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Path-integral solutions for a class of 2D systems with local degeneracies

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Résumé. — Le potentiel \( V(r) = \alpha r^{d-2} - \beta r^{d-2} \) est étudié par les Intégrales de Parcours. La fonction de Green est calculée pour \( E = 0 \) et \( d \) quelconque, en coordonnées polaires et dans des coordonnées généralisant les coordonnées de Lévi-Civita. Il est montré que cette fonction de Green est la somme d'une partie discrète mais fine et d'une partie continue. Des cas limites et des cas particuliers sont étudiés.

Abstract. — The potential \( V(r) = \alpha r^{d-2} - \beta r^{d-2} \) is studied via the path integral approach. The Green function is calculated for \( E = 0 \) and for any value of \( d \) in polar coordinates as well as in coordinates generalizing the Levi-Civita transformation. It is shown that this Green function is the sum of a discrete but finite part and of a continuous part. Limiting cases will be investigated.

1. Introduction.

The aim of this paper is to use the formalism of path integrals to calculate the Green function relative to the potential class defined by

\[
V(r) = r^{2d-2} - \beta r^{d-2},
\]

where \( \alpha \) and \( \beta \) are positive parameters and \( d \) is a positive or nil rational number. So, for \( d = 1 \), \( V(r) = \alpha - \beta/r \) is the Coulomb potential, and for \( d = 2 \), \( V(r) = \alpha r^2 - \beta \) is the harmonic oscillator.

This two dimensional potential class has been recently studied [1-3], and it was proved that the state corresponding to \( E = 0 \) featured a multiple order degeneracy because of certain hidden symmetry properties.

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It is shown that the Green function can be calculated for any \( d \) and for \( E = 0 \) by first using the polar coordinates.

It is well known that, say, for the Coulomb potential, the Green function can be written in compact form via the Levi-Civita transformation. Thus we generalize this transformation for any \( d \) in order to calculate the Green function. We sum up the series obtained in polar coordinates by showing that it is composed of two parts, namely a discrete and finite sum of Green functions as well as a continuous one. We shall also consider a few limiting cases for \( E = 0 \) and deduce the energy spectrum and the wave functions for any \( E \) and for \( d = 1 \) and 2.

2. Green functions in polar coordinates.

In the canonical formulation of the path integrals, the propagator is written formally as follows, in standard notation,

\[
K(\mathbf{r}_i, \mathbf{r}_f; T) = \int Dx \, Dp_x \, Dy \, Dp_y \exp \left\{ \frac{i}{\hbar} \int_0^T \left[ P_x \dot{x} + P_y \dot{y} - \frac{P_x^2 + P_y^2}{2m} - V(x, y) \right] dt \right\}. \tag{2}
\]

The radial potential class (1) presents singularities for all values of the rational number \( d \in [0, 2] \).

Thus let us use the Duru-Kleinert transformation [4-6], which consists in choosing an adequate regulating function \( f(r) > 0 \), in order to avoid a path collapsing.

Let us use a \( t \rightarrow s \) time transformation defined by

\[
dt = ds \, f(r) = ds \, f_\ell(r) \, f_\tau(r), \tag{3}
\]

or in the discrete version,

\[
e = \varepsilon_S \, f_\ell(r_n) \, f_\tau(r_{n-1}), \quad T = (N + 1) \varepsilon, \quad S = (N + 1) \varepsilon_S,
\]

where \( f_\ell \) and \( f_\tau \) are two regulating functions chosen adequately so as to stabilize the path integral.

Using the Green function, we now introduce the energy \( E \), setting \( m = \hbar = 1 \) for all that follows,

\[
G(\mathbf{r}_i, \mathbf{r}_f; E) = \int_0^\infty dT \, K(\mathbf{r}_i, \mathbf{r}_f; T) \exp(iET) = \int_0^\infty dS \, P_E(\mathbf{r}_i, \mathbf{r}_f; S), \tag{4}
\]

the new form of the \( P_E \) Kernel being

\[
P_E^N(\mathbf{r}_i, \mathbf{r}_f; S) = \frac{f_\ell(r_i) f_\ell(r_f)}{2 \pi i \varepsilon_S f_\ell(r_i) f_\ell(r_f)} \prod_{n=1}^{N} \int \frac{d^2r_n}{2 \pi i \varepsilon_S f(r)} \exp(iA_E^N), \tag{5a}
\]

where

\[
A_E^N = \sum_{n=1}^{N+1} \left\{ \frac{(r_n - r_{n-1})^2}{2 \varepsilon_S f_\ell(r_n) f_\ell(r_{n-1})} - \varepsilon_S f_\ell(r_n) \left[ V(r_n) - E \right] f_\tau(r_{n-1}) \right\}, \tag{5b}
\]

is the new action.

An appropriate family of regulating functions is given by

\[
f_\ell(r) = [f(r)]^{1-\lambda} \quad \text{and} \quad f_\tau(r) = [f(r)]^\lambda
\]
Set $\lambda = 0$ to simplify the post-point prescription calculation. The stabilizing functions are now

$$f_\ell = f \quad \text{and} \quad f_r = 1. $$

Having a central two-dimensional potential, it may seem convenient to go over to the usual polar coordinate system, $x = r \cos \theta$ and $y = r \sin \theta$; yet, these coordinates cannot prevent path collapsing. We therefore shift the singularities to infinity with the transformation $(x, y) \rightarrow (q, \theta)$ defined by the following equations:

$$r = e^q, \quad -\infty < q < +\infty \quad \text{and} \quad 0 \leq \theta \leq 2\pi. \quad (6)$$

With these new variables, the action can be written in terms of the following expressions

$$\Delta x_n = x_n - x_{n-1} = r_n \cos \theta_n - r_{n-1} \cos \theta_{n-1},$$
$$\Delta y_n = y_n - y_{n-1} = r_n \sin \theta_n - r_{n-1} \sin \theta_{n-1},$$
$$\Delta r_n = e^{q_n} - e^{q_{n-1}}, \quad \Delta q_n = q_n - q_{n-1}, \quad \Delta \theta_n = \theta_n - \theta_{n-1}$$

The regulating function being now defined as

$$f = e^{2q},$$

the action is then

$$A_E^N = \sum_{n=1}^{N+1} \left\{ \frac{1}{2r} [ (\Delta q_n)^2 + (\Delta \theta_n)^2 ] - \varepsilon_S [ \alpha e^{2\Delta q_n} - \beta e^{\Delta q_n} - E e^{2\Delta q_n} ] + \Delta A_n \right\}, \quad (7a)$$

where the correction $\Delta A_n$ is explicitly given by

$$\Delta A_n = \frac{1}{2} \varepsilon_S \left\{ - (\Delta q_n)^3 + \frac{7}{12} (\Delta q_n)^4 - (\Delta q_n) (\Delta \theta_n)^2 + \frac{(\Delta q_n)^2 (\Delta \theta_n)^2}{2} - \frac{1}{12} (\Delta \theta_n)^4 \right\}. \quad (7b)$$

Let us now integrate with respect to the intervals, instead of the $x_n$ and $y_n$ positions,

$$\prod_{n=1}^{N} \int dx_n \, dy_n = \prod_{n=2}^{N+1} d(\Delta x_n) \, d(\Delta y_n).$$

The Jacobian of this transformation being equal to

$$f_{n-1} = \frac{\partial (\Delta x_n, \Delta y_n)}{\partial (\Delta q_n, \Delta \theta_n)} = e^{2q_{n-1}},$$

the measure of equation (5a) is given by

$$\frac{1}{2\pi i \varepsilon_S} \prod_{n=2}^{N+1} d(\Delta q_n) \, d(\Delta \theta_n) \prod_{n=1}^{N+1} e^{-2\Delta q_n} = \frac{1}{2\pi i \varepsilon_S} \prod_{n=2}^{N+1} \int d(\Delta q_n) \, d(\Delta \theta_n) \prod_{n=1}^{N+1} (1 + C_{\text{meas}}), \quad (8)$$

wherein

$$C_{\text{meas}} = -2 \Delta q_n + 2(\Delta q_n)^2,$$

is the correction relative to the measure.
Taking into account the two corrections originating in the action and in the measure, we now come to the total correction \( C_T \),

\[
C_T = \frac{\Delta A_N}{2} + \frac{\Delta A_N}{2} C_{\text{meas}} (1 + i \Delta A_N). \tag{9}
\]

