Caustics and symmetries in optical imaging. The example of convective flow visualization
A. Joets, R. Ribotta

To cite this version:
<jpa-00246961>

HAL Id: jpa-00246961
https://hal.archives-ouvertes.fr/jpa-00246961
Submitted on 1 Jan 1994

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Caustics and symmetries in optical imaging. The example of convective flow visualization

A. Joets and R. Ribotta

Laboratoire de Physique des Solides, Bât. 510, Université de Paris Sud, 91405 Orsay Cedex, France

(Received 19 January 1994, accepted 18 March 1994)

Abstract. — It is shown that qualitative and quantitative information can be collected from the study of the full caustic formed by light rays deflected by some medium. The study necessitates the calculation of the ray trajectories along with a detailed balance of the symmetries of the whole set-up. As an example we consider the case of the caustic produced by parallel rays interacting with a periodic convective flow in a liquid crystal. The amplitude of the flow and all its essential features can be deduced, indicating that a caustic carries more information than a simple image.

1. Introduction.

Light rays deflected either by a transparent inhomogeneous medium or by a deformed surface can provide information on the inhomogeneities or the deformations. This is the basis of imaging techniques that are widely used in macroscopic experiments. Usually, transmitted rays are deviated because of index change in space, while reflection images are used to analyse surface deformations. Among the most familiar examples is the visualization of flows (open flows, confined periodic flows, turbulence...). Another case is provided by the optical techniques developed to study the elastic deformations in a liquid crystal subjected to some external field, with applications to optical devices. In general, these techniques can give qualitative information about the structures, but it is difficult to extract quantitative measurements, except may be concerning geometrical characteristics (distances, angles). And, actually, there is a need to increase the information contained in the images and for instance, it is important to trace back to the amplitude of the deformation.

What is called an image is the section by some plane of the family of emerging rays. Generally, this family admits a surface envelope, the caustic, named so because at every of its points the light intensity is sharply increased. It is this surface which instead, contains the information concerning the structure of the deformation under study. This surface may itself contain singularities of lower dimension that appear brighter. Now, a focussed image of a given object is obtained when the plane is positioned in such a way that it contains these
brightest singularities. An example is given by figure 1. One may presume that a better knowledge of the full caustic, including all the possible singularities, is necessary to have a more correct interpretation of the images. As an illustration of this approach we shall consider a rather simple case of 2-D periodic flow structure developed in a liquid crystal. This study was first motivated by the interpretation of the different light patterns caused by the various flow structures developed on the way to the chaos.

![Fig. 1](image1.png)

**Fig. 1** — a) Photograph of an image of the convective layer. The bright thick lines are the sections of the virtual cones by the focal plane of the microscope, located slightly below the cusps. The twinning is indicated by the different separation distances A and B. Some lines are destabilized into some points close to a defect of periodicity (dislocation), where the flow is complex and 3-D (see text). b) Sketch of the caustic in 3-D.

2. Basic elements on caustics in periodic flows.

An optical caustic is defined as the envelope of a given family of nonparallel rays [1] (Fig. 2). If the rays are parallel the caustic is formally a point rejected to infinity. Any system that has the property of deflecting incident parallel rays for instance, will produce a caustic in the near field, i.e. this caustic is pulled back from infinity. The first example that comes naturally to mind is the focal point of a converging lens. The process of ideal image formation through a perfect lens associates a point caustic with every source point. However, this caustic is unstable against infinitesimal deviations from perfect focalisation, producing then the well-known effects of spherical aberration, coma, astigmatism.

![Fig. 2](image2.png)

**Fig. 2.** — A caustic is the envelope of a family of non-parallel rays.
Other familiar examples of caustics are the dancing bright lines at the bottom of a pool due to the focalisation of sun rays by the irregular surface of the water, and also the sparkling of the reflected sun rays by the same surface. Less obvious examples in the atmosphere are the formation of mirages and of rainbows.

From a mathematical standpoint caustics are singularities that may be either stable or unstable [2-4]. It is shown that the stable singularities belong to a small number of classes: folds, cusps, swallowtails, umbilics... However, the focal point of an ideal converging lens does not appear in this classification because it is unstable (degenerate) and the effect of a perturbation is to decompose it into a set of those stable singularities.

