Monte Carlo simulations of aggregation phenomena
N. Menci, S. Colafrancesco, L. Biferale

To cite this version:
N. Menci, S. Colafrancesco, L. Biferale. Monte Carlo simulations of aggregation phenomena. Journal de Physique I, EDP Sciences, 1993, 3 (5), pp.1105-1118. <10.1051/jp1:1993259>. <jpa-00246783>

HAL Id: jpa-00246783
https://hal.archives-ouvertes.fr/jpa-00246783
Submitted on 1 Jan 1993

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Monte Carlo simulations of aggregation phenomena

N. Menci (1), S. Colafrancesco (2) and L. Biferale (1)

(1) Dipartimento di Fisica II Università di Roma, via della Ricerca Scientifica 1, I-00133 Roma, Italy
(2) Osservatorio Astronomico di Roma, via dell'Osservatorio, I-00040 Monteporzio, Italy

(Received 16 July 1992, revised 23 December 1992, accepted 22 January 1993)

Abstract. — We discuss the evolution of many-body systems under merging interactions between their components; the realizations that are considered here can be related to various physical systems from the evolution of systems of galaxies to the growth of structures in colloids. We run Monte Carlo simulations of aggregation processes of clusters with mass $M$, for various merging probabilities $K \sim M^\lambda$, and study the evolution of the resulting mass distribution (MD). We find out evidence for the known phase transition around the critical value $\lambda = 1$. We extend the analysis of the aggregation to the case of spatial fluctuations of the cluster distribution by means of a multifractal analysis. The distribution of clusters turns out to be fractal in the mass space and multifractal in the real space. The phase transition produces an enhancement of the spatial dishomogeneity for large masses, that is quantified in terms of different values of the generalized fractal dimension $D_r(q)$. A correspondence between the effects of the phase transition in the mass space and in the coordinates is also found.

1. Introduction.

The kinetics of aggregation processes have been studied in a variety of fields, from statistical physics to chemistry [1, 2] to planetary formation [3] and also in astrophysics and cosmology [4, 5]: specifically, there has been a renewed interest for the study of merging phenomena occurring between galaxies in groups or in the field, or between substructures in forming clusters of galaxies [6, 7].

The classical approach to merging phenomena is based on the Smoluchowski [8] aggregation equation

$$\dot{N}(M,t) = \frac{1}{2} \sum_{M' + M'' = M} K(M', M'', t) N(M', t) N(M'', t)$$

$$-N(M,t) \sum_{M' ' = 1}^{\infty} K(M, M', t) N(M', t)$$

Send offprint requests to: S. Colafrancesco.
describing the evolution with time $t$ of the mass distribution $N(M,t)$ of clusters with mass $M$ under binary collisions. The quantity $K(M, M', t)$ is the aggregation rate and is given by

$$K(M, M', t) = \rho \langle \Sigma(M', M'', t)V \rangle,$$

where $\rho$ is the average density of aggregating clusters, $\Sigma$ is their cross section for aggregation averaged over their relative velocity $V$. Analytical studies of aggregation phenomena described by equation (1.1) have been performed for separable homogeneous interaction kernels verifying

$$K(aM, aM', t) = a^\lambda K(M, M')F(t)$$

where $F(t) \sim t^f$. The behaviour of the solutions depends critically on the homogeneity degree $\lambda$ [1, 9] and on the exponent $f$ [6, 7].

Specifically, for $\lambda < 1$ a self-similar scaling form

$$N(M, t) \sim M_*^t(t)^{-2} \phi(M/M_*)$$

is obtained, where the mean cluster size $M_*(t)$ remains finite at every time $t$. In terms of the adimensional variable $m \equiv M/M_*$, one can verify [1] that $\phi(m) \sim m^{-\xi}$ (with $\xi > 0$) for $m \ll 1$, while for $m \gg 1$, $\phi(m) \sim m^{-\lambda} \exp(-m)$ holds. This leads to the conservation of the total mass $\mathcal{M}$, as