If we miss out the terms \( \Delta q_n \Delta \theta_n \) which do not contribute within the \( \varepsilon_S \to 0 \) limit, we obtain

\[
(\Delta A_N)^2 = \frac{1}{4 \varepsilon_S^2} [(\Delta q_n)^6 + 2 (\Delta q_n)^4 (\Delta \theta_n)^2 + (\Delta q_n)^2 (\Delta \theta_n)^4],
\]

\[
C_{\text{meas}} \Delta A_N = \frac{1}{2 \varepsilon_S} [2 (\Delta q_n)^4 + 2 (\Delta q_n)^2 (\Delta \theta_n)^2].
\]

By taking into account the relations,

\[
\langle (\Delta q)^2 \rangle = i \varepsilon_S, \quad \langle (\Delta \theta)^2 \rangle = i \varepsilon_S, \quad \langle (\Delta q)^2 (\Delta \theta)^2 \rangle = (i \varepsilon_S)^2, \quad \langle (\Delta q)^4 \rangle = 3 (i \varepsilon_S)^2
\]

\[
\langle (\Delta \theta)^4 \rangle = 3 (i \varepsilon_S)^2, \quad \langle (\Delta q)^6 \rangle = 15 (i \varepsilon_S)^3, \quad \langle (\Delta q)^2 (\Delta \theta)^2 \rangle = 3 (i \varepsilon_S)^3,
\]

\[
\langle (\Delta q)^4 (\Delta \theta)^2 \rangle = 3 (i \varepsilon_S)^3,
\]

correction (9) thus leads to an effective nil potential, \( V_{\text{eff}} = 0 \).

Thus, from the above results we can readily obtain

\[
P_E^N (r_f, r_i; (N + 1) \varepsilon_S) = \frac{1}{2 \pi i \varepsilon_S} \int \frac{d(\Delta q_n) d(\Delta \theta_n)}{2 \pi i \varepsilon_S} \times
\]

\[
\times \exp \left\{ i \sum_{n=1}^{N+1} \left[ \frac{1}{2 \varepsilon_S} [(\Delta q_n)^2 + (\Delta \theta_n)^2] - \varepsilon_S [\alpha e^{2 dq_n} - \beta e^{dq_n} - E e^{2 q_n}] \right] \right\}, \tag{10}
\]

and it follows that

\[
P_E (r_f, r_i; S) = \int \mathcal{D} q \mathcal{D} p \mathcal{D} \theta \mathcal{D} \psi \exp \left\{ i \int_0^S dS \left[ p \dot{q} + p_\theta \dot{\theta} - \frac{p_\theta^2}{2} - \frac{p^2}{2} + \alpha e^{2 dq} + \beta e^{dq} + E e^{2 q} \right] \right\}. \tag{11}
\]

Integrating over the angular variable we are led to

\[
G(r_f, r_i; E) = \sum_{\ell = -\infty}^{\infty} \frac{e^{i \ell (\theta_f - \theta_i)}}{2 \pi} \int_0^\infty dS e^{-i \ell^2 S/2} \int \mathcal{D} q \mathcal{D} p \exp \left[ i \int_0^S (p \dot{q} - H) dS \right], \tag{12a}
\]

where the Hamiltonian \( H \) is a function of the \((q, s)\) coordinates,

\[
H = \frac{p^2}{2} + \alpha e^{2 dq} - \beta e^{dq} - E e^{2 q} \tag{12b}
\]

Thus, we have reduced the motion of the particle subjected to the action of potential (1), to a motion governed by the sum of three exponential potentials. It is obvious that the Green function can be analytically calculated only for a combination of two exponentials, i.e. a Morse like potential. One can see that there are four cases admitting an exact solution:

1) First case : the centrifugal barrier, \( d = 0 \).
In the \((q, s)\) coordinate system, the problem of the barrier is described by

\[
H = \frac{p^2}{2} - E e^{2q} + (\alpha - \beta).
\] (13)

ii) Second case: the Coulomb problem, \(d = 1\).
In the \((q, s)\) coordinate system, the Coulomb problem is described by the Hamiltonian related to the Morse potential [7],

\[
H = \frac{p^2}{2} + (\alpha - E) e^{2q} - \beta e^q
\] (14)

iii) Third case: the harmonic oscillator, \(d = 2\).
The Hamiltonian describing the H.O. in the \((q, s)\) coordinate system is also related to the Morse Potential [7],

\[
H = \frac{p^2}{2} + \alpha e^{4q} - (\beta + E) e^{2q}
\] (15)

iv) Fourth case: any \(d\) and \(E = 0\).
In this case, the Hamiltonian

\[
H = \frac{p^2}{2} + \alpha e^{2dq} - \beta e^{dq}
\] (16)
is also related to the Morse potential.
Let us analyze this important 4th case: the Green function can be written

\[
G(r_f, r_i; E) = \sum_{\ell = -\infty}^{\infty} \frac{1}{2\pi} \exp[i\ell(\theta_f - \theta_i)] \langle q_f | q_i \rangle,
\] (17a)

with

\[
\langle q_f | q_i \rangle = \int_0^\infty dS \int Dq Dp \exp \left\{ i \int_0^s dS \left[ p\dot{q} - \frac{p^2}{2} - \alpha e^{2dq} + \beta e^{dq} - \frac{p^2}{2} \right] \right\}.
\] (17b)

With the help of \(w = r^{d/2} = \exp[dq/2]\) transformation, we see that there is a simple relation between \(\langle q_f | q_i \rangle\) and \(\langle w_f | w_i \rangle\) given by [4], namely,

\[
\frac{d}{2} \exp \left[ \frac{d}{4} (q_f + q_i) \right] \langle q_f | q_i \rangle = \langle w_f | w_i \rangle,
\] (18)

where

\[
\langle w_f | w_i \rangle = \int_0^\infty dS \exp \left[ iS \left( \frac{2}{d} \right)^2 \beta \right] \left\{ \int Dw Dp w \exp[iA_{osc}] \right\},
\] (19)

with

\[
A_{osc} = \int_0^s dS \left[ P_w \dot{w} - \frac{P_w^2}{2} - \left( \frac{2}{d} \right)^2 \alpha w^2 - \frac{1}{2} \frac{(2 \ell/d)^2 - 1/4}{w^2} \right],
\] (20)

being the amplitude related to the radial harmonic oscillator.
As we know the expression of $\langle w_1 | w_1 \rangle$ [4], the Green function can finally be given by

$$G(r_r, r_i; 0) = \sum_{\ell = -\infty}^{+\infty} \frac{1}{2\pi} e^{i\ell(\theta_r - \theta_i)} \int_0^\infty dS \exp\left[iS\left(\frac{2}{d}\right)\beta\right] \left[\frac{2\sqrt{2}\alpha}{i\sin(2S\sqrt{2\alpha/d})}\right] \times$$

$$\times \exp\left[i\sqrt{2\alpha \frac{2\alpha}{d}} \left(r_r^2 + r_i^2\right) \cot(2S\sqrt{2\alpha/d})\right] I_2 |\ell| \left(\frac{2\sqrt{2}(\alpha - E) r_r^2 r_i^2}{i\sin(2S\sqrt{2\alpha/d})}\right).$$

We shall notice that, at this stage, equation (21) allows us to deduce, for any $E$, the Green functions relative to

