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To cite this version:
Maria Trache. Fluctuation theory for an equilibrium superradiant model. Journal de Physique I, EDP Sciences, 1993, 3 (4), pp.957-967. <10.1051/jp1:1993177>. <jpa-00246776>

HAL Id: jpa-00246776
https://hal.archives-ouvertes.fr/jpa-00246776
Submitted on 1 Jan 1993

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Classification

Physics Abstracts
64.60

Fluctuation theory for an equilibrium superradiant model

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(Received 19 August 1992, accepted in final form 14 December 1992)

Abstract. — Some critical properties for static and dynamic equilibrium superradiant models are discussed, within a Gaussian fluctuation theory. The temperature critical region and the λ-critical region are calculated and the phase diagram of the superradiant phase transition is obtained.

1. Introduction.

Large fluctuations of the field variables are expected in the neighborhood of a phase transition. They influence the behavior of the thermodynamic quantities and lead to deviations of the critical exponents from their mean field values, as predicted for low dimensional models [1, 2]. The Gaussian Theory of Fluctuations (GTF) is an usual approach to obtain information about the critical region: the interval of external parameters of the system for which the Mean Field Theory (MFT) fails and the fluctuations cannot be neglected. The Ginsburg criterion or similar analogous statements can be applied to calculate this critical region [1-4].

The superradiant model belongs to a large class of models used in Quantum Optics [5]. Its critical properties have been previously studied within a Mean Field Theory [6-11]. Beyond the mean field behavior, the model presents statistical [12, 13] and quantum fluctuations [14, 15]. For an equilibrium model, the quantum fluctuations of the field variables are of thermal origin. The Method of Functional Integration (MFI) [11, 12, 16] is a suitable formalism to include the effect of both types of fluctuations.

The aim of this paper is to discuss comparatively the validity of the MFT for a static and a dynamic superradiant model and to estimate the critical regions. Thus, in section 2, the formal interpretation of the superradiant phase transition is used, in order to calculate the specific heat, for both models, within the Gaussian approximation [6-10, 13, 15]. This section contains a necessary review of results previously obtained.

In section 3 the Ginsburg criterion is applied, both for the static and the dynamic superradiant model [13, 17], to estimate the T-critical regions (where T is the temperature of the system).

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In section 4, the formal application of the second order phase transition theory to a superradiant model is replaced with a new description supported by the experiments on the superradiant effect in cavities [18]. Thus, the $N$-critical region (where $N$ is the number of excited atoms in the cavity) or equivalently, the $\lambda$-critical region (where $\lambda$ is the coupling constant between atoms and radiation) are calculated, at a fixed temperature of the cavity.

Section 5 contains conclusions about the phase diagram of the superradiant model.

### 2. The superradiant phase transition.

Superradiance is the collective effect of spontaneous emission exhibited by a system of $N$ two-level atoms prepared in a fully excited state.

We shall discuss a quasipunctiform model whose characteristic length is less than the wavelength of the spontaneous emitted field and for which the propagation phenomena are neglected. Within the electric dipole approximation and coupling the atoms with a single resonant mode of the electromagnetic field, the Hamiltonian is expressed in the second quantization formalism as:

$$ H = \omega a^+ a + \frac{e}{2} \sum_j (c^+_j c_j - c^+_j c^+_j) - \frac{i \lambda}{\sqrt{2} \omega N} \sum_j (a - a^+)(c^+_j c_j + c^+_j c^+_j) $$

(1)

where

- $a^+ (a)$ are the bosonic operators of the resonant mode of energy $\omega$,
- $c^+_j, c_j$ are the fermionic operators associated with the $j$-th atom of energies $\pm e/2$,
- $\lambda = D_{12} \omega \left( \frac{N}{\epsilon_0 V} \right)^{1/2}$ is the coupling parameter, $D_{12}$ is the electric dipole matrix element,
- $\epsilon_0$ is the permittivity and $V$ is the effective volume which contains $N$ fixed emitters.