An index modulation in space is commonly assimilated to a lens. Periodic convective flows realise a periodical set of such lenses. Indeed, a prototype is the Rayleigh-Bénard convection of a layer of isotropic fluid where the thermal gradients modulate the index of refraction. Another example is given by liquid crystals in which the index of refraction is spatially modulated by convective rolls. These rolls are produced under the action of a transverse electric field across a layer, and are periodic in one direction of space and invariant by translation along their axis [5, 6]. The problem of caustic formation reduces to a 2-D one corresponding to the basic experimental situation. We shall give a complete description of the caustics formed by the emergent rays and also by their continuation: the real and virtual caustics. Then we shall calculate the ray trajectories inside the nematic layer and we deduce the envelope of the rays. By comparing the simulation with our experimental values we are able to estimate the amplitude of the convection. We shall also study the relationship between the symmetry of the caustic and the symmetry of the index modulation. As an example, the so-called « squint effect » shall be reinterpreted.

As the structure of the flow becomes more complicated (at higher values of the electric field) one observes that the caustic becomes unstable and now contains other types of singularities that necessitate a 3-D analysis which will be presented in another publication.

3. Observation of caustics in the electroconvection of a nematic layer.

Nematics are composed of elongated molecules, whose local orientation defines a new macroscopic variable, the director usually denoted by a unit vector \( \mathbf{m} \). It is understood that directors of opposite sense are equivalent: \( \mathbf{m} = -\mathbf{m} \). Optically, the nematics are uniaxial materials, the optical axis lying along the director. The nematic (Merck Phase V) has an ordinary refractive index \( n_0 = 1.65 \) and an extraordinary refractive index \( n_\ell = 1.94 \). The liquid crystal is sandwiched between two conducting semi-transparent plates (plane \( \{x, y\} \) ) separated by a typical distance of \( d \approx 100 \mu \text{m} \). By an appropriate treatment of the plates one imposes a uniform molecular alignment in the plane of the plates (planar anchoring) along, say, the \( x \)-axis. In the rest state the nematic layer constitutes a homogeneous uniaxial medium of optical axis \( \mathbf{m} = x \).

When an AC voltage \( V \), of typical frequency \( f = 100 \text{ Hz} \) (« conduction regime » [5]), is applied across the layer and fixed at some value just above the threshold \( V_c \approx 6 \text{ V} \), the rest state is destabilized and the convection develops in the form of a stationary structure of periodic rolls, of period \( A \approx 2d \). The axis of the rolls is normal to the initial direction \( x \) of the molecules and this is the « normal roll structure » [6]. Because of the coupling between the hydrodynamic velocity and the director, the molecular alignment is periodically distorted along \( x \), the maximum of the distortion being at the roll centers (Fig. 4). A light beam, polarized along \( x \), is sent normally to the plates. Inside the layer, the effective local index for each incoming (extraordinary) ray is periodically modulated along \( x \). In a rather primary description, the layer can be reduced to a set of alternately diverging and converging lenses [7]. After traversing the nematic the rays refract at the liquid crystal-glass interface.
Fig. 3. — Microphotograph of the caustic. The sample is tilted around $x$ by 40°. The image is topologically equivalent to a vertical section of the caustics. Note that the virtual cusps are destabilized into unfolded butterflies.

$z = d$ (contrary to the incident rays which are normal both to the glass-liquid crystal interface $z = 0$ and to the optical axis $m = x$). The outgoing rays are focalized along a real caustic and their prolongation below the upper interface appear focalized along a virtual caustic. The light pattern is shown using a polarizing microscope, which gives the section of the caustic in its focal plane (Fig. 1).