1) the Coulomb potential: we just have to set $d = 1$ and replace $\alpha$ with $(\alpha - E)$:

$$G_{\text{coul}}(r_r, r_i; E) = \sum_{\ell = -\infty}^{+\infty} \frac{1}{2\pi} e^{i\ell(\theta_r - \theta_i)} \int_0^\infty dS \exp(4i\beta S) \left(\frac{2\sqrt{2}(\alpha - E)}{i\sin(2S\sqrt{2(\alpha - E)})}\right) \times$$

$$\times \exp[i\sqrt{2(\alpha - E)}(r_r + r_i) \cot(2S\sqrt{2(\alpha - E)})] I_2 |\ell| \left(\frac{2\sqrt{2}(\alpha - E) (r_r r_i)^{1/2}}{i\sin(2S\sqrt{2(\alpha - E)})}\right).$$

ii) The two-dimensional harmonic oscillator: we just have to set $d = 2$ and to replace $\beta$ with $\beta + E$:

$$G_{\text{OH}}(r_r, r_i; E) = \sum_{\ell = -\infty}^{+\infty} \frac{1}{2\pi} e^{i\ell(\theta_r - \theta_i)} \int_0^\infty dS e^{iS(\beta - E)} \frac{\sqrt{2\alpha}}{i\sin(S\sqrt{2\alpha})} \times$$

$$\times \exp\left[i\frac{\sqrt{2\alpha}}{2} (r_r^2 + r_i^2) \right] \cot(S\sqrt{2\alpha}) I_2 |\ell| \left(\frac{\sqrt{2\alpha} r_r r_i}{i\sin(S\sqrt{2\alpha})}\right).$$

Now it is well known that there is a compact form for the green function of the two-dimensional Coulomb potential [4]. This compact form can be obtained via a Levi-Civita type transformation generalized for any $d$.

3. The Green function via the Levi-Civita type transformation.

3.1 Green function in $(u, v)$ coordinates. — Let us consider the well known transformation $(r, \theta) \rightarrow (z, z^*)$,

$$z = re^{i\theta} \quad \text{and} \quad z^* = re^{-i\theta},$$

as well as a $(z, z^*) \rightarrow (w, w^*)$ transformation, defined by

$$w = \sqrt{2/}\!\!\!\!d z = u + iv, \quad w^* = \sqrt{2/}\!\!\!\!d z^* = u - iv,$$

The passage from $(x, y)$ coordinates to the new $(u, v)$ coordinates is given by the following relations

$$x = \left(\frac{d}{2}\right)^{1/2d} \frac{(u + iv)^{2d} - (u - iv)^{2d}}{2i},$$

$$y = \left(\frac{d}{2}\right)^{1/2d} \frac{(u + iv)^{2d} + (u - iv)^{2d}}{2i}.$$
Thus,

\[
\begin{align*}
  u &= \sqrt{\frac{2}{d}} e^{\frac{d \theta}{2}} \cos \left( \frac{d \theta}{2} \right) \\
  v &= \sqrt{\frac{2}{d}} e^{\frac{d \theta}{2}} \sin \left( \frac{d \theta}{2} \right)
\end{align*}
\]

The areas of variation for \((u, v)\) depend upon \(d\) and \(\theta\). In matrix notation the system of linear equations (25) may be written as

\[
\mathbf{r} = B \mathbf{v}, \quad \mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} u \\ v \end{bmatrix},
\]

where

\[
B = \left( \frac{d}{2} \right)^{1d} \begin{bmatrix} (u + iv)^{\frac{d}{2}} - (u - iv)^{\frac{d}{2}} & 2i (u + iv)^{\frac{d}{2}} - (u - iv)^{\frac{d}{2}} \\ (u + iv)^{\frac{d}{2}} - (u - iv)^{\frac{d}{2}} & 2 (u + iv)^{\frac{d}{2}} - (u - iv)^{\frac{d}{2}} \end{bmatrix}.
\]

For \(d = 1\), the Levi-Civita transformation

\[
x = \frac{u^2 - v^2}{2}, \quad y = uv,
\]

can be recognized.

With these new \((u, v)\) coordinates, the Kernel \(P_E(V_i, V_{i+1}); S)\) of the Green function (4) can now be written,

\[
P_E(V_i, V_{i+1}; S) = \frac{g_r(V_i) g_t(V_{i+1})}{2 \pi \varepsilon s} \frac{1}{g(V_{i+1}) g_r(V_{i+1})} \lim_{N \to \infty} \prod_{n=1}^{N} \frac{1}{2 \pi \varepsilon s} g(V_n) \sum_{n=2}^{N+1} d(\Delta u_n) d(\Delta v_n) J_{n-1} \times
\]

\[
\times \exp \left\{ \sum_{n=1}^{N+1} \frac{1}{2 \varepsilon s} g_t(V_n) g_r(V_{n+1}) \left[ \left( \frac{d}{2} \right)^{2d} (u_n + iv_n)^{2d} - (u_{n-1} + iv_{n-1})^{2d} \right] - \varepsilon s \left( \frac{d}{2} \right)^{2 - 2d} (u_n^2 + v_n^2)^{-1/2} g_t(V_n) \right\} \times
\]

\[
\times \left\{ \alpha (u_n^2 + v_n^2) - 2b/d - E \left( \frac{d}{2} \right)^{2d-2} (u_n^2 + v_n^2)^{-1/2} g_r(V_{n-1}) \right\},
\]

where

\[
J_{n-1} = \left| \frac{\partial (\Delta x_n, \Delta y_n)}{\partial (\Delta u_n, \Delta v_n)} \right| = \left( \frac{d}{2} \right)^{2d-2} (u_{n-1}^2 + v_{n-1}^2)^{2d-1},
\]

is the Jacobian of the \((x, y) \to (u, v)\) transformation and \(g = g_t g_r\) is a product of stabilizing functions.

Let us eliminate the term \(g_r(V_i) g_t(V_{i+1})[g_t(V_i) g_r(V_{i+1})]^{-1}\) within the mid-point prescription,

\[
g = g_t^2 = g_r^2 = J_n.
\]
Thus,
\[
(u_n + i v_n)^2 - (u_{n-1} + i v_{n-1})^2 = \left( \frac{d}{2} \right)^2 (\tilde{u}_n^2 + \tilde{v}_n^2)^{2d-1} (\Delta u_n^2 + \Delta v_n^2) \times
\]
\[
\times \left\{ 1 + \frac{(2-d)(2-3d)}{24 d^2} \left[ \left( \frac{\Delta u_n + \Delta v_n}{\tilde{u}_n + i \tilde{v}_n} \right)^2 + \left( \frac{\Delta u_n - i \Delta v_n}{\tilde{u}_n - i \tilde{v}_n} \right)^2 \right] \right\} + \cdots. \tag{30}
\]
and
\[
\theta_t(V_n) g_r(V_{n-1}) = \theta_t(V_n) \theta_t(V_{n-1}) = \left( \frac{d}{2} \right)^{2d-2} (\tilde{u}_n^2 + \tilde{v}_n^2)^{2d-1} \times
\]
\[
\times \left\{ 1 + \frac{d-2}{8d} \left[ \left( \frac{\Delta u_n + \Delta v_n}{\tilde{u}_n + i \tilde{v}_n} \right)^2 + \left( \frac{\Delta u_n - i \Delta v_n}{\tilde{u}_n - i \tilde{v}_n} \right)^2 \right] \right\} + \cdots, \tag{31}
\]
wherein
\[
\tilde{u}_n = \frac{u_n + u_{n-1}}{2} \quad \text{and} \quad \tilde{v}_n = \frac{v_n + v_{n-1}}{2}.
\]

Thus, using (28) and (29), we obtain the two following quantities in the exponent of exponential from equation (28):

i) the action
\[
A_N = \sum_{n=1}^{N+1} \left\{ \frac{1}{2} \varepsilon_S (\Delta u_n^2 + \Delta v_n^2) + \varepsilon_S \left[ \left( \frac{d}{2} \right)^{2d-2} E (\tilde{u}_n^2 + \tilde{v}_n^2)^{2d-1} + \frac{2\beta}{d} - \alpha (\tilde{u}_n^2 + \tilde{v}_n^2) \right] \right\}. \tag{32a}
\]
Let us notice that this action is related, for $E = 0$, to a two dimensional harmonic oscillator whose motion is spatially constrained.

ii) a term grouping together all the fourth order terms in $\Delta u_n$ and $\Delta v_n$,
\[
\Delta A_N = \frac{1}{2} \varepsilon_S \sum_{n=1}^{N+1} \left\{ (\Delta u_n^2 + \Delta v_n^2) + \frac{(4-d^2)}{32 d^2} \left[ \left( \frac{\Delta u_n + \Delta v_n}{\tilde{u}_n + i \tilde{v}_n} \right)^2 + \mathrm{c.c.} \right] \right\}, \tag{32b}
\]
namely the correction to the action $A_N$.

