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The magnetic response of chaotic mesoscopic systems

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Abstract. — The magnetic response of mesoscopic systems with ballistic motion of electrons that is chaotic in the classical limit is considered. Semiclassical methods are applied in order to derive a formula for the susceptibility which is expressed in terms of a finite number of classical periodic orbits. This formula is used to study the fluctuations of the susceptibility in comparison with the fluctuations of random systems. Some of the mechanisms which lead to these fluctuations are discussed. At relatively high temperatures the formula for the susceptibility reduces to a simple expression in which only few short periodic orbits dominate the behavior. An experiment based on this result is proposed.

1. Introduction.

Mesoscopic systems were extensively investigated in recent years [1, 2]. A mesoscopic system is usually large enough to contain a number of electrons that may be considered as thermodynamically large, but still sufficiently small to maintain phase coherence over distances comparable to the size of the sample. The elastic scattering of the particles from the impurities and from the boundary generates behavior which is usually chaotic in its classical limit. Presently, there is a growing interest in understanding the imprints of the underlying classical dynamics on the quantum behavior of chaotic systems. The investigations of such systems form the field of "quantum chaos" [3-8]. In the nonlocalized regime the high level wave functions of chaotic systems spread over all the system [9, 10], therefore, they are sensitive to the precise arrangement of the impurities as well as to the shape of the boundary. The energy spectrum, on the other hand, displays rigidity and long range correlations. These correlations are the subject of several recent investigations.

Many quantum effects of mesoscopic systems are directly related to the spectral correlation function. For diffusing electrons, this function was calculated by Altshuler and Shklovskii [11] using a perturbative diagrammatic method. An alternative derivation [12] applies Gutzwiller's trace formula [13, 14] which expresses the density of states in terms of a sum over classical periodic orbits. Following Berry [15] and applying a generalization of Hannay and Ozorno de Almeida sum rule [16] one is able to recover the same results obtained by diagrammatic
expansion. Recent theoretical [17-20] and experimental [21] studies show also that statistical properties of the fluctuations of transport in irregular mesoscopic structures, may be explained in the framework of the periodic orbit theory.

This paper focuses on the orbital magnetic susceptibility of mesoscopic systems. Several authors investigated this problem for random systems [22-30]. One of the conclusions which emerged from these studies is that the magnetic response in a specific sample is sensitive to the exact configuration of the impurities, and that the sample to sample fluctuations are proportional to $k_F l$ where $k_F$ is the Fermi wavenumber and $l$ is the elastic mean free path. This result is somewhat counterintuitive since as the disorder decreases the fluctuations increase. In order to examine this problem it is instructive to study the susceptibility of clean mesoscopic systems. The sample to sample fluctuations of the magnetic response of such systems arise because of the different shape of their boundaries. These are defined as $\delta \chi = \sqrt{\langle (\chi - \langle \chi \rangle)^2 \rangle}$, where $\chi$ is the susceptibility of some individual sample, and $\langle \cdots \rangle$ represents ensemble averaging associated with the roughness of the boundaries. One expects that as $l$ becomes sufficiently large compared to the linear size of the system $L$, the fluctuations will cease to increase and will remain proportional to $k_F L$.

In the clean limit, it is important to distinguish between two extreme cases. One is the situation in which the classical dynamics of one particle in the corresponding system is integrable, while the other is when it is completely chaotic. In between these two cases are the systems with mixed dynamics. It turns out that the structure of their classical phase space is directly related to the magnitude of the fluctuations of the susceptibility. The fluctuations of a system in which some of the corresponding periodic orbits appear in continuous families of $d$ parameters (e.g. integrable systems with $d$ degrees of freedom) are in general proportional to $(k_F L)^{\frac{1}{2}(1+d)}$ [31]. For completely chaotic systems the periodic orbits on the energy shell are isolated and the fluctuations in the susceptibility are proportional to $k_F L$. One purpose of this paper is to present a detailed derivation of this result. The impurities in the random systems generate a chaotic classical motion. Therefore, a proper comparison between them and the clean systems requires the consideration of systems which in the classical limit exhibit chaotic dynamics.

In an early work on this subject [32], Dingle considered the magnetic response of a noninteracting electron gas confined to move in a cylinder of radius $R$ where a uniform magnetic field is applied parallel to the axis of the cylinder. He showed that the behavior of this system depends on the ratio of the cyclotron radius at the Fermi energy, $R_c$, to the linear size of the system $R$. When $R \gg R_c$ ("large systems") there is a small steady diamagnetism together with some small terms periodic in the field which are significant only at low temperature. The latter correspond to de-Haas van-Alphen oscillations [33, 34]. At the other limit $R \ll R_c$ ("small systems") he found a large steady diamagnetism and some oscillatory terms which become significant at very low temperature. (In fact, the rotational symmetry of the system may account for perfect diamagnetism [29].) A more general case of large systems where the electrons are confined to move in a cylinder of arbitrary section was considered by Robnik at high temperatures [35]. Sivan and Imry investigated large two dimensional systems at very low temperature [36], while Nakamura and Thomas studied numerically billiard systems with both circular and elliptic shapes [37].

The model investigated in the present paper consists in a noninteracting electron gas confined to in a small two dimensional chaotic billiard of arbitrary shape. The electronic motion is assumed to be completely ballistic, namely, the elastic mean free path due to the (static) impurities is infinite. The levels broadening because of inelastic processes is supposed to be negligible compared to the thermal energy, i.e. the inelastic mean free path is also assumed to be infinite. The spin orbit coupling will be ignored.
The problem will be analyzed using semiclassical methods. It turns out that this approach clarifies the mechanism which leads to the large fluctuations in the susceptibility. Recent resummation techniques which enable to extract from Gutzwiller’s trace formula the semiclassical information regarding individual energy levels of chaotic systems will be used. This issue and some other related topics are summarized in section 2. In section 3 we analyze magnetic response due to a single Aharonov-Bohm flux line. The main result there is a general formula for the susceptibility expressed in terms of a finite number of periodic orbits. At high temperatures it reduces to a simple expression in which only the shortest periodic orbits have substantial contribution. It is clear from the latter that the susceptibility is an oscillating function of the flux and of the Fermi wavenumber with a typical amplitude proportional to $k_F L$. A formula for the low temperature regime is obtained applying the Berry-Keating resummation method \[38\]. The fluctuations in this limit are shown to be, in general, proportional to $k_F L \sqrt{\ln k_F L}$. Some of the mechanisms which lead to anomalously large fluctuations of the susceptibility in this energy regime will be related to “scars” and to avoided crossings. In section 4, these results are generalized to the case of a uniform magnetic field. In particular it is shown that for a wide range of field strengths, the response is very similar to that of a magnetic flux line. Finally, a proposed experiment, and some related open problems are discussed in section 5.

2. Semiclassical quantization.

The semiclassical approximation concerns the leading terms of expansions in which $\hbar$ plays the role of a small parameter, therefore, it reflects the underlying dynamics of the corresponding classical system. The characterization of this dynamics is crucial since the appropriate semiclassical method of quantization depends on the nature of this dynamics.

Integrable systems of $N$ degrees of freedom contain $N$ independent constants of motion in involution. Their phase space motion is confined to a $N$ dimensional torus, and semiclassical quantization of these systems may be obtained using Einstein-Brillouin-Keller (E.B.K) theory \[39\]. In this case, each energy level of the spectrum arises from a set of values of the irreducible action integrals associated with the different periods of the motion.

Chaotic systems, on the other hand, do not contain besides the energy any constant of motion. A typical trajectory of these systems covers the whole energy shell ergodically, therefore, E.B.K method of quantization is inadequate for this case. Yet, chaos introduces another sort of simplicity, and semiclassical quantization may be formulated, rather simply, in terms of the isolated periodic orbits of the classical dynamics. Gutzwiller’s trace formula \[13, 14\], which expresses the density of states as a sum over these orbits, is the main tool in the field of chaos quantization.

Generic systems, however, are neither chaotic nor integrable. Their phase space structure is a mixture of regular and chaotic domains, consequently, semiclassical quantization of these systems is more complicated. In some cases one may consider the regular and the chaotic parts of the phase space separately, and apply correspondingly the E.B.K or the Gutzwiller’s quantization methods. A remarkable example of this approach is the semiclassical quantization of the helium atom \[40\]. A different route for quantizing such systems is based on time domain techniques which where recently developed \[41\]. According to this method the spectrum is obtained from the Fourier transform of the autocorrelation function, $< \psi(0) | \psi(t) >$, expressed in terms of the Van-Vleck propagator, where $| \psi(0) >$ is some initial Gaussian wave packet.

The systems this paper is dealing with are two dimensional closed chaotic billiards. The semiclassical approximation for their density of states is obtained by tracing the Green function using the stationary phase approximation. The result may be written as a sum of a smooth
background and oscillatory corrections,
\[
\rho(E) \simeq \tilde{\rho}(E) + \rho_{\text{osc}}(E).
\]
(2.1)
The first term originates from short orbits, i.e. orbits with actions which are small relative to \( \hbar \). It is given approximately by the Thomas-Fermi expression,
\[
\tilde{\rho}(E) \simeq \frac{1}{\hbar^2} \int \int dp dq \delta(E - \mathcal{H}(q,p)),
\]
(2.2)
where \( \mathcal{H}(q,p) \) is the Hamiltonian of the system, and \((q,p)\) are the coordinates and momenta. For closed billiards, the analytic expressions for the higher corrections of \( \tilde{\rho}(E) \) are also known [42]. The oscillatory contribution, \( \rho_{\text{osc}}(E) \), is related to the classical periodic orbits of the system [13]:
\[
\rho_{\text{osc}}(E) = \frac{1}{\pi \hbar} \operatorname{Re} \sum_j A_j e^{i S_j/\hbar - i \gamma_j}.
\]
(2.3)
The sum over \( j \) includes all the periodic orbits, \( S_j \) is the action of the \( j \)-th orbit, \( A_j \) is an amplitude which characterizes the behavior of the trajectories in the vicinity of that orbit, and \( \gamma_j \) is the Maslov phase determined by the focusing paths close to \( j \). In the case where all the periodic orbits (on the energy shell) are isolated, this contribution may be conveniently expressed in terms of the dynamical zeta function [43], \( \zeta_s(E) \), as
\[
\rho_{\text{osc}}(E) = -\frac{1}{\pi} \text{Im} \frac{\partial}{\partial E} \log(\zeta_s(E)).
\]
(2.4)
This function, also called the Selberg zeta function, is defined as a product over primitive periodic orbits (i.e. orbits which do not retrace themselves),
\[
\zeta_s(E) = \prod_{p.p.o.} \prod_j \left( 1 - \exp\left( \frac{i}{\hbar} S_p(E) - i \gamma_p - \left( \frac{1}{2} + j \right) u_p \right) \right),
\]
(2.5)
where the subscript \( p \) denotes these orbits, \( S_p(E) \) is the corresponding action, \( \gamma_p \) is the Maslov phase, and \( u_p \) is the instability exponent associated with the linearized motion in the vicinity of the periodic orbit.

The introduction of \( \zeta_s(E) \) is extremely useful. For example, the semiclassical quantization rule for chaotic systems may be formulated as
\[
\zeta_s(E) = 0.
\]
(2.6)
Unfortunately, the dynamical zeta function suffers from severe convergence problems due to the exponential growth in the number of periodic orbits as their period increases. In order to clarify this point, it is convenient to represent \( \zeta_s(E) \) as a sum over pseudo-orbits,
\[
\zeta_s(E) = \sum_{\mu} c_{\mu} e^{i S_{\mu}}. \tag{2.7}
\]
Each pseudo-orbit, labeled here by \( \mu \), is a linear combination of primitive orbits where \( \mu \) represents the vector of the repetition numbers of the primitive orbits, \( \{ r_p \}_{\mu} = (r_1, r_2, r_3, \ldots) \). Thus the pseudo-orbit action is
\[
S_{\mu} = \sum_{\{ r_p \}_{\mu}} r_p S_p, \tag{2.8}
\]
and the corresponding period is given by \( T_\mu = \partial S_\mu / \partial E \). For long orbits and pseudo-orbits of ergodic systems \( T_\mu \) and \( S_\mu \) are proportional and satisfy the relation

\[
S_\mu \simeq \frac{D\mu E}{\Omega'(E)},
\]

where \( D \) is the number of degrees of freedom of the system (two in our case), \( \Omega(E) \) is the classical phase space volume with energy less than \( E \),

\[
\Omega(E) = \int \int dp dq \Theta[E - \mathcal{H}(q,p)],
\]

and \( \Omega'(E) \) is its derivative with respect to the energy. In the above equation, \( \Theta \) is the unit step function.

