us to establish an important equivalence between the $M$-Brownian motion ensemble and a more microscopic approach using the non linear $\sigma$-model.
4.1. Exact Calculation of the Correlation Functions For $\beta = 2$. — We start with the exact result given in reference [33] for the joint probability distribution of the variables $\lambda_j = (1 - T_j)/T_j = \sinh^2 x_j$ which satisfies the Fokker-Planck equation of the $M$-Brownian motion ensemble:

$$p(\lambda, t) \propto \prod_{i,j} (\lambda_i - \lambda_j) \det(g_m(\lambda_j, t)) e^{CNt},$$

(4.2)

with

$$g_m(\lambda, t) = \int_0^\infty dk (k^2)^m \frac{k}{2} \tanh \left( \frac{\pi k}{2} \right) P_{\frac{1}{2}(m-1)}(1 + 2\lambda)e^{-k^2t},$$

(4.3)

where $C_N$ is just given by (2.12) for $\beta = 2$. We underline that the changed sign in the exponential factor of (4.2) if compared with (2.21) is correct. The variable $t$ is now $t = L/(4\xi l) = L/(2\xi)$ ($l$ is the elastic mean free path, $L$ is the length of the conductor, and $\xi = 2\pi l$ is the quasi one-dimensional localization length for the unitary case). The index $m$ in (4.3) takes the values $m = 0, 1, \ldots, N - 1$. For later use we mention that the generalized Legendre function can be expressed as a Hypergeometric function via

$$P_{\frac{1}{2}(m-1)}(1 + 2\lambda) = F \left( \frac{1}{2} - i\frac{k}{2}, \frac{1}{2} + i\frac{k}{2}; 1; -\lambda \right)$$

(4.4)

This function is an eigenfunction of the differential operator

$$D = - \left( 4 \frac{\partial}{\partial \lambda} \lambda (1 + \lambda) \frac{\partial}{\partial \lambda} + 1 \right)$$

(4.5)

with eigenvalue $k^2$. The operator $D$ is of course related with the one-particle Hamiltonian $\mathcal{H}_0$ [33] by a suitable transformation. We can now state the following properties of $g_m(\lambda, t)$

$$g_m(\lambda, t) = D^m g_0(\lambda, t),$$

(4.6)

$$\frac{\partial}{\partial t} g_0(\lambda, t) = -D g_0(\lambda, t),$$

(4.7)

$$g_0(\lambda, 0) = \delta(\lambda)$$

(4.8)

Equations (4.6) and (4.7) follow directly from equation (4.3). Equation (4.8) reflects the initial condition for the one-particle Green’s function [33] and was already given in reference [42], where the expression (4.2) was found for the one channel case.

We are now interested in integrals of the form

$$a_{nm}(t) = \int_0^\infty d\lambda P_n(1 + 2\lambda) g_m(\lambda, t)$$

(4.9)

where $P_n(\cdots)$ again denotes the Legendre polynomial of degree $n$. We consider first the case $m = 0$. From (4.8) we find $a_{n0}(0) = P_n(1) = 1$ and from (4.7) we get the differential equation

$$\frac{\partial}{\partial t} a_{n0}(t) = - \int_0^\infty d\lambda P_n(1 + 2\lambda) D g_0(\lambda, t)$$

$$= \int_0^\infty d\lambda g_0(\lambda, t) D P_n(1 + 2\lambda) = (1 + 2n)^2 a_{n0}(t) = \varepsilon_n a_{n0}(t)$$

(4.10)

which has the solution $a_{n0}(t) = \exp(\varepsilon_n t)$. In equation (4.10) we have used the fact that $P_n(1 + 2\lambda)$ is an eigenfunction of $D$ with eigenvalue $-\varepsilon_n = -(1 + 2n)^2$, a property following
directly from Legendre's differential equation. From equations (4.6), (4.9) we get by a similar calculation the general result

\[ a_{nm}(t) = (-\varepsilon_n)^m a_{n0}(t) = (-\varepsilon_n)^m e^{\varepsilon_n t} \quad . \]

(4.11)

As in Section 3.1 we can replace the exponential factors \((k^2)^m\) in equation (4.3) by an arbitrary set of linear independent Polynomials of (maximal) degree \((N - 1)\) in the variable \(k^2\) because the determinant in equation (4.2) yields only a constant factor after such a transformation. We now choose the replacement

\[ (k^2)^m \rightarrow L_m \left( -\frac{1}{4} k^2 \right) \quad (4.12) \]

where \(L_m(u)\) are just the Lagrangian interpolation polynomials already given by equation (3.10). We introduce the following notations

\[ Q_n(\lambda, t) = P_n(1 + 2\lambda) e^{-\varepsilon_n t} \quad , \]

(4.13)

and

\[ h_m(\lambda, t) = \int_0^\infty dk L_m \left( -\frac{1}{4} k^2 \right) \frac{k}{4} \tanh \left( \frac{\pi k}{2} \right) P_{\frac{1}{2}(k-1)}(1 + 2\lambda) e^{-k^2 t} \]

