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Directed polymers in the presence of columnar disorder

Joachim Krug(1) and Timothy Halpin-Healy(2)

(1) Institut für Festkörperforschung, Forschungszentrum Jülich, P.O. Box 1913, D-52425 Jülich, Germany
(2) Physics Departement, Barnard College, Columbia University, New York, NY 10027-6598, U.S.A.

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Abstract. — We consider directed polymers in a random landscape that is completely correlated in the time direction. This problem is closely related to diffusion-reproduction processes and undirected Gaussian polymers in a disordered environment. In contrast to the case of uncorrelated disorder, we find the behavior to be very different at zero temperature, where the scaling exponents depend on the details of the random energy distribution, and at finite temperature, where the transverse wandering is subballistic, \(x \sim t/(\log t)^\gamma\) with \(\gamma = 1 + 2/d\) for bounded distributions in \(d+1\) dimensions. Numerically, these strong logarithmic corrections give rise to apparently nontrivial effective exponents. Our analytic results are based on appropriate Flory expressions for the (free) energy at \(T = 0\) and \(T > 0\). Some universal statistical properties of the evolutionary hopping of the optimal path are also derived.

1. Introduction.

The problem of directed polymers in a random potential [1–3] is one of the few cases in the statistical mechanics of disordered systems where the properties of the disorder-dominated low-temperature phase can be analyzed in considerable detail [4, 5]. In addition to its value as a testing ground for concepts and methods in the theory of ill-condensed matter, this system has attracted much interest due to its ramifications into other classes of problems, most notably higher-dimensional manifolds in random media [6] and, somewhat surprisingly, the nonequilibrium fluctuations of moving interfaces [7–9]. In the latter case the directed paths encode the growth sequence of the interface, and the disorder average becomes an average over growth histories [9]. Physical realizations of the directed polymer model itself include interfaces in two-dimensional systems with bond disorder [1] and flux lines in dirty type II superconductors [10].

Our understanding is rather complete for the case of a single transverse direction (the \(1+1\)-dimensional directed polymer), where the behavior is disorder-dominated at all temperatures.
Of primary interest is the fluctuation in the transverse coordinate, $x$, as a function of the time coordinate $t$ along which the polymer is directed, given that its starting point $(t = 0)$ is pinned at $x = 0$. One expects a scaling law

$$x_c(t) \equiv \langle |x|^2 \rangle^{1/2} \sim t^\zeta$$

(1)

where the angular brackets denote a thermal average and the overbar represents the subsequent average over the quenched disorder. The disorder-induced fluctuations are generally superdiffusive, so the wandering exponent $\zeta > 1/2$. A second scaling exponent describes the fluctuations in the free energy about its extensive disorder average $\bar{F} \sim t$,

$$[\bar{F}^2 - \bar{F}_0^2]^{1/2} \sim t^\theta.$$  

(2)

For a random energy landscape that is uncorrelated both in $x$ and $t$ the value $\zeta = 2/3$ of the wandering exponent is known to be exact in $1 + 1$ dimensions [1, 7]. Moreover, an average Hookes law holds in the sense that [5]

$$\langle x^2 \rangle - \langle x \rangle^2 \sim t,$$  

(3)

which enforces the exponent identity [1, 2, 8]

$$\theta = 2\zeta - 1$$  

(4)

and thus $\theta = 1/3$. Remarkably, these results apply both at finite temperature and at $T = 0$ [1] (where the thermal average is replaced by a single optimal path), a statement that can be extended beyond the values of the scaling exponents to include amplitudes and scaling functions [11].

A natural question concerns the dependence of the scaling exponents on the statistical properties of the random energy landscape. Zhang [12] has pointed out that non-Gaussian tails in the distribution of the disorder may alter the scaling of the polymers, a behavior which is due to the dominance of exceptional energy values over typical ones [13]. The influence of long-ranged disorder correlations in the spatial ($x$-) direction is also well understood [6, 8, 14]; the identity (4) remains valid in this case, and the values of the exponents can be obtained exactly. In contrast, (3) and (4) break down in the presence of long-ranged temporal correlations, and both analytic [8] and numerical [15] treatments become more difficult. In the present paper we consider the extreme case of temporally correlated disorder, when the random energies become independent of $t$. This type of columnar disorder has attracted much interest in recent studies of flux line pinning in high-temperature superconductors [16], where it can be induced experimentally by ion irradiation [17]. As will be mentioned in section 3, this variant of the directed polymer problem has been addressed previously in different contexts. Nevertheless, we have encountered a number of novel and unexpected features, which we describe in the following. We will mostly be working in $1 + 1$ dimensions, however many of our results are easily extended to a general number $d$ of transverse dimensions.

2. Model and main results.

We consider directed paths starting at the origin of a square lattice and proceeding upwards into the half plane $t > 0$. The transverse displacement is restricted to at most one lattice spacing per time step. Apart from this restriction, no additional energy cost is associated with horizontal bonds. Quenched random energies $\epsilon(x)$ are assigned independently to the columns
of the lattice. At a given temperature \( T \), the restricted partition function \( Z(x,t) \) for all paths running from the origin to the point \( (x, t) \) then evolves recursively according to [1]

\[
Z(x, t + 1) = e^{-\varepsilon(x)/T} [Z(x, t) + Z(x - 1, t) + Z(x + 1, t)]
\]  

(5)

with initial condition \( Z(x,0) = \delta_{x,0} \). Note that due to the restriction on transverse steps \( Z(x,t) \) vanishes outside of the wedge \(-t \leq x \leq t\). We will refer to this setup as the \emph{wedge geometry}. The total partition function is obtained from (5) by summing over \( x \), and the free energy is given by

\[
F(t) = -T \ln \left( \sum_{x=-t}^{t} Z(x,t) \right).
\]  