Formula (2.7) is obtained by expanding the product (2.5) and collecting the various terms. The resulting amplitudes are given by [43]

\[
c_\mu = \prod_{\{\tau_p\}_\mu} \left( -1 \right)^{\tau_p} e^{-\frac{E}{2} \tau_p^2} e^{-ir_p \gamma_p} \prod_{p=1}^{\tau_p} (1 - e^{-u_p k}).
\]

They decrease exponentially with the period \( \tau \) as \( e^{-\frac{h}{2} \tau} \) where \( \lambda \) is the metric entropy associated with the average value of the instability exponents. On the other hand, the number of pseudo-orbits proliferates exponentially with their period as \( e^{hT \tau} \) where \( hT \) is the topological entropy. Since \( hT > \lambda \), absolute convergence of the sum (2.7) is obtained only for complex values of \( 1/h \) satisfying [38]

\[
\text{Im} \frac{1}{h} > \left( hT - \frac{\lambda}{2} \right) \frac{\Omega'(E)}{2\Omega(E)}.
\]

Convergence of the semiclassical expansion for complex values of the energy was discussed by Eckhardt and Aurell [44].

To be able to obtain real eigenenergies from (2.6), an analytic continuation of \( \zeta_\mu(E) \) into the real \( 1/h \) axis is required. The main conclusion from studies concerning this issue is that the sum (2.7) over pseudo-orbits should be effectively truncated [38, 45-49]. Generally, it will include all pseudo-orbits of period smaller than the Heisenberg time. Yet, a considerable reduction in the number of pseudo-orbits may be obtained using functional relations that exist within the quantum theory, and the longest orbit required (labeled here by \( \bar{\mu} \)) has a period equal to half of the Heisenberg time,

\[
T_{\bar{\mu}} = \frac{\pi h}{\Delta},
\]

where \( \Delta = 1/\rho(E) \) is the mean level spacing. The resummed formula which will be presented here was recently obtained by Berry and Keating [38]. They considered the semiclassical spectral determinant, \( \mathcal{D}(E) \), which is related to the dynamical zeta function by

\[
\mathcal{D}(E) = e^{-i\pi \bar{N}(E)} \zeta_\mu(E),
\]

where \( \bar{N}(E) \) is the smooth level staircase approximately given by the classical phase space volume of energy less than \( E \) divided by \( h^2 \), namely,

\[
\bar{N}(E) \simeq \frac{\Omega(E)}{h^2} = \frac{mAE}{2\pi h^2}.
\]
Fig. 1. — (a) The billiard together with a typical periodic orbit. (b)+(c) the corresponding spectral determinant for $u=1.5$, and $L=2m=\hbar=1$. It is calculated using (2.16) from 17347 periodic orbits. The tuning parameter is set to $K^2 = 5 \cdot \frac{\pi \partial^2 \tilde{N}(E)}{\hbar \partial (1/\hbar)^2}$. The vertical bars along the energy axis mark the exact eigenvalues.

Here $A$ is the area of the billiard, and $m$ is the particle mass. Like for $\zeta_a(E)$, the zeros of the spectral determinant provide the semiclassical approximation for the eigenenergies. The advantage of this function is that, semiclassically, it is real for real values of $E$ [43], and unlike $\zeta_a(E)$ it is invariant under the change $\hbar \rightarrow -\hbar$ [38]. Exploiting these properties, Berry and Keating were able to continue analytically the spectral determinant. The leading term of the resulting asymptotic formula is

$$D(E) \simeq \text{Re} \sum_{\mu} c_{\mu} e^{-i\pi \tilde{N}(E)} + k_{\mu} \text{Erfc} \left\{ \frac{\xi(\mu, \hbar, E)}{Q(K, \hbar, E) \sqrt{2\hbar}} \right\}, \quad (2.16)$$

where

$$\xi(\mu, \hbar, E) = S_{\mu} - \pi \frac{\partial}{\partial (1/\hbar)} \tilde{N}(E), \quad (2.17)$$

and

$$Q(K, \hbar, E)^2 = K^2 + i \frac{\pi}{\hbar} \frac{\partial}{\partial (1/\hbar)^2} \tilde{N}(E). \quad (2.18)$$

Inspection of this formula reveals that the complementary error function term cuts smoothly the infinite sum over pseudo-orbits. The center of the smoothed cutoff (2.13) may be found from $\xi(\tilde{\mu}, \hbar, E) = 0$, and the width of the smoothing region is determined by $K$ which is a free fine tuning parameter. This width is minimal when $K^2 = \frac{\pi}{\hbar} \frac{\partial^2}{\partial (1/\hbar)^2} \tilde{N}(E)$.

The applicability of formula (2.16) is demonstrated in figure 1. The system consist in a particle of mass $m$ moving freely inside a two dimensional chaotic billiard, which is a closed version of the open hyperbola billiard studied by Sieber and Steiner [50]. figure 1a shows the billiard together with a typical periodic orbit. The corresponding spectral determinant, calculated in the semiclassical approximation from 17347 periodic orbits (the shortest among those which bounce the hyperbola boundary segment up to 13 times), is drawn in figures 1b and 1c. The exact eigenvalues of the system, obtained by solving the Schrödinger equation numerically, are indicated by vertical bars along the energy axis, and the zeros of $D(E)$ give the corresponding semiclassical approximation. In this figure, 122 of them are shown and found in good agreement with the exact eigenenergies.
Even after the resummation, the number of periodic orbits required for the calculation of the spectral determinant proliferates exponentially with the energy. This proliferation is the major obstacle in calculating the spectral determinant at high energies. Using arguments related to the pruning of pseudo-orbits, Bogomolny proposed a way for reducing their number for a given accuracy [51]. However, it is still not clear whether this approach, indeed, improves the accuracy to work ratio [52].

Part of the analysis presented in the following section, is formulated in terms of the dynamical zeta function rather than in terms of the spectral determinant. A resummed formula for this function may be easily obtained from (2.14) and (2.16). A simplified expression, in which the complementary error function is replaced by a sharp cutoff, is given by

$$
\zeta_{\delta}(E) \approx \sum_{n \leq \frac{E}{\delta}} \left[ c_\mu e^{\frac{i}{\delta} S_\mu} + c_\mu^* e^{i2\pi(E-n)\delta} - \frac{i}{\delta} S_\mu \right].
$$

(2.19)

3. The magnetic response of Aharonov-Bohm billiards.

In this section, we investigate the magnetic response of a two dimensional noninteracting electron gas, due to a single Aharonov-Bohm flux line. The confining potential of the electrons is a chaotic billiard of area $A$ lying in the $xy$ plane. The magnetic flux $\phi$, located at some point $(x_0, y_0)$ within the billiard, is related to the vector potential $A$ satisfying

$$
\vec{\phi}(x-x_0)\delta(y-y_0) = \nabla \times A.
$$

(3.1)

Its components expressed in polar coordinates $(r, \theta, z)$ where $(x_0, y_0)$ is chosen as the origin are $A_r=A_z=0$, and $A_\theta=\phi/2\pi r$.

The thermodynamic properties of the system are determined by the corresponding one particle density of states $\rho(E)$, and may be derived from the grand potential [53]

$$
\Omega = -T \int dE \rho(E) \ln \left( 1 + e^{\frac{E-E_F}{T}} \right),
$$

(3.2)

where the temperature units chosen here and henceforth are such that the Boltzman constant $k_B$ is unity, and $\eta$ is the chemical potential. For two dimensional systems at low temperature ($T \ll \eta$) here considered it may be approximated by the Fermi energy, $\eta \approx E_F$. Equation (3.2) is correct for spinless particles. Nevertheless, in Aharonov-Bohm billiards, the effect of the spin amounts to multiply by a factor of 2 the integral in (3.2). In what follows this factor will be ignored.

Consider first the case of zero temperature. To the ground state of the system with $\phi \neq 0$ corresponds nonzero planar current density $j_\phi(r)$. It generates a magnetic moment $M$ perpendicular to the plane of the billiard, given by Biot-Savart law $M = \frac{1}{2c} \int d^2rr \times j_\phi$. For noninteracting electrons this moment is built up from moments associated with the single electron states $\psi_n$ of energy $E_n$ below the Fermi energy. Namely, $M = \sum_{E_n < E_F} M_n$ with $M_n = -\frac{e}{2c} < \psi_n | r \psi_n | \psi_n >$, where $r$ and $\theta$ are polar coordinates with the origin located at the flux line, and $\psi_n$ is the tangential component of the velocity operator $v$. The thermodynamic current which flows around the flux line is $I_n = -c \partial E_n / \partial \phi$. Let us define the quantity: $\tilde{M}_n = -A \frac{\partial E_n}{\partial \phi} = -\frac{e}{c} < \psi_n | \frac{A}{2\pi T} v_\phi | \psi_n >$. In general $M_n$ and $\tilde{M}_n$ are different, however, for the case of a uniform magnetic field where $\phi$ is taken as the total flux through the two dimensional sample
they coincide. In analogy with the standard definition of the susceptibility as the derivative of the magnetic moment with respect to the field, we define the response due to a flux line by

$$
\chi = \frac{\partial}{\partial \phi} \sum_{E_n < E_P} \tilde{M}_n.
$$

For nonzero temperature it is

$$
\chi = -\mathcal{A} \left( \frac{\partial^2 \Omega}{\partial \phi^2} \right)_{E_P, T}.
$$

(3.3)

The starting point for the semiclassical analysis is the Gutzwiller approximation for \( \rho(E) \) expressed in terms of the classical periodic orbits of the system [13]. Substituting (2.1) in (3.2) and integrating by parts yields the grand potential as a sum of smooth and oscillatory terms analogous to the density of states:

$$
\Omega \simeq \tilde{\Omega} + \Omega_{osc}.
$$

(3.4)

The smooth part corresponding to the contribution of short orbits is

$$
\tilde{\Omega} = - \int dE \frac{\tilde{N}(E)}{1 + e^{\frac{E - E_P}{T}}},
$$

(3.5)

where \( \tilde{N}(E) \) is the smooth level staircase (2.15). The oscillatory term coming from the periodic orbits is expressed in terms of the dynamical zeta function (2.5) as

$$
\Omega_{osc} = \frac{1}{\pi} \text{Im} \int dE \frac{\log \zeta_s(E)}{1 + e^{\frac{E - E_P}{T}}}
$$

(3.6)

The semiclassical approach requires the exact dependence of various classical quantities in the flux. Here it is simple, since the classical dynamics is not affected by the flux line. In particular, the phase space volume is independent of the flux. This leads to the invariance of smooth level staircase, \( \tilde{N}(E, \phi) = \tilde{N}(E) \) [54]. Consequently, the only contribution to the susceptibility, in the semiclassical limit, comes from the classical periodic orbits of the system. Substituting (3.6) in (3.3), differentiating with respect to the flux, and converting the integral into a Matsubara sum (by integrating along the contour shown in Fig. 2), we obtain a general formula for the susceptibility of chaotic billiards:

$$
\chi \simeq 12 \chi_0 \frac{T}{\Delta} \text{Re} \sum_{n=0}^{\infty} \left[ \left( \frac{\zeta_s^{(1)}(E; \varphi)}{\zeta_s(E; \varphi)} \right)^2 \frac{\zeta_s^{(2)}(E; \varphi)}{\zeta_s(E; \varphi)} \right]_{E = E_P + \imath \varepsilon_n}
$$

(3.7)

Here \( \varphi = \phi/\phi_0 \) denotes the reduced flux i.e. the magnetic flux measured in units of the flux quantum \( \phi_0 = \frac{hc}{e} \); \( \chi_0 = \frac{-e^2}{12\pi m c^2} \) is the Landau susceptibility in two dimensions, \( \Delta = \frac{2\pi \hbar^2}{m \mathcal{A}} \) is the mean level spacing, and \( \zeta_s^{(j)}(E; \varphi) \) denotes the \( j \)-th derivative of the dynamical zeta function with respect to \( \varphi \). The Matsubara frequencies are \( \varepsilon_n / \hbar \) with

$$
\varepsilon_n = (2n + 1)\pi T.
$$

(3.8)

The use of Cauchy theorem for the conversion of the integral (3.6) into a sum over the residues (3.7) is justified by the following properties of the dynamical zeta function and its derivatives: (i) \( \lim_{n \to -\infty} \zeta_s(e^{e^\theta}; \varphi) = 1 \) for \( \pi/2 > \theta > 0 \); (ii) in the same limit the derivatives \( \zeta_s^{(j)}(e^{e^\theta}; \varphi) \) decay exponentially; (iii) the contribution from the integral along the imaginary axis may be
shown (integrating by parts) to be of the order $e^{-E_F/T}$, therefore, it may be neglected when $T \ll E_F$.