$$\dot{\mathcal{M}} = \sum M \dot{N}(M, t) = 0.$$  

A different situation arises when $\lambda > 1$ and $f > -1$; in this case, a substitution of the simple scaling (1.3) in equation (1.1) leads to a divergence of $M_*(t)$ after a critical finite time $t_c \sim 1/\mathcal{M}$ (see [10], [11]). This implies a scaling form $\phi(m) \sim m^{-(\lambda+3)/2}$ at all masses which, in turn, yields

$$\dot{\mathcal{M}} = \sum M \dot{N}(M, t) < 0$$

for $t > t_c$. This phenomenon may be regarded as a phase transition (known as gelation) occurring at the critical time $t_c$, with order parameter $1 - \mathcal{M}(t)/\mathcal{M}(0)$. Numerical solutions of the mean field equation (1.1) lead to the same results [7].

The usual interpretation of the result in equation (1.5) is that a part of the system is no longer described by equation (1.1); this part consists of a large merger which gains mass from the rest of the system with rate $\mathcal{M}$.

In chemistry, this result has been interpreted as responsible for the formation of an ‘infinite’ cluster in suspensions of particles (aerosol), see [1, 12] and references therein. It has also been considered as one possible mechanism for the formation of massive (cD-like) galaxies present in the central regions of dense groups of galaxies as well as for the erasure of substructures developing in the late phases of the overall collapse of gravitationally bound structures [6, 7]. In this context, the colliding bodies are galaxies in groups or in clusters. For such systems, the kernel is characterized by a mixed cross section [13] which is a sum of two terms, whose scalings are parametrized by $\lambda = 2/3$ and $\lambda = 4/3$, respectively. The relative importance of the two terms depends on the detailed characteristics of the environment [6, 7].

The previous analytical approach based on the solutions of equation (1.1) reveals several limitations in the description of the system at all times.

Firstly, it does not describe the evolution of the whole system for $t \geq t_c$ in the cases with $\lambda > 1$, $f > -1$ (gelling systems). The absence of detailed dynamical information about the total mass distribution (including the merger) is replaced by integrated information, namely, the appearance of a finite mass flux $\mathcal{M} < 0$.

Secondly, the mass distribution obtained as a solution of equation (1.1) is averaged over the ensemble of the single realizations of the aggregation process. The peculiar aspects of each realization can be revealed only through a more detailed analysis.
Thirdly, the spatial fluctuations are not accounted for in equation (1.1); it is important to know whether they affect or not the evolutionary behaviour of the mass distributions.

In this paper we deal with the previous points by studying the evolution of aggregating systems by means of Monte Carlo simulations of merging processes. The simulations are constructed by an inverse-cascading scheme (Fig. 1) that reproduces the aggregation mechanism and the resulting kinetics described (only on average) by equation (1.1). We perform the analysis for various interaction kernels with different values of $\lambda$ and $f$. We follow the evolution of the system with $\lambda > 1$ beyond the critical time $t_c$ and extract detailed information about the growing dishomogeneity in the mass distribution by means of a fractal analysis. The role of spatial fluctuation is also investigated by assigning a spatial structure to the system of interacting particles.
2. Monte Carlo simulation of cascading systems.

The structure of equation (1.1) suggests the possibility of describing an aggregation process by means of the more general formalism of random cascading models. The first term on the right-hand side of equation (1.1) describes the increase in \( N(M,t) \) due to the coalescence of clusters of masses \( M' \) and \( M'' \) with clusters of mass \( M' + M'' = M \), while the second term describes the decrease on \( N(M,t) \) due to the coalescence of clusters of mass \( M \) with other clusters of different masses. Thus, considering discrete time steps, it is possible to look at the evolution of the system as a possible configuration of a tree structure (see Fig. 1), where links are drawn everytime that two clusters coalesce, with a probability proportional to the aggregation rate \( K(M, M', t) \).

The picture of the aggregation processes we are discussing is similar to that of random cascading models, and the relation between these two formalisms can also be studied analytically; this leads to a correspondence of the solutions of equation (1.1) with the generating functionals of the partition function of a Cayley tree *averaged* over all the possible configurations of the tree [14]. This approach, although very attractive, provides again only an *average* description of the system evolution, and thus contains the problems previously discussed.

Alternatively, one can use the cascading approach and study the configurations of the tree by a Monte Carlo simulation. In this paper we follow this second approach.