(6)

When only the free energy fluctuations (2) are of interest, it may be more efficient computationally to use the \emph{substrate} initial condition \( Z(x,0) = 1 \) with periodic boundary conditions on a transverse strip of width \( L \), corresponding to the simultaneous propagation of \( L \) directed paths. The disorder average can then be replaced by an average over \( x \) [11]. The price to be payed is the introduction of finite size effects when \( t^c \) becomes comparable to \( L \). At zero temperature (5) reduces to a recursion for the restricted ground state energy \( E(x,t) = -\lim_{T \to 0} T \ln Z(x,t) \) [1],

\[
E(x, t + 1) = \min [E(x,t), E(x - 1,t), E(x + 1,t)] + \varepsilon(x).
\]  

(7)

In this case both the transverse positional and the energy fluctuations can be obtained in the substrate geometry [18].

The focus of this work is on \emph{bounded} distributions for the random energies \( \varepsilon(x) \). In particular we have considered a one-parameter family of continuous distributions on the unit interval

\[
P_\nu(\varepsilon) = (\nu + 1)e^\nu
\]  

(8)

with \( \nu > -1 \), as well as the binary distribution

\[
P_b(\varepsilon) = b\delta(\varepsilon) + (1 - b)\delta(\varepsilon - 1),
\]  

(9)

\( 0 < b < 1 \). In a certain sense \( P_b \) can be regarded as the \( \nu \to -1 \) limit of \( P_\nu \), since in that limit \( P_\nu \) acquires a finite weight at \( \varepsilon = 0 \). We will also present some \( T = 0 \) results for the family of power-law distributions [12, 13]

\[
P_\mu(\varepsilon) = |\varepsilon|^{-(\mu+1)}, \quad \varepsilon < -1,
\]  

(10)

with \( \mu > 2 \).

Our central result is the \emph{breakdown of the equivalence between the \( T = 0 \) and the \( T > 0 \) scaling properties} familiar from the case of uncorrelated disorder. At \( T = 0 \) the scaling exponents are nonuniversal, showing a sensitive dependence on the details of the energy distribution (Sect. 4). In particular, we show that

\[
\zeta = \theta = \frac{\nu + 1}{\nu + 2}
\]  

(11)

for the bounded distributions (8), \( \zeta = \theta = 0 \) for the binary distribution (9) (note that \( \zeta \to 0 \) for \( \nu \to -1 \) in (11)), and

\[
\zeta = 1, \quad \theta = 1 + 1/\mu
\]  

(12)

for the power law distribution (10). Qualitatively, the absence of universality at \( T = 0 \) is due to the fact that optimal configurations of the directed polymer are localized at individual columns associated with exceptionally low energies. In contrast, in the problem with uncorrelated
disorder the optimal path encounters an extensive ($\sim t$) number of impurities, and a type of central limit theorem ensures the convergence to a universal distribution.

At $T > 0$ universality is largely restored, however the simple power law scaling (1) is replaced by an asymptotic behavior of the form

$$x_c(t) \sim \frac{t}{\log(t)^\gamma}$$

where $\gamma = 3$ for bounded distributions (Sect. 6). While (13) implies formally that $\zeta = 1$, our simulations show that the large value of $\gamma$ leads to effective wandering exponents which remain significantly different from unity on numerically accessible scales.

In a single realization of disorder the time evolution of the thermally averaged position $\langle x \rangle$ or, at $T = 0$, the position of the optimal path, shows an intermittent pattern with the total transverse displacement occurring during a few abrupt jumps. In section 5 we derive two universal relations for this kind of evolutionary hopping. First, the cumulative number of jumps up to time $t$ is proportional to $\ln t$. Second, the distribution of the range $R$ covered by the first jump decays as $R^{-2}$

### 3. Summary of previous work.

The recursion (5) lends itself to a number of interpretations. Nieuwenhuizen [19] studied it (with the binary distribution (9) and an additional boundary condition) as a model for wetting in a two-dimensional system with bulk disorder fully correlated in the direction of the substrate. If the time $t$ is thought to label the monomers along a polymer chain, (5) describes the conformations of an undirected Gaussian polymer, with one end fixed at the origin, in a one-dimensional random potential [20, 21]. In other applications one may regard $Z(x,t)$ as the population density of some (chemical or biological) species which migrates diffusively and reproduces (or is trapped) at a position-dependent random rate [21–23]. Here much interest has focused on the asymptotic behavior of the moments of $Z$ [24]. In the biological context the $x$-coordinate could represent either real space, or a one-dimensional phenotype space, in which each point corresponds to a set of organismic properties [22]. The analogy to biological evolution has inspired the use of (5) as an optimization strategy in complex landscapes [25]. Finally, in the zero temperature limit (7) the recursion becomes equivalent to the one-dimensional polynuclear model of crystal growth [9] with time-independent random local growth rates.

These (and several other) versions of the problem have generally been treated within the continuum approximation to (5), which takes the form of an imaginary-time Schrödinger equation

$$\frac{\partial}{\partial t} \mathcal{Z}(x,t) = D \frac{\partial^2}{\partial x^2} \mathcal{Z} + \mathcal{V}(x) \mathcal{Z}$$

with a Gaussian random potential $\mathcal{V}$ characterized by $\overline{\mathcal{V}} = 0$ and

$$\mathcal{V}(x)\mathcal{V}(x') = V^2 \delta_a(x-x'),$$

where $\delta_a(x)$ is the $\delta$-function smeared over some microscopic distance $a$. The free energy $\mathcal{F}(x,t) = -\ln \mathcal{Z}(x,t)$ then satisfies

$$\frac{\partial}{\partial t} \mathcal{F}(x,t) = D \frac{\partial^2}{\partial x^2} \mathcal{F} - D \left( \frac{\partial}{\partial x} \mathcal{F} \right)^2 - V$$
which is the Kardar-Parisi-Zhang equation for a moving interface [7, 9] subject to time-independent random growth rates [20, 21, 26]. The energy landscapes of the models considered here can therefore be viewed as evolving, rough surfaces, the height of which at a given site \(x\) and time \(t\) is the (ground state or free) energy of all directed paths of length \(t\) which end at \(x\) and start anywhere on the substrate \(t = 0\) [11].

