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Dynamics of a Neural Network Composed by two Hopfield Subnetworks Interconnected Unidirectionally

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Abstract. — By means of numerical simulations, the dynamical behaviour of a Neural Network composed of two Hopfield Subnetworks interconnected unidirectionally and updated synchronically (at zero temperature \( T = 0 \)) is studied. These connections are such that each of the \( N \) neurons on the first subnet sends information to exclusively one of the \( N \) neurons on the second. A set of \( p \) patterns, composed by a subpattern for each of the subnets, is stored according to a Hebb-like rule. The recoverability of one particular subpattern by the second subnet is studied as a function of \( N \) and the load parameter \( \alpha = p/N \) in the case when the first subnet has already recovered its corresponding subpattern.

1. Introduction

During the last few years, there has been a growing interest in the study of Ising-like models of Neural Networks (NN), that is, the study of the collective behaviour of a large number \( N \) of simple interacting two-state elements called neurons. These systems have become the prototype model for the storage and retrieval of information in a content-addressable way, as stored information acts as attractors of the dynamics of the system. One of the most studied models is the Hopfield model [1], which consists of \( N \) elements connected to each other with symmetrical interactions between them as given by Hebb’s learning rule [2] for \( p \) patterns. It has been found that in the thermodynamical limit \( (N \to \infty) \), for \( \alpha = (p/N) \to 0 \), all \( p \) patterns can be stored exactly [3]; on the other hand, for \( \alpha \approx 0 \), the information is stored with a few mistakes, the number of which increases with \( \alpha \), until for \( \alpha = \alpha_c \approx 0.138 \), there is a sudden catastrophe and information can no longer be retrieved [4].

In this paper we study the dynamical behaviour at zero temperature \( (T = 0) \) of a finite NN composed by two subnetworks, each with internal symmetric connections between every pair of neurons given by the Hebb rule. These subnetworks are unidirectionally interconnected to each other, by means of strong Hebb-like connections in such a way that each of the \( N \) neurons in the first net (subnetwork 1) sends information to one, and only one, of the \( N \) neurons in the
We considered two finite coupled subnetworks interconnected unidirectionally, in such a way that each of the $N$ elements in subnetwork 1 sends information to one, and only one, of the $N$ elements in subnetwork 2.

second one (subnetwork 2) (see Fig. 1). In this system, we store $p$ random unbiased patterns or states (memories) $\{\xi_i^m\}$ ($i = 1, \ldots, 2N$), each composed by two subpatterns or submemories and stored in its respective subnet ($i = 1, \ldots, N$ and $i = N + 1, \ldots, 2N$, respectively). An appropriate value for the connections between the two subnetworks is expected to allow for the recovery of a complete pattern, provided the initial state of the first subnet is "close enough" to the corresponding subpattern. This is the issue we address in this paper: by doing computer simulations we will study the recoverability of a complete pattern when subnetwork 1 has already recovered its corresponding subpattern, as a function of the number of stored patterns.

Studying this kind of models is of interest as we know that complex behaviour is constituted by a series of simpler operations. For example, in order to read, it is important to recognize letters, but we need to know that a string of letters represents a word. In order to exemplify the problem to be studied, we could make an analogy consisting in storing 2-letter words (patterns), formed by letters (subpatterns). In this way, letters would be stored in the subnets while words would be stored in the whole network. For example, we could store the word "si" (pattern number 1) consisting of letters "s" (subpattern1) and "i" (subpattern2), where upper index refers to pattern number and the lower one to subnetwork number. In a similar way, we could also store other 2-letter words, such as "no" (pattern number 2). If we intended to recover the word "si" by putting the initial state of the first subnetwork within the basis of attraction of the letter "s", we could retrieve the word "si", but perhaps, we could also retrieve the word "so" or the letter "s" together with any nominated (another letter) or not nominated state (a spurious state) [5]. For the sake of clarity and without losing any generality, throughout this paper we will always refer to the retrievability of pattern1 by subnetwork 2 (although the results apply to any of the patterns), therefore eliminating the need of a script to indicate pattern number in the variables notation.