The technical procedure is as follows:

a) We distribute a total number \( N \) of clusters in the mass space, assigning a mass \( M \) to each cluster according to the initial distribution function \( N(M, t_0) \).

b) We select a couple of clusters with masses \( M' \) and \( M'' \) with a probability proportional to \( N(M', t_0) N(M'', t_0) \). The selected couple then aggregates to form a larger cluster, with mass \( M = M' + M'' \), with a probability \( p(M', M'') \propto K(M', M'', t_0) \). Thus, at the next time step, we will find one cluster with mass \( M = M' + M'' \) with a probability \( p \), and the two selected clusters \( M' \) and \( M'' \) with probability \( 1 - p \). The step b) is repeated until every cluster present at \( t = t_0 \) has been selected.

c) We re-define the mass-distribution simply by counting the masses of the clusters that we have generated. We then iterate the procedure of steps b) and c) in the whole time interval considered.

Doing this, we obtain configurations of a tree structure that are reproduced, on average, by equation (1.1). In this approach, we can study - in principle - aggregation mechanisms characterized by the more general kind of kernels and we can investigate the properties of the *single* realizations of the aggregation process.

3. Analysis in the mass space.

We have performed simulations with an initial number of particles \( N = 5 \times 10^4 \) and with different initial distributions of masses in the range \( M_0 < M < 50M_0 \), with \( M_0 \) - the unit mass - taken to be \( 1 \).

We have checked that already for values of \( N \sim O(1 \times 10^3) \) the main statistical features of our results are independent on the size of the system for all values of \( \lambda \) and \( f \) chosen hereafter. The first feature we want to investigate is the mechanism leading to the phase transition at the critical time \( t_c \), characteristic of kernels with \( \lambda > 1 \) and \( f > -1 \). The plots in figure 2 show the quantity \( \frac{N(M,t)}{M} \), normalized to the total mass \( M \), as a function of the mass formed at different times \( t \). The left column of figure 2 illustrates the behaviour of the system for a
Fig. 2. — The mass distribution of the system at different times. The plots show the fraction of total mass $N(M,t) M/\mathcal{M}$ (y-axis) found at the mass $M$ (x-axis) at three different times $t = 10$, $t = 35$ and $t = 100$, in units of the time step used in the numerical calculations. a) The three plots in the left column show the behaviour of the system for a kernel with $\lambda = 2/3$, b) The three plots in the right column show the evolution of the system for a kernel with $\lambda = 4/3$. The mass axis has been chosen different in the two cases in order to show more details of the mass distribution in the case $\lambda = 2/3$ that evolves at slower rates. In both cases (represented in the two columns) we kept $f = 0$ (the aggregation rate does not depend explicitly on time).
kernel with $\lambda = 2/3$, while the right column refers to $\lambda = 4/3$. In both cases we kept $f = 0$ (the aggregation rate does not depend explicitly on time).

The analytical result of the phase transition appears here as a stronger dishomogeneity of the gelling system ($\lambda > 1$) compared with the relatively homogeneous distribution produced by non-gelling systems ($\lambda < 1$). In particular, gelling systems are characterized by a branching in the mass distribution after a short time $t \approx t_c$ (the critical time), with the high-mass branch (gel-phase) rapidly concentrating the main part of the total mass of the system. The low-mass branch (sol-phase) evolves smoothly while the gel phase evolves through a nearly self-similar series of branchings. The time evolution of the MD is characterized by an initial transient for $t \lesssim \tau \sim (N\langle\Sigma V\rangle)^{-1}$ (the typical timescale for the evolution of the system) where the relative increase of the gel mass is small, because the flux of clusters from the sol to the gel phase consists predominantly of small masses; the further branchings take longer and longer to take place.

