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► To cite this version:

J. de Miranda-Neto, Fernando Gehm Moraes. Vitreous B₂O₃: a geometrical study. Journal de Physique I, 1993, 3 (5), pp.1119-1130. 10.1051/jp1:1993260 . jpa-00246784

HAL Id: jpa-00246784

<https://hal.science/jpa-00246784>

Submitted on 1 Jan 1993

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Classification

Physics Abstracts

61.40D — 64.60C — 02.40

Vitreous B_2O_3 : a geometrical study

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(Received 18 September 1992, revised 22 December 1992, accepted 15 January 1993)

Abstract. — Vitreous boron trioxide (B_2O_3) is an intrinsic noncrystalline material with planar equilateral triangles (BO_3 triangles) as structural units, presenting an overall two-dimensional character, a solid-like membrane structure. The local structural similarities between that glass and the negatively curved Bethe lattice motivated us to build an ideal model for vitreous B_2O_3 , propagating its local order on a surface of constant negative Gaussian curvature (the hyperbolic plane H^2) and using non-Euclidean hierarchical lattices as structural substrates. Based on the metric and symmetry properties of such lattices, we make an analytical investigation of the structure of the ideal glass model. This way, we obtain the peaks of the geometrical radial distribution function for the ideal glass structure, which are in good agreement with experimental data and theoretical studies for vitreous B_2O_3 . Those facts suggest the evidence of non-Euclidean local order for that amorphous solid, indicating the arrangement of planar triangular units on a negatively curved surface, forming few or no six-membered (boroxol) rings.

1. Introduction.

Boron trioxide (B_2O_3) may be called an « ideal » glass-former, since it presents a remarkable tendency to vitrify. B_2O_3 glass never recrystallises before melting on heating and vitrifies from the melt even at very low rates of cooling. As this inherent glass forming feature and other important physical properties may be associated with structural details, the study of the structure of that material assumes a very important role in the understanding of its various properties.

Two basic types of structural models, which are inconsistent with each other, have been proposed up to now generating an intense debate and considerable uncertainty on the structure of this amorphous material. The generally accepted model for the structure of vitreous B_2O_3 is a network with a planar equilateral BO_3 triangle as the basic structural unit (Fig. 1a) ; the threefold coordination of boron and the two-dimensional character of that triangular structural unit are firmly evidenced by NMR experiments [1]. On the other hand, it has long been thought that there may be a certain fraction of a larger unit in the form of a six-membered ring, the so-called boroxol ring B_2O_6 , consisting of three triangular BO_3 units (Fig. 1b). In the literature one finds a consensus about the existence of planar equilateral BO_3 triangles, but there is no consensus about the necessity of introducing boroxol rings and in what fraction they

Fig. 1. — Possible structural units of glassy B_2O_3 . (a) Planar BO_3 triangle ; (b) Planar B_3O_6 boroxol ring. The open circles represent boron atoms and the filled circles, oxygen atoms.

would be present. Krogh-Moe [2] in an earlier article has proposed that the structure of the glass consists solely of randomly connected boroxol rings, Mozzi and Warren [3] performed X-ray diffraction experiments and interpreted their results as being in accordance with a glass composed basically by boroxol rings. Guha and Walrafen [4] using Raman spectroscopy postulated that the structure of the glass is comprised of an equal proportion of boroxol rings and isolated BO_3 triangles. Johnson *et al.* [5, 6] performed neutron diffraction experiments and suggested a mixture of six-membered boroxol rings and simple BO_3 triangles with a fraction of boron atoms in boroxol rings equal to 0.6 ± 0.2 [7]. Prabakar *et al.* [8] through a NMR study suggest that around 66 % of boron atoms is likely to be present in the boroxol units, the rest being present in BO_3 units. Finally, continuous random network (CRN) models based in computer simulations [9] or in ball-and-stick models [10] and molecular dynamics calculations [11-14], show that the structure of the glass can be explained by randomly connected BO_3 triangles, where no or almost no boroxol rings are formed. Despite the diversity of interpretations of various experimental results and theoretical models, there are two well established facts about the B_2O_3 glass : (a) the presence of planar BO_3 triangular units with the O-B-O bond angle equal to 120° and (b) an overall two-dimensional character for the glass, making it a good example of a solid-like membrane [15].