At the limit $\varepsilon_S \to 0$, it is easy to make sure, thanks to
\[
\langle \Delta u_n^4 \rangle = \langle \Delta v_n^4 \rangle; \quad \langle \Delta u_n^3 \Delta v_n \rangle = \langle \Delta u_n \Delta v_n \rangle = 0,
\]
that $\langle \Delta A_N \rangle = 0$, which means that no effective potential is induced by these changes. So finally, the new kernel is the following:
\[
P_E(V_f; \ V_i; \ S) = \int \mathcal{D}u \mathcal{D}p_u \mathcal{D}v \mathcal{D}p_v \ e^{iA_E}, \tag{33a}
\]
where
\[
A_E = \int_0^{S'} dS' \left[ p_u \dot{u} + p_v \dot{v} - \frac{p_u^2 + p_v^2}{2} + \alpha (u^2 + v^2) + \frac{2\beta}{d} + E \right]. \tag{33b}
\]
Let us now briefly tackle two particular cases:

i) Ford $d = 1$, the action is relative to the Coulomb potential,
\[
A_E^{\text{Coul}} = \int_0^{S'} dS' \left[ p_u \dot{u} + p_v \dot{v} - \frac{p_u^2 + p_v^2}{2} - (\alpha - E) (u^2 + v^2) + 2\beta \right], \tag{34}
\]
obtained via the well-known Levi-Civita transformation

\[ x = \frac{u^2 - v^2}{2}, \]
\[ y = uv. \]

The Green function is then given by

\[ G^{\text{Coul}}(r_f, r_i; E) = \int_0^\infty dS' e^{2iBS'} \left[ \langle V_f | V_i \rangle + \langle -V_f | V_i \rangle \right], \quad (35a) \]

where

\[ \langle V_f | V_i \rangle = \frac{\sqrt{2(\alpha - E)}}{2 \pi i \sin(\sqrt{2}(\alpha - E))} \times \]
\[ \times \exp \left\{ \frac{i \sqrt{2(\alpha - E)}}{\sin(\sqrt{2}(\alpha - E))} \left[ (V_f^2 + V_i^2) \cos(\sqrt{2}(\alpha - E)) - 2 V_f \cdot V_i \right] \right\}, \quad (35b) \]

is the well-known [4] amplitude relative to the harmonic oscillator.

ii) For \( d = 2 \), the action is relative to the harmonic oscillator potential,

\[ A^{\text{HO}}_E = \int_0^S dS' \left[ p_u \dot{u} + p_v \dot{v} - \frac{p_u^2 + p_v^2}{2} - \alpha (u^2 + v^2) + (E + \beta) \right], \quad (36a) \]

obtained via relations

\[ x = u, \]
\[ y = v. \]

The corresponding Green function takes the form

\[ G^{\text{HO}}(r_f, r_i; E) = \int_0^\infty dS' e^{i(\beta + E)S'} \langle r_f | r_i \rangle, \quad (36b) \]

where \( \langle r_f | r_i \rangle \) is the usual propagator relative to the bi-dimensional harmonic oscillator, (Eq. (35b)).

In these two cases (\( d = 1 \) and \( 2 \)), the Green functions can be easily calculated. In the general case, for any \( d \), the Green functions can be analytically calculated for \( E = 0 \) (Sect. 2).

When going from the \((x, y)\) coordinate system over to the \((u, v)\) system, we can see that

— transformation (25) is not univocal,
— the area of variation of \( u \) and \( v \) is not the whole plane: it depends upon parameter \( d \) and angle \( \theta \).

It is thus not easy to find \( G(r_f, r_i; 0) \) for any \( d \), according to equations (31, 33a, 33b).

To bypass this difficulty, let us go back to equation (21) and sum up the series to obtain \( G(r_f, r_i; 0) \) as a function of the new \((u, v)\) coordinates.

3.2 Summation of the Green function for \( E = 0 \). — Let us first notice that, here, \( E \) is understood as \( E + i0 \). According to the expression of the action (12c), for \( E = 0 \), the imaginary term \(-i0 e^{i\ell^2/2} \approx -i0 \) can be combined with \(-\ell^2/2 \) to give

\[ -\ell^2/2 - i0 \approx -(\ell + i0)^2/2 \approx -\ell^2/2 - i0 |\ell|. \]
Let us now take the integral representation of the modified Bessel function [8],

\[ I_\mu(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos(\mu \theta) \, d\theta - \frac{\sin(\mu \pi)}{\pi} \int_0^{\infty} e^{-z \cos t - \mu t} \, dt, \]

with \(|\arg(z)| \leq \frac{\pi}{2}\) and \(\text{Re}(\mu) > 0\).

We can easily prove that it can be written

\[ I_\mu(z) = \frac{1}{2\pi} \int_C dt \, e^{z \cos t + i\mu}, \]

where \(C\) represents the following integration contour (Fig. 1).

![Integration contour](image)

Fig. 1. — Integration contour for Bessel functions.

Let us consider the sum

\[ \sum_{\ell = -\infty}^{+\infty} e^{i\ell \Delta \theta} I_2 |\ell|/2(z) = \sum_{\ell = -\infty}^{+\infty} \frac{1}{2} \int_C dt \, e^{z \cos t - 2\ell t |\ell|/d + i\ell \Delta \theta}, \]

where

\[ z = \frac{2 \sqrt{2} \alpha r_{\ell/2}^d}{id \sin(2S \sqrt{2} \alpha/d)}, \quad \Delta \theta = \theta_1 - \theta_2. \]

One has

\[ \sum_{\ell = -\infty}^{+\infty} e^{i\ell (\Delta \theta + 2t |\ell|/d)} = \sum_{\ell = -\infty}^{+1} e^{i\ell(\Delta \theta - 2 \ell d) + \ell 0} + \sum_{\ell = 1}^{+\infty} e^{i\ell(\Delta \theta - 2 \ell d) - \ell 0} + 1, \]

where the terms ±\(\ell 0\), introduced beforehand, have been added in order to regularize the summations, and therefore

\[ \sum_{\ell = -\infty}^{+\infty} \exp[i(\ell \Delta \theta + 2t |\ell|/d) - |\ell| \cdot 0] = \frac{1}{2} \left\{ \cot \left( \frac{\Delta \theta}{2} - \frac{t}{d} + i0 \right) + \cot \left( \frac{\Delta \theta}{2} + \frac{t}{d} - i0 \right) \right\}. \]

Thus

\[ \sum_{\ell = -\infty}^{+\infty} e^{i\ell \Delta \theta} I_2 |\ell|/d(z) = \frac{1}{4\pi} \int_C dt \, e^{z \cos t} \left\{ \cot \left( \frac{\Delta \theta}{2} - \frac{t}{d} + i0 \right) + \cot \left( \frac{\Delta \theta}{2} + \frac{t}{d} - i0 \right) \right\}. \]
Let us define

\[ J_{\pm} = \frac{i}{4} \pi \int_{C} dt \ e^{t \cot \theta} \cot \left( \frac{\Delta \theta}{2} \pm \frac{t}{d} \pm i0 \right). \] (42)

If we change \( t \to t + i0 \) within \( J_+ \), we shall obtain

\[ J_+ = \frac{i}{4} \pi \int_{C_1} dt \ e^{t \cot \theta} \cot \left( \frac{\Delta \theta}{2} - \frac{t}{d} \right), \] (43a)

where \( C_1 \) is the following contour (Fig. 2).