We now investigate the dependence of $\zeta_n(E; \varphi)$ on the flux. For this purpose it is sufficient to examine the dependence for an arbitrary periodic orbit as the magnetic flux changes. Consider first the zero flux limit $\varphi = 0$. In this case the system exhibits time reversal invariance, and this symmetry is reflected in the periodic orbits structure by the fact that there are two types of orbits. There are orbits that trace themselves back along the same path. These exhibit a time reversal invariance. On the other hand orbits which do not have this symmetry may be traced in two opposite directions, and therefore to every such orbit corresponds a time reversed counterpart. The time reversal symmetry presents itself in the Gutzwiller formula by the property that such pairs of orbits are characterized by exactly the same classical quantities.

When $\varphi \neq 0$ this degeneracy is lifted. The magnetic flux line is introduced in the equations by replacing the original momentum $p = mv$ by $p = mv - eA/c$, where $A$ is the vector potential which satisfies (3.1). This replacement leads to a flux dependence of the periodic orbits actions of the form [55]

$$S_p(\varphi) = \oint pdq = S_p - \hbar W_p \varphi, \quad (3.9)$$

where $S_p$ is independent of $\varphi$ and corresponds to the zero flux value, and $W_p$ is the winding number of the orbit around the flux line at $(x_0, y_0)$. Note that the other relevant classical quantities, namely, the instability exponent and the Maslov phase are not affected by the flux line.

The change in the time reversal properties of the system is related to the possible change in the actions of pairs of time reversed orbits. The opposite signs of their winding numbers lead to a difference in their actions. Yet, whenever $\varphi$ is an integer (i.e $\phi$ is an integer multiple of $\phi_0$), there will be a phase coherence between the orbit and its time reversed counterpart. Therefore, the various thermodynamic properties of this system are periodic functions of $\varphi$ [56].

Inserting expression (3.9) for the action into the approximate formula (2.19) and taking the
derivatives with respect to $\varphi$ yields

$$
\zeta_\mu^{(j)}(E; \varphi) \simeq \sum_{T_\mu \leq \frac{2\pi}{h}} \left[ c_\mu^{(j)} e^{i k S_\mu(\varphi)} + c_\mu^{(j)*} e^{-i 2\pi F(E) - i k S_\mu(\varphi)} \right],
$$

(3.10)

where the pseudo-orbits actions, $S_\mu(\varphi)$, are given by formula (2.8) with $S_p$ replaced by $S_\mu(\varphi)$, and the amplitudes are

$$
c_\mu^{(j)} = c_\mu (-2\pi i W_\mu)^j,
$$

(3.11)

where $c_\mu$ is given by (2.11), and $W_\mu$ is the pseudo-orbit winding number,

$$
W_\mu = \sum_{\{r_p\}_\mu} r_p W_p.
$$

(3.12)

Note that at $\varphi = 0$, $\zeta_\mu^{(1)}(E; \varphi)$ vanishes. It follows from the linear dependence of the amplitudes $c_\mu^{(1)}$ on the winding numbers, which implies that the contributions from a pair of time reversed pseudo-orbits add up destructively. At $\varphi = 0$ these exactly cancel each other.

Formula (3.7) for the susceptibility of chaotic billiards will now be investigated. We assume a large enough Fermi energy (i.e. $k_F L \gg 1$ where $k_F$ is the Fermi wave number), much larger than the thermal excitations ($E_F \gg T$). Under this assumption, the pseudo-orbits actions evaluated at $E_F + i \epsilon_n$ may be expanded to first order in $\epsilon_n/E_F$ as

$$
S_\mu(E_F + i \epsilon_n) \approx S_\mu(E_F) + i T_\mu(2n + 1)\pi T,
$$

(3.13)

where $T_\mu = \partial S_\mu/\partial E$ is the pseudo-orbit period. This expansion holds only for the low Matsubara frequencies for which $n \ll \frac{E_F}{2\pi T}$. It turns out that contributions from large $n$ terms are negligibly small as will be argued when the contribution of the various Matsubara frequencies will be calculated (see discussion following (3.29)). The imaginary part of (3.13) results in an exponential damping of the corresponding pseudo-orbit contribution. For the lowest frequency ($n = 0$) it is proportional to $e^{-\frac{E_F}{k_F} T_\mu}$, while for higher frequencies it is much stronger. This damping implies that at high temperature only the shortest pseudo-orbits contribute significantly to the susceptibility. One may define a threshold temperature, $T_{th}$, according to the requirement that the upper bound on these damping factors (which corresponds to the shortest periodic orbit) is of order $e^{-1}$. Since the shortest periodic orbit length is of order $2L$ this condition leads to the definition of the threshold temperature:

$$
T_{th} = \frac{\beta^2}{(2\pi)^2} k_F L \Delta,
$$

(3.14)

where $\beta = \sqrt{A}/L$ is a geometrical factor of order unity. There are, therefore, three relevant energy scales in the problem: (i) the mean level spacing $\Delta$; (ii) the threshold temperature $T_{th}$ associated with the time it takes for a particle at the Fermi energy to cross the sample; (iii) the Fermi energy $E_F \approx \Delta 4\pi (k_F L)^2$. When the Fermi energy is sufficiently high, so that $k_F L \gg (2\pi)^2$, these scales are well separated, namely, $\Delta \ll T_{th} \ll E_F$.

At high temperatures ($T \geq T_{th}$), the thermal fluctuations wash out structures in the energy scale of the mean level spacing, and only short periodic orbits contribute effectively to the susceptibility. We then expect a simple dependence on the parameters. As the temperature decreases below $T_{th}$ longer periodic orbits become more important, and at very low temperatures ($T \ll \Delta$) all orbits (of period smaller than half of the Heisenberg time) have significant contributions.
Fig. 3. — A schematic illustration of a billiard with rough boundary (thick line), and some of the shortest periodic orbits (thin lines), which almost cover the energy shell uniformly. \( L \) is the linear dimension of the billiard, \( A \) is its area, and \( L_R \) is the scale associated with the roughness of the boundary. The latter is assumed to be large compared to the Fermi wavelength.

In what follows, an estimation for the magnitude of the susceptibility, in two ranges of temperature, \( \Delta > T \) and \( T_{th} \leq T \), is calculated. For this purpose, some statistical properties of the periodic orbits are required (cf. Appendix A), and the following discussion will be confined to billiards for which the boundary is complicated enough so that one may consider the corresponding set of the shortest periodic orbits (e.g. the orbits which bounce the boundary twice) as covering the energy shell almost uniformly in an ensemble of billiards. An example of such billiard together with some of the shortest periodic orbits is illustrated in figure 3. In addition, it will be assumed that the scale associated with the roughness of the boundary, \( L_R \), is large compared with the Fermi wavelength. We shall also introduce two types of ensemble averaging, \( \langle \cdot \cdot \rangle \) and \( \langle \langle \cdot \cdot \rangle \rangle \). The first represents an averaging over some external parameter related to the roughness of the boundary and is an average over an ensemble of billiards, while the second includes also an additional averaging over the flux.

\section{3.1 The high temperature regime \( T \geq T_{th} \).} — It is clear from the above discussion that at temperatures sufficiently high so that \( T \geq T_{th} \) is satisfied, the shortest periodic orbits dominate the behavior. In this case one may exploit the natural cutoff of the periodic orbit sum (2.3) due to the temperature in order to derive a simpler formula for the susceptibility which does not require the resummed expression of the zeta function. Substituting (2.3) in (3.2) and using (3.3) leads to

\[
\chi \simeq 12\chi_0 \frac{T}{\Delta} \text{Re} \sum_{n=0}^{\infty} \sum_j \frac{A_j}{T_j} (2\pi W_j)^2 e^{i \frac{s_j(E_F + \pi n)}{h} - i \gamma_j},
\]

where the sum is over all the periodic orbits (primitive and repeated), and \( A_j, T_j, \) and \( W_j \) are the \( j \)-th periodic orbit amplitude, period, and winding number respectively. In the range of temperature here considered, the main contribution comes from the lowest Matsubara frequency \( n = 0 \), and the higher frequencies may be neglected. Then using first order expansion of the actions similar to (3.13) we obtain:

\[
\chi \simeq 12\chi_0 \frac{T}{\Delta} \text{Re} \sum_j \frac{A_j}{T_j} (2\pi W_j)^2 e^{-\frac{T_j x T}{h} + \frac{s_j(E_F + \pi n)}{h} - i \gamma_j}, \quad T_{th} \leq T \ll E_F \quad (3.16)
\]

A simpler form of this formula, where the dependence on the various parameters is clearer, is obtained by substituting the explicit form of the actions (at zero flux) \( S_j(E_F) = p_F L_j \) where
$p_F = h k_F$ is the Fermi momentum and $L_j$ is the length of the orbit, as well as the flux dependent part given by (3.9). Then one finds

$$
\chi \approx 24\chi_0 \frac{T}{\Delta} \sum_j \frac{A_j}{T_j} e^{-\frac{L_j T}{\pi T_{th}} (2\pi W_j)^2} \cos(k_F L_j - \gamma_j) \cos(2\pi W_j \varphi),
$$

(3.17)

where here $\sum_j$ denotes a sum over pairs of time reversed orbits. This formula shows that, in general, $\chi$ is an oscillatory function of the Fermi wavenumber and of the flux. For the shortest orbits $A_j/T_j$ is of order unity and $L_j \approx 2L$, therefore, the amplitude of these oscillations at $T = T_{th}$ is proportional to $k_F L$ and it decreases exponentially as $e^{-T/T_{th}}$ for $T > T_{th}$. In order to estimate the magnitude of the sample to sample fluctuations we shall consider

$$
\delta \chi = \sqrt{\langle \chi^2 \rangle}.
$$

(3.18)

Its calculation (given in Appendix A) is based on two results: one is the Hannay and Ozorio de Almeida sum rule [16] which expresses the density of the periodic orbits weighted by their amplitudes as a simple function of their periods, and the other is a result of Berry and Robnik [55] according to which the winding numbers, $W_j$, of periodic orbits of chaotic billiards satisfy a discrete Gaussian distribution. The corresponding variance $\sigma^2$ is found to be proportional to the orbits period $t$, namely

$$
\sigma^2 = C \frac{v_F}{\sqrt{A}} t,
$$

(3.19)

where $v_F$ is the Fermi velocity, and $C$ is a constant approximately given by $C \approx 0.215$. The average $\langle \ldots \rangle$ over the roughness of the fluctuation function, $\delta \chi$, takes the form

$$
\frac{\delta \chi}{\chi_0} \approx \sqrt{\frac{2\pi}{T_{th}}} \frac{T}{\Delta} I(T, \varphi) \quad T_{th} \leq T \ll E_F
$$

(3.20)

where $I(T, \varphi)$ is a periodic function of $\varphi$ for which an exact formula is given in Appendix A (Eq. (A.14)). It is presented in figure 4. Note that $\delta \chi$ is a periodic function of $\varphi$ with periodicity $1/2$. This periodicity is a consequence of the coherent interference of the orbits and their time reversed counterparts which takes place not only at integer values of $\varphi$ but also at half integers which correspond to false time breaking symmetry [55]. In order to estimate the magnitude of
the fluctuations it is instructive to consider $\delta \chi$ which is the same function defined in (3.18) but with $\langle \cdots \rangle$ replaced by $(\langle \cdots \rangle)$. It is straightforward to show that $\delta \chi$ satisfies a similar formula but with $I(T, \varphi)$ replaced by a simpler function $I(T)$ given by (A.17). In particular, as shown at the end of Appendix A, at $T = T_{th}$ it yields

$$\frac{\delta \chi}{|\chi_0|} = 3\sqrt{2}e^{-1}C k_F \sqrt{A} \approx 2k_F L.$$  \hspace{1cm} (3.21)

Note that $k_F L$ is of the order of the square root of the number of electrons in the system, thus at the threshold temperature the fluctuations in the susceptibility are in general very large compared to $|\chi_0|$. The asymptotic behavior for $T \gg T_{th}$ may be also deduced from (A.17), and in this regime the average of the fluctuations decreases exponentially as

$$\frac{\delta \chi}{|\chi_0|} \approx 4.5k_F \sqrt{A} \sqrt{\frac{T}{T_{th}}} e^{-\frac{T}{T_{th}}} \hspace{1cm} T_{th} \ll T \ll E_F.$$  \hspace{1cm} (3.22)

The temperature for which the fluctuations are of order of the Landau susceptibility $\chi_0$ is approximately $T \approx T_{th} \ln(4.5k_F L)$ i.e. typically larger than $T_{th}$.