In the literature these caustics are usually reduced to a set of « foci », or « focal points » (in the $\{x, z\}$ plane). A more careful study of these points [8] has shown by a reconstruction technique that they are in fact the tips of a small portion of the caustic, in the form of a cone. In order to obtain at once an image of the caustic we simply rotate the sample around the $x$-axis by an angle of about 40°. Due to the invariance of the normal roll along $y$, the resulting image is topologically equivalent to the section of the caustics in the vertical $\{x, z\}$ plane (Fig. 3). We see clearly that the caustic is composed of a sequence of cones alternately oriented upwards and downwards. In the language of singularities, all points of the caustic are « fold points », except the tips of the cones which are « cusp points ». In practice, the term cusp means rather some finite portion of the caustics around a cusp point. According to the theory of singularities [9], the two fold-curves of a cone connect tangentially at the cusp-point, as can be seen in figure 3. The cusps are the so-called « focal points » where the light intensity is much higher than anywhere else on the fold-curve. We check that the real cusps are located above the separation lines between two adjacent rolls (Fig. 4). We name them up-cusps or down-cusps.
Fig. 4. — Simulation of the ray trajectories (thin lines) and of the caustic (thick lines). The model for the molecular distortion is: \( \varphi = \varphi_0 \sin kx \sin qz \) (with \( \varphi_0 = 21.2^\circ \), \( A = 2 \pi/\lambda = 393 \mu m \), \( d = \pi/q = 308 \mu m \)). The arrows represent the director in the midplane of the layer. The convection is indicated by the dashed lines. The small circles indicate the experimental points from the observed caustics. The fit is made by adjusting at once the majority of the experimental points on the theoretical caustic curve. The typical error bar is indicated on the lowest point on the left side of the figure. The mirrors \( M \) passing through each real cusp are intercalated with the virtual cusps.

depending on whether the flow moves along \( z \) or \(-z\). The cones are not limited along \( z \). They can easily be observed up to distances of the order of \( 20 \ d \). The caustic presents remarkable geometrical characteristics. For instance, the down-cones are not identical to the up-cones: their cusp-points have a distinct height. This is not expected at first from a rapid consideration of the model for the flow. We also observe that the virtual cusps are not equidistant along \( x \). This effect was named « squint » [10] and attributed to some nonlinearity in the electroconvec-
tive mechanism, although it is observed even at or near the threshold. In the following model, we shall show that this effect can be explained differently.

It is observed that, for values of the constraint \( c = (V - V_0)/V_1 \approx 0.15 \), each virtual cusp is replaced by an ensemble of three cusps (unfolded butterfly singularity) (Fig. 3). This observation will be discussed later.

4. Equations for the rays.

We first calculate the ray trajectories inside the nematic layer. This problem can be solved by several general methods [11-13]. Here, we apply Fermat's principle which can be shown to be valid in an anisotropic medium [12]. It states that the rays are the extremals of the optical path \( \ell = \int n \, ds \) where \( n \) is the local ray index and \( ds \) is the curvilinear abscissa along the path. Here, \( n \) is given by [1].

\[
n = \sqrt{n_0^2 \sin^2 \beta + n_0^2 \cos^2 \beta}
\]  

(1)

where \( \beta \) is the angle between the local optical axis \( m \) and the local tangent vector \( r = (dx/ds, dy/ds) \) to the ray. Assuming that all the rays pass through the layer, which is checked in the experiment, we can choose \( z \) as the independent variable and \( x \) as the unknown function: \( \nu = x(z) \). The optical path now takes the form:

\[
\ell = \int_{z_0}^{z_1} L \, dz = \int_{z_0}^{z_1} n \left( z, x(z), \frac{dx}{dz} \right) \sqrt{1 + \left( \frac{dx}{dz} \right)^2} \, dz.
\]  

(2)

It is convenient to introduce the angle \( \varphi \) between the \( x \)-axis and \( m \), so that \( \beta(z, x, \dot{x}) = -\varphi(x, z) + \arctan 1/\dot{x} \) (the dot indicates the derivation with respect to \( z \)). As usual, we assume that the director is harmonically distorted: \( \varphi = \varphi_0 \sin kx \sin qz \) \( (k = 2 \pi/\lambda, q = \pi/d) \). The equation for the rays is obtained by writing the Euler-Lagrange equations for the functional \( \ell \), i.e. that the first variation of the optical path is zero for the actual rays. The equation for the rays is then:

\[
\frac{d^2 x}{dz^2} = \frac{1 + \dot{x}^2}{n + n_{\varphi \beta}} \left[ n_{\varphi} (\dot{x} \varphi_\beta - \varphi_\beta) - n_{\beta \beta} (\dot{x} \varphi_\varphi + \varphi_\beta) \right]
\]  

(3)

where the partial derivatives are noted by subscripts: \( \varphi_\beta = \partial \varphi / \partial x \), \( n_\beta = \partial n / \partial \beta \), etc. An equivalent form of equation (3) was given by Kosmopoulos and Zenginoglou, who analyzed the grating action of such a nematic layer [14].

