A plaqet representation of ruptures and models for earthquakes

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Abstract. — We present a model for ruptures in an elastic medium. The basic unit is a plaquet on which local strain and stress tensors are assigned. We show that our discretization naturally leads to a double-couple representation of local ruptures. The stress redistribution due to ruptures is conveniently calculated using lattice Green’s functions. This description can be immediately applied to the study of earthquakes, which are a sequence of ruptures generated by continuous stress increase in a medium. We show that our model is suitable for incorporating some basic properties of rocks (e.g. static fatigue and healing of the fractured surfaces), as well as for studying the spatial patterns of earthquake activities.

1. Introduction.

In an earlier paper [1] we introduced a discretization of continuum elastic theory with a double-couple representation of local ruptures. We applied this discretization technique to the study of earthquakes and demonstrated how the tensor character of the stresses can be taken into account in a dynamical model. We also showed how to incorporate some basic properties of rocks (e.g. static fatigue and healing of the fractured surface) in the local dynamics of the model. The purpose of the present paper is (i) to describe the details of the discretization and lattice Green’s function approach for calculating the stress redistribution due to ruptures and (ii) to present a study of the earthquake model and discuss various phenomenological laws of earthquakes generated in our models.

One of the major challenges of material science research is to understand the breakdown process of a heterogeneous solid. This is not only technologically important, but also crucial in understanding natural phenomena such as earthquakes. There are a few standard approaches in the study of this fracture process. Molecular dynamics can simulate the formation and propagation of a crack on a microscopic level. The method, however, only has limited applications, as it is computationally too demanding to study the medium with more than a few

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cracks. A formal approach widely used in engineering is the finite element method, due to its simplicity and flexibility in dealing with very different problems. However, the description of a real elastic solid will be accurate only if the displacement field varies slowly over the size of the elements used. If there is crack in a medium, the displacement field around the crack will be accurate only if the elements are much smaller than the size of the crack itself. Also, in modeling a heterogeneous solid, the size of the heterogeneities should be larger than the size of the elements. This is a serious drawback if one deals with many microcracks in a disordered medium. Recently, a few simple lattice models, e.g., the central force model and the beam model, have been proposed by physicists. For a review of these models, see reference [2]. In these approaches, the medium consists of a network of bonds (or beams) and a local rupture is simply the breaking of a bond. The approach we use is different. The basic unit of our discretization is a plaquet; this has the advantage that the stress and strain tensors can be defined directly. Also, local physics such as static fatigue, healing, etc. can be incorporated rather naturally in our model. This is particularly useful for modeling the dynamics of earthquakes, which we will discuss as an application of our discretization approach.

Earthquakes can be viewed as sequences of ruptures which release the elastic stress built up gradually by the movement of tectonic plates. The global dynamics of earthquakes is quite rich and complex. There have been many attempts at earthquake modeling, yet there are still many unsolved problems. Part of the reason is the large amount of phenomenology needed to model, a more important reason is that one does not know how to model the physics well. One of the outstanding problems is to incorporate the tensor character of the stresses satisfactorily into a dynamical model of earthquakes [3]. This is the problem we attempt to solve in our model of earthquakes. We also show that a few empirical laws regarding healing of fractured surfaces [4] (for ensuring a steady state) and static fatigue of rocks [5, 6] (responsible for generating aftershocks) can be incorporated naturally in our model. In addition to reproducing a number of phenomenological laws such as the Gutenberg-Richter law [7] (as in a number of previous earthquake models [8–14]), our model can also dynamically generate a relatively stable fault structure, made up of regions that have been weakened by earthquakes.

2. Lattice representation of ruptures in an elastic medium.

In this section we will concentrate on a detailed description of our discretization procedure and the representation of local ruptures. The first step is to construct a discrete derivative. For simplicity let us restrict our attention to ruptures due to shear stress in the $x$-$y$ plane. Generalization to three dimensions is straightforward. In our work we use the following forms of the discretized derivative (say for a function $g(r)$):

$$D_x g(r) \equiv \frac{1}{2}[g(r + b) + g(r + d) - g(r - b) - g(r - d)],$$

(1)

and

$$D_y g(r) \equiv \frac{1}{2}[g(r + b) - g(r + d) - g(r - b) + g(r - d)],$$

(2)

where $\hat{e}_x, \hat{e}_y$ are unit vectors (the lattice spacing is taken to be unity), $d = (\hat{e}_x - \hat{e}_y)/2$ and $b = (\hat{e}_x + \hat{e}_y)/2$. Note that if $g$ sits on the nodes of a square lattice, then $D_x g$ sits on the centers of the plaquets formed by the nodes of this square lattice, and vice versa. A consequence of this definition is that the basic unit of our discretization is a plaquet rather than a bond.