(4.14)

From equation (4.11) we get directly for \(n, m = 0, 1, \ldots, N - 1\)

\[ \int_0^\infty d\lambda \, Q_n(\lambda, t) h_m(\lambda, t) = L_m(\frac{1}{4} \varepsilon_n) = \delta_{nm} \quad (4.15) \]

The joint probability distribution (4.2) can then be written in the form

\[ p(\lambda, t) = \frac{1}{N!} \det(Q_n(\lambda_j, t)) \det(h_m(\lambda_j, t)) = \frac{1}{N!} \det(\mathbb{K}_N(\lambda_i, \lambda_j; t)_{1 \leq i, j \leq N}) \]

(4.16)

with

\[ \mathbb{K}_N(\lambda, \tilde{\lambda}; t) = \sum_{m=0}^{N-1} Q_m(\lambda, t) h_m(\tilde{\lambda}, t) \]

(4.17)

Similarly as in Section 3.1 we find from (4.15)

\[ \int_0^\infty d\lambda \, \mathbb{K}_N(\lambda, \lambda; t) = N \]

(4.18)

and

\[ \int_0^\infty d\mu \, \mathbb{K}_N(\lambda, \mu; t) \mathbb{K}_N(\mu, \tilde{\lambda}; t) = \mathbb{K}_N(\lambda, \tilde{\lambda}; t) \quad . \]

(4.19)

These properties lead, as in equation (3.18), to the \(m\)-point correlation functions

\[ R_m(\lambda_1, \ldots, \lambda_m; t) = \det(\mathbb{K}_N(\lambda_i, \lambda_j; t)_{1 \leq i, j \leq m}) \]

(4.20)

where \(\mathbb{K}_N(\lambda, \tilde{\lambda}; t)\) can also be expressed by the more explicit formula

\[ \mathbb{K}_N(\lambda, \tilde{\lambda}; t) = \int_0^\infty dk \frac{k}{2} \tanh \left( \frac{\pi k}{2} \right) P_{\frac{1}{2}(k-1)}(1 + 2\tilde{\lambda}) e^{-k^2 t} \]

\[ \times \sum_{m=0}^{N-1} L_m \left( -\frac{1}{4} k^2 \right) P_m(1 + 2\lambda) e^{-\varepsilon_m t} \]

(4.21)
4.2. **Equivalence With the Microscopic Non-Linear σ-Model Approach.** — As a first application we calculate an exact expression for the average conductance, when the length of the wire increases. This expression is valid from the disordered conductor towards the Anderson insulator. Since the conductance is a linear statistics of the \( \lambda_i, g = \sum_i (1 + \lambda_i)^{-1} \), we have to evaluate the corresponding integral over the density:

\[
\langle g \rangle = \int_0^\infty d\lambda \frac{1}{1 + \lambda} \sum_{m=0}^{N-1} Q_m(\lambda, t) h_m(\lambda, t)
\]  (4.22)

Since \( Q_m(\lambda, t) \) is a polynomial of degree \( m \) in the variable \( 1 + \lambda \), we can write

\[
\frac{1}{1 + \lambda} Q_m(\lambda, t) = \frac{1}{1 + \lambda} Q_m(-1, t) + r_{m-1}(\lambda) = \frac{(-1)^m}{1 + \lambda} e^{-\varepsilon m t} + r_{m-1}(\lambda)
\]  (4.23)

where \( r_{m-1}(\lambda) \) is a polynomial of degree \( m - 1 \) which does not contribute in the integral (4.22) due to the quasi orthogonality relation (4.15). We find therefore

\[
\langle g \rangle = \sum_{m=0}^{N-1} \int_0^\infty d\lambda e^{-(\varepsilon m + k^2) t} (-1)^m L_m(-\frac{1}{4} k^2) \frac{k}{2} \tanh \left( \frac{\pi k}{2} \right) \cdot I(k)
\]  (4.24)

where \( I(k) \) stands for the integral

\[
I(k) = \int_0^\infty \frac{1}{1 + \lambda} F \left( \frac{1}{2} - i \frac{k}{2}, \frac{1}{2} + i \frac{k}{2}; 1; -\lambda \right)
\]  (4.25)

We use the transformation formula (15.3.4) in reference [39] for the Hypergeometric function in (4.25) and expand afterwards the latter in its power series. The integral over \( \lambda \) can then be done and gives the sum

\[
I(k) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\frac{1}{2} + n - i \frac{k}{2}} \left( -\frac{1}{2} + i \frac{k}{2} \right)^n
\]  (4.26)

where \( \binom{a}{n} \) denotes the generalized Binomial coefficient which is defined for arbitrary complex values of \( a \). The evaluation of the sum (4.26) yields

\[
I(k) = \int_0^1 du \left( 1 - u \right)^{-\frac{1}{2} + i \frac{k}{2}} u^{-\frac{1}{2} - i \frac{k}{2}} = \Gamma \left( \frac{1}{2} + i \frac{k}{2} \right) \Gamma \left( \frac{1}{2} - i \frac{k}{2} \right)
\]  (4.27)