The differential equation of second order (3) is integrated with the classical Runge-Kutta method of order 4. The step of integration \( dz \) is taken equal to \( d/200 \). The incident rays are parametrized by the \( x \)-coordinate \( \lambda \) of their entry point. The rays being normal to the layer, the initial data are \( x(0) = \lambda, \dot{x}(0) = 0 \) (the layer is defined by \( 0 \leq z \leq d \)). The abscissa \( \lambda \) of the entry point is taken between 0 and \( \lambda \). Figure 4 shows an example of such a simulation of the ray trajectories.

Before leaving the nematic layer, the rays are refracted on the liquid crystal-glass interface \( z = d \). In order to test the model quantitatively this refraction must be included in the calculation. The relation between the angle of incidence \( \iota = -\arctan \dot{x}(d) \) and the angle of
refraction $\alpha$ can be extracted from the general laws of crystals optics [1]

$$\sin \alpha = \frac{n_g^2 \tan \theta}{n_g (n_g^2 + n_0^2 \tan^2 \theta)^{1/2}}$$

(4)

where $n_g$ is the refractive index of the glass.

We now consider a set of incident rays parametrized by $\lambda$. After passing through the layer and refracting at the liquid crystal-glass interface, the rays follow a rectilinear trajectory characterized by two parameters: the abscissa $x_0 = x(d)$ of the exit point and the angle of refraction $\alpha$. Of course, $x_0$ and $\alpha$ depend on $\lambda$.

5. Calculation of the caustic curve.

The set of the emerging rays constitutes a one-parameter family of straight lines. Its equation is

$$x = x_0(\lambda) + (d - z) \tan \alpha(\lambda).$$

(5)

The caustic is by definition the envelope of the emerging rays, that is, the locus where «two neighbouring rays intersect». This is expressed by zeroing the partial derivative of $x$ with respect to $\lambda$ (see [15] and appendix):

$$\frac{\partial x}{\partial \lambda} = 0.$$  
(6)

Equations (6) and (5) define the caustic as a curve parametrized by $\lambda$. Explicitly, we obtain:

$$x(\lambda) = x_0(\lambda) - \frac{x_0'(\lambda) \cos \alpha(\lambda) \sin \alpha(\lambda)}{\alpha'(\lambda)}$$

and

$$z(\lambda) = d + \frac{x_0'(\lambda) \cos^2 \alpha(\lambda)}{\alpha'(\lambda)}$$

(7)

where the prime denotes the derivation with respect to $\lambda$. For each value of $\lambda$, the values $x_0(\lambda)$ and $\alpha(\lambda)$ are calculated by the method described in the preceding section. Then, the caustic curve is deduced by equation (7) (the derivatives with respect to $\lambda$ are numerically obtained by the central-difference method). The result is shown in figure 4. Experimentally, the position of points of the caustic is determined by varying the height of the local plane of the microscope. These are reported in the same figure. A fit is obtained by adjusting only one parameter: the amplitude $\varphi_0$ of the molecular distortion so that the majority of the points lie on the curves. Since the caustic is the ensemble of the cones the fit is not made for each cone separately, but rather for all the cones at once. A rather good fit is obtained for $\varphi_0 = 21.2^\circ$. The asymptotic branches of the caustic correspond to $\alpha'(\lambda) = 0$. The real and virtual cusp-points correspond respectively to the local minima and to the local maxima of the height $z$. They are then given by

$$\frac{dz}{d\lambda} = 0$$

(8)

(see the appendix for a rigorous presentation). In previous articles [7, 16, 17], in which approximations were made, some analytical formula was given for the height of the cusps. Here, on the contrary, the height can be deduced directly from the calculation of the whole caustic and without any approximation. However another more powerful method can be derived if one is interested only in calculating the height of the cusps. This method will be described in a forthcoming paper.
6. Caustics and symmetries.