Once $D_{x,y} g$ is defined, the discretization of elastic theory is straightforward. First we define a displacement vector $u$ on each node (corner of a plaquet). The distortion of the plaquet is
then characterized by the strain tensor (defined at the center of each plaquet)

\[ u_{ij}(r) = \frac{1}{2} (D_i u_j(r) + D_j u_i(r)). \] (3)

When all deformation are elastic (no ruptures exist) the stress tensor is related to the strain tensor through the generalized Hooke’s law:

\[ \sigma_{ij}(r) = K u_{ii}(r) \delta_{ij} + 2\mu (u_{ij}(r) - \frac{1}{3} \delta_{ij} u_{ll}(r)), \] (4)

where \( K, \mu \) are bulk and shear module, respectively, and summation over repeated indices is implied. The discretized elastic equation is given by the force balance condition, which, in discrete form, is

\[ F_i = D_j \sigma_{ij} = 0. \] (5)

We further restrict our consideration to the shear mode of rupture (most earthquakes are shear mode fractures). In this case only two components of the stress are needed in order to determine where a rupture will occur: \( \sigma_1 = \sigma_{xy} \) — the shear stress at the \( \hat{e}_{x,y} \) direction, and \( \sigma_2 = (\sigma_{yy} - \sigma_{xx})/2 \) — the shear stress at the directions of \( b \) or \( d \). In this paper we only consider ruptures in the \( \hat{x}, \hat{y} \) and \( b \) or \( d \) directions. The ruptures in an arbitrary direction can be represented as a linear combination of these two cases. (The shear stress in the direction which makes an angle \( \theta \) with \( x \)-direction is simply \( \sigma_1 \cos 2\theta + \sigma_2 \sin 2\theta \).

Let us consider the effect of a local rupture in an infinite medium, say a shear fracture in the \( \hat{e}_x \) direction at plaquet centered at \( r_0 \). After the break the original shear stress on the fractured surface is reduced to a percentage \( x \) of the original stress. The stress reduction causes a force imbalance and a subsequent stress redistribution occurs. We denote the stress after the rupture as \( \sigma_{ij}^{\text{new}} \) and the stress before the rupture as \( \sigma_{ij}^{\text{old}} \). We have \( \sigma_{ij}^{\text{new}} = \sigma_{ij}^{\text{old}} + \sigma_{ij}' \), where \( \sigma_{ij}' \) is the additional stress caused by the rupture. At equilibrium \( \sigma_{ij}^{\text{new}} \) must also satisfy the force-balance equation, and we must have

\[ D_j \sigma_{ij}' = 0. \] (6)

Let \( u'_i(r) \) denote the additional displacement induced by the rupture, \( \sigma_{ij}' \) can be related to \( u'_i \) via the generalized Hooke’s law throughout the system except for \( \sigma_{xy}'(r_0) \) at the ruptured site. To separate the violation of Hooke’s law at the ruptured site, we write

\[ \sigma_{xy}'(r) = \sigma_{xy}^e(r) + \sigma_{xy}^{ne}(\delta_{r,r_0}), \] (7)

\[ \sigma_{xx}'(r) = \sigma_{xx}^e(r), \] (8)

\[ \sigma_{yy}'(r) = \sigma_{yy}^e(r), \] (9)

where \( \sigma_{ij}^e \) is the part of stress that is related to \( u'_i \) via the generalized Hooke’s law. Substituting these expressions in (6) we obtain the equations for \( \sigma^{ne} \).

\[ D_j \sigma_{xx}^e(r) + \sigma_{ee}^{ne} D_y (\delta_{r,r_0}) = 0, \] (10)

\[ D_j \sigma_{yy}^e(r) + \sigma_{ee}^{ne} D_x (\delta_{r,r_0}) = 0. \] (11)