We now use the expression (3.10) for the polynomials \( L_m(\cdots) \) and find for the averaged conductance the expression

\[
\langle g \rangle = 2 \sum_{m=0}^{N-1} \int_0^\infty d\lambda e^{-((1+2m)^2+k^2) t} k \tanh \left( \frac{\pi k}{2} \right) \frac{2m+1}{k^2 + (1+2m)^2} a(N, m, k)
\]  (4.28)

where we have introduced the coefficient

\[
a(N, m, k) = \frac{k^2 + (1 + 2m)^2}{4(2m + 1)} (-1)^m L_m(-\frac{1}{4} k^2) \Gamma \left( \frac{1}{2} + i \frac{k}{2} \right) \Gamma \left( \frac{1}{2} - i \frac{k}{2} \right)
\]  (4.29)

\[
a(N, m, k) = \frac{\Gamma \left( N + \frac{1}{2} + i \frac{k}{2} \right) \Gamma \left( N + \frac{1}{2} - i \frac{k}{2} \right)}{\Gamma(N-m) \Gamma(N+m+1)}
\]  (4.30)
The result \((4.28)\) is exact for all values of \(N\) and \(t = L/(2\xi)\). In the limit \(N \to \infty\) the coefficient \(a(N, m, k)\) is just 1 and the expression \((4.28)\) becomes identical with the microscopic result of Zirnbauer et al. [35] based on the supersymmetric non-linear \(\sigma\)-model, in the unitary case.

A similar calculation yields the second moment of \(g\)

\[
\langle g^2 \rangle = \frac{1}{2} \sum_{m=0}^{N-1} \int_0^{\infty} dk \ e^{-(1+2m)^2+k^2} \ t \ \tanh \left( \frac{\pi k}{2} \right) \ (2m + 1) \ a(N, m, k) \quad (4.31)
\]

We omit the details, since this result can also be derived in a more direct way by the general identity

\[
\langle g^2 \rangle = -\frac{1}{4} \frac{\partial \langle g \rangle}{\partial t} \quad (4.32)
\]

which follows directly from the original Fokker-Planck equation [40] for the case \(\beta = 2\).

This gives the proof that the microscopic supersymmetric approach and the \(M\)-Brownian motion ensemble of references [28,40] are equivalent in the limit \(N \to \infty\), for arbitrary values of \(L\), at least as far as the average and the variance of the conductance are concerned.

4.3. Quasi-1D Localized Limit. — In the localized limit \(t \gg 1\) the integral in equation \((4.14)\) can be evaluated by a saddle point approximation. In order to do this, we have to consider the oscillatory behavior of the Legendre function for large values of the variable \(\lambda\). We use now the transformation formula \((15.3.8)\) of reference [39] and express the \(k\)-integration measure as a product of Gamma functions. Instead of equation \((4.14)\) we get then the modified expression

\[
h_m(\lambda, t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dk \ e^{-k^2t} \ (1 + \lambda)^{-\frac{1}{2} - \frac{i}{2} \frac{k}{L}} H \left( k, \frac{1}{1+\lambda} \right) \quad (4.33)
\]

with the abbreviation

\[
H \left( k, \frac{1}{1+\lambda} \right) = \frac{\Gamma \left( \frac{1}{2} + i \frac{k}{L} \right)^2}{\Gamma \left( ik \right)} \ F \left( \frac{1}{2} + i \frac{k}{L}, \frac{1}{2} + i \frac{k}{L}; 1 + ik; \frac{1}{1+\lambda} \right) \ L_m \left( -\frac{1}{4} k^2 \right) \quad (4.34)
\]

It is more convenient to replace \(\lambda\) by the new variable \(z = \frac{1}{2} \ln(1+\lambda)\). A simple substitution then leads to the expression

\[
h_m(e^{2z} - 1, t) = e^{-z^2/(4t-z)} \ \frac{1}{4\pi \sqrt{t}} \int_{-\infty}^{\infty} dq \ e^{-q^2} H \left( \frac{q}{\sqrt{t}} - \frac{z}{2t}, e^{-2z} \right) \quad (4.35)
\]

which is still exact. We mention that this expression is rather useful for an efficient numerical evaluation since the difficult oscillatory behavior of the Legendre functions has been taken into account by a shift of the integration path into the complex plane. In the limits \(t \gg 1, z \gg 0\), the integral can be done and we obtain up to corrections of the order \(1/t\) or \(e^{-z^2}\), respectively

\[
h_m(e^{2z} - 1, t) \simeq \frac{1}{2\sqrt{4\pi t}} e^{-z^2/(4t-z)} \ \frac{\Gamma \left( \frac{1}{2} (1 + \frac{z}{2t}) \right)^2}{\Gamma \left( \frac{z}{2t} \right)} \ L_m \left( \frac{1}{4} \left( \frac{z}{2t} \right)^2 \right) \quad (4.36)
\]