To make the previous considerations more quantitative, we analyse the moments $\langle M^q \rangle = \sum N(M) M^q$ of the mass distribution. In figure 3 we plot the ratio $\langle M^2 \rangle / \langle M^1 \rangle^2$ as a function of time for gelling systems with $\lambda = 4/3$, $f = 0$ (solid line), for non-gelling systems with $\lambda = 2/3$, $f = 0$ (dotted line). If the mass distribution had a self similar form of the kind (1.3), the ratio $\langle M^2 \rangle / \langle M^1 \rangle^2$ would stay constant in time. From the behaviour of $\langle M^2 \rangle / \langle M^1 \rangle^2$ plotted in figure 3, one can conclude that this is actually the case for the non-gelling systems. A different situation holds for gelling systems after the time $t_c$, because of the formation of a large merger. We also investigated the role of the explicit $t$-dependence of the interaction kernel. In general, the larger is $f$, the faster is the evolution. The value $f = -1$ is critical, the evolution being appreciable only for substantially larger values of $f$. In this respect, we verified that the gelling phase transition occurs even for a time decreasing kernel provided the limiting condition $f > -1$ holds.

![Figure 3](image)

Fig. 3. — The ratio $\langle (M - \bar{M})^2 \rangle / \bar{M}^2$ is shown as a function of time for gelling (solid line) and non-gelling (dotted line) systems. A value $f = 0$ is used throughout.
To analyze quantitatively the actual mass distribution of a single realization (without making any kind of average), we study the degree of dishomogeneity by means of a fractal analysis in the mass space (see [15] for a review). To this aim, we divide the whole mass range in boxes of size $\epsilon$, and we define the measure of the $i$-th box as

$$
\mu_i(\epsilon) \equiv N_i/N,
$$

where $N_i$ is the number of clusters in the $i$-th box and $N$ is the total number of clusters. The set of generalized dimension is then defined by

$$
(q - 1)D(q) \equiv \lim_{\epsilon \to 0} \frac{\ln Z(q, \epsilon)}{\ln \epsilon},
$$

with

$$
Z(q, \epsilon) = \sum_i \mu_i(\epsilon)^q.
$$

For the fractal analysis of the mass distribution of clusters, the number of boxes with $N_i \neq 0$ needed to cover the $M-$axis scales as $\epsilon^{D_m(0)}$, where $D_m(0)$ is the fractal dimension in the mass space. Such a dimension is a measure of the homogeneity of the distribution of particles in the considered mass range.

From the analysis of the distribution produced by the aggregation at each fixed time, we find that, for every $q$, the function $Z(q)$ scales with $\epsilon$ as a power law $\sim \epsilon^{(q - 1)D_m(q)}$ (in the limit $\epsilon \ll 1$) with $D_m(0) < 1$. Thus, the realizations of aggregation processes seem to distribute the mass according to a fractal distribution in the mass space.

To understand whether the distribution follows a pure fractal or a multifractal scaling, we study the $q$-behaviour of the function $\tau(q) \equiv (q - 1)D_m(q)$. According to a general theorem [15], a multifractal distribution produces a convex function of $q$ while a linear dependence is the signature of a purely fractal behaviour. We find that the previous function shows a linear trend, providing evidence for a simple fractal scaling, so that, for every $q$, $D_m(q) = D_m(0) \equiv D_m$.

The dimension $D_m$ changes with time following the developing dishomogeneity of the system. Its time behaviour will depend on the strength of the interaction, i.e., on $\lambda$ and $f$. We illustrate in figure 4 the evolution of $D_m(t)$ as a function of time for $\lambda = 2/3$ and $\lambda = 4/3$ (with $f = 0$).

For increasing time, a gap develops between the characteristic dimensions $D_m$ for $\lambda < 1$ and $\lambda > 1$. This indicates a different growth of the dishomogeneity for the two cases that reflects two different ways of producing a large mass distribution starting from a uniform initial condition. In the non-gelling case high masses are produced by a diffusion-like homogeneous process, while the gelling phase tends to create large attractive clusters at the expense of many smaller interacting bodies.

After the initial transient, the systems evolves toward a stable state, characterized by the plateau of $D_m(0)$, that defines the asymptotic mass distribution.

A final remark concerns the reliability of the statistic: with the time going on, the number of particles decreases, but always stays large enough to allow a feasible estimate of the smaller moments of mass distribution, in both gelling and non-gelling phases (see [1, 6, 7] for a direct calculation of the total number of clusters as a function of time).
4. The role of spatial fluctuations.