Given \( \sigma^{ne} \) we can obtain a solution for \( \sigma^e \), which then can be used to obtain \( \sigma^{ne} \) self-consistently by applying the boundary condition for the ruptured site

\[ \sigma_{xy}^{\text{new}}(r_0) = x \sigma_{xy}^{\text{old}}(r_0) \] (12)
Fig. 1. — Schematic illustration of a region subjected to external shear stress. The system is divided into many plaquetts, and is driven by slowly increasing shear stress. When a local stress is larger than the corresponding threshold stress, the plaquet fractures, causing a long-range redistribution of elastic forces. The force distribution of local double-couple sources are also illustrated: (a) fracture in \( \hat{e}_x \) or \( \hat{e}_y \) direction; (b) fracture in \( \hat{e}_x \pm \hat{e}_y \) direction.

i.e. the stress is reduced to a fraction \( x \) of the original stress.

The equation for \( \sigma^{el} \) can be cast (substituting \( \sigma^{ne} = -\sqrt{2}f \)) as

\[
D_f \sigma_{ij}^{el} - F_{i}^{db} = 0,
\]

where

\[
F_{x}^{db} = \frac{f}{\sqrt{2}}[\delta_{r,r_0+b} - \delta_{r,r_0+d} - \delta_{r,r_0-b} + \delta_{r,r_0-d}],
\]

and

\[
F_{y}^{db} = \frac{f}{\sqrt{2}}[\delta_{r,r_0+b} - \delta_{r,r_0+d} - \delta_{r,r_0-b} + \delta_{r,r_0-d}].
\]

Therefore, \( \sigma^{el} \) can be viewed as generated by the external source \( F^{db} \), which is only nonzero at the corners of the fractured surface, and is shown in figure 1. This force distribution satisfies the condition that its net force and net torque are zero, and thus is a double couple (a term used in seismology to describe the earthquake source). Our discretization is in complete agreement with the well-established fact that a shear rupture can be modeled by a double-couple force redistribution [15].

Next, we want to determine \( \sigma_{ij}^{el} \), given this double-couple force distribution. The solution of this problem can be easily obtained in Fourier space. Since the side of the unit cell is unity we define the Fourier transform as

\[
g(\mathbf{r}) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \text{d}k_x \text{d}k_y g(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}}
\]

The Fourier transform of the discrete derivative is then given by

\[
(D_x g)(\mathbf{k}) = 2i \sin(k_x/2) \cos(k_y/2)g(\mathbf{k}) \equiv D_x(\mathbf{k})g(\mathbf{k}),
\]

\[
(D_y g)(\mathbf{k}) = 2i \sin(k_y/2) \cos(k_x/2)g(\mathbf{k}) \equiv D_y(\mathbf{k})g(\mathbf{k}).
\]
Combining equations (3) and (4) and replacing \( \sigma \) with \( \sigma^{el} \) and \( u \) with \( u' \) we obtain

\[
D_j \sigma^{el}_{x_j} = [(K + 4\mu/3)D^2_x + \mu D^2_y]u'_x + (K + \mu/3)D_x D_y u'_y,
\]

and

\[
D_j \sigma^{el}_{y_j} = [(K + 4\mu/3)D^2_y + \mu D^2_x]u'_y + (K + \mu/3)D_x D_y u'_y.
\]

Fourier transformation of equation (13) yields

\[
[(K + 4\mu/3)D^2_x + \mu D^2_y]u'_x(k) + (K + \mu/3)D_x D_y u'_y(k) - \sqrt{2}f D_y e^{-i\mathbf{k} \cdot \mathbf{r}_o} = 0,
\]

and

\[
[(K + 4\mu/3)D^2_y + \mu D^2_x]u'_y(k) + (K + \mu/3)D_x D_y u'_x(k) - \sqrt{2}f D_x e^{-i\mathbf{k} \cdot \mathbf{r}_o} = 0.
\]