The corresponding limit for the \(Q_m(\lambda, t)\) in equation \((4.13)\) is

\[
Q_m(e^{2z} - 1, t) \simeq \left( \frac{2m!}{m!^2} \right) e^{-(1+2m)^2t} e^{2x_m} \quad (4.37)
\]
The density $\tilde{R}_1(z; t)$ and the two-point function $\tilde{R}_2(z, \tilde{z}; t)$ for the variable $z$ can be obtained from equations (4.20) and (4.17). The result (containing additional factors $2e^{2z}$ or $2e^{2\tilde{z}}$ due to the Jacobian of the variable transformation $\lambda \rightarrow z$) is

$$
\tilde{R}_1(z; t) \simeq \sum_{m=0}^{N-1} G_m(z; t), \quad \tilde{R}_2(z, \tilde{z}; t) \simeq \sum_{m, \tilde{m}=0, m \neq \tilde{m}} G_m(z; t) G_{\tilde{m}}(\tilde{z}; t)
$$

(4.38)

where we have used the abbreviation

$$
G_m(z; t) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{1}{4t}(z - 2t(1 + 2m))^2 \right)
$$

(4.39)

for the Gaussian distribution. We mention that for (4.38) we have replaced the argument $z/2t$ of the Gamma functions and of the polynomial $L_m(\cdots)$ in equation (4.36) by the most probable values $z/2t = 1 + 2m$ or $1 + 2\tilde{m}$ due to the Gaussian factors. We recover the simple sum of Gaussian distributions with mean values $2t(1 + 2m)$ and a variance $2t$ already obtained by a direct simplification of the Fokker Planck equation valid in the localized limit [30, 31, 41]. The above calculation shows that the general expressions (4.20) and (4.21) of Section 4.1 are indeed consistent with known results concerning the quasi-1D insulators.

5. **Weakly Coupled Chaotic Cavities Versus Disordered Lines**

We describe now the differences between transmission through two ballistic cavities coupled by a narrow constriction (Section 3) and transmission through a homogeneously disordered wire of constant transverse section (Section 4). This can be of practical interest for many purposes which are not restricted only to mesoscopic electrical conductances, but which could be relevant for other systems as wave guide communication lines. One can imagine that the signal from an antenna to a receiver is anomalously weak and that the radio-engineering problem is to know if the weakness of the signal is due to an homogeneous deterioration of the transmission line or only to a local deterioration resulting from an accidental narrow constriction.

Characteristic behaviors of the S-Brownian motion ensemble are first given, characterizing the coupled ballistic dots under a small applied magnetic field. In Figure 2, the density of the variable $\nu = \left( \arcsinh(\lambda) \right)^{1/2}$ is shown for the one channel case ($N = 1$) at four different values of $t/t_c = 10.0, 1.0, 0.1, 0.01$ ($t_c = 1/(8N)$ is the typical time scale introduced in Section 2.2). The variables $\lambda = (1 - T)/T = (1 + x)/(1 - x)$ are the radial parameters of the transfer matrix, which are usual in random transfer matrix theory [15, 28, 41]. The curves are calculated from equations (3.19) and (3.21). Figure 3 gives this density for the same values of $t$ but for $N = 5$ channels. In addition for $t/t_c = 0.1, 0.01$ the asymptotic expression (3.40) that arises from the Laguerre approximation is shown, too. The latter is valid if $N \gg 1$ and $t \ll t_c$. In both figures one can see that the typical values for $\nu$ increase (on a logarithmic scale) if $t$ decrease. One can identify a rather sharp minimum value $\nu_{\min}$ for $\nu$ if $t \ll t_c$. A simple estimation from equation (3.39) yields

$$
\nu_{\min} \approx -\frac{1}{2} \ln \left( \frac{t}{2t_c} \right)
$$

(5.1)

For values $\nu \geq \nu_{\min}$ the distribution is rather broad and the overall form is (nearly) independent of $N$ (apart from the case $t \gg t_c$). A reduction of the time leads essentially to a translation of the density. For finite values of $N$ (see $N = 5$), one can see typical but rather small oscillations around the limiting distribution for $N \rightarrow \infty$. The value $t/t_c = 10.0$ corresponds within the accuracy of the plots to the stationary limit $t/t_c \rightarrow \infty$, i.e., the single CUE behavior. For this
Fig. 2. — Density of the variable \( \nu = (\text{arsinh}(\lambda))^{1/2} \) for the crossover \( 2 \times \text{CUE} \rightarrow \text{CUE} \) with \( N = 1 \) at different values of \( t/t_c = 10.0, 1.0, 0.1, 0.01 \). The curves are calculated from equations (3.19) and (3.21).

