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Dynamics of a ball on a vibrating plate

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Abstract. — We study both analytically and numerically the problem of a partially inelastic ball on a platform vibrating with frequency ω . Two parameters control the dynamics, the restitution coefficient η , and the reduced acceleration $\Gamma = \gamma/g$, where γ is the maximum acceleration of the motion of the plate and g the acceleration of gravity. When η exceeds some value, simple stable fixed points no longer exist, but it is shown that generic trajectories are periodic with period T_B which behaves as $T_B \sim (1 - \eta)^{-5}$ for η close to 1

Introduction.

In this paper, we revisit the problem of one ball on a platform vibrating with period $T = 2\pi/\omega$ and maximum amplitude a [1-6], which was recently studied in reference [6]. In the following, we show that all the periodic motions, in some parameter regions, include the rest state, the state where the ball moves with the plate. The period between a rest state and the next rest state, scales as $(1 - \eta)^{-5}$, η is the restitution coefficient.

1. Description of the model.

Instead of taking the movement of the platform to be sinusoidal, as in reference [6], we take it piecewise parabolic, with period T . Namely, the height $s(t)$ of the platform in the laboratory frame is, on each half period, an arch of a parabola, whose concavity is alternatively up and down (except at the very beginning, for $t \leq \frac{T}{2}$), as depicted in figure 1. We have :

$$s(t) = \frac{\gamma}{2} \left(t - \frac{T}{2} \right) (T - t) \quad \text{for} \quad \frac{T}{2} \leq t \leq T, \quad (1a)$$

$$s(t) = \frac{\gamma}{2} (T - t) \left(\frac{3T}{2} - t \right) \quad \text{for} \quad T \leq t \leq \frac{3T}{2}, \quad (1b)$$

$$s(t + nT) = s(t) \quad \text{for} \quad t \geq \frac{T}{2}, \quad n \in \mathbb{N}. \quad (1c)$$

Fig. 1 — Position $s(t)$ of the vibrating platform as a function of time t . For $t \geq \frac{T}{4}$, $s(t)$ is made of arches of parabola with second derivative $\frac{d^2s}{dt^2}$ equal to γ or $-\gamma$.

γ is an acceleration which relates to the amplitude a of the motion by

$$a = \frac{\gamma T^2}{32}, \quad (2a)$$

and the reduced acceleration Γ will be defined by

$$\Gamma = \gamma/g, \quad (2b)$$

where g is the acceleration of gravity.

From now on, positions and velocities of the ball will not be counted in the laboratory frame, but in the reference frame of the moving platform, so that the position of the ball is also its height with respect to the platform and is always positive. In the reference frame of the platform, the force felt by a ball of mass one, when not in contact with the platform, is $-\frac{d^2s}{dt^2} - g$. It takes alternatively the values $-\gamma - g$ and $\gamma - g$. We assume that we have a perfectly hard ball, and that Γ is larger than 1. We denote by W^+ and W^- the relative velocities of the ball with respect to the platform respectively immediately before and after a bounce with the platform. The coefficient of restitution η is defined by

$$W^+ = -\eta W^- \quad (3)$$

2. Description of the dynamics.

We shall use two alternative ways of describing the dynamics. One is the standard way in the literature. The time τ_l where the l -th bounce occurs and W_l^+ , the relative velocity of the ball just after the bounce, are recorded. A two dimensional map $(W_{l+1}^+, \tau_{l+1}) = f(W_l^+, \tau_l)$ is obtained, where f is a function. This, however, is not convenient when the ball performs infinitely many hops and comes to rest on the platform. In this case, as in reference [6], we say that the ball lands in an absorbing region. It is relaunched later, when the force felt by the ball

becomes positive, namely, when the driving acceleration, which is equal to $-\frac{d^2s}{dt^2}$, overcomes gravity.