These are coupled linear equations and can be solved straightforwardly. The solution is

\[
u'_x(k) = \frac{\sqrt{2}f}{\Delta(k)}[(K + 4\mu/3)D^3_y - (K - 2\mu/3)D^2_x D_y]e^{-i\mathbf{k} \cdot \mathbf{r}_o},
\]

\[
u'_y(k) = \frac{\sqrt{2}f}{\Delta(k)}[(K + 4\mu/3)D^3_x - (K - 2\mu/3)D^2_x D_y]e^{-i\mathbf{k} \cdot \mathbf{r}_o},
\]

where \( \Delta(k) = \mu(K + 4\mu/3)(D^2_x + D^2_y)^2 \). We can now obtain the stress tensor \( \sigma^{el}(k) \) using the generalized Hooke's law:

\[
\sigma^{el}_{x x}(k) = (K - 2\mu/3)(D_x u'_x(k) + D_y u'_y(k)) + 2\mu D_x u'_x(k) = \frac{\tilde{f}}{\tilde{\Delta}} \frac{4D_x^2 D^3_y}{(D^2_x + D^2_y)^2} e^{-i\mathbf{k} \cdot \mathbf{r}_o},
\]

\[
\sigma^{el}_{y y}(k) = (K - 2\mu/3)(D_x u'_x(k) + D_y u'_y(k)) + 2\mu D_y u'_y(k) = \frac{\tilde{f}}{\tilde{\Delta}} \frac{4D_y^2 D^3_x}{(D^2_x + D^2_y)^2} e^{-i\mathbf{k} \cdot \mathbf{r}_o},
\]

and

\[
\sigma^{el}_{x y}(k) = \mu(D_x u'_x(k) + D_y u'_y(k)) = \frac{\tilde{f}}{\tilde{\Delta}} \frac{4D^2_x D^2_y}{(D^2_x + D^2_y)^2} e^{-i\mathbf{k} \cdot \mathbf{r}_o} + \sqrt{2}f e^{-i\mathbf{k} \cdot \mathbf{r}_o},
\]

where \( \tilde{\Delta} = \mu(K + \mu/3)/(K + 4\mu/3) \). We only monitor the shear stresses \( \sigma_1 \) and \( \sigma_2 \). The additional shear stresses are given by

\[
\sigma'_1(k) = \sigma'_x(k) = \sigma^{el}_{x y}(k) + \sigma^{ne}_{x y} e^{-i\mathbf{k} \cdot \mathbf{r}_o},
\]

\[
\sigma'_2(k) = \frac{1}{2}(\sigma'_y(k) - \sigma'_z(k)) = \frac{1}{2}(\sigma^{el}_{y y}(k) - \sigma^{el}_{z z}(k)).
\]

Using the fact that \( \sigma^{ne}_{x y} = -\sqrt{2}f \) we obtain

\[
\sigma'_1(k) = -\tilde{f} \frac{4D^2_x D^2_y}{(D^2_x + D^2_y)^2} e^{-i\mathbf{k} \cdot \mathbf{r}_o}.
\]
and
\[ \sigma_2'(k) = \int \frac{2D_x D_y (D_x^2 - D_y^2)}{(D_x^2 + D_y^2)^2} e^{-ik \cdot r_0} \]  
(34)

In real space we have
\[ \sigma_1'(r) = -\tilde{f} G_1(r - r_0), \]  
(35)
and
\[ \sigma_2'(r) = -\tilde{f} G_2(r - r_0), \]  
(36)

where \( G_1 \) and \( G_2 \) are lattice Green's functions:
\[ G_1(r) = \int_{-\pi}^{\pi} \frac{d k_x}{2\pi} \int_{-\pi}^{\pi} \frac{d k_y}{2\pi} \frac{\sin^2 k_x \sin^2 k_y}{(1 - \cos k_x \cos k_y)^2} e^{i \mathbf{k} \cdot \mathbf{r}}, \]  
(37)
\[ G_2(r) = \int_{-\pi}^{\pi} \frac{d k_x}{2\pi} \int_{-\pi}^{\pi} \frac{d k_y}{2\pi} \frac{\sin k_x \sin k_y (\cos k_x - \cos k_y)}{(1 - \cos k_x \cos k_y)^2} e^{i \mathbf{k} \cdot \mathbf{r}}. \]  
(38)