The importance of the disorder can be estimated [21] by comparing the two terms on the right hand side of (14) on a scale \(R \gg a\). Their ratio defines a dimensionless coupling constant \(g(R) = (V/D)R^{2-d/2}\) which increases with increasing length scale in transverse dimensions \(d < 4\), showing that the large-scale behavior is dominated by the disorder. In the weak coupling regime \(g \ll 1\), studied e.g. by Cates and Ball [20], the polymer averages over many impurity sites and the effective disorder distribution becomes Gaussian by virtue of the central limit theorem. However, in the strong coupling (large scale) regime \(g \gg 1\) the polymer is pinned by individual impurities, and their energy distribution, which is a microscopic, nonuniversal feature of the system, becomes of crucial importance. In most of the work based on the continuum equation (14) the on-site energy distribution has, explicitly or implicitly, been assumed to be Gaussian. It so happens that a Gaussian disorder distribution gives rise to weakly subballistic scaling of the form (13) both at \(T = 0\) and at \(T > 0\). As a consequence, the difference between the zero and finite temperature cases, as well as the possibility of nonuniversal scaling exponents at \(T = 0\) has been obscured in much of the earlier discussion.

The problem was recognized by Zhang [23], who pointed out that \(\zeta = 1/2\) for a uniform energy distribution at \(T = 0\) (the case \(\nu = 0\) in (11)), and observed, in simulations of a discrete model similar to ours, a crossover to a behavior of the type (13) at finite temperature. However, as we shall argue below, the finite temperature scaling cannot be attributed to the central limit theorem or the fact that the system effectively averages over many impurities. Rather, our analysis based on the connection to Anderson localization [22] reveals the Lifshitz phenomenon occurring at the band edges of random Schrödinger operators with bounded disorder [27] to be responsible for the universal behavior at \(T > 0\). We show that the correct behavior both at zero and at finite temperature can be extracted from appropriate Flory-type expressions for the (free) energy; in this sense we propose a unified picture [28] which should cover arbitrary on-site distributions of disorder. In order to make contact with the bulk of previous work, we will also apply our approach to the case of a Gaussian distribution.


The optimal path has a simple form [25]: Starting from the origin, it travels to a favorable site at some transverse distance \(R\) and remains there up to time \(t\). In order to maximize the time spent at \(R\), the transverse displacement occurs at maximal speed, i.e. the path moves sideways one lattice spacing per time step. The energy of the transverse portion of the path is \(\bar{e}R\), where \(\bar{e}\) is the average of the random energy distribution. The total energy of the path can therefore be estimated as

\[
\Phi(R, t) = \bar{e}R + (t - R)E_{\min}(R)
\]

where \(E_{\min}(R)\) denotes the lowest energy expected to be encountered in the spatial interval \([-R, R]\). This quantity is related to the extremal statistics [29] of the energy distribution. For a given \(P(\epsilon)\), the distribution of the smallest among \(N\) independently chosen random energies is

\[
\mathcal{P}_N(\epsilon) = NP(\epsilon)[1 - \int_{-\infty}^{\epsilon} d\delta P(\delta)]^{N-1},
\]

(18)
and, since a transverse displacement \( R \) implies that the path explores \( 2R + 1 \) sites, 

\[
E_{\text{min}}(R) = \int_{-\infty}^{\infty} \text{d}\epsilon \, \epsilon P_{2R+1}(\epsilon).
\]  

(19)

For the family of bounded distributions \( P_\nu(\epsilon) \) defined in (8) this yields, asymptotically for large \( R \), 

\[
E_{\text{min}}(R) \approx \Gamma(1/\zeta)(2R)^{-1/(\nu+1)}
\]  

(20)

with \( \zeta \) given by (11), while the power law distributions (10) give 

\[
E_{\text{min}}(R) \approx -\Gamma(1 - 1/\mu)(2R)^{1/\mu}
\]  

(21)

Quite generally, \(|E_{\text{min}}|\) is a decreasing (increasing) function of \( R \) for bounded (unbounded) energy distributions. Consequently the dominant energy cost, which has to be balanced against the energy gain \( tE_{\text{min}} \) in (17), is given by the first term \( R \) for bounded distributions, but by the third term \(-RE_{\text{min}}\) for unbounded distributions (note that then \( E_{\text{min}} < 0 \)). In the bounded case this implies that \( \Phi \sim R \) for the optimal path, showing that the energy fluctuations are of the same order as the positional fluctuations and hence \( \theta = \zeta \). In contrast, for unbounded distributions \( \Phi \sim E_{\text{min}}(R)R \) which, for the power law distributions described by (21), leads to \( \theta = \zeta(1 + 1/\mu) \).

To determine the actual value of the wandering exponent, we use the appropriate expression for \( E_{\text{min}} \) in (17) and minimize with respect to \( R \). For the bounded distributions \( P_\nu \) we obtain 

\[
R \approx \frac{1}{2} \left[ \frac{2\Gamma(1/\zeta)}{\zeta(\nu + 1)} \right]^{\zeta} t^\zeta
\]  

(22)

with \( \zeta = (\nu + 1)/(\nu + 2) \) as claimed in (11), while the result for the power law distributions \( P_\mu \) reads 

\[
R \approx \frac{2}{\nu + 1/\mu} t
\]  

(23)

so \( \zeta = 1 \) and \( \theta = 1 + 1/\mu \). We note that \( \theta > 1 \), which implies that the ground state energy per unit length diverges as \( t^{1/\mu} \) (see Fig. 2). For the binary distribution (9) the zero temperature problem is obviously trivial: Once the path has found a site with \( \epsilon = 0 \), which is expected to require only a small transverse displacement, there is no incentive to venture further away, hence \( \zeta = 0 \).