In this work we infer some information on the structure of B_2O_3 glass from a geometrical study, propagating its local order (planar equilateral triangle BO_3) on a surface of constant negative Gaussian curvature (the hyperbolic plane H^2). Very recently, surfaces of negative Gaussian curvature have been used to study graphitic sheets [16, 17], analogue forms of C_{60} [18], amorphous carbon [19], surfaces and films [20] ; furthermore, non-Euclidean hyperbolic symmetries were recently found in hexagonal and cubic-close-packed crystals [21]. Use of non-Euclidean surfaces for the study of disordered (amorphous) systems was proposed some years ago by Kléman and Sadoc [22] ; since that seminal work, a variety of frustrated solids (disordered systems) have been modelled by ideal lattices in curved spaces and surfaces. Many authors have used the space of constant positive Gaussian curvature (three-dimensional spherical space S^3) to study metallic [23-25] and covalent [26, 27] glasses, liquid crystals (blue phases) [28-30], polymer packings [31], quasicrystals [32], etc. In spite of the success of curved crystal approach [33], models on spaces and surfaces of constant negative Gaussian curvature have not received as much attention as those on S^3 . In part, this neglect could be due to the lack of familiarity with hyperbolic geometry. Contrary to what happened to spherical

models, only a few authors have explored the geometry of negatively curved spaces and surfaces to analyse disordered systems : Nelson [23, 34] and Rubinstein and Nelson [35] studied dense packing of hard discs and the statistical mechanics of dense liquids on $\mathbb{H}^2$, Kléman [36, 37] and Kléman and Donnadiou [38], studied defects on $\mathbb{H}^2$ and suggested structural description of covalent glasses on spaces of constant negative curvature. An interesting work using surfaces of negative curvature is the article of Mosseri and Sadoc [39], where they pointed out that the Bethe lattice [40] (an infinite ramified network containing no rings, widely used in solid state physics and statistical mechanics), can be represented as a regular lattice (a perfect crystal) on the hyperbolic plane $\mathbb{H}^2$. Recently we used that idea to study the metric [41] and the symmetry [42] properties of the Bethe lattice and other hierarchical structures on $\mathbb{H}^2$. In this paper we analyse the similarities between those hierarchical lattices and the structure of B_2O_3 glass and draw structural information based on its negatively curved crystal counterpart.

At this point we would like to call the attention of the reader to the fact that the negatively curved crystallographic approach for the Bethe lattice permits its use in the investigation of structural properties of amorphous systems. In $\mathbb{H}^2$ that lattice has well defined bond angles and lengths, what makes possible the definition of a metric (geodesic) distance [41], a concept completely meaningless in conventional uses of hierarchical lattices [43]. Obviously, the ordinary Bethe lattice used in condensed matter physics, e.g., in cluster-Bethe lattice models [44] or in statistical mechanics [45], is a poor model for the structure of amorphous solids [46]. In our earlier work [41, 42], in addition to introducing a metric distance, we investigated the intrinsic symmetries of the hyperbolic Bethe lattice, which help finding the spatial localisation of each vertex of the structure. Those metric and symmetry properties are used in this work to explore an ideal infinite regular structure based on the Bethe lattice which contains the local symmetries of B_2O_3 glass. The model structure preserves the intrinsic structural aspects of the glass and produces results in good agreement with experimental and theoretical data. In summary, we obtain information about the structural properties of an amorphous material (B_2O_3 glass), propagating its local order on a negatively curved surface (hyperbolic Bethe lattice substrate), using hyperbolic geometry as a mathematical tool.

In part II, we analyse the metric and the symmetry properties of the negatively curved hierarchical lattices, connecting those properties with the structural aspects of glassy B_2O_3 . Then, we build an ideal model for that glass on such structures and compare our theoretical results with experimental data and with a continuous random network (CRN) model. Finally in part III we discuss the results and present our conclusions.

2. Geometrical investigation of vitreous B_2O_3 .

As stated in the introduction, the boron trioxide glass presents planar BO_3 triangles with the O-B-O bond angle equal to 120° , arranged together to form a solid-like membrane. In this section, we build an ideal model that preserves the structural requirements cited above and permits an analytical investigation that points out some details of the structure of that amorphous solid. Our ideal model does not include boroxol and other rings, providing, in this first theoretical approximation, a totally open structure containing only BO_3 triangles. This ideal situation provides an analytical tool for the study of the local and intermediate order in glassy B_2O_3 , furnishing new insights on the structure of the glass.

