Long-range correlated percolation and flow and transport in heterogeneous porous media

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Long-range correlated percolation and flow and transport in heterogeneous porous media

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Abstract. — We introduce a percolation model with long-range correlations, and investigate its scaling properties. Using this model, we provide a theoretical explanation for experimental data for hydrodynamic dispersion in heterogeneous rock formations and aquifers that had remained unexplained for a long time.

Transport processes in disordered media are relevant to modelling a wide variety of phenomena, such as flow in porous media [1]. Many of such processes involve a *critical point* phenomenon, and to model them the disordered medium is usually represented by a percolation network [2, 3] in which a randomly-selected fraction p of the bonds allow transport processes to occur, while the rest of the bonds do not allow them. However, although random percolation has provided much insight into transport in disordered media, natural systems are *not* usually random and contain some correlations. In some cases, e.g., packing of particles, there are only short-range correlations, while in other cases, e.g., heterogeneous rocks and aquifers (see below), there are long-range correlations that make them very different from random media. The goal of this paper is to introduce a physically-motivated model of percolation with long-range correlations, and to demonstrate its relevance to some problems of practical importance.

An important property of percolation systems is their correlation length ξ_p which diverges as the percolation threshold p_c is approached as $\xi_p \sim (p - p_c)^{-\nu}$. For any length scale $L \gg \xi_p$ the system is macroscopically homogeneous, and thus the classical equations of transport with constant transport coefficients are applicable, while for $L \ll \xi_p$ the system is not homogeneous, such equations are not applicable, and the sample-spanning cluster is a self-similar and fractal object with a fractal dimension d_p. The sample-spanning cluster can be divided into two parts: the dead-end part that carries no flow or current, and the *backbone* through which flow and transport take place. Near p_c the backbone fraction X^B vanishes as $X^B \sim (p - p_c)^{\beta_B} \sim \xi_p^{-\beta_B/\nu}$, while for $L \ll \xi_p$ the backbone is a fractal object with a fractal dimension $d_B = d - \beta_B/\nu$, where d is the dimensionality of the system. For random percolation, these critical exponents are universal, and in particular, $\nu = 4/3$, $d_p = 91/48$, and $d_B \simeq 1.64$ for $d = 2$. If the conductance or permeability of the bonds is selected randomly from a distribution function $f(x)$, then the conductivity g of the system vanishes as $g \sim (p - p_c)^{\mu} \sim \xi_p^{-\mu/\nu}$, and

the hydrodynamic permeability k vanishes as $k \sim (p - p_c)^e \sim \xi_p^{-e/\nu}$. If $f(0)$ is nonsingular, then $e = \mu$. However, if $f(x)$ is very broad, e and μ are not necessarily equal [4]. For $L \ll \xi_p$ one should replace ξ_p with L to obtain scale-dependent properties.

Now consider a stationary stochastic process $B_H(\mathbf{r})$, called *fractional Brownian motion* (fBm) [5], with the following properties

$$\langle B_H(\mathbf{r}) - B_H(\mathbf{r}_0)\rangle = 0 \,, \tag{1}$$

$$\langle [B_H(\mathbf{r}) - B_H(\mathbf{r}_0)]^2\rangle \sim |\mathbf{r} - \mathbf{r}_0|^{2H} \,, \tag{2}$$

where $\mathbf{r} = (x, y, z)$ and $\mathbf{r}_0 = (x_0, y_0, z_0)$ are two arbitrary points, and H is called the Hurst exponent. A remarkable property of fBm is that it generates correlations whose extent is *infinite*. For example, if we define a correlation function $C(r)$ by

$$C(r) = \frac{\langle -B_H(-r)B_H(r)\rangle}{\langle B_H(r)^2\rangle} \,, \tag{3}$$

then one finds that $C(r) = 2^{2H-1} - 1$, *independent* of r. Moreover, the type of correlations can be tuned by varying H. If $H > 1/2$, then $C(r) > 0$ and fBm displays *persistence*, i.e., a trend (for example, a high or low value) at r is likely to be followed by a similar trend at $r + \Delta r$. If $H < 1/2$, then $C(r) < 0$ and fBm generates *antipersistence*, i.e., a trend at r is not likely to be followed by a similar trend at $r + \Delta r$. For $H = 1/2$ the trace of $B_H(\mathbf{r})$ is similar to that of a random walk. Fractional Brownian motion has found many applications. For example, it has been shown [6] that successive increments in the cardiac beat-to-beat intervals of healthy subjects follow fBm with $H < 1/2$.