![Integration contour for Bessel function \( J_+ \).](c1)

If now we change \( t \to -t + i0 \) within \( J_- \), we shall obtain

\[ J_- = \frac{i}{4} \pi \int_{C_2} dt \ e^{t \cot \theta} \cot \left( \frac{\Delta \theta}{2} - \frac{t}{d} \right), \] (43b)

where \( C_2 \) is the following contour (Fig. 3).

![Integration contour for Bessel function \( J_- \).](C2)
Eventually, we have

\[ \sum_{t=-\infty}^{t=\infty} e^{i t \Delta \theta} I_2 \mid_{t=1/d}(z) = \frac{i}{4 \pi} \int_{\mathcal{C}_1 \cup \mathcal{C}_2} \text{dt} \cot \left( \frac{\Delta \theta - t}{d} \right) e^{i \cos t} = I_{C_1 \cup C_2}, \]  

(44)

where \( C_1 \cup C_2 \) is the following contour (Fig. 4).

![Integration contour for Bessel function \( I_{C_1 \cup C_2} \)]

Given

\[ F(t) = e^{i \cos t} \cot \left( \frac{\Delta \theta}{2} - \frac{t}{d} \right), \]  

(45)

we can write

\[ \int_{\mathcal{C}_1 \cup \mathcal{C}_2} F(t) \text{dt} = \int_{D_1 + E_1 + F_1 + D_2 + E_2 + F_2} F(t) \text{dt} = \]  

\[ = \int_{ABCD} F(t) \text{dt} + \int_{-\pi - 10^o}^{-\pi - 10^o} F(t) \text{dt} + \int_{\pi + 10^o}^{\pi + 10^o} F(t) \text{dt}, \]  

(46)

where

\[ \frac{i}{4 \pi} \int_{ABCD} F(t) \text{dt} = \frac{i}{4 \pi} \int \text{dt} \cot \left( \frac{\Delta \theta}{2} - \frac{t}{d} \right) e^{i \cos t} = -\frac{1}{2} \sum \text{Residues}. \]

The poles \( t_n \) within the closed contour are given by the equation

\[ \sin \left( \frac{\Delta \theta}{2} - \frac{t}{d} \right) = 0, \]

their positions on the real axis being \( t_n = (\Delta \theta - 2n\pi)d/2, n \) integer.
The poles fulfill the condition
\[-\pi < t_n < \pi \leftrightarrow -\pi < (\Delta \theta - 2 n \pi) \frac{d}{2} < \pi.\]

Thus
\[\lim_{t \to t_n} (t - t_n) \cot \left( \frac{\Delta \theta}{2} - \frac{t}{d} \right) e^{z \cos t} = -d e^{z \cos t_n},\]

From this formulation it becomes clear that
\[
\sum_{t = -\infty}^{+\infty} e^{t \Delta \theta} I_2 \left| \frac{z}{id} \right| (z) = \frac{d}{2} \sum e^{z \cos t_n} + \frac{t}{4 \pi} \left\{ \int_{-\infty}^{+\infty} dt \cot \left( \frac{\Delta \theta}{2} - \frac{t}{d} \right) e^{z \cos t} + \right.
\]
\[\left. + \int_{-\infty}^{+\infty} dt \cot \left( \frac{\Delta \theta}{2} - \frac{t}{d} \right) e^{z \cos t} \right\}, \quad (47)
\]

Set \( t = \mp \pi + ip \), then
\[\int_{-\pi}^{+\pi} dt \cot \left( \frac{\Delta \theta}{2} - \frac{t}{d} \right) e^{z \cos t} = -i \int_{-\infty}^{+\infty} dp \cot \left( \frac{\Delta \theta}{2} + \frac{\pi}{d} + \frac{ip}{d} \right) e^{-z \cosh p},\]

This gives
\[\sum_{t = -\infty}^{+\infty} e^{t \Delta \theta} I_2 \left| \frac{z}{id} \right| (z) = \frac{d}{2} \sum e^{z \cos t_n} + \frac{1}{4 \pi} \int_{-\infty}^{+\infty} dp \exp \left[ \cot \left( \frac{\Delta \theta}{2} - \frac{\pi}{d} - \frac{ip}{d} \right) - \right.
\]
\[\left. - \cot \left( \frac{\Delta \theta}{2} + \frac{\pi}{d} - \frac{ip}{d} \right) \right]. \quad (48)
\]

It can readily be seen that the Green function (21) can be brought into the form
\[G(r_n, r_1; 0) = \frac{1}{2 \pi} \int_0^{+\infty} dS \exp \left[ iS \left( \frac{2}{d} \right)^2 \beta \right] \frac{2 \sqrt{2} \alpha}{i d \sin \left( 2 S \sqrt{2} \alpha / d \right)} \times
\]
\[\times \exp \left[ \frac{i \sqrt{2} \alpha}{d} (r_1^d + r_1^d) \cot \left( 2 S \sqrt{2} \alpha / d \right) \right] \times
\]
\[\times \left\{ \sum_n \exp \left[ \frac{2 \sqrt{2} \alpha \cos t_n}{i d \sin \left( 2 S \sqrt{2} \alpha / d \right)} (r_1 r_n)^{\phi_2} \right] \right\} +
\]
\[+ \frac{1}{d \pi} \int_{-\infty}^{+\infty} dp \exp \left[ \frac{-2 \sqrt{2} \alpha}{i d \sin \left( 2 S \sqrt{2} \alpha / d \right)} \cosh p (r_1 r_n)^{\phi_2} \right] \times
\]
\[\times \frac{\sin \left( 2 \pi / d \right)}{\cos \left( 2 \pi / d \right) - \cos \left( \Delta \theta - 2 \pi / d \right)} \right\}. \quad (49)
\]

This set of equations (48, 49) becomes more transparent if we introduce the new variables \( \phi_i, \phi_f \) defined by
\[t_n = (\Delta \theta - 2 n \pi) \frac{d}{2} = \Delta \phi = \phi_f - \phi_i, \quad \phi_i = \frac{d}{2} \theta_i \quad \text{and} \quad \phi^f = (\theta_f - 2 n \pi) \frac{d}{2}, \]
\( \phi_n \) depends upon \( n \) and if we note that we can always find an axis system such that

\[
V = \sqrt{\frac{2}{d}} r^d \begin{bmatrix} \cos \frac{d \theta}{2} \\ \sin \frac{d \theta}{2} \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad V_i^0 = \sqrt{\frac{2}{d}} r^d \begin{bmatrix} \cos \phi_i \\ \sin \phi_i \end{bmatrix} = \begin{bmatrix} u_i^0 \\ v_i \end{bmatrix};
\]

\[
V_i = \sqrt{\frac{2}{d}} r^d \begin{bmatrix} \cos \phi_i \\ \sin \phi_i \end{bmatrix} = \begin{bmatrix} u_i \\ v_i \end{bmatrix}. \quad (50)
\]

Eventually, the Green function can be written as a function of these new \((u, v)\) coordinates:

\[
G(r_0, r_1 ; 0) = \frac{1}{2 \pi} \int_{0}^{\infty} dS e^{it_4 t_2 / d^2} \sum_{n} \frac{2 \sqrt{2} \alpha}{i d \sin (2 S \sqrt{2} \alpha / d)} \times \\
\times \left\{ \exp \left[ \frac{i \sqrt{2} \alpha}{2 \sin (2 S \sqrt{2} \alpha / d)} \left( (u_i^n)^2 + (v_i^n)^2 + u_i^2 + v_i^2 \right) \cos (2 S \sqrt{2} \alpha / d) - V_i^n \cdot V_i \right] + \right. \\
\left. + \frac{1}{d \pi} \int_{-\infty}^{\infty} dp \ e^{-z \cosh p} \frac{\sin (2 \pi / d)}{\cos (2 \pi / d) - \cos (\Delta \theta - 2 i p / d)} \right\}, \quad (51)
\]

with

\[
z = \frac{2 \sqrt{2} \alpha (r_0 r_1)^{d/2}}{d \sin (2 \sqrt{2} \alpha / d)}
\]

This equation (51) is our main result.