We now turn to considering the effect of spatial fluctuations of the cluster density \( \rho = \rho(r) \). In order to obtain a spatially resolved representation of the merging process, we distribute the \( N \) masses in a 2-dimensional box of size \( R \). We also assign coordinates to each particle. The Monte Carlo procedure described in §2 is then modified as follows:
- To select a couple of interacting clusters, we extract first one particle, and we explore a region inside a sphere with radius \( V \Delta t \), centered on such a particle. The cluster velocity \( V \) is here assumed constant and \( \Delta t \) is the time step.
- A second particle is selected only if it is inside such a region.
- In case of aggregation, the new cluster will be placed in the center of the mass of the two interacting particles.

The remaining criteria for the particle selection and aggregation are those described in §2; the cluster positions are changed at every interaction by an amount \( V \Delta t \) in a randomly chosen direction.

The described procedure will allow to interact only clusters which are able to meet in the sample time interval \( \Delta t \); the interaction probability in the time \( \Delta t \) will be proportional to the interaction volume \( \Sigma V \Delta t \) (as for the homogeneous case), thus reproducing the aggregation rate entering the Smoluchowski equation. The point is that, for given \( \Sigma V \Delta t \), the aggregation rate will be proportional to the number of particles which is inside such a volume, i.e. to the density \( \rho \). However, the density here is not constant, so that the merging rate for the homogeneous case \( \alpha \Sigma V \Delta t \) turns out now to be modulated by the local density

\[
\rho(r) = \bar{\rho} \left[ 1 + \xi(r) \right]
\] (3.4)
The initial form of the two-point correlation function $\xi(r)$ is provided by the physical background of the aggregation process. We investigated different scenarios characterized by different initial correlation functions: homogeneous ($\xi_0(r) = \text{const}$.), or power-law correlations, $\xi_0(r) \propto r^{-\gamma}$ with $\gamma = 0.8$ and $\gamma = 1.5$. The value of $\gamma$ depends on the physics of the particular system considered. For instance, in the case of the distribution of cosmic structures (from galaxies to clusters and larger structures), a value $\gamma \approx 1.8$ (with a flattening towards $\gamma \approx 1$ on large scales) is indicated by cosmogonical theories and relevant observations [16, 17].

The correlation function will evolve according to the aggregation mechanism and it is studied by a multifractal analysis. We have verified that, as for the simulations with no spatial structure, the initial total number of clusters $N = 5 \times 10^4$ is large enough to prevent detectable finite size effects. A discussion about the dependence of our results on size is given, in quantitative terms, at the end of this section. The detailed shape of the initial mass distribution, $N(M, t_0)$, turns out to be unimportant for the asymptotic time evolution.

RESULTS. — A first result is that, the initial spatial correlations do not produce appreciable variation as for the appearance of the phase transition in the $\lambda > 1$ cases, independently of the particular initial density distribution. On the other hand, the correlations tend to shorten the critical time $t_c$ by a few percent.

Moreover, the spatial fluctuations affect the asymptotic spatial distribution of clusters. To perform a fractal analysis of the space distribution of the clusters, we divide the total volume in boxes of size $\epsilon$, and we define a local spatial measure as the mass density in the $i$-th box normalized to the total mass, i.e., $\mu_i = \mu_i/\mu_{\text{tot}}$. The generalized dimensions, $D_r(q)$, in the coordinate space depend on the initial configuration.

As a first aim, we study the time evolution of $D_r(q)$. In the homogeneous case (no initial correlations) $D_r(q)$ stays constant as a function of both $q$ and time. So, in absence of correlations, the initial random fluctuations are not sufficient to produce an increasing disomogeneity during the time evolution for every value of $\lambda$.

A different situation arises for the initially correlated configuration, provided the condition $\lambda > 1$ holds. The behavior of $D_r(q)$ for $q = 0, 2, 4$ is plotted in figure 5 as a function of time for an initial distribution defined by a correlation function $\xi_0(r) \propto r^{-\gamma}$ with $\gamma = 0.8$, in the case $\lambda = 4/3$. A transition from the initial to the asymptotic values of $D_r(q)$ appears after a short time, similarly to what previously resulted in the mass space analysis (cf. Fig. 4). The values of $D_r(q)$ in the plateau thus define the asymptotic spatial distribution of the clusters. Similar results are obtained for different initial values of $\gamma > 0$ and $\lambda > 1$.