Fig. 3. — Density of the variable \( \nu = (\text{arsinh}(\lambda))^{1/2} \) for the crossover \( 2 \times \text{CUE} \rightarrow \text{CUE} \) with \( N = 5 \) at different values of \( t/t_c = 10.0, 1.0, 0.1, 0.01 \). The curves are calculated from equations (3.19) and (3.21). In addition, for \( t/t_c = 0.1, 0.01 \) the Laguerre approximation (3.40) is included.

case the position of the first maximum decreases and becomes sharper when \( N \) increases. This gives rise to a conductance \( g \sim N/2 \) (cp. Eq. (3.27)).

The corresponding behaviors of the \( M \)-Brownian motion ensemble exhibit essential differences from the previous case. Figure 4 shows the \( \nu \)-density for the disordered wire with \( N = 5 \) channels in three different regimes, i.e., the localized regime (\( L/\xi = 10.0 \)), the crossover regime (\( L/\xi = 1.0 \)) and the metallic regime (\( L/\xi = 0.2 \)). Figures 5a-c contain the same density curves in a more readable scale than in Figure 4. The Gaussian approximation (4.38) (dotted line)
which fits very well the exact density at $L/\xi = 10.0$ is indicated in Figure 5c (2). The results shown in Figures 4 and 5 were obtained by a numerical evaluation of the integral (4.35) except for the limits $t \ll 1$, $\nu \leq 1$ where the integral (4.14) is better suited. The substitution $z = \ln(\cosh \nu)$ was of course exactly considered.

One can see that the $\nu$-density of the disordered line differs essentially from the density shown in Figure 3 for the two weakly coupled cavities. For small transmission (i.e., the localized limit), the $\nu$-density is mainly concentrated around maximal values $\nu_{\text{max},m} = (1 + 2m)L/\xi$ ($m = 0, 1, \ldots, N - 1$) with a relative variance $\langle (\nu - \nu_{\text{max},m})^2 \rangle^{1/2}/\nu_{\text{max},m} \sim \sqrt{\xi/L}$. Between the maxima, the density nearly vanishes. This effect can be seen in Figure 5c and is still increased if the ratio $L/\xi$ is increased. But in the crossover regime $L \approx \xi$ (Fig. 5b), the minima are no longer negligible compared to the maxima. In Figure 5a (metallic regime), the amplitude of the finite $N$ oscillations are small but still clearly observable. Further calculations for different values of $N$ show that the amplitude of the oscillations remains unchanged if $N$ is increased for a fixed ratio $L/\xi$. In order to reach a metallic regime where the density is really a smooth function of $\nu$, one has to increase $N$ to very high values and at the same time to decrease $L/\xi$ (within the limit $L/\xi \gg 1/N$).

Finally, we compare $\langle g \rangle$ and of $\langle \delta g^2 \rangle$ between the two cases in the limit of small transmission, i.e., $t \ll t_c$ for the $2 \times \text{CUE} \rightarrow \text{CUE}$ crossover or $L \gg \xi$ for the disordered line. For the two weakly coupled cavities, we get in the lowest order of $t/t_c$ from equations (3.27) and (3.28)

$$\langle g(t) \rangle \simeq \frac{N}{2} \frac{t}{t_c} \simeq n$$

$$\langle \delta g^2(t) \rangle \simeq \frac{1}{4} \left( \frac{t}{t_c} \right)^2 \simeq \left( \frac{n}{N} \right)^2$$

where $n \simeq (N/2)$ ($t/t_c$) is the effective number of coupling channels introduced in Section 1.

---

(2) The variable $z$ used in Section 4.3 is only approximately equal to $\nu$ (i.e., $z = \ln(\cosh \nu) \simeq \nu$ when $\nu \gg 1$).
Fig. 5. — Same as Figure 4, but with a modified scale, for a) $L/\xi = 0.2$; b) $L/\xi = 1.0$; c) $L/\xi = 10.0$. The Gaussian approximation (Eq. (4.38), dotted line) is included for comparison.
These results lead to the limit
\[
\lim_{t \to 0} \frac{\langle \delta g^2(t) \rangle}{\langle g(t) \rangle^2} = \frac{1}{N^2}
\]
(5.4)
for the relative variance of \( g \). The corresponding expressions for the quasi one-dimensional limit
in the localized limit \( L \gg \xi \) can be obtained from equations (4.28) and (4.32) by an evaluation
of the \( k \)-integral in a saddle point approximation
\[
\langle g(L) \rangle \simeq \frac{\pi^{3/2}}{4} \frac{\Gamma(N + \frac{1}{2})^2}{\Gamma(N)\Gamma(N + 1)} \left( \frac{L}{2\xi} \right)^{-3/2} e^{-L/(2\xi)} ,
\]
(5.5)
\[
\langle \delta g^2(L) \rangle \simeq \langle g^2(L) \rangle \simeq \frac{1}{4} \langle g(L) \rangle .
\]
(5.6)
The relative variance of \( g \) is now infinite and the infinite length limit is characterized by
\[
\lim_{L \to \infty} \frac{\langle \delta g^2(L) \rangle}{\langle g(L) \rangle} = \frac{1}{4} .
\]
(5.7)
The comparison with (5.4) shows again a qualitative difference in the statistics of the conduc-
tance between the two cases. Equations (5.5) and (5.6) illustrate the well-known fact that the
average of the conductance does not significantly characterize the conductance properties in
the localized limit. On the other hand, the limit (5.4) for the two weakly coupled ballistic
cavities is a small quantity for a large number of channels.