The presence of symmetries is, among the essential geometrical characteristics of the caustics, the most noticeable feature. Here, the caustic is periodical along \( x \), with the same periodicity \( \Lambda \) as that of the flow pattern (Fig. 4), measured by another method [6]. We notice from the experiment that the real cusps are equidistant with period \( \Lambda/2 \) along \( x \), while the virtual cusps are twinned (twins separated by \( \Lambda \)) and each pair separated by \( B \) such that \( A + B = \Lambda \). The whole set of cones (real and virtual ones) has the mirror symmetry: the mirror is vertical and passes through a real cusp (a vertical line in the plane of the figure). Obviously there is a mirror for each real cusp and no mirror for the virtual ones. This may indicate a symmetry breaking, as proposed by Ben-Abraham [10]. One also notices that the down-cusp and the up-cusp are not equivalent by superposition and are at different heights. In order to understand the light pattern, one must examine all the possible symmetries of the full system.

Owing to Curie's principle [18] one must include in this description, the liquid crystal, the light beam, the interaction light-matter and the boundary conditions. This principle states that «the symmetry elements of the causes must be present in the effects produced» In our system, it is clear that we may take the light pattern as the «effect», and the director field, conjointly with boundary conditions and the incident beam, as the «cause». Now the problem to solve is the relationship between the effect and its cause. In particular, is the symmetry of the flow directly reflected in the caustic?

The nematic liquid crystal is characterized by its director field which bears the local optical axis. A simple way to define a symmetry of the director field is to consider how it is transformed by a displacement \( f \) (punctual affine transformation of the \( \{x, z\} \)-plane preserving the distance). By the transformation \( f \), a point \( P \) is transformed into the point \( f(P) \) and any vector attached to \( P \) is transformed by the first derivative \( f'(P) \) (i.e. the Jacobian matrix) of \( f \) into a vector attached to \( f(P) \) (Fig. 5). We note that the derivative \( f'(P) \) transforms also the directions attached to \( P \) into directions attached to \( f(P) \) (this results from the linearity of the operator \( f'(P) \)). One can say that the director field \( m(P) \) is invariant by the transformation \( f \), if \( m(f(P)) = f'(m(P)) \) [19].

Let us now assume that the director field \( m \) has a symmetry \( f \), and let us compare the optical path along some curve \( C \), parametrized by \( s \), and the optical path along the transformed curve \( f(C) \), parametrized by \( s' \). Observing that the local index \( n \) depends only on the angle \( \beta \) between the director \( m \) and the local direction \( ds = (dx, dz) \) of the curve, and also that \( f'(P) \) preserves this angle (or at least its magnitude), we can write:

\[
\int_{f(P)}^{f(Q)} n(\beta') ds' = \int_{P}^{Q} n(\beta) ds
\]

where \( \beta' = (m(s'), ds') \) (Fig. 5). We have also used the fact that \( ds \) is preserved by \( f \). The optical paths along \( C \) and \( f(C) \) are equal. As a consequence, the rays, which are the extremals of the optical path, are exchanged into one another by the transformation \( f \). If now the boundary conditions and the incident beam both have the symmetry of the director field, then the ray trajectories form a set globally invariant under the same transformation. In this case, the envelope of the rays, i.e. the caustic, is also invariant under the transformation. More generally, the symmetries of the caustic reveal the symmetry elements common to the director field, the incident light beam and the boundary conditions.