The complete multifractal spectrum in the asymptotic region has been also evaluated as a function of the initial spatial configuration. The results for $\lambda = 4/3$ are illustrated in figure 6 for different initial distributions, and compared with the spectrum of the initial condition. This is not flat because of the non-zero initial correlation (the exponent $\gamma$ is actually related to the fractal dimension $D(2)$ of the initial distribution, computed from the measure $\mu_i(\epsilon) = N_i/N$, by the standard relationship $\gamma = D_e - D(2)$, where $D_e$ is the dimension of the embedding space, 2 in our case).

An evolution from the initial condition is present, but it is appreciable only for $q \gtrsim 2$. This is because, while $D_r(0)$ represents the simple counting of boxes with a non-zero measure, $D_r(q)$ with $q \neq 0$ weights preferentially the peaks of the mass distribution describing the intermittent behaviour of the system. Thus the time evolution changes mainly the distribution of massive clusters, leaving the distribution of the smaller background particles unchanged. This is a further consequence of the strong dishomogeneity in the mass distribution, resulting after the phase transition.

In the cases with $\lambda < 1$, no significant evolution of the multifractal spectrum is observed for any initial condition, which implies the same self-similar clustering for small and large clusters.
Fig. 5. — The time evolution for three generalized fractal dimensions, $D_r(q)$, evaluated for $\lambda = 4/3$ and for $q = 0$ (solid line), $q = 2$ (dashed line), $q = 4$ (dotted line). The critical time $t_c$ corresponds to the beginning of the asymptotic region.

In order to test our statistics, we have run the same simulations for a total number of particles from $N = 5 \times 10^8$ up to $N = 5 \times 10^5$. The scalings with log($\epsilon$) of the logarithm of the partition function $Z(q, \epsilon)$ defined in equation (3.3), are plotted in figure 7 for $q = 0$ and $q = 4$, for different values of $N$. The scalings refer to the case $\lambda = 4/3$ with an initial $\gamma = 0.8$. As appears from the figure (and from the reported variance of the data from the least-square fit), the scalings for $N = O(10^4)$ are already quite good. They show the same slopes (i.e. fractal dimensions) for the various values of $N$, with an error on the dimension estimate decreasing for larger values of $N$. We have verified that, at least for moments up to order $q = 5 - 6$, the scalings are again well stable and reach a good reliability, so that the previously presented results are robust relative to variations of $N$.

The results shown in figure 7 are actually very common in multifractal systems where only high moment distributions are affected from the convergence problem due to the possible presence of strong-fluctuations.

5. Discussion.

In this paper we show that a statistical numerical approach to the study of aggregation processes allows us to describe in detail the statistics of the mass distribution that is represented only on average by the mean field Smoluchowski equation. With such an approach, we also extended the study to the role of spatial correlations.

To analyze the different behaviour for non-gelling and gelling cases, we performed a fractal analysis, both in the mass and in the coordinate spaces.

From the analysis in the mass space, we find that the mechanism driving the evolution of the
Fig. 6. — The comparison between initial and asymptotic values of some generalized dimensions for different initial conditions, and for $\lambda = 4/3$.

(a) The solid line represents $D_r(q|t = 0)$, while the dotted line is the same quantity in the stationary regime $t \gg t_c$, with initial condition $\gamma = 0.8$.

(b) The same as in (a), but for the case with initial condition $\gamma = 1.5$.

system depends on the strength of the interaction (parametrized by the homogeneity degree $\lambda$ of the Kernel) in a critical way. For $\lambda < 1$ (non-gelling systems), the system tends to create still larger objects by a diffusion-like process among clusters of comparable size. For $\lambda > 1$ (gelling systems) instead, the evolution is driven by the accretion of large clusters (formed at a finite time $t = t_c$) at the expense of smaller interacting bodies. Thus, in the latter case, a mass gap is created between two phases, resulting in a highly dishomogeneous mass distribution. By means of our numerical approach, we follow the behaviour of the system beyond the phase
Fig. 7. — The logarithm of $Z(q, \epsilon)/(q - 1)$ versus $\log(\epsilon)$. The slopes of the resulting curves in the scaling regions give the generalized fractal dimensions. The three plots on the left column correspond to $q = 0$ for $N = 5 \times 10^3$, $5 \times 10^4$, $5 \times 10^5$, from top to bottom. On the right column the scalings for $q = 4$ are plotted for the same values of $N$. The least-squares fit (dotted lines) are evaluated in the linear region. The resulting slope (i.e. $D_r(q)$) and the variance on their estimate are also given.
transition, where the MD evolves through a repeating series of branchings (Figs. 2 and 3).