3.2 THE LOW TEMPERATURE REGIME $T \ll \Delta$. — Consider now the low temperature regime $T \ll \Delta$. It is important to note that unlike high temperatures where the flux dependence of the chemical potential is very weak (and therefore was ignored), here it is very strong. The reason is that in order to maintain the number of particles constant, the chemical potential must lie in the middle between two energy levels, therefore it must follow the variations of the energy levels due to changes in the flux. At high temperatures, on the other hand, the chemical potential is affected by the occupation of many levels in the energy interval of size $T$. The flux does not affect the mean density of states and therefore it does not affect the chemical potential as well. Note also that fixing the number of electrons while considering the grand potential (3.2) is justified in the limit of large number of particles where the differences between the thermodynamic properties derived from the canonical and from the grand canonical ensembles are small.

In order to obtain a closed formula for the susceptibility in the low temperature regime, we shall apply Berry and Keating resummation method [38]. Since this method was already described (see also Ref. [57]), here it will only be outlined briefly. Let $\hat{\Omega}$ be defined as

$$\hat{\Omega} = \frac{T}{\pi} \int dE \mathcal{R}(E) \ln(1 + e^{\frac{E_F - E}{T}}),$$  \hspace{1cm} (3.23)

where $\mathcal{R}(E)$ is the resolvent given by

$$\mathcal{R}(E) = \sum_{\alpha} \frac{1}{E + i\varepsilon - E_{\alpha}},$$  \hspace{1cm} (3.24)

and $E_{\alpha}$ are the eigenenergies of the system ($\varepsilon$ is an infinitesimal positive number). The imaginary part of $\hat{\Omega}$ is the grand potential defined in (3.2). An important relation satisfied by $\hat{\Omega}$ is the exact functional equation:

$$\hat{\Omega}(-\hbar) = \hat{\Omega}^*(\hbar).$$  \hspace{1cm} (3.25)

It is due to the fact that the eigenvalues $E_{\alpha}$ are invariant while $\varepsilon$ changes sign under $\hbar$ reversal [38, 57]. Let $\tilde{\chi}$ be related to $\hat{\Omega}$ as in (3.3), namely

$$\tilde{\chi} = -\mathcal{A} \left( \frac{\partial^2 \hat{\Omega}}{\partial \phi^2} \right)_{E_F, T},$$  \hspace{1cm} (3.26)
where $E_F$ is the chemical potential. The susceptibility is merely the imaginary part of $\tilde{\chi}$. Using (3.23) and (3.26) and introducing the semiclassical approximation for the resolvent, 
\[ \mathcal{R}(E) = \frac{\partial}{\partial E} \ln \mathcal{D}(E), \]
one obtains
\[ \tilde{\chi} \simeq \frac{6\chi_0 T Am}{\pi \hbar^2} \sum_{n=0}^{\infty} \sum_{p,j} (2\pi W_p)^2 \left[ \frac{t_{pj}(E)}{1 - t_{pj}(E)} + \left( \frac{t_{pj}(E)}{1 - t_{pj}(E)} \right)^2 \right], \tag{3.27} \]
where
\[ t_{pj}(E) = e^{-u_p(\frac{1}{2} + j) + \frac{T_{pj}}{2\pi} \varepsilon_p E_F - i\gamma_p}. \tag{3.28} \]
Here the sum over $p$ includes all the primitive periodic orbits, $W_p$ are the corresponding winding numbers, and $n$ counts the Matsubara frequencies (3.8). The derivation of the above formula is performed using the infinite product representation of the zeta function (2.5), and by converting the integral over the energy to a sum over Matsubara frequencies as explained before. For real values of $1/\hbar$ the periodic orbit sum (3.27) diverges in the limit $T \to 0$. To avoid this problem it is implicitly assumed that the imaginary part of $1/\hbar$ is sufficiently large (see (2.12)) so that all the periodic orbit sums and products converge.

In the limit $T \to 0$, the sum over Matsubara frequencies may be approximated by an integral. The integral is performed using the approximation
\[ t_{pj}(E_F + i\varepsilon_n) \simeq e^{-u_p(\frac{1}{2} + j) + \frac{T_{pj}}{2\pi} \varepsilon_p E_F - i\gamma_p}, \tag{3.29} \]
which is obtained by expanding the action to first order in $\varepsilon_n / E_F$ as in (3.13). This approximation is valid only for sufficiently low values of $n$. However, the contributions from the higher Matsubara frequencies decay exponentially, and become negligible for values of $n$ where $\varepsilon_n / E_F \ll 1$ and the expansion (3.13) leading to (3.29) is still valid. The result is
\[ \tilde{\chi} \simeq \frac{3i\chi_0 m A}{\pi^2} \sum_{p,j} (2\pi W_p)^2 \frac{t_{pj}(E_F)}{T_p \hbar (1 - t_{pj}(E_F))}. \tag{3.30} \]
The common denominator of all the terms in this sum is precisely the zeta function. Multiplying the numerator and the denominator by $e^{-i\pi \eta(1/\hbar)}$, where $\eta(1/\hbar)$ is the mean level staircase (given approximately by the Weyl term (2.15) and written explicitly as function of $1/\hbar$), one obtains the spectral determinant, $D(E_F)$, in the denominator, and $\tilde{\chi}$ may now be written alternatively as
\[ \tilde{\chi} \simeq \frac{3i\chi_0 m A}{\pi^2} \mathcal{N}(E_F; \frac{1}{\hbar}) \mathcal{D}(E_F), \tag{3.31} \]
where
\[ \mathcal{N}(E_F; \frac{1}{\hbar}) = \frac{i}{\hbar} \sum_p (2\pi W_p)^2 T_p \Lambda_p(E_F; \varphi), \tag{3.32} \]
while
\[ \Lambda_p(E_F; \varphi) = e^{-i\pi \eta(\frac{1}{\hbar})} \sum_{j} t_{pj}(E_F) \prod_{(p',j') \neq (p,j)} (1 - t_{pj'}). \tag{3.33} \]
Notice that $\chi_0$ is independent of $\hbar$ unlike $\mathcal{N}(E_F; 1/\hbar)$ and $\mathcal{D}(E_F)$. The analytic continuation of these functions is performed for the numerator and the denominator separately. The resummed formula of $\mathcal{D}(E_F)$ was discussed in section 2 following reference [38]. Thus we are left with the
problem of the analytic continuation of $\mathcal{N}(E_F;1/h)$. To this purpose one has first to express it as a sum over pseudo-orbits. It is of the form of a triple sum,

$$\mathcal{N}(E_F;1/h) = \int_0^1 \sum_{p,\omega} \frac{2\pi W_p^2}{T_p} c_{p\omega} \exp^{i\lambda S_{p\omega} - i\pi R(1/h)},$$

(3.34)

where $S_{p\omega} = S_p + S_{\omega}$,

(3.35)

with $S_p$ given by (2.8), $S_{\omega}$ is the action of the primitive periodic orbit $p$, and $c_{p\omega}$ are amplitudes which are similar, although different, to those of the spectral determinant (2.11). The construction of these amplitudes is discussed in reference [57].

$\mathcal{N}(E_F;1/h)$ satisfies the functional equation

$$\mathcal{N}(E_F;1/h) = \mathcal{N}^*(E_F;1/h),$$

(3.36)

which follows directly from (3.35) and the invariance property of the energy levels which implies that $\mathcal{D}(E_F + i\varepsilon)$ transforms into $\mathcal{D}(E_F - i\varepsilon) = \mathcal{D}^*(E_F + i\varepsilon)$ under $h$ reversal.

By means of Cauchy theorem $\mathcal{N}(E_F;1/h)$ is now expressed as a contour integral of the form

$$\mathcal{N}(E_F;1/h) = \frac{1}{2\pi i} \int_{C_+} \frac{dz}{z} \gamma(z,h) \mathcal{N}(E_F;1/h + z),$$

(3.37)

where $C_\pm$ are the contours shown in figure 5, and the function $\gamma(z,h)$ is even in $z$, analytic within the integration strip, and satisfying $\gamma(0,h) = 1$. This is an analytic continuation of $\mathcal{N}$ from the region of complex $1/h$, where the sum is absolutely convergent, to the real $1/h$ axis. It assumes the analyticity of $\mathcal{N}$ in a sufficiently wide strip around the real axis. With the aid of relation (3.36) one obtains

$$\mathcal{N}(E_F;1/h) = \frac{1}{2\pi i} \int_{C_+} \frac{dz}{z} \gamma(z,h) \left[ \mathcal{N}(E_F; z + 1/h) + \mathcal{N}^*(E_F; z - 1/h) \right].$$

(3.38)

Choosing the integration path $C_+$ sufficiently far from the real axis of $1/h$ so that condition (2.12) holds, ensures that the sum (3.34) converges everywhere on $C_+$. Therefore, one may substitute (3.34) into (3.38) to obtain

$$\mathcal{N}(E_F;1/h) = \sum_{p,\omega} \frac{2\pi W_p^2}{T_p} \left[ U_{p,\omega}(1/h) + U_{p,\omega}^*(-1/h) \right],$$

(3.39)
where

\[
U_{p,j,\mu}\left(\frac{1}{\hbar}\right) = \frac{1}{2\pi i} \int_{C_+} \frac{dz}{z} \gamma(z, \hbar) \left(\frac{1}{\hbar} + z\right) e^{i [\pi \hat{N}(E, \frac{1}{\hbar}) - \pi \hat{N}(E, \frac{1}{\hbar} + z) + \pi \hat{S}_{\mu,p}]} \times
\]

By means of arguments similar to those presented in references [38, 57] one may show that the pseudo-orbits in (3.34) satisfy the relation:

\[
c_{\mu}^{(p,j)} e^{-\pi \hat{N}(E) + \frac{i}{\hbar} \hat{S}_{\mu,p}} \rightarrow \left[c_{\mu}^{(p,j)} e^{-\pi \hat{N}(E) + \frac{i}{\hbar} \hat{S}_{\mu,p}}\right]^*,
\]

if \(h \rightarrow -h\).

From this property and by an argument which involves deformation of the integration path \(C_+\) to the real axis one concludes that

\[
U_{p,j,\mu}\left(-\frac{1}{\hbar}\right) = U_{p,j,\mu}\left(\frac{1}{\hbar}\right).
\]

Therefore, using (3.39) one may express the triple sum of (3.34) as

\[
\mathcal{N} \left(E_F; \frac{1}{\hbar}\right) = \sum_{p,j,\mu} \left(\frac{2\pi W_p}{T_p}\right)^2 U_{p,j,\mu}\left(\frac{1}{\hbar}\right).
\]

This formula makes sense only if all the sums and integrals converge. This may be achieved by choosing [38]

\[
\gamma(z, \hbar) = e^{-\frac{1}{2} K^2 z^2 |z|},
\]

where \(K\) is a constant which plays a similar role to the one introduced for the spectral determinant (see Sect. 2).