Let us now interpret the geometrical features of the caustics observed in our experiment. The periodicity of the caustics along the \( x \)-axis is that of the director field. The presence of the vertical mirrors passing through the real cusps indicates that the director field has this mirror symmetry. Examination of figure 3 does not reveal any other symmetry. Now let us introduce
the symmetry elements of the boundary conditions and of the incident light beam. The boundary conditions concern the lower and upper separation planes between the liquid crystal and the holding plates. There, in addition to the previous vertical mirrors, one has the horizontal mirror located in the middle of the layer and consequently an inversion point. The boundary conditions are also invariant by continuous translations along x. Indeed, they have the maximal possible symmetry. For a source located only on one side of the layer, as it is in the experiment, the incident beam is symmetrical by all translations along x and in every vertical mirror. But, in order to be complete, one must also consider an identical source located on the other side. There is indeed no a priori reason for the structure in the layer to have the reversal symmetry, for example the mirror symmetry about the median line. This is achieved experimentally, by simply turning the layer upside down. In this operation, the new caustic is clearly deduced from the previous one by an inversion relative to a point located in the centre of the rolls. At some abscissa x, a up-cusp of the former caustic is replaced on the other side of the layer by a down-cusp (Fig. 6). The up-cusp is recovered by translating the caustic by

Fig. 6. — The full caustic produced by the incident beam coming from below (resp. above) is indicated by the thick (resp. dotted) lines. The director field corresponds to the standard model. It shows the symmetry 5, generated by a mirror (the dashed vertical line), and an inversion around the center of the point located in the mid-plane of the layer. The nematic layer is confined between the two horizontal lines. The full caustic does have the same symmetries.
Moreover the twinning effect is still present and undergoes the same transformation. In conclusion, the knowledge of the full caustic (the effect), and that of the two components of the cause, enables us to deduce the symmetry elements of the last component of the cause, i.e. the structure.

This study can be supplemented, in this case, where we know some of the features of the flow structure, by using the classification of the symmetry groups for a 2-D layer. Here, there is a periodicity along an axis \( x \), and then, the classification gives seven symmetry groups [20]. These are the groups of the patterns obtained by repeating some motifs on a line. Here we choose to represent the motifs by appropriate alphabet characters. Thus the seven patterns are:

1) \( \cdots \text{LLLL} \cdots \) 2) \( \cdots \text{LTL} \cdots \) 3) \( \cdots \text{VVVV} \cdots \) 4) \( \cdots \text{NNNN} \cdots \) 5) \( \cdots \text{AVAV} \cdots \) 6) \( \cdots \text{DDDD} \cdots \) 7) \( \cdots \text{HHHH} \cdots \)

We notice that the symmetry groups 6 and 7 are not compatible with a non-zero molecular distortion at the roll centers and that the symmetry groups 1, 2 and 4 do not contain the mirror symmetry. The choice is between the symmetry group 3, generated by two vertical parallel mirrors, and the symmetry group 5, generated by one vertical mirror and one inversion (Fig. 6). In order to reveal a possible inversion point, one must insure that the two components of the cause under control have this inversion point. This is what has been made by reversing the sample. One is left with group 5. This is the maximal symmetry corresponding to a convective flow in form of rolls filling the whole thickness of the sample.

Let us now examine the particular distribution of the virtual cusps. First, one notices that the real cusps possess the symmetry with respect to the vertical mirror passing through them. This is associated with their equidistance along \( x \) (periodicity \( \lambda/2 \)). On the contrary, there is no mirror passing through the virtual cusps, but rather there is a mirror passing in between with the same periodicity \( \lambda/2 \). This does not impose an equidistance between the virtual cusps, in which case the periodicity of the cusps is \( \lambda \). This particular distribution of the virtual cusps is due to the convective flow cell characterized by an inversion point, the centre, and the absence of any vertical mirror. It might appear in experiments that the virtual cusps are very close to equidistant, but this would be rather due to some particular values of the parameters. The generality is that those cusps are not equidistant but are rather twinned.

Then, there is no need to invoke the effect of the nonlinearities, contrary to what is assumed in reference [21] in order to interpret this result. In the paper cited here, the non-equidistance was ascribed to some particular flow structure (squint) that would imply the absence of any inversion point. On the contrary, the experiment clearly shows the presence of such symmetry. Moreover, it can hardly be the result of nonlinear effects, since the twinning is observed even within 1% above threshold.

The same reason, i.e. the absence of any vertical mirror passing through the centre of the roll, implies that the up and down-cusps are not necessarily identical. Indeed the experiment indicates that they appear at different height (Fig. 3).