Geometrically speaking, we need a threefold coordinated two-dimensional lattice with angles of 120° and presenting no rings ; but this ideal structure is exactly the negatively curved Bethe lattice [39, 41, 42]. In that sense, the Bethe lattice is a regular tiling of a surface of constant negative curvature (the hyperbolic plane $\mathbb{H}^2$), a totally open two-dimensional structure where the angle between adjacent edges is precisely 120° . This motivated us to model the

structure of B_2O_3 glass on such negatively curved ideal lattice, decorating it with atoms, satisfying the requirements of coordination three for boron and two for oxygen. With the model thus constructed, we obtain the position of the peaks of the radial distribution function (the angular average of the density-density correlation function) for that structure, using the metric [41] and the symmetry [42] properties of negatively curved hierarchical lattices. We compare our results with experimental data [5, 6] and with a continuous random network model [10] also without boroxol rings.

Before getting into the details of the calculation we give a quick review of the metric [41] and the symmetry [42] properties of non-Euclidean hierarchical lattices, using basic facts of the geometry of constant negatively curved surfaces (hyperbolic geometry). The hyperbolic geometry is the geometry of the hyperbolic or Lobachevsky plane $\mathbb{H}^2$ [47], which cannot be wholly embedded in ordinary three-dimensional Euclidean space [48] and hence can only be visualised through suitable projective models. One such a model is a conformal mapping of the whole hyperbolic plane onto a disc, the Poincaré disc, which will be used throughout this work. In the projective model of Poincaré, $\mathbb{H}^2$ is represented by a disc whose boundary is denominated the Absolute. This limiting circle represents the points of $\mathbb{H}^2$ at infinity and the orthogonal arcs to the Absolute, the geodesics. While angles are preserved in this representation (conformal mapping), distances are more and more deformed as one goes from the centre of the disc towards the boundary. The Bethe lattice [40] appears naturally in $\mathbb{H}^2$ as a limit case of its tiling by regular polygons. In particular, the threefold coordinated Bethe lattice is a tiling of $\mathbb{H}^2$ by regular apeirogons (polygons of an infinite number of edges), three of them around each vertex defining the bond angles as 120° . In figure 2 we show the Poincaré disc representation of the threefold coordinated hyperbolic Bethe lattice and the structure obtained by joining its mid edges, the Husimi cactus [49]. In this representation those structures can have coordinates assigned to their vertices and distances can thus be measured with the help of non-Euclidean hyperbolic geometry ; the spatial localisation of the vertices (atoms) and the calculation of the distances among them, will be necessary in the comparison with experimental and theoretical radial distribution functions for B_2O_3 glass. In the following we will show how to calculate the coordinates of each vertex (atom) in the negatively curved Bethe lattice and in the Husimi cactus, and also show how to measure the distances between a given vertex and another one taken as the origin, for both structures.

We use barycentric coordinates [50] on the projective plane. This coordinate system requires the use of a triangle of reference on whose vertices we put « masses » t_0 , t_1 and t_2 . Fixing the vertices of that triangle, any point of the projective plane can be described by a set (t_0, t_1, t_2) . The Absolute is represented on the projective plane by a conic (a quadratic

Fig. 2. — Negatively curved hierarchical lattices conformally represented in the Poincaré disc model. (a) The threefold coordinated Bethe lattice and (b) its corresponding Husimi cactus.

equation of the coordinates). For simplicity, we take $\Omega = x_0 x_1 + x_1 x_2 + x_0 x_2 = 0$. Choosing an equilateral triangle of reference with the coordinates (1, 0, 0), (0, 1, 0) and (0, 0, 1) for its vertices, that expression represents a circle, the boundary of the Poincaré disc. Points satisfying $x_0 x_1 + x_1 x_2 + x_0 x_2 > 0$ are inside the disc and hence represent points of $\mathbb{H}^2$.

In this coordinate system the vertices of the threefold coordinated Bethe lattice and its associated Husimi cactus are given by solutions of diophantine equations [41]. The vertices of the threefold negatively curved Bethe lattice are given by the odd integral solutions of

$$x_0 x_1 + x_1 x_2 + x_0 x_2 = 3, \quad (1)$$

and the vertices of its associated Husimi cactus are given by the integral solutions of

$$x_0 x_1 + x_1 x_2 + x_0 x_2 = 1. \quad (2)$$

The rotational and translational symmetries of such negatively curved hierarchical lattices [42] are in this coordinate system given respectively by the transformations

$$S: (x_0, x_1, x_2) \rightarrow (x_1, x_2, x_0) \quad (3)$$

and

$$T: (x_0, x_1, x_2) \rightarrow (2x_0 + x_1, -x_0, 2x_0 + x_2), \quad (4)$$

which leave equations (1) and (2) invariant.