The integral (3.40) is now performed noticing that for two dimensional closed billiards the exponent factor has the form [42]

\[
e^{i [\pi \hat{N}(\frac{1}{\hbar}) - \pi \hat{N}(\frac{1}{\hbar} + z)]} = e^{-i \pi [\hat{N}_1 z + \frac{1}{2} \hat{N}_2 z^2]},
\]

where

\[
\hat{N} \left(\frac{1}{\hbar}\right) \approx \frac{\Omega(E)}{\hbar^2},
\]

\[
\hat{N}_1 = \frac{\partial}{\partial \left(\frac{1}{\hbar}\right)} \hat{N} \left(\frac{1}{\hbar}\right) \approx \frac{\Omega(E)}{\pi \hbar},
\]

\[
\hat{N}_2 = \frac{\partial^2}{\partial \left(\frac{1}{\hbar}\right)^2} \hat{N} \left(\frac{1}{\hbar}\right) \approx \frac{\Omega(E)}{2\pi^2}.
\]

Thus

\[
U_{p,j,\mu} \left(\frac{1}{\hbar}\right) = \frac{1}{2\hbar} c_{\mu}^{(p,j)} e^{-i \pi \hat{N}(E, \frac{1}{\hbar}) + \frac{i}{\hbar} \hat{S}_{\mu,p}} (J_{p,\mu} + I_{p,\mu}),
\]
where
\[
J_{p,\mu} = \text{Erfc} \left\{ \frac{\xi_p(\mu, E_F)}{Q(K, \hbar, E) \sqrt{2\hbar}} \right\},
\]
and
\[
I_{p,\mu} = \frac{i\sqrt{\hbar}}{\pi Q(K, \hbar, E)} \exp \left\{ -\frac{\xi_p^2(\mu, E_F)}{2\hbar Q^2(K, \hbar, E)} \right\}.
\]
Here \(Q(K, \hbar, E)\) is given by (2.18) with \(K\) as a free tuning parameter, and
\[
\xi_p(\mu, E_F) \simeq S_{\mu,p}(E_F) - \frac{\Omega(E)}{2\pi\hbar}.
\]
The terms associated with \(I_{p,\mu}\) pick their contribution only from small number of pseudo-orbits in the vicinity of the cutoff. Their contribution is small compared to that corresponding to the \(J_{p,\mu}\) terms since the latter includes all pseudo-orbits with period smaller than half of the Heisenberg time. Henceforth, we shall neglect this contribution. Thus altogether we obtain the resummed expression for \(N(E_F; 1/\hbar)\), namely
\[
N(E_F; \frac{1}{\hbar}) = \frac{i}{\hbar} \sum_{p,\mu} \frac{(2\pi W_p)^2}{T_p} c_\mu^{(p,j)} e^{\hbar S_{\mu,p} - \pi \hbar} \text{Erfc} \left\{ \frac{\xi_p(\mu, E_F)}{Q(K, \hbar, E) \sqrt{2\hbar}} \right\}.
\]
The susceptibility is the imaginary part of \(\chi\). Substituting (3.51) into (3.31) and taking the imaginary part one finally obtains:
\[
\chi \simeq \frac{6\chi_0}{\pi D(E_F)} \text{Re} \sum_p \frac{\hbar}{\Delta T_p} (2\pi W_p)^2 \Lambda_p(E_F; \varphi) \quad T \ll \Delta
\]
where the functions \(\Lambda_p(E_F; \varphi)\) are now given by their corresponding resummed versions,
\[
\Lambda_p(E_F; \varphi) = \sum_{j=0}^{\infty} \sum_\mu c_\mu^{(p,j)} e^{\hbar S_{\mu,p} - \pi \hbar} \text{Erfc} \left\{ \frac{\xi_p(\mu, E_F)}{Q(K, \hbar, E) \sqrt{2\hbar}} \right\}.
\]
Formula (3.52) is an approximate semiclassical expression for the susceptibility at low temperatures. It is expressed in terms of sum over finite number classical periodic orbits.

We turn now to estimate the average of the fluctuations \(\delta \chi\) in the low temperature regime. It is difficult to perform this calculation using formula (3.52) directly, since the numerator and the denominator are correlated. Therefore, a simplified version of this formula is required. For this purpose we shall investigate the possibility that \(\Lambda_p(E_F; \varphi)\) as well as \(\zeta_n(E_F)\) may be approximated by truncated products corresponding to (3.33) and (2.5) respectively. There are some indications that the spectral determinant may be approximated in this way [58, 59]. Under this assumption
\[
\frac{\Lambda_p(E_F; \varphi)}{D(E_F)} \approx \sum_j \frac{t_{pj}}{1 - t_{pj}},
\]
and choosing the cutoff of the truncated products to be at the orbit with period of the Heisenberg time, one deduces from (3.53) that
\[
\chi \approx \frac{3}{\pi^2} \chi_0 m A \text{Re} \sum_{T_p \leq \Delta} \frac{(2\pi W_p)^2}{T_p \hbar} \sum_j \frac{t_{pj}(E_F)}{1 - t_{pj}(E_F)},
\]
which as expected corresponds to the imaginary part of the truncated sum of (3.30). Note that here the spectral determinant is evaluated far from its zeros at values of $E_F$ which lie in the middle between two energy levels, therefore, the expression on the LHS of (3.54) does not diverge as seen also from the RHS.

The sum (3.55) consists in two sorts of contributions. One, which actually includes almost all the orbits, is associated with the long unstable orbits where $|t_{p0}| \ll 1$ and therefore $\sum_j t_{pj} / (1 - t_{pj}) \approx t_{p0}$. The other originates from the shortest periodic orbits for which the instability exponent $u_p$ is of order unity so that this approximation does not hold. First, we shall ignore the latter contribution. This may be justified when the shortest orbits are very unstable, for example due to the roughness of the boundary as shown in figure 3. Under these assumptions one may estimate the average of the fluctuations $\delta \overline{X}$. The calculation is carried out in Appendix B where it is shown that

$$\frac{\delta \overline{X}}{|X_0|} \approx \sqrt{216 \cdot \ln (k_F L/2)} C k_F \sqrt{A} \quad \text{at} \quad T \ll \Delta$$

(3.56)

Thus the fluctuations of the susceptibility as $T \to 0$ are larger than those at $T = T_{\text{th}}$, but since the square root of the logarithm exhibits only a very weak dependence on $k_F L$, also in this case the fluctuations are practically proportional to $k_F L$. The factor $\sqrt{\ln (k_F L/2)}$ is an enhancement of the fluctuations resulting from the wide range of periodic orbits contributing to the susceptibility in the low temperature regime, and it is of semiclassical origin.

Let us now investigate the possibility of an additional contribution from the shortest periodic orbits for which $u_p$ is of order one. Unlike the contribution from long orbits that cover the energy shell ergodically, this contribution, if it exists, gives rise to nonuniversal individual properties of the system. In order to understand the nature of this contribution we now briefly discuss the closely related phenomenon of “scars”.

For some chaotic systems it was found that some eigenstates are strongly peaked near unstable periodic orbits. These imprints of the periodic orbits were termed as “scars” by Heller [60, 61]. An example of a scarred wave function is shown in figure 6. The system is a two dimensional billiard with boundaries drawn by thick line. The probability density is concentrated near the unstable periodic orbit drawn by thin solid line.

Heller gave a heuristic argument for the existence of scars, and an estimate of their magnitude. The scar phenomenon was investigated numerically [62-68] and experimentally [69, 70] for a variety of systems. It was analyzed in the framework of periodic orbit theory by Bogomolny [71] in the configuration space, and by Berry [72, 73] in phase space. Recently, a resummed formula for the semiclassical approximation of the Wigner functions corresponding to eigenstates of chaotic systems was obtained [57]. In that formula the functions $\Lambda_p(E)$, which are equal to the imaginary part of $\Lambda_p(E)$, were shown to be related to the occurrence of scars. The wave function of energy $E_\alpha$ is scarred along some particular periodic orbit $p^*$ when $\Lambda_{p^*}(E_\alpha)$ is large compared to all $\Lambda_p(E_\alpha)$ with $p \neq p^*$. This happens when the term

$$F_{p^*} = 1 - t_{p^*0} = 1 - e^{\frac{1}{2} S_{p^*}(E_\alpha)} e^{\frac{1}{2} \gamma_{p^*} - \frac{1}{2} u_{p^*}},$$

(3.57)

which appears in the product representation of the dynamical zeta function (2.5), is small compared with all the other. Such a situation is possible when the instability exponent $u_{p^*}$ is of order (or smaller) than one, that is when $p^*$ corresponds to a short periodic orbit. Then one can show that while all $\Lambda_p(E_\alpha)$ with $p \neq p^*$ are proportional to $u_{p^*}/2$, the function $\Lambda_{p^*}(E_\alpha)$ is of order one [57].
Fig. 6. — A contour plot of the probability density corresponding to a scarred wave function of a billiard system. The thick line marks the boundary of the billiard, while the thin line corresponds to the unstable periodic orbit which scars the wave function.

Exactly the same line of reasoning may be applied here for the functions $\Lambda_p(E_F; \varphi)$. Consider a situation where the Fermi energy $E_F$ and the flux $\varphi$ are such that the term (3.57) approaches its minimal absolute value i.e. $E_{F*} \approx u_{p*}/2$. In this situation it is not justified to approximate $\Lambda_{p*}(E_F; \varphi)/D(E_F)$ by $t_{p*0}$ since, as for scars

$$\frac{\Lambda_{p*}(E_F; \varphi)}{D(E_F)} \approx \frac{t_{p*0}}{1 - t_{p*0}} \approx \frac{2}{u_{p*}}. \tag{3.58}$$

The corresponding contribution from this orbit to the susceptibility, denoted here by $\chi_{\text{scar}}$, at these values of $E_F$ and $\varphi$ is

$$\chi_{\text{scar}} \approx \frac{12\chi_0 \hbar^2 (2\pi W_{p*})^2}{\pi T_{p*} u_{p*} \Delta} \approx 12\chi_0 k_F L, \tag{3.59}$$

where for the second estimation on the RHS we assumed that $W_{p*} = 1$, $T_{p*} \approx 2L/v_F$, and $u_{p*} \approx 1$ since the orbit $p^*$ is very short. Note that the contribution to the susceptibility due to strong scars is diamagnetic. This is expected since the electron spends a large portion of the time moving along one unstable periodic orbit, consequently this orbit behaves as a classical loop that carries electric current. Another important result which this calculation reveals is that although the number of shortest orbits is generally very small, the contribution associated with them is of the same order of magnitude as that resulting from all the long orbits. One cannot rule out the possibility that in some circumstances it might be even the most important one.

At this point it is instructive to re-examine the resummed formula for the susceptibility (3.52). The arguments that were given above show that anomalous fluctuations in the susceptibility are associated with the small denominators in terms such as (3.58) corresponding to
short periodic orbits. On the other hand the spectral determinant is approximately proportional to these denominators, thus as also clear from (3.52) the large fluctuations appear when the $D(E_F)$ is small. Since $E_F$ lies between two energy levels, such situation usually happens near avoided crossings (see for example Fig. 1). Thus anomalous fluctuations in the susceptibility are related to avoided crossings of the energy levels from both sides of the Fermi energy. This conclusion is supported by direct numerical calculation of the susceptibility performed by Nakamura and Thomas [37]. The behavior of scarred wave functions in the vicinity of avoided crossings was studied by Takami [67].

4. The susceptibility for a uniform magnetic field.

The calculation of the susceptibility of a noninteracting electron gas, confined to move inside a two dimensional billiard, and subjected to a uniform magnetic field $B$, differs from that corresponding to the flux line. One difference results from the coupling between the electron spin and the magnetic field. Since this effect is small its discussion will be postponed to the end of this section, and it will be ignored in the meanwhile. The important difference is that the classical dynamics depends on the field, and it is no longer a simple free motion between bounces. Similar to the previous case of a single flux line, the phase space volume with energy less than $E$ is independent of the magnetic field. Consequently, the mean level staircase $\bar{N}(E)$ is independent of the magnetic field [54], and the susceptibility is again given by the contribution of the periodic orbits (3.7). However, the dependence of the periodic orbits on the reduced flux $\varphi = BA/\phi_0$ is now different, and the main purpose of this section is to investigate this dependence.

In general, the dependence of the periodic orbits on the magnetic field is rather complicated. This is reflected in the density of states $\rho(E)$, and therefore also in the grand potential $\Omega$. For instance, in the limit of strong magnetic field, where $\varphi \gg \phi_0$, many of the eigenenergies correspond to Landau levels and therefore have high degeneracy. On the other hand, in the limit $B \to 0$ the only possible degeneracy is accidental or results from the symmetry of the system. From the semiclassical point of view, when the magnetic field is strong, the cyclotron radius,

$$R_c = \frac{c\sqrt{2mE}}{eB}, \quad (4.1)$$

is much smaller than the linear dimension of the billiard, $R_c \ll L$. Consequently, many orbits do not bounce the boundary at all, and the corresponding part of the phase space is of regular motion. Complications in the calculation of the periodic orbits may also arise from bifurcations of certain periodic orbits as $B$ changes [74]. Yet, in the weak field limit where $BA \ll \phi_0$, one may expect simple dependence on $\varphi$, because of the structural stability of the topology of the phase space in the vicinity of unstable periodic orbit with respect to small perturbations. To clarify this point, note that under uniform magnetic field the particle trajectory between two consequent bounces is a curved line with constant radius of curvature given by the cyclotron radius (4.1). The condition $BA \ll \phi_0$ implies that even at the lowest energy level of the system, $R_c \ll L$ is much larger than the linear dimension of the billiard. Therefore, the particle trajectory is built of almost straight segments, similar to the zero field case. This similarity improves at higher energies because the cyclotron radius is proportional to $\sqrt{E}$. Henceforth, we shall consider the regime of weak magnetic field, and assume that bifurcations of the relevant periodic orbits have a negligible effect.