The \( G(r_0, r_1 ; 0) \) Green function consists of two parts: a finite sum of Green functions relative to the bidimensional oscillators, and a second part as given by the continuous integral: let us notice that this second part disappears, of course, for \( d = 1 \) and \( d = 2 \).

Let us check briefly our result (51) on two particular cases:

Case 1: \( d = 1 \) : Coulomb potential.

In this case, the number of \( t_n \) poles is two. Let us repeat here that \( t_n = \frac{\Delta \theta}{2} - n \pi \), and as \( 0 \leq \Delta \theta \leq 2 \pi \), the condition is satisfied for \( n = 0 \) and \( 1 \). Thus,

\[
\phi_n = \frac{\theta_n}{2}, \quad \phi_0 = \frac{\theta_0}{2}, \quad \phi_1 = (\theta_1 - 2 \pi) \frac{1}{2} = \frac{\theta_1}{2} - \pi,
\]

and

\[
G(r_0, r_1 ; 0) = \sum_{n = 0, 1} \int_{0}^{\infty} dS e^{it_4 S \beta} \frac{\sqrt{2} \alpha}{i \pi \sin (2 S \sqrt{2} \alpha)} \times \\
\times \exp \left[ \frac{i \sqrt{2} \alpha}{2 \sin (2 S \sqrt{2} \alpha)} \left( (u_i^n)^2 + (v_i^n)^2 + u_i^2 + v_i^2 \right) \cos (2 S \sqrt{2} \alpha / d) - V_i^n \cdot V_i \right], \quad (52)
\]

or

\[
G(r_0, r_1 ; 0) = \int_{0}^{\infty} dS e^{it_4 S} \left[ \langle \nu_i^0 | V_i \rangle + \langle V_i^1 | V_i \rangle \right], \quad (53)
\]

with

\[
V_i^1 = - V_i^0
\]

Knowing \( G(r_0, r_1 ; 0) \), it is possible to deduce \( G(r_0, r_1 ; E) \) for any \( E \).
We only have to change \( \alpha \) into \( \alpha - E \). The obtained Green function will correspond to the Levi-Civita transformation.

Case 2 : \( d = 2 \) : bidimensional harmonic oscillator.

In this case, the number of poles is one. As \(- \pi \leq \Delta \theta < \pi\), the \(- \pi < \tau_n < \pi\) condition will only be satisfied for \( n = 0 \).

Thus
\[
\phi_1 = \frac{\theta_1}{2}, \quad \phi_\ell = \frac{\theta_\ell}{2},
\]
and
\[
G(r_\ell, r_1; 0) = \int_0^\infty dS e^{i\beta S} \frac{\sqrt{2}}{2 i \pi \sin \left( S \sqrt{2} \alpha \right)} \times \exp \left[ \frac{t \sqrt{2}}{2 \sin \left( S \sqrt{2} \alpha \right)} \left( (u_\ell^0)^2 + (v_\ell^0)^2 + u_\ell^2 + v_\ell^2 \right) \cos \left( S \sqrt{2} \alpha \right) - V_\ell^0 V_1 \right]. \tag{54}
\]

The Green function for any \( E \) will be deduced by changing \( \beta \) into \( \beta + E \),
\[
G(r_\ell, r_1; E) = \int_0^\infty dS e^{iS(\beta + E)} \langle r_\ell | r_1 \rangle,
\]
where \( \langle r_\ell | r_1 \rangle \) is the usual propagator of the 2D harmonic oscillator (See Kleinert [4]).

Let us now go over to the study of limiting cases.

4. Limiting cases.

Let us set \( E = 0 \), with \( E = E + i0 \), and taking (12c) into account let us write (17b) in the following form:
\[
\langle q_\ell | q_i \rangle_{E=0, t} = \frac{2}{d} \int_0^\infty dS \int Dq Dp \exp \left\{ i \left[ \frac{p^2}{2} - \alpha e^{dq} + \beta e^{dq} + (0 + i0) e^{2q} - \frac{P^2}{2} \right] \right\}. \tag{56}
\]

Under the \( w = r^{d/2} = e^{4d/2} \) transformation the \( \langle q_\ell | q_i \rangle \) amplitude can be obtained as follows:
\[
\langle r_\ell | r_i \rangle_{E=0, t} = \frac{2}{d} \int_0^\infty dS e^{iS(\frac{d}{2})^2 \beta} \frac{2 \sqrt{2}}{id \sin \left( 2 S \sqrt{2} \alpha/d \right)} \times \exp \left[ i \frac{\sqrt{2}}{d} (r_\ell^d + r_i^d) \cot \left( 2 S \sqrt{2} \alpha/d \right) \right] I_2 \left| \ell \right| \left( \frac{2 \sqrt{2}}{id \sin \left( 2 S \sqrt{2} \alpha/d \right)} \right). \tag{57}
\]

To evaluate this integral, we use the standard formula [9]
\[
\int_0^\infty dx \left[ \cot \left( \frac{x}{2} \right) \right]^2 e^{-\beta \cosh x} J_\mu(\alpha \sinh x) = \frac{\Gamma((1 + \mu)/(2 - \nu))}{\alpha \Gamma(\mu + 1)} W_{\gamma, \mu/2}(\sqrt{\alpha^2 + \beta^2 + \beta}) M_{-\gamma, \mu/2}(\sqrt{\alpha^2 + \beta^2 - \beta}),
\]
where \( W_{\gamma, \mu/2}(z) \) and \( M_{-\gamma, \mu/2}(z) \) are the Whittaker functions. The formula is valid for \( \text{Re} \beta > |\text{Re} \alpha|, \text{Re} \left( \mu/2 - \nu \right) > -1/2 \).
The following variable changes
\[ \sqrt{\alpha^2 + \beta^2} \pm \beta = t \gamma_f, \quad \sinh x = (\sinh y)^{-1}, \quad \cosh x = \coth y, \]
\[ \coth (x/2) = e^y, \quad \cosh x = \sinh y. \]

allow us to write
\[ \int_{0}^{\infty} \frac{\, dy}{\sinh y} e^{2\lambda y} \exp \left\{ - \frac{t}{2} (\gamma_f + \gamma_i) \coth y \right\} I_{\mu} \left( \frac{t \sqrt{\gamma_f \gamma_i}}{\sinh y} \right) = \]
\[ = \frac{\Gamma((1 + \mu)\gamma_f - \lambda)}{t^\gamma (\gamma_f \gamma_i)^\lambda} W_{\lambda, \mu} (t \gamma_f) M_{\lambda, \mu} (t \gamma_i). \quad (58) \]

where \( W_{\lambda, \mu} (t \gamma_f) \) and \( M_{\lambda, \mu} (t \gamma_i) \) are the Whittaker functions with \( \gamma_f > \gamma_i > 0, \)
\[ \Re t > 0, \quad |\Arg t| < \pi \quad \text{and} \quad \Re [(1 + \mu) \gamma_f - \lambda] > 0. \]

Setting \( y = \frac{2iS \sqrt{2\alpha}}{d}, \lambda = \frac{\beta}{d \sqrt{2\alpha}}, \frac{2\sqrt{2\alpha}}{d} \rho^4 = t \gamma, \)
\[ \frac{2\sqrt{2\alpha}}{d} (r_i r_f)^\mu^2 = t \sqrt{\gamma_f \gamma_i}, \quad \mu = 2 |\ell|/d, \]

it follows that
\[ \langle r_t | r_i \rangle_{E=0, \ell} = \]
\[ = -\frac{2i}{d \sqrt{2\alpha}} \left( \frac{1}{2} + \frac{|\ell|}{d} - \frac{\beta}{d \sqrt{2\alpha}} \right) W_{\frac{\beta}{d \sqrt{2\alpha}}, \frac{|\ell|}{d}} M_{\frac{\beta}{d \sqrt{2\alpha}}, \frac{|\ell|}{d}} \left( \frac{2\sqrt{2\alpha}}{d} r_i^\mu d \right) \left( \frac{2\sqrt{2\alpha}}{d} r_f^\mu d \right). \quad (59) \]

5. Continuous states.

As we noticed before, the \( (i0 e^{i\gamma_f} - \ell^2/2) \) term can be written \( (- (\ell + i0)^2)/2 \), which means that \( i0 \) is combined with \( \ell \).