The statistical properties of the model are obtained from the partition function written as a functional integral over the complex bosonic and the Grassmann fermionic variables [16]:

$$ Z/Z_0 = \frac{\int \mathcal{D}a^+ \mathcal{D}a \mathcal{D}c^+ \mathcal{D}c \exp S}{\int \mathcal{D}a^+ \mathcal{D}a \mathcal{D}c^+ \mathcal{D}c \exp S_0} $$

(2)

The functional $S$ is the «quasiclassical» action, defined by:

$$ S = \int_0^\beta d\tau \left[ a^+ (\tau) \partial_\tau a(\tau) + \sum_{j,a} c^+_j (\tau) \partial_\tau c_j (\tau) - H(\tau) \right] $$

(3)

where $\tau$ is the Matsubara imaginary time and $\beta = 1/(kT)$, $T$ is the temperature and $k$ is the Boltzmann constant. In equation (3), $\alpha = 1, 2$ and $S_0 = S(\lambda = 0)$. Because explicit calculations of the effective actions in MFI, both for the static and the dynamic model are already done by many authors [11-13, 15], only a few steps leading to the results will be pointed out further.

In order to discuss the phase transition supposed to occur as an ordered state over the bosonic microstates associated with the electromagnetic field, the guess of the relevant variable for the critical behavior is

$$ A(\tau) = \frac{i}{\sqrt{2} \omega N} [a(\tau) - a^+ (\tau)] $$

(4)
The integration over all the other variables [11, 12, 16] leads to an effective « action » of the critical model

\[ S_{\text{eff}} \{ A(\tau) \} = -\frac{N}{2} \int_0^\beta d\tau \left[ \sigma^2 A^2(\tau) + (\partial_A A)^2 \right] + N \text{Tr}_\tau \ln \left[ 1 + \lambda^2 G_2 A G_1 A \right] \]

(5)

where \( G_{1,2}(\tau - \tau') \) are the fermionic Green functions and the trace \( \text{Tr}_\tau \) indicates the integration over all the « quasiclassical » trajectories described by the continuous parameter \( \tau (\tau \in [0, \beta]) \). The Fourier transforms \( b(\omega_n) \) of the bosonic field variable \( A(\tau) \) are introduced and the trace \( \text{Tr}_\tau \) is estimated up to the second order of the coupling parameter \( \lambda [15] \).

If the contributions of the static mode \( \eta = b(\omega_n = 0) \) and of the dynamic ones \( b(\omega_n \neq 0) \) are separated and all the correlations between field variables are neglected, the approximate equation of the effective action is obtained:

\[ S_{\text{eff}}(\eta, \{ b(\omega_n) \}) = -\frac{N \beta \omega^2 \eta^2}{2} + 2N \ln \text{ch} (\beta \omega_F/2) - \frac{N}{2} \sum_{n \neq 0} [\omega^2 + \omega_n^2 + (2 \lambda^2 / \beta) K(\omega_n)] b(\omega_n) b(-\omega_n) + \]

(6)

where

\[ \omega_F = \left( (\epsilon/2)^2 + \lambda^2 \eta^2 \right)^{\frac{1}{2}} \]

(7)

and \( K(\omega_n) \) is calculated as a sum over the fermionic frequencies. As such, equation (6) is valid for the dynamic model. It has the form of a Ginzburg Landau functional in the dynamic modes, while the perturbation series in the static mode are exactly performed. The static model keeps only the \( \omega_n = 0 \) term of the Fourier expansion of the field variables [13] and therefore, the corresponding equation for the action is exact

\[ S_{\text{eff}}(\eta) = -\frac{N \beta \omega^2 \eta^2}{2} + 2N \ln \text{ch} (\beta \omega_F/2) . \]

(8)

The mean field equations for the dynamic model are

\[ b(\omega_n) = 0 \quad \eta = 0 \quad \text{for} \quad T > T_c \]

and

\[ b(\omega_n) = 0 \quad \text{th} \frac{\beta \omega_F}{2} = \frac{\omega^2 \omega_F}{\lambda} \quad \text{for} \quad T < T_c , \]

(9)

solutions which assure the stability condition for the functional \( S_{\text{eff}} \) in the specified intervals of temperature. The critical temperature \( T_c \) is defined as the solution of the equation:

\[ \text{th} \frac{\beta_c \epsilon}{4} = \frac{\omega^2 \epsilon}{2 \lambda^2} \]

(10)

as usually obtained in the theory of phase transitions.