Let $p$ denote some periodic orbit, and $p = mv - \frac{e}{c}A$ be the generalized momentum, where
A is the magnetic vector potential which satisfies

\[ B \dot{z} = \nabla \times A. \]  

(4.2)

The periodic orbit action is given by

\[ S_p(\varphi) = \oint (mv - \frac{e}{c} A) dq = \tilde{S}_p(\varphi) - \hbar \varphi a_p(\varphi), \]  

(4.3)

where \( \varphi = B A / \phi_0 \) is the reduced flux, and

\[ \tilde{S}_p(\varphi) = \oint mv dq. \]  

(4.4)

The reduced oriented area, \( a_p(\varphi) \), is the oriented area enclosed by the orbit divided by the billiard area \( A \), namely

\[ a_p(\varphi) = \frac{1}{A} \oint xy dq, \]  

(4.5)

where the integration is along the periodic orbit projection on the configuration space.

The action \( S_p(\varphi) \) will now be expanded at fixed energy in the small parameter \( \varphi \) around the zero field action \( S_p \). For this purpose we shall use the relation

\[ \left( \frac{\partial S_p}{\partial \varphi} \right)_E = - \int_0^{T_p} \frac{\partial \mathbf{H}}{\partial \varphi} dt, \]  

(4.6)

where the derivative of the action with respect to the flux is taken at fixed energy and on the manifold of periodic orbits which correspond to \( p \). It is straightforward to obtain this relation from \( \left( \frac{\partial S_p}{\partial \varphi} \right)_E = \oint \left( \frac{\partial p}{\partial \varphi} \right)_E dq \) when substituting \( \left( \frac{\partial p}{\partial \varphi} \right)_E dq \) which is solved from

\[ 0 = \frac{dH}{d\varphi} = \frac{\partial H}{\partial p} \frac{\partial p}{\partial \varphi} + \frac{\partial H}{\partial \varphi}, \]  

and changing the integration variable from the coordinate \( q \) to the time \( t \) along the orbit noticing that \( \dot{q} = \frac{\partial H}{\partial p} \). The integral (4.6) is easily calculated and may be explicitly written as

\[ \frac{\partial S_p}{\partial \varphi} \left| _E = -\hbar a_p(\varphi). \right. \]  

(4.7)

Thus the first order in the expansion of the action may be obtained from (4.3) by replacing \( \tilde{S}_p(\varphi) \) and \( a_p(\varphi) \) by their corresponding values at zero magnetic field. To calculate the second order corrections we differentiate (4.7) with respect to \( \varphi \) to obtain

\[ \frac{\partial^2 S_p}{\partial \varphi^2} \left| _E = -\hbar \frac{\partial a_p(\varphi)}{\partial \varphi}. \right. \]  

(4.8)

A first order expansion of \( a_p(\varphi) \) around the zero field value \( a_p \) is now required. For this calculation an orbit of the following geometry is considered. At \( B = 0 \) the periodic orbit consists \( N_p \) straight cords of lengths \( \{l_i\}^{N_p}_{i=1} \); pairs of these cords are joined together at the points \( \{b_i\}^{N_p}_{i=1} \) on the boundary, and the total length of the orbit is \( L_p = \sum_{i=1}^{N_p} l_i \). As explained before when \( B \neq 0 \) the straight cords deform into curved lines with constant radius of curvature (4.1). For calculation in the leading order, one may assume that the points \( b_i \) of the deformed periodic orbit almost do not move. This assumption will be justified later on. In figure 7
Fig. 7. — An illustration of a part of some periodic orbit. At zero magnetic field (dashed line) it is composed of \( N_p \) straight cords of lengths \( \{l_i\}_{i=1}^{N_p} \). These are joined together at the points \( \{b_i\}_{i=1}^{N_p} \) on the boundary. At finite small magnetic field the cords deform into curved lines with constant radius of curvature given by the cyclotron radius (4.1). To the leading order one may assume that the points \( \{b_i\}_{i=1}^{N_p} \) are independent of the magnetic field (see text).

A schematic illustration of a part of a periodic orbit with the geometry considered above is depicted for a field \( B \) pointing upwards. As this illustration shows, when the magnetic field increases, a small area difference \( \bar{a}_i \) develops near each cord \( i \). Simple trigonometrics yields

\[
\bar{a}_i \approx \frac{1}{16} \frac{l_i^3}{R_c},
\]

which holds to the leading order in \( l_i/R_c \). The total area difference is, therefore, \( \frac{1}{16} \sum_i l_i^3/R_c \). It is convenient to introduce a dimensionless parameter \( G_p \) which is associated with each periodic orbit, and defined as

\[
G_p = \frac{1}{L_p A} \sum_{i=1}^{N_p} l_i^3.
\]

It is of order one since \( l_i^3 \) is of order of the billiard area \( A \). With this definition, the lowest order expansion for the reduced oriented area \( a_p(\varphi) \) may be written as

\[
a_p(\varphi) \approx a_p + \frac{1}{16} \frac{G_p L_p}{A \sqrt{2mE}} \varphi,
\]

where \( a_p = a_p(0) \) is the reduced oriented area at zero field. Substituting this expression in (4.8) gives the second derivative of the action with respect to the flux. The expansion of the action to second order in \( \varphi \) may now be obtained, and with the help of the approximate expression (2.15) for the mean level staircase one finds

\[
S_p(\varphi) = S_p - h a_p \varphi - \frac{\pi}{32 N(E)} \varphi^2 + O(\varphi^3).
\]

Let us now return to examine the assumption concerning the displacements of the points \( b_i \). In general they do move from their zero field positions. However, these small displacements lead to a change in the parameter \( G_p \), which corresponds to corrections of higher order than that considered in (4.12).
The formula for the susceptibility in the case of a uniform magnetic field is identical to (3.7), but various quantities of the periodic orbits in the expansion of the dynamical zeta function and its derivatives are different. The pseudo-orbits actions are still given by (2.8) but $S_p$ should be replaced by $S_p(\varphi)$ of (4.12). The derivatives $\zeta^{(3)}(E; \varphi)$ are given by (3.10) with action of (4.12) and the new amplitudes

$$c^{(1)}_{\mu} = -i2\pi c_{\mu} \sum_{\{r_p\}_\mu} r_p \left[ a_p + \frac{G_p S_p}{32\hbar N(E_F)} \varphi \right],$$  \hspace{1cm} (4.13)

and

$$c^{(2)}_{\mu} = \frac{1}{c_{\mu}} (c^{(1)}_{\mu})^2 - \frac{i\pi c_{\mu}}{16\hbar N(E_F)} \sum_{\{r_p\}_\mu} r_p G_p S_p,$$  \hspace{1cm} (4.14)

where $c_{\mu}$ are given by (2.11), and $\{r_p\}_\mu$ represents the set of the repetition numbers corresponding to the pseudo-orbit $\mu$.

Consider the expansion of the actions of the periodic orbits in the presence of weak magnetic field (4.12). It contains three terms: The first is the periodic orbit action at zero field. The second is proportional to the ratio between the oriented area enclosed by the orbit (at $B=0$) and the total area of the billiard. It may be interpreted as the contribution to the action due to a uniform distribution of flux lines in the billiard which have a total reduced flux $\varphi$. In the third term $G_p$ is of order one, and since $S_p$ is bounded between the shortest and the longest periodic orbits actions (see (2.13) and (2.9)) one may easily verify that up to constants of order unity,

$$\frac{\hbar}{k_p L} < \frac{\pi G_p S_p}{32N(E_F)} < \hbar.$$  \hspace{1cm} (4.15)

To estimate the influence of this term one has to compare it with the first order correction in (4.12) i.e. with a typical value of the reduced oriented area, $a_p$, multiplied by Planck's constant. In Appendix C it is shown that the reduced oriented areas satisfy a Gaussian distribution. It has zero mean value and a variance given by (3.19) but with the constant $C$ replaced by a different constant $C'$ which depends on the shape of the billiard via

$$C' = \frac{2}{A^{1/2}} \sum_{j=1}^{\infty} \frac{1}{k_j^2} \int \int \mathrm{d}r_1 \mathrm{d}r_2 \ J_0(k_j |r_1 - r_2|).$$  \hspace{1cm} (4.16)

Here, the integrals over $r_1$ and $r_2$ cover the billiard domain, and $k_j$ is the wave number corresponding to the $j$-th eigenstate of the billiard. The latter may be replaced by the Weyl approximation $j - \frac{1}{2} \approx \frac{k_j^2 A}{4\pi}$. An upper limit on $C'$ in this approximation is obtained by replacing the Bessel function with its maximal value (that is one). The result is $C' \leq C$.

Assuming now that a typical value of the reduced oriented area for long orbits is given by the standard deviation $\sigma$, one may estimate the ratio between the second and the first order contributions to the action (4.12) at a given orbit length $L_p$. It turns out to be of the order of

$$\frac{\pi \varphi}{16\sqrt{C' \hbar k_p L}} \sqrt{\frac{L_p}{L}},$$  \hspace{1cm} (4.17)

where $G_p$ was assumed to be of order one. The period of the longest contributing orbits is of order of the Heisenberg time, therefore, for these orbits this ratio is of the order $1/\sqrt{k_p L}$. For the shortest orbits $a_p \approx 1$ and $L_p \approx L$ and one concludes by direct inspection of (4.12) that this
ratio is of order $1/k_F L$. Thus if $C'$ is not very small the second order term of the action for $|\varphi|<1$ is usually negligible compared with the first order term. Therefore it may be neglected in the pseudo-orbits amplitudes (4.13) and (4.14) which may be approximated by

$$c_{\mu}^{(j)} \approx c_{\mu} \left(-2\pi i \sum_{\{r_p\}} r_p a_p \right)^j. \quad (4.18)$$

At low temperatures long orbits have a significant contribution. For these the second order contribution is of order $\hbar$, therefore, it cannot be neglected from the phases of the long pseudo-orbits unless $\varphi \ll 1$. At high temperatures where only the shortest periodic orbits contribute effectively, this term may be neglected from the phases as well.

From these arguments one concludes that for $T \geq T_{th}$ the susceptibility is given by a formula (3.16) but with the winding numbers replaced by the reduced oriented areas. The crucial point is that the latter are not integers and may acquire a continuous set of values. Thus the susceptibility in general will be a complicated non-periodic function of the flux. The calculation of the average of the fluctuations, $\delta\chi$, is performed along the same lines as for single flux line. In Appendix C it is shown that in analogy with (3.20) it satisfies

$$\frac{\delta\chi}{|\chi_0|} \approx \sqrt{2/\Delta} \frac{T}{E_F} I'(T, \varphi) \quad T_{th} \leq T \ll E_F \quad (4.19)$$

where $I'(T, \varphi)$ is given by (C.10). As explained above, $\chi$ and therefore also $\delta\chi$, are not periodic functions of the flux. Moreover, $\delta\chi$ depends on the shape of the billiard via $C'$ given by (4.16). To illustrate the nature of this dependence, $\delta\chi(\varphi)$ for various values of $C'$ at $T=T_{th}$ is presented in figure 8.

Consider now the low temperature regime. One may easily show that for the calculation of $\delta\chi$ the periodic orbit actions do not play any role (see Appendix A). Therefore, $\delta\chi$ depends solely on the pseudo-orbits amplitudes for which a good approximation is given by (4.18). This implies that the averaged fluctuations in this regime are given by the same expression as (3.56) but with $C$ replaced by $C'$. Similarly the contribution due to scars is given by (3.59) but with $W_p$ replaced by the reduced oriented area $a_p$. 

---

**Figure 8.** The fluctuations of the susceptibility for the case of uniform magnetic field. $\delta\chi$, in units of $|\chi_0|k_F L$, is plotted as a function of the reduced flux $\varphi = BA/\phi_0$ for various values of $C'$ at $T = T_{th}$. The solid line corresponds to $C' = C$, the dashed line to $C' = 0.5C$, and the dotted line to $C' = 0.1C$. $C = 0.215$. 