If we assume the boundary condition at the ruptured site, that after the rupture the shear stress \( \sigma_1 \) on the fractured segment is reduced to a percentage \( x \) of the original value, then \( \tilde{f} \) can be determined by \( \sigma_0 + \sigma_1'(r_0) = x \sigma_0 \), where \( \sigma_0 \) is the original shear stress just before the rupture. This leads to \( -\tilde{f} G_1(0) + (1 - x) \sigma_0 = 0 \), and \( \sigma_1, \sigma_2 \) can be written as,
\[ \sigma_1', \sigma_2'(r) = -(1 - x) \sigma_0 G_{1,2}(r - r_0)/G_1(0). \]  
(39)

We have checked that the Green functions reduce to correct continuum limit for the double-couple stress redistribution. At long distances the Green functions decay as \( 1/r^2 \) in our two dimensional model, where \( d \) is the spatial dimension. At short distances the greatest stress increase is near the tip of a crack.

For a fracture in the \( \hat{b} \) or \( \hat{d} \) direction the double-couple force distribution is shown in plaquet (b) of figure 1, and is given below,
\[ \mathbf{F}^d(r) = f \sqrt{2} [-d \delta_{r, r_0 + \hat{b}} - b \delta_{r, r_0 + \hat{d}} + d \delta_{r, r_0 - \hat{b}} + b \delta_{r, r_0 - \hat{d}}]. \]  
(40)

We can obtain similarly the additional stress \( \sigma_1', \sigma_2' \) due to a rupture in the direction of \( \hat{b} \) or \( \hat{d} \),
\[ \sigma_1'(r) = -(1 - x) \sigma_0 G_2(r - r_0)/(1 - G_1(0)), \]  
(41)
\[ \sigma_2'(r) = -(1 - x) \sigma_0 (\delta_{r, 0} - G_1(r - r_0))/(1 - G_1(0)). \]  
(42)

Again, \( \sigma_0 \) is the original shear stress \( \sigma_2 \) at the ruptured site just before the rupture.

We can also calculate the stress redistribution when more than one site ruptures. For simplicity, let us consider the case that all ruptures occur in \( \hat{e}_x \) direction. Let \( r_l, l = 1, \ldots, N_b \) (\( N_b \) is the total number of ruptures in the system) be the locations of ruptures. Again we can represent the ruptures by double-couple force distribution. We assume the corresponding strength of the double couples be \( f_l, l = 1, \ldots, N_b \). Because of the crack-crack interaction we can not determine \( f_l \) the same way as in the case of a single rupture. Instead, the boundary condition \( \sigma^\text{new}_{1}(r_l) = x \sigma^\text{old}_{1}(r_l) \) corresponds to a set of linear equations which have to be solved to obtain \( f_l \):
\[ (1 - x) \sigma^\text{old}_{1}(r_l) - \sum_{m} f_m G_1(r_l - r_m) = 0, \]  
(43)
for \( l = 1, \ldots, N_b \). After \( f_l \) is obtained we can determine the additional stress \( \sigma_{1,l} \) caused by the ruptures in the entire medium.
Fig. 2. — The shear stress $\sigma_{xy}$ redistribution calculated from the Green's function method in the text. The length of the fault at the center is 100 lattice units. The shear stress is reduced by 100 units uniformly along the fault. This distribution is in good agreement with the analytical results (see, for example, Fig. 8 in Ref. [16]).

As a test study we consider the stress redistribution caused by a segment of consecutive ruptures in $\hat{e}_x$ direction (with $x = 0$ corresponding to complete stress release). The corresponding continuum case — a segment in the interior of the medium is ruptured and the shear stress on it reduced to zero — can be solved analytically [16]. In figure 2 we plot the additional stress $\sigma'_i$ generated by the ruptures from the procedure described above. The agreement is excellent (see e.g. Fig. 8 of Ref. [16]). This again gives us confidence that our discretization is a very good representation of the elastic theory.

3. Applications to earthquakes.

We now discuss the application of our discretization scheme to earthquake dynamics. The model we study in this paper is the same as the model we presented in our previous paper except that in the present version we also incorporate an empirical law describing the healing of a fractured surface. In this section we will present our model in more detail.