Within a mean field theory, the dynamic and the static models give the same results for the thermodynamic functions and for their critical behavior, as the field variable involved is only the static one. Thus, a typical result for a second order phase transition which is obtained is the jump of the specific heat at \( T = T_c \):
\[ \Delta C = N e^3 \left[ 32 k^2 T_c^2 \text{ch}^2 z \left( \text{th} \frac{z}{\text{ch}^2 z} \right) \right]^{-1} \]  

(11)

where \( \frac{z}{4} \)

The role of fluctuations in low dimensional models is suggested by the results of a Gaussian theory, which is further discussed for the superradiant model.

For both sides of the critical temperature, the effective actions (Eq. (6) and Eq. (8)) are expanded around the mean field solutions of the field variables up to the second order. The effective actions will yield the form of Ginsburg-Landau functionals in all the field variables and the last integrals in the partition functions can be performed. The corresponding thermodynamic potential will consequently contain several contributions:

- a mean field term showing an extensive behavior \( \sim N \), the same for both models;
- nonsingular Gaussian contributions due to the higher modes \( \omega_n \neq 0 \), which become negligible in the thermodynamic limit and are present only in the dynamic model;
- singular contributions of the lower modes \( \omega_n = 0 \) which proceed from the neighborhood of the static one; this vicinity contains both the \( \omega_n = 0 \) term \( n = 0 \) and small terms \( \omega_n = 2 \pi n k T \neq 0 \), obtained from \( n \neq 0 \), when \( T \rightarrow 0 \). This last contribution would also be negligible in the thermodynamic limit if \( T \) were far from the critical temperature but becomes large inside the critical region, even for large values of the number of particles \( N \).

The singularities are differently introduced for the two models, in connection with their dimension. If the static critical model has the behavior of a 0-dimensional model, the dynamic model will be a 1-dimensional one, due to the supplementary dimension introduced by the Matsubara imaginary time [14, 15].

For both models, the singular contribution in the specific heat becomes, respectively:

\[ C_s^{(0)} = k/2 T_c^2 |T - T_c|^{-2} \]  

(12)

as in reference [13] and

\[ C_s^{(1)} = \frac{A}{16 \sqrt{2k}} (\text{ch} z)^{-1} |T - T_c|^{-3/2} \]  

(13)

as in reference [15].

Therefore, the critical exponent of the specific heat changes from \( \alpha_\delta^{(0)} = 2 \) in the static model, to \( \alpha_\delta^{(1)} = 3/2 \) in the dynamic model, both Gaussian results satisfying the Josephson rule [2]:

\[ \nu d = 2 - \alpha \]  

(14)

where \( \nu = 1/2 \) is the critical exponent of the correlation length.


The critical region is the interval of the parameter values where the MFT is no longer valid. It can be evaluated by the Ginsburg criterion [1, 3]: if the temperature is chosen as a control parameter, the temperature critical region is expressed as:

\[ (\Delta T)_{ct} = \zeta T_c \]  

The parameter \( \zeta_T \) can be calculated as the ratio between the singular term (introduced by
fluctuations) in the specific heat \( C_s \) and the jump of the same quantity \( \Delta C \) at \( T = T_c \). The 
equation for \( C_s \) will include only the term of maximum singularity of the specific heat, as 
already considered in equation (12) and equation (13). The same critical exponents are 
obtained for both sides of the critical point.

For a static model, the Ginsburg criterion states:

\[
\frac{C_s^{(0)}}{\Delta C} = \left( \frac{\xi_T^{(0)}}{|T - T_c|/T_c} \right)^{2} \tag{15}
\]

and for a dynamic model:

\[
\frac{C_s^{(1)}}{\Delta C} = \left( \frac{\xi_T^{(1)}}{|T - T_c|/T_c} \right)^{3/2} \tag{16}
\]

We intend to identify the parameter \( \xi_T \) from the direct calculation of the specific heat, 
already done. In reference [13], only a maximum value of the critical region for a static model 
was calculated and some conclusions about the role of fluctuations were drawn. In this paper, 
a systematic evaluation of fluctuations in the second order of the perturbation theory and 
comparative results for both models are expected.