Expressing (x_0, x_1, x_2) as a column vector, we find that S and T assume the matrix forms

$$S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad (5)$$

and

$$T = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}. \quad (6)$$

The matrices S and T suffice to generate the coordinates of all the vertices of both negatively curved hierarchical lattices; they actually form a discrete group with presentation $\langle S, T; S^3, (TS)^2 \rangle$ (notice that $S^3 = (TS)^2 = \mathbb{1}$), the so-called modular group [51]. The vertices of both lattices are obtained by successive applications of the matrices S and T to a starting point in a hierarchical way given by the algorithm [42]

$$\boxed{\begin{array}{ccc} & X_n & \\ TS^2 X_n & & TX_n \end{array}} \quad (7)$$

followed by iteration. X_n represents all the $2^{(n-1)}$ points of a given generation n of vertices. Each new generation of points is obtained hierarchically by application of operators TS^2 and T to all points of the previous generation. The starting points are (1, 1, 1) for the Bethe lattice (the centre of the Poincaré disc) and (1, 0, 1) for the Husimi cactus. This process generates only one of the branches of either lattice, the order two branches can be obtained applying the rotational operators S and S^2 to the points generated above.

With the coordinates of the vertices of the negatively curved Bethe lattice and Husimi cactus, the geodesic distances between pairs of vertices on those structures can be obtained from the expressions derived in reference [41].

The geodesic distance d between two generic points (x_0, x_1, x_2) and (y_0, y_1, y_2) in $\mathbb{H}^2$ is found to be

$$d = k^{-1} \operatorname{arc} \cosh \left(\frac{x_0(y_1 + y_2) + x_1(y_0 + y_2) + x_2(y_0 + y_1)}{2 \sqrt{(x_0 x_1 + x_1 x_2 + x_0 x_2)(y_0 y_1 + y_1 y_2 + y_0 y_2)}} \right). \quad (8)$$

The real constant k , related to the Gaussian curvature K by $k = \sqrt{-K}$, sets the unit length in $\mathbb{H}^2$. In the following, we detail the expressions of distances between given vertices on both hierarchical structures ; these compact expressions will be useful in the explicit calculation of the peak positions of the partial radial distribution functions B-B, O-O and B-O for the ideal B_2O_3 model.

The geodesic distance between a generic vertex (x_0, x_1, x_2) and the lattice point $(1, 1, 1)$, in the Bethe lattice, is then

$$d = k^{-1} \operatorname{arc} \cosh \left(\frac{x_0 + x_1 + x_2}{3} \right); \quad (9)$$

(where the x_i are related by Eq. (1)) on the other hand, the geodesic distance between a generic vertex (x_0, x_1, x_2) and the lattice point $(1, 0, 1)$, in the Husimi cactus, is

$$d = k^{-1} \operatorname{arc} \cosh \left(\frac{x_0 + 2x_1 + x_2}{2} \right) \quad (10)$$

(here the x_i are related by Eq. (2)).

The distance between the point $(1, 1, 1)$ of the Bethe lattice, the origin of the Poincaré disc, and a generic vertex (x_0, x_1, x_2) of the Husimi cactus is found to be

$$d = k^{-1} \operatorname{arc} \cosh \left(\frac{x_0 + x_1 + x_2}{\sqrt{3}} \right); \quad (11)$$

the distances calculated above in expression (11) can also be obtained by

$$d = k^{-1} \operatorname{arc} \cosh \left(\frac{x_0 + 2x_1 + x_2}{2\sqrt{3}} \right), \quad (12)$$

where d represents the distance between the point $(1, 0, 1)$ of the Husimi cactus and a generic vertex (x_0, x_1, x_2) of the Bethe lattice. Since in both lattices all the vertices are equivalent, instead of measuring distances between two generic vertices we conveniently choose a given vertex as the origin and measure the distances to it. Thus the choice $(1, 1, 1)$ for the Bethe lattice and $(1, 0, 1)$ for the Husimi cactus.