Throughout all the derivation it was assumed that all orbits are chaotic and therefore the cyclotron radius, $R_c$, is much larger than the linear dimension of the system $L$ even for energies as low as the ground state, leading to the condition $|\varphi| \ll 1$. This condition is, however, too restrictive. If it is violated for the low energy states, i.e., $R_c(k) < L$ where $k$ is the corresponding wave number, then the contribution of these states will be just the Landau susceptibility, $\chi_0$, that is much smaller than the contribution of the regime where the orbits are chaotic. Therefore, the main contribution to the susceptibility comes from the energy regime where the motion is chaotic. The values of the susceptibility will be close to those found in the present paper if the condition $R_c(k) \gg L$ holds for a restricted range below the Fermi energy that is sufficiently large, say of wave numbers $\beta' k_F < k < k_F$ where $\beta'$ is of order 1/2. This leads to the requirement

$$|\varphi| \ll \frac{\beta'}{2\pi} k_F L.$$  \hfill (4.20)

The contribution from the intermediate energy regime $R_c(k) \approx L$ (assumed to be small) is ignored in our considerations.

The condition (4.20) turns out to be also sufficient to insure that the contribution of order $\varphi^2$ in (4.12) is not important for the short orbits and therefore for high temperatures. It implies that the effect of the magnetic field is similar to that of a flux line for a wide range of the magnetic field amplitude where many oscillations of the susceptibility as a function of $\varphi$ may be observed. For low temperatures the requirement that the ratio (4.17) is small for all orbits leads to a stronger condition:

$$|\varphi| \ll \frac{16\sqrt{C'}}{\pi} \sqrt{k_F L}. \hfill (4.21)$$

The sensitivity of a given periodic orbit to the applied magnetic field usually grows exponentially with its length. One may argue then that at large values of $\varphi$, such as the above formula permits, many periodic orbits disappear or bifurcate, and that this effect should be taken into account. Generally, this will happen for long periodic orbits which are ergodic and behave statistically. Therefore, these bifurcations are not expected to have strong influence on the Hannay and Ozorio de Almeida sum rule which takes into account only statistical properties of these orbits (see Appendix A, Eq. (A.1)). Then the main conclusion regarding the amplitude of fluctuations will not change.

Finally, we comment about the effect of Zeeman splitting due to the electron spin, ignoring the spin orbit coupling. It may be incorporated in the formalism by considering the grand potential

$$\Omega_{\text{spin}}(E_F) = \Omega \left(E_F + \frac{e\hbar B}{2m_0c}\right) + \Omega \left(E_F - \frac{e\hbar B}{2m_0c}\right), \hfill (4.22)$$

where $m_0$ is the actual mass of the electron, and $\Omega$ is the grand potential calculated by ignoring the spin. Up to corrections of the order of the Pauli susceptibility (which is of the order of the Landau susceptibility) it can be assumed that $\Omega_{\text{spin}} = 2\Omega$. Thus the spin effect is not expected to change our previous conclusions.

5. Discussion.

The main results of the first part of this paper, concerned with the magnetic response due to a single Aharonov-Bohm flux line, are equations (3.7), (3.17) and (3.52). They express the susceptibility of a clean mesoscopic system, chaotic in its classical limit, in terms of finite sums over classical periodic orbits. At high temperatures, of the order of $T_{\text{th}} \approx \frac{k_F L}{4\pi^2} \Delta$, thermal
excitations wash out almost all the microscopic structure of the density of states and only the long range correlation of the spectrum are of any importance. They are determined by the shortest periodic orbits of the system. Thus for $T > T_{th}$ a very simple formula (3.17), in which only few short orbits have a significant contribution, characterizes the magnetic response of the system. In this temperature regime the periodic orbit expansion is a very efficient way of calculating the susceptibility. At $T = T_{th}$ the average amplitude of the fluctuations of the magnetic response is found to be of the order of $k_F L |\chi_0|$ where $\chi_0$ is the Landau susceptibility in two dimensions, and to decrease exponentially as $e^{-T/T_{th}}$ for temperatures $T$ that are sufficiently higher.

For temperatures lower than the mean level spacing ($T < \Delta$), the periodic orbits required for the calculation of the susceptibility are exactly those needed in order to obtain the energy spectrum. The longest orbit has a period equal to half of the Heisenberg time [38]. Since the number of periodic orbits proliferates exponentially with their length, in practice, (3.52) enables calculation of the susceptibility for a relatively low Fermi energy. The magnitude of the fluctuations in this regime is found to be of the order of $k_F L \sqrt{\ln(k_F L/2)} |\chi_0|$. Anomalous fluctuations of the susceptibility in the low temperature regime were associated with the existence of "scars". They correspond to wave functions concentrated near one (or few) classical unstable periodic orbit. This orbit, then, behaves like a classical current carrying loop. It was also shown that large fluctuations in the susceptibility are related to avoided crossings of the energy levels from both sides of the Fermi energy. This result is supported by previous numerical calculations [37], and may be related to calculations of curvature statistics [75-82]. A more detailed characterization of the fluctuations of the susceptibility should probably include a study of the the statistical properties of avoided crossings in relation to the localization of wave functions near unstable periodic orbits as observed in reference [67].

In the second part of this paper, the response due to a weak uniform magnetic field was investigated. The main difference between this case and the previous one is that as the magnetic field increases, Landau levels, associated with bulk states, are gradually appearing. The degeneracy of each Landau level, disregarding the spin, is $BA/\phi_0$ where $B$ is the magnetic field, $A$ is the area of the billiard, and $\phi_0$ is the flux quantum. Therefore, whenever the flux through the system increases by $\phi_0$ one electron per occupied Landau level is transferred from the surface to the bulk, where electrons are referred as belonging to the surface or the bulk if they are scattered or not scattered by the surface respectively. This process diminishes the fluctuations of the susceptibility since the contribution from the bulk electrons is of the order of Landau susceptibility. However, when the magnetic field is sufficiently weak, only a small fraction of the electrons occupies the Landau levels and the response is dominated by the surface states associated with the chaotic orbits. The condition on the magnetic field is such that the total flux through the system satisfies $\phi \ll \frac{k_F L}{2\pi} \phi_0$ (see (4.20)). For this range of field strengths the fluctuations of the susceptibility are of the same order of magnitude as those corresponding to a single flux line. Moreover, it was shown that at high temperatures, when this condition is satisfied, the response due to the magnetic field behaves in a similar way as the response due to a uniform density of flux lines with the total flux $\phi = BA$. This implies that, many oscillations of the susceptibility as function of the flux may be characterized in terms of the properties of a few periodic orbits at zero field.

The last result and the simplicity of the magnetic response in the high temperature regime suggest an experimental test. By measuring the susceptibility of a clean mesoscopic sample as a function of the flux at $T \approx T_{th}$, and then by Fourier transforming the result one should be able to identify the areas enclosed by the shortest orbits. Since at this temperature only few orbits contribute effectively to the susceptibility, it is characterized by a small number of
parameters (see (3.17)).

Our calculations were performed assuming that the motion is completely ballistic. Introducing disorder in the system has two major effects. One is the destruction of periodic orbits, while the other is the formation of new unstable periodic orbits which are scattered by impurities. The shortest among them will typically be shorter than those of the clean system. When the elastic mean free path is sufficiently large, \( l \gg L \), most of the short periodic orbits of the clean system will not be destroyed. On the other hand, the typical area enclosed by orbits which are scattered from a small number of impurities is very small, otherwise they are very unstable and consequently their contribution at high temperatures is small. Therefore, the period of oscillation in \( B \) associated with them is very large, and in the experiment proposed above they are expected to add a low frequency noise.

The estimations for the fluctuations in the susceptibility were found only in two limiting regions, namely, \( T \ll \Delta \) and \( T \geq T_{th} \). It was shown that they do not differ very much. Yet, the exact behavior of the fluctuations as function of the temperature in the wide range \( \Delta < T < T_{th} \) is still lacking, and will be subject of further studies.

The use of Gutzwiller's trace formula in the present paper is valid under the assumption that all the classical periodic orbits are unstable. However, generic billiards exhibit mixed dynamics. Then, there may be orbits of marginal stability (e.g. such as the bouncing ball orbits in the stadium billiard), stable periodic orbits corresponding to parts of the phase space where the motion is regular, and even families of stable periodic orbits (as in integrable systems). The influence of such orbits on the spectral properties of the system is usually much stronger than that of unstable orbits [31, 83]. For instance, using a formula derived by Berry and Tabor [39] in order to estimate the contribution associated with families of orbits shows that in two dimensions it is of order \( (k_F L)^{\frac{3}{2}}|\chi_0| \). This is much larger than the contribution associated with scars along unstable periodic orbits and other contributions of unstable orbits. The general behavior of the susceptibility in mixed systems was not investigated in the present paper and is left for future studies.

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Appendix A.

Fluctuations of \( \chi \) for \( T \geq T_{th} \).

The purpose of this appendix is to calculate the average \( \langle \chi^2 \rangle \), where \( \langle \cdots \rangle \) denotes an averaging over some external parameter associated with the roughness of the boundary. The main tool for this calculation is the classical sum rule derived by Hannay and Ozorio de Almeida (H&OA) [16]. According to this rule, the density of the periodic orbits weighted by their corresponding amplitudes is given by

\[
\sum_j |A_j|^2 \delta(t - T_j) \approx t, \tag{A.1}
\]
where $A_j$ is the amplitude associated with the $j$-th periodic orbit, and $T_j$ is its period. The derivation of (A.1) is based on the uniformity principle which states that long periodic orbits of chaotic systems covers the energy shell ergodically. Therefore, in general, this formula is correct for $t > t_L$ where $t_L$ is the time which takes for a particle to explore the energy shell. By dimensional analysis argument it is reasonable that in simply connected billiards with ballistic motion of electrons without trapping or sticking, the time $t_L$ is inversely proportional to the topological entropy. Thus when the instability is large, $t_L$ is close to the period of the shortest periodic orbit, $t_{\text{min}} \approx 2L/v_F$. The following calculations involve a sample averaging over billiards similar to that illustrated in figure 3. The motion in these is considerably unstable due to the roughness of the boundary, therefore, it will be assumed that (A.1) gives a reasonable approximation for time as small as $t = t_{\text{min}}$.

From the approximate equation (3.16) for the susceptibility at high temperatures one finds

$$
\chi^2 \simeq \frac{1}{4} \left(48\pi^2 \frac{T}{\Delta^2} \right)^2 \sum_{j,j'} \frac{W_j^2 W_{j'}^2}{T_j T_{j'}} e^{-\frac{4\pi k}{3} (T_j + T_{j'})} \times
$$

$$
\times \left[ A_j A_{j'} e^{i(S_j - S_{j'})} e^{2\pi \phi (W_j - W_{j'})} + A_j A_{j'} e^{i(S_j + S_{j'})} e^{2\pi \phi (W_j + W_{j'})} \right] + C.C.
$$

(A.2)

Here $S_j$ and $S_{j'}$ denote the parts of the periodic orbits actions which do not depend on the flux, and the Maslov phases $\gamma_j$ where absorbed into the amplitudes $A_j$. The flux dependence is written explicitly in terms of the winding numbers $W_j$ and $W_{j'}$. To each orbit in this sum corresponds a time reversed counterpart with the same value of action at $\varphi = 0$ but with a winding number of opposite sign. Upon ensemble averaging the second double sum in the parenthesis of (A.2) drops out due to the wildly fluctuating phases, while in the first sum, the diagonal part survives the averaging. The contributions from each pair of time reversed orbits interfere and the result one obtains is

$$
\langle \chi^2 \rangle \simeq \frac{1}{2} \left(48\pi^2 \frac{T}{\Delta^2} \right)^2 \sum_j \left| \frac{A_j}{T_j} \right|^2 \langle W_j^4 (1 + e^{4\pi \phi W_j}) e^{-\frac{2\pi k}{3} T_j} \rangle.
$$

(A.3)

The average quantity $\langle W_j^4 e^{4\pi \phi W_j} \rangle$, where $\alpha = 4\pi \phi$, is now required. It will be calculated using the result of Berry and Robnik [55] that the winding numbers of chaotic systems satisfy a discrete Gaussian distribution, namely, $P(W) = \mathcal{N} e^{-\frac{W^2}{2A^2}}$ where $\mathcal{N}$ is normalization constant, and $\sigma^2$ is the second moment given by

$$
\sigma^2 = \Lambda t,
$$

(A.4)

where the period of the periodic orbit is $t$, and

$$
\Lambda = C \frac{v_F}{\sqrt{A}}
$$

(A.5)

where

$$
C = \frac{1 - 2^{-3/2}}{\sqrt{2\pi^3}} \zeta \left( \frac{3}{2} \right) \approx 0.215.
$$