In figures 1 and 2 we show numerical results for the bounded distributions \( P_\nu \), and the power law distributions \( P_\mu \), respectively. While the predictions for the exponents \( \zeta \) and \( \theta \) are in excellent agreement with the simulations, the prefactors in (22) and (23) do not appear to be exact. This is not surprising, since in our derivation the disorder average, leading to the function \( E_{\text{min}}(R) \) in (17), was performed before minimizing with respect to \( R \).

The above arguments are easily generalized to \( d \) transverse dimensions. The number of sites, \( N \), among which the minimum energy is sought, then scales as \( R^d \), which implies that \( E_{\text{min}}(R) \sim R^{-d/(\nu+1)} \) for the bounded distributions \( P_\nu \), and \( E_{\text{min}}(R) \sim R^{d/\mu} \) for the power law distributions \( P_\mu \). Inserting into (17) and minimizing yields the exponents 

\[
\zeta = \theta = \frac{\nu + 1}{\nu + 1 + d}
\]  

(24)

for the bounded case, and 

\[
\zeta = 1, \quad \theta = 1 + d/\mu
\]  

(25)
Fig. 1. — a) Positional fluctuation $|x_{\text{opt}}(t)|$ (upper curve) and ground state energy fluctuation $[E_{\text{opt}}^2 - \langle E_{\text{opt}}^2 \rangle]^{1/2}$ (lower curve) for bounded disorder with $\nu = -1/2$, obtained in the substrate geometry with a transverse system size $L = 5 \times 10^5$. The dashed line indicates the prediction $\zeta = \theta = 1/3$; b) Same as a) for $\nu = 1$. These data were obtained from $10^4$ independent runs in the wedge geometry.

Fig. 2. — a) Positional fluctuation $|x_{\text{opt}}(t)|$ (lower curve) and ground state energy fluctuation $[E_{\text{opt}}^2 - \langle E_{\text{opt}}^2 \rangle]^{1/2}$ (upper curve) for power law disorder with $\mu = 3$, obtained from $10^4$ disorder realizations in the wedge geometry. The dashed lines indicate the predictions for $\zeta$ and $\theta$; b) Divergence of the ground state energy per unit length, obtained from the same simulations as a).
for the power law distributions.

It is amusing to note that the nonuniversality with respect to the distribution may, in some instances, appear as a nonuniversal behavior with respect to the precise definition of the model. Consider, for example, a modified model [30] in which the paths are not allowed to remain at the same site for two subsequent time steps. Rather than looking for a single favorable site, the optimal path now has to find the pair of neighboring sites of lowest energy within the distance $R$, between which it can then alternate. The distribution of the total energy of the pair is the self-convolution of the original energy distribution. For the $P_\nu$ this implies a renormalization $\nu \rightarrow \bar{\nu} = 2\nu + 1$, hence the wandering exponent for the modified model is increased to $\zeta = (\bar{\nu} + 1)/(\bar{\nu} + 2) = (2\nu + 2)/(2\nu + 3)$.

To end this section we apply our approach to a Gaussian energy distribution

$$ P_G(e) = \frac{1}{\sqrt{\pi}} e^{-e^2} \quad (26) $$

The extremal statistics calculation yields [21]

$$ E_{\text{min}}(R) \approx -\frac{\sqrt{2\pi}}{e} (\ln(R))^{1/2} \quad (27) $$

Inserting in (17) and minimizing [31] yields

$$ 2R \ln R \approx t \quad (28) $$

which, for long times, leads to an asymptotic behavior of the form (13) with $\gamma = 1$. Here we differ from Nattermann and Renz [21], who found $\gamma = 3/4$ for the same problem. The discrepancy arises because they, working in a continuum model, assumed no limit to the transverse speed of the optimal path, i.e. to the angle between the transverse portion of the path and the $t$-axis [25]. If the speed is optimized (using a conventional elastic energy cost $R^2/t$ for the transverse stretching instead of the $\bar{e}R$ term in (17)) one finds that it diverges as $(\ln R)^{1/4}$, which accounts for the smaller value of $\gamma$. Possibly this case is realized in a lattice model with no explicit restriction on the range of transverse steps.

5. Evolutionary hopping.

It is of some interest to consider in detail how the transverse displacement (1) accumulates for a single realization of disorder. This aspect is of particular importance when assessing the efficiency of the transfer matrix (7) (or (5)) as an optimization strategy used to locate minima of the random potential [25]. In figure 3 we show the time dependence of the end point of the optimal path $x_{\text{opt}}(t)$ at $T = 0$, along with the evolution of the corresponding energy $e(x_{\text{opt}})$. The behavior is highly intermittent: The optimal end point remains fixed for long stretches of time, and the transverse displacement occurs during a few abrupt jumps, corresponding to a switch between two optimal paths whose energies cross. In the language of current evolutionary theory, long periods of stasis are punctuated by short episodes of rapid change, a pattern suggested by the fossil record to be typical of species evolution [32].