The calculation of coordinates and distances on both negatively curved hierarchical lattices and the local structural similarities between the glass and the hyperbolic Bethe lattice, motivated us to model the structure of the glass on the overlap of the threefold coordinated Bethe lattice and its associated cactus along the lines set forward by Kléman and Sadoc [22]. As shown in figure 3, we put boron atoms on the vertices of the threefold coordinated Bethe lattice and the oxygen atoms on the vertices of the cactus, the chemical bonds being the geodesic segments of the Bethe lattice. This satisfies the requirements of coordination three for boron and two for oxygen, propagating BO_3 triangles with O-B-O angles of 120° in a

Fig. 3. — (a) Structural ideal model proposed for vitreous B_2O_3 in the hyperbolic plane $\mathbb{H}^2$ (conformal projection in the Poincaré disc), the continuous lines represent the Bethe lattice and the dashed lines, the Husimi cactus ; (b) Finite portion of the hyperbolic plane $\mathbb{H}^2$ isometrically immersed in $\mathbb{R}^3$, containing a piece of the ideal B_2O_3 model. The open circles represent boron atoms and the filled circles, oxygen atoms.

negatively curved surface. The spatial localisation of boron and oxygen atoms is done using the algorithm (7) applied respectively to the starting points $(1, 1, 1)$ and $(1, 0, 1)$. This development permits the calculus of the peak positions of the partial radial distribution functions for atoms of a given type in an analytical form : B-B pair functions can be computed by equation (9) ; O-O by equation (10) and finally B-O pair functions with the help of equation (11) or (12).

In figures 4 and 5 one can see that the calculated peak positions for our negatively curved model for the B_2O_3 glass is in good agreement with the continuous random network model proposed by reference [10] (which also does not include any boroxol ring) and with the experimental radial distribution function (neutron diffraction scattering) obtained by references [5] and [6]. We emphasise that there are no adjustable parameters in our negatively curved Bethe lattice ideal model for this glass ; the constant k was obtained using the experimental value $d = 1.363 \text{ \AA}$ [6] for the first distance B-O and the distance between the points $(1, 1, 1)$ and $(1, 0, 1)$, $kd = \ln \sqrt{3}$, from either equation (11) or (12).

3. Discussion and conclusions.

In this paper we constructed an ideal crystalline model for B_2O_3 glass based in a negatively curved Bethe lattice that shares structural similarities with that amorphous solid. We propagated the local structural unit of the glass (the BO_3 triangles), decorating the vertices of negatively curved hierarchical lattices with atoms of boron and oxygen. The peaks of the partial radial distribution function were analytically obtained for this model performing the calculation with the help of the metric and symmetry properties of the substrate lattices of the ideal glass structure. We would like to emphasize that we make an analytical study of the glass structure through an ordered crystal counterpart, contrary to other theoretical analysis, where the ideal model is also disordered ; the curvature of the ideal crystal may be associated with the glass intrinsic frustration.

Fig. 4. — Partial radial distribution functions for (a) B-B correlations ; (b) O-O correlations and (c) B-O correlations. The continuous curves represent the partial RDF calculated in reference [10] (continuous random network model) and the vertical lines indicate the peak positions for our geometric model for vitreous B_2O_3 .

One important advantage of simulating the structure of glassy B_2O_3 by means of the negatively curved Bethe lattice model is that the partial radial distribution functions (RDF) peaks for atoms of a given type may be individually computed (in an analytical form), providing a way of identifying the origin of the experimental peaks, i.e., whether a given peak comes from B-B, B-O or O-O correlations (Fig. 5), which cannot otherwise be done for a single diffraction experiment. The peak positions exactly calculated by our model result to be in good agreement with neutron diffraction experiments for the glass and with a theoretical continuous random network model containing no boroxol rings. At this point we call the attention of the reader to figure 4a : notice that there is a visible difference between the position of the first B-B peak calculated from our geometrical model and the theoretical CRN calculation [10]. On the other hand, the first B-B peak position computed by our model agrees with the experimental neutron correlation function [5], corresponding to the third peak shown in figure 5. These facts support our geometrical computation.

Fig. 5. — Experimental neutron correlation function for vitreous B_2O_3 from reference [6] (continuous curve) and the calculated peak positions (vertical lines) for our geometric negatively curved model.