(A.6)

Like H&OA sum rule, this result was also derived under the assumption of ergodicity of the periodic orbits. Therefore, generally, it holds only for long periodic orbits. For the subset of billiards here considered, the set of the shortest periodic orbits already covers the energy shell.
uniformly, and it is plausible that the Gaussian distribution of the winding numbers gives a reasonable approximation also for short periodic orbits. Thus

$$
\langle W^4 e^{iaW} \rangle = \mathcal{M} \sum_{n=-\infty}^{\infty} n^4 e^{ian} e^{-\frac{n^2}{2\sigma^2}} = \mathcal{M} \frac{\partial^4}{\partial \alpha^4} \sum_{n=-\infty}^{\infty} e^{ian} e^{-\frac{n^2}{2\sigma^2}}
$$

(A.7)

Using Poisson summation formula, this sum may be written as

$$
\langle W^4 e^{iaW} \rangle = \mathcal{M} \frac{\partial^4}{\partial \alpha^4} \sum_{m=-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2(\alpha+2\pi m)^2},
$$

where

$$
\mathcal{M}^{-1} = \sum_{m=-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2(2\pi m)^2}
$$

(A.9)

It will be shown now that a good approximation is $\mathcal{M} = 1$, which is obtained by taking only the contribution from $m = 0$. For this purpose, let us evaluate the $m = 1$ term. It is large when $\sigma$ is small. However, the minimal value of $\sigma$ which corresponds to the shortest periodic orbit is, $\sigma_{\min}^2 = \Delta t_{\min} \approx 0.43$. Therefore, the maximal value of this term is approximately $2 \times 10^{-4}$, which is small compared with 1. For longer orbits the approximation becomes very rapidly more accurate. Setting $\mathcal{M} = 1$ in (A.8) and taking the derivatives with respect to $\alpha$ one finds

$$
\langle W^4 e^{iaW} \rangle = \sum_m (3\sigma^4 - 6\alpha_m^2 \sigma^2 + \alpha_m^4 \sigma^2) e^{-\frac{1}{2}\sigma^2 \alpha_m^2},
$$

(A.10)

where $\alpha_m$ is related to the reduced flux $\varphi$ according to

$$
\alpha_m = 4\pi \left( \varphi + \frac{m}{2} \right).
$$

(A.11)

In particular $\langle W^4 \rangle = 3\sigma^4$. Inserting (A.10) and (A.4) into (A.3), and using H&OA sum rule (A.1) yields

$$
\langle \chi^2 \rangle \simeq 72x_0^2 \left( \frac{T}{\Delta} \right)^2 I^2(T, \varphi),
$$

(A.12)

where

$$
I^2(T, \varphi) = (2\pi)^4 \int \frac{dt}{t} \left[ 3(\Lambda t)^2 + \sum_m (3(\Lambda t)^2 - 6\alpha_m^2 (\Lambda t)^3 + \alpha_m^4 (\Lambda t)^4) e^{-\frac{1}{2}\alpha_m^2 \Lambda t} \right] e^{-\frac{2\sigma}{\pi} t}
$$

(A.13)

A straightforward integration over the time from $t_{\min}$ to $\infty$ leads to the required result:

$$
I(T, \varphi) = (2\pi)^2 \left[ 3 \left( \frac{\Lambda}{2x_0} \right)^2 F_1 \left( \frac{2x_0}{\hbar} \right) \right] +
$$

$$
+ \sum_{m=-\infty}^{\infty} \left( 3 \left( \frac{\Lambda}{\xi_m} \right)^2 F_1(\xi_m) - 6\alpha_m^2 \left( \frac{\Lambda}{\xi_m} \right)^3 F_2(\xi_m) + \alpha_m^4 \left( \frac{\Lambda}{\xi_m} \right)^4 F_3(\xi_m) \right)^{\frac{1}{2}},
$$

(A.14)

where the functions $F_\nu(\kappa)$ are defined through the integral

$$
F_\nu(\kappa) = \int_{\kappa t_{\min}}^{\infty} t^\nu e^{-t} dt,
$$

(A.15)
and
\[ \xi_m = \frac{1}{2} \Lambda \alpha_m^2 + \frac{2 \varepsilon_0}{\hbar}. \] (A.16)

One sees from (A.3) that by an additional averaging over the flux, i.e. replacing the \( \langle \cdots \rangle \) averaging by \( \langle \cdots \rangle \) in equation (A.12), terms which correspond to the \( m \) sum in equation (A.14) drop out. The result in this case is the same as (A.12) but with \( I(T, \varphi) \) replaced by the function
\[ \tilde{I}(T) = (2\pi)^2 \frac{\Lambda \hbar}{2 \varepsilon_0} \sqrt{3F_1\left(\frac{2 \varepsilon_0}{\hbar}\right)}. \] (A.17)

The asymptotic form of this functions for \( T \gg T_{\text{th}} \) is obtained by substituting the limiting expressions for the functions \( F_\nu(\kappa) \), namely \( F_\nu(\kappa) \propto (\kappa t_{\text{min}})^\nu e^{-\kappa t_{\text{min}}} \) for \( \kappa t_{\text{min}} \to \infty \). This leads to equation (3.22). For \( T = T_{\text{th}} \) one finds that \( 2 \varepsilon_0/\hbar = 2 \) and \( F_1(2) = 3 \) leading to (3.21).

**Appendix B.**

**Fluctuations of \( \chi \) at \( T \ll \Delta \).**

In this Appendix, the average of the fluctuations in the susceptibility, \( \delta \bar{\chi} \), for low temperatures \( T \ll \Delta \), is calculated. The starting point is the approximate formula (3.55). Additional approximations will be: (a) to consider only the leading term of the \( j \) sum, i.e. to neglect the contributions from \( j \neq 0 \), and (b) to approximate \( t_p/1 - t_p \) by \( t_p \). These are good approximations for orbits with \( u_p \gg 1 \). The average of \( \chi^2 \) is now calculated as in Appendix A (Eqs. (A.2) and (A.3)) and the result is
\[ \langle (\chi^2) \rangle = \frac{1}{2} (12 \varepsilon_0 m A \hbar)^2 \sum_{T_p \leq 2A^2} \frac{1}{T_p^2} \langle W_p^4 \rangle e^{-u_p} \] (B.1)

The sum over periodic orbits is now replaced by an integral over the time using Hannay an Ozorio de Almeida sum rule (A.1). This is possible since \( e^{-u_p} \approx \left| \frac{A_p}{T_p} \right|^2 \), and because due to the exponential proliferation in the number of periodic orbits with their period, the main contribution for the sum in (A.1) comes from the primitive and not from the repeated orbits [16]. Substituting also \( \langle W_p^4 \rangle = 3 \Lambda^2 t^2 \) leads to
\[ \langle (\chi^2) \rangle = \frac{1}{2} (12 \varepsilon_0 m A \hbar)^2 \int_{t_{\text{mm}}}^{2 \Lambda \hbar} \frac{dt}{t^3} 3 \Lambda^2 t^2, \] (B.2)

where \( t_{\text{mm}} \) is the period of the shortest orbit which is approximated by \( \frac{2L}{v_F} \approx \frac{4\pi \hbar}{\Delta k_F L} \). The integration is straightforward, and substituting (A.5) for \( \Lambda \) yields
\[ \langle (\chi^2) \rangle = \frac{3}{2} (12 \varepsilon_0 m A \hbar)^2 C^2 \frac{v_F^2}{\Lambda} \ln \left( \frac{k_F L}{2} \right). \] (B.3)

Finally, using the definition \( \delta \bar{\chi} = \sqrt{\langle (\chi^2) \rangle} \) one obtains (3.56).
Appendix C.

Averages related to the case of a uniform magnetic field.

The purpose of this appendix is to calculate averages similar to those calculated in Appendix A, but for the case of a uniform magnetic field. In particular we will be interested in the formula which is analogous to (A.14). A first step towards this aim is to calculate the distribution of the reduced oriented areas, \( a_p \).

Let \( W(r) \) denote the winding number of some long ergodic periodic orbit with respect to a flux line at \( r \). Berry and Robnik [55] showed that it is a Gaussian variable, which vanishes on the boundary, and behaves like a Brownian variable as it changes by \( \pm 1 \) whenever the flux line \( r \) crosses the orbit. The reduced oriented area is given by

\[
a = \frac{1}{\mathcal{A}} \int dr W(r),
\]

therefore, it may also be considered as a Gaussian variable. It has zero mean value, and for its full characterization it is sufficient to calculate the second moment:

\[
\langle a^2 \rangle = \frac{1}{\mathcal{A}^2} \int \int dr_1 dr_2 (W(r_1)W(r_2)),
\]

where the average is over all orbits of about the same length. It is convenient to expand \( W(r) \) in the basis of the billiard eigenfunctions \( \psi_j(r) \) as

\[
W(r) = \sum_{j=1}^{\infty} \frac{f_j}{k_j^{3/2}} \psi_j(r),
\]

where \( k_j \) is the magnitude of the wave number corresponding to \( \psi_j(r) \). For this form of expansion it may be shown [55] that the amplitudes \( f_j \) satisfy the relation

\[
\langle f_i f_j \rangle = \frac{2vt}{\mathcal{A}} \delta_{ij},
\]

where \( v \) is the velocity of the particle, and \( t \) is the period. Substituting (C.3) into (C.2) one obtains

\[
\langle a^2 \rangle = \frac{1}{\mathcal{A}^2} \int \int dr_1 dr_2 \sum_{j=1}^{\infty} \frac{(f_j^2)}{k_j^{3/2}} \langle \psi_j(r_1)\psi_j(r_2) \rangle,
\]

where in this formula we have also introduced a local space averaging near \( r_1 \) and \( r_2 \), which is classically very small but semiclassically large [84]. The correlation function appearing in (C.5) was calculated by Berry [84] and it is found to be

\[
\langle \psi_j(r_1)\psi_j(r_2) \rangle = \frac{1}{\mathcal{A}} J_0(k_j|r_1 - r_2|),
\]

where \( J_0(x) \) is the Bessel function of zero order. Substituting this result, in (C.5) yields

\[
\langle a^2 \rangle = \Lambda' t,
\]

with

\[
\Lambda' = C' \frac{vt}{\sqrt{\mathcal{A}}},
\]
and $C'$ given by (4.16).

We proceed by calculating the average $\langle \chi^2 \rangle$ at high temperatures, for the case of uniform magnetic field. As argued in section 4, the second order term in the expansion of the phase (4.12) has negligible effect regarding this calculation, therefore, it may be neglected. The calculation is then very similar to the one presented in Appendix A. The difference is that the winding numbers $W_j$ are replaced by the reduced oriented areas $a_j$ which satisfy a continuous Gaussian distribution. The analog of equation (A.10) turns out to be

$$\langle a^4 e^{\alpha a} \rangle = (3\alpha^4 - 6\alpha^2 \sigma^2 + \alpha^4 \sigma^8) e^{-1/2 \alpha^2 \sigma^2},$$

where $\sigma^2 = \langle a^2 \rangle$. Using this result and following the same derivation presented in Appendix A one obtains

$$I'(T, \varphi) = (2\pi)^2 \left[ 3 \left( \frac{\Lambda' \hbar}{2\varepsilon_0} \right)^2 F_1(\frac{2\varepsilon_0}{\hbar}) + \right.$$

$$+ 3 \left( \frac{\Lambda'}{\xi} \right)^2 F_1(\xi) - 6\alpha^2 \left( \frac{\Lambda'}{\xi} \right)^3 F_2(\xi) + \alpha^4 \left( \frac{\Lambda'}{\xi} \right)^4 F_3(\xi) \right]^{1/2},$$

where the functions $F_\nu(\kappa)$ are defined in (A.15), $\alpha=4\pi\varphi$ is proportional to the magnetic field, $\Lambda' = C' \frac{\eta_0}{\sqrt{A}}$ is related to the variance of the reduced oriented areas, while $\xi = \frac{1}{2} \Lambda' \alpha^2 + \frac{2\varepsilon_0}{\hbar}$.

References


[54] The independence of $\bar{N}(E)$ on the magnetic flux is closely related to the Bohr van-Leeuwen theorem concerning the vanishing of the susceptibility in classical statistical mechanics. See Ref. [34] pp. 99-105.


[56] This is an exact statement not limited to the semiclassical approximation, See Byers N. and Yang C. N., *Phys. Rev. Lett.* 7 (1961) 46.


[73] Berry M. V., in Ref. [8]