7. Discussion.

Let us point out that these results can be generalised. We first consider the effects of an anharmonicity of the molecular distortion along \( x \).

As already mentioned, far above the convection threshold nonlinearities come into play and the virtual cusps evolve towards more complex shapes (Fig. 3). This effect can be expected from the presence of higher harmonics in the molecular distortion. The numerical simulation of the caustic is now made after introducing two harmonics in the previous model for the molecular distortion \( \varphi = (\varphi_0 \sin kx + a \sin 2kx + b \sin 3kx) \sin qz \). The result shown in figure 7 agrees quite well with the experimental observation. Then the anharmonicity can be reflected by changes (instabilities) in the topology of the caustic. The symmetry is conserved,
because of the symmetry of the chosen harmonics. In the general case, it is likely that another type of harmonic with a different symmetry would induce a symmetry breaking of the caustic.

We have studied the case where all the rays are parallel to the \( \{ x, z \} \)-plane and the problem is essentially 2-D. The basic equations for the caustic (6) and (8) have there a simple interpretation and do not justify a mathematical demonstration. However, for other convective structures that break the continuous translational invariance along the \( y \)-axis, for instance the varicose structure [6], the problem becomes 3-D. It needs a mathematical formulation based on the theory of the singularities. Such an analysis is sketched in the appendix for the 2-D light patterns, and will be developed in detail for 3-D light patterns in a forthcoming paper. However, our statement that the full caustic has the symmetries common to the director field, the boundary conditions and the incident light beam, is independent of the dimension of the physical space. Consequently, it remains valid in the much more general case of the 3-D-light patterns. It is also valid for other types of materials, provided that the refractive index depends only on angles, so that equation (9) holds. In particular, our results apply to the case of biaxial liquid crystals.

The symmetries considered in the previous section are generated by punctual transformations, namely displacements. They act on the points, the vectors and the directors. We may also define local transformations which keep the points invariant and which act only on the vectors or on the directors. For example let us consider the standard model for the director field \( \varphi = \varphi_0 \sin kx \sin qz \) and the local mirror symmetry \( t \) around the \( x \)-axis (Fig. 8). This director field is not invariant by \( t \). Composing \( t \) with the mirror \( \ell \) in the mid-plane of the layer, we define the transformation \( \ell^* = t \circ \ell \). The director field is invariant by \( \ell^* \) while the optical path is not conserved by \( \ell^* \). Then, the reasoning that we have used throughout this article is no longer valid, and the symmetry \( \ell^* \) cannot be evidenced in the caustic. Generally, the caustic will not necessarily contain these local symmetries. This implies that the caustic might not reveal the fine structure, for instance of the flow or of the index distribution, thus restricting the amount of information that one can extract from a caustic.

A wider class of symmetries is provided by diffeomorphisms, i.e. bijective and differentiable point transformations. It is obvious that here too, our conclusions on the symmetries of the caustic are identical, provided that equation (9) holds. This problem would deserve a deeper
Fig. 8.—a) Definition of the local mirror symmetry \( t \) around the \( x \) axis, and of the global mirror symmetry \( \ell \) around the \( x \) axis. b) The director field in the convection flow does not possess both symmetries \( t \) and \( \ell \), but the symmetry \( \ell^* = t \circ \ell \).

study. We note only that it can be formalized in terms of isometries of the metric space, whose line element is defined by the optical path element \( nds \) (cf. [13]).

For the state of comparison with our example, let us finally consider the much simpler case of thermal convection in isotropic fluids (Rayleigh-Bénard convection) close to the threshold. There, the index is a function of the temperature distribution. Its period along \( x \) is the period \( A \) of the flow. The real caustics now correspond to only one direction of the vertical flow (downwards) while the virtual one corresponds to the upwards motion. There are no other cusps. With each cusp is associated a vertical mirror. Thus all cusps are equidistant.