6. Summary

The \( S \) and \( M \) Brownian motion ensembles which we have used are characterized by two Fokker-
Planck equations expressing isotropic diffusion equations on the compact space of unitary
matrices (\( S \)) or on a non-compact symmetric space of pseudo-unitary transfer matrices [43],
respectively. The differential operators which appear in the Fokker-Planck equations are just
the radial part of the Laplace-Beltrami operators of the corresponding spaces, describing free
isotropic diffusion on those curved spaces. In the case of the \( M \)-Brownian motion ensemble,
the variable \( \nu \) is the good coordinate since it measures in some way the geodesic distance
on the space of pseudo-unitary matrices (cp. [43]). Figure 4 shows very clearly the free diffusion
of \( \nu \). Since the range of \( \nu \) is non compact the rather strong level repulsion gives well-separated
maxima. For the compact space of unitary matrices, it is the variable \( \varphi \) used in Sections 2.1
and 2.2 which measures the geodesic distance. The range for \( \varphi \) is compact and diffusion leads
after a long time to an essentially uniform distribution. The level repulsion only causes some
small oscillations around this distribution.

Apart from the (important) differences which we have reviewed, we can also notice nice
mathematical similarities between the two Fokker-Planck approaches. Many formulae and
results of Section 2 coincide with the results of reference [33] and Section 4, after replacing
the variables \( \varphi_j \) by the \( \nu_j \) (= \( x_j \) in [33]) and \( \sin, \cos \) by \( \sinh, \cosh \). This is partly related to the
similarity of the parametrization (2.1) with the corresponding parametrization of the transfer
matrix [23] (with the substitution \( T \to -\Lambda \)). This occurs only for the unitary case (\( \beta = 2 \)).
For the orthogonal and unitary cases (\( \beta = 1, 4 \)) this similarity does not exist, since the unitary
matrices \( u_{1,2}, v_{1,2} \in (2.1) \) are related in a different way than for the transfer matrix [23] by
time reversal symmetry.

Our results (Eqs. (5.2)-(5.7)) are valid for an arbitrary number \( N(= 1, 2, 3, \ldots) \) of channels.
In the case of a single channel, the quotient of the Gamma functions takes the value \( \pi/4 \) and
equation (5.5) becomes identical with the result of reference [44] obtained using a supervector model for the one-dimensional white noise potential at small disorder. Indeed, it was known that the one-dimensional white noise model for small disorder leads [45, 46] to the same Fokker-Planck equation as the one of references [27, 28, 42] for one channel. In the opposite limit $N \rightarrow \infty$, the Gamma function quotient reduces to one and we recover the corresponding expression [35] obtained by the supersymmetric non-linear $\sigma$-model: the expressions (4.28) and (4.31) for $\langle g \rangle$ and $\langle g^2 \rangle$ coinciding exactly for all length scales $0 < L/\xi < \infty$ with those of reference [35]. The non-linear $\sigma$-model for quasi one-dimensional disordered conductors can at least be derived assuming three different microscopic Hamiltonians, i.e., the white noise potential [47], Wegner’s $N$ orbital model [48], and a random banded matrix Hamiltonian [49]. It is then very remarkable that in the unitary case, these microscopic approaches become equivalent to the $M$-Brownian motion ensemble of reference [28]. This behavior is apparently not true for the symplectic case, since the non-linear $\sigma$-model predicts [35] finite values of $\langle g \rangle$ and $\langle \delta g^2 \rangle$ in the localized regime, while the Fokker-Planck equation gives an exponential decrease of the typical conductance for each $\beta = 1, 2, 4$ [30]. In principle, it could still be possible that in the large $N$ limit the average $\lim_{L \rightarrow \infty} \langle g(L) \rangle$ takes a non-vanishing value because of some very subtle effects concerning the tails of the normal distribution of $\ln(g)$. But even this cannot happen, as it can be easily seen from the generalization [40] of equation (4.32) for arbitrary $\beta$

$$\frac{\partial \langle g \rangle}{\partial L} = -\frac{2}{\xi_\beta} \left( \langle g^2 \rangle + \frac{\beta}{2} (\langle g^2 \rangle - \langle g \rangle) \right)$$

(6.1)

Here $\xi_\beta \sim \beta \xi_1$ denotes the localization length and $g_2$ is just the trace of the squared transmission matrix $(tt^\dagger)^2$, i.e., $g_2 = \sum \lambda_i (1 + \lambda_i)^{-2}$. From the inequality $0 \leq g_2 \leq g^2$ one finds directly that in the limit $L \rightarrow \infty$ the behavior $\langle g \rangle = \text{const.} \neq 0$ (as predicted in Ref. [35] for $\beta = 4$) is not possible since $\langle g^2 \rangle$ must vanish, which yields directly $\langle g \rangle = 0$. This contradiction when $\beta = 4$ between the two approaches is surely worth to be considered in future work.