In order to describe a dynamical model of earthquakes, several ingredients are necessary. First, we need a fracture criterion. This can be achieved by assigning a distribution of breaking thresholds (when the local shear stress is larger than the threshold, a rupture will occur). After each rupture, however, the fractured plaquet is weakened: the stress threshold is reduced to a lower value $\theta_l$. (Initially we assign the thresholds to be random numbers between $\theta_l$ and $\theta_u$). In reality the threshold for shear failure of previously fractured surface is better described by
\[ \theta_i = \theta^0 + \nu \sigma_n, \]
where \( \sigma_n \) is the normal stress, \( \theta^0 \) represents cohesion in the material, and \( \nu \) can be interpreted as the internal friction coefficient. As a first step we only consider the special case \( \nu = 0 \) in this paper. The weakening of fractured surfaces is responsible for generating a fault zone. In order that a steady state can be established, the fractured surface is allowed to anneal slowly. The healing process is not well understood. In this paper, we assume [4] that the strength of the fractured surface will grow as

\[ \theta_i = \theta_i + A \log(t - t_i), \]

where \( t_i \) denotes the time of the last fracture and \( \theta_i \) is the lower cutoff of the threshold (to indicate the fact that after the rupture, the fractured surfaces are most likely still in contact, parts are not completely ruptured, and the fractured surface will be able to support a small applied stress). The healing takes effect after a given event. The proposed healing law is taken as purely empirical and other forms could equally well be proposed [1, 9]. In practice healing will take place over a hierarchy of different timescales, equation (44) then takes into account that there is no single characteristic time for healing. Second, we need to choose boundary conditions as we only can simulate a finite region. In earthquakes it makes sense to use open (absorbing) boundary conditions: we consider an infinite medium, but neglect the effect of ruptures outside the region on which we concentrate. This also has an advantage computationally, because we can use the Green’s functions for the infinite medium. Third, in order to obtain earthquake sequences in the model containing aftershocks, some time delay mechanism for ruptures must also be included, this will be discussed later.

The system is driven by uniformly increasing the shear stress at a constant rate \( \rho \), with zero initial stress. Eventually the stress on one of the plaquet will exceed its threshold and it will rupture. The stress redistribution from this rupture is computed and we check if this causes the stress on other plaquet to exceed their thresholds. This produces a list of plaquet which will rupture in the next step. In the simulations presented here the stress redistribution from these plaquet are simply calculated individually and summed, rather than using (43). This process is the same as was used in [14]. It could be mentioned that, although the time scale for breaking and stress redistribution are fast, these processes are not truly instantaneous. One could easily implement different selection schemes for the order in which the plaquet break. The scheme used assumes that the time for restoration of mechanical equilibrium is much shorter than other time scales in the problem equations are satisfied we do not believe that these details are important for the overall picture.

In the calculations presented here we restrict ourselves to the case where the stress is increased for the component \( \sigma_2 \). As discussed elsewhere [17], a large fraction of the events take place near the edges, if we instead uniformly increase the component \( \sigma_{xy} \) in a square region with sides parallel to \( x \) and \( y \). We only consider the shear mode of rupture and begin by considering the model without static fatigue.

We measure the size of an earthquake by the total number \( s \) of sites that have ruptured following the initial instability. If a given site ruptures more than once during an event every occurrence is counted. We also define the “seismic moment” which is proportional to the summed stress drop over the ruptured sites. We have calculated the distribution \( N(s) \) and found that it is a power law \( N(s) \propto 1/s^\tau \). The exponent \( \tau \) is approximately 1.4 (see Fig. 3). We have also investigated the distribution of the seismic moment as defined above and found it to follow the same power law as \( N(s) \). We have not detected any dependence of the exponent on the value of \( A \) in (44) [17]. Also it does not seem to matter if we put \( \theta_i \) to be constant or random after rupture as long as the initial value of \( \theta_i \) is random. The exponent for \( N(s) \) agrees with that found in [14] within computational errors.
Ho,

In physics tips.

a)

\( s = \) combining will the plaquet exponent an respective time when

\[ \text{approximately } 1.41 \]

The parameters used in the simulation are \( p = 10^{-9}, \theta_1 = 0.25, x = 0.5 \) and \( \theta_a = 1.0 \). (b) Size distribution of earthquakes generated in the stationary state of the model for a 100 \( \times \) 100 lattice in the case of weakening and healing. 10000 earthquakes are included to obtain the statistics. The linear behavior of the log-log plot indicates a Gutenberg-Richter law with an exponent approximately 1.38. The parameters used in the simulation are \( p = 10^{-9}, \theta_1 = 0.25, x = 0.5 \) and \( \theta_a = 1.0 \). The healing constant is \( A = 10^{-4} \)

Figure 4 shows some typical large and a medium size events. The pattern of ruptured sites associated with a single event does not depend a great deal on the healing law, but is largely determined by the direction of stress enhancement after rupture which is largest near the crack tips.