We now use fBm to propose a percolation model with long-range correlations. Although a few percolation models with long-range correlations have already been proposed [7, 8], these models are different from ours, which has clear physical motivation and applications. Hewett [9] analyzed the permeability of heterogeneous rock formations at large length scales (of order of several kilometers), and showed that it follows a fBm with $H > 1/2$. An fBm may also be relevant to modelling fracture network of rock [10]. Thus, we propose the following percolation model with long-range correlations. To each bond of a network we assign a number selected from an fBm, and interpret it as its permeability. To construct a percolation network and to preserve the correlations between the bonds, we remove those bonds that have been assigned the *smallest* numbers. The idea is that in rock with a broad distribution (such as the fBm) of the permeabilities a finite fraction of the rock should have very small permeability, and therefore their contribution to the macroscopic properties of the system would be negligible. Figure 1 shows a square network in which the permeabilities have been selected according to an fBm with $H = 0.8$, and a fraction of the bonds with the smallest permeabilities have been removed. For comparison, we also show the same network in which the same fraction of the bonds have been removed *at random*. The striking difference between the two systems is due to the positive correlations, as a result of which most bonds with large or small permeabilities are clustered together. Moreover, as we can see in figure 1, the percolation cluster generated by this model does not have many dead-end bonds and is close to its backbone. This assertion is confirmed by our numerical results discussed below. On the other hand, if we consider the percolation cluster for, e.g., $H = 0.2$, shown in figure 2 (the anticorrelated case), it contains more randomness.

To see the difference between our correlated percolation model and that of [8], we should look at the correlation functions of the two models. Prakash *et al.* [8] considered a percolation model in which the correlation function $C(r)$, defined by $C(r) = \langle u(r')u(r + r')\rangle$, where $u(r)$

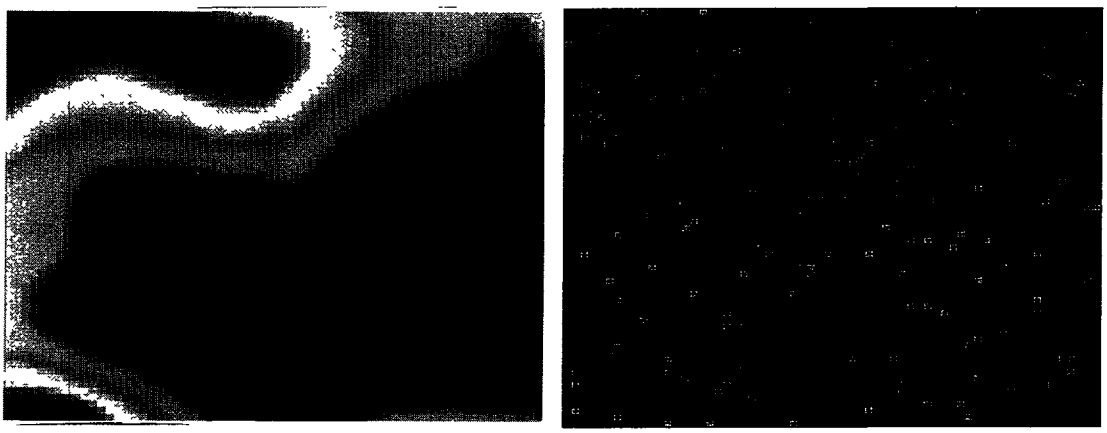

Fig. 1. — a) The structure of the correlated and b) random percolation clusters for $H = 0.8$. In the correlated network 40% of the bonds with the lowest permeabilities have been removed, whereas in the random network the same fraction of the bonds has been removed *randomly*. Lightest area denotes the bonds with the highest permeabilities, whereas black area denotes those with zero permeability (removed bonds).

Fig. 2. — The structure of the percolation network with anticorrelations for $H = 0.2$.

is a random variable obeying the distribution with long-range correlations, and $\langle\cdot\rangle$ denotes an average over all values of r', in a d-dimensional system is given by

$$C(r) \sim r^{-(d-\lambda)} , \tag{4}$$

where $-2 \leq \lambda \leq 2$ is a parameter of their model, such that $0 \leq \lambda \leq 2$ represents positive correlations, while $-2 \leq \lambda \leq 0$ corresponds to negative correlations. Note that, according to equation (4) $C(r)$ *always* decays if $\lambda < d$. In contrast, if in our model we define the correlation function $C(r)$ by the same equation, then

$$\Delta C(r) = C(0) - C(r) \sim r^{2H} , \tag{5}$$

which indicate that $C(r)$ does not decay, unless H is negative. Moreover, in our model, $C(r)$ as defined by equation (3) is independent of r.