Now \( \sqrt{(\ell + i0)^2} = \pm |\ell| \). Let us thus determine the wave function by writing
\[ \psi (r_f) \psi^* (r_i) = \frac{1}{2\pi} [G(r_f, r_i ; E + i0)_{E=0} - G(r_f, r_i ; E - i0)_{E=0}] = \]
\[ = \sum_{\ell = -\infty}^{\infty} \frac{e^{i\ell (\theta_f - \theta_i)}}{(2\pi)^2} \left\{ \langle r_f | r_i \rangle_{0+i0} - \langle r_f | r_i \rangle_{0-i0} \right\}, \]

and let us use the following relations \[10\]
\[ W_{\lambda, \mu} (z) = W_{\lambda, -\mu} (z), \quad W_{\lambda, \mu} (z) = \frac{\Gamma(-2\mu)}{\Gamma(1/2 - \mu - \lambda)} M_{\lambda, \mu} (z) + \frac{\Gamma(2\mu)}{\Gamma(1/2 + \mu - \lambda)} M_{\lambda, -\mu} (z). \]
Then we have

$$
|\psi(r_t)\psi^*(r_i)|_{E=0, \ell} = \frac{1}{(2\pi)^2 \ell \sqrt{2\alpha}} e^{i\ell(\theta_i - \theta_t)} \times
\left| \frac{\Gamma\left(\frac{1}{2} - \frac{|\ell|}{d} - \frac{\beta}{d \sqrt{2\alpha}}\right) \Gamma\left(\frac{1}{2} + \frac{|\ell|}{d} - \frac{\beta}{d \sqrt{2\alpha}}\right)}{\Gamma\left(2 |\ell|/d\right) \Gamma\left(1 - 2 |\ell|/d\right)} \right|
\times
W_{-\frac{\beta}{d \sqrt{2\alpha}}, \frac{\ell}{d}} \left(\frac{2\sqrt{2\alpha}}{d} r^d\right) W_{\frac{\beta}{d \sqrt{2\alpha}}, -\frac{\ell}{d}} \left(\frac{2\sqrt{2\alpha}}{d} r^d\right).
$$

We thus obtain the wave function for \( E = 0 \)

$$
\psi(r)_{E=0, \ell} = \frac{e^{i\ell\theta}}{2 \pi \sqrt{2\alpha}} \left[ \frac{\Gamma\left(\frac{1}{2} - \frac{|\ell|}{d} - \frac{\beta}{d \sqrt{2\alpha}}\right) \Gamma\left(\frac{1}{2} + \frac{|\ell|}{d} - \frac{\beta}{d \sqrt{2\alpha}}\right)}{\ell \Gamma\left(2 |\ell|/d\right) \Gamma\left(- 2 |\ell|/d\right)} \right]^{1/2}
\times
\frac{1}{r^{d/2}} W_{-\frac{\beta}{d \sqrt{2\alpha}}, \frac{\ell}{d}} \left(\frac{2\sqrt{2\alpha}}{d} r^d\right) W_{\frac{\beta}{d \sqrt{2\alpha}}, -\frac{\ell}{d}} \left(\frac{2\sqrt{2\alpha}}{d} r^d\right).
$$

Or \( W_{-\beta/d \sqrt{2\alpha}, \ell/d} \left(\frac{2\sqrt{2\alpha}}{d} r^d\right) \) is a linear combination [10] of

$$
W_{-\beta/d \sqrt{2\alpha}, \ell/d} \left(\frac{2\sqrt{2\alpha}}{d} r^d\right) \text{ and } W_{\beta/d \sqrt{2\alpha}, -\ell/d} \left(\frac{2\sqrt{2\alpha}}{d} r^d\right).
$$

As we have [11]

$$
M_{\lambda, \mu}(z) = z^{\mu + 1/2} e^{-z^{1/2}} {}_1F_1(\mu - \lambda + 1/2, 2\mu + 1; z),
$$
$$
M_{\lambda, -\mu}(z) = z^{-\mu + 1/2} e^{-z^{1/2}} {}_1F_1(-\mu - \lambda + 1/2, -2\mu + 1; z),
$$

there are two states for \( E = 0 \) whatever \( d \):

$$
\psi^{(+)}(r) = e^{i\ell\theta} r^{\ell} e^{-\sqrt{2\alpha} r^d} {}_1F_1\left(\frac{-\beta}{d \sqrt{2\alpha}} + \frac{|\ell|}{d} + \frac{1}{2}; \frac{2|\ell|}{d} + 1; \frac{2\sqrt{2\alpha}}{d} r^d\right),
$$
$$
\psi^{(-)}(r) = e^{i\ell\theta} r^{\ell} e^{-\sqrt{2\alpha} r^d} {}_1F_1\left(\frac{-\beta}{d \sqrt{2\alpha}} - \frac{|\ell|}{d} + \frac{1}{2}; -\frac{2|\ell|}{d} + 1; \frac{2\sqrt{2\alpha}}{d} r^d\right),
$$

where \( {}_1F_1(\alpha, \beta ; z) \) is the standard hypergeometric function.

These states \( \psi^{(+)}(r) \) and \( \psi^{(-)}(r) \) are linearly independent unless \( 2|\ell|/d \in \mathbb{N} \).

Taking into account the following formulae [12] in standard notation,

$$
{}_1F_1(\alpha, \gamma ; z) = e^z {}_1F_1(\gamma - \alpha, \gamma ; -z),
$$
$$
\lim_{\gamma \to -n} \frac{1}{\Gamma(\gamma)} {}_1F_1(\alpha, \gamma ; z) = z^{n+1} \binom{\alpha + n}{n+1} {}_1F_1(\alpha + n + 1, n + 2; -z),
$$

we can express the wave function in terms of these hypergeometric functions.
(which implies that \(d\) takes defined values) and [13]

\[
W_{n+\mu+1/2, \mu}(z) = \frac{(-)^n z^{\mu+1/2} e^{-z/2} (2 \mu + 1) (2 \mu + 2) \cdots (2 \mu + n)}{\Gamma(n, 2 \mu + 1)} = \\
\left( - \right)^n z^{\mu+n/2} e^{-z/2} L_n^{2\mu}(z),
\]

(64)

where \((n + 1)\) is a natural number and \(L_n^{2\mu}(z)\) a Laguerre polynomial, it appears that in equation (60)

\[
F \left( \frac{1}{2} + \frac{|\ell|}{d} - \frac{\beta}{d \sqrt{2} \alpha} \right) = \Gamma(- n) = \pm \infty.
\]

so that the constant in equation (60) diverges,

\[
\psi(r_f) \psi(r_i) \bigg|_{E=0} \to \pm \infty.
\]

\(\psi(r)\) being not normalisable, it will describe a discrete bound state.

The single \(\psi(r)\) wave function behaves like \(L_n^{2\mu}(\ell d) \left( \frac{2 \sqrt{2} \alpha r}{d} \right)\), and in the cases where \(d = 1\) and \(2\), it is calculable.