On a more quantitative level, we observe that the average number of jumps up to time $t$ increases as $\ln t$ (Fig. 4). This behavior was previously found by Kauffman in a study of long-range adaptive walks on rugged fitness landscapes [33]. It follows from a well-known order statistics argument [29, 34] if the full optimization problem is replaced by a simpler process in which the endpoint always occupies the site with the smallest energy available at time $t$,

$$ e(x_{\text{opt}}(t)) = \min_{-t \leq x \leq t} [e(x)]. \quad (29) $$
Fig. 3. — a) Position of the end point of the optimal path for a single realization of bounded disorder with $\nu = 2$; b) Energies of the sites visited by the optimal path.

Fig. 4. — Cumulative number of evolutionary jumps up to time $t$ for power law disorder with $\mu = 3$ (upper curve) and bounded disorder with $\nu = 1$ (lower curve). Each data set is the average over $10^4$ realizations.

A jump occurs at time $t$ if one of the two sites $x = \pm t$ which become newly available is more favorable than all the previously available sites $-(t - 1) \leq x \leq t - 1$. By symmetry, the probability for this event is $2/(2t + 1)$, and therefore the expected number of jumps up to time
\( t \) is
\[
\overline{\tau}(t) = \sum_{s=1}^{t} \frac{2}{2t + 1} \approx \ln t,
\]
(30)
a universal result valid for any continuous disorder distribution. The additional correlations present in the full optimization problem show up in the prefactor of the logarithm, which is less than unity and distribution dependent (Fig. 4). A rough interpretation of this observation is that the effective number of independent paths available to the optimization process grows as some sublinear power of \( t \).

The simplified process discussed above suggests the following approximate picture for the evolution of the energies of the sites visited by \( x_{opt} \). We assume that given the energy \( \epsilon(x_{opt}) = \epsilon_n \) for the endpoint of the optimal path, the next jump occurs to a site with an energy \( \epsilon_{n+1} \) chosen at random from the disorder distribution, subject only to the constraint that \( \epsilon_{n+1} < \epsilon_n \). We therefore write, for an arbitrary disorder distribution \( P(\epsilon) \),
\[
\overline{\tau}_{n+1} = \int_{\epsilon_{n}}^{\epsilon} \frac{d\epsilon}{\int_{\epsilon_{n}}^{\epsilon}} e^\frac{d}{P(\epsilon)}
\]
(31)
where the disorder average is conditioned on the value of \( \epsilon_n \). For the \( P_\nu \) this implies that the average site energy decreases exponentially with the number of jumps, \( \overline{\tau}_n = \zeta^n \) where \( \zeta = (\nu + 1)/(\nu + 2) \) is the wandering exponent. Equating this with the power law decay \( \overline{\tau}(x_{opt}) \sim E_{\min}(R) \sim t^{-(1-\zeta)} \) we obtain
\[
\overline{\tau}(t) \approx \frac{1-\zeta}{\ln(1/\zeta)} \ln t.
\]
(32)
While the expression for the prefactor in (32) is not quantitatively correct, the argument does indicate how a prefactor different from unity may arise.

The corresponding calculation for the power law distributions \( P_\mu \) yields an exponential divergence \( \overline{\tau}_n = -(\mu/(\mu - 1))^n \), which, equated to the time dependence \( \overline{\tau}(x_{opt}) \sim t^{1/\mu} \), also results in \( \overline{\tau} \sim \ln t \) (cf. Fig. 4). For the Gaussian distribution (26), (31) reduces to
\[
\overline{\tau}_{n+1} = -\frac{1}{2} e^{-\epsilon_n} \text{erfc}(-\epsilon_n)^{-1}
\]
(33)
where \( \text{erfc}(x) \) is the complementary error function. Expanding this function for large arguments we recover the known result [22, 23] \( \overline{\tau}_{n+1} - \epsilon_n \approx -1/(2\epsilon_n) \), which implies that \( \overline{\tau}_n \approx -\sqrt{n} \) for large \( n \). According to (27), the site energies decrease with time as \( -(\ln t)^{1/2} \), so, again, we obtain the relationship \( \overline{\tau} \sim \ln t \). These arguments apply in arbitrary transverse dimensionality \( d \). Related considerations for the finite temperature case will be given in section 6.

Exact results are available for the statistics of the first transverse jump in the full optimization process. Consider the probability \( r(t) \) that no jump has occurred up to time \( t \), for the family of bounded disorder distributions \( P_\nu(\epsilon) \). This implies that the energies of all optimal paths which can be reached at time \( t \) are larger than the energy \( \epsilon_0 t \) of the path that remains at the origin, i.e.
\[
\overline{\epsilon}x + (t-x)\epsilon(x) > \epsilon_0 t
\]
(34)
for \( x = \pm 1, \ldots, \pm t \). Here we have assumed that the energy of the transverse portion of the path can be replaced, for large \( x \) and \( t \), by its average \( \overline{\epsilon}x \). Using the independence of the \( \epsilon(x) \) and the simple form of the distribution \( P_\nu \) this leads to
\[
r(t) \approx \int_0^1 d\epsilon_0 \left[ \prod_{z=1}^{x^*} (1 - (\epsilon_0 - \overline{\epsilon}x/t)^{\nu+1}) \right]^2
\]
(35)
is bounded of \( \sim \)

Fig. 5. — a) Distribution of the time \( t \) of the first evolutionary jump, obtained from \( 10^6 \) realizations of bounded disorder with \( \nu = 0 \) (uniform distribution), using a maximal path length of \( 10^3 \). The full line indicates the predicted \( t^{-3/2} \) decay; b) Distribution of the range \( R \) of the first jump for the same disorder distribution as in a). The data were averaged over \( 4.2 \times 10^5 \) realizations with a maximal path length of \( t = 10^4 \). The full line is the rigorous lower bound (37). The saturation at large \( R \) is due to finite size effects.

with \( x^* = (\epsilon_0/\bar{\epsilon})t \). Evaluating this expression asymptotically for large \( t \) results in

\[
r(t) \approx \Gamma(\zeta)[(\nu + 1)/2]^\zeta t^{-\zeta}
\]

and the distribution of jump times decays as \( -dr/dt \sim t^{-(1+\zeta)} \) Simulation results for the jump time distribution are shown in figure 5a. We also note that for the simplified process (29) the argument leading to (30) shows that \( r(t) \sim 1/t \), which is consistent with (36) and the fact that \( \zeta = 1 \) for this process.