Using equations (11, 12) and (13), one obtains

\[
\xi_T^{(0)} = \frac{\sqrt{2}}{\sqrt{N}} \left[ \ln \frac{1 + x}{1 - x} \right]^{-3/2} (1 - x^2)^{-1} \left[ x - \frac{1 - x^2}{2} \ln \frac{1 + x}{1 - x} \right]^{1/2} \tag{17}
\]

where

\[
x = \text{th} \frac{\beta_c \varepsilon}{4} = \frac{N_0}{N}. \tag{18}
\]

We have defined \( N_0 \) as a critical number of atoms whose minimum value assures the 
occurrence of the phase transition at the minimum possible temperature, \( T_c = 0 \) K:

\[
N_0 = \frac{\varepsilon_0 \varepsilon V}{2 D_{12}^2}
\]

For the dynamic model, the same quantity \( \xi_T \) is given by the equation

\[
\xi_T^{(1)} = \frac{(xN^2)^{-1/3}}{2} \left[ \ln \frac{1 + x}{1 - x} \right]^{-1} (1 - x^2)^{-1} \left[ x - \frac{1 - x^2}{2} \ln \frac{1 + x}{1 - x} \right]^{2/3} \tag{19}
\]

The parameters \( \xi_T \) depend not only on the adimensional quantity \( x \), but explicitly on the 
number of particles \( N \).

If the equation of the critical temperature expressed with the same parameter \( x \) is used:

\[
T_c = \frac{\varepsilon}{2k} \left[ \ln \frac{1 + x}{1 - x} \right]^{-1}, \tag{20}
\]

the temperature critical regions are obtained:

\[
(\Delta T)_{ct}^{(0)} = \frac{\varepsilon}{k \sqrt{2N}} \left[ \ln \frac{1 + x}{1 - x} \right]^{-5/2} (1 - x^2)^{-1} \left[ x - \frac{1 - x^2}{2} \ln \frac{1 + x}{1 - x} \right]^{1/2} \tag{21}
\]

\[
(\Delta T)_{ct}^{(1)} = \frac{\varepsilon}{4kN^{2/3}} (x)^{-1/3} \left[ \ln \frac{1 + x}{1 - x} \right]^{-2} (1 - x^2)^{-1} \left[ x - \frac{1 - x^2}{2} \ln \frac{1 + x}{1 - x} \right]^{2/3} \tag{22}
\]
Approximate dependences on the number of particles $N$ are obtained for extreme values of $x$, namely:

- in the static model:
  
  $$(\Delta T)^{(0)}_{cr} \sim \frac{\epsilon}{k} \frac{\sqrt{N}}{N_0} \quad \text{for } x \to 0, \quad \text{and}$$  
  
  $$(\Delta T)^{(0)}_{cr} \sim \frac{\epsilon}{k} \frac{\sqrt{N}}{N - N_0} \quad \text{for } x \to 1$$

and

- in the dynamic model:
  
  $$(\Delta T)^{(1)}_{cr} \sim \frac{\epsilon}{k} (N N_0)^{-1/3} \quad \text{for } x \approx 0, \quad \text{and}$$  
  
  $$(\Delta T)^{(1)}_{cr} \sim \frac{\epsilon}{k} \frac{N^{1/3}}{N - N_0} \quad \text{for } x \approx 1.$$  

While the critical temperature $T_c$ is only a function of $x$, which is already a result of the MFT, the temperature critical region depends explicitly on the number of particles and on the dimension of the model. This conclusion is particularly important for finite systems, which have to show large fluctuations over a wide temperature interval around the critical point. The mean field results for different physical quantities are no longer dominant, as soon as $(\Delta T) < (\Delta T)_{cr}$.

For macroscopic systems ($N \gg \infty$), excepting extreme values of $x$, the critical region tends to zero and the MFT is valid.

If one compares the size of the temperature critical regions with the critical temperature itself, one sees that in the neighborhood of $x = 1$ ($T_c \approx 0$, or $N \approx N_0$) both models show large critical regions. The fluctuations however seem to decrease for the dynamic model, because

$$\frac{(\Delta T)^{(0)}_{cr}}{(\Delta T)^{(1)}_{cr}} = \frac{N^{1/6}}{[\ln (1 - x)]^{1/2}} \quad (x \approx 1).$$

Otherwise, for $x \approx 0$ ($N \gg N_0, T_c \gg \infty$), the critical region remains large for the static model, but is strongly diminished for the dynamic one. The ratio:

$$\frac{(\Delta T)^{(0)}_{cr}}{(\Delta T)^{(1)}_{cr}} = \frac{N^{5/6}}{N_0^{1/3}} \gg 1.$$  

The increase of the dimension of the model leads to the reduction of the effect of fluctuations, excepting the range of small critical temperatures, when the number of particles $N$ tends also to its minimum value $N_0$. Figure 1 shows both the dependence of the critical temperature and of the temperature critical regions versus parameter $x$.