We investigated various properties of this percolation model and calculated p_c, ν, $\hat{e} = e/\nu$, $\hat{\mu} = \mu/\nu$, d_p and d_B; see table I. We found that for $1/2 < H < 1$ p_c decreases with increasing

Table I. — *Values of the critical exponents for the two-dimensional percolation with long-range correlation, as a function of the Hurst exponent H.*

H	$\hat{e}$	$\hat{\mu}$	d_B
0.50	0.98	0.98	1.64
0.60	0.91	0.95	1.82
0.75	0.86	0.80	1.85
0.90	0.82	0.50	1.89
0.98	0.76	0.32	1.96

H, while the reverse is true for $0 < H < 1/2$. Moreover, ν and d_p essentially retain their value for random percolation, except when $H \simeq 1$, where $d_p \to 2$. We also found that for $H > 1/2$ d_B increases with H, that $d_B \to 2$ as $H \to 1$, and that $d_p \simeq d_B$, confirming our assertion that for $H > 1/2$ the cluster and its backbone are similar. More details about various properties of our correlated percolation model for the entire range $0 < H < 1$ for both $d = 2$ and 3 will be given elsewhere [11].

We now show how our correlated percolation model can explain experimental data on hydrodynamic dispersion in heterogeneous porous media. A complete discussion of our results will be given elsewhere [12]. Dispersion, the unsteady mixing of two miscible fluids in a porous medium, is caused by a chaotic velocity field in the pore space. It can be modified by molecular diffusion which transfers the solute out of the stagnant regions of the pore space and the slow boundary layer zones near the pore walls. Dispersion is important to enhanced recovery of oil, salt-water intrusion in coastal aquifers, pollution of groundwater flow, and several other phenomena [1]. Dispersion in homogeneous porous media is modeled by the convective-diffusion equation (CDE), $\partial C/\partial t + \mathbf{v} \cdot \nabla C = D_L \partial^2 C/\partial x^2 + D_T \nabla_2^2 C$, where C is the solute concentration, $\mathbf{v}$ the average flow velocity, D_L the longitudinal dispersion coefficient (in the direction of macroscopic flow x), and D_T and ∇_2^2 are the dispersion coefficient and the Laplacian in the transverse (perpendicular to the macroscopic flow) directions, respectively. An important characteristic of dispersion is the dispersivity $\alpha_L = D_L/\mathbf{v}$, which is the length scale above which a description of dispersion by a CDE is valid. Such a description, which assumes that $\mathbf{v}$, D_L, D_T and α_L are independent of length scale and time, has been reasonably successful for porous media at *small* length scales (of order of at most a few meters) [1].

However, there have been several *field* studies of dispersion [13-16] indicating that D_L and α_L are scale and time dependent, and that D_L depends linearly on $\mathbf{v}$. Arya *et al.* [16] analyzed over 130 greatly-varying field dispersivities, collected on length scales up to 100 km, and showed that most of the data follow the scaling law

$$\alpha_L \sim L^{\delta} , \tag{6}$$

where L is the length scale of the measurements or the distance from the source (where the solute is injected into the flowing fluid in the rock), with $\delta \simeq 3/4$. However, a reanalysis of these and other data [17] indicated that, for the data that are collected at large length scales, δ may be close to 1. The analysis of various field data has also indicated that [16-19]

$$\alpha_L \sim t^{\zeta} , \tag{7}$$

where a *nonuniversal* $\zeta \simeq 0.5 - 0.6$ has been found to provide accurate fit of the data. Up to now, equations (6) and (7) have not found a satisfactory explanation. We now show that our correlated percolation model provides a consistent theoretical explanation for these results.

As discussed above, if rock permeabilities are distributed according to a fBm, then the pore space in which flow and transport take place is similar to the backbone of a correlated percolation system. Therefore, since the fraction of the dead-end pores or stagnant regions in the system is very small, molecular diffusion that transfers the solute into and out of such regions plays no significant role. This means that dispersion is dominated by the stochastic velocity field imposed on the medium by the permeability distribution and, consistent with the field data, D_{L} depends on the average flow velocity $\mathbf{v}$ as

$$D_{\mathrm{L}} \sim \xi\, \mathbf{v}\,, \tag{8}$$

where ξ is some appropriate length scale. Under such conditions, the role of diffusion is to transfer the solute out of the slow boundary layer zones near the pore surfaces, and its effect appears only as a logarithmic correction to equation (8). That is, if we include this effect, we obtain [20, 21] $D_{\mathrm{L}} \sim \xi \mathbf{v} \ln(a\mathbf{v})$, where a is a constant, but since we are interested only in the *scaling* of D_{L} and α_{L} with t (or L), such logarithmic corrections do not affect the scaling. Note that in the presence of the low permeability zones, diffusion into and out of such zones would be important, and we would have [21] $D_{\mathrm{L}} \sim \mathbf{v}^2$, contradicting the field data.