8. Conclusion.

We have analysed in detail the geometry of the light pattern produced by a periodic flow in a layer. We have taken as an example the convection in a nematic layer mainly because it also includes an anisotropy, thus making the example richer and of a wider interest than the isotropic case. The light pattern is the section of a surface, the caustic which is singular and which contains singularities of lower dimension. The equations for the rays have been derived, and the full caustic has been calculated. It is found to be in good agreement with the experimental observations. The amplitude of the molecular distortion has been directly deduced by fitting with the experiment. The inverse problem of interpretation of the director field starting from the light pattern necessitates a detailed account of the symmetries which are present in the whole set-up: the light pattern, the light source and the boundary conditions. This gives a powerful method and the symmetry of the director field has been identified. Moreover, we have given a satisfactory interpretation of the pairing of the virtual cusps. This approach is rather general and may be valid also to caustics formed after reflections on deformed surfaces. However, in most cases it has to be supplemented by other methods in order to give a fully detailed description of the deformations that produce the light pattern. Inversely, since a caustic carries more information, it might be misleading to restrict the study to the image which is a mere section of it. Our purpose is now to extend this study to the more realistic 3-D case.

Acknowledgments.

This work was supported by the Direction des Recherches et Etudes Techniques under the contract DRET/ERS/901616.
Appendix.

Here, the general formalism of the theory of singularities [22, 9] is used to write the equations of the fold and cusp-points. We start from the equation for the rays:

$$\Phi (x, z, \lambda ) = (x - x_0(\lambda) + (z - d) \tan \alpha(\lambda) = 0$$  \hspace{1cm} (10)

given in section 5. This equation defines a 2-D surface $$M$$ in the 3-D space $$\{x, z, \lambda\}$$. The surface $$M$$ is globally parametrized by $$z$$ and $$\lambda$$, equation (10) giving the coordinate $$x$$ as a function of $$z$$ and $$\lambda$$: $$x(z, \lambda) = x_0(\lambda) + (d - z) \tan \alpha(\lambda)$$. The caustic points correspond to the singularities of the projection $$\pi$$ from $$M$$ onto the physical 2-D space $$\{x, z\}$$, that is to say to the points where the Jacobian matrix $$\pi^*$$ of $$\pi$$ is not of maximum rank (here 2). More precisely, the fold-points are defined by the condition: corank $$\pi^*(z, \lambda) = 1$$, where the corank (at the source) of $$\pi^*$$ is defined as the difference between the dimension of $$M$$ and its rank. The matrix $$\pi^*$$ reads

$$\pi^* = \begin{pmatrix} \partial x/\partial z & \partial x/\partial \lambda \\ \partial z/\partial \lambda & \partial z/\partial \lambda \end{pmatrix} = \begin{pmatrix} \partial x/\partial z & \partial x/\partial \lambda \\ 1 & 0 \end{pmatrix}$$  \hspace{1cm} (11)

so that the condition for the corank of $$\pi^*$$ to be equal to 1, is written $$\partial x/\partial \lambda = 0$$, and we recover formula (6). It defines the curve $$\Sigma^1$$ of the fold-points.

In order to find the cusp-points, we repeat the same reasoning, the only difference being that $$M$$ is replaced here by $$\Sigma^1$$. The curve $$\Sigma^1$$ is parametrized by $$\lambda$$, and the restriction of $$\pi$$ to $$\Sigma^1$$ is $$\pi |_{\Sigma^1}: \lambda \rightarrow (x(z(\lambda)), z(\lambda))$$.

The condition giving the cusp-points is that the corank of the Jacobian matrix $$(\pi |_{\Sigma^1})^*$$ should be equal to 1, or in other words, that its rank should be equal to 0. On the other hand we have

$$\left((\pi |_{\Sigma^1})^* = \begin{pmatrix} \partial x/\partial z \cdot dz/d\lambda + \partial x/\partial \lambda \\ dz/d\lambda \end{pmatrix} = \frac{dz}{d\lambda} \begin{pmatrix} \partial x/\partial z \\ 1 \end{pmatrix} \right)$$  \hspace{1cm} (12)

(using the fact that $$\partial x/\partial \lambda = 0$$ on $$\Sigma^1$$). Then the equation that gives the set $$\Sigma^{1,1}$$ of the cusp-points is $$dz/d\lambda = 0$$, and we recover formula (8).

References