Though the two Fokker-Planck approaches are very similar from a mathematical point of view, they nevertheless describe two different physical situations where the Fokker-Planck time controls the crossover between systems with a high and a low conductance. For both approaches we have found (rather) closed expressions (valid for a finite channel number) for any $m$-point correlation functions of the transmission eigenvalues, allowing us to calculate the first and second moment of the conductance. This complete analytical solution is essentially restricted to the unitary case where both Fokker Planck approaches can be solved in terms of fermionic one particle Green’s functions. At the moment, the extension of this solution for arbitrary $N$ to the orthogonal and symplectic cases looks difficult, since the Hamilton operator obtained after the Sutherland transformation does not reduces to the sum of one particle operators.

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Appendix A

Fokker-Planck Equation For the Transmission Eigenvalues In the S-Brownian Motion Ensemble

In this Appendix, we derive the Fokker-Planck equation (2.7) for the case $\beta = 2$. In order to do this, we consider the matrix

$$ Q(t) = S(t)\Sigma_z S(t)\dagger \Sigma_z = \begin{pmatrix} v_1 & 0 \\ 0 & u_1 \end{pmatrix} \begin{pmatrix} \cos(2\varphi) & i\sin(2\varphi) \\ i\sin(2\varphi) & \cos(2\varphi) \end{pmatrix} \begin{pmatrix} v_1\dagger & 0 \\ 0 & u_1\dagger \end{pmatrix} $$

(A.1)

where

$$ \Sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} $$

(A.2)

The matrix $Q(t + \delta t)$ then has the form

$$ Q(t + \delta t) = \begin{pmatrix} v_1(t + \delta t) & 0 \\ 0 & u_1(t + \delta t) \end{pmatrix} \begin{pmatrix} \cos(2(\varphi + \delta \varphi)) & i\sin(2(\varphi + \delta \varphi)) \\ i\sin(2(\varphi + \delta \varphi)) & \cos(2(\varphi + \delta \varphi)) \end{pmatrix} \begin{pmatrix} v_1\dagger(t + \delta t) & 0 \\ 0 & u_1\dagger(t + \delta t) \end{pmatrix} $$

(A.3)

On the other hand, the evolution equation $S(t + \delta t) = S(t)e^{\delta X}$ implies

$$ Q(t + \delta t) = S(t)e^{\delta X}e^{-i\Sigma_z\delta X\Sigma_z}S(t)\dagger \Sigma_z S(t)\dagger \Sigma_z = S(t)e^{\delta Q} \Sigma_z S(t)\dagger \Sigma_z $$

(A.4)

with

$$ i\delta Q = i(\delta X - \Sigma_z\delta X\Sigma_z) + \frac{1}{2} [\delta X, \Sigma_z\delta X\Sigma_z] + O(\delta X^2) $$

(A.5)

The matrix $\delta Q$ has the form

$$ \delta Q = \begin{pmatrix} 0 & \delta m \dagger \\ \delta m & 0 \end{pmatrix} $$

(A.6)

where $\delta m$ is an arbitrary complex $N \times N$-random matrix with

$$ \langle \delta m_{ij} \rangle = 0 \quad , \quad \langle \delta m_{ij}\delta m_{kl} \rangle = 4D\delta t \delta_{ik}\delta_{jl} \quad , \quad \langle \delta m_{ij}\delta m_{kl} \rangle = 0 \quad . $$

(A.7)

The matrix $S(t)$ in (A.4) contains still the unitary matrices $v_2$ and $u_2$ which can be taken into account by the replacement $v_2\delta m_{ij}u_2\dagger = \delta \tilde{m}$. This gives for $\delta \tilde{m}$ the same statistics as that described by equation (A.7).

At this point, we can make an important remark concerning the case where the matrix $\delta X$ obeys a more general distribution than that of equation (2.6). In this case, the property (A.7) for the matrix $\delta \tilde{m}$ is of course no longer valid. But let us assume that the two unitary matrices $u_2$ and $v_2$ are independently distributed according to the invariant measure for unitary $N \times N$-matrices, i.e., $u_2$ and $v_2$ are described by two independent $N \times N$ CUEs. Then the average over these two unitary matrices yields just the distribution (A.7) for the transformed matrix $\delta \tilde{m} = v_2\delta m_{ij}u_2\dagger$. The only information that we need from $\delta m$ is therefore the invariant quantity

$$ \langle \text{tr}(\delta m_{ij}\delta m) \rangle = 4D N^2 \delta t $$

(A.8)