Similar results concerning the equations of the temperature critical regions are obtained by comparing the mean field value of the order parameter (the slowest field variable $\eta$) with its corresponding dispersions [4], calculated for each model, in the neighborhood of the critical temperature. The dispersion of the order parameter being expressed with the second order derivative of the effective action

$$\langle (\Delta \eta)^2 \rangle = \left| \frac{\partial^2 S_{eff}}{\partial \eta^2} \right|^{-1}$$  

(27)
Fig. 1. — a) The critical temperature $kT_c/\varepsilon$ (curve 1). The static critical region $k(\Delta T)_c^{(0)}/\varepsilon$ (curve 2). The dynamic critical region $k(\Delta T)_c^{(1)}/\varepsilon$ (curve 3); as a function of the variable $x = N_s/N$, for $N = 2$; b) Static critical region for $N = 2$ (curve 1); for $N = 200$ (curve 2); Dynamic critical region for $N = 2$ (curve 3) for $N = 200$ (curve 4); c) A detail of figure 1b and of the critical temperature curve (the crossing curve) in the neighborhood of $x = 1$. 
it shows large singularities on both sides of the critical point. As soon as the dispersion exceeds the mean value of the order parameter, the temperature critical region is reached (Fig. 2).

Fig. 2. — The temperature dependence of the order parameter \( \eta \) and of its dispersion \( \langle (\Delta \eta)^2 \rangle \) for different values of the coupling constant \( \lambda \).

4: The \( \lambda \)-critical region.

The theoretical question on the validity of the MFT and the role of fluctuations is naturally correlated with the experimental observation of the phase transition. Only this one is able to suggest how the critical behavior is reached and which are the natural parameters to check.

The second order phase transitions, typically met in condensed matter physics, use the temperature as a natural control parameter. The behavior of response functions, as the specific heat or susceptibilities, announces the instability and the occurrence of a phase transition.

The occurrence of the superradiant effect [18, 19], as well as that of other collective effects studied in Quantum Optics, is quite different. A macroscopic superradiant state is maintained in a cavity at a fixed temperature. The ordered state (which is observed as an oscillatory dynamic behavior of the spontaneous emitted field) is reached only if a sufficiently large number of emitters are present in the cavity of a given volume. This implies a necessary large coupling constant \( \lambda \), between atoms and the emitted field. Therefore, a more natural « external » parameter seems to be the number of atoms \( N \), or the coupling parameter \( \lambda \). This leads to an equivalent description of the superradiant phase transition. For the purpose of this paper it is sufficient to formulate the theory in the \( \lambda \) representation, that is, to express both the order parameter of the transition and the critical region using \( \lambda \) as a control parameter.

For a given temperature \( T (= T_{\text{cav}}) \), if \( \lambda \Rightarrow \lambda_c \) the expansion of the nontrivial mean field equation (Eq. (9)) around \( \eta = 0 \) is obtained:

\[
\eta = \frac{\omega e^{3/2}}{\sqrt{2} \lambda_c^{5/2}} \left[ \frac{\theta((\beta e/4) - \frac{\beta e}{4 \cosh^2(\beta e/4)})}{(\lambda - \lambda_c)^{1/2}} \right]^{-1/2} \tag{28}
\]

The critical coupling is obtained from equation 9, for \( \eta = 0 \):

\[
\lambda_c^2 = \frac{\varepsilon \omega^2}{2 \theta(\beta e/4)} \tag{29}
\]
and has a minimum value $\lambda_c^{\text{min}} = (\varepsilon \omega^2/2)^{1/2}$, if $T_{\text{cav}} = 0$ K. The critical number of particles $N_0$ introduced by equation (18) corresponds to the minimum critical value of the coupling. For finite temperatures of the cavity, the critical coupling increases, so that the condition imposed on the number of particles assuring the occurrence of the phase transition is more restrictive.