Because flow and dispersion take place only in the backbone of a correlated percolation system, the average flow velocity $\mathbf{v}$ is proportional to k/X^{B}, and near p_{c} we have $\mathbf{v} \sim (p-p_{\mathrm{c}})^{e-\beta_{\mathrm{B}}} \sim \xi_{\mathrm{p}}^{-\theta}$, where $\theta = (e-\beta_{\mathrm{B}})/\nu = \hat{e} + d_{\mathrm{B}} - d$. Although dispersion in *random* percolation networks has been analyzed before [22, 23], it has been investigated neither in percolation structures with long-range correlations, nor in the regime we discuss here. Since the permeabilities are infinitely correlated, their correlation length is larger than any other relevant length scale of the system, and therefore the only dominant length scale of the system is its linear size L, implying that the system is a backbone fractal for *any* L and that $\mathbf{v} \sim L^{-\theta}$, so that

$$D_{\mathrm{L}} \sim L^{1-\theta} \tag{9}$$

We now define $\langle \Delta x^2 \rangle = \langle [x - \langle x \rangle]^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$, where $\langle \cdot \rangle$ represents an average over all values of x. Then, as all length scales of the system must be proportional to each other (and to L), equation (9) can be rewritten as $D_{\mathrm{L}} \sim \langle \Delta x^2 \rangle^{(1-\theta)/2}$ Since in general $D_{\mathrm{L}} \sim \mathrm{d}\langle \Delta x^2 \rangle / \mathrm{d}t$, we obtain $\mathrm{d}\langle \Delta x^2 \rangle / \mathrm{d}t \sim \langle \Delta x^2 \rangle^{(1-\theta)/2}$, which means that

$$\langle \Delta x^2 \rangle \sim t^{2/(1+\theta)}\,, \tag{10}$$

and that

$$D_{\mathrm{L}} \sim t^{(1-\theta)/(1+\theta)}\,. \tag{11}$$

On the other hand, we can also write $\mathbf{v} \sim \langle x \rangle^{-\theta}$, and since $\mathbf{v} = d\langle x \rangle / dt$, we obtain $\langle x \rangle \sim t^{1/(1+\theta)}$, implying that

$$\mathbf{v} \sim t^{-\theta/(1+\theta)}\,, \tag{12}$$

in sharp contrast with homogeneous porous media for which $\mathbf{v}$ is constant. Finally, since $\alpha_{\mathrm{L}} = D_{\mathrm{L}}/\mathbf{v}$, we combine equations (11) and (12) to obtain

$$\alpha_{\mathrm{L}} \sim t^{1/(1+\theta)}\,, \tag{13}$$

which means that $\zeta = 1/(1+\theta)$.

We argue that it is our *two-dimensional* correlated percolation that is relevant to the field-scale data, since such data are obtained at large distances from the source (up to several tens of kilometers), whereas the thickness of such porous media is at most a few hundred meters, and therefore such porous media are long and thin, and thus essentially two dimensional. Analysis of various field-scale permeability data by Hewett [9] and others [16-19] have indicated that

$H > 1/2$ (and mostly $0.7 < H < 0.9$), and from table I we obtain a *nonuniversal* $\zeta \simeq 0.53-0.62$ for this range, consistent with the experimental data discussed above. Note that had we not removed the low permeability zones, we would have obtained [11] $\zeta = (2+\theta_d)/[(1+\theta_a)(2\theta_a-\theta_d)]$, where $\theta_a = \hat{e} + d_p - d$ and $\theta_d = \hat{\mu} + d_p - d$. Thus, for example, $\zeta(H = 0.6) \simeq 2$, completely inconsistent with the field data.

We thus propose that percolation with long-range correlation is relevant to flow phenomena in field-scale porous media and aquifers, as it provides a consistent explanation for the experimental data on dispersion coefficients and dispersivities collected in field experiments. Separately, we have investigated the effect of such long-range correlations on growth phenomena [24], and on miscible displacements [25], and have shown that the results are consistent with field-scale experimental data.

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