An alternative way is to simply record the height and velocity of the ball at each time the acceleration changes sign (at each half period). The acceleration felt by the ball will be alternatively $-\gamma - g$ and $\gamma - g$. In this description, g will be set to one, the length of the half-period will be set to 1, and accelerations will be counted in units of $\gamma + g$. The acceleration seen by the ball will thus alternatively take the values -1 and

$$A = \frac{\Gamma - 1}{\Gamma + 1}. \quad (4)$$

Time is denoted by t' in this description. For any integer l , with $l \leq t' \leq l + 1$, the acceleration felt by the ball will be A . For $l + 1 \leq t' \leq l + 2$, it will be -1 . For every time t' equal to an even integer $2n$, the positions and velocities at this time will be denoted by x_n and v_n respectively. It will turn out to be convenient to use the variables

$$Y_n = v_n, \quad (5a)$$

and

$$X_n = \sqrt{v_n^2 + 2x_n}. \quad (5b)$$

We also introduce x'_n and v'_n , the positions and velocities of the ball at times $t' = 2n + 1$ (i.e. every time the acceleration switches from positive to negative). X'_n and Y'_n are defined in analogy to X_n and Y_n . The dynamics is now described by a map:

$$(X_n, Y_n) \xrightarrow{b} (X'_n, Y'_n) \xrightarrow{h} (X_{n+1}, Y_{n+1}). \quad (6)$$

The motion of the representative points (X_n, Y_n) and (X'_n, Y'_n) in phase space is restricted to the cones $(X_n \geq 0, |Y_n| \leq X_n)$ and $(X'_n \geq 0, |Y'_n| \leq X'_n)$ respectively.

The precise form of the mapping b and h is a little complicated and not very informative, it is given in the appendix. One property of h however is important. For any (X'_n, Y'_n) such that

$$Y'_n < 1 - rX'_n \quad (7a)$$

with

$$r = \frac{1 + \eta}{1 - \eta}, \quad (7b)$$

$$X_{n+1} = Y_{n+1} = 0. \quad (7c)$$

That is to say, h is identically zero. This corresponds to the absorbing region, the ball comes to rest on the platform.

3. Scaling of the period of the motion for large restitution coefficient.

In this section, the period of the periodic motion for large restitution coefficient is estimated. Starting from any initial condition, the ball always goes to the rest state, moving together with the plate. The regime before the ball has reached the rest state for the first time will be called the transient regime. After it has reached the rest state for the first time, its movement will then be periodic. This is called the permanent regime. The number of iterations between two rest states, the period of the motion in this parameter region, is calculated as a function of both the restitution coefficient and the size of the permanent regime. The size of the permanent regime in turn, is also related to the value of the restitution coefficient.

Having set the notations, we now give results about the dynamics. One issue is whether the ball generically lands in an absorbing region. If this occurs, then the subsequent motion will be periodic. It is simpler to discuss first a situation where a ball does not generically land in an absorbing region. Let us consider the mapping $(W_{l+1}^+, \tau_{l+1}) = f(W_l^+, \tau_l)$, which relates times and velocities just after the $(l+1)$ -th bounce of the ball with the platform to times and velocities just after the l -th bounce. There may exist fixed points of the map, i.e. : $(W_{l+1}^+ = W_l^+ \text{ and } \tau_{l+1} = \tau_l)$. A fixed point with $W_{l+1}^+ = W_l^+ = 0$ corresponds to the ball landing in the absorbing region. We call this kind of fixed point a null velocity simple fixed point. A fixed point with $W_l^+ \neq 0$ will be called a non-zero velocity simple fixed point. If such a non-zero velocity simple fixed point exists, with $W_l^+ = W_0 \neq 0$, for example, and if it is stable, then there will be a neighbourhood of this point for which the ball will be attracted to it. For large l , W_l^+ tends to W_0 and the ball will never land in an absorbing region. It is possible to enumerate all non-zero velocity simple fixed points, for a given value of η and F , but the analytic expressions are moderately complicated and not particularly illuminating. One result is that for

$$(1 + \eta)^2 (1 + \eta^2)^{-1} > 2 F^{-1}, \quad (8)$$

no stable non-zero velocity simple fixed points exist. Although the absence of such stable fixed points with $W_l^+ \neq 0$ does not mean that a generic trajectory lands in an absorbing region, other stable attractors, fixed cycles, strange attractors, etc... are not observed in the numerical simulations for η larger than 0.6 and F larger than 2. Thus, we shall assume that the only stable attractor, in this parameter region, is the rest state.

We now calculate the period of the motion (from one rest state to another), as a function of the restitution coefficient and of the size of the permanent regime. Later, we shall see that the permanent regime dynamics is confined to a region where $W^+ < W_{\text{sup}}$, where W_{sup} depends on η , if the dynamics is described by mapping f . If the description of the dynamics is done using mappings b and h (Eq. (6)), the permanent regime dynamics is confined to a region where $\sqrt{X'^2 + Y'^2} < R_0$, where R_0 is a number which depends on η .