Despite a theorem due to Hilbert [48], which states that a complete two-dimensional Riemannian manifold of constant negative curvature cannot be isometrically immersed in three-dimensional Euclidean space, we are allowed to compare our negatively curved model results with three-dimensional real experiments and theoretical models built in real space : Cartan [52] showed that a n -dimensional hyperbolic form (i.e., a finite portion of the hyperbolic n -dimensional space) can be isometrically immersed in a $(2n - 1)$ -dimensional Euclidean space. In our case $n = 2$ and a finite portion of H^2 can be isometrically immersed in R^3 , the three-dimensional Euclidean space. Then, this finite portion of H^2 appears as a wrinkled surface in R^3 (Fig. 3b), reinforcing the relation between curvature (negatively curved ideal model) and frustration. Notice that as the local order detected in radial distribution functions (experiments of neutron diffraction and X-ray scattering) varies in the range 6-8 Å [3, 5], then we need just a finite portion of H^2 to make the comparison ; in addition such portion is perfectly (isometrically) immersed in R^3 . The reader may argue that instead of measuring interatomic geodesic distances on the immersed portion of H^2 we should calculate the Euclidean 3D distances after the immersion (with the shortest chords connecting the atoms). That is correct. Unfortunately, the immersed surface is only one-time differentiable [53], what forbids its description in terms of analytical functions. That makes the identification of the 3D coordinates of the atoms and consequently the computation of distances among them a very complicated matter. On the other hand, since the RDF of a glass involves only the first few shells of atoms and since H^2 is locally Euclidean, the distances as measured in H^2 will not be much distorted in comparison to the Euclidean ones : consider for example a geodesic segment of length d in H^2 ; after the immersion, the curvature $1/R$ of this segment in R^3 is, in each point, less or equal than the maximum of the two principal curvatures of the immersed surface [53], which are no longer constant. In the worst case, the distance d between the extremities of the segment will be the length of an arc of circle of radius R . In this case the corresponding 3D distance is $2R \sin \left(\frac{d}{2R} \right)$, where $d < 2\pi R$ and $R \geq \frac{1}{k}$. The error in taking

the hyperbolic distance instead of the 3D one is then less than $d - 2R \sin\left(\frac{d}{2R}\right) \approx \frac{d^3}{24R^2} + O\left(\frac{d^5}{R^4}\right)$. Since the relevant distances for comparison with the experiment are at most of order $2/k \sim 5 \text{ \AA}$ our approach is justified.

The agreement between our model and theoretical and experimental data gives the evidence of a negatively curved local order for the glass, similarly to what was suggested recently in the literature for various carbon forms [16-19], implying non-Euclidean local symmetries in the glass, a feature also recently detected in Euclidean crystals [21] and in amphiphile films [20]. Also, molecular dynamics studies for vitreous B_2O_3 [11] suggest that steric forces cause adjacent BO_3 triangles to be twisted with respect to each other, excluding the formation of boroxol rings having the preference to be not in the same plane, inducing the connection between curvature and frustration. We would like to stress that even not including any kind of rings, our ideal model presents good agreement with experimental data of the real glass structure and with a continuous random network (CRN) containing no boroxol rings [10]. Such facts support the hypothesis of a network containing a great quantity of BO_3 triangular units and perhaps a low fraction of boroxol and other larger rings [9-14]. Boroxol rings have been suggested as responsible for the anomalous Raman spectrum of vitreous B_2O_3 [54], however, as Williams and Elliott [55] have pointed out, this connection is not conclusive.

A refinement of this ideal model may be obtained by the introduction of defects associated to the intrinsic hyperbolic symmetries (disclinations [38, 56] and disvections [38]). This procedure on the negatively curved Bethe lattice, besides providing a way of adding dangling bonds and rings, may provide also additional disorder to the network. Thus, glassy boron trioxide can be seen as an overall two-dimensional network of BO_3 triangles, connected together, exhibiting a certain topological ordering; an open and low density structure, presenting negatively curved hyperbolic domains, interrupted by defect-rich tissue that may contain dangling bonds, boroxol rings, etc. Further refinement of the model could be obtained by using the procedure devised by Sadoc and Rivier [57, 58] to decurve a curved crystal while introducing disorder in a hierarchical way. This procedure has been used to obtain both partial RDF and structure factors for a metallic glass [59].

The local geometry is primarily responsible for the character of many physical properties of a solid [46]. The study of electronic and acoustic properties for the ideal negatively curved model, through a group theoretical approach, in analogy to what happened in well known ideal spherical crystals [60], would reveal physical properties reminiscent of the real amorphous solid. We leave to a forthcoming publication [61] a study of such physical properties. We finalise by remarking that our model on H^2 does not resolve the uncertainty on the structure of B_2O_3 glass. It does, however, contribute with some new ideas to the controversy.

Acknowledgements.

We are very much indebted to the referees and to Dr. S. Levy (Department of Math.-U. C. Berkeley) for helpful comments and suggestions. We also thank CNPq and FINEP for financial support and Drs. Adrian C. Wright and Roger N. Sinclair for kindly supplying their numerical neutron diffraction data.

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