Together with the scaling relation \( R \sim t^\zeta \) (36) suggests that the probability distribution for the range \( R \) covered by the first jump should decay as \( R^{-2} \) independent of \( \zeta \). This follows from a simple order statistics argument similar to that used in the derivation of (30). The end point of the optimal path jumps from the origin, with energy \( \epsilon_0 \) selected at random from the disorder distribution, to a site at a distance \( R \) with energy \( \epsilon_1 < \epsilon_0 \). This site need not be the closest site from the origin with an energy less than \( \epsilon_0 \); rather, it is the site with the shortest transition time [22, 25]. Nevertheless, if the energies of all the \( N = 2R \) sites within range \( R \) from the origin have energies larger than \( \epsilon_0 \), the first jump necessarily has to cover a distance larger than \( R \). The probability for this event is equal to \( 1/(N+1) \), i.e. the probability that \( \epsilon_0 \) is the smallest among \( N + 1 \) independent random variables [29, 34]. This implies a rigorous, universal bound on the cumulative distribution of the range of the first transverse jump,

\[
 P_{\text{cum}}(R) \equiv \sum_{R'=R+1}^{\infty} P(R') \geq \frac{1}{2R+1}
\]

In particular, it follows that \( P(R) \) cannot decay more rapidly than as \( R^{-2} \), and the expected jump length is infinite. Our numerical data shown in figure 5b indicate that the bound provides
a good approximation to the actual distribution (note that in a finite system of length \( t \) the cumulative distribution saturates at \( R \sim t^k \), since jumps of larger size would not have had time to occur). In \( d \) transverse dimensions the number of sites within distance \( R \) scales as \( N \sim R^d \) and consequently the decay of \( P(R) \) is bounded by \( R^{-(d+1)} \)

6. Finite temperature.

In figure 6 we show the finite temperature equivalent of the position of the optimal path, the thermally averaged transverse displacement \( \langle x \rangle \), for a single realization of disorder. The punctuated pattern is seen to be very similar to the \( T = 0 \) case; the extent of thermal smoothening of the jumps can be assessed from the peaks in the width \( (\langle x^2 \rangle - \langle x \rangle^2)^{1/2} \) of the thermal wave packet, which develop because \( Z(x, t) \) briefly becomes bimodal as the average \( \langle x \rangle \) shifts between two positions [25] (Fig. 6b).

Looking at these pictures, one is lead to inquire about the nature of the attractors that define the preferential locations of the wavepacket. In contrast to the \( T = 0 \) situation, they do not coincide with individual lattice sites, and cannot, therefore, be directly associated with local minima of the energy landscape. Following Ebeling et al. [22] and Zhang [23], we identify the attractors with the localization centers of the eigenstates of the time-independent version of the discrete, imaginary time Schrödinger equation (5), which satisfy

\[
\psi_j(x) + \psi_j(x + 1) + \psi_j(x - 1) = e^{\epsilon(x)/T - \lambda_j} \psi_j(x),
\]

where \( \lambda_j \) denote the quasienergies. For bounded disorder distributions with \( \epsilon(x) \geq 0 \) solutions of (38) exist for \( -\ln 3 < \lambda_j < \infty \), and the long time behavior of \( Z(x, t) \) is governed by the low-lying states close to \( \lambda_0 = -\ln 3 \). Equation (38) belongs to a large class of disordered systems for which the low-lying states are exponentially localized [35],

\[
\psi_j(x) \approx e^{-|x - x_j|/\xi_j},
\]

where \( x_j \) is the localization center and \( \xi_j \) is the localization length. Expanding \( Z(x, t) \) in the \( \psi_j \) and using the initial condition \( Z(x, 0) = \delta_{x,0} \) we obtain [22]

\[
Z(x, t) \approx \sum_j \exp[-(|x_j|/\xi_j + |x - x_j|/\xi_j + \lambda_j t)].
\]

Clearly, the localization centers correspond to local maxima of \( Z \). To decide which state dominates the sum at a given time \( t \), the argument of the exponentials in (40) has to be maximized. This involves knowing the smallest value of \( \lambda_j \) likely to be encountered in a region of given size. We therefore introduce a function \( \lambda_{\text{min}}(R) \), which is the analogue of the lowest energy on the scale \( R, E_{\text{min}}(R) \), used in the zero temperature case. It can be computed from the extremal statistics formula (18), if the bare disorder distribution \( P(\epsilon) \) is replaced by the density of states \( \rho(\lambda) \) of the Anderson problem (38). Once \( \lambda_{\text{min}}(R) \) and the localization length \( \xi \) are known, the typical distance of the dominant localization center in (40) can be estimated by minimizing the effective Flory expression

\[
\Phi(R, t) = R/\xi + t\lambda_{\text{min}}(R).
\]

At this level of generality, we conclude that the introduction of a finite temperature has two consequences: First, the bare disorder distribution \( P \) is renormalized to the density of states \( \rho \). Second, the zero temperature Flory expression (17) is replaced by (41).
Fig. 6. — a) Motion of the thermally averaged position for a single realization of binary disorder with \( b = 0.3 \), at temperature \( T = 0.5 \); b) Width \( [\langle x^2 \rangle - \langle x \rangle^2]^{1/2} \) of the thermal wavepacket. Note the correspondence of the peaks with the jumps in a); c) Decrease of the free energy per unit length \( \phi(t) = -T \ln Z/t \) for the same disorder realization and temperature as in a) and b). Our analysis shows that \( \phi(t + 1) - \phi(t) \approx T\lambda_j \) where \( \lambda_j \) is the eigenvalue of the localized state at which the wavepacket resides at time \( t \). For long times \( T\lambda_j \to T\lambda_0 = -T \ln 3 \approx -0.549306... \)