From equations (7a) to (7c), we see that a ball lands in the absorbing region if, and only if, it has not been absorbed before and $Y'_n \leq 1 - rX'_n$. The region $Y'_n \leq 1 - rX'_n$, and $|Y'_n| < X'_n$ defines a triangle in (X', Y') plane. In principle, it suffices to find the inverse iterates of this triangle via the applications b and h . We try to evaluate the area in (X', Y') plane made by the points which are going to land in the absorbing region after n periods of vibration of the platform, and not before. More precisely, if one starts with a point of coordinates (X_0, Y_0) , it transforms under b into a point of coordinates (X'_0, Y'_0) and lands in the absorbing region if $Y'_0 \leq 1 - rX'_0$. We say that the 0th order absorbing zone, later denoted by $\mathcal{A}_0$ is made by the rectangle triangle delimited by the straight lines : $Y'_0 = 1 - rX'_0$, $Y'_0 = -X'_0$ and $Y'_0 = X'_0$. The n -th order absorption zone $\mathcal{A}_n$, is defined in an analogous way as the points X'_n, Y'_n which land in the absorbing region after n steps, but have not landed there before ; i.e., $Y'_n \leq 1 - rX'_n$ and $Y'_s > 1 - rX'_s$ for any s , $0 \leq s < n$.

The area of $\mathcal{A}_0$ is simply $(1 - \eta)^2 / (4 \eta)$. Since b and h are defined in a piecewise fashion, $\mathcal{A}_n$ will be made in general of many patches. The Jacobian of b^{-1} , taken at the value (X'_n, Y'_n) , with $(X_n, Y_n) = b^{-1}(X'_n, Y'_n)$, will be denoted by $J(b^{-1})(X'_n, Y'_n)$. It takes the value

$$J(b^{-1})(X'_n, Y'_n) = X'_n / X_n, \quad \text{if } Y'_n \leq k(X'_n), \quad (9a)$$

$$\text{where} \quad k(X'_n) = \sqrt{\frac{A}{A+1}} X'_n, \quad \text{if } X'_n < \sqrt{A(A+1)} \quad (9b)$$

$$\text{and} \quad k(X'_n) = \sqrt{X_n'^2 + A + 1} - 1, \quad \text{if } X'_n \geq \sqrt{A(A+1)}. \quad (9c)$$

$$\mathfrak{J}(h^{-1})(X'_n, Y'_n) = \eta^{-2} X'_n / X_n, \quad \text{if } Y'_n > k(X'_n). \quad (9d)$$

We have also $\mathfrak{J}(h^{-1}) = \eta^{-m}$, where m is defined in equation (A.2f) of the appendix. As η tends to 1, the area of $\mathcal{A}_0$ shrinks as $(1 - \eta)^2$ and the available phase space grows like R_0^2 , where R_0 is the size of the region where the permanent regime dynamics takes place.

We only give here a simplified backbone of the argument. The $\mathcal{A}_n$ zones gradually cover all the available phase space as n increase. By definition, the $\mathcal{A}_n$ zones are disjoint. Otherwise, if two $\mathcal{A}_n$ and $\mathcal{A}_m$ with $n \neq m$ overlapped, the ball would land for the first time in the absorbing region after n steps and also after m steps, which is impossible.

If $\mathcal{A}_n$ is mainly built up by points located at a distance of order z from the origin in (X', Y') plane, the presence of the factor X'_n/X_n in the Jacobian (Eq. (9a)) implies that the area of the $\mathcal{A}_n$ zone shrinks by a factor $\mathcal{F}_n = \frac{X'_n X'_{n-1} X'_{n-2} \dots X'_0}{X_n X_{n-1} X_{n-2} \dots X_0}$, with $X'_n \sim z$ and $X_n \sim z$. In the regions where the main part of the dynamics takes place, we have $X'_n = X_{n-1}$. This can be seen from equations (A.2d) to (A.2g) of the appendix, giving the precise form of the mapping h ; we have almost all the time $m = 0$ in equation (A.2f). The shrink factor $\mathcal{F}_n$ will thus be of order $\frac{X'_0}{X_n}$, which behaves as z^{-1} .