In this problem, subnetwork 1 is a typical Hopfield Neural Network, which evolves independently of any dynamical process taking place in subnetwork 2. The sizes of the basins of attraction for such a NN have been calculated by using several techniques, so we will not repeat such calculations. By doing computer simulations, Forrest [6] evaluated the mean fraction $f(q_o)$ of states which are recalled with less than $N/16$ errors as a function of $q_o$ and $p = \alpha/N$, where $q_o$ is the overlap between the initial state of the network and the related nominated pattern. This criterion is equivalent to demanding a final overlap $q_f \geq 7/8$, or a fraction of mistakes lower than 0.0625, to consider the information as correctly recalled. Another group of methods used to calculate the size of the basins of attraction for the Hopfield NN, in the thermodynamical limit, consists in iterating numerically the flux equations for the overlaps between the state of the system and each of the $p$ stored patterns until a fixed point is reached [7];
following this approach, Viana and Coolen [8] calculated, for the Hebb model, the fraction of all initial states which are within any of the \((p = 1, \ldots, 18)\) basins of attraction related to the nominated patterns. By dividing such fraction by \(p\), we can obtain the mean fraction of all initial states which lie within the basin of attraction of one (any) of the patterns.

Knowing these previous results, we can concentrate herein on studying the behaviour of subnetwork 2 provided subnetwork 1 has already recovered the information related to \(\pi\). We would like to mention that there is another group working on coupled networks from a different perspective: they are modelling semantic memory by considering the case of biased patterns stored on several coupled subnetworks with diluted connections between them [9].

2. The Model

We consider a Neural Network composed by two subnetworks, each constituted by \(N\) interacting two-state neurons with symmetric interactions. Both nets are connected unidirectionally in such a way that \(i\)th neuron in subnetwork 1 sends information exclusively to the \((i + N)\)th neuron in subnetwork 2. The dynamical evolution is assumed to take place synchronically at zero \((T = 0)\) noise; therefore, these two subnetworks evolve in time according to the updating rule

\[
S_i(t + 1) = \text{sign}(h^k_i(t)),
\]

where \(S_i(t) = \pm 1\) denotes the dynamical state of the \(i\)th neuron at time \(t\) with \(i = 1, \ldots, N\) corresponding to subnetwork 1 and \(i = N + 1, \ldots, 2N\) to subnetwork 2; \(h^k_i\) is the field at site \(i\) on the \(k\)th subnetwork, due to the interaction with the rest of the neurons. For the first network this field is given by

\[
h^1_i(t) = \sum_{j=1}^{N} J_{ij} S_j,
\]

with \(i = 1, \ldots, N\), and for the second network

\[
h^2_i(t) = \sum_{j=N+1}^{2N} J_{ij} S_j + \sum_{j=1}^{N} W_{ij} S_j \delta_i, j+N,
\]

with \(i = N + 1, \ldots, 2N\). The interactions \(J_{ij} = J_{ji}\) within each subnetwork are such that the first \(N\) components of each of \(p\) random unbiased patterns \(\{\xi^\mu\}, (\xi^\mu = \pm 1\) with equal probability\), have been stored in subnetwork 1, and the remaining \(N\) components in subnetwork 2; these connections are given by

\[
J_{ij} = \frac{1}{N} \sum_{\mu=1}^{p} \xi^\mu_i \xi^\mu_j,
\]

for \(i \neq j\) and within the same subnet, and \(J_{ii} = 0\). \(W_{ij}\) is an asymmetrical Hebb-like connection sending information unidirectionally from spin \(j\) in subnetwork 1 to spin \(i\) in subnetwork 2, and is given by

\[
W_{ij} = \sum_{\mu=1}^{p} \xi^\mu_i \xi^\mu_j \delta_{i,j+N},
\]
for \( j = 1, \ldots, N \) and \( i = N + 1, \ldots, 2N \). In the last expression the normalization was chosen in a way that subnetworks do not get disconnected as \( N \) grows.

In order to characterize how close the system is, at any time, to the pattern to be recalled, we define \( m^k \) to be the overlap between the dynamical state of subnetwork \( k \) \( \{S_i\} \) and its corresponding subpattern \( \{\xi_i^1\} \), given by

\[
m^k \equiv \frac{1}{N} \sum_{i=(k-1)N+1}^{kN} \xi_i^1 S_i,
\]

with \( k = 1, 2 \) for the first and second subnets. The use of the subscripts \( (f) \) and \( (o) \), in \( m^k_f \) and \( m^k_o \), will account for final and initial values, respectively. Notice that the maximum value for the overlaps has been normalized to 1 within the subnets, so the fraction of correct bits recovered by each of the subnetworks is given by \( \frac{1}{2} (1 + m^k) \). Therefore, the overlaps \( \{m^k\} \) are related to the total overlap \( q \), between the complete pattern and the state of the whole system, as follows

\[
q = \frac{1}{2N} \sum_{i=1}^{2N} \xi_i^1 S_i = \frac{1}{2} [m^1 + m^2]
\]