Bounded disorder distributions are universally characterized by a Lifshitz tail in the integrated density of states \( k(\lambda) = \int_{\lambda_0}^{\lambda} d\lambda \rho(\lambda) \) near the band edge \( \lambda_0 \), i.e. in \( d \) dimensions.

\[
k(\lambda) \approx e^{-C(\lambda-\lambda_0)^{-d/2}}
\] (42)
with a positive constant $C$. This relation holds rigorously in the sense that [27]

$$\lim_{\lambda \to \lambda_0} \frac{\ln \ln k(\lambda)}{\ln(\lambda - \lambda_0)} = -\frac{d}{2}$$  \hspace{1cm} (43)

Using (42) in the extremal statistics estimate (18) and (19), we obtain

$$\lambda_{\text{min}}(R) - \lambda_0 \sim (\ln R)^{-2/d}$$  \hspace{1cm} (44)

Since the kinetic energy of a state of spatial extent $\ell$ is of the order $1/\ell^2$, the states which dominate the partition function (40) occupy regions of size $(\ln R)^{1/d}$. This scale is small compared to the distance $R$ from the origin. Hence the overlap of such a state with the origin, which gives rise to the first term in (41), decays exponentially in $R$ with a localization length $\xi = O(1)$ which is independent of $R$ [36]. Using this and (44) in (41), and minimizing with respect to $R$ [31] we obtain $R \sim t/(\ln t)^{\gamma}$ with

$$\gamma = 1 + 2/d. \hspace{1cm} (45)$$

The crossover from the $T = 0$ behavior described in section 4 occurs when the typical energy differences between competing optimal paths become of the order of $T$. Since these energy differences decrease as $t^{-(1-\theta)}$, where $\theta < 1$ is the zero temperature exponent, the crossover time scale is given by

$$t^* \sim T^{-1/(1-\theta)}.$$  \hspace{1cm} (46)

The resulting picture for the time evolution of $Z(x,t)$ is confirmed by recent rigorous results due to Sznitman [37]. Studying continuous Brownian motion in the presence of Poissonian obstacles, he proves that at long times the process is concentrated in "clearings" of size $(\ln t)^{1/d}$ associated with low local eigenvalues of the corresponding Schrödinger operator which are of the order $(\ln t)^{-2/d}$. Moreover, his work shows that $\zeta = 1$ in the sense that $\lim_{t \to \infty} t^{\zeta}/x_c(t) = 0$ for any $\zeta < 1$, and provides rigorous bounds on the logarithmic correction exponent $\gamma$, viz. $\gamma > 2/d$ for $d \geq 1$ and $\gamma < 6$ in $d = 1$, consistent with our estimate (45).

Severe corrections to the Lifshitz behavior (42) arise for continuous, bounded disorder distributions such as the $P(x)$ defined in (8) [38]. To leading order, (42) is replaced by

$$k(\lambda) \approx e^{-C|\ln(\lambda - \lambda_0)||\lambda - \lambda_0|^{-d/2}}$$  \hspace{1cm} (47)

Note that this does not affect the rigorous result (43), only the rate at which the limit is approached. The corresponding correction to the transverse wandering is of the form

$$R/t \sim \frac{(\ln(\ln t))^{2/d}}{(\ln t)^{1+2/d}}.$$  \hspace{1cm} (48)

Needless to say, such corrections are extremely difficult to detect numerically [38, 39]. We therefore begin our discussion of numerical results at finite temperature with the binary disorder distribution (9), for which a pure Lifshitz tail is expected.

In figure 7a we show that the data for $x_c(t) = [\langle x^2 \rangle]^{1/2}$, as well as the free energy fluctuations, are reasonably represented by conventional power law scaling with a value $\zeta \approx \theta \approx 0.58$ of the scaling exponents that is very far from the asymptotic prediction $\zeta = \theta = 1$. However, plotting $x_c(t)/t$ vs. $t$ we find a long time behavior of the type (13), with an exponent $\gamma$ approaching the predicted value $\gamma = 3$ (Fig. 7b); a least squares fit for $t = 500 - 1000$ yields $\gamma = 2.74$ for $x_c(t)$, and $\gamma = 2.86$ for the second moment $x_2(t) = \langle x^2 \rangle^{1/2}$. In particular, our data rule out
the value $\gamma = 3/2$ proposed by Nieuwenhuizen [19] based on a conventional elastic energy cost $R^2/t$ instead of the term $R/\xi$ in the energy estimate (41).

Another prediction of our approach concerns the free energy per unit length $f(t) = -T \ln Z/t$, which is directly related to the eigenvalues $\lambda_j$ of the localization centers visited by the thermal wave packet (Fig. 6c). Asymptotically $\lim_{t \to -\infty} f(t) \approx f_\infty = T \lambda_0 = -T \ln 3$ and the leading correction is expected, from (44), to decay as (see also [37])

$$\Delta f(t) \equiv f(t) - f_\infty \sim (\ln x_c)^{-2} \sim (\ln t)^{-2}$$

(49)

As in the case of the positional fluctuations, our data is quite consistent with a power law behavior $\Delta f \sim t^{-(1-\theta)}$ with $\theta \approx 0.7$ (Fig. 8a), but a plot of $\Delta f$ vs. $1/(\ln t)^2$ also shows a linear region for large $t$ (Fig. 8b). In summary, while our numerical results for the binary disorder distribution would not necessarily require an analysis in terms of $\xi(=\theta) = 1$ with strong logarithmic corrections (rather than in terms of simple power laws), they are certainly consistent with such an interpretation.

As expected from the $\ln(\ln t)$ correction in (48), for the continuous disorder distributions $P_\nu$, a plot of $x_c/t$ vs. $\ln t$ would suggest a value of $\gamma$ smaller than 3, $\gamma \approx 1.9$ (Fig. 9). Similarly the effective wandering exponent, determined from a conventional power law fit $x_c \sim t^\zeta$, shows a spurious dependence on $\nu$, apparently increasing with increasing $\nu$. For example, we estimate $\zeta \approx \theta \approx 0.68$, for $\nu = -1/2$ (Fig. 10) and $\zeta \approx 0.79, \theta \approx 0.85$ at $\nu = 2$, both at temperature $T = 0.5$ (recall that the zero temperature exponents for these two values of $\nu$ are $\zeta = \theta = 1/3$, and $\zeta = \theta = 3/4$, respectively). Qualitatively, this trend can be associated with the fact that the coefficient $C$ in (47) is proportional to $\nu$ [38]. The clearest numerical indication of the logarithmic factor in (47) shows up in the finite size correction to the free energy per unit
Fig. 8. — a) Convergence of the free energy per unit length to its asymptotic value, obtained from the simulation shown in figure 7. The dashed line indicates an effective power law with exponent $1 - \theta \approx 0.3$; b) The data of a) plotted in the form $\Delta f \sim (\ln t)^{-2}$ predicted by the analysis. The dashed line demonstrates the linearity of the extrapolation to $\Delta f = 0$.

Fig. 9. — Positional fluctuations for a uniform disorder distribution at temperature $T = 0.5$, obtained from $10^4$ runs in the wedge geometry, plotted in the form predicted by the analysis. The dashed line indicates the effective value of the logarithmic correction exponent $\gamma$.

length, which is expected to behave as

$$\Delta f / (\ln \Delta f)^2 \sim (\ln t)^{-2}$$  \hspace{1cm} (50)
Fig. 10. — Positional fluctuation $\sigma_c(t)$ (upper curve) and free energy fluctuation (lower curve) for bounded disorder with $\nu = -1/2$. The data were obtained from $10^4$ runs in the wedge geometry. The dashed line indicates the effective exponent.

Fig. 11. — Convergence of the free energy per unit length to its asymptotic value, plotted in the form suggested by the analysis for continuous bounded disorder distributions. These data were obtained in the substrate geometry, with a transverse system size of $L = 2 \times 10^5$ and a path length of $t = 10^4$. The disorder distribution was uniform ($\nu = 0$) and the temperature $T = 0.5$.

rather than as (49). This prediction is well confirmed by the simulations (Fig. 11).

To gain some insight into the statistics of finite temperature evolutionary hopping as shown in figure 6, we use the arguments developed in section 5 for the zero temperature case. Let $\lambda_n$
denote the expected quasienergy after $n$ jumps (no confusion should arise with the labeling of the eigenstates in (38)), and $\delta_n = \lambda_n - \lambda_0$ the distance to the band edge. Applying (31) to the Lifshitz density of states $\rho(\delta) \approx \delta^{-3/2} \exp(-C\delta^{-1/2})$, we obtain

$$\bar{\delta}_{n+1} = C \frac{\delta_n}{2} \int_0^{\delta_n} \delta^{-1/2} e^{-C(\delta^{1/2} - \delta_n^{1/2})} d\delta = F(\delta_n).$$

Noting that $F'(\delta) = (C/2)[\delta^{-1/2} - \delta^{-3/2}F(\delta)]$ it follows that $F(\delta) \approx \delta - (2/C)\delta^{3/2}$ for $\delta \to 0$. Hence $\bar{\delta}_{n+1} - \delta_n \sim -\delta_n^{3/2}$ and the band edge is approached as $\bar{\delta}_n \sim n^{-2}$. Comparing with the decay of the free energy per unit length as a function of time (49), we find that the cumulative number of jumps scales as $\bar{n} \sim \ln t$, as in the zero temperature case. We have not attempted a numerical verification of these predictions.

Finally, we apply our approach to the case of a Gaussian disorder distribution (26). The density of states remains Gaussian [22], so $\lambda_{\min}(R)$ is given by (27). In contrast to the case of bounded distributions, however, the localization length decreases with decreasing energy (low lying states are more strongly localized), as $\xi(\lambda) \sim |\lambda|^{-1/2}$ [22]. Inserting into (41) the expression to be minimized is now

$$\Phi(R, t) = R(\ln R)^{1/4} - t(\ln R)^{1/2}$$

which yields

$$R(\ln R)^{3/4} \sim t,$$

i.e. $\gamma = 3/4$. This differs from the result $\gamma = 1/2$ given by Engel and Ebeling [23], however we have checked that the original approach of Ebeling et al. also leads to $\gamma = 3/4$. It is conceivable that, in a lattice model, the decrease of the localization length would saturate at some microscopic length scale, which indeed implies a crossover to $\gamma = 1/2$.

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Note added in proof

After the final version of this article was submitted, we became aware of related work on the zero temperature problem by A. Hansen, E.L. Hinrichsen and S. Roux, J. Phys I France 3 (1993) 1569.
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This modification was suggested to us by Bernard Derrida.

It should be pointed out that in the presence of logarithmic factors the minimization of \( \Phi \) is not equivalent to merely balancing the competing terms.

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The relation to the type of mathematical models considered here is discussed in reference [22], and in Newman C.M., Cohen J.E. and Kipnis C., Nature 315 (1985) 400.


We are grateful to Th. M. Nieuwenhuizen for clarifying this point.