which relates the time step $\delta t$ with the perturbation $\delta X$ or $\delta m$ respectively. This proves that the Fokker-Planck equation derived below is also valid for a more general perturbation $\delta X$,
provided that the two unitary matrices \( u_2 \) and \( v_2 \) are for each time \( t \) independently CUE distributed. The latter assumption is of course of crucial importance. In the case of an initial condition of one \( 2N \times 2N \) COE or two \( N \times N \) CUEs for \( S(0) \), this assumption is at least valid at \( t = 0 \) and we assume that it remains correct for arbitrary \( t > 0 \). We conclude that the Fokker-Planck equation derived for a perturbation \( \delta X \) obeying (2.6), still holds if one can relate the Fokker-Planck time to the perturbation via (A.8).

Since we are interested in the statistics of \( \delta \varphi_j \) in (A.3), we have to express the \( \delta \varphi_j \) in terms of \( \delta \bar{m} \). This corresponds to an eigenvalue problem after the transformation

\[
\bar{Q} = U^\dagger QU, \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\]

which implies

\[
\bar{Q}(t) = W \begin{pmatrix} e^{2i\varphi} & 0 \\ 0 & e^{-2i\varphi} \end{pmatrix} W^\dagger, \quad (A.10)
\]

\[
\bar{Q}(t + \delta t) = W \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} e^{i\delta \bar{Q}} \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} W^\dagger, \quad (A.11)
\]

\[
\delta \bar{Q} = \frac{1}{2} \begin{pmatrix} \delta \bar{m}^\dagger + \delta \bar{m} & \delta \bar{m}^\dagger - \delta \bar{m} \\ -(\delta \bar{m}^\dagger - \delta \bar{m}) & -(\delta \bar{m}^\dagger + \delta \bar{m}) \end{pmatrix} \quad (A.12)
\]

The matrix \( W \) which appears in (A.10) and (A.11) is just the unitary matrix

\[
W = U^\dagger \begin{pmatrix} v_1 & 0 \\ 0 & u_1 \end{pmatrix} U
\]

where \( U \) is defined in (A.9). The matrix \( W \) results only in a similarity transformation which does not change the eigenvalues. Equation (A.3) implies that \( \bar{Q}(t + \delta t) \) has just the eigenvalues \( e^{2i(\varphi_j + \delta \varphi_j)} \) and \( e^{-2i(\varphi_j + \delta \varphi_j)} \), which can be obtained by second order perturbation theory

\[
e^{2i(\varphi_j + \delta \varphi_j)} = e^{2i\varphi_j} + V_{jj} + \sum_{k(\neq j)} \frac{V_{jk} V_{kj}}{e^{2i\varphi_j} - e^{2i\varphi_k}} + \sum_k \frac{V_{2,j+k+N} V_{k+N,j}}{e^{2i\varphi_j} - e^{-2i\varphi_k}} \quad (A.14)
\]

The perturbing matrix \( V \) in equation (A.14) is just

\[
V = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \left( i\delta \bar{Q} - \frac{1}{2} \delta \bar{Q}^2 \right) \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \quad (A.15)
\]

Equation (A.14) leads after some algebra to

\[
\delta \varphi_j = \frac{1}{2} \delta \bar{Q}_{jj} + \frac{1}{4} \left( \sum_{k(\neq j)} \cot(\varphi_j - \varphi_k) \delta \bar{Q}_{jk} \delta \bar{Q}_{kj} + \sum_k \cot(\varphi_j + \varphi_k) \delta \bar{Q}_{j+k,N} \delta \bar{Q}_{N+k,j} \right) \quad (A.16)
\]

We insert equation (A.12) in equation (A.16) and apply the average (A.7) for the matrix \( \delta \bar{m} \) that results in

\[
\langle \delta \varphi_j \rangle = \frac{D}{4} f_j(\varphi) \delta t \quad , \quad \langle \delta \varphi_j \delta \varphi_k \rangle = \frac{D}{2} \delta \delta \varphi_j \delta \varphi_k \quad (A.17)
\]

with

\[
f_j(\varphi) = 2 \sum_{k(\neq j)} \left( \cot(\varphi_j - \varphi_k) + \cot(\varphi_j + \varphi_k) \right) + 2 \cot(2\varphi_j) \quad (A.18)
\]
The corresponding Fokker-Planck equation has the form

\[
\frac{\partial p(\varphi, t)}{\partial t} = \frac{D}{4} \sum_j \frac{\partial}{\partial \varphi_j} \left( \left( \frac{\partial}{\partial \varphi_j} - f_j(\varphi) \right) p(\varphi, t) \right)
\]

(A.19)

which is just equation (2.7) for \( \beta = 2 \) because \( f_j(\varphi) = \frac{\partial}{\partial \varphi_j} \ln F_2(\varphi) \), with \( F_2(\varphi) \) given by equation (2.4).

References

[34] Beenakker C.W.J., private communication.