The dispersion of the order parameter can also be identified, using the Ginsburg-Landau expansion of the effective action around the mean field solution. It is easier to write the second derivative coming from the disordered state ($\lambda < \lambda_c$); the dispersion of the order parameter will show a singular behavior when $\lambda \Rightarrow \lambda_c$.

\[
\frac{\delta^2 S_{\text{eff}}}{\delta \eta^2} \bigg|_{\eta = 0} = -\frac{N \beta \omega^2}{\lambda^2} (\lambda_c^2 - \lambda^2) \quad \text{and} \\
\langle (\Delta \eta)^2 \rangle |_{\lambda = \lambda_c} = \frac{\lambda_c}{2 N \beta \omega^2 \Delta \lambda}
\]

In order to obtain the $\lambda$-critical region, we impose the condition of the validity of MFT, that is:

\[
\eta^2 \gg \langle (\Delta \eta)^2 \rangle.
\]

One obtains:

\[
\Delta \lambda \gg \frac{\lambda_c^3}{\sqrt{N \beta \omega^2} \varepsilon^{1/2}} \left[ \text{th} \left( \frac{\beta \varepsilon}{4} \right) - \frac{\beta \varepsilon}{4 \text{ch}^2(\beta \varepsilon/4)} \right]^{1/2}
\]

and identify the $\lambda$-critical region, as the r.h.s. of the inequality (33).

With the parameter $z = \beta \varepsilon/4$ already introduced, one obtains:

\[
(\Delta \lambda)_{\text{cr}} = \frac{\lambda_c}{4 \sqrt{N z}} \left( \text{th} z \right)^{-1} \left[ \text{th} z - \frac{z}{\text{ch}^2 z} \right]^{1/2}
\]

(the connection between $\lambda_c$ and $z$ is given by Eq. (29)).

In figure 3 the curves of the order parameter for different temperatures $T$ and the fluctuations in the neighborhood of the critical coupling value are shown. The family of curves is limited by

![Fig. 3. — The $\lambda$-dependence of the order parameter $\eta$ and of its dispersion $\langle (\Delta \eta)^2 \rangle$, for different values of the cavity temperature.](image-url)

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that of $T_{\text{cav}} = 0$, which starts at $\lambda_{\text{c}}^{\text{min}}$. The curves showing the $\lambda$-dependence of the order parameter are going to infinity and there is a region of the parameter $\lambda < \lambda_{\text{c}}^{\text{min}}$ where a nontrivial solution can never exist.

The $\lambda$-critical region (Eq. (34)) still depends on the temperature of the cavity and on the number of particles $N$, as the $T$-critical region was dependent on the coupling constant $\lambda$ and the number of particles $N$.

Equations (10) and (29) which are similar, represent equations of an equilibrium curve. Using any of the two, a direct equation between the variations of the temperature $T$ and coupling constant $\lambda$ could be obtained, as:

$$\Delta \lambda = \frac{\lambda^2}{4 kT^2 \omega^2 \text{ch}^2(\beta \epsilon/4)} |\Delta T| .$$  \hfill (35)

It gives the connection between the two critical regions. The equation (35) has the form of a Clapeyron-Clausius equation.

5. Conclusions.

The above discussion suggests two equivalent descriptions of the superradiant phase transition. The temperature parameter $T$ or the number of particles $N$ (connected with the $\lambda$ coupling parameter) are possible control parameters of the critical behavior. The experimental conditions lead to the choice of the more natural description and we emphasize that, for the superradiant effect in cavities, the $\lambda$-description is more suitable. A $N$-description is generally convenient for systems with a variable and finite number of particles.

The phase diagram of the superradiant phase transition is viewed in figure 4; the equilibrium curve $\lambda = \lambda (T)$ is separating the ordered and disordered states. The curve is extended to infinity, but has a critical end point at $\lambda_{\text{c}}^{\text{min}} = (\omega^2 \epsilon/2)^{1/2}$ and $T_{\text{c}}^{\text{min}} = 0$. The infinite extension of the curve is an argument for a second order phase transition, as the jump of the specific heat and the continuity of entropy have also been indicated. The plane of parameters is thus separated and the transition between the ordered and disordered states is possible only crossing the equilibrium curve.

![Fig. 4. — The $\lambda$-$T$ phase diagram for the superradiant phase transition: (o) ordered states; (d) disordered states.](image-url)
Acknowledgment.

The hospitality of the laboratory Aimé Cotton (Orsay) during the final stage of this work is warmly acknowledged.

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