3. Numerical Simulations

We study the dynamical behaviour of the network for synchronous updating of the spin states as given by equations (1-2). As we mentioned before, we are interested in evaluating the "size" of the basin of attraction of a subpattern, provided subnetwork 1 has already recovered its corresponding part. We considered values of \( N \) ranging between 32 and 1024, and several values of \( p \), the highest of them being 144 for the largest network \( (N = 1024) \). For each set of values \( p \) and \( N \) considered, we studied a collection of 100 different neural networks where the patterns were stored according to the learning rules given by equations (3). In all cases, the initial state of subnetwork 1 was set to be \( S_i = \xi_i^1 \), for \( i = 1, \ldots, N \) \( (m^1_o \equiv 1) \), while various initial states for subnetwork 2 were considered; so we knew subnetwork 1 was at, or at least very close to, the minimum related to the subpattern. After the whole network reached a steady state, the criterion used to identify whether or not it had recovered it, was having a final overlap \( m^2_f \) with this memory of order 1, while the absolute value of the other overlaps being significantly lower (of order \( 1/\sqrt{N} \)).

The first thing we noticed was that the basin of attraction related to the subpattern, if subnetwork 1 has already recovered its part, is huge (much bigger than the size it would have for a Hopfield network with the same \( N \) and \( \alpha \) and without an external signal); except for a small \( \alpha \)-dependent fraction of cases, shown in Figure 2, the following happens:

If subnetwork 1 is put in an initial state given by \( S_i = \xi_i^1 \) for \( i = 1, \ldots, N \), so \( m^1_o = 1 \), then the persistent signal coming from subnetwork 1 drives subnetwork 2 towards the minimum related to the subpattern, independently of the value of \( m^2_o \). Furthermore, the value of the final overlap \( m^2_f \) is generally the same for any initial state of subnetwork 2 (having different values for different realizations \( \{J_{ij}\} \)). Crosses (××) in Figure 3 show a typical relationship between the initial and final overlaps for an arbitrary network with \( N = 1024 \) and \( p = 127 \). Notice that even for an initial state with \( m^2_o = -1 \) the system evolves towards the same final state as it does when values \( m^2_o \approx +1 \) are used. Similar results were found for most \( N \) and \( p \) considered, even if the initial state of subnetwork 2 was identical to other nominated pattern; however, we found a few isolated cases difficult to quantify (always for even \( p \), and the lowest values of \( N \).
Fig. 2. — Percentage of times that subnetwork 2 recovered a given subpattern if \( m^1_0 = 1 \), for a set of 100 different realizations \( \{J_x\} \) for each \( p \) value, with \( N = 1024 \). A dotted (solid) line connects even (odd) values for the number \( p = \alpha N \) of stored patterns.

Fig. 3. — Final overlap \( m^2_n \) as a function of the initial overlap \( m^2_0 \), for a typical and an untypical cases: Crosses (××) represent the usual relation between these parameters, for one particular case with \( N = 1024 \) and \( p = 5 \). An exception to this behaviour, where although information is still recovered its quality depends on \( m^2_0 \), is represented by diamonds (○○) for a case with \( N = 64 \) and \( p = 4 \).
Fig. 4.—Statistical study over 100 realizations in each case, of the shift in the positions of the attractors, for \( N = 1024 \) and various even values of \( p \). The figure in the right upper corner indicates the percentage of cases in which subnetwork 2 did not recover the pattern. For odd \( p \), there is a similar behaviour, but the shift is bigger, as can be seen in Figure 5.

considered) where, even when the information was still recovered, its quality depended on the initial overlap \( m_f^2 \). This is illustrated for a particular case with \( N = 64 \) and \( p = 4 \) by diamonds (\( \infty \)) in Figure 3.

These above-mentioned results are interesting, as they imply that, if \( m_f^2 = 1 \), a particular network is able to recall a complete pattern, or it is not, with an \( \alpha \)-dependent probability. But if this happens, then the basin of attraction covers the whole space of states, since any initial state for subnetwork 2 will evolve towards a minimum related to the subpattern. However, as can be seen in Figure 4, the positions of these minima are considerably displaced from the nominated patterns (as \( m_f^2 < 1 \)) which reflects a reduction in the quality of the information recovered.