The area of the $\mathcal{A}_n$ zone shrinks thus as z^{-1} . Since every point belongs to only one $\mathcal{A}_n$ zone, the number l of iterations necessary to cover all the accessible phase space, where the permanent regime takes place, whose area is of order R_0^2 , behaves as

$$l \sim \int_0^{R_0} \frac{2 \pi z}{(1 - \eta)^2 z^{-1}} dz \sim (1 - \eta)^{-2} R_0^3. \quad (10)$$

We give a few words of explanation about this formula. One starts with $\mathcal{A}_0$, whose area is already of order $(1 - \eta)^2$. The elementary portion of phase space, in (X', Y') plane limited by the two diagonals $Y' = X'$, $Y' = -X'$ and by the two circles centered at the origin and of radii z and $z + dz$, will have to be covered by $\mathcal{A}_n$ zones. The area of one particular $\mathcal{A}_n$ zone is of order $(1 - \eta)^2 z^{-1}$ and the area to be covered is $2 \pi z dz$. It thus takes $\frac{2 \pi z dz}{(1 - \eta)^2 z^{-1}}$ iterations to cover this elementary portion of phase space. Since all the accessible phase space must eventually be covered, equation (10) follows. We are thus left with the task of estimating R_0 .

The typical size of the region of phase space where the permanent regime takes place has been already studied for a model slightly different from ours, in reference [6]. Applying their method to our case, the permanent regime dynamics will be confined to the region $W^+ < W_{\text{sup}}$, with

$$W_{\text{sup}} = \frac{\eta \gamma T}{4(1 - \eta)(1 + \eta)} \left(\eta + 3 + \sqrt{(1 + 3\eta)^2 + \frac{4}{T}(1 - \eta^2)} \right). \quad (11)$$

We have the limiting behaviour

$$W_{\text{sup}} \sim (1 - \eta)^{-1} \quad \text{for } \eta \rightarrow 1. \quad (12)$$

This means that an upper bound on the velocities is of order $(1 - \eta)^{-1}$. Integration of the equations of motion between successive bounces with the platform show that time between

two consecutive bounces will be at most of order $(1 - \eta)^{-1}$, and that maximum heights reached by the ball will be at most of order $(1 - \eta)^{-2}$. The maximum values of the velocity of the ball, taken at times where the acceleration changes sign will also be at most of order $(1 - \eta)^{-1}$. These are precisely the values of v_n and v'_n , used in the description of the dynamics by mappings b and h . Thus an upper bound on v_n or v'_n is of order $(1 - \eta)^{-1}$. An upper bound on x_n , or x'_n is of order $(1 - \eta)^{-2}$. By definition of X'_n and Y'_n (Eqs. (5a) and (5b)), the maximum values of X'_n and Y'_n behave at most as $(1 - \eta)^{-1}$. Thus, the dynamics is confined to a region of (X', Y') plane where $X' < X'_{\text{sup}}$, with $X'_{\text{sup}} \sim (1 - \eta)^{-1}$.

Since this is only an upper bound, we checked numerically that the typical size R_0 of the domain where the dynamics takes place in the permanent regime scales as $(1 - \eta)^{-1}$ for $\eta \rightarrow 1$. We take a ball at the origin $(X', Y') = (0, 0)$ and integrate the dynamics. We study R_0 , the maximum value of $\sqrt{X'^2 + Y'^2}$ reached by the ball. Since the values of R_0 strongly depend upon A , averages have been taken over typically 10^2 trajectories, with different values of A in intervals $[A_{\text{min}}, A_{\text{max}}]$. We did not have to average on η . Figure 2 shows a Log-log plot of R_0 as a function of $(1 - \eta)^{-1} (1 - A)^{-1/2}$, for η between 0.5 and 0.9 and $(1 - A)$ between 10^{-4} and 10^{-1} . R_0 behaves thus as $(1 - \eta)^{-1}$ for $\eta \rightarrow 1$.

Fig. 2. — Log-log plot of R_0 the typical size of the phase space region visited in the permanent regime (see text) versus $(1 - A)^{-1/2} (1 - \eta)^{-1}$, for values of $\eta = 0.5$ ($\diamond$), $\eta = 0.6$ ($+$), $\eta = 0.8$ ($\square$), and $\eta = 0.9$ ($\times$). Averages have been taken over typically 10^2 values of A . For example, for the point centered at $(1 - A) = 7.5 \times 10^{-4}$, and $\eta = 0.6$, averages have been taken over 100 values of $(1 - A)$ ranging from $(1 - A) = 5 \times 10^{-4}$ to $(1 - A) = 1.1 \times 10^{-3}$. A least square fit to the data gives a slope 1.01. The uncertainty on the slope is about 0.1.