**Particular cases**

**Case 1:** for \(d = 1\) : Coulomb potential

\[
\frac{2}{d} |\ell| = 2 |\ell| = \text{integer}.
\]

In this case \(\psi^+\) and \(\psi^-\) are not linearly independent any more and equation (21) takes the following form

\[
G_{\text{Coul}}(r_f, r_i; 0) = \sum_{\ell = -\infty}^{\infty} e^{i\ell(\theta_f - \theta_i)} \frac{2}{\sin (2 \sqrt{2} \alpha)} \times
\]

\[
\times \exp \left\{ i \sqrt{2} \alpha (r_f + r_i) \cot (2 \sqrt{2} \alpha) \right\} I_2 |\ell| \left( \frac{8 \alpha (r_f r_i)^{1/2}}{i \sin (2 \sqrt{2} \alpha)} \right) \right\}.
\]

(65b)

Let us make use of the standard formula [9] in equation (58) and let us set

\[
y = 2 i \sqrt{2} \alpha, \quad \lambda = \frac{\beta}{\sqrt{2} \alpha}, \quad 2 \sqrt{2} \alpha r = t \gamma, \quad 2 \sqrt{2} \alpha (r_f r_i)^{1/2} = t \sqrt{\gamma_f \gamma_i}, \quad \mu = 2 |\ell|,
\]

we will obtain:

\[
\langle r_f | r_i \rangle = \frac{\Gamma \left( - \frac{\beta}{\sqrt{2} \alpha} + |\ell| + \frac{1}{2} \right)}{\sqrt{2} \alpha (r_f + r_i)^{1/2} \Gamma(2 |\ell| + 1)} W_{\frac{\beta}{\sqrt{2} \alpha}, |\ell|} (2 \sqrt{2} \alpha r_f) W_{\frac{\beta}{\sqrt{2} \alpha}, |\ell|} (2 \sqrt{2} \alpha r_i).
\]

(66)
The poles of the gamma function are given by
\[ \frac{\beta}{\sqrt{2} \alpha} = \left( \frac{1}{2} + |\ell| + n \right) \quad \text{or} \quad \alpha = \frac{\beta^2}{2(n + |\ell| + 1/2)^2}, \]
where \( \alpha \) is the energy.

To extract the discrete states, we use the following formula given by Kleinert [5]
\[ \lim_{E \to E_n} (E - E_n) \frac{\Gamma(f(E))}{\Gamma(f'(E_n))} n! = \frac{1}{f'(E_n)} (-1)^n, \]
which is valid for \( f(E_n) = -n \).

We thus obtain
\[ \psi(r_\ell) \psi(r_i) = \frac{(-1)^n e^{i(\ell \varphi_i - \varphi_j)}}{n! \pi \beta (r_\ell r_i)^{1/2}} \Gamma(2|\ell| + 1) W(2\sqrt{2} \alpha r_\ell) M \frac{\beta}{\sqrt{2} \alpha}, |r_\ell| (2\sqrt{2} \alpha r_i). \quad (67) \]

If we use the formula of reference [4] again
\[ W_{(1 - \mu)2 + n, \mu/2}(z_\ell) M_{(1 + \mu)2 + n, \mu/2}(z_i) = \frac{\Gamma(-\mu)}{\Gamma(-n - \mu)} \exp \left[ -\frac{z_\ell + z_i}{2} \right] (z_\ell + z_i)^{(1 + \mu)2} M(-n, 1 + \mu, z_\ell) M(-n, 1 + \mu, z_i), \quad (68) \]
and insert the identity
\[ \frac{(-1)^n \Gamma(-\mu)}{\Gamma(-n - \mu)} = \frac{\Gamma(n + 1 + \mu)}{\Gamma(1 + \mu)}, \quad (69) \]
into equation (68), we eventually obtain the wave function
\[ \psi(r) = \frac{(2\sqrt{2} \alpha)^{|\ell| + 1}}{\sqrt{2} \pi} \frac{\Gamma(n + 1)}{(2n + 2|\ell| + 1) \Gamma(n + 2|\ell| + 1)^{1/2}} \times \]
\[ \times e^{i\ell \varphi} r^{|\ell|} e^{-\sqrt{2} \alpha r} L_n^\mu(z)^{1/2} (2\sqrt{2} \alpha r), \quad (70) \]
where the \( L_n^\mu(z) \) functions are the usual Laguerre polynomials [14]
\[ L_n^\mu(z) = \frac{(n + \mu)!}{n! \mu!} M(-n, \mu + 1; z). \quad (71) \]

Case 2: for \( d = 2 \): harmonic oscillator potential.
In this case, equation (21) becomes
\[ G_{\text{HO}}(r_\ell, r_i; 0) = \sum_{i = -\infty}^{+\infty} \frac{e^{i\ell(\varphi_i - \varphi_j)}}{2 \pi} \langle r_\ell | r_i \rangle, \quad (72a) \]
where
\[ \langle r_\ell | r_i \rangle = \int_0^{+\infty} dS \frac{\sqrt{2} \alpha}{i \sin (S \sqrt{2} \alpha)} \times \]
\[ \times \exp \left\{ i \sqrt{2} \alpha (r_\ell^2 + r_i^2) \cot (S \sqrt{2} \alpha) \right\} I_{|\ell|} \left( \frac{2 \sqrt{2} \alpha r_\ell r_i}{i \sin (S \sqrt{2} \alpha)} \right). \quad (72b) \]
If we use the standard formula [9] in equation (58) with the following notation
\[ y = i S \sqrt{2\alpha}, \quad \lambda = \frac{\beta}{2 \sqrt{2\alpha}}, \quad \sqrt{2\alpha} r = t \gamma, \quad \sqrt{2\alpha} r_i r_i = t (\gamma_i \gamma_i)^{1/2}, \quad \mu = |\ell|, \]
we obtain
\[ \langle r_i | r_i \rangle = -i \frac{\Gamma\left(-\frac{\beta}{2 \sqrt{2\alpha}} + \frac{|\ell|}{2} + \frac{1}{2}\right)}{\sqrt{2\alpha}} \frac{W_{\frac{\beta}{2 \sqrt{2\alpha}}, |\ell| (\sqrt{2\alpha} r_i)}}{\Gamma(|\ell| + 1)} W_{\frac{\beta}{2 \sqrt{2\alpha}}, |\ell| (\sqrt{2\alpha} r_i)} \Gamma\left(|\ell| + 1\right). \]  
(73)

The gamma function has poles given by
\[ \frac{\beta}{2 \sqrt{2\alpha}} = \left(\frac{1}{2} + \frac{|\ell|}{2} + n\right), \quad \text{or} \quad \beta = \sqrt{2\alpha} \left(1 + |\ell| + 2n\right). \]

Proceeding like in case 1, we eventually obtain the wave function corresponding to \( E = 0 \).
\[ \psi_{n, \ell}(r) = (-)^n (2\alpha)^{\frac{|\ell| + 1}{4}} \left[ \frac{n!}{\pi (n + |\ell| + 1)!} \right]^{1/2} e^{i\ell \theta} r |\ell| e^{-\frac{\sqrt{2\alpha} r^2}{2}} L_n^{|\ell|} (2 \sqrt{2\alpha} r^2). \]  
(74)

The spectrum and the associated wave functions, corresponding to any \( E \), can be deduced by changing \( \beta \) into \( \beta + E \):
\[ E = \sqrt{2\alpha} (2n + |\ell| + 1) - \beta, \]  
(75)

and
\[ \psi_{n, \ell}(r) = (-)^n (2\alpha)^{\frac{|\ell| + 1}{4}} \left[ \frac{n!}{\pi (n + |\ell|)!} \right]^{1/2} e^{i\ell \theta} r |\ell| e^{-\frac{\sqrt{2\alpha} r^2}{2}} L_n^{|\ell|} (\sqrt{2\alpha} r^2). \]  
(76)

The degeneracy of the state corresponding to \( E = 0 \), versus the values of \( d \), has been discussed in references [1, 3, 4].

6. Conclusion.

For \( E = 0 \), we have determined the Green function in polar as well as in the generalized Levi-Civita coordinates. As it is not possible to determine the Green function as a function of \((u, v)\), because \((u, v)\) do not generate the whole plane, except for \( d = 1 \) and \( 2 \), we have determined it by summing the series, which constitutes our main result. Our Green function has been written in two parts:

- a discrete and finite sum of Green functions depending upon the \( d \) parameter and the \( \theta \) angle
- and a continuous sum.

A few limiting cases have also been investigated.

References