Figure 4 shows a statistical study of the shift in the positions of the minima \( (m_f^2) \), for \( N = 1024 \), and various values of \( p (\alpha) \). These graphs show the percentage of times % in which \( m_f^2 \) was obtained as the final overlap, with respect to the total of cases in which the subpattern was recalled. The figure in the right upper corner indicates the percentage of cases in which the subpattern was not recovered; in all those cases, subnetwork 2 never reached a steady state but kept switching, due to frustration, among several states with the same value \( m_f^2 \). It is interesting to note that the behaviour of this model depends on the parity of \( p \); for even values the percentage of times the information is recovered is slightly smaller than it is for odd values; however the quality of the information, when recovered, is better for even \( p \) than it is for odd \( p \) (Fig. 5). As \( \alpha \) increases, the quality of the recalled information diminishes and the shift in the mean value \( <m_f^2> \) grows; this effect is stronger for small \( p \) values, and becomes smoother as \( p \) grows. It is not possible to give a critical value for \( \alpha \), for finite \( N \), since these effects increase
4. Discussion

It is very important to emphasize that if the initial state of subnetwork 1 is within the basin of attraction of one pattern, subnetwork 2 will evolve towards the attractor related to that same pattern with a high $\alpha$-dependent probability [10]. Therefore, in these cases, neither other memory-nominated or not—will play a role in the recovery of information by subnetwork 2, as the basin of attraction covers the whole space of states. This is very different from what happens in a usual Hopfield model, where the average size of the basins of attraction of each of the $p$ stored memories is the same, so the fraction of states leading to each of them diminishes as $p$ grows.

In these two coupled networks, the strength of the incoming signals in subnetwork 2 is so important, that it induces the latter into the minimum required, no matter its initial state. This finding is consistent with the observation that nervous systems show a different set of activity patterns when observed in freely moving animals than when observed in vitro, where many important afferent inputs are eliminated [11]. It also explains the repertoire of different collective responses, of a group of interacting neurons, depending on the afferent signals.

It is difficult to compare the load capacity of this system with that of the Hopfield model $\alpha_c \approx 0.138$ for $N \rightarrow \infty$, since many different criteria can be used to make such an analysis. Notice that as subnetwork 1 is an $N$-element Hopfield Network, we cannot store more subpatterns (and therefore more patterns) than $p \approx \alpha_c N$; therefore, the relation between the number $p$ of complete patterns and the total number $2N$ of units gives us as a storage limit of the complete network half that of the common Hopfield Network. However, in order to recover information, we also need half the percentage of bits required to recover it in a Hopfield NN. For it is only necessary for the initial state of subnetwork 1 to be within the basin of attraction of a given subpattern, independently of the initial state of subnetwork 2 (it could even coincide with any
of the other subpatterns). On the other hand, each of the $2p$ subpatterns could be considered as a pattern itself, so by taking this position, for large $N$ this system would equal the capacity limit of the Hopfield network, but with half the total number of connections, as these would be $2N^2$ for a Hopfield NN with $2N$ elements and $N^2$ in this case; but we would still need only half the bits which are necessary to recover each of the $p$ patterns in the usual Hopfield model.

The results that we found for this system composed by two finite coupled subnetworks are very promising. They could be useful to implement chains of coupled networks storing information composed of meaningful subsets, which are to be recalled sequentially in time and space. It is important to point out that in this model, the fraction of the total information needed to make the recovery possible is much lower than in the usual Hopfield model, since the persistent signal coming from the previous subnetwork - when it has reached a stable state - acts as a threshold on the neurons of the second subnetwork drawing collectively its dynamic state into the basin of attraction related to the stored memory in consideration, thus playing a role similar to that of an "initial state". However, one of the main problems found is that information recalled by the second subnetwork has many errors with respect to the stored information (low $m_1^2$ values). This suggests to us, as a next step, to study a dynamical process composed by as many stages as subnetworks are coupled, where, after subnetwork $n$ reaches a steady state, the signal coming from subnetwork $(n - 1)$ is suppressed. In this way, subnetwork $n$ would be allowed to search for a new minimum taking as an initial state the one obtained when still connected to its feeding partner. This possibility is currently under investigation [12].

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References

[4] This value for $\alpha_c$ was calculated within the replica symmetric approximation, in the limit $N \rightarrow \infty$.
[5] One word of caution about this analogy must be said: the analogy has its pitfalls, when used for the visual representation of letters, since here we are considering non-correlated patterns with zero mean and without any spatial structure.
[10] This $\alpha$-dependent probability has its origin in the cross-talk noise.