From equation (10), the number l of iterations necessary to cover all the accessible phase space behaves as

$$l \sim (1 - \eta)^{-5}. \quad (13)$$

We performed numerical simulations, starting from a ball at rest. We determined the first time T_B for which the ball comes to rest on the platform. Because T_B very strongly depends on Γ , results have been averaged over typically 10^3 values of Γ in intervals $[\Gamma - \delta\Gamma, \Gamma + \delta\Gamma]$, with $\delta\Gamma < \Gamma$. A Log-log of T_B versus $(1 - \eta)^{-5} \Gamma^{-1}$, for two values of Γ is shown in figure 3. Least square fits to the data give slopes 1.002 ± 0.02 for Γ around 7.8 and 0.985 ± 0.03 for Γ around 15.8. Moreover, data seem to collapse on a single curve.

Fig. 3. — Log-log plot of the ratio $T_B \Gamma^{-1}$, where T_B is the period of the movement of the ball and Γ the reduced acceleration versus $(1 - \eta)^{-5}$ for two values of Γ , ($\diamond$) $7.4 \leq \Gamma \leq 8.2$ and (+) $15.4 \leq \Gamma \leq 16.2$. Each point corresponds to averages over typically 10^3 values of Γ and one definite value of η . The straight line is a least square fit to the data for $7.4 \leq \Gamma \leq 8.2$ and has a slope 1.002.

4. Conclusion.

We have introduced a simplified version of the partially inelastic bouncing ball on a vibrating platform which enables the calculation of fixed points. For an acceleration larger than $2g$, beyond some restitution coefficient $\eta = 0.6$, all trajectories seem to end up to be periodic. However, the period can be shown to diverge as $(1 - \eta)^{-5}$ for $\eta \rightarrow 1$, and as Γ for $\Gamma \rightarrow \infty$.

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Appendix.

We give below the exact expressions for the mappings b and h which appear in (6).

The mapping b is defined by :

$$X'_n = \sqrt{X_n^2 + 2(A+1)Y_n + A(A+1)}, \quad (\text{A.1a})$$

$$Y'_n = Y_n + A, \quad (\text{A.1b})$$

$$\text{if } X_n < \sqrt{A(A+1)} \quad \text{and} \quad Y_n > -\sqrt{\frac{A}{A+1}} X_n, \quad (\text{A.1c})$$

$$\text{or if } X_n > \sqrt{A(A+1)} \quad \text{and} \quad Y_n > 1 - \sqrt{A+1+X_n^2}, \quad (\text{A.1d})$$

and, otherwise, by

$$X'_n = \sqrt{\frac{A+1}{A} Y_n'^2 - \frac{\eta^2}{A} [(A+1)Y_n'^2 - AX_n'^2]}, \quad (\text{A.1e})$$

$$Y'_n = Y_n + A + (1+\eta) \sqrt{(A+1)Y_n'^2 - AX_n'^2}. \quad (\text{A.1f})$$

The mapping h is defined, if $X'_n \neq 0$ by :

$$X_{n+1} = Y_{n+1} = 0, \quad (\text{A.2a})$$

$$\text{if } Y'_n < 1 - rX'_n, \quad (\text{A.2b})$$

where

$$r = \frac{1+\eta}{1-\eta} \quad (\text{A.2c})$$

This corresponds to the absorbing region.

Otherwise, h is defined by :

$$X_{n+1} = \eta^m X'_n, \quad (\text{A.2d})$$

$$Y_{n+1} = Y'_n - 1 + \left\{ 1 + 2 \frac{\eta - \eta^m}{1 - \eta} + \eta^m \right\} X'_n, \quad (\text{A.2e})$$

with

$$m = \text{Int} \left\{ \text{Ln} \left[\left(\frac{1+\eta}{2} \right) - \left(\frac{1-\eta}{2} \right) Q' \right] / \text{Ln } \eta \right\}, \quad (\text{A.2f})$$

and

$$Q' = \frac{1 - Y'_n}{X'_n} \quad (\text{A.2g})$$

« Int » denotes the integer part of a real number.

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