The model which we have described so far has only one time scale: the slow geological time scale which determines the accumulation of shear stress. This time scale is very large. In a real earthquake sequence, however, there appears to be a second and a smaller time scale. Although the main shock lasts only for about a minute, the entire earthquake sequence combining foreshocks, mainshock, and aftershocks can last for many weeks. The aftershock sequences are found to obey Omori's law [18]: the rate of aftershocks following the main shock decays as \( 1/t^\xi \) as a function of time, with the exponent \( \xi \) close to one. The most popular explanation of Omori's law is static fatigue [19, 6], that is the rock strength is time-dependent. When the applied stress \( \sigma \) is below the instantaneous breaking strength but is above the stress-corrosion limit \( \theta_0 \), rocks generally break with probability per unit time proportional to \( e^{\alpha \sigma} \), where \( \alpha \) is a constant.

To illustrate this general physical picture, we have modified our model to incorporate the physics of static fatigue [1]. Consider a specific plaquet at the position \( r_0 \) in our model. If the shear stress \( \sigma_i(r_0) \) is above the instantaneous breaking threshold of the plaquet \( \theta_i(r_0) \), the plaquet will fracture as described above. However, when the stress is below \( \theta_i \) but is still above \( \theta_0 \), the plaquet is set to break with probability per unit time given by \( e^{\alpha(\sigma - \theta_i)} \)

Now, an earthquake sequence can be defined as follows. When all local stresses are below the respective instantaneous breaking thresholds as well as the stress corrosion limit \( \theta_0 \), nothing will happen: this is in the period between earthquake sequences, which we call the rest state.
Fig. 4. — Some typical spatial patterns of simulated earthquakes.

Fig. 5. — Spatial pattern of a main quake (full squares) and aftershocks (open circles). The parameters used in the simulation are the same as those in figure 3b.
Whenever there is a place where the stress is above the local instantaneous breaking threshold or the stress corrosion limit \( \theta_0 \), ruptures in the systems are expected, and an earthquake sequence begins. The return of the system to the rest state signals the end of this earthquake sequence. Each earthquake sequence can contain many shocks, separated by periods of no activity with some sites remaining to be fractured by static fatigue. The largest shock in a sequence is defined to be the main shock; the shocks before the main shock are foreshocks and the shocks after it are aftershocks. As shown in our previous paper [1], this model will obey Omori's law, with an exponent which is close to 1. The spatial pattern associated with a typical series of events is shown in figure 5.

4. Discussion.

We have derived in detail a novel discretization scheme to represent ruptures in elastic media. The method has the advantage of being fast and easy to implement, and it is quite accurate in cases where we have been able to compare with analytical results. The method can easily incorporate local disorder in yield strength and can deal with phenomenological corrections associated with static fatigue and healing. We have illustrated the method on two-dimensional square lattices with open boundary conditions. The boundary conditions can easily be modified in the case where the active region has an arbitrary shape, but the use of "non open" boundary conditions would complicate the calculation by destroying the translational invariance of the Green's functions. Similar difficulties occur if we attempt to deal with a medium with a heterogeneous elastic modulus. A modification of the model which would make it more realistic without causing a great deal of complication, is to allow ruptures in arbitrary directions and we expect to present results of such simulations elsewhere. The generalization to three dimensions is also quite straightforward.

When applied to earthquakes the model will, just as several other models reproduce some of the size and temporal scaling laws. What is new is that there is also a possibility of studying spatial patterns. Since the model is rather flexible, we have concentrated on showing how to implement it and giving some examples. We therefore have by no means been able to explore the full parameter space.

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References