The fractal analysis quantifies such results. A pure fractal scaling for the dimension \( D_m(q) \), related to the mass distribution, results. The fractal dimension reaches a stationary value (depending on \( \lambda \)) after a transient time, thus describing the asymptotic mass distribution. A gap between the asymptotic values of \( D_m \) for gelling and non-gelling systems is found, due to different dishomogeneities in the mass distributions.

We have investigated the role of spatial fluctuations for different initial spatial configurations and at different times, and found some interesting results.

In the case of gelling systems, the multifractal spectrum (i.e., the generalized dimension \( D_r(q) \) as a function of \( q \)) reflects the different behaviour and spatial distribution of large massive clusters and small background particles. In fact, an evolution with respect to the initial conditions is present only for \( q \geq 2 \), that is, only when large masses dominate the statistics.

The dependence from the initial condition is as expected; increasing the initial correlation, produces a stronger dishomogeneity of the system, expressed by the lower values of \( D_r(q) \) at large \( q \).

As for the time behaviour, the system reaches an asymptotic regime characterized by the presence of a plateau for \( D_r(q) \) with \( q \geq 2 \). Such evolution is analogous to that of the fractal dimension \( D_m \) in the mass space on the same time scale.

Thus, the phase transition characteristic of gelling systems is marked by analogous behaviours in the mass and in the coordinate space. Such behaviours affect each other: the concentration of particles in strongly correlated regions enhances the probability of producing large clusters; on the other hand, the presence of large clusters enhances the merging probability (which is proportional to \( M^\lambda \)) with other clusters, thus changing the local spatial distribution.

The non-gelling systems show a smoother behaviour both in the mass and in the coordinate space. In view of the previous considerations, this is due to the fact that the lower \( M \)-dependence of the aggregation probability not only produce less massive objects, but also affects the spatial distribution in a minor way.

Thus, in this case, the mass distribution is almost homogeneous, while no time evolution of the space distribution is present. Such a result is the signature of similar aggregation behaviour and clustering of large and small clusters, which have, however, always comparable masses.

A few comments about the relation with physical systems are in order: the physical time scale is given by \( 1/ \Sigma V = t_e/(N/R^3) \Sigma R \) with \( t_e = R/V \), where \( R \) and \( V \) are now the physical size of the system and the physical particle velocity, respectively. The assignment of some computing units to \( V \) and \( R \), for a given number of particles \( N \), provides a relationship between the computing and the physical time scale, relative to the particular system in consideration. As an example, the critical time \( t_c \) resulting from the simulations for a of group of galaxies with \( R \approx 250 \) kpc and galaxy velocity dispersion \( V \approx 500 \) km/s, containing a number \( N = 10^3 \) galaxies, is of the order of \( 5 \times 10^5 \) yrs.

Our results compare interestingly to some astrophysical problems.

For example, they are analogous to those usually found in cosmology for the different matter and light distributions [18, 19, 20]. In this case, the luminous visible baryonic material in fully evolved cosmic structures, is located preferentially at the peaks of their density distribution and is segregated from the underlying distribution of dark collisionless material. Actually, a multifractal distribution of the luminous matter seems to fit the observations [21, 22] and the result of N-body cosmological simulations [23] are in qualitative agreement with our picture that predicts a larger clustering for more massive clusters. A closer, quantitative relation of the described picture of aggregation with the referred astrophysical problems may be proven
by extending our study to three dimensional distributions.

We believe that the present numerical approach could be easily extended also to study the statistics of very general aggregation mechanisms based on cascading (or inverse-